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INDISTINGUISHABILITY OF THE COMPONENTS OF RANDOM SPANNING FORESTS[1]

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We prove that the infinite components of the Free Uniform Spanning Forest (FUSF) of a Cayley graph are indistinguishable by any invariant property, given that the forest is different from its wired counterpart. Similar result is obtained for the Free Minimal Spanning Forest (FMSF). We also show that with the above assumptions there can only be 0, 1 or infinitely many components, which solves the problem for the FUSF of Caylay graphs completely. These answer questions by Benjamini, Lyons, Peres and Schramm for Cayley graphs, which have been open up to now. Our methods apply to a more general class of percolations, those satisfying "weak insertion tolerance", and work beyond Cayley graphs, in the more general setting of unimodular random graphs.

1. Introduction. We prove indistinguishability and 1-infinity laws for the components (*clusters*) of random spanning forests of Cayley graphs, given that the forest has a property that we call weak insertion tolerance (see Definition 1), and it has a tree with infinitely many ends. The perhaps most important examples of random forests that satisfy weak insertion tolerance are the Free and the Wired Uniform Spanning Forest (FUSF and WUSF) and the Free and the Wired Minimal Spanning Forest (FMSF and WMSF); see Definitions 2, 3. The importance and some main properties of the uniform and minimal spanning forests are explained in [8].

Say that a graph G is quasitransitive, if its group of automorphisms has finitely many orbits on G. Say that G is unimodular, if for any two vertices $x, y \in V(G)$ on the same orbit, $|S_x(y)| = |S_y(x)|$, where S_x is the set of all automorphisms fixing x, and $S_x(y)$ is the orbit of y by S_x. In particular, every Cayley graph is transitive and unimodular; see [8] for more details.

In particular, the following theorems are proved.

THEOREM 1.1. *Suppose that the FUSF and WUSF are different for some unimodular quasitransitive graph G. Then the following hold*:

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1. *The FUSF has either* 1 *or infinitely many components*.
2. *Every component of the FUSF has infinitely many ends*.
3. *More generally*, *no two components of the FUSF can be distinguished by any invariantly defined property.*

The condition FUSF $\neq$ WUSF is equivalent to that there exist nonconstant harmonic Dirichlet functions on G, or, in different terms, that the first L^2 Betti number is nonzero. This was shown by Benjamini, Lyons, Peres and Schramm; see [4].

THEOREM 1.2. *Suppose that the FMSF and WMSF are different for some unimodular quasitransitive graph G. Then the following hold*:

1. *The FMSF has either* 1 *or infinitely many components*.
2. *Every component of the FMSF has infinitely many ends*.
3. *More generally*, *no two components of the FMSF can be distinguished by any invariantly defined property.*

The condition FMSF $\neq$ WMSF is equivalent to $p_c < p_u$, as shown by Lyons, Peres and Schramm. Here, p_c and p_u are respectively the critical probability and uniqueness critical probability for Bernoulli percolation on G. The condition $p_c < p_u$ is conjecturally equivalent to G being nonamenable, and is known to hold for *some* Cayley graph of every nonamenable group; see [9] for more details.

All the results of this paper, including Theorems 1.1 and 1.2, remain valid if G is a *unimodular random graph*. See [1] for the definition of this notion, which includes all unimodular quasitransitive graphs (or more generally, invariant random subgraphs of a unimodular quasitransitive graph). We present the proofs for unimodular quasitransitive graphs because this setting is more widely known. Remark 4.4 describes the extra details needed for the proofs to be applied to a unimodular random G.

The above theorems follow from Lemma 1.3, Corollary 1.5 and Theorem 3.3 (Part 1), Theorem 3.1 (Part 2), Theorem 4.3 (Part 3). One needs that the uniform and the minimal spanning forests are ergodic, which were proved in [4] and [9], respectively. What needs to be further added is that FUSF $\neq$ WUSF implies that some tree of FUSF has infinitely many ends; and similarly for the minimal spanning forest. For the uniform spanning forest this is true by Proposition 10.11 in [4] and for the minimal spanning forest this is part (e) of Proposition 3.5 in [9].

Theorems 1.1 and 1.2 resolve questions asked by Benjamini, Lyons, Peres and Schramm [4] and by Lyons, Peres and Schramm [9]. Part 1 in Theorem 1.1 answers Question 15.6 in [4], Part 2 answers Question 15.8 for the case when the transitive graph is unimodular, while Part 3 confirms Conjecture 15.9 in the same paper for the case of FUSF when FUSF $\neq$ WUSF. Part 2 of Theorem 1.2 was Question 6.7 in [9] and was answered in [13] using a different method as here. Parts 1 and 3 answer Question 6.10 and Conjecture 6.11, respectively, for the case of FMSF when FMSF $\neq$ WMSF. Chifan and Ioana ([5], Corollary 9) proved that there are at most countably many types of indistinguishability for the components, using operator algebraic techniques and the result of [13] that the number of ends is the same in every component. The conjecture on the indistinguishability of FMSF-clusters was restated by Gaboriau and Lyons in [6], because in case of a positive answer (as provided by Theorem 1.2), the FMSF can serve as the treeable ergodic subrelation in their construction (Proposition 13 of [6]) for some Cayley graph of the given group. After finishing the first draft of this manuscript, we learned that Hutchcroft and Nachmias gave an independent proof, about the same time, to the results of Theorem 1.1, for transitive unimodular graphs [7]. Their paper further shows the same conclusions for the WUSF.

Let G be the underlying unimodular quasitransitive graph (such as a Cayley graph), with vertex set $V(G) = V$ and edge set $E(G) = E$. Denote by d the (maximal) degree in G. Let $\mathrm{dist}(u, v)$ be the distance between u and v in G, where $u, v \in V \cup E$. Denote by $B(x, r)$ the ball of radius r around x in G, that is, the set of points at distance at most r from x and all the edges induced by them. Use notation $B_\Gamma(x, r)$ for the ball of radius r around x in some given subgraph Γ of G. Denote the edge-boundary of an induced subgraph $H \subset G$ by ∂H: this is the set of edges with one endpoint in H and the other endpoint outside of it. Given some percolation (random subgraph) ω on G, the component of a given vertex x will be denoted by C_x. (We hide the dependence of C_x on ω for the ease of notation.) Given $e \in E$, $f \in E \cup \{\varnothing\}$ and a configuration $\omega \in 2^E$, let $\pi_e^f \omega := \omega \cup e \setminus f$. For an event A, let $\pi_e^f A := \{\omega \cup e \setminus f : \omega \in A\}$. Denote the complement of a set A within some superset (clear from the context) by A^c. We will use $\mathbf{P}$ for different probability measures in the paper, but its meaning will be always clear from the context. We will use $\mathbf{E}$ for the expectation of $\mathbf{P}$.

A percolation is called insertion tolerant (see [10]), if one can insert a fixed edge to each configuration of a given event A and obtain an event of positive probability after the insertion, provided that the original event had positive probability. The key property needed for our proofs is a weak form of insertion tolerance, as given in the next definition. Informally, this is the following modification of the "usual" notion of insertion tolerance. First, one may assume that the event A is such that the endpoints of the edge e are in distinct components on A. (This is not a real constraint, since in applications one usually wants to insert e if it is between two components.) Then we can insert e to the configurations in A, but at the cost of possibly deleting another edge f. Furthermore, this f can be chosen for any fixed $r \geq 0$ to be at distance greater than r from e, and it can be chosen so that it is in the component of a previously fixed endpoint x of e.

DEFINITION 1 (Weak insertion tolerance, WIT). Let $\mathcal{F}$ be a random forest of a unimodular quasitransitive graph G. Suppose that for any $\{x, y\} = e \in E(G)$, r nonnegative integer and configuration ω such that x and y are in different components, there exists an $f = f(\omega, e, x, r) \in E(G) \cup \{\varnothing\}$ such that the following properties hold. Fixing e, x, r and looking at f as a function of ω, it is measurable. If A is such that $\mathbf{P}(A) > 0$ and for almost every configuration in A, $C_x \neq C_y$, then $\mathbf{P}(\pi_e^f A) > 0$. Furthermore, if $f \neq \varnothing$ then f is in $C_x \cap B(x,r)^c$ almost surely. Then we say that $\mathcal{F}$ is weakly insertion tolerant (WIT).

Suppose that G is an infinite graph and $G_n \subset G$ is an exhausting sequence of connected finite graphs. Let $\hat{G}_n$ be obtained from G_n by adding an extra vertex z_n to it, and replacing every edge of the form $\{x, y\} \in E(G)$, $x \in V(G_n)$, $y \notin V(G_n)$, by a copy of the edge $\{x, z_n\}$.

DEFINITION 2 (Uniform Spanning Forest). Let G be an infinite graph and $G_n \subset G$ be an exhausting sequence of connected finite induced subgraphs. Let T_n be a uniformly chosen spanning tree of G_n, and $\hat{T}_n$ be a uniformly chosen spanning tree of $\hat{G}_n$. Pemantle showed that the weak limits of T_n and of $\hat{T}_n$ exists [12]. The first one is called the Free Uniform Spanning Forest (FUSF) of G; the second one is called the Wired Uniform Spanning Forest (WUSF) of G.

DEFINITION 3 (Minimal Spanning Forest). Let G be an infinite graph and λ be an i.i.d. labelling of its edges by Lebesgue$[0,1]$ labels. Delete each edge from G if its label is maximal in some (finite) cycle of G, and call the remaining almost sure random forest $F_{\text{free}}(\lambda)$, the Free Minimal Spanning Forest (FMSF) of G. Alternatively, delete each edge from G if its label is maximal in some cycle or biinfinite path of G, and call the remaining random forest $F_{\text{wired}}(\lambda)$, the Wired Minimal Spanning Forest (WMSF) of G.

