The near-critical Gibbs measure of the branching random walk

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Abstract. Consider the supercritical branching random walk on the real line in the boundary case and the associated Gibbs measure $\nu_{n,\beta}$ on the nth generation, which is also the polymer measure on a disordered tree with inverse temperature β. The convergence of the partition function $W_{n,\beta}$, after rescaling, towards a nontrivial limit has been proved by Aïdékon and Shi (*Ann. Probab.* **42** (3) (2014) 959–993) in the critical case $\beta = 1$ and by Madaule (*J. Theoret. Probab.* **30** (1) (2017) 27–63) when $\beta > 1$. We study here the near-critical case, where $\beta_n \to 1$, and prove the convergence of W_{n,β_n}, after rescaling, towards a constant multiple of the limit of the derivative martingale. Moreover, trajectories of particles chosen according to the Gibbs measure $\nu_{n,\beta}$ have been studied by Madaule (*Stochastic Process. Appl.* **126** (2) (2016) 470–502) in the critical case, with convergence towards the Brownian meander, and by Chen, Madaule and Mallein (On the trajectory of an individual chosen according to supercritical gibbs measure in the branching random walk (2015) Preprint) in the strong disorder regime, with convergence towards the normalized Brownian excursion. We prove here the convergence for trajectories of particles chosen according to the near-critical Gibbs measure and display continuous families of processes from the meander to the excursion or to the Brownian motion.

Résumé. Considérons une marche aléatoire branchante surcritique réelle dans le cas frontière et la mesure de Gibbs associée $\nu_{n,\beta}$ sur la n-ième génération, qui est aussi la mesure de polymère sur un arbre désordonné avec température inverse β. La convergence de la fonction de partition $W_{n,\beta}$, après renormalisation, vers une limite non-triviale a été démontrée par Aïdékon et Shi (*Ann. Probab.* **42** (3) (2014) 959–993) dans le cas critique $\beta = 1$ et par Madaule (*J. Theoret. Probab.* **30** (1) (2017) 27–63) pour $\beta > 1$. On s'intéresse ici au cas presque-critique, où $\beta_n \to 1$, et on montre la convergence de W_{n,β_n}, après renormalisation, vers la limite de la martingale dérivée à un facteur multiplicatif près. D'autre part, les trajectoires de particules tirées selon la mesure de Gibbs $\nu_{n,\beta}$ ont été étudiées par Madaule (*Stochastic Process. Appl.* **126** (2) (2016) 470–502) dans le cas critique, avec convergence vers le méandre brownien, et par Chen, Madaule et Mallein (On the trajectory of an individual chosen according to supercritical gibbs measure in the branching random walk (2015) Preprint) dans le régime de désordre fort, avec convergence vers l'excursion brownienne. On montre ici la convergence des trajectoires de particules tirées selon la mesure de Gibbs presque-critique et cela fait apparaître une famille continue de processus allant du méandre jusqu'à l'excursion ou jusqu'au mouvement brownien.

1. Introduction and main results

1.1. *Definitions and assumptions*

The branching random walk on the real line is a natural extension of the Galton–Watson process, with addition of a position to each individual, and is defined as follows. Initially, there is a single particle at the origin, forming the 0th generation. It gives birth to children, scattered in $\mathbb{R}$ according to some point process $\mathcal{L}$ and forming the 1st generation. Then, each particle of the 1st generation produces its own children disposed around its position according to the law of $\mathcal{L}$ independently of others: this set of children forms the 2nd generation. The system goes on indefinitely, unless there is no particle at some generation.

The genealogical tree of the branching random walk, denoted by $\mathbb{T}$, is a Galton–Watson tree (where an individual can have an infinity of children). For $z \in \mathbb{T}$, we denote by $|z|$ the generation of the particle z and by $V(z)$ its position in $\mathbb{R}$. We denote by Ψ the log-Laplace transform of $\mathcal{L}$: we set, for each $\beta \in \mathbb{R}_+$,

$$\Psi(\beta) := \log \mathbb{E}\left[\sum_{|z|=1} \mathrm{e}^{-\beta V(z)} \right] \in (-\infty, \infty],$$

noting that $\mathcal{L}$ has the same law as $(V(z), |z| = 1)$.

Throughout the paper, we assume the following integrability conditions on the reproduction law $\mathcal{L}$. First of all, we need to assume that the Galton–Watson tree $\mathbb{T}$ is *supercritical*, that is

$$\mathbb{E}\left[\sum_{|z|=1} 1 \right] > 1, \tag{1.1}$$

so that the survival event S has positive probability and, thus, we can introduce the new probability $\mathbb{P}^* := \mathbb{P}(\cdot | S)$. Moreover, we work in the *boundary case* (Biggins and Kyprianou [12]) by assuming

$$\mathbb{E}\left[\sum_{|z|=1} \mathrm{e}^{-V(z)} \right] = 1 \quad \text{and} \quad \mathbb{E}\left[\sum_{|z|=1} V(z)\mathrm{e}^{-V(z)} \right] = 0, \tag{1.2}$$

which means $\Psi(1) = 0$ and $\Psi'(1) = 0$. See the arXiv version of Jaffuel [30] for discussion on the cases where the branching random walk can be reduced to assumption (1.2). We assume also that

$$\sigma^2 := \mathbb{E}\left[\sum_{|z|=1} V(z)^2 \mathrm{e}^{-V(z)} \right] \in (0, \infty) \quad \text{and} \quad \mathbb{E}\left[X(\log_+ X)^2 + \widetilde{X} \log_+ \widetilde{X} \right] < \infty, \tag{1.3}$$

where we set, for $y \geq 0$, $\log_+ y := \max(0, \log y)$ and

$$X := \sum_{|z|=1} \mathrm{e}^{-V(z)} \quad \text{and} \quad \widetilde{X} := \sum_{|z|=1} V(z)_+ \mathrm{e}^{-V(z)} \tag{1.4}$$

with $V(z)_+ := \max(0, V(z))$. The first part of (1.3) gives $\Psi''(1) = \sigma^2$. We will say that $\mathcal{L}$ is (h, a)-*lattice* if $h > 0$ is the largest real number such that the support of $\mathcal{L}$ is contained by $a + h\mathbb{Z}$ and, then, h is called the *span* of $\mathcal{L}$. In this paper, we work in both lattice and nonlattice cases, but we will need sometime to distinguish these cases. Finally, we set two last assumptions that are not supposed to hold in the whole paper, but only in specific cases of the results. The following assumption,

$$\exists 0 < \delta_0 < 1 : \mathbb{E}\left[\sum_{|z|=1} \mathrm{e}^{-(1-\delta_0)V(z)} \right] < \infty, \tag{1.5}$$

means that Ψ is finite on $[1 - \delta_0, 1]$ and, thus, analytic on $(1 - \delta_0, 1)$ and, using (1.2) and (1.3), we have $\Psi(\beta) = \frac{\sigma^2}{2}(\beta - 1)^2 + o((\beta - 1)^2)$ as $\beta \uparrow 1$ by a Taylor expansion. The second one,

$$\exists 0 < \delta_1 < \delta_2 < 1/4 : \mathbb{E}\left[\left(\sum_{|z|=1} \mathrm{e}^{-(1-\delta_1)V(z)} \right)^{1+2\delta_2} \right] < \infty, \tag{1.6}$$

comes from Madaule [34] and is probably not optimal for the results where it is used. Under assumption (1.6), Ψ is finite on $[1 - \delta_1, 1 + \delta_2/2]$ and, therefore, analytic on a neighbourhood of 1, so we can improve the Taylor expansion: $\Psi(\beta) = \frac{\sigma^2}{2}(\beta - 1)^2 + O((\beta - 1)^3)$ as $\beta \to 1$.

For $n \in \mathbb{N}$ and $\beta \in \mathbb{R}_+$, we set

$$W_{n,\beta} := \sum_{|z|=n} \mathrm{e}^{-\beta V(z)} \quad \text{and} \quad \nu_{n,\beta} := \frac{1}{W_{n,\beta}} \sum_{|z|=n} \mathrm{e}^{-\beta V(z)} \delta_z,$$

as soon as $W_{n,\beta} < \infty$, which holds a.s. for $\beta \geq 1$ under assumption (1.2). Then, $\nu_{n,\beta}$ is a random probability measure on $\{z \in \mathbb{T} : |z| = n\}$, which is called the *Gibbs measure* of parameter β on the nth generation of the branching random walk. It is also the law of a directed polymer on the tree $\mathbb{T}$ in a random environment, introduced by Derrida and Spohn [24] as a mean field limit for directed polymer on a lattice as dimension goes to infinity: with this terminology, $V(z)$ is the energy of the path leading from the root to particle z, β is the inverse temperature and $W_{n,\beta}$ is the *partition function*. In the case $\beta = \infty$, we can define $\nu_{n,\infty}$ as the uniform measure on the random set $\{|x| = n : V(x) = \min_{|z|=n} V(z)\}$.

According to Derrida and Spohn [24], there is a critical parameter $\beta_c > 0$ for the directed polymer on a disordered tree (with our setting $\beta_c = 1$, see Section 1.2 for more details) and our aim in this paper is to study the near-critical case, where β depends on n and tends from above and below to $\beta_c = 1$ as $n \to \infty$. The near-critical case has been recently studied for the directed polymer on the lattice in dimension $1 + 1$ and $1 + 2$ by Alberts, Khanin and Quastel [4] and Caravenna, Sun and Zygouras [17,18], with the emergence of the so-called intermediate disorder regime. For the polymer on a tree, some work near criticality has been done by Alberts and Ortgiese [5] and Madaule [34], mostly on the partition function.

Before stating our results, we recall some well-known properties of the branching random walk, that hold under assumptions (1.1), (1.2) and (1.3). First, the sequence

$$D_n := \sum_{|z|=n} V(z) \mathrm{e}^{-V(z)}, \quad n \in \mathbb{N},$$

is a martingale, called the *derivative martingale*, and Biggins and Kyprianou [11] (under slightly stronger assumptions) and Aïdékon [1] showed that we have

$$D_n \xrightarrow[n \to \infty]{} D_\infty > 0, \quad \mathbb{P}^*\text{-a.s.} \tag{1.7}$$

Moreover, Chen [20] proved that these assumptions are optimal for the nontriviality of D_∞. Furthermore, Aïdékon [1] also showed that, in the nonlattice case, $\min_{|z|=n} V(z) - \frac{3}{2} \log n$ converges in law under $\mathbb{P}^*$ and described the limit as a random shift (depending on D_∞) of a Gumbel distribution. In the lattice case, we do not have this convergence, but the tightness still holds (see Equation (4.20) of Chen [21] or Mallein [36]): for each $\varepsilon > 0$, it exists $C > 0$ such that

$$\mathbb{P}\left(\min_{|z|=n} V(z) - \frac{3}{2} \log n \in [-C, C] \right) \geq 1 - \varepsilon, \tag{1.8}$$

for n large enough.

1.2. *The partition function*

The process $(W_{n,\beta})_{n \in \mathbb{N}}$ for some fixed $\beta \in \mathbb{R}_+$ has been intensively studied because, if $\Psi(\beta)$ is finite, then the renormalized process $(\widetilde{W}_{n,\beta})_{n \in \mathbb{N}} := (\mathrm{e}^{-n\Psi(\beta)} W_{n,\beta})_{n \in \mathbb{N}}$ is a nonnegative martingale, called *additive martingale*, and, therefore, converges a.s. to some limit $\widetilde{W}_{\infty,\beta}$. Kahane and Peyrière [31], Biggins [7] and Lyons [33] have determined when this limit is nontrivial: under the additional assumption that the expectation $\mathbb{E}[W_{1,\beta} \log_+ W_{1,\beta}]$ is finite, we have the following dichotomy

$$\begin{cases} \widetilde{W}_{\infty,\beta} > 0 & \mathbb{P}^*\text{-a.s. if } \beta \in [0, 1), \\ \widetilde{W}_{\infty,\beta} = 0 & \mathbb{P}^*\text{-a.s. if } \beta \geq 1. \end{cases} \tag{1.9}$$

With the terminology of polymers' literature (see [23]), the region $\beta \in [0, 1)$ is thus called the *weak disorder regime* and the region $\beta \geq 1$ the *strong disorder regime*.

In the strong disorder regime $\beta \geq 1$, it is natural to seek a proper renormalization of $W_{n,\beta}$, so that it converges towards a nontrivial limit. This question has already been answered when β does not depend on n. In the critical case $\beta = 1$, Aïdékon and Shi [3] proved that we have

$$\sqrt{n} W_{n,1} \xrightarrow[n \to \infty]{} \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} D_\infty, \quad \text{in } \mathbb{P}^*\text{-probability}, \tag{1.10}$$

and that the particles that contribute mainly to $W_{n,1}$ are those of order $\sqrt{n}$. On the other hand, Theorem 2.3 of Madaule [35] shows (in the nonlattice case) that, in the case $\beta > 1$, we have

$$n^{3\beta/2} W_{n,\beta} \xrightarrow[n \to \infty]{} Z_\beta D_\infty, \quad \text{in law},$$

where Z_β is a random variable independent of D_∞, whose law have been described by Barral, Rhodes and Vargas [6]. Moreover, the particles that contribute mainly to $W_{n,\beta}$ are in $[\frac{3}{2} \log n - C, \frac{3}{2} \log n + C]$ with C some large constant and, thus, are close to the lowest particle at time n (see (1.8)).

Since there is a discontinuity in the size of the partition function $W_{n,\beta}$ as $\beta \downarrow 1$ between $n^{-3\beta/2}$ and $n^{-1/2}$, and also as $\beta \uparrow 1$ between exponential and polynomial size, we try to understand this transition by considering the near-critical case where $\beta_n \to 1$ as $n \to \infty$. Some work has been done to this end by Alberts and Ortgiese [5], who considered the case where $\beta_n = 1 \pm n^{-\delta}$ for $\delta > 0$ and proved, under stronger assumptions, that

$$W_{n,\beta_n} = \begin{cases} n^{2\delta - 3/2 + o(1)} & \text{if } \beta_n = 1 + n^{-\delta} \text{ with } 0 < \delta < 1/2, \\ n^{-1/2 + o(1)} & \text{if } \beta_n = 1 \pm n^{-\delta} \text{ with } \delta \geq 1/2, \\ \exp(\frac{\sigma^2}{2} n^{1-2\delta}(1 + o(1))) & \text{if } \beta_n = 1 - n^{-\delta} \text{ with } 0 < \delta < 1/2, \end{cases} \quad \text{in } \mathbb{P}^*\text{-probability}.$$

Moreover, the behavior of the limit $\widetilde{W}_{\infty,\beta}$ of the additive martingale has been studied near criticality by Madaule [34], who showed under assumption (1.6) the following convergence

$$\frac{\widetilde{W}_{\infty,\beta}}{1 - \beta} \xrightarrow[\beta \uparrow 1]{} 2 D_\infty, \quad \text{in } \mathbb{P}^*\text{-probability}, \tag{1.11}$$

and, although this is not exactly our setting where $n \to \infty$ and $\beta \to 1$ simultaneously, this result and its proof are useful in this paper. The following theorem improves Alberts and Ortgiese's result, showing convergence to a nontrivial limit of W_{n,β_n} after rescaling, for every sequence $\beta_n := 1 \pm 1/\alpha_n$.

Theorem 1.1. *Assume* (1.1), (1.2), (1.3) *and that* $\alpha_n \to \infty$ *as* $n \to \infty$. *Let* $\mathcal{M}$ *denotes the Brownian meander of length* 1.