We mention that the WMSF and FMSF can equivalently be defined using an exhausting sequence of finite graphs; see [9] for the details.

LEMMA 1.3. *The Free Uniform Spanning Forest and the Wired Uniform Spanning Forest are weakly insertion tolerant. Moreover, there exists a uniform* $\delta(r)$ *such that for any* A, e, $\mathbf{P}(\pi_e^f A) > \delta(r)\mathbf{P}(A)$. *Here,* f *is defined as in Definition* 1.

For a transitive graph G and $p \in [0,1]$, denote by $\theta(p, G)$ the probability that o is in an infinite component of Bernoulli(p) edge percolation, where $o \in V(G)$ is some fixed vertex. Whether $\theta(p_c, G) = 0$, is a central open problem, known to be true when G is nonamenable unimodular [3]. This, together with the fact that FMSF$\neq$WMSF can only happen when G is nonamenable [9], implies the corollary of Lemma 1.4 below.

LEMMA 1.4. *Let* G *be unimodular and quasitransitive, such that* $\theta(p_c, G) = 0$. *The Free Minimal Spanning Forest and the Wired Minimal Spanning Forest are weakly insertion tolerant.*

COROLLARY 1.5. *Let G be unimodular and quasitransitive. Suppose* FMSF $\neq$ WMSF. *Then the FMSF and the WMSF are weakly insertion tolerant.*

A standard tool in the study of percolations on transitive unimodular graphs (such as Cayley graphs) is the so-called Mass Transport Principle (MTP). In brief, it says that if x sends mass $\phi(\omega, x, y)$ to y and this mass transport function is diagonally invariant, then the expected total mass $\mathbf{E}\sum_y \phi(\omega, x, y)$ sent out by x is the same as the expected total mass $\mathbf{E}\sum_y \phi(\omega, y, x)$ received by x. The most typical use of the MTP is that there is no way to assign some vertex to each vertex in an invariant way such that some vertex is assigned to infinitely many other vertices with positive probability; see Section 8.1 in [8] for more details and the history of the MTP.

2. Uniform and Minimal Spanning Forests are WIT. The perhaps most important examples of weakly insertion tolerant forests are the Uniform and the Minimal Spanning Forests. For the latter case, we are only able to prove WIT if we assume $\theta(p_c, G) = 0$, although a weaker version of WIT holds in full generality, namely, if we do not require $f(\omega, e, x, r)$ to be in C_x. See Remark 2.2 for an explanation of what benefits and losses there would have been of such an alternative definition of WIT.

First, we give some intuitive reasoning about what makes the statements be true for the USF and the MSF.

In case of the USF, the proof is based on the following observation. Suppose that G is not infinite, just a large finite graph, and that we consider its uniform spanning *tree* T. Let e be an edge, P a simple path between its endpoints. Suppose that P exits the ball of radius r around e, and let f be an edge of P outside of this ball. Let A be an event such that $e \notin T$ and $P \subset T$ on A. Then $\pi_e^f A = \{\omega \cup \{e\} \setminus \{f\} : \omega \in A\}$ has the same probability as A, using the uniformity of T. The same phenomenon extends to the infinite graph.

For the MSF, we refer to a simple lemma from [9], which says that by changing the label of one edge, at most this edge and another one changes status (see our Proposition 2.1). It is not surprising that by changing the label of an edge e to a value very close to 0, e will become part of the forest, that is, we are able to insert e. By the quoted lemma, when doing this change, at most one other edge "drops out" of the forest. This is almost the f that we need in the definition of WIT. However, to ensure that f is also r-far from e, we have to work more, and possibly modify the labels of several edges. This can be thought of as a chain of edges that are forced to remain in the forest, so that the "final" and only f that drops out of the forest is far enough from e. The changes of labels mentioned above, can be performed in such a way that probablilities are distorted by at most a positive constant factor, so the condition on positive probabilities (in the definition of WIT) will be satisfied. There are some further technical difficulties that complicate the proof, but this is the main idea behind it.

PROOF OF LEMMA 1.3. Fix $e = \{x, y\}$. Denote a random forest on G by F. First, consider the case of FUSF.

Fix r. Let G_j be an exhaustion of G by finite graphs. Let T_j be a uniformly chosen random spanning tree of G_j. We may assume that every G_j contains the ball $B(x, r+1)$ of radius r around x. Denote the path from x to y in T_j by P_j. Let $D_j := \{x$ and y are in different components of $\{B(x, r+1) \cap T_j\}\}$, and let $D := \{x$ and y are in different components of $\{B(x, r+1) \cap F\}\}$. By the convergence of T_j to F, $\mathbf{P}(D_j)$ converges to $\mathbf{P}(D)$.

Recall that one can define a metric on (rooted equivalence classes of) rooted graphs, where the distance between rooted graphs (Γ_1, o_1) and (Γ_2, o_2) is $1/r$ if r is the maximal integer such that the r-neighborhoods of o_1 in Γ_1 and o_2 in Γ_2 are rooted isomorphic. This metric defines a Polish space. By the Skorokhod representation theorem, the weak convergence of T_j to F implies the existence of a coupling between $(T_n)_n$ and F such that T_n converges to F a.s. Conditioned on D_j, $P_j \cap \partial B(x, r+1) \neq \varnothing$. Let f_j be the first edge of $\partial B(x, r+1)$ when going along P_j starting from x. Since T_j converges to F, its restriction to $B(x, r+1)$ also converges to that of F a.s. In particular, f_j has a limit as $j \to \infty$. Let f be the random edge given by this limiting distribution. Note that $f \in C_x \cap \partial B(x, r+1)$.

Now assume that F is given by the FUSF. Under the map $\phi_j : T_j \mapsto T_j \cup e \setminus f_j$, every configuration has at most $|\partial B(x, r+1)|$ preimages [at most one for each potential $f_j \in \partial B(x, r+1)$]. It follows that for any event $A_j \subset D_j$, $\mathbf{P}(\phi_j(A_j)) \geq |\partial B(x, r+1)|^{-1}\mathbf{P}(A_j)$. For any $A \subset D$, one can choose an approximating sequence $A_j \to A$, $A_j \subset D_j$. Hence $\mathbf{P}(\pi_e^f A) = \lim_j \mathbf{P}(\phi_j(A_j)) \geq |\partial B(x, r+1)|^{-1} \lim_j \mathbf{P}(A_j) = |\partial B(x, r+1)|^{-1}\mathbf{P}(A)$. This completes the proof for FUSF.

A similar argument works for the WUSF with G_j replaced by $\hat{G}_j$. $\square$

Given a labelling $\lambda : E(G) \to [0, 1]$ and $e \in E(G)$, define $Z_\lambda(e) = Z(e) = \inf_C \sup\{\lambda(e') : e' \in C \setminus \{e\}\}$, where the infimum is over all cycles C in G that contain e. Depending on the context, by a cycle we may mean only finite cycles (in case of the FMSF) or finite cycles and biinfinite paths (in case of WMSF). If the infimum of the sup is attained in the definition of $Z(e)$, denote the edge e' by $\phi(e, \lambda)$ [then $\lambda(\phi(e, \lambda)) = Z(e)$], otherwise let $\phi(e, \lambda) = \varnothing$.

In the edge labellings considered, we always assume that all labels are different. This holds with probability 1 when the labels are i.i.d. Lebesgue[0, 1]. For the proof of Lemma 1.4, we will need the following observation.

PROPOSITION 2.1. *Let $e \in E(G)$, $\lambda : E(G) \to [0, 1]$ be a labelling and λ' be another labelling that agrees with λ for every edge other than e*:

1. *Suppose that $e \in E(G)$ is in $F_{\text{free}}(\lambda)$. If $\lambda'(e) < Z_\lambda(e)$, then $F_{\text{free}}(\lambda) = F_{\text{free}}(\lambda')$. If $\lambda'(e) > Z_\lambda(e)$, then $F_{\text{free}}(\lambda') = \{\phi(e, \lambda)\} \cup F_{\text{free}}(\lambda) \setminus \{e\}$ if $\phi(e, \lambda) \neq \varnothing$, otherwise $F_{\text{free}}(\lambda') = F_{\text{free}}(\lambda) \setminus \{e\}$.*

2. *Suppose that $e \in E(G)$ is not in $F_{\text{free}}(\lambda)$. If $\lambda'(e) > Z_\lambda(e)$, then $F_{\text{free}}(\lambda) = F_{\text{free}}(\lambda')$. If $\lambda'(e) < Z_\lambda(e)$ $F_{\text{free}}(\lambda') = \{e\} \cup F_{\text{free}}(\lambda) \setminus \{\phi(e,\lambda)\}$.*

Statements similar to (1) *and* (2) *hold with F_{free} replaced by F_{wired} above* (*with Z_λ changed to the wired version*).

PROOF. The proposition follows from the proof of Lemma 3.15 in [9]. There it is shown that $\lambda|_{E\setminus\{e\}}$ determines $F(\lambda)|_{E\setminus\{e,\phi(e,\lambda)\}}$ [note $\phi(e,\lambda)$ is determined by $\lambda|_{E\setminus\{e\}}$], and that one has either $e \in F(\lambda)$, $\phi(e,\lambda) \notin F(\lambda)$ [iff $\lambda(e) < Z(e) = \lambda(\phi(e,\lambda))$ or if $\phi(e,\lambda) = \varnothing$], or $e \notin F(\lambda)$, $\phi(e,\lambda) \in F(\lambda)$ [iff $\lambda(e) > \lambda(\phi(e,\lambda))$]. These imply the claim. $\square$

PROOF OF LEMMA 1.4. First, we show the claim for the WMSF. In [9], it is proved that in case of $\theta(p_C, G) = 0$, every component of the WMSF has one end. (In general, WMSF components can have at most two ends.)

Let λ be the random labelling and the corresponding spanning forest be $\mathcal{F}(\lambda) = F_{\text{wired}}(\lambda)$. Given $e \in E(G)$, we will use $\mathcal{C}(e)$ for the set of all cycles containing e, where *cycles are understood to be finite cycles or biinfinite paths* in the rest of this proof. Fix e, r and A as in Definition 1. We may assume that A is a $\mathcal{F}^{-1}$(WMSF)-measurable set of labellings, one that arises as the preimage of some measurable set of forests; in particular, if $\lambda \in A$ and $\mathcal{F}(\lambda) = \mathcal{F}(\lambda')$ then $\lambda' \in A$. Recall the notation of $f(\omega, e, x, r)$, and note that ω here stands for a realization $\mathcal{F}(\lambda)$; condition on A, that is, suppose that $\lambda \in A$.

To prove the claim, for any labelling λ we will define a labelling λ' whose properties are explained next. We will have $\mathcal{F}(\lambda) = \mathcal{F}(\lambda')$. If we change $\lambda'(e)$ to a $\lambda''(e) < Z_{\lambda'}(e)$ leaving all other labels unchanged, then Proposition 2.1 applies, and hence $\mathcal{F}(\lambda'') = \mathcal{F}(\lambda') \cup e \setminus \phi(e,\lambda')$. Now, $\phi(e,\lambda')$ will satisfy the requirements for $f(\mathcal{F}(\lambda'), e, x, r)$: if $\phi(e,\lambda') \neq \varnothing$, then its distance from x is at least r, and it is in C_x. Furthermore, the map $\lambda \mapsto \lambda'$ will be measurable and it will take sets of positive probability to sets of positive probability. This will prove the lemma.

Let $Z_1(e) := Z(e)$ and $f_1(\lambda) = f_1 := \phi(e,\lambda)$. Let $i \in \mathbb{Z}^+$, $i \geq 2$, and suppose that $Z_1(e), \dots, Z_i(e)$ and $f_1, \dots, f_i$ have been defined and that $i < k$, where k is to be determined later. Then define $Z_{i+1}(e) := \inf_{C \in \mathcal{C}(e)} \sup\{\lambda(e') : e' \in C \setminus \{e, f_1, \dots, f_i\}\}$ and let $f_{i+1}(\lambda) = f_{i+1}$ be the edge where this inf sup is attained, if there is any, otherwise let $f_{i+1} := \varnothing$.

Let k be the smallest number such that $f_k \notin C_y \cup (B(x,r) \cap C_x)$ (including the case when $f_k = \varnothing$). If such a k does not exist, define k to be infinity. The set $C_y \cup (B(x,r) \cap C_x)$ does not contain any element of $\mathcal{C}(e)$ as a subset, thus the sup in the definition of $Z_i(e)$ is always taken over some nonempty set as long as $i \leq k$, $i \in \mathbb{N}$.

Suppose first that k is finite. Define the following labelling λ' from λ:

(i) $\lambda'(f_i) := \lambda(f_i) Z_k(e)$ for all $i < k$,
(ii) leave all other labels unchanged.

Call the above relabelling operation Rel, that is, $\mathrm{Rel}(\lambda) = \lambda'$. We will show that λ' satisfies the properties that we proposed above.

By the choice of k, in (i) we only decrease labels of edges in $\mathcal{F}(\lambda)$, hence the minimal spanning forest is not changed by these changes of labels, using Proposition 2.1. It is easy to check that the map Rel is measurable. We show that Rel takes A to a set of positive probability. There exist a $k' \in \mathbb{Z}^+$ and $h_1, \dots, h_{k'-1}$ in $E(G)$ such that on A, $k = k'$, $f_i = h_i$ for every $i \in \{1, \dots, k'-1\}$ with positive probability (this is because there are countably many choices for the collection of h_i). Fix such $h_1, \dots, h_{k'-1} \in E(G)$, and call this subevent of A as A_1. (Note that $k', h_i, \dots, h_{k'-1}$ are not dependent on λ. In fact, we introduce them in order to have a fixed set of edges instead of the random one.) Using the dual definition of $Z(e)$ and $\phi(e, \lambda)$ (see Lemma 3.2 in [9]), applied to $G \setminus \{h_1, \dots, h_{i-1}\}$, there exists a finite minimal cut $\Pi_i \subset E(G)$ such that $e \in \Pi_i$, f_i is minimal in $\Pi_i \setminus \{e, h_1, \dots, h_{i-1}\}$, and $Z_i(e) = \sup_\Pi \min\{\lambda(f) : f \in \Pi \setminus \{e, h_1, \dots, h_{i-1}\}\}$ is attained for $\Pi = \Pi_i$ and $f = h_i$. (In the supremum, Π is ranging over all finite minimal cuts containing e.)

For $\varepsilon > 0$, define $A_2(\varepsilon)$ to be the subevent of A_1 such that the following properties hold:

- $\min_{g,h \in \cup \Pi_i, g \neq h}\{|\lambda(g) - \lambda(h)|\} > \varepsilon$,
- for every h_i, and every $h \in E(G)$ such that $\phi(h, \lambda) = h_i$ or $\phi(h_i, \lambda) = h$, we have $|\lambda(h_i) - \lambda(h)| > \varepsilon$,
- $Z_{k'}(e) > \varepsilon$.

Such an $\varepsilon > 0$ exists with $\mathbf{P}(A_2(\varepsilon)) > 0$ because letting $\varepsilon \to 0$, $A_2(\varepsilon)$ tends to A_1 a.s. (Note that in the second bullet point, the set of h's satisfying the requirement is a.s. finite, by the MTP.) Fix such an ε and the corresponding $A_2 = A_2(\varepsilon)$. Suppose that λ_0 is an arbitrary labelling that agrees with $\lambda \in A_2$ outside of $\{h_1, \dots, h_{k'-1}\} =: H$ and such that for every i we have $0 < \lambda_0(h_i) - \lambda(h_i) < \varepsilon$. Then $\mathcal{F}(\lambda_0) = \mathcal{F}(\lambda)$, and $f_i(\lambda_0) = h_i$ for every i, for the following reason. [The first fact will imply that $\lambda_0 \in A$; the second will imply that $Z_{k'}(\lambda_0) = Z_{k'}(\lambda)$.] The only places where λ_0 and λ differ are in $H = \{h_1, \dots, h_{k'-1}\}$, and the label of each element of H is larger by λ_0 than by λ. Consider the series of labellings λ_i from λ to λ_0 such that λ_i agrees with λ_0 on $\{h_1, \dots, h_i\}$ and with λ elsewhere. Then any two consecutive elements of this series only differ in the label of an edge h_i, which gets increased by $< \varepsilon$. The second bullet point above shows that the new label of h_i hence will not exceed $\lambda(\phi(h_i, \lambda)) = Z_\lambda(h_i)$, therefore, the forest is unchanged, by Proposition 2.1. Hence $\mathcal{F}(\lambda_0) = \mathcal{F}(\lambda)$. To see the other claim, for any two edges in $\bigcup_j \Pi_j$, the order of the labels of these two edges is the same in λ and λ_0, by the first bullet point. Therefore, f_i is still minimal in Π_i. On the other hand, the min of labels [as in the dual definition of $Z_i(e)$] is still maximized by the cutset Π_i, because we increased the max label of Π_i when going from λ to λ_0 (and no other cutset can "take over", because of our observation that the ordering on $\bigcup \Pi_i$ given by the labels is the same by λ and by λ_0). To summarize, conditioned on $\lambda \in A_2$, an increase of λ on the elements of H arbitrarily by at most ε will result in a labelling λ_0 that gives rise to the same WMSF as λ, and furthermore, still satisfies $f_i = h_i$, and in particular, $Z_{k'}(\lambda_0) = Z_{k'}(\lambda)$. Hence the λ_0 are in A on one hand, and the set of $Z_{k'}(e)$ over the λ_0's is the same as the set over the $\lambda \in A_2$. Now, let λ^- be a random labelling of $E(G) \setminus H$ by i.i.d. [0, 1] labels, and say that λ^- *extends* if there is an element in A_2 whose restriction to $E(G) \setminus H$ is λ^-. Then

$$\mathbf{P}(A) \geq \mathbf{P}(\lambda^- \text{ extends})\varepsilon^{k'-1},$$

by Fubini's theorem and our observation above on the λ_0's. Similarly [and recalling that $Z_{k'}(e)$ is a function of the labelling, which we hid in notation],

$$\mathbf{P}(\mathrm{Rel}(A)) \geq \mathbf{P}(\lambda^- \text{ extends})\Big(\varepsilon \inf_{\lambda \in A_2} Z_{k'}(e)\Big)^{k'-1} \geq \mathbf{P}(\lambda^- \text{ extends})\varepsilon^{2k'-2}.$$

Using Fubini's theorem again, we have $\mathbf{P}(\lambda^- \text{ extends}) \geq \mathbf{P}(A_2) > 0$, and the claim follows: Rel takes $\mathcal{F}^{-1}$(WMSF)-measurable sets A of positive probability to sets of positive probability.