(i) *If* $\beta_n := 1 + 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \infty$ *as* $n \to \infty$, *then we have*

$$\frac{n^{3\beta_n/2}}{\alpha_n^2} W_{n,\beta_n} \xrightarrow[n \to \infty]{} \frac{1}{\sigma^3} \sqrt{\frac{2}{\pi}} D_\infty, \quad \text{in } \mathbb{P}^*\text{-probability}.$$

(ii) *If* $\beta_n := 1 + 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \gamma \in [0, \infty)$ *as* $n \to \infty$, *then we have*

$$\sqrt{n} W_{n,\beta_n} \xrightarrow[n \to \infty]{} \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} \mathbb{E}[e^{-\sigma \gamma \mathcal{M}(1)}] D_\infty, \quad \text{in } \mathbb{P}^*\text{-probability}.$$

(iii) *If* (1.5) *holds,* $\beta_n := 1 - 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \gamma \in [0, \infty)$ *as* $n \to \infty$, *then we have*

$$\sqrt{n} W_{n,\beta_n} \xrightarrow[n \to \infty]{} \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} \mathbb{E}[e^{\sigma \gamma \mathcal{M}(1)}] D_\infty, \quad \text{in } \mathbb{P}^*\text{-probability}.$$

(iv) *If* (1.6) *holds,* $\beta_n := 1 - 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \infty$ *as* $n \to \infty$, *then we have*

$$\alpha_n \mathrm{e}^{-n\Psi(\beta_n)} W_{n,\beta_n} \xrightarrow[n\to\infty]{} 2D_\infty, \quad \textit{in } \mathbb{P}^*\textit{-probability.}$$

Note that, in case (i), the size of W_{n,β_n} can be a $o(n^{-3/2})$ as soon as $\alpha_n \log \alpha_n \ll \log n$: this possibility does not appear in Alberts and Ortgiese's result.

1.3. *Trajectory of particles under the Gibbs measure*

The second main result of this paper concerns the trajectory of particles chosen according to the Gibbs measure. We first need to introduce some additional notation. For a particle z at the nth generation and $0 \le i \le n$, we denote by z_i its ancestor at the ith generation and we set

$$\mathbf{V}(z) := \left(\frac{V(z_{\lfloor tn \rfloor})}{\sigma\sqrt{n}}, t \in [0, 1] \right)$$

the rescaled trajectory of z's lineage. We work here in the set $\mathcal{D}([0, 1])$ of the càdlàg functions from $[0, 1]$ to $\mathbb{R}$, with the Skorokhod distance (see Section A.2). For $n \in \mathbb{N}$ and $\beta \ge 1$, we denote by $\mu_{n,\beta}$ the image measure of the Gibbs measure $\nu_{n,\beta}$ by $\mathbf{V}$, that is the random measure on $\mathcal{D}([0, 1])$ such that, for each $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$, we have

$$\mu_{n,\beta}(F) := \frac{1}{W_{n,\beta}} \sum_{|z|=n} \mathrm{e}^{-\beta V(z)} F\big(\mathbf{V}(z)\big),$$

where $\mathcal{C}_b(\mathcal{D}([0, 1]))$ denotes the set of continuous bounded functions from $\mathcal{D}([0, 1])$ to $\mathbb{R}$.

Convergence of $\mu_{n,\beta}$ has already been studied in the strong disorder regime when β does not depend on n, under assumptions (1.1), (1.2) and (1.3). In the critical case $\beta = 1$, Theorem 1.2 of Madaule [34] shows[1] that, for all $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$,

$$\mu_{n,1}(F) \xrightarrow[n\to\infty]{} \mathbb{E}\big[F(\mathcal{M})\big], \quad \text{in } \mathbb{P}^*\text{-probability.} \tag{1.12}$$

On the other hand, in the case $\beta > 1$, Chen, Madaule and Mallein [22] proved (in the nonlattice case) that, under $\mathbb{P}^*$, we have, for all uniformly continuous $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$,

$$\mu_{n,\beta}(F) \xrightarrow[n\to\infty]{} \sum_{k\in\mathbb{N}} p_k F(\mathfrak{e}_k), \quad \text{in law,} \tag{1.13}$$

where $(\mathfrak{e}_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. normalized Brownian excursions and $(p_k)_{k\in\mathbb{N}}$ follows an independent Poisson–Dirichlet distribution with parameter $(\beta^{-1}, 0)$. The convergence in (1.13) is believed to hold for all $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$. Moreover, Chen [21] considered the case $\beta = \infty$ and showed that, for all $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$, we have $\mathbb{E}^*[\mu_{n,\infty}(F)] \to \mathbb{E}[F(\mathfrak{e})]$ as $n \to \infty$, where $\mathfrak{e}$ denotes the normalized Brownian excursion.

Finally, in the weak disorder regime $\beta < 1$, if there is some $p > 1$ such that $\mathbb{E}[W_{1,\beta}^p] < \infty$, we have the following convergence, with $\sigma_\beta^2 := \Psi''(\beta)$, for all $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$,

$$\frac{1}{W_{n,\beta}} \sum_{|z|=n} \mathrm{e}^{-\beta V(z)} F\left(\frac{V(z_{\lfloor tn \rfloor}) + tn\Psi'(\beta)}{\sigma_\beta\sqrt{n}}, t \in [0, 1] \right) \xrightarrow[n\to\infty]{} \mathbb{E}\big[F(B)\big], \tag{1.14}$$

in $\mathbb{P}^*$-probability.[2] It means that the trajectory is a straight line of slope $-\Psi'(\beta) > 0$ at first order and around which Brownian fluctuations occur at second order.

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[1]Actually, Madaule [34] considers the linear interpolation of the trajectory, instead of $\mathbf{V}$, and the convergence on $\mathcal{C}([0, 1])$, instead of $\mathcal{D}([0, 1])$. But the convergence (1.12) follows from Madaule's result.

[2]To our knowledge, this result has not been proved yet (except when F only depends on the final position, see [8]). The proof is very similar to the proof of case (iv) of Theorem 1.2, with no need to introduce the lower barrier and using Theorem 1 of [9] to get that $(\widetilde{W}_{n,\beta})_{n\ge 0}$ converges in L^p. See the arXiv version of this paper for more details.

Our aim is to prove the convergence for trajectories of particles chosen according to the Gibbs measure in the near-critical case, in order to explain how happens the transition between the Brownian excursion, the Brownian meander and the straight line with Brownian fluctuations.

Theorem 1.2. *Assume* (1.1), (1.2), (1.3) *and that* $\alpha_n \to \infty$ *as* $n \to \infty$. *Let* $\mathfrak{e}$ *denotes the normalized Brownian excursion,* $\mathcal{M}$ *the Brownian meander of length* 1 *and* B *the Brownian motion.*

(i) *If* $\beta_n := 1 + 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \infty$ *as* $n \to \infty$, *then we have, for all* $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$,

$$\mu_{n,\beta_n}(F) \xrightarrow[n\to\infty]{} \mathbb{E}\big[F(\mathfrak{e})\big], \quad \text{in } \mathbb{P}^*\text{-probability.}$$

(ii) *If* $\beta_n := 1 + 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \gamma \in [0, \infty)$ *as* $n \to \infty$, *then we have, for all* $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$,

$$\mu_{n,\beta_n}(F) \xrightarrow[n\to\infty]{} \frac{1}{\mathbb{E}[e^{-\sigma\gamma\mathcal{M}(1)}]}\mathbb{E}\big[e^{-\sigma\gamma\mathcal{M}(1)}F(\mathcal{M})\big], \quad \text{in } \mathbb{P}^*\text{-probability.}$$

(iii) *If* (1.5) *holds,* $\beta_n := 1 - 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \gamma \in [0, \infty)$ *as* $n \to \infty$, *then we have, for all* $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$,

$$\mu_{n,\beta_n}(F) \xrightarrow[n\to\infty]{} \frac{1}{\mathbb{E}[e^{\sigma\gamma\mathcal{M}(1)}]}\mathbb{E}\big[e^{\sigma\gamma\mathcal{M}(1)}F(\mathcal{M})\big], \quad \text{in } \mathbb{P}^*\text{-probability.}$$

(iv) *If* (1.6) *holds,* $\beta_n := 1 - 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \infty$ *as* $n \to \infty$, *then we have, for all* $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$,

$$\frac{1}{W_{n,\beta_n}} \sum_{|z|=n} e^{-\beta_n V(z)} F\left(\frac{V(z_{\lfloor tn \rfloor}) + tn\Psi'(\beta_n)}{\sigma\sqrt{n}}, t \in [0, 1] \right) \xrightarrow[n\to\infty]{} \mathbb{E}\big[F(B)\big], \quad \text{in } \mathbb{P}^*\text{-probability.}$$

We now state a corollary of this theorem, concerning the location of the mass of the Gibbs measure. Note that, for the terminology of the polymers' literature, this position is the typical energy of the polymer in the near-critical case.

Corollary 1.3. *Assume* (1.1), (1.2), (1.3) *and that* $\alpha_n \to \infty$ *as* $n \to \infty$.

(i) *If* $\beta_n := 1 + 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \infty$ *as* $n \to \infty$, *then, for all* $\varepsilon > 0$, *it exists* $C > 0$ *such that for n large enough*

$$\mathbb{P}^*\left(\nu_{n,\beta_n}\left(\left[\frac{3}{2}\log n + C^{-1}\alpha_n, \frac{3}{2}\log n + C\alpha_n \right] \right) \geq 1 - \varepsilon \right) \geq 1 - \varepsilon.$$

(ii) *If* $\beta_n := 1 + 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \gamma \in [0, \infty)$ *as* $n \to \infty$, *then, for all* $\varepsilon > 0$, *it exists* $C > 0$ *such that for n large enough*

$$\mathbb{P}^*\left(\nu_{n,\beta_n}\left(\left[C^{-1}\sqrt{n}, C\sqrt{n} \right] \right) \geq 1 - \varepsilon \right) \geq 1 - \varepsilon.$$

(iii) *If* (1.5) *holds,* $\beta_n := 1 - 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \gamma \in [0, \infty)$ *as* $n \to \infty$, *then the same property as in case* (ii) *holds.*

(iv) *If* (1.6) *holds,* $\beta_n := 1 - 1/\alpha_n$ *and* $\sqrt{n}/\alpha_n \to \infty$ *as* $n \to \infty$, *then, for all* $\varepsilon > 0$, *it exists* $C > 0$ *such that for n large enough*

$$\mathbb{P}^*\left(\nu_{n,\beta_n}\left(\left[-\Psi'(\beta_n)n - C\sqrt{n}, -\Psi'(\beta_n)n + C\sqrt{n} \right] \right) \geq 1 - \varepsilon \right) \geq 1 - \varepsilon.$$

Proof. In cases (ii), (iii) and (iv), it is a direct consequence of Theorem 1.2. For case (i), the assertion will be proved at the end of Section 5.2. $\qquad\square$

1.4. *Genealogy under the Gibbs measure*

We state here a direct consequence of Theorem 1.2 concerning the overlap in the branching random walk, introduced by Derrida and Spohn [24] in the context of polymers on trees. We set, for $x, y \in \mathbb{T}$,

$$|x \wedge y| := \max\{k \leq \min(|x|, |y|) : x_k = y_k\},$$

that is the generation of the most recent common ancestor of x and y. For some couple of particles (x, y) chosen according to $\nu_{n,\beta}^{\otimes 2}$, we are interested in the *overlap* between x and y defined by $|x \wedge y|/n$. Thus, we set, for a Borel set $A \subset [0, 1]$,

$$\omega_{n,\beta}(A) := \nu_{n,\beta}^{\otimes 2}\left(\frac{|x \wedge y|}{n} \in A\right)$$

so that $\omega_{n,\beta}$ is a random probability measure on $[0, 1]$.

Madaule [34] gives the following consequence of (1.12) in the case $\beta = 1$:

$$\omega_{n,1} \xrightarrow[n\to\infty]{} \delta_0, \quad \text{in } \mathbb{P}^*\text{-probability.}$$

For the other extremal case $\beta = \infty$, one can prove in the nonlattice case only[3] that,

$$\omega_{n,\infty} \xrightarrow[n\to\infty]{} \delta_1, \quad \text{in } \mathbb{P}^*\text{-probability.} \tag{1.15}$$

The transition between this two cases appears with case $\beta \in (1, \infty)$, with which Chen, Madaule and Mallein [22] deal, but their result (1.13) is only proved for $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$ and, thus, the convergence in law of $\omega_{n,\beta}$ cannot be obtained as a corollary. However, Mallein [37] shows that, under $\mathbb{P}^*$, we have

$$\omega_{n,\beta} \xrightarrow[n\to\infty]{} (1 - \pi_\beta)\delta_0 + \pi_\beta\delta_1, \quad \text{in law,}$$

where $\pi_\beta := \sum_{k \in \mathbb{N}} p_k^2$ and $(p_k)_{k \in \mathbb{N}}$ follows a Poisson–Dirichlet distribution with parameter $(\beta^{-1}, 0)$. It confirms a conjecture of Derrida and Spohn [24].

In the near critical case, we can state the following consequence of Theorem 1.2.

Corollary 1.4. *In each case of Theorem* 1.1, *under the same assumptions, we have*

$$\omega_{n,\beta_n} \xrightarrow[n\to\infty]{} \delta_0, \quad \text{in } \mathbb{P}^*\text{-probability.}$$

Proof. We give the proof for the case (i), but it is exactly the same for other cases (only the limiting trajectory changes). First note that, for all $F \in \mathcal{C}_b(\mathcal{D}([0, 1])^2)$, we have, with $\mathfrak{e}$ and $\mathfrak{e}'$ independent normalized Brownian excursions,

$$\mu_{n,\beta_n}^{\otimes 2}(F) = \frac{1}{W_{n,\beta_n}^2} \sum_{|x|=|y|=n} e^{-\beta_n V(x) - \beta_n V(y)} F\big(\mathbf{V}(x), \mathbf{V}(y)\big) \xrightarrow[n\to\infty]{} \mathbb{E}\big[F\big(\mathfrak{e}, \mathfrak{e}'\big)\big], \tag{1.16}$$

in $\mathbb{P}^*$-probability and therefore in L^1, because $\mu_{n,\beta_n}^{\otimes 2}(F)$ is bounded. Indeed, by Theorem 1.2, (1.16) holds when F is of the form $F(x, y) = G_1(x)G_2(y)$ for some $G_1, G_2 \in \mathcal{C}_b(\mathcal{D}([0, 1]))$ and, the general case follows. Then, we consider some $\varepsilon > 0$ and the closed set $A := \{(x, y) \in \mathcal{D}([0, 1])^2 : \forall t \leq \varepsilon, x_t = y_t\}$ of $\mathcal{D}([0, 1])^2$. We have

$$\mathbb{E}\big[\omega_{n,\beta_n}\big([\varepsilon, 1]\big)\big] \leq \mathbb{E}\big[\mu_{n,\beta_n}^{\otimes 2}(A)\big] \xrightarrow[n\to\infty]{} \mathbb{P}\big(\forall t \leq \varepsilon, \mathfrak{e}_t = \mathfrak{e}'_t\big) = 0,$$

using (1.16) and the Portmanteau theorem. $\square$

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[3]For example, using the stopping line $\mathcal{Z}[A] := \{z \in \mathbb{T} : V(z) \geq A > \max_{k<|z|} V(z_k)\}$ for large A, it is possible to show that, with high probability for n large, all particles under $\nu_{n,\infty}$ have the same ancestor in $\mathcal{Z}[A]$. Then (1.15) follows from Lemma 3.3 of Chen [21]. In the lattice case, one can see that (1.15) cannot hold: for example, if there are two particles at generation 1, each branching random walk iniated from these particles has a positive probability to have its minimum at time n at the same given position around $\frac{3}{2}\log n$.