Recalling the notation $\mathrm{Rel}(\lambda) = \lambda'$, we claim that

$$\phi(e, \lambda') = f_k. \tag{2.1}$$

Suppose to the contrary that it is not true, and suppose first that $f_k \neq \varnothing$. Then there is some cycle C such that the supremum of λ' on $C \setminus \{e\}$ is less than $\lambda'(f_k)$. Since $\lambda|_{C\setminus\{e, f_1, \dots, f_{k-1}\}} = \lambda'|_{C\setminus\{e, f_1, \dots, f_{k-1}\}}$, we have

$$\begin{aligned}\lambda(f_k) = \lambda'(f_k) &> \sup\{\lambda'(h) : h \in C \setminus \{e\}\} \\ &= \max\big(\sup\{\lambda(h) : h \in C \setminus \{e, f_1, \dots, f_{k-1}\}\}, \lambda'(f_1), \dots, \lambda'(f_{k-1})\big) \\ &\geq \sup\{\lambda(h) : h \in C \setminus \{e, f_1, \dots, f_{k-1}\}\},\end{aligned}$$

contradicting the choice of f_k. For $f_k = \varnothing$, the same argument works, using $Z_k(e)$ and $Z(e)$ instead of $\lambda'(f_k)$ and $\lambda(f_k)$. So, in fact $\phi(e, \lambda') = f_k$.

Now, $\phi(e, \lambda') = f_k$ implies $f_k \in C_x \cup C_y$, for the following reason. Otherwise, if we decrease the label of e in λ' (as in the definition of λ'' above), so that e becomes part of the forest $[\lambda''(e) < Z_k(e)]$, the edge $\phi(e, \lambda') = f_k$ that drops out of the forest is outside of $C_x \cup C_y \cup \{e\}$. Then $C_x \cup C_y \cup \{e\}$ is one of the new components, with two ends. "Decreasing the label of e" could be done in a measurable way so that this happens with positive probability, a contradiction. We conclude using the definition of k that if k is finite then $\phi(e, \lambda') = f_k$ is in $(C_y \cup (B(x, r) \cap C_x))^c \cap (C_x \cup C_y) = C_x \cap B(x, r)^c$, as we wanted.

Finally, suppose that k is infinity. Then for some $K > 0$ we have $f_i \in C_y$ for every $i > K$.

For $\alpha \in [0, 1]$, define $G_\alpha = G_\alpha(\lambda) = \{e' \in E(G) : \lambda(e') \leq \alpha\}$. Let $i \geq 1$ be arbitrary. Let $O_i \in \mathcal{C}(e)$ be a cycle such that f_i is maximal in $O_i \setminus \{e, f_1, \dots, f_{i-1}\}$.

By definition of f_i, there exists such a cycle. It has the property that $O_i \setminus \{e, f_1, \dots, f_{i-1}\} \subset G_{\lambda(f_i)}$ a.s. It follows from the choice of O_i (and O_{i-1}) that

$$\begin{aligned}\lambda(f_i) &= \max\{\lambda(f) : f \in O_i \setminus \{e, f_1, \dots, f_{i-1}\}\}\\ &= \inf_{C \in \mathcal{C}(e)} \sup\{\lambda(e') : e' \in C \setminus \{e, f_1, \dots, f_{i-1}\}\}\\ &\le \sup\{\lambda(f) : f \in O_{i-1} \setminus \{e, f_1, \dots, f_{i-1}\}\} < \lambda(f_{i-1}).\end{aligned} \tag{2.2}$$

Note that for the first inequality we need that $O_{i-1} \setminus \{e, f_1, \dots, f_{i-1}\} \neq \varnothing$, but this is guaranteed by the fact that $f_1, \dots, f_{i-1} \in C_x \cup C_y$ and that $|O_{i-1} \setminus (C_x \cup C_y)| \ge 2$ or O_{i-1} is infinite. We obtained that λ is monotone decreasing on the sequence f_i. By this monotonicity, if a cycle in $G_{\lambda(f_j)} \cup \{e, f_1, \dots, f_{j-1}\}$ contains e and f_j, then it also contains $f_1, \dots, f_{j-1}$. Otherwise, if O is such a cycle and $i < j$ is the smallest index with $f_i \notin O$, then $\sup\{\lambda(h) : h \in O \setminus \{e, f_1, \dots, f_{i-1}\}\} \le \sup\{\lambda(h) : h \in G_{\lambda(f_j)} \cup \{f_{i+1}, \dots, f_{j-1}\}\} < \lambda(f_i)$, which would contradict the choice of the f_i.

Next, we show that all the f_i $(i > K)$ are on the infinite ray from y in C_y. Proving by contradiction, suppose that this is not the case, and let $h > K$ be an index such that f_h is not on the infinite ray from y in C_y. Define a labelling λ' as in (i) and (ii) above, with h playing the role of k. This relabelling operation preserves positive measure and produces a labelling λ' that generates the same WMSF as λ. Furthermore, as in (2.1) with k replaced by h, $\phi(e, \lambda') = f_h$. But then f_h is necessarily on the infinite ray from y in C_y. Otherwise, if we insert e and delete f_h from the configuration, the new configuration would still contain the infinite ray from y in C_y and the infinite ray from x in C_x [where by C_x and C_y here we mean components of WMSF(λ')], that is, we would obtain a tree with at least two ends, which is a contradiction. Therefore, all the f_i $(i > K)$ are on the infinite ray from y in C_y.

For any j, $\lambda(f_j)$ is greater than the max in $\bigcup_{i>j} O_i \setminus \{e, f_1, \dots, f_j\}$, by (2.2). The set $\bigcup_{i>j} O_i \setminus \{e, f_1, \dots, f_j\}$ has an infinite component, because $\bigcup_{i>j} O_i$ is infinite and connected. We conclude that $\lambda(f_j) > p_c$.

In the definition of the f_i we are having an inf sup, which, if attained, has to be attained in a simple cycle C_i (meaning that C_i is connected and every vertex of it has degree 2). Therefore, we may assume that every O_i is a simple cycle. Take a subsequential limit of the O_i, call the resulting biinfinite path B. For every $p > p_c$, every component of the WMSF intersects the cluster $G_p(\lambda)$ in an infinite component (Lemma 3.11 in [9]). This implies

$$\lim \lambda(f_i) = p_c,$$

using again that the trees of the WMSF are 1-ended and that all the f_i $(i > K)$ are on an infinite ray within one of these trees (C_y). All the f_i are on one side of e in B for $i > K$, hence there is an infinite path $P \subset B$ such that $P \cap \{e, f_1, f_2, \dots\} = \varnothing$. Take an arbitrary edge $g \in P$. If $\lambda(g) > p_c$, then if i is large enough then we have $\lambda(f_i) < \lambda(g)$. By definition of B, if i is large enough, then further O_i contains g. But this contradicts the choice of f_i, because then $\lambda(f_i)$ is not maximal in $O_i \setminus \{e, f_1, \ldots, f_{i-1}\}$. We conclude that every $g \in P$ has $\lambda(g) \leq p_c$, contradicting $\theta(p_c) = 0$. This final contradiction shows that k cannot be infinite, and the proof is complete.

The case of FMSF follows from the previous proof: note that we constructed λ' by lowering the labels of some edges in $F_{\mathrm{wired}}(\lambda) \subset F_{\mathrm{free}}(\lambda)$. Hence $F_{\mathrm{wired}}(\lambda') = F_{\mathrm{wired}}(\lambda) \subset F_{\mathrm{free}}(\lambda) = F_{\mathrm{free}}(\lambda')$. When reducing the label of e in λ' [below $Z_{\mathrm{Free}}(e) := \inf_C \sup\{\lambda(e') : e' \in C \setminus \{e\}\}$, C finite cycle containing e], then e becomes part of the forest, and either no edge drops out of it, or the edge that drops out is $\phi(e, \lambda')$. Now $\phi(e, \lambda')$ satisfies the requirements for $f(\mathcal{F}(\lambda'), e, x, r)$, because it did satisfy it for the WMSF. The fact that it is in $C_x \cap B(x, r)^c$, where C_x is the WMSF component of x, implies that it is also in the FMSF component of x. $\square$

REMARK 2.2. In the definition of WIT, the requirement that $f \in C_x$ is needed only for the proof of Theorems 3.1 and 3.3. Theorem 4.3 is true without this assumption, if we know that the conclusion of Theorem 3.1 holds. Along the lines of the above proof, one could show that WMSF and FMSF are weakly insertion tolerant without the assumption $\theta(p_c, G) = 0$, if we had chosen the less restrictive form of weak insertion tolerance, where $f(\omega, e, x, r)$ need not be in C_x.

3. Number of components, number of ends.

THEOREM 3.1. *Let G be a unimodular quasitransitive graph, and $\mathcal{F}$ an ergodic random spanning forest of G. If $\mathcal{F}$ satisfies weak insertion tolerance and one of its components has infinitely many ends, then every component has infinitely many ends.*

PROOF. Suppose by contradiction that there is also some component with finitely many ends. Then there is a vertex x, edge $e = \{x, y\}$ and event A_0 with $\mathbf{P}(A_0) > 0$ such that conditioned on A_0, x is in a component C with infinitely many ends, and y is in a different component C' with finitely many ends. To see this, note that with probability 1 there exist adjacent components such that one of them has infinitely many ends and the other one has finitely many ends. Then for some fixed edge e, there is a positive probability that e connects two such components—otherwise, summing up over the countably many edges, we would get an event of 0 probability, contradicting the previous sentence. A similar argument will be used later several times without explicit mention. Namely, if there exists an edge of a certain property with positive probability, then there exists a fixed edge e that has this property with positive probability.