1.5. *Comments on the results*

Theorem 1.1 fully describes the transition from the size $n^{-3\beta/2}$ when $\beta > 1$ to the size $e^{n\Psi(\beta)}$ when $\beta < 1$ and Corollary 1.3 shows the transition in the location of the mass of the Gibbs measure from $[\frac{3}{2}\log n - C, \frac{3}{2}\log n + C]$ when $\beta > 1$, to $[-\Psi'(\beta)n - C\sqrt{n}, -\Psi'(\beta)n + C\sqrt{n}]$ when $\beta < 1$ (this follows from (1.14)). Secondly, Theorem 1.2 describes the transition between the Brownian excursion in case (i) and the straight line with Brownian fluctuations in case (iv). Note that, since $\min_{z\in\mathbb{T}} V(z) > -\infty$ a.s., it is natural that the limiting trajectory stays nonnegative on $[0, 1]$ in cases (i) to (iii). In case (iv), this constraint disappears in the limit due to the drift. Indeed, staying above a constant for a random walk with drift approximately $\sigma\sqrt{n}/\alpha_n$ needs effort until times of order α_n^2, so it disappears in the trajectory after scaling by n. However, this effort has a cost of order $1/\alpha_n$, which explains the presence of this factor in the size $e^{n\Psi(\beta_n)}/\alpha_n$ of the partition function.

In cases (ii) and (iii), when α_n is of order $\sqrt{n}$ or larger, the perturbation is too small to change radically the behavior of the Gibbs measure in regards to the critical case $\beta = 1$: the size of the partition function is still $n^{-1/2}$, the typical energy is of order $\sqrt{n}$ and the limiting trajectory has a density w.r.t. the Brownian meander. Therefore, cases (ii) and (iii) are called the *critical window* by Alberts and Ortgiese [5]. It brings to light a family of laws $(P^\gamma)_{\gamma\in\mathbb{R}}$ on $\mathcal{D}([0, 1])$ defined by $P^\gamma(F) := \mathbb{E}[e^{-\sigma\gamma\mathcal{M}(1)}F(\mathcal{M})]/\mathbb{E}[e^{-\sigma\gamma\mathcal{M}(1)}]$ for all $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$, including the law of the Brownian meander of length 1 for $\gamma = 0$ and such that $P^\gamma \Rightarrow \mathcal{L}(\mathfrak{e})$ as $\gamma \to \infty$, where $\mathcal{L}(\mathfrak{e})$ denotes the law of $\mathfrak{e}$. But there is no convergence as $\gamma \to -\infty$, because the trajectory is sent to infinity. However, we can consider another family $(Q^\gamma)_{\gamma\in\mathbb{R}}$ defined by $Q^\gamma(F) := \mathbb{E}[e^{\sigma\gamma\mathcal{M}(1)}F(\mathcal{M}_t - \sigma\gamma t, t \in [0, 1])]/\mathbb{E}[e^{\sigma\gamma\mathcal{M}(1)}]$, so that, in case (iii), we have

$$\frac{1}{W_{n,\beta_n}} \sum_{|z|=n} e^{-\beta_n V(z)} F\left(\frac{V(z_{\lfloor tn\rfloor}) + tn\Psi'(\beta_n)}{\sigma\sqrt{n}}, t \in [0, 1]\right) \xrightarrow[n\to\infty]{} Q^\gamma(F), \quad \text{in } \mathbb{P}^*\text{-probability,}$$

This family includes also the Brownian meander's law and we have $Q^\gamma \Rightarrow \mathcal{L}(B)$ as $\gamma \to \infty$.

As opposed to this, cases (i) and (iv) are called the *near-critical window* and highlight some new behaviors. In case (i), the transition between the critical size $n^{-1/2}$ and the strong disorder size $n^{-3\beta/2}$ appears. The factor $n^{3\beta_n/2}$ starts to behave differently than $n^{3/2}$ when $\alpha_n = O(\log n)$ and this is also the region where the particles mainly contributing to W_{n,β_n} are not simply those in $[C^{-1}\alpha_n, C\alpha_n]$: the fact that the lowest particle at time n is around $\frac{3}{2}\log n$ plays a role only in this region (see Lemma 5.2). Nevertheless, in the near-critical regime, the lowest particle at time n never has a positive weight in ν_{n,β_n} in the limit, unlike in the case $\beta > 1$. Since the particles mainly contributing to W_{n,β_n} are far below $\sqrt{n}$, the endpoint of the limiting trajectory has to be 0 and, therefore, the excursion appears.

In case (iv), since β_n tends to 1 slowly enough, we find the same asymptotic behavior for the partition function as in (1.11) when we first take $n \to \infty$ and then $\beta \uparrow 1$. For the limiting trajectory, the result is also similar to (1.14) in the case $\beta > 1$. Moreover, when α_n is not too small, the different results in case (iv) can be rewritten only in terms of σ^2: if $\alpha_n \gg n^{1/3}$, the size of the partition function is $e^{\sigma^2 n/2\alpha_n^2}/\alpha_n$ and, if $n^{1/4} = O(\alpha_n)$, the location of the mass is in $[\sigma^2 n/\alpha_n - C\sqrt{n}, \sigma^2 n/\alpha_n + C\sqrt{n}]$. But, on the contrary, if α_n is too small, there is a break of universality.

Finally, we stress that there is no discontinuity between the different cases of the results. Indeed, using that $\mathbb{E}[e^{-\sigma\gamma\mathcal{M}(1)}] \sim 1/(\sigma\gamma)^2$ and $\mathbb{E}[e^{\sigma\gamma\mathcal{M}(1)}] \sim \sqrt{2\pi}\sigma\gamma e^{(\sigma\gamma)^2/2}$ as $\gamma \to \infty$, all cases of Theorem 1.1 (requiring $\alpha_n \gg n^{1/3}$ in case (iv)) can be written

$$n^{3(1-\beta_n)/2} \frac{\sqrt{n}}{\mathbb{E}[e^{(1-\beta_n)\sigma\mathcal{M}(1)\sqrt{n}}]} W_{n,\beta_n} \xrightarrow[n\to\infty]{} \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} D_\infty, \quad \text{in } \mathbb{P}^*\text{-probability,} \tag{1.17}$$

noting that $n^{3(1-\beta_n)/2} \to 1$ as soon as $\alpha_n \gg \log n$. For Theorem 1.2, the continuity between the different cases follows from the convergences $P^\gamma \Rightarrow \mathcal{L}(\mathfrak{e})$ and $Q^\gamma \Rightarrow \mathcal{L}(B)$ as $\gamma \to \infty$.

1.6. *Organization of the paper*

Sections of this paper correspond to the different cases in the results: case (i) is treated in Section 5, cases (ii) and (iii) in Section 3 and case (iv) in Section 4. The behavior in the critical window (cases (ii) and (iii)) is a direct consequence of the analogue results (1.10) and (1.12) in the critical case (apart from some technical details). The near-critical

window in the weak disorder regime (case (iv)) needs slightly more work, but relies mainly on Madaule's [34] results and on L^p inequalities techniques (see [9]). Finally, the main part of this paper is dedicated to the proof of case (ii), which follows some ideas of Aïdékon and Shi [3], with change of measure and spine decomposition techniques. One main difference with the previous literature on the branching random walk is that we need here to consider particles that are far below $\sqrt{n}$ but also far above the lowest particle. Note that we prove in Section 5.6 a new version of the so-called peeling lemma (see Shi [40]) with a more general setting than what is needed for the aim of this paper and that could be of independent interest.

On the other hand, Section 2 regroups some well-known results on the branching random walk and on classical random walk. Some new results are stated in this section and proved in the Appendix. Note that none of the results of Section 2 are needed for the proof of cases (ii) and (iii) and only a few of them for case (iv). The Appendix contains some other technical results.

Throughout the paper, the c_i's denote positive constants, we set $\mathbb{N} := \{0, 1, 2, \dots\}$ and, for $a, b \in \mathbb{N}$, $[\![a, b]\!] := [a, b] \cap \mathbb{N}$. For two sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ of positive real numbers, we say that $u_n \sim v_n$ as $n \to \infty$ if $\lim_{n \to \infty} u_n/v_n = 1$, that $u_n = O(v_n)$ as $n \to \infty$ if $\limsup_{n \to \infty} u_n/v_n < \infty$, and that $u_n = o(v_n)$ or $u_n \ll v_n$ if $\lim_{n \to \infty} u_n/v_n = 0$. For (S, d) a metric space, let $\mathcal{C}_b(S)$ be the set of bounded continuous functions from $S \to \mathbb{R}$ and $\mathcal{C}_b^u(S)$ be its subset of uniformly continuous functions. For $F \in \mathcal{C}_b(S)$, we set $\|F\| := \sup_{x \in S} |f(x)|$. For $F \in \mathcal{C}_b^u(S)$, we will denote by ω_F a modulus of continuity for function F: ω_F is a continuous bounded nondecreasing function from $\mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega_F(0) = 0$ and $\forall x, y \in S$, $|F(x) - F(y)| \le \omega_F(d(x, y))$.

2. Preliminary results

In this section, we state some preliminary results that are mostly needed in Section 5. In Section 2.1, we present some well-known tools to study the branching random walk and the next subsections contain results concerning one-dimensional random walk.

2.1. *Many-to-one lemma and changes of probabilities*

For $a \in \mathbb{R}$, let $\mathbb{P}_a$ denote a probability measure under which $(V(z), z \in \mathbb{T})$ is the branching random walk starting from a, and $\mathbb{E}_a$ the associated expectation (for brevity we will write $\mathbb{P}$ and $\mathbb{E}$ instead of $\mathbb{P}_0$ and $\mathbb{E}_0$). We define a random walk $(S_n)_{n \ge 0}$ associated to the branching random walk: under $\mathbb{P}_a$, $S_0 = a$ a.s. and the law of the increments is given by

$$\mathbb{E}_a\big[h(S_1 - S_0)\big] = \mathbb{E}\left[\sum_{|z|=1} h\big(V(z)\big) \mathrm{e}^{-V(z)}\right],$$

for all measurable $h \colon \mathbb{R} \to \mathbb{R}_+$. This random walk is well-defined and centred thanks to assumption (1.2). Moreover, by assumption (1.3), we have $\mathbb{E}[S_1^2] = \sigma^2 \in (0, \infty)$. Then, by induction, one gets the following result (see Biggins and Kyprianou [10]). It is also a corollary of the forthcoming Proposition 2.2.

Lemma 2.1 (Many-to-one lemma). *For all $n \ge 1$, $a \in \mathbb{R}$ and all measurable function $g \colon \mathbb{R}^{n+1} \to \mathbb{R}_+$, we have*

$$\mathbb{E}_a\left[\sum_{|z|=n} g\big(V(z_0), \dots, V(z_n)\big)\right] = \mathbb{E}_a\big[\mathrm{e}^{S_n - a} g(S_0, \dots, S_n)\big].$$

We now state some well-known change of probabilities and spinal decomposition results. This method dates back at least to Kahane and Peyrière [31], Rouault [38] and Chauvin and Rouault [19]. See also Biggins and Kyprianou [11] for spinal decomposition in more general type of branching structures and Shi [40] for a survey on this topic. Let $\mathscr{F}_n$ denote the σ-algebra generated by $(V(z), |z| \le n)$ and $\mathscr{F}_\infty := \sigma(\bigvee_{n \in \mathbb{N}} \mathscr{F}_n)$.

We first introduce Lyons' change of measure [33]: since $(W_{n,1})_{n \in \mathbb{N}}$ is a nonnegative martingale of mean e^{-a} under $\mathbb{P}_a$, we can define a new probability measure $\mathbb{Q}_a$ on $\mathscr{F}_\infty$, by letting for all $n \in \mathbb{N}$,

$$\mathbb{Q}_a|_{\mathscr{F}_n} := \mathrm{e}^a W_{n,1} \bullet \mathbb{P}_a|_{\mathscr{F}_n}.$$

We will denote by $\mathbb{E}_{\mathbb{Q}_a}$ the associated expectation and we will write $\mathbb{Q}$ and $\mathbb{E}_{\mathbb{Q}}$ instead of $\mathbb{Q}_0$ and $\mathbb{E}_{\mathbb{Q}_0}$. Let $\hat{\mathcal{L}}$ be a point process on $\mathbb{R}$ which has the law of $(V(z), |z| = 1)$ under $\mathbb{Q}$.

Lyons [33] proved the following description for the branching random walk under $\mathbb{Q}_a$, with a decomposition along a *spine* $(w_n)_{n \in \mathbb{N}}$ which is a marked ray in the the genealogical tree $\mathbb{T}$ (in order to be mathematically rigorous, one should enlarge the probability space and work on a product space, see Lyons [33]). The system starts with one particle w_0 at position a, forming the 0th generation. For each $n \in \mathbb{N}$, individuals of the nth generation give birth independently of each other and of the foregoing. Individuals other than w_n generate offspring around their position according to the law of $\mathcal{L}$ and w_n breeds according to the law of $\hat{\mathcal{L}}$. Then, w_{n+1} is chosen independently among w_n's children, with probability proportional to $e^{-V(z)}$ for each child z. Moreover, Lyons showed the following result concerning the spine $(w_n)_{n \in \mathbb{N}}$ under $\mathbb{Q}_a$.

Proposition 2.2. *Let $a \in \mathbb{R}$.*

(i) *For each $n \in \mathbb{N}$ and $|z| = n$, we have*

$$\mathbb{Q}_a(w_n = z | \mathscr{F}_n) = \frac{e^{-V(z)}}{W_{n,1}}.$$

(ii) *The process $(V(w_n))_{n \in \mathbb{N}}$ under $\mathbb{Q}_a$ has the same law as $(S_n)_{n \in \mathbb{N}}$ under $\mathbb{P}_a$.*

Now, we present another change of measure, that was first introduced by Biggins and Kyprianou [11]. For this, we need to define R the renewal function in the first strict descending ladder height process of the random walk $(S_n)_{n \in \mathbb{N}}$. For $u \geq 0$,

$$R(u) := \sum_{k=0}^{\infty} \mathbb{P}(H_k \leq u),$$

where $(H_k)_{k \in \mathbb{N}}$ is the first strict descending ladder height process: we set $\tau_0 := 0$, $H_0 := 0$ and, for $k \geq 1$, $\tau_k := \inf\{n > \tau_{k-1} : S_n < S_{\tau_{k-1}}\}$ and $H_k := -S_{\tau_k}$. Then, we introduce the truncated derivative martingale: for $L \geq 0$ and $n \in \mathbb{N}$,

$$D_n^{(L)} := \sum_{|z|=n} R_L(V(z)) e^{-V(z)} \mathbb{1}_{\underline{V}(z) \geq -L},$$

where, for $u \geq -L$, $R_L(u) := R(L + u)$ and, for $|z| = n$, $\underline{V}(z) := \min_{0 \leq i \leq n} V(z_i)$. Fix now some $L \geq 0$. For $a \geq -L$, under $\mathbb{P}_a$, $(D_n^{(L)})_{n \in \mathbb{N}}$ is a nonnegative martingale of mean $R_L(a)$ and therefore we can define another probability measure $\mathbb{Q}_a^{(L)}$ by

$$\mathbb{Q}_a^{(L)}|_{\mathscr{F}_n} := \frac{D_n^{(L)}}{R_L(a)} \bullet \mathbb{P}_a|_{\mathscr{F}_n}.$$

We will denote by $\mathbb{E}_{\mathbb{Q}^{(L)}}$ the associated expectation and write $\mathbb{Q}^{(L)}$ and $\mathbb{E}_{\mathbb{Q}}^{(L)}$ instead of $\mathbb{Q}_0^{(L)}$ and $\mathbb{E}_{\mathbb{Q}_0^{(L)}}$. For $a \geq -L$, let $\hat{\mathcal{L}}_a^{(L)}$ be a point process on $\mathbb{R}$ with the law of $(V(z), |z| = 1)$ under $\mathbb{Q}_a^{(L)}$.

Biggins and Kyprianou [11] proved the following spinal decomposition description for the branching random walk under $\mathbb{Q}_a^{(L)}$, where the spine is denoted by $(w_n^{(L)})_{n \in \mathbb{N}}$. The description is similar to the previous one, but here $w_n^{(L)}$ have offspring according to $\hat{\mathcal{L}}_{V(w_n^{(L)})}^{(L)}$ and $w_{n+1}^{(L)}$ is chosen among these children, with probability proportional to $R_L(V(z)) e^{-V(z)} \mathbb{1}_{\underline{V}(z) \geq -L}$ for each child z. Moreover, we get the following analogue of Proposition 2.2.

Proposition 2.3. *Let $L \geq 0$ and $a \geq -L$.*

(i) *For each $n \in \mathbb{N}$ and $|z| = n$, we have*

$$\mathbb{Q}^{(L)}(w_n^{(L)} = z | \mathscr{F}_n) = \frac{R_L(V(z)) e^{-V(z)} \mathbb{1}_{\underline{V}(z) \geq -L}}{D_n^{(L)}}.$$

(ii) *The process $(V(w_n^{(L)}))_{n \in \mathbb{N}}$ under $\mathbb{Q}_a^{(L)}$ has the same law as $(S_n)_{n \in \mathbb{N}}$ under $\mathbb{P}_a$ conditioned to stay in $[-L, \infty)$: for all $n \in \mathbb{N}$ and all measurable function $g \colon \mathbb{R}^{n+1} \to \mathbb{R}_+$,*

$$\mathbb{E}_{\mathbb{Q}_a^{(L)}}\big[g\big(V\big(w_0^{(L)}\big), \ldots, V\big(w_n^{(L)}\big)\big)\big] = \frac{1}{R_L(a)} \mathbb{E}_a\big[g(S_0, \ldots, S_n) R_L(S_n) \mathbb{1}_{\underline{S}_n \geq -L}\big].$$

2.2. *One-dimensional random walks*

Up to the end of this section, we consider a centred random walk $(S_n)_{n \in \mathbb{N}}$ with finite variance $\mathbb{E}[S_1^2] = \sigma^2 \in (0, \infty)$. In this subsection, we state various results concerning this one-dimensional random walk and the associated renewal function R. For $n \in \mathbb{N}$, we set $\underline{S}_n := \min_{0 \leq i \leq n} S_i$.

Recall that R is the renewal function associated to the first strict descending ladder height process $(H_k)_{k \in \mathbb{N}}$. Since $\mathbb{E}[S_1] = 0$ and $\mathbb{E}[S_1^2] < \infty$, by Feller [27, Theorem XVIII.5.1 (5.2)], we have $\mathbb{E}[H_1] < \infty$. Thus, it follows from Feller's [27, p. 360] renewal theorem that it exists a constant $c_0 > 0$ such that

$$\frac{R(u)}{u} \xrightarrow[u \to \infty]{} c_0 \tag{2.1}$$

and so there exist also $c_1, c_2 > 0$ such that, for all $u \geq 0$,

$$c_1(1 + u) \leq R(u) \leq c_2(1 + u). \tag{2.2}$$

We are interested in the behavior of random walks staying above a barrier. First, we recall the following estimate for the probability of a random walk to stay above $-a$: by Kozlov [32, Theorem A], it exists a constant $\theta > 0$ such that for all $u \geq 0$,

$$\mathbb{P}(\underline{S}_n \geq -u) \underset{n \to \infty}{\sim} \frac{\theta R(u)}{\sqrt{n}}, \tag{2.3}$$

and it exists $c_3 > 0$ such that, for all $n \in \mathbb{N}$ and $u \geq 0$, we have the uniform bound

$$\mathbb{P}(\underline{S}_n \geq -u) \leq \frac{c_3(1 + u)}{\sqrt{n}}. \tag{2.4}$$

Constants c_0 and θ will appear all along the paper and they are related by the following equation, from Aïdékon and Shi [3, Lemma 2.1],

$$\theta c_0 = \left(\frac{2}{\pi \sigma^2}\right)^{1/2}. \tag{2.5}$$

The following result states the convergence in law of $S_n / \sigma \sqrt{n}$ conditioned to stay above $-u$ towards a Rayleigh distribution, with uniformity in the position of the barrier $-u$: by Aïdékon and Jaffuel [2, Lemma 2.2], if $(\gamma_n)_{n \in \mathbb{N}}$ is a sequence of positive numbers such that $\gamma_n \ll \sqrt{n}$ as $n \to \infty$, then we have, for all continuous bounded function $g \colon \mathbb{R}_+ \to \mathbb{R}$,

$$\mathbb{E}\left[g\left(\frac{S_n + u}{\sigma \sqrt{n}}\right) \mathbb{1}_{\underline{S}_n \geq -u}\right] = \frac{\theta R(u)}{\sqrt{n}}\left(\int_0^\infty g(t) t e^{-t^2/2} \, dt + o(1)\right), \tag{2.6}$$

uniformly in $u \in [0, \gamma_n]$.

From Lemmas 2.2 and 2.4 of Aïdékon and Shi [3], we have the following inequalities, sometimes called ballot theorems: it exists $c_4 > 0$ such that, for all $b > a \geq 0$, $u \geq 0$ and $n \geq 1$,

$$\mathbb{P}\big(S_n \in [a - u, b - u], \underline{S}_n \geq -u\big) \leq c_4 \frac{(u + 1)(b + 1)(b - a + 1)}{n^{3/2}}, \tag{2.7}$$

and, for $\lambda \in (0, 1)$, it exists $c_5 = c_5(\lambda) > 0$ such that for all $b > a \geq 0$, $u \geq 0$, $v \in \mathbb{R}$ and $n \geq 1$, we have

$$\mathbb{P}\Big(S_n \in [a + v, b + v], \underline{S}_{\lfloor \lambda n \rfloor} \geq -u, \min_{\lfloor \lambda n \rfloor \leq j \leq n} S_j \geq v\Big) \leq c_5 \frac{(u + 1)(b + 1)(b - a + 1)}{n^{3/2}}, \qquad (2.8)$$

where we added a second barrier between times $\lfloor \lambda n \rfloor$ and n. From the previous results, it follows (see Aïdékon [1, Lemma B.2]) that it exists $c_6 > 0$ such that, for all $a, u \geq 0$,

$$\sum_{i \geq 0} \mathbb{P}(\underline{S}_i \geq -u, S_i \leq a - u) \leq c_6(1 + a)\big(1 + \min(a, u)\big). \qquad (2.9)$$

Finally, we state a slight extension of Lemma A.6 of Shi [40]. It will be used for the proof of the peeling lemma in Section 5.6.

Lemma 2.4. *We set, for $0 \leq i \leq n$, $u, v, \mu \geq 0$,*

$$m_i^{(n)} := \begin{cases} -u + 2i^{1/7} - \mu & \text{if } 0 \leq i < \lfloor n/2 \rfloor, \\ v + 2(n - i)^{1/7} - \mu & \text{if } \lfloor n/2 \rfloor \leq i \leq n. \end{cases}$$

For each $\varepsilon > 0$, it exists $\mu > 0$ such that for all $b, u, v \geq 0$ and $n \in \mathbb{N}^*$,

$$\mathbb{P}\Big(\underline{S}_{\lfloor n/2 \rfloor} \geq -u, \min_{\lfloor n/2 \rfloor \leq j \leq n} S_j \geq v, S_n \in [b + v, b + v + 1], \exists i \in [\![0, n]\!] : S_i \leq m_i^{(n)}\Big)$$

$$\leq (1 + u)(1 + b)\left(\frac{\varepsilon}{n^{3/2}} + c_7 \frac{(n^{1/7} + v)^2}{n^{2 - (1/7)}}\right),$$

where c_7 is a positive constant depending only on the law of the random walk.

Proof. Lemma A.6 of Shi [40] deals with the event $\{S_n \in [v, b + v]\}$ instead of $\{S_n \in [b + v, b + v + 1]\}$ and, therefore, yields a factor $(1 + b)^2$ instead of our factor $(1 + b)$. However, the proof is exactly the same, except at two points, that we specify thereafter with Shi's notation (where his $(1 + u)$ corresponds to our $(1 + b)$). Firstly, at the equation before (A.11), the use of his Lemma A.4 gives a factor $(1 + u)$ instead of $(1 + u)^2$. Secondly, at the equation after (A.11), by using (2.7) instead of his Lemma A.1, we also get a factor $(1 + u)$ instead of $(1 + u)^2$. $\square$

Remark 2.5. This kind of result is usually proved with an envelope of order $i^\alpha \wedge (n - i)^\alpha$, where $\alpha \in (0, 1/6)$ (see also Aïdékon [1, Lemma B.3] and Madaule [35, Lemma 6.1]). Indeed, with their arguments, a term $(n^\alpha + v)^2/n^{2-\alpha}$ appears and it is a $o(n^{-3/2})$ iff $\alpha < 1/6$ and $v = o(n^{(1-2\alpha)/4})$. Therefore, an anonymous referee asked us whether a similar result holds for any $\alpha \in (0, 1/2)$. The answer is yes (and moreover it holds for any $v \in \mathbb{R}$), but, since this extension is not useful in the sequel, we let the proof in the arXiv version of this paper.

2.3. *Convergence towards the Brownian meander*

We define the rescaled trajectory of the random walk until time n: for each $n \in \mathbb{N}^*$,

$$\mathbf{S}^{(n)} := \left(\frac{S_{\lfloor nt \rfloor}}{\sigma \sqrt{n}}, t \in [0, 1]\right).$$

We state the following convergence result for the trajectory $\mathbf{S}^{(n)}$ conditioned to stay nonnegative, with uniformity in the starting point of the random walk.

Proposition 2.6. *Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\gamma_n \ll \sqrt{n}$ as $n \to \infty$. Then, for all $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$, we have*

$$\mathbb{E}_u\big[F\big(\mathbf{S}^{(n)}\big)\mathbb{1}_{\underline{S}_n \geq 0}\big] = \frac{\theta R(u)}{\sqrt{n}}\big(\mathbb{E}\big[F(\mathcal{M})\big] + o(1)\big),$$

as $n \to \infty$, uniformly in $u \in [0, \gamma_n]$, where $\mathcal{M}$ denotes the Brownian meander of length 1.

This invariance principle has been proved in the case $u = 0$ by Iglehart [28], Bolthausen [14] and Doney [25]. The case where F is a function of the terminal value of the trajectory is already showed in (2.6). We give a short proof of this generalization in Section B.1, that relies on the invariance principle by Caravenna and Chaumont [15], for random walk conditioned to stay nonnegative for all time.

We present also a corollary of Proposition 2.6, which holds under an additional assumption on the random walk S that is equivalent to assumption (1.5) by the many-to-one lemma. It will be used in Section 4 and its proof is postponed to Section B.1.

Corollary 2.7. *Assume that it exists $\delta_0 > 0$ such that $\mathbb{E}[e^{\delta_0 S_1}] < \infty$. Then, for all $C, L > 0$ and $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$, we have*

$$\mathbb{E}\big[e^{C S_n/\sqrt{n}} F\big(\mathbf{S}^{(n)}\big) \mathbb{1}_{\underline{S}_n \geq -L}\big] = \frac{\theta R(L)}{\sqrt{n}} \big(\mathbb{E}\big[e^{C\sigma \mathcal{M}(1)} F(\mathcal{M})\big] + o(1)\big),$$

as $n \to \infty$, where $\mathcal{M}$ denotes the Brownian meander of length 1.

2.4. *Convergence towards the Brownian excursion*

In this subsection, our interest is the convergence of $\mathbf{S}^{(n)}$, conditioned to stay above two successive barriers and to end up in a small interval, towards the normalized Brownian excursion, with uniformity with respect to the barriers' positions and to the endpoint as long as they are much smaller than $\sqrt{n}$. This result will be used repetitively in Section 5 for the proof of part (ii) of Theorems 1.1 and 1.2.

At this scale for the endpoint S_n, the random walk behaves differently in the lattice and nonlattice cases, so they have to be distinguished. Moreover, we need some new notation: let $(S_n^-)_{n \in \mathbb{N}}$ be a random walk such that, under $\mathbb{P}_a$, $S_0^- = a$ a.s. and $S_1^- - S_0^-$ has the same law as $S_0 - S_1$. All objects referring to S ($R, c_0, \theta, \dots$) have their analogue for S^- denoted with a $-$ superscript ($R^-, c_0^-, \theta^-, \dots$).

We can now state our result, which generalizes Lemma 2.6 of Chen, Madaule and Mallein [22] and is proved in Section B.4.

Proposition 2.8. *Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\gamma_n \ll \sqrt{n}$ as $n \to \infty$ and $\mathfrak{e}$ be the normalized Brownian excursion.*

(i) *If the law of S_1 is nonlattice, then for all $h > 0$, $\lambda \in (0, 1)$ and $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$,*

$$\mathbb{E}\big[F\big(\mathbf{S}^{(n)}\big) \mathbb{1}_{\underline{S}_{\lfloor \lambda n \rfloor} \geq -u, \min_{\lfloor \lambda n \rfloor \leq i \leq n} S_i \geq v, S_n \in [v+b, v+b+h]}\big] = \sqrt{\frac{\pi}{2}} \frac{\theta \theta^-}{\sigma} \frac{R(u)}{n^{3/2}} \big(\mathbb{E}\big[F(\mathfrak{e})\big] + o(1)\big) \int_b^{b+h} R^-(t) \, dt,$$

as $n \to \infty$, uniformly in $b, u \in [0, \gamma_n]$ and $v \in [-\gamma_n, \gamma_n]$.

(ii) *If the law of S_1 is (h, a)-lattice, then for all $\lambda \in (0, 1)$ and $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$,*

$$\mathbb{E}\big[F\big(\mathbf{S}^{(n)}\big) \mathbb{1}_{\underline{S}_{\lfloor \lambda n \rfloor} \geq -u, \min_{\lfloor \lambda n \rfloor \leq i \leq n} S_i \geq v, S_n = v+b}\big] = \sqrt{\frac{\pi}{2}} \frac{\theta \theta^-}{\sigma} \frac{R(u)}{n^{3/2}} \big(\mathbb{E}\big[F(\mathfrak{e})\big] + o(1)\big) h R^-(b),$$

as $n \to \infty$, uniformly in $u \in [0, \gamma_n]$, $v \in [-\gamma_n, \gamma_n]$ and $b \in [0, \gamma_n] \cap (-v + an + h\mathbb{Z})$.

3. The critical window

We prove here cases (ii) and (iii) of Theorems 1.1 and 1.2, where $\beta_n = 1 \pm 1/\alpha_n$ with $\sqrt{n}/\alpha_n \to \gamma \in [0, \infty)$. This proof is based on Theorem 1.2 of Madaule [34], recalled in (1.12).