Choose $r > 0$ such that $C \setminus B(x, r)$ has at least 3 infinite components with probability at least $\mathbf{P}(A_0)/2$. Such an r exists by the assumption on C. Let A be the subevent of A_0 that $C \setminus B(x, r)$ has at least 3 infinite components. In particular, $\mathbf{P}(A) \geq \mathbf{P}(A_0)/2 > 0$. Now consider $f = f(\omega, e, x, r)$ as in the definition of weak insertion tolerance and take $\pi_e^f A$. Then for every $\omega \in A$, the component of x in $\omega \cup \{e\} \setminus \{f\} \in \pi_e^f A$ contains C', e and at least 2 ends from C. In particular, it has an isolated end (in C'), which is impossible by a standard MTP argument (see Proposition 3.9 in [10]). By WIT, $\mathbf{P}(\pi_e^f A) > 0$, giving a contradiction. □

The next lemma summarizes some well-known claims that we will need later. All our applications of this lemma will be for the component of the "root" at some random invariant subforest of a quasitransitive graph G. By a *root*, we mean a randomly chosen vertex o of G that makes (G, o) a unimodular random graph. More precisely, if $t_1, \dots, t_m$ is a set of representatives from the classes of transitivity of $V(G)$, then the root is t_i with probability $p(t_i)$, where p is the probability mass function that satisfies $p(t_i)/p(t_j) = |\operatorname{Stab}_{t_i}(t_j)/|\operatorname{Stab}_{t_j}(t_i)|$. (This is understood up to rooted isomorphisms, therefore, the choice of the t_i does not matter.) See [1], especially Theorem 3.1, for more details.

LEMMA 3.2. *Let (T, o) be an ergodic unimodular random tree. Then the following hold with probability* 1:

(i) *If T has infinitely many ends, then it is transient.*

(ii) *If the expected degree of o is strictly greater than* 2, *then T has infinitely many ends. Conversely, if T has infinitely many ends, then the expected degree of o is greater than* 2.

(iii) *If T has infinitely many ends, then it has exponential growth.*

(iv) *If T has infinitely many ends, then for any finite subset S of edges and vertices every infinite component of $T \setminus S$ has infinitely many ends.*

PROOF. An argument similar to the proof of Proposition 3.11 in [10] shows (i). Part (ii) follows from Theorem 6.2 in [1]. For (iii), one has to use the fact that the existence of infinitely many ends implies $p_c < 1$ (see Lemma 6.12 in [1]). Hence $\underline{\mathrm{gr}} \geq \mathrm{br} = p_c^{-1} > 1$, where br is the branching number, $\underline{\mathrm{gr}}$ is the lower exponential growth rate and p_c is the critical percolation probability (see Sections 1.5 and 3.3 in [8] for the equality and inequality, which hold for arbitrary trees). Part (iv) is true because otherwise there would be an isolated end in C_o. This is impossible; see, for example, Proposition 6.10 in [1]. □

THEOREM 3.3. *Let G be a unimodular quasitransitive graph. Suppose that $\mathcal{F}$ is an ergodic random spanning forest of G that satisfies weak insertion tolerance and one of its components has infinitely many ends. Then it either has one component, or it has infinitely many components.*

We mention that the proof of the same fact for Bernoulli percolation cannot be generalized to our setting directly. In case of Bernoulli percolation, one assumes, proving by contradiction, that there are k components, $1 < k < \infty$. Then one inserts an edge, to derive that the probabilities of having k components or having $k-1$ components are both positive. This contradicts ergodicity. In our case, when we apply weak insertion tolerance, even though we reduce the number of components when inserting an edge e, we increase it when deleting edge f. Hence there is no direct contradiction to the fact that the number of components is a constant a.s.

PROOF. We will prove by contradiction. Suppose that there are more than one, but finitely many components.

The proof will loosely follow the method in [14], with insertion tolerance replaced by weak insertion tolerance. Some arguments become simpler because the components are trees and also because of the assumption that there are only finitely many components. We say that two components C and C' *touch each other* at x if there is an edge $\{x, y\} \in E$, $\{x, y\} \notin \mathcal{F}$, with $x \in C$ and $y \in C'$.

There exist distinct components C and C' such that C has infinitely many ends, and further, C and C' touch each other at infinitely many places (because the outer boundary of a cluster C is infinite, and there are finitely many neighboring components). Choose C and C' with these properties uniformly at random, from the finitely many possible pairs. Hence there exist adjacent vertices x and y and an event A_0 such that $\mathbf{P}(A_0) > 0$, and such that conditioned on A_0, $C_x = C$, $C_y = C'$. (In particular, C_x has infinitely many ends, and it touches C_y at infinitely many places on A_0.) Fix such vertices x and y, let $e = \{x, y\}$ $(\in E)$, and condition on A_0. Let the set of such touching points be $\mathcal{T} = \{v \in C : \text{there is a } u \in C' \text{ such that } \{v, u\} \in E(G)\}$. For any $v \in C$ and infinite component C^- of $C \setminus v$, C^- has infinitely many ends [by (iv) of Lemma 3.2] and $C^- \cap \mathcal{T} \neq \varnothing$. (This latter can be shown by a standard mass transport argument. To sketch it: one could assign to each point of C the element of $\mathcal{T}$ that is closest to it in C. If the claim were not true, there would be a point that is assigned to infinitely many points of C^- with positive probability, which is impossible.)

Fix $r > 0$ such that given A_0, $C_x \setminus B(x, r)$ has at least 3 infinite components with probability at least $1/2$. (Such an r exists because C_x has infinitely many ends on A_0.) Let A be the subevent of A_0 when this holds. We have $\mathbf{P}(A) \geq \mathbf{P}(A_0)/2 > 0$. Note that on A, $x \in \mathcal{T}$ (because this holds on A_0 already).

Let us sketch the rest of the proof before going into the details. We will define the following mass transport. For each v, w in the same $\mathcal{F}$-component such that v and w are adjacent in G, take the minimal path $P_{v,w}$ within the $\mathcal{F}$-component between them. For each such pair v, w, let v send mass i^{-2} to the vertex of $P_{v,w}$ that has distance i from v in $P_{v,w}$. Then the expected mass sent out is at most $d\pi^2/6$, where d is the maximum degree in G. However, the expected mass received is infinite, because of the way we constructed C_x on $\pi_e^f A$, with an exponentially growing set of touching pairs.

Now we give the detailed proof. Let $P(C, C') \subset \mathcal{T}$ be the set of all v in $\mathcal{T}$ with $C \setminus B(v,r)$ having at least 3 infinite components. For the x and r that we fixed above, $x \in P(C, C')$ conditioned on A. Thus $P(C, C') \neq \varnothing$ with positive probability, and hence by ergodicity and the mass transport principle, $P(C, C')$ is infinite a.s. Condition on A and let $C_1, \dots, C_m$ be the infinite components of $C \setminus B(x,r)$ $(m \geq 3)$. We will show that $P(C, C') \cap C_i$ has exponential growth within C_i for every i. Define $T(C, C')$ as the minimal subtree of C that contains every vertex of $P(C, C')$. In other words, $T(C, C')$ is the union of all simple paths with both endpoints in $P(C, C')$. The graph $C \setminus T(C, C')$ only has finite components, as can be easily seen by a mass transport argument. [Otherwise, let each vertex send mass 1 to a uniformly chosen element of $P(C, C')$ that is closest to it...] Hence $T(C, C') \cap C_i$ is a (unimodular) tree with infinitely many ends (using the fact that C_i has infinitely many ends), thus the growth of $T(C, C')$ is in fact exponential [Lemma 3.2(iii)]. Define the subtree $T_\ell(C, C')$ of $T(C, C')$ as the union of all minimal paths between two points of $P(C, C')$ such that the path has length at most ℓ. The tree $T_\ell(C, C')$ converges to $T(C, C')$, and so does the expected degree within it. By (ii) in Lemma 3.2, the expected degree in $T(C, C')$ is greater than 2. Hence it is greater than 2 in $T_\ell(C, C')$ as well for large enough ℓ. It follows that some component of $T_\ell(C, C')$ has exponential growth for ℓ large enough, using again Lemma 3.2. Consequently, conditioned on A, for large enough ℓ and some $c > 1$, the inequality $|B_{T_\ell(C,C')}(x,r) \cap C_i \cap P(C, C')| \geq c^r$ is satisfied for each r large enough. Thus $|B_{T(C,C')}(x,r) \cap C_i \cap P(C, C')| \geq c^r$ also holds.

Consider the infinite components $C_1, \dots, C_m$ of $C \setminus B(x,r)$. Conditioned on A, we have $m \geq 3$. On the other hand, we have seen that for each C_i, the set $P(C, C') \cap C_i$ has exponential growth in C. All but at most one of $C_1, \dots, C_m$ are in the same component of $\pi_e^f \omega$ as x $(\omega \in A)$. We may assume that C_1 is the exceptional one (if any).

Define the following mass transport. For every v adjacent to some w in G, choose the minimal path in C between v and w if $v, w \in C$, and let v send mass i^{-2} to the ith vertex on this path. The expected mass sent out is finite, because v has a bounded number of neighbors. To compute the expected mass received, note that on $\pi_e^f A$, x will receive mass i^{-2} from every vertex of $S_{C_x}(x,i) \cap C_x \cap (P(C, C') \setminus C_1)$. Because of the exponential growth of $P(C, C')$ in C_i, the expected mass received is hence infinite. This contradiction completes the proof. $\square$

REMARK 3.4. One is tempted to think that the above arguments may work to show (similar to [14]) that there are no infinitely touching clusters when the percolation is weakly insertion tolerant and each component has infinitely many ends. However, this claim is not true: consider FUSF on the free product of $\mathbb{Z}^5$ and $\mathbb{Z}$, and use the result of [2] that any two of the infinitely many FUSF-clusters in $\mathbb{Z}^5$

touch each other at infinitely many places, and the fact that $\mathrm{FUSF}(\mathbb{Z}^5 * \mathbb{Z})|_{\mathbb{Z}^5}$ has the same distribution as $\mathrm{FUSF}(\mathbb{Z}^5)$. Weak insertion tolerance is not enough in this setting to make the argument of [14] work, because deleting f may make a part of the cluster "fall off" that contains all the touching points $\mathcal{T}$.