Proof of part (ii) of Theorems 1.1 and 1.2. Our aim is to show that, for all $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$, we have

$$\frac{1}{W_{n,1}} \sum_{|z|=n} e^{-\beta_n V(z)} F\big(\mathbf{V}(z)\big) \xrightarrow[n \to \infty]{} \mathbb{E}\big[e^{-\sigma \gamma \mathcal{M}(1)} F(\mathcal{M})\big], \quad \text{in } \mathbb{P}^*\text{-probability.} \tag{3.1}$$

Then, using (1.10) and the case $F \equiv 1$, part (ii) of Theorems 1.1 and 1.2 follows. Note also that it is sufficient to show (3.1) for F nonnegative.

We first deal with the case $\gamma \in (0, \infty)$. We consider some nonnegative function $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$ and $\varepsilon > 0$. By dominated convergence, we have that $\mathbb{E}[e^{-\sigma \gamma' \mathcal{M}^{(1)}} F(\mathcal{M})]$ tends towards $\mathbb{E}[e^{-\sigma \gamma \mathcal{M}^{(1)}} F(\mathcal{M})]$ as $\gamma' \to \gamma$, so we can choose $0 < \gamma^- < \gamma < \gamma^+$ such that

$$\mathbb{E}\big[e^{-\sigma \gamma \mathcal{M}^{(1)}} F(\mathcal{M})\big] - \frac{\varepsilon}{2} \leq \mathbb{E}\big[e^{-\sigma \gamma^{\pm} \mathcal{M}^{(1)}} F(\mathcal{M})\big] \leq \mathbb{E}\big[e^{-\sigma \gamma \mathcal{M}^{(1)}} F(\mathcal{M})\big] + \frac{\varepsilon}{2}. \tag{3.2}$$

Since $\sqrt{n}/\alpha_n \to \gamma$ and, under $\mathbb{P}^*$, $\min_{|z|=n} V(z) \to \infty$ a.s., it exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have $\sqrt{n}/\alpha_n \in [\gamma^-, \gamma^+]$ and also $\mathbb{P}^*(\min_{|z|=n} V(z) < 0) \leq \varepsilon$. Moreover, we set, for $x \in \mathcal{D}([0, 1])$, $G^+(x) := F(x)(e^{-\sigma \gamma^+ x_1} \wedge 1)$ and $G^-(x) := F(x)(e^{-\sigma \gamma^- x_1} \wedge 1)$: if $x_1 \geq 0$, then we have $G^+(x) \leq e^{(1-\beta_n)x_1} F(x) \leq G^-(x)$. Thus, we get, for all $n \geq n_0$,

$$\mathbb{P}^*\bigg(\frac{1}{W_{n,1}} \sum_{|z|=n} e^{-\beta_n V(z)} F\big(\mathbf{V}(z)\big) \notin \big[\mu_{n,1}(G^+), \mu_{n,1}(G^-)\big]\bigg) \leq \mathbb{P}^*\bigg(\min_{|z|=n} V(z) < 0\bigg) \leq \varepsilon.$$

Therefore, we have

$$\mathbb{P}^*\bigg(\bigg|\frac{1}{W_{n,1}} \sum_{|z|=n} e^{-\beta_n V(z)} F\big(\mathbf{V}(z)\big) - \mathbb{E}\big[e^{-\sigma \gamma \mathcal{M}^{(1)}} F(\mathcal{M})\big]\bigg| \leq \varepsilon\bigg)$$

$$\leq \varepsilon + \mathbb{P}^*\big(\mu_{n,1}(G^-) \geq \mathbb{E}\big[e^{-\sigma \gamma \mathcal{M}^{(1)}} F(\mathcal{M})\big] + \varepsilon\big) + \mathbb{P}^*\big(\mu_{n,1}(G^+) \leq \mathbb{E}\big[e^{-\sigma \gamma \mathcal{M}^{(1)}} F(\mathcal{M})\big] - \varepsilon\big)$$

$$\leq \varepsilon + \mathbb{P}^*\bigg(\mu_{n,1}(G^-) \geq \mathbb{E}\big[G^+(\mathcal{M})\big] + \frac{\varepsilon}{2}\bigg) + \mathbb{P}^*\bigg(\mu_{n,1}(G^+) \leq \mathbb{E}\big[G^-(\mathcal{M})\big] - \frac{\varepsilon}{2}\bigg), \tag{3.3}$$

using (3.2). Then, applying (1.12), we get that both probabilities in (3.3) tends to 0, because $G^-, G^+ \in \mathcal{C}_b(\mathcal{D}([0, 1]))$ and it concludes the proof of (3.1) in the case $\gamma \in (0, \infty)$.

Finally, for the case $\gamma = 0$, we proceed in the same way as for $\gamma \in (0, \infty)$, taking here $\gamma^+ > 0$ such that $\mathbb{E}[F(\mathcal{M})] - \frac{\varepsilon}{2} \leq \mathbb{E}[e^{-\sigma \gamma^+ \mathcal{M}^{(1)}} F(\mathcal{M})]$, G^+ defined as before and $G^- := F$. Then, the same inequalities hold. $\square$

Remark 3.1. In case (iii), we work under assumption (1.5) and we will use Proposition 3.8 of Madaule [34], whereas Madaule works in [34] under the stronger assumption (1.6). But, for the proof of his Proposition 3.8, he only uses Assumption (1.6) in the proof of his Lemma A.2, in order to have that Ψ is finite on a left-neighbourhood of 1 and $\Psi(\beta) = \frac{\sigma^2}{2}(\beta - 1)^2 + o((\beta - 1)^2)$ as $\beta \uparrow 1$, and this holds also under our assumption (1.5).

Proof of part (iii) of Theorems 1.1 and 1.2. Applying Proposition 3.8 of Madaule [34] (see Remark 3.1 above), we get that $\mu_{n,1}(G) \to \mathbb{E}[G(\mathcal{M})]$ in $\mathbb{P}^*$-probability with $G: x \in \mathcal{D}([0, 1]) \mapsto e^{Cx_1}$ for any $C > 0$, although G is not bounded. Combining this with (1.12), it is straightforward to extend this convergence to functions of the type $G: x \in \mathcal{D}([0, 1]) \mapsto e^{Cx_1} F(x)$ with $C > 0$ and $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$.

Then, we prove that, for all $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$ nonnegative,

$$\frac{1}{W_{n,1}} \sum_{|z|=n} e^{-\beta_n V(z)} F\big(\mathbf{V}(z)\big) \xrightarrow[n \to \infty]{} \mathbb{E}\big[e^{-\sigma \gamma \mathcal{M}^{(1)}} F(\mathcal{M})\big], \quad \text{in } \mathbb{P}^*\text{-probability}, \tag{3.4}$$

using the same method as for the proof of case (ii): we approach function $x \mapsto e^{\sigma x_1 \sqrt{n}/\alpha_n} F(x)$ from above and from below, by considering here $G^+(x) := e^{\sigma \gamma^+ x_1} F(x)$ and $G^-(x) := e^{\sigma \gamma^- x_1} F(x)$ when $\gamma \in (0, \infty)$ and the same function G^+ but with $G^- := F$ when $\gamma = 0$. Finally, part (iii) of Theorems 1.1 and 1.2 follows from (3.4). $\square$

4. The near-critical window in the weak disorder regime

In this section, we deal with the case where $\beta_n = 1 - 1/\alpha_n$ and $\alpha_n \ll \sqrt{n}$ and prove successively convergence of the rescaled partition function and then convergence of the trajectories. We work here under assumption (1.6) so Ψ is

analytic on an open interval I containing 1. Moreover, by Lemma 4.3 of Madaule [34], assumption (1.6) implies that there exist $c_8 > 0$ and $\eta_0 < 1$ such that $\mathbb{E}[(\widetilde{W}_{\infty,\beta})^{1+\frac{\eta}{2}}] \leq c_8$ for any $0 < \eta < \eta_0$ and $\beta = 1 - \eta$. Thus, for any $n \in \mathbb{N}$ and $0 < \eta < \eta_0$, we have, with $\beta = 1 - \eta$,

$$\mathbb{E}\big[(\widetilde{W}_{n,\beta})^{1+\frac{\eta}{2}}\big] = \mathbb{E}\big[\mathbb{E}[\widetilde{W}_{\infty,\beta}|\mathscr{F}_n]^{1+\frac{\eta}{2}}\big] \leq \mathbb{E}\big[(\widetilde{W}_{\infty,\beta})^{1+\frac{\eta}{2}}\big] \leq c_8, \tag{4.1}$$

by using Jensen's inequality.

We need to introduce a more general statement of the many-to-one lemma. For $\beta \in I$, we define another random walk $(S_{n,\beta})_{n\in\mathbb{N}}$ starting at 0 under $\mathbb{P}$ and such that

$$\mathbb{E}\big[h(S_{1,\beta})\big] = e^{-\Psi(\beta)}\mathbb{E}\left[\sum_{|z|=1} h\big(V(z) + \Psi'(\beta)\big)e^{-\beta V(z)}\right].$$

Then, $S_{1,\beta}$ is centred and has variance $\sigma_\beta^2 := \mathbb{E}[S_{1,\beta}^2] = \Psi''(\beta) \in (0,\infty)$. Moreover, we have the following analogue of the many-to-one lemma (see Shi [40]): for all $n \geq 1$ and all measurable function $g\colon \mathbb{R}^{n+1} \to \mathbb{R}_+$, we have

$$\mathbb{E}\left[\sum_{|z|=n} g\big(V(z_i), i \in [\![0,n]\!]\big)\right] = e^{n\Psi(\beta)}\mathbb{E}\big[e^{\beta(S_{n,\beta}-n\Psi'(\beta))}g\big(S_{i,\beta} - i\Psi'(\beta), i \in [\![0,n]\!]\big)\big]. \tag{4.2}$$

One can see $(S_{n,\beta})_{n\in\mathbb{N}}$ as a discrete Girsanov transform of $(S_n)_{n\in\mathbb{N}}$.

We first establish a preliminary lemma.

Lemma 4.1. *Let $(\beta_n')_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ be sequences such that $\beta_n' \to 1$ and $k_n \geq 1/(1-\beta_n')^2$ for any $n \in \mathbb{N}$. Then, for any $L > 0$,*

$$\limsup_{n\to\infty} \frac{1}{1-\beta_n'}\mathbb{E}[\widetilde{W}_{k_n,\beta_n'}\mathbb{1}_{\inf_{x\in\mathbb{T}} V(x)\geq -L}] \leq \theta R(L)\mathbb{E}\big[e^{\sigma\mathcal{M}(1)}\big].$$

Proof. We first apply the many-to-one lemma to get that

$$\frac{1}{1-\beta_n'}\mathbb{E}[\widetilde{W}_{k_n,\beta_n'}\mathbb{1}_{\inf_{x\in\mathbb{T}} V(x)\geq -L}] \leq \frac{1}{1-\beta_n'}e^{-k_n\Psi(\beta_n')}\mathbb{E}\big[e^{(1-\beta_n')S_{k_n}}\mathbb{1}_{\underline{S}_{k_n}\geq -L}\big].$$

Then, we set $m_n := \lfloor(1-\beta_n')^{-2}\rfloor \leq k_n$ and, applying the Markov property at time m_n, we have

$$\mathbb{E}\big[e^{(1-\beta_n')S_{k_n}}\mathbb{1}_{\underline{S}_{k_n}\geq -L}\big] \leq \mathbb{E}\big[e^{(1-\beta_n')S_{m_n}}\mathbb{1}_{\underline{S}_{m_n}\geq -L}\big]\mathbb{E}\big[e^{(1-\beta_n')S_{k_n-m_n}}\big]$$

$$= \frac{\theta R(L)}{\sqrt{m_n}}\big(\mathbb{E}\big[e^{\sigma\mathcal{M}(1)}\big] + o(1)\big)e^{(k_n-m_n)\Psi(\beta_n')}, \tag{4.3}$$

using Corollary 2.7 to bound the first expectation in the middle part of (4.3) and the many-to-one lemma for the second expectation. Using that $e^{-m_n\Psi(\beta_n')} \leq 1$, it proves the lemma. $\qquad\square$

Proof of part (iv) of Theorem 1.1. We set $\xi_n := \alpha_n|\widetilde{W}_{\infty,\beta_n} - \widetilde{W}_{n,\beta_n}|$ and want to show that $\xi_n \to 0$ in $\mathbb{P}^*$-probability. It will prove part (iv) of Theorem 1.1, since $\alpha_n\widetilde{W}_{\infty,\beta_n} \to 2D_\infty$ in $\mathbb{P}^*$-probability by (1.11).

We first follow the proof of Lemma 4.2 of Madaule [34]. We set $p_n := 1 + 1/2\alpha_n$ and $\xi_n' := \mathbb{E}[\xi_n^{p_n}|\mathscr{F}_n]$. For $\varepsilon > 0$, we have

$$\mathbb{P}^*(\xi_n \geq \varepsilon) \leq \frac{1}{\mathbb{P}(S)}\mathbb{E}[\mathbb{1}_{\xi_n^{p_n}\geq\varepsilon^{p_n}}\mathbb{1}_{\xi_n'<\varepsilon^{1+p_n}}] + \mathbb{P}^*\big(\xi_n' \geq \varepsilon^{1+p_n}\big) \leq \frac{\varepsilon}{\mathbb{P}(S)} + \mathbb{P}^*\big(\xi_n' \geq \varepsilon^{1+p_n}\big), \tag{4.4}$$

using that $\mathbb{P}(\xi_n^{p_n} \geq \varepsilon^{p_n}|\mathscr{F}_n) \leq \varepsilon^{-p_n}\mathbb{E}[\xi_n^{p_n}|\mathscr{F}_n] = \varepsilon^{-p_n}\xi_n'$. By the branching property at time n, we have

$$\xi_n = \alpha_n \sum_{|x|=n} e^{-\beta_n V(x) - n\Psi(\beta_n)}\big(\widetilde{W}_{\infty,\beta_n}^{(x)} - 1\big),$$

where, conditionally on $\mathscr{F}_n$, the $\widetilde{W}^{(x)}_{\infty,\beta_n}$ for $|x| = n$ are independent variables with the same law as $\widetilde{W}_{\infty,\beta_n}$. Then, using that for any sequence $(X_i)_{i\in\mathbb{N}}$ of independent centred variables and any $\gamma \in [1, 2]$ we have $\mathbb{E}[|\sum X_i|^\gamma] \leq 2\sum \mathbb{E}[|X_i|^\gamma]$ (see [42]), we get

$$\xi'_n = \mathbb{E}\big[\xi_n^{p_n}|\mathscr{F}_n\big] \leq 2\alpha_n^{p_n} \sum_{|x|=n} e^{-p_n\beta_n V(x) - np_n\Psi(\beta_n)} \mathbb{E}\big[|\widetilde{W}_{\infty,\beta_n} - 1|^{p_n}\big]$$

$$\leq 2\alpha_n^{p_n} \widetilde{W}_{n,p_n\beta_n} e^{n[\Psi(p_n\beta_n) - p_n\Psi(\beta_n)]} 2^{p_n}(c_8 + 1), \tag{4.5}$$

by using (4.1) for n large enough such that $1 - \eta_0 < 1 - \beta_n < 1$. Now, we choose $L > 0$ such that $\mathbb{P}^*(\inf_{x\in\mathbb{T}} V(x) < -L) \leq \varepsilon$ and, by (4.5) and Markov's inequality, we get

$$\mathbb{P}^*\big(\xi'_n \geq \varepsilon^{1+p_n}\big) \leq \varepsilon + \varepsilon^{-(1+p_n)} \frac{c_9}{\mathbb{P}(S)}(2\alpha_n)^{p_n} e^{n[\Psi(p_n\beta_n) - p_n\Psi(\beta_n)]} \mathbb{E}[\widetilde{W}_{n,p_n\beta_n} \mathbb{1}_{\inf_{x\in\mathbb{T}} V(x) \geq -L}].$$

As $n \to \infty$, we have $(2\alpha_n)^{p_n} \sim 2\alpha_n$, $1 - p_n\beta_n \sim 1/2\alpha_n$ and, by a Taylor expansion, $n[\Psi(p_n\beta_n) - p_n\Psi(\beta_n)] \sim -3\sigma^2 n/8\alpha_n^2 \to -\infty$. Thus, applying Lemma 4.1 with $k_n = n$ and $\beta'_n = p_n\beta_n$, we showed that $\limsup_{n\to\infty} \mathbb{P}^*(\xi'_n \geq \varepsilon^{1+p_n}) \leq \varepsilon$. Coming back to (4.4), it concludes the proof. $\square$

Proof of part (iv) of Theorem 1.2. By Lemma A.1, we can reduce the proof to the case $F \in \mathcal{C}^u_b(\mathcal{D}([0,1]))$. Moreover, by considering $F - \mathbb{E}[F(B)]$ instead of F, we can assume that $\mathbb{E}[F(B)] = 0$. By Theorem 1.1, we have $\alpha_n e^{-n\Psi(\beta_n)} W_{n,\beta_n} \to 2D_\infty$ in $\mathbb{P}^*$-probability with $D_\infty > 0$ $\mathbb{P}^*$-a.s., so it is sufficient to prove that

$$U_n(F) := \alpha_n e^{-n\Psi(\beta_n)} \sum_{|z|=n} e^{-\beta_n V(z)} F\big(\widetilde{\mathbf{V}}^{(n)}(z)\big) \xrightarrow[n\to\infty]{} 2D_\infty \mathbb{E}\big[F(B)\big] = 0, \tag{4.6}$$

in $\mathbb{P}^*$-probability, where $\widetilde{\mathbf{V}}^{(n)}_t(z) := [V(z_{\lfloor tn\rfloor}) + tn\Psi'(\beta_n)]/\sigma\sqrt{n}$ for $t \in [0, 1]$. For some $C > 0$, we set $k_n := \lfloor C\alpha_n\rfloor^2$ and, for each $x \in \mathcal{D}([0,1])$, $F_n(x) := F(x_{((n-k_n)t+k_n)/n} - x_{k_n/n}, t \in [0,1])$. Let $\varepsilon > 0$. In order to prove (4.6), it is sufficient to prove that we can choose C such that

$$\limsup_{n\to\infty} \mathbb{P}^*\big(\big|U_n(F) - U_n(F_n)\big| \geq \varepsilon\big) \leq 2\varepsilon, \tag{4.7}$$

$$\limsup_{n\to\infty} \mathbb{P}^*\big(\big|U_n(F_n) - \mathbb{E}\big[U_n(F_n)|\mathscr{F}_{k_n}\big]\big| \geq \varepsilon\big) \leq \big(2 + \mathbb{P}(S)^{-1}\big)\varepsilon, \tag{4.8}$$

$$\limsup_{n\to\infty} \mathbb{P}^*\big(\big|\mathbb{E}\big[U_n(F_n)|\mathscr{F}_{k_n}\big]\big| \geq \varepsilon\big) \leq \varepsilon. \tag{4.9}$$

The assumption that $\mathbb{E}[F(B)] = 0$ is only needed for (4.9). For the sequel, we fix some $L > 0$ such that $\mathbb{P}^*(\inf_{x\in\mathbb{T}} V(x) < -L) \leq \varepsilon$.

We first prove (4.8). We set $\zeta_n := U_n(F_n) - \mathbb{E}[U_n(F_n)|\mathscr{F}_{k_n}]$ and proceed in a similar way as for the proof of part (iv) of Theorem 1.1, by setting $p_n := 1 + 1/2\alpha_n$ and $\zeta'_n := \mathbb{E}[|\zeta_n|^{p_n}|\mathscr{F}_n]$. By (4.4), we have $\mathbb{P}^*(|\zeta_n| \geq \varepsilon) \leq \varepsilon\mathbb{P}(S)^{-1} + \mathbb{P}^*(\zeta'_n \geq \varepsilon^{1+p_n})$. By the branching property at time k_n, we have

$$\zeta_n = \alpha_n \sum_{|x|=k_n} e^{-\beta_n V(x) - k_n\Psi(\beta_n)}\big(\Upsilon_n^{(x)} - \mathbb{E}[\Upsilon_n]\big), \tag{4.10}$$

where, conditionally on $\mathscr{F}_{k_n}$, the $\Upsilon_n^{(x)}$ for $|x| = k_n$ are independent variables with the same law as Υ_n defined by

$$\Upsilon_n := \sum_{|z|=n-k_n} e^{-\beta_n V(z) - (n-k_n)\Psi(\beta_n)} F\left(\frac{V(z_{\lfloor t(n-k_n)\rfloor}) + t(n-k_n)\Psi'(\beta_n)}{\sigma\sqrt{n}}, t \in [0,1]\right). \tag{4.11}$$

Since the $\Upsilon_n^{(x)} - \mathbb{E}[\Upsilon_n]$ are also centred, we get, in the same way as for (4.5),

$$\zeta'_n = \mathbb{E}\big[|\zeta_n|^{p_n}|\mathscr{F}_{k_n}\big] \leq 2\alpha_n^{p_n} \sum_{|x|=k_n} e^{-p_n\beta_n V(x) - k_n p_n\Psi(\beta_n)} \mathbb{E}\big[\big|\Upsilon_n - \mathbb{E}[\Upsilon_n]\big|^{p_n}\big]$$

$$\leq 4c_8 \big(2\|F\|\alpha_n \big)^{p_n} \widetilde{W}_{k_n, p_n \beta_n} e^{k_n (\Psi(p_n \beta_n) - p_n \Psi(\beta_n))}, \tag{4.12}$$

by using the following bound

$$\mathbb{E}\big[\big| \Upsilon_n - \mathbb{E}[\Upsilon_n] \big|^{p_n} \big] \leq 2^{p_n+1} \mathbb{E}\big[|\Upsilon_n|^{p_n} \big] \leq 2^{p_n+1} \|F\|^{p_n} \mathbb{E}\big[|\widetilde{W}_{n-k_n, \beta_n}|^{p_n} \big] \leq 2^{p_n+1} \|F\|^{p_n} c_8,$$

where we used (4.1) for n large enough such that $1 - \eta_0 < 1 - \beta_n < 1$. Now, using that $\mathbb{P}^*(\inf_{x \in \mathbb{T}} V(x) < -L) \leq \varepsilon$ and Markov's inequality, we get

$$\mathbb{P}^*\big(\zeta_n' \geq \varepsilon^{1+p_n} \big) \leq \varepsilon + \varepsilon^{-(1+p_n)} \frac{c_{10}}{\mathbb{P}(S)} \big(2\|F\|\alpha_n \big)^{p_n} e^{k_n [\Psi(p_n \beta_n) - p_n \Psi(\beta_n)]} \mathbb{E}[\widetilde{W}_{k_n, p_n \beta_n} \mathbb{1}_{\inf_{x \in \mathbb{T}} V(x) \geq -L}]$$

so, using that, as $n \to \infty$, we have $(2\|F\|\alpha_n/\varepsilon)^{p_n} \sim 2\|F\|\alpha_n/\varepsilon$, $1 - p_n \beta_n \sim 1/2\alpha_n$ and $k_n [\Psi(p_n \beta_n) - p_n \Psi(\beta_n)] \sim -3\sigma^2 k_n/8\alpha_n^2 \to -3\sigma^2 C^2/8$, and applying Lemma 4.1 with $\beta_n' = p_n \beta_n$, we get

$$\limsup_{n \to \infty} \mathbb{P}^*\big(\zeta_n' \geq \varepsilon^{1+p_n} \big) \leq \varepsilon + c_{11} \frac{R(L)}{\varepsilon^2} e^{-3\sigma^2 C^2/8} \leq 2\varepsilon,$$

by choosing C large enough. This proves (4.8). The constant C is now fixed.

We now prove (4.7). Using that $\mathbb{P}^*(\inf_{x \in \mathbb{T}} V(x) < -L) \leq \varepsilon$ and the Markov inequality, we get, with $c_{12} := 1/\mathbb{P}(S)$,

$$\mathbb{P}^*\big(|U_n(F) - U_n(F_n)| \geq \varepsilon \big) \leq \varepsilon + \frac{c_{12}\alpha_n}{\varepsilon e^{n\Psi(\beta_n)}} \mathbb{E}\left[\sum_{|z|=n} e^{-\beta_n V(z)} \mathbb{1}_{\underline{V}(z) \geq -L} |F - F_n|(\widetilde{\mathbf{V}}^{(n)}(z)) \right]. \tag{4.13}$$

For $t \in [0, 1]$, we define $\widetilde{\mathbf{S}}_t^{(n)} := [S_{\lfloor tn \rfloor} + tn\Psi'(\beta_n)]/\sigma\sqrt{n}$. Then, using the many-to-one lemma and the triangle inequality, we get that, for any $M > 0$, the expectation in (4.13) is smaller than

$$\mathbb{E}\big[e^{S_n/\alpha_n} \mathbb{1}_{\underline{S}_n \geq -L, \max_{0 \leq k \leq k_n} S_k \leq M\alpha_n} |F - F_n|(\widetilde{\mathbf{S}}^{(n)}) \big] + 2\|F\| \mathbb{E}\big[e^{S_n/\alpha_n} \mathbb{1}_{\underline{S}_n \geq -L, \max_{0 \leq k \leq k_n} S_k > M\alpha_n} \big]. \tag{4.14}$$

Note that, using (A.7) with here $\kappa_n = k_n/n$, we have, for any $x \in \mathcal{D}([0, 1])$,

$$|F - F_n|(x) \leq \omega_F \left(\frac{k_n}{n} \vee 3 \max_{[0, k_n/n]} |x| \right)$$

and, thus, the first term in (4.14) is smaller than

$$\omega_F \left(\frac{\lfloor C\alpha_n \rfloor^2}{n} \vee \frac{3M\alpha_n}{\sigma\sqrt{n}} \right) \mathbb{E}\big[e^{S_n/\alpha_n} \mathbb{1}_{\underline{S}_n \geq -L} \big] = o\left(\frac{e^{n\Psi(\beta_n)}}{\alpha_n} \right),$$

using (4.3) to bound the expectation and recalling that $\alpha_n \ll \sqrt{n}$. On the other hand, using Markov property at time k_n, the second term in (4.14) is smaller than

$$2\|F\| \mathbb{E}\big[e^{S_{k_n}/\alpha_n} \mathbb{1}_{\underline{S}_{k_n} \geq -L, \max_{0 \leq k \leq k_n} S_k > M\alpha_n} \big] \mathbb{E}\big[e^{S_{n-k_n}/\alpha_n} \big]$$

$$= 2\|F\| \frac{\theta R(L)}{\sqrt{k_n}} \big(\mathbb{E}\big[e^{C\sigma \mathcal{M}(1)} \mathbb{1}_{\max \mathcal{M} > M/(C\sigma)} \big] + o(1) \big) e^{(n-k_n)\Psi(\beta_n)}$$

applying Corollary 2.7 and recalling that $\sqrt{k_n} = \lfloor C\alpha_n \rfloor$. Coming back to (4.13), we finally get

$$\limsup_{n \to \infty} \mathbb{P}^*\big(|U_n(F) - U_n(F_n)| \geq \varepsilon \big) \leq \varepsilon + c_{13} \frac{R(L)}{\varepsilon C} \mathbb{E}\big[e^{C\sigma \mathcal{M}(1)} \mathbb{1}_{\max \mathcal{M} > M/(C\sigma)} \big] \leq 2\varepsilon, \tag{4.15}$$

by choosing M large enough (L and C being fixed). It proves (4.7).

Finally, we prove (4.9). Using the branching property in the same way as for (4.10), we have $\mathbb{E}[U_n(F_n)|\mathscr{F}_{k_n}] = \alpha_n \widetilde{W}_{k_n,\beta_n} \mathbb{E}[\Upsilon_n]$, where Υ_n is defined in (4.11). By (4.2), we get

$$\mathbb{E}[\Upsilon_n] = \mathbb{E}\left[F\left(\frac{S_{\lfloor t(n-k_n)\rfloor,\beta_n} + (t(n-k_n) - \lfloor t(n-k_n)\rfloor)\Psi'(\beta_n)}{\sigma\sqrt{n}}, t \in [0,1] \right) \right]$$

$$= \mathbb{E}\left[F\left(u_n \mathbf{S}^{(n-k_n,\beta_n)} + v_n \right) \right],$$

where we introduced $\mathbf{S}^{(n,\beta)} := (S_{\lfloor tn\rfloor,\beta}/\sigma_\beta\sqrt{n})_{t\in[0,1]}$ and where $(u_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}} \in \mathcal{D}([0,1])^{\mathbb{N}}$ satisfy $u_n \to 1$ and $\|v_n\|_\infty \to 0$, using that $k_n \ll n$, $\Psi'(\beta_n) \to 0$ and $\sigma_{\beta_n} \to \sigma$. Now, note that $\mathbf{S}^{(n-k_n,\beta_n)} \to B$ in law. This is not a direct consequence of Donsker's theorem because here the law of the random walk changes for each n. However, we can apply a strong invariance principle like Equation (11) of Sakhanenko [39]: it exists $c_{14} > 0$ such that, for any $n \geq 1$ and $\beta \in I$, it exists a Brownian motion $B^{(n,\beta)}$ such that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |\mathbf{S}_t^{(n,\beta)} - B_t^{(n,\beta)}| \geq c_{14}\frac{n^{2/5}}{\sigma_\beta\sqrt{n}} \right) \leq \frac{n\mathbb{E}[|S_{1,\beta}|^3]}{(n^{2/5})^3}.$$

Since $\sigma_{\beta_n} \to \sigma$ and $\mathbb{E}[|S_{1,\beta_n}|^3] \to \mathbb{E}[|S_1|^3] < \infty$ as $n \to \infty$, this proves that $\mathbf{S}^{(n-k_n,\beta_n)} \to B$ in law. Then, applying Lemma A.3, we get that $\mathbb{E}[\Upsilon_n] \to \mathbb{E}[F(B)] = 0$ as $n \to \infty$. On the other hand, using that $\mathbb{P}^*(\inf_{x\in\mathbb{T}} V(x) < -L) \leq \varepsilon$ and the Markov inequality, we get

$$\mathbb{P}^*\left(|\mathbb{E}[U_n(F_n)|\mathscr{F}_{k_n}]| \geq \varepsilon \right) \leq \varepsilon + \frac{1}{\varepsilon}\alpha_n \mathbb{E}[\widetilde{W}_{k_n,\beta_n} \mathbb{1}_{\inf_{x\in\mathbb{T}} V(x) \geq -L}] |\mathbb{E}[\Upsilon_n]|,$$

and it proves (4.9) by using Lemma 4.1 and that $\mathbb{E}[\Upsilon_n] \to 0$ as $n \to \infty$. $\qquad\square$

5. The near-critical window in the strong disorder regime

In this section, we prove case (i) of Theorems 1.1 and 1.2 and of Corollary 1.3, where $\beta_n = 1 + 1/\alpha_n$ with $\alpha_n \ll \sqrt{n}$. This case constitutes the main part of this paper.