4. Indistinguishability of clusters. In this section, we will prove the indistinguishability of clusters. By this, we mean that for any invariant measurable $\mathcal{A} \subset 2^{E(G)}$, a.s. either every infinite component belongs to $\mathcal{A}$ or none of them. When an invariant measurable $\mathcal{A} \subset 2^{E(G)}$ is given, we will refer to $\mathcal{A}$ and to $\mathcal{A}^c$ as a *type*. If $C_o \in \mathcal{A}$, we say that $\mathcal{A}$ is the type of C_o or (with a slight abuse of terminology) that $\mathcal{A}$ is the type of o; similarly for $\mathcal{A}^c$.

The following lemmas will be needed for the proof. The first one was shown in [10] for unimodular transitive graphs, and generalized for unimodular random graphs in Theorem 4.1 of [1]. The version that we are stating for unimodular quasitransitive graphs is essentially Theorem 3.1 in [11]. Delayed simple random walk on a transitive graph G is defined as follows. At each time, the walker chooses one of its G-neighbors uniformly at random, and takes a step to this neighbor if and only if they are neighbors in the percolation, too; see [1] for the more formal and more general definition. Informally speaking, the next lemma says that an invariant percolation process looks the same from a "fixed" vertex as from the vertex where a delayed simple random walker is after one step within the cluster starting from the fixed vertex. (Here, the "fixed" vertex is in fact fixed in the transitive case, and randomly chosen from the representatives of the finitely many orbits in the quasi-tranistive case, so that the resulting rooted graph is unimodular.)

LEMMA 4.1. *Let G be a quasitransitive unimodular graph, and o be a random root so that (G, o) is a unimodular random graph. Consider an invariant edge-percolation process on G. Let $\hat{\mathbf{P}}$ be the joint distribution of ω and the two-sided delayed simple random walk on C_o started from the (random) root o. Then the restriction of $\hat{\mathbf{P}}$ to the* $\mathrm{Aut}(G)$*-invariant σ-field is stationary. More precisely, $\hat{\mathbf{P}}(\mathcal{A}) = \hat{\mathbf{P}}(\mathcal{S}\mathcal{A})$, where $\mathcal{S}$ is the shift operator by the random walk step, and $\mathcal{A}$ is any* $\mathrm{Aut}(G)$*-invariant subset of $V^{\mathbb{Z}} \times 2^E$.*

DEFINITION 4 (Pivotal pairs). Let $r \geq 0$ be an integer, $e, f \in E$, and let $\mathcal{A}$ be a type. Say that (e, f), is an *r-pivotal pair*, if $f = f(\omega, e, x, r)$ for an endpoint x of e (as in the definition of WIT), and if the type of at least one of the endpoints of e is different in $\pi_e^f \omega$ than in ω (meaning that it changes from type $\mathcal{A}$ to $\neg\mathcal{A}$ or vice versa). Define $z(e, f) = x$ if the type of x is different in $\pi_e^f \omega$ than in ω, otherwise define $z(e, f) = y$ (where y is the other endpoint of e). If (e, f) is an r-pivotal pair for some r, then we call (e, f) *pivotal*.

Note that x can be recovered from f because f is in C_x, so $z(e, f)$ is in fact determined by e and f if (e, f) is an r-pivotal pair.

DEFINITION 5. For each edge $\{x, y\} = e \in E$, $r > 0$, $f = f(\omega, e, x, r)$, define a measure $\mathbf{P}_{e,x,r} = \mathbf{P}_e$ on $A_e := \{\omega : C_x \neq C_y\}$ as $\mathbf{P}_e(A) := \mathbf{P}(\pi_e^f A)$ for $A \subset A_e$ arbitrary measurable. By weak insertion tolerance, the restriction of $\mathbf{P}$ to A_e is absolutely continuous with respect to $\mathbf{P}_e$. Let $\Delta_{e,x,r}(\omega) = \frac{d\mathbf{P}}{d\mathbf{P}_e}(\omega)$ be the Radon–Nikodym derivative of $\mathbf{P}$ with respect to $\mathbf{P}_e$ on A_e.

The next lemma slightly extends similar claims in [10].

LEMMA 4.2. *Let $\mathcal{F}$ be some invariant ergodic random forest of G. Suppose that there exists a type $\mathcal{A}$ such that a.s. some cluster belongs to $\mathcal{A}$ and some other belongs to $\mathcal{A}^c$ (i.e., suppose that indistinguishability fails). Suppose that $\mathcal{F}$ is weakly insertion tolerant and it has a component with infinitely many ends a.s. Then there are numbers $\delta > 0$, $r \geq 0$, $p_{\mathcal{A}} > 0$ such that with probability at least $p_{\mathcal{A}}$, the following event* $\mathrm{Piv}(e, x, \mathcal{A}, \delta)$ *holds*:

- *there exists an edge e with an endpoint x such that the pair (e, f) is pivotal with $f = f(\omega, e, x, r)$,*
- $\Delta_{e,x,r}(\omega) < \delta^{-1}$,
- *delayed simple random walk $(W(1), W(2), \dots)$ started from $W(0) = z(e, f)$ avoids the endpoints of e and f.*

PROOF. There is a cluster with infinitely many ends, hence by Theorem 3.1, all clusters have infinitely many ends. By the assumption, with positive probability $C_x \in \mathcal{A}$, and $C_y \in \mathcal{A}^c$ for some $e = \{x, y\} \in E$. Fix such an x, y, e, and call the event just described as A. Let $A_{r,\delta} \subset A$ be the event that the following hold: $\Delta_{e,x,r}(\omega) < \delta^{-1}$, $C_x \setminus B(x, r)$ has at least 3 infinite components and $C_y \setminus B(x, r)$ has at least 3 infinite components. If r is large enough and $\delta > 0$ is small enough, then $\mathbf{P}(A_{r,\delta}) \geq \mathbf{P}(A)/2$. Condition on $A_{r,\delta}$. Both C_x and C_y are transient by Lemma 3.2. Consequently, any $f = \{u, v\} \in C_x$ with $\mathrm{dist}(x, f) \geq r$ is such that $C_x \setminus \{u, v\}$ has an infinite and transient component, and similarly for $C_y \setminus \{u, v\}$. Hence the last statement of the lemma holds with positive probability. What remains is to prove that $(e, f(\omega, e, x, r))$ is a pivotal pair.

Let C'_x be the component of x in $\pi_e^f \omega$, and C_y the component of y in ω. Let $\mathcal{B} \in \{\mathcal{A}, \mathcal{A}^c\}$ be the type of C'_x. Since $C_y \cap C'_x$ and $C_x \cap C'_x (\neq \emptyset)$ are contained in clusters of different types in ω, either the type of the points in $C_x \cap C'_x$ changed (from $\mathcal{A}$ to $\mathcal{A}^c$ if $\mathcal{B} = \mathcal{A}^c$), or the type of the points in $C_y \cap C'_x$ changed (from $\mathcal{A}^c$ to $\mathcal{A}$ if $\mathcal{B} = \mathcal{A}$) when going from ω to ω'. If the former happens with positive probability, the proof is completed with $z(e, f) = x$, otherwise with $z(e, f) = y$. $\square$

THEOREM 4.3. *Let G be a unimodular quasitransitive graph. Let $\mathcal{F}$ be an invariant ergodic random forest that is weakly insertion tolerant, and such that some cluster has infinitely many ends. Then for every invariant measurable $\mathcal{A} \subset$*

$2^{E(G)}$, *either every infinite component belongs to* $\mathcal{A}$ *a.s., or none of them. In other words, infinite clusters are indistinguishable.*

Similar to Lemma 4.2, our proof will follow that of Theorem 3.3 in [10], with some significant modifications. Let us mention the most important difference here. The starting idea of that proof is that the existence of pivotal edges (in that setup meaning edges that connect clusters of different types) implies that by the insertion of one of them, the type of an infinite cluster changes. Hence the type of a cluster depends on the status of each of these single edges. Such edges exist arbitrarily far from the "root" o of the cluster. This makes it impossible to determine the type of the cluster of a vertex o from a large enough neighborhood up to arbitrary precision, giving a contradiction. One difficulty in this sketch is that the pivotal edges are random (dependent on the configuration), hence one cannot directly apply the insertion tolerance property to bound the probabilities after insertion. This is overcome by the use of a random walk to choose the pivotal edge, at arbitrary distance, in such a way that by inserting that edge, the probability will be distorted up to some uniform factor, regardless of its distance from the root. In our setup, when only weak insertion tolerance is assumed, a further difficulty is that an edge f is removed while an edge e is inserted. This may not be sufficient to guarantee that the type of o changes. However, by proper conditioning and choosing e and f far enough from each other (i.e., r large enough, as in the proof of Lemma 4.2), one can guarantee that infinitely many vertices change their type, including o, without distorting the probability of the event too much.