5.1. *Change of probabilities*

We introduce a first barrier by setting, for $L > 0$ and $F \in \mathcal{C}_b(\mathcal{D}([0,1]))$,

$$W_{n,\beta_n}(F) := \sum_{|z|=n} \mathrm{e}^{-\beta_n V(z)} F(\mathbf{V}(z)),$$

$$W_{n,\beta_n}^{(L)}(F) := \sum_{|z|=n} \mathrm{e}^{-\beta_n V(z)} F(\mathbf{V}(z)) \mathbb{1}_{\underline{V}(z)\geq -L},$$

so that for L large this two variables are equal with high probability. Moreover, recall that, for $u \geq 0$, $R_L(u) = R(L+u)$ and

$$D_n^{(L)} = \sum_{|z|=n} \mathrm{e}^{-V(z)} R_L(V(z)) \mathbb{1}_{\underline{V}(z)\geq -L}$$

and that, using the martingale $(D_n^{(L)})_{n\in\mathbb{N}}$, we defined the modified probability measure $\mathbb{Q}^{(L)}$. We will work under this measure to study the asymptotic behaviour of $W_{n,\beta_n}(F)$.

Proposition 5.1. *For all $L > 0$ and $F \in \mathcal{C}_b^u(\mathcal{D}([0,1]))$, we have the convergence*

$$\frac{n^{3\beta_n/2}}{\alpha_n^2} \frac{W_{n,\beta_n}^{(L)}(F)}{D_n^{(L)}} \xrightarrow[n\to\infty]{} \frac{\theta}{\sigma^2}\mathbb{E}[F(\mathfrak{e})], \quad in \ \mathbb{Q}^{(L)}\text{-probability}.$$

This proposition will be proved in the following subsections, using a second moment argument similar to the one used by Aïdekon and Shi [3].

Proof of part (i) of Theorems 1.1 and 1.2. We consider $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$ and we are going to show here that

$$\frac{n^{3\beta_n/2}}{\alpha_n^2} W_{n,\beta_n}(F) \xrightarrow[n\to\infty]{} \frac{c_0\theta}{\sigma^2}\mathbb{E}[F(\mathfrak{e})]D_\infty, \quad \text{in } \mathbb{P}^*\text{-probability.} \tag{5.1}$$

Using (2.5), it proves part (i) of Theorem 1.1 by taking $F \equiv 1$. Moreover, noting that $\mu_{n,\beta_n}(F) = W_{n,\beta_n}(F)/W_{n,\beta_n}(1)$ and $D_\infty > 0$ $\mathbb{P}^*$-a.s., it proves part (i) of Theorem 1.2 in the case $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$. The general case follows by Lemma A.1. Thus, it is now sufficient to prove (5.1) and we can for this purpose assume that F is nonnegative.

Let $\varepsilon, \eta > 0$. We first fix $L > 0$ such that $\mathbb{P}^*(\inf_{x\in\mathbb{T}} V(x) < -L) \leq \eta$. Combining that $\min_{|x|=n} V(x) \to \infty$ $\mathbb{P}^*$-a.s. as $n \to \infty$ and that $R_L(u) \sim c_0 u$ as $u \to \infty$, we get that, on the event $\{\inf_{x\in\mathbb{T}} V(x) \geq -L\}$, $\lim_{n\to\infty} D_n^{(L)} = c_0 D_\infty > 0$ $\mathbb{P}^*$-a.s. Thus, considering the event

$$\Omega_0 := S \cap \left\{\inf_{x\in\mathbb{T}} V(x) \geq -L\right\} \cap \left\{\forall n \geq n_0, 0 < c_0(1-\varepsilon)D_\infty \leq D_n^{(L)} \leq c_0(1+\varepsilon)D_\infty\right\},$$

we can fix $n_0 \in \mathbb{N}$ such that $\mathbb{P}^*(\Omega_0) \geq 1 - 2\eta$. We now introduce the event

$$E_n := \left\{\frac{n^{3\beta_n/2}}{\alpha_n^2} W_{n,\beta_n}(F) \notin \left[\frac{c_0\theta}{\sigma^2}D_\infty\big(\mathbb{E}[F(\mathfrak{e})] - \varepsilon\big)(1-\varepsilon), \frac{c_0\theta}{\sigma^2}D_\infty\big(\mathbb{E}[F(\mathfrak{e})] + \varepsilon\big)(1+\varepsilon)\right]\right\}.$$

Then, using that on Ω_0 we have $W_{n,\beta_n}^{(L)}(F) = W_{n,\beta_n}(F)$, we get, for $n \geq n_0$,

$$\mathbb{E}\big[D_n^{(L)}\mathbb{1}_{E_n\cap\Omega_0}\big] = \mathbb{Q}^{(L)}(E_n \cap \Omega_0) \leq \mathbb{Q}^{(L)}\left(\left|\frac{n^{3\beta_n/2}}{\alpha_n^2}\frac{W_{n,\beta_n}^{(L)}(F)}{D_n^{(L)}} - \frac{\theta}{\sigma^2}\mathbb{E}[F(\mathfrak{e})]\right| > \frac{\theta\varepsilon}{\sigma^2}\right) \xrightarrow[n\to\infty]{} 0,$$

applying Proposition 5.1. Since $D_n^{(L)}\mathbb{1}_{E_n\cap\Omega_0} \geq 0$, it follows that $D_n^{(L)}\mathbb{1}_{E_n\cap\Omega_0} \to 0$ in $\mathbb{P}$-probability. Using again that, on Ω_0, $\lim_{n\to\infty} D_n^{(L)} = c_0 D_\infty > 0$ $\mathbb{P}^*$-a.s., we get that $\mathbb{P}^*(E_n \cap \Omega_0) \to 0$. Recalling that $\mathbb{P}^*(\Omega_0) \geq 1 - 2\eta$, we showed that $\limsup_{n\to\infty} \mathbb{P}^*(E_n) \leq 2\eta$ and, therefore, it proves (5.1). $\qquad\square$

5.2. *Proof of Proposition 5.1 and of part* (i) *of Corollary* 1.3

The aim of this section is to break down the proof of Proposition 5.1 into the proof of several lemmas. Our goal is to use a second moment argument, but the first moment of $W_{n,\beta_n}^{(L)}(F)/D_n^{(L)}$ under $\mathbb{Q}^{(L)}$ does not have the right order and the second moment is not even necessarily finite. Thus, we first need to come down to another random variable $Y_n'(F)$ that is close to $W_{n,\beta_n}^{(L)}(F)$ with high probability and that has first and second moments of the right order, by eliminating some rare particles with a too strong weight in the expectations. The lemmas stated in this subsection will also allow us to prove part (i) of Corollary 1.3.

Until the end of the paper, L is a fixed positive constant. We first add a second barrier between times $\lfloor n/2 \rfloor$ and n at position $(3/2)\log n - K$, by setting, for $K > 0$,

$$W_{n,\beta_n}^{(L,K)}(F) := \sum_{|z|=n} e^{-\beta_n V(z)} F\big(\mathbf{V}(z)\big)\mathbb{1}_{\underline{V}(z)\geq -L, \min_{\lfloor n/2\rfloor\leq j\leq n} V(z_j)\geq \frac{3}{2}\log n - K}.$$

This first lemma shows that $W_{n,\beta_n}^{(L)}(F)$ and $W_{n,\beta_n}^{(L,K)}(F)$ are close with high probability and will be proved in Section 5.4. Note that this step is superfluous when $\log n \ll \alpha_n \ll \sqrt{n}$: since the particles contributing to W_{n,β_n} are of order α_n, it is approximately the same difficulty for them to stay above the two barriers than only above the first (see Proposition 2.8) and, thus, $W_{n,\beta_n}^{(L)}(F)$ and $W_{n,\beta_n}^{(L,K)}(F)$ have asymptotically the same first moment in that case.
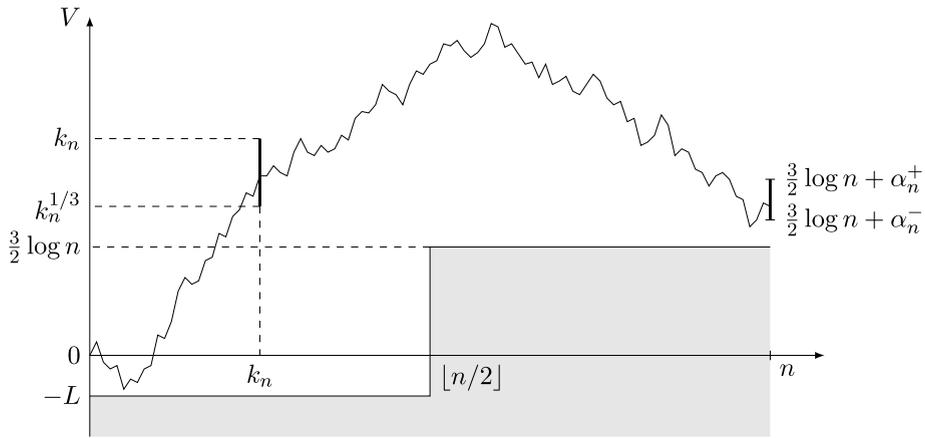
Fig. 1. Representation of the trajectory of a particle in A_n. It has to stay above the gray area and to pass through both thick segments.

Lemma 5.2. *For all $L > 0$ and $\varepsilon, \eta > 0$, it exists $K > 0$ such that, for each $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$,*

$$\limsup_{n \to \infty} \mathbb{Q}^{(L)}\left(\frac{n^{3\beta_n/2}}{\alpha_n^2} \frac{|W_{n,\beta_n}^{(L)}(F) - W_{n,\beta_n}^{(L,K)}(F)|}{D_n^{(L)}} > \varepsilon \|F\| \right) \leq \eta.$$

We now consider a fixed $K > 0$. We will see in Lemma 5.4 that $W_{n,\beta_n}^{(L,K)}(F)$ has the right first moment, but we still need to remove other particles for the second moment. Let $(\alpha_n^+)_{n \in \mathbb{N}}$ and $(\alpha_n^-)_{n \in \mathbb{N}}$ be sequences of positive real numbers and $(k_n)_{n \in \mathbb{N}}$ be a sequence of integers such that

$$1 \ll \alpha_n^- \ll \alpha_n \ll \alpha_n^+ \ll \sqrt{n},$$
$$(\log n)^6 \ll k_n \ll \sqrt{n},$$

when $n \to \infty$. We add some controls on the trajectory of the particle's lineage, by considering

$$Y_n(F) := \sum_{|z|=n} e^{-\beta_n V(z)} F(\mathbf{V}(z)) \mathbb{1}_{z \in A_n},$$

where we set $A_n := \{|z| = n : \forall j \in [\![0, n]\!], V(z_j) \in I_{n,j}\}$ (see Figure 1), with

$$I_{n,j} := \begin{cases} [-L, \infty) & \text{if } 0 \leq j < \lfloor n/2 \rfloor \text{ and } j \neq k_n, \\ [k_n^{1/3}, k_n] & \text{if } j = k_n, \\ [(3/2)\log n, \infty) & \text{if } \lfloor n/2 \rfloor \leq j \leq n, \\ [(3/2)\log n + \alpha_n^-, (3/2)\log n + \alpha_n^+] & \text{if } j = n. \end{cases}$$

Note that the second barrier is here simply at $(3/2)\log n$: indeed, Lemma 5.4 shows that it does not change the first moment (and we could even have taken $(3/2)\log n + C$ with any $C > 0$).

But, in order to compute the first moment of $Y_n(F)$, we will first need to take the conditional expectation given $\mathscr{F}_{k_n}$ and, thus, we want to show that $F(\mathbf{V}(z))$ does not asymptotically depend on what happen before time k_n. For this, we consider for each $n \in \mathbb{N}^*$ a slight modification F_n of function F such that

$$F_n(\mathbf{V}(z)) := F\left(\frac{V(z_{\lfloor (n-k_n)t \rfloor + k_n}) - V(z_{k_n})}{\sigma \sqrt{n - k_n}}, t \in [0, 1] \right).$$

More formally, we set, for each $x \in \mathcal{D}([0, 1])$, $F_n(x) := F([\sqrt{n}/\sqrt{n - k_n}] \cdot [x_{((n-k_n)t+k_n)/n} - x_{k_n/n}], t \in [0, 1])$. The following lemma, proved in Section 5.5, shows that we can replace F with F_n. It does not play a crucial role, but makes the calculations easier for the next lemmas.

Lemma 5.3. *For all $L, K > 0$ and $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$, we have*

$$\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{W_{n,\beta_n}^{(L,K)}(|F - F_n|)}{D_n^{(L)}}\right] = o\left(\frac{\alpha_n^2}{n^{3\beta_n/2}}\right),$$

as $n \to \infty$.

Noting that, for F nonnegative, $Y_n(F) \leq W_{n,\beta_n}^{(L,K)}(F)$, Lemma 5.3 combined with the following lemma shows that the first moments under $\mathbb{Q}^{(L)}$ of $W_{n,\beta_n}^{(L,K)}(F)$, $W_{n,\beta_n}^{(L,K)}(F_n)$, $Y_n(F)$ and $Y_n(F_n)$ (divided by $D_n^{(L)}$) have the same equivalent as $n \to \infty$. It will be proved in Section 5.3.

Lemma 5.4. *For all $L, K > 0$ and $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$ nonnegative, we have*

$$\limsup_{n\to\infty} \frac{n^{3\beta_n/2}}{\alpha_n^2} \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{W_{n,\beta_n}^{(L,K)}(F)}{D_n^{(L)}}\right] \leq \frac{\theta}{\sigma^2}\mathbb{E}[F(\mathfrak{e})] \leq \liminf_{n\to\infty} \frac{n^{3\beta_n/2}}{\alpha_n^2}\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n(F_n)}{D_n^{(L)}}\right].$$

However, this is still not sufficient for controlling the second moment of $Y_n(F)$ and we need to introduce a new random variable $Y_n'(F)$. We consider the sequences

$$a_j^{(n)} := \begin{cases} -L & \text{if } 0 \leq j < \lfloor n/2 \rfloor, \\ \frac{3}{2}\log n & \text{if } \lfloor n/2 \rfloor \leq j \leq n, \end{cases} \quad \text{and} \quad \ell_j^{(n)} := \begin{cases} j^{1/7} & \text{if } 0 \leq j < \lfloor n/2 \rfloor, \\ (n-j)^{1/7} & \text{if } \lfloor n/2 \rfloor \leq j \leq n. \end{cases}$$

Then, for some fixed sequence $(\rho_n)_{n\in\mathbb{N}}$ that tends to infinity such that $\rho_n \ll \alpha_n^2$, we define the following set, that will allow us to control the offspring of the spine in the second moment calculation (see Aïdékon [1] for the first use of this method),

$$B_n := \left\{|z| = n : \forall j \in [\![0, n-1]\!], \sum_{y\in\Omega(z_{j+1})} e^{-[V(y) - a_j^{(n)}]} \leq \rho_n e^{-\ell_j^{(n)}}\right\},$$

where $\Omega(x)$ is the set of brothers of x, and we set

$$Y_n'(F) := \sum_{|z|=n} e^{-\beta_n V(z)} F_n(\mathbf{V}(z)) \mathbb{1}_{z \in A_n \cap B_n}.$$

The following lemma shows that this new random variable $Y_n'(F)$ is close to $Y_n(F)$. Its proof relies on the peeling lemma stated in Section 5.6 in a more general feature.

Lemma 5.5. *For all $L, K > 0$ and $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$, we have*

$$\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{|Y_n(F_n) - Y_n'(F_n)|}{D_n^{(L)}}\right] = o\left(\frac{\alpha_n^2}{n^{3\beta_n/2}}\right),$$

as $n \to \infty$.

By considering $Y_n'(F_n)$, we can now control the second moment properly, as stated in this last lemma, proved in Section 5.7. This second moment is exactly the square of the first moment in Lemma 5.4.

Lemma 5.6. *For all $L, K > 0$ and $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$ nonnegative, we have*

$$\limsup_{n\to\infty}\left(\frac{n^{3\beta_n/2}}{\alpha_n^2}\right)^2 \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\left(\frac{Y_n'(F_n)}{D_n^{(L)}}\right)^2\right] \leq \left(\frac{\theta}{\sigma^2}\mathbb{E}[F(\mathfrak{e})]\right)^2.$$

Using these lemmas, we can now prove Proposition 5.1.

Proof of Proposition 5.1. We consider here the case $F \geq 0$ and the general case follows by taking the positive and negative parts of F. We first fix $K > 0$. Using Lemmas 5.4 and 5.5, we have

$$\liminf_{n \to \infty} \frac{n^{3\beta_n/2}}{\alpha_n^2} \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n'(F_n)}{D_n^{(L)}}\right] \geq \frac{\theta}{\sigma^2}\mathbb{E}[F(\mathfrak{e})], \tag{5.2}$$

and so, by applying Bienaymé–Chebyshev inequality and Lemma 5.6, we get that

$$\frac{n^{3\beta_n/2}}{\alpha_n^2}\frac{Y_n'(F_n)}{D_n^{(L)}} \xrightarrow[n \to \infty]{} \frac{\theta}{\sigma^2}\mathbb{E}[F(\mathfrak{e})], \quad \text{in } \mathbb{Q}^{(L)}\text{-probability}. \tag{5.3}$$

Combining (5.3) with Lemmas 5.4, 5.3 and 5.5, we deduce that for all $K > 0$,

$$\frac{n^{3\beta_n/2}}{\alpha_n^2}\frac{W_{n,\beta_n}^{(L,K)}(F)}{D_n^{(L)}} \xrightarrow[n \to \infty]{} \frac{\theta}{\sigma^2}\mathbb{E}[F(\mathfrak{e})], \quad \text{in } \mathbb{Q}^{(L)}\text{-probability}. \tag{5.4}$$

Now, for $\varepsilon, \eta > 0$, by Lemma 5.2, we can choose $K > 0$ such that

$$\limsup_{n \to \infty} \mathbb{Q}^{(L)}\left(\frac{n^{3\beta_n/2}}{\alpha_n^2}\frac{|W_{n,\beta_n}^{(L)}(F) - W_{n,\beta_n}^{(L,K)}(F)|}{D_n^{(L)}} > \frac{\varepsilon}{2}\right) \leq \eta \tag{5.5}$$

and, thus, we get, using the triangle inequality, (5.4) and (5.5),

$$\limsup_{n \to \infty} \mathbb{Q}^{(L)}\left(\left|\frac{n^{3\beta_n/2}}{\alpha_n^2}\frac{W_{n,\beta_n}^{(L)}(F)}{D_n^{(L)}} - \frac{\theta}{\sigma^2}\mathbb{E}[F(\mathfrak{e})]\right| > \varepsilon\right) \leq \eta.$$

This concludes the proof of Proposition 5.1. $\qquad \square$

We conclude this subsection by proving part (i) of Corollary 1.3, which is a consequence of the fact that considering only particles in A_n does not change the first moment asymptotic.

Proof of part (i) of Corollary 1.3. First note that it is sufficient to prove that, for all sequences $(\alpha_n^-)_{n \in \mathbb{N}}$ and $(\alpha_n^+)_{n \in \mathbb{N}}$ such that $1 \ll \alpha_n^- \ll \alpha_n \ll \alpha_n^+ \ll \sqrt{n}$, we have

$$\nu_{n,\beta_n}\left([(3/2)\log n + \alpha_n^-, (3/2)\log n + \alpha_n^+]\right) \xrightarrow[n \to \infty]{} 1, \quad \text{in } \mathbb{P}^*\text{-probability}. \tag{5.6}$$

Indeed, if we assume that part (i) of Corollary 1.3 is false, then it exists $\varepsilon > 0$ such that for any $k \geq 1$, it exists $n_k > n_{k-1}$ (with $n_0 := 0$) such that

$$\mathbb{P}^*\left(\nu_{n_k,\beta_{n_k}}\left([(3/2)\log n_k + k^{-1}\alpha_{n_k}, (3/2)\log n_k + k\alpha_{n_k}]\right) \leq 1 - \varepsilon\right) \geq 1 - \varepsilon. \tag{5.7}$$

Setting $e_n := \inf\{k \in \mathbb{N} : n_k \geq n\}$, we have $e_n \to \infty$ and $e_{n_k} = k$. Thus, with $\alpha_n^+ := e_n\alpha_n$ and $\alpha_n^- := e_n^{-1}\alpha_n$, (5.7) implies the negation of (5.6). Therefore, we now want to prove (5.6).

The left-hand side of (5.6) is larger than $Y_n(1)/W_{n,\beta_n}$, therefore, it is sufficient to show that $Y_n(1)/W_{n,\beta_n} \to 1$ in $\mathbb{P}^*$-probability. Combining (5.3) with Lemma 5.5, we first have

$$\frac{n^{3\beta_n/2}}{\alpha_n^2}\frac{Y_n(1)}{D_n^{(L)}} \xrightarrow[n \to \infty]{} \frac{\theta}{\sigma^2}, \quad \text{in } \mathbb{Q}^{(L)}\text{-probability}, \tag{5.8}$$

and, in the same way as in the proof of part (i) of Theorems 1.1 and 1.2, where we showed (5.1) from Proposition 5.1, it follows from (5.8) that

$$\frac{n^{3\beta_n/2}}{\alpha_n^2} \frac{Y_n(1)}{D_n} \xrightarrow[n\to\infty]{} \frac{c_0\theta}{\sigma^2}, \quad \text{in } \mathbb{P}^*\text{-probability.} \tag{5.9}$$

Using (5.9) and (5.1) with $F \equiv 1$, we get that $Y_n(1)/W_{n,\beta_n} \to 1$ in $\mathbb{P}^*$-probability and so (5.6) is proved. $\qquad\square$

5.3. First moments of $W_{n,\beta_n}^{(L,K)}(F)$ and $Y_n(F_n)$

We start with the proof of Lemma 5.4 in this subsection, because this first moment calculation will be a kind of routine at which we will refer for the proof of other lemmas. In this calculations, sums of general term $(i + C)e^{-i/\gamma_n}$ appear regularly, with $C \in \mathbb{R}$ a constant and $\gamma_n \to \infty$ as $n \to \infty$. Therefore, we state the following result, proved by explicit computation: if $(\gamma_n^+)_{n\in\mathbb{N}}$ and $(\gamma_n^-)_{n\in\mathbb{N}}$ are sequences with values in $\mathbb{R}_+ \cup \{\infty\}$ such that $\gamma_n^- \ll \gamma_n \ll \gamma_n^+$ as $n \to \infty$, then we have, for any constant $C \in \mathbb{R}$,

$$\sum_{i\in[\gamma_n^-,\gamma_n^+]\cap\mathbb{N}} (i + C)e^{-i/\gamma_n} \underset{n\to\infty}{\sim} \gamma_n^2 \quad \text{and} \quad \sum_{i\notin[\gamma_n^-,\gamma_n^+]\cap\mathbb{N}} (i + C)e^{-i/\gamma_n} = o(\gamma_n^2). \tag{5.10}$$

Proof of Lemma 5.4. Recall that we consider $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$ nonnegative. We begin with the control of $\mathbb{E}_{\mathbb{Q}^{(L)}}[W_{n,\beta_n}^{(L,K)}(F)/D_n^{(L)}]$. Note that $\mathbb{E}_{\mathbb{Q}^{(L)}}[W_{n,\beta_n}^{(L,K)}(F)/D_n^{(L)}] = \mathbb{E}[W_{n,\beta_n}^{(L,K)}(F)]/R(L)$. Then, using the many-to-one lemma, we get

$$\mathbb{E}[W_{n,\beta_n}^{(L,K)}(F)] = \mathbb{E}[e^{-S_n/\alpha_n} F(\mathbf{S}^{(n)}) \mathbb{1}_{\underline{S}_n \geq -L, \min_{\lfloor n/2\rfloor \leq j \leq n} V(z_j) \geq \frac{3}{2}\log n - K}].$$

We cut this expectation in two pieces depending on whether $S_n \leq \frac{3}{2}\log n + \alpha_n^+$ or $S_n > \frac{3}{2}\log n + \alpha_n^+$. Let start with the case $S_n > \frac{3}{2}\log n + \alpha_n^+$: we have, by cutting the interval $[\alpha_n^+ + K, \infty)$ in pieces of length 1,

$$\mathbb{E}[e^{-S_n/\alpha_n} F(\mathbf{S}^{(n)}) \mathbb{1}_{\underline{S}_n \geq -L, \min_{\lfloor n/2\rfloor \leq j \leq n} S_j \geq \frac{3}{2}\log n - K, S_n > \frac{3}{2}\log n + \alpha_n^+}]$$

$$\leq \sum_{i\geq\lfloor\alpha_n^+ + K\rfloor} \mathbb{E}[e^{-(\frac{3}{2}\log n - K + i)/\alpha_n} \|F\| \mathbb{1}_{\underline{S}_n \geq -L, \min_{\lfloor n/2\rfloor \leq j \leq n} S_j \geq \frac{3}{2}\log n - K, S_n - (\frac{3}{2}\log n - K) \in [i,i+1)}]$$

$$\leq \|F\| \frac{e^{K/\alpha_n}}{n^{3/2\alpha_n}} \sum_{i\geq\lfloor\alpha_n^+ + K\rfloor} e^{-i/\alpha_n} c_5 \frac{(L + 1)2(i + 2)}{n^{3/2}}, \tag{5.11}$$

by using (2.8). Since $\lfloor\alpha_n^+ + K\rfloor \gg \alpha_n$, it follows from (5.10) that the right-hand side of (5.11) is a $o(\alpha_n^2/n^{3\beta_n/2})$. We now control the term corresponding to the case $S_n \leq \frac{3}{2}\log n + \alpha_n^+$, which is the dominant term in $\mathbb{E}[W_{n,\beta_n}^{(L,K)}(F)]$. This time, we cut the segment $[0, \alpha_n^+ + K]$ in pieces of length $h > 0$, where h is any real number if S_1 is nonlattice and is the span of the lattice if S_1 is lattice, and thus we get

$$\mathbb{E}[e^{-S_n/\alpha_n} F(\mathbf{S}^{(n)}) \mathbb{1}_{\underline{S}_n \geq -L, \min_{\lfloor n/2\rfloor \leq j \leq n} S_j \geq \frac{3}{2}\log n - K, S_n \leq \frac{3}{2}\log n + \alpha_n^+}]$$

$$\leq \sum_{i=0}^{\lceil(\alpha_n^+ + K)/h\rceil - 1} \frac{e^{(K-ih)/\alpha_n}}{n^{3/2\alpha_n}} \mathbb{E}[F(\mathbf{S}^{(n)}) \mathbb{1}_{\underline{S}_n \geq -L, \min_{\lfloor n/2\rfloor \leq j \leq n} S_j \geq \frac{3}{2}\log n - K, S_n - (\frac{3}{2}\log n - K) \in [ih,(i+1)h)}]$$

$$\leq \sum_{i=0}^{\lceil(\alpha_n^+ + K)/h\rceil - 1} \frac{e^{-ih/\alpha_n}}{n^{3/2\alpha_n}} \sqrt{\frac{\pi}{2}} \frac{\theta\theta^-}{\sigma} \frac{R(L)}{n^{3/2}} (\mathbb{E}[F(\mathfrak{e})] + o(1)) h R^-((i + 1)h), \tag{5.12}$$

by using Proposition 2.8 in both lattice and nonlattice cases, with uniformity in i because $h(\lceil (\alpha_n^+ + K)/h \rceil - 1) \ll \sqrt{n}$. Then, by applying (2.1) to R^-, for $\varepsilon > 0$, it exists $M > 0$ such that, for all $u \geq 0$, $R^-(u) \leq c_0^-(1 + \varepsilon)(M + u)$ and thus we get

$$
\sum_{i=0}^{\lceil (\alpha_n^+ + K)/h \rceil - 1} R^-\big((i+1)h\big) e^{-ih/\alpha_n} \leq \sum_{i=0}^{\infty} c_0^-(1+\varepsilon)\left(i + 1 + \frac{M}{h}\right) h e^{-ih/\alpha_n}
$$

$$
= \big(1 + o(1)\big) c_0^-(1+\varepsilon) h \left(\frac{\alpha_n}{h}\right)^2, \tag{5.13}
$$

by applying (5.10). Coming back to (5.12), we get

$$
\limsup_{n \to \infty} \frac{n^{3\beta_n/2}}{\alpha_n^2} \mathbb{E}\big[e^{-S_n/\alpha_n} F(\mathbf{S}^{(n)}) \mathbb{1}_{\underline{S_n} \geq -L, \min_{\lfloor n/2 \rfloor \leq j \leq n} S_j \geq \frac{3}{2}\log n - K, S_n \leq \frac{3}{2}\log n + \alpha_n^+} \big]
$$

$$
\leq \sqrt{\frac{\pi}{2}} \frac{\theta \theta^-}{\sigma} R(L) \mathbb{E}\big[F(\mathfrak{e})\big] h \frac{c_0^-(1+\varepsilon)}{h} \xrightarrow[\varepsilon \to 0]{} \frac{\theta}{\sigma^2} R(L) \mathbb{E}\big[F(\mathfrak{e})\big], \tag{5.14}
$$

by applying (2.5) to constants c_0^- and θ^-. Combining (5.11) and (5.14), we conclude that

$$
\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{W_{n,\beta_n}^{(L,K)}}{D_n^{(L)}} \right] = \frac{1}{R(L)} \mathbb{E}\big[W_{n,\beta_n}^{(L,K)}\big] \leq \frac{\theta}{\sigma^2} \frac{\alpha_n^2}{n^{3\beta_n/2}} \big(\mathbb{E}[F(\mathfrak{e})] + o(1) \big),
$$

and it shows the first part of Lemma 5.4.

We now want to prove the lower bound for $\mathbb{E}_{\mathbb{Q}^{(L)}}[Y_n(F_n)/D_n^{(L)}] = \mathbb{E}[Y_n(F_n)]/R(L)$. We use the branching property at time k_n to get

$$
\mathbb{E}\big[Y_n(F_n)\big] = \mathbb{E}\left[\sum_{|x| = k_n} \mathbb{1}_{\underline{V}(x) \geq -L, V(x) \in [k_n^{1/3}, k_n]} \psi\big(V(x)\big) \right], \tag{5.15}
$$

where we set, for all $b \in [k_n^{1/3}, k_n]$,

$$
\psi(b) := \mathbb{E}_b\left[\sum_{|z| = n - k_n} e^{-\beta_n V(z)} F\left(\mathbf{V}(z) - \frac{b}{\sigma \sqrt{n - k_n}}\right) \mathbb{1}_{\forall 0 \leq i \leq n - k_n, V(z_i) \in I_{n, i + k_n}} \right].
$$

We note $m := n - k_n$ and fix $\lambda \in (0, 1/2)$, then we have $\lfloor \lambda m \rfloor \leq \lfloor n/2 \rfloor - k_n$ for n large enough because $k_n \ll n$. Using the many-to-one lemma, we