PROOF. Fix a type $\mathcal{A}$ and an $r \geq 0$, $\delta > 0$, $p_{\mathcal{A}} > 0$ such that Lemma 4.2 holds with $C_{z(e,f)} \in \mathcal{A}$. Fix some vertex o. Define $\mathcal{A}_o$ as the event that C_o is of type $\mathcal{A}$. Given $e = \{x, y\} \in E$, $f \in E$ and $w \in \{x, y\}$, let $\mathcal{P}^f_{e,x}$ be the event that (e, f) is r-pivotal with $f = f(\omega, e, w, r)$ and $z(e, f) = x$, $\Delta_{e,x,r}(\omega) < \delta^{-1}$, $C_x \in \mathcal{A}$. [Note that w can be recovered from f, because f is in the component of w, so there is no information lost in the notation $\mathcal{P}^f_{e,x}$. Let us point at the little unpleasant technicality here, why we need to distinguish x and w: $w \in \{x, y\}$ is the vertex whose component in ω contains f, and x is the vertex whose component changes type if we insert e and delete f. We also make the remark that $\Delta_{e,x,r}(\omega)$ is defined whenever (e, f) is pivotal.] For an arbitrary $\varepsilon > 0$, let $\mathcal{A}'_o(\varepsilon) = \mathcal{A}'_o$ be some fixed event that depends on only finitely many edges and satisfies $\mathbf{P}(\mathcal{A}_o \Delta \mathcal{A}'_o) < \varepsilon$. Fix $R = R(\varepsilon)$ such that $\mathcal{A}'_o$ only depends on edges in $B(o, R)$.

Let $\{\mu_x : x \in V(G)\}$ be a collection of probability measures on $E(G) \times E(G)$ that is equivariant under the automorphisms of G, and that $\mu_x(e, f) > 0$ for every $e, f \in E(G)$. Such a measure exists for the following reason. Fix an origin o in one of the (finitely many) orbits of the automorphism group of G on $V(G)$. Let μ be an arbitrary probability measure on the set of ordered pairs of edges $(e, f) \in E(G) \times E(G)$ such that $0 < \mu(e, f) = \mu(\gamma e, \gamma f)$ for every automorphism γ of G

that fixes o. Since the orbit of (e, f) is finite by all such γ's, there exists such a μ. Then for every point of x of the orbit of o we can define $\mu_x(e, f) = \mu(\gamma_x e, \gamma_x f)$, where γ_x is any automorphism that takes the origin to x. It is easy to check that μ_x does not depend on the choice of γ_x, hence it is well defined. Doing the above construction as o ranges through a (finite) transversal of the orbits, we define μ_x for every $x \in V(G)$ as claimed.

Let o be a random root of G with distribution that makes (G, o) unimodular. (If G is transitive, we can make o to be a fixed vertex.) Let $\omega \in \mathcal{F}$, and $\mathcal{W}_\omega = (W_\omega(j))_{j=-\infty}^{\infty}$ be the biinfinite 2-sided delayed random walk on C_o with $W_\omega(0) = o$. Define (e_n, f_n) to be a pair of edges chosen according to distribution $\mu_{W_\omega(n)}$, and let w_n be a uniformly chosen endpoint of e_n. Let $\mathcal{W}_\omega^+ = (W_\omega(j), e_j, f_j, w_j)_{j=-\infty}^{\infty}$ be the random walk path together with the choices we make on the way. Denote by $\Psi = \Xi \times E(G) \times E(G) \times V(G)$ (where Ξ is the set of all biinfinite paths started from o) the space where all possible $\mathcal{W}_\omega^+$ are located. Given some ω, where Ω is the space of subgraphs of G, we denote the probability measure on Ψ as defined above by $\hat{\mathbf{P}}_\omega^o = \hat{\mathbf{P}}_\omega$. Let $\hat{\mathbf{P}}^o = \hat{\mathbf{P}}$ be the joint distribution of the random forest and the $\mathcal{W}_\omega^+$. So $\hat{\mathbf{P}}$ is defined on the space $\Omega \times \Psi$. For an event $F \subset \Omega$, we will also use notation F for $F \times \Psi \subset \Omega \times \Psi$, and similarly for events in Ψ.

For fixed edges $e, f \in E$, $e = \{x, y\}$, and fixed ω, denote by $\mathcal{E}_{x,e,f}^n(\omega)$ the event that $e_n = e$, $f_n = f$, $W_\omega(n) = x$, and that $W_\omega(j)$ is not an endpoint of e or f whenever $-\infty < j < n$. Observe that the $\mathcal{E}_{x,e,f}^n(\omega)$ are pairwise disjoint as n varies. We denote by $\mathcal{E}_{x,e,f}^n$ the union of all the $\mathcal{E}_{x,e,f}^n(\omega)$ over $\omega \in \Omega$. We mention (though we will only use this fact later) that $\mathcal{E}_{x,e,f}^n$ has positive probability for certain pairs (e, f) and n by Lemma 4.2, with the last bullet point in the lemma applied to $(W(-1), W(-2), \ldots)$. For a fixed $\omega \in \Omega$ and fixed $e, f \in E$, the probabilities $\hat{\mathbf{P}}_\omega(\mathcal{E}_{x,e,f}^n(\omega))$ and $\hat{\mathbf{P}}_{\pi_e^f \omega}(\mathcal{E}_{x,e,f}^n(\pi_e^f \omega))$ are the same. For any $n \geq 0$ and measurable $\mathcal{B} \subset \mathcal{P}_{e,x}^f$ such that $\mathbf{P}(\mathcal{B}) > 0$ and $\mathbf{P}(\pi_e^f \mathcal{B}) > 0$, we deduce the following equality:

$$\begin{aligned}\hat{\mathbf{P}}(\mathcal{E}_{x,e,f}^n \cap \pi_e^f \mathcal{B}) &= \int_{\mathcal{B}} \hat{\mathbf{P}}_{\pi_e^f \omega}(\mathcal{E}_{x,e,f}^n(\pi_e^f \omega))\, d\mathbf{P}_e \\ &= \int_{\mathcal{B}} \hat{\mathbf{P}}_\omega(\mathcal{E}_{x,e,f}^n(\omega))\, d\mathbf{P}_e \\ &= \int_{\mathcal{B}} \hat{\mathbf{P}}_\omega(\mathcal{E}_{x,e,f}^n(\omega)) \Delta_{e,x,r}(\omega)^{-1}\, d\mathbf{P}.\end{aligned}$$

(Recall $\mathbf{P}_e$ from Definition 5.) Apply the previous equality to $\mathcal{B} = \mathcal{A}'_o \cap \mathcal{P}_{e,x}^f$ if $\mathbf{P}(\mathcal{B}) > 0$ [which implies $\mathbf{P}(\pi_e^f \mathcal{B}) > 0$ by the definition of $\mathcal{P}_{e,x}^f$]. Using that $\Delta_{e,x,r}(\omega) < \delta^{-1}$ on $\mathcal{P}_{e,x}^f$ we obtain

$$\hat{\mathbf{P}}(\mathcal{E}_{x,e,f}^n \cap \pi_e^f(\mathcal{A}'_o \cap \mathcal{P}_{e,x}^f)) \geq \delta \hat{\mathbf{P}}(\mathcal{E}_{x,e,f}^n \cap \mathcal{A}'_o \cap \mathcal{P}_{e,x}^f). \tag{4.1}$$

The inequality is trivially true if $\mathcal{B} = \mathcal{A}'_o \cap \mathcal{P}^f_{e,x}$ has probability 0, so it holds in general (for any triple e, f, x).

Note that if $(\omega, \mathcal{W}^+_\omega) \in \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x}$, then $C_{W(n)} = C_o$ and the components of o and $W(n)$ also coincide in $\pi^f_e \omega$. We have obtained in (4.1) that by opening e and closing f we distort the probability of our event by at most a factor of δ, where "our event" is, vaguely speaking, the event that the random walk on C_o hits the endpoint x of e in the nth step, (e, f) is pivotal, and by changing the status of e and f, the type of x (and hence the type of o) will change. This suggests that the status of e and f has a high impact on the type of o on this event. In what follows, we will apply this observation to all possible pivotal pairs (e, f), and use the fact that the random walk hits infinitely many of them eventually. If n is large enough, both e and f are outside of the cylinder that determines $\mathcal{A}'_o$, leading to a conclusion that the type of o is not determined by $\mathcal{A}'_o$ up to a small error, a contradiction. We make this argument precise in the rest of the proof.

For $e \notin B(o, R)$, $f \notin B(o, R)$, if $(\omega, \mathcal{W}^+_\omega) \in \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x}$, then $\pi^f_e \omega \notin \mathcal{A}_{W(n)}$ (by definition of $\mathcal{P}^f_{e,x}$), and thus $\pi^f_e \omega \notin \mathcal{A}_o$. Consequently, for such e and f, $\mathcal{A}_o \cap \pi^f_e(\mathcal{A}'_o \cap \mathcal{P}^f_{e,x}) \cap \mathcal{E}^n_{x,e,f} = \varnothing$ up to measure 0. On the other hand, for $e \notin B(o, R)$, $f \notin B(o, R)$, we have $\mathcal{A}'_o \supset \pi^f_e(\mathcal{A}'_o \cap \mathcal{P}^f_{e,x}) \cap \mathcal{E}^n_{x,e,f}$, because $\mathcal{A}'_o$ is determined by the edges in $B(o, R)$. These observations show that $\mathcal{A}'_o \setminus \mathcal{A}_o \supset \bigcup_{e \notin B(o,R), f \notin B(o,R), e=\{x,y\}} \pi^f_e(\mathcal{A}'_o \cap \mathcal{P}^f_{e,x}) \cap \mathcal{E}^n_{x,e,f}$, up to measure 0. This implies the first inequality below, while the second one is by (4.1):

$$\begin{aligned}
\mathbf{P}(\mathcal{A}'_o \setminus \mathcal{A}_o) &\geq \hat{\mathbf{P}}\Big(\bigcup_{e \notin B(o,R), f \notin B(o,R), e=\{x,y\}} \pi^f_e(\mathcal{A}'_o \cap \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x}) \Big) \\
&= \sum_{e \notin B(o,R), f \notin B(o,R), e=\{x,y\}} \hat{\mathbf{P}}(\pi^f_e(\mathcal{A}'_o \cap \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x})) \\
&\geq \delta \sum_{e \notin B(o,R), f \notin B(o,R), e=\{x,y\}} \hat{\mathbf{P}}(\mathcal{A}'_o \cap \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x}) \\
&\geq -\delta\varepsilon + \delta \sum_{e \notin B(o,R), f \notin B(o,R), e=\{x,y\}} \hat{\mathbf{P}}(\mathcal{A}_o \cap \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x}),
\end{aligned}$$

for every $n \geq 0$. We may assume $\delta < 1$. This implies right away the following inequalities for $n \geq 0$:

$$2\varepsilon \geq \varepsilon + \mathbf{P}(\mathcal{A}'_o \setminus \mathcal{A}_o) \geq \delta \sum_{e \notin B(o,R), f \notin B(o,R), e=\{x,y\}} \hat{\mathbf{P}}(\mathcal{A}_o \cap \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x}).$$

By choosing n large enough, the right-hand side is arbitrarily close to $\delta \sum_{e,f \in E, e=\{x,y\}} \hat{\mathbf{P}}(\mathcal{A}_o \cap \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x})$, using that C_o is infinite. So fix $n(\varepsilon)$ such that

$$\left|\delta \sum_{e,f\in E, e=\{x,y\}} \hat{\mathbf{P}}(\mathcal{A}_o \cap \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x}) \right.$$
$$\left. - \delta \sum_{e\notin B(o,R), f\notin B(o,R), e=\{x,y\}} \hat{\mathbf{P}}(\mathcal{A}_o \cap \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x}) \right| < \varepsilon$$

holds for every $n \geq n(\varepsilon)$. Then for $n \geq n(\varepsilon)$, we have

$$\begin{aligned} 3\varepsilon &\geq \delta \sum_{e,f\in E, e=\{x,y\}} \hat{\mathbf{P}}(\mathcal{A}_o \cap \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x}) \\ &= \delta \sum_{e,f\in E, e=\{x,y\}} \hat{\mathbf{P}}(\mathcal{A}_o \cap \mathcal{E}^0_{x,e,f} \cap \mathcal{P}^f_{e,x}). \end{aligned}$$

Here, the last equation holds because $\sum_{e,f\in E,e=\{x,y\}} \hat{\mathbf{P}}(\mathcal{A}_o \cap \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x})$ is the same as $\sum_{e,f\in E,e=\{x,y\}} \hat{\mathbf{P}}(\mathcal{A}_{W(n)} \cap \mathcal{E}^n_{x,e,f} \cap \mathcal{P}^f_{e,x})$, and the latter is equal to $\sum_{e,f\in E,e=\{x,y\}} \hat{\mathbf{P}}(\mathcal{A}_o \cap \mathcal{E}^0_{x,e,f} \cap \mathcal{P}^f_{e,x})$ by stationarity, given in Lemma 4.1. The disjoint union $\bigcup_{e,f\in E,e=\{x,y\}} \mathcal{A}_o \cap \mathcal{E}^0_{x,e,f} \cap \mathcal{P}^f_{e,x}$ is contained in $\mathrm{Piv}(e, x, \mathcal{A}, \delta)$, whose probability is above some uniform positive constant for every ε by Lemma 4.2. Letting ε tend to zero, we arrive to a contradiction. □

EXAMPLE 4.1. The next example shows that the condition that $\mathcal{F}$ has a tree with infinitely many ends cannot be removed with all other conditions unchanged, that is, there exists a weakly insertion tolerant random forest $\mathcal{F}$ with all components infinite, but such that its components can be distinguished. We first present an example when G is quasitransitive, then sketch an example with G transitive. Let $G' := \mathbb{Z}^5$ and $\mathcal{F}'$ be the WUSF (= FUSF) on G'. Let G be the quasitransitive graph as follows. For each $v \in V(G')$, define two new vertices v' and v'', and let $V(G) = \bigcup_{v\in V(G')}\{v, v', v''\}$. Add all edges $\{v, v'\}$, $\{v, v''\}$, $\{v', v''\}$ besides the edges of G' [so $E(G) = \bigcup_{v\in V(G')}\{\{v, v'\}, \{v, v''\}, \{v', v''\}\} \cup E(G')$]. Define $\mathcal{F}$ from $\mathcal{F}'$ by first taking $\mathcal{F}'$ on $E(G') \subset E(G)$. For each cluster C of $\mathcal{F}'$, flip a coin. If it comes up head, for each $v \in C$ add one of the pairs $\{v, v'\}$, $\{v, v''\}$ or $\{v, v'\}$, $\{v', v''\}$ or $\{v, v''\}$, $\{v'', v'\}$ to the edge set of $\mathcal{F}$, and decide which one to add uniformly, and independently over the v. If the coin came up tail, then do the same thing, but now the probability of adding edge $\{v, v'\}$, $\{v, v''\}$ is $1/2$, while the probabilities for adding edges $\{v, v'\}$, $\{v', v''\}$ or edges $\{v, v''\}$, $\{v'', v'\}$ are $1/4$. This way we defined an invariant spanning forest $\mathcal{F}$ on G. Clusters containing trees of $\mathcal{F}'$ where the coin tosses came up head are distinguishable from those where it came up tail, from the densities of the 2-paths and "cherries" hanging off the vertices in $V(G')$. On the other hand, using the fact that the components of $\mathcal{F}'$ are one-ended and that the WUSF is weakly insertion tolerant, one can check that $\mathcal{F}$

is also weakly insertion tolerant. (Note however that if we applied the same construction for an $\mathcal{F}'$ where every cluster has infinitely many ends, then the resulting $\mathcal{F}$ would not be weakly insertion tolerant.)

To have a transitive example, we will use $G = \mathbb{Z}^5 \square K_4$, that is, the Cartesian product of Z^5 and the complete graph K_4 on vertex set $\{v_1, v_2, v_3, v_4\}$. Start with the USF $\mathcal{F}'$ on $\mathbb{Z}^5$ as before, and for each edge $\{x, y\} \in \mathcal{F}'$, choose one of the four edges $\{(x, v_i), (y, v_i)\}$ uniformly at random and independently. Let the chosen edges be part of the forest $\mathcal{F}$. For each component of $\mathcal{F}'$, flip a coin. For each vertex $x \in \mathbb{Z}^5$, we will add a random spanning subgraph of $x \times K_4$ to $\mathcal{F}$. The spanning subgraph will be chosen to be a uniform spanning path with probability p and a uniform spanning star with probability $1 - p$, where $p = 1/2$ if the coin flip for the cluster of x came up heads, and $p = 1/3$ if it came up tail.

REMARK 4.4. All results in the paper are valid in the more general setting when G is a unimodular random network. More precisely, let (G, o) be an ergodic unimodular random network, as defined in [1]. The definitions of the uniform and minimal spanning forests can be extended to this setting; see Section 7 of [1]. The definition of weak insertion tolerance has to be modified by requiring the properties in Definition 1 to hold for every edge e of almost every (G, o). Expectation in the proofs is understood jointly with respect to the distribution of the unimodular random graph and the random forest. To apply the proof of Theorem 3.3 directly, one needs to have finite expected degree for (G, o). However, by using cutoff (applying mass transport only when the vertex has a degree below some properly chosen bound), one can extend the proof to an arbitrary unimodular graph.

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REFERENCES

[1] ALDOUS, D. and LYONS, R. (2007). Processes on unimodular random networks. *Electron. J. Probab.* **12** 1454–1508. MR2354165

[2] BENJAMINI, I., KESTEN, H., PERES, Y. and SCHRAMM, O. (2004). Geometry of the uniform spanning forest: Transitions in dimensions 4, 8, 12, ... *Ann. of Math.* (2) **160** 465–491. MR2123930

[3] BENJAMINI, I., LYONS, R., PERES, Y. and SCHRAMM, O. (1999). Group-invariant percolation on graphs. *Geom. Funct. Anal.* **9** 29–66. MR1675890

[4] BENJAMINI, I., LYONS, R., PERES, Y. and SCHRAMM, O. (2001). Uniform spanning forests. *Ann. Probab.* **29** 1–65. MR1825141

[5] CHIFAN, I. and IOANA, A. (2010). Ergodic subequivalence relations induced by a Bernoulli action. *Geom. Funct. Anal.* **20** 53–67.

[6] GABORIAU, D. and LYONS, R. (2009). A measurable-group-theoretic solution to von Neumann's problem. *Invent. Math.* **177** 533–540. MR2534099

[7] HUTCHCROFT, T. and NACHMIAS, A. (2017). Indistinguishability of trees in uniform spanning forests. *Probab. Theory Related Fields* **168** 113–152. MR3651050

[8] LYONS, R. and PERES, Y. (2016). *Probability on Trees and Networks*. Cambridge Univ. Press, Cambridge. MR3616205

[9] LYONS, R., PERES, Y. and SCHRAMM, O. (2006). Minimal spanning forests. *Ann. Probab.* **34** 1665–1692. MR2271476

[10] LYONS, R. and SCHRAMM, O. (1999). Indistinguishability of percolation clusters. *Ann. Probab.* **27** 1809–1836. MR1742889

[11] LYONS, R. and SCHRAMM, O. (1999). Stationary measures for random walks in a random environment with random scenery. *New York J. Math.* **5** 107–113. MR1703207

[12] PEMANTLE, R. (1991). Choosing a spanning tree for the integer lattice uniformly. *Ann. Probab.* **19** 1559–1574. MR1127715

[13] TIMÁR, Á. (2006). Ends in free minimal spanning forests. *Ann. Probab.* **34** 865–869. MR2243871

[14] TIMÁR, A. (2006). Neighboring clusters in Bernoulli percolation. *Ann. Probab.* **34** 2332–2343. MR2294984

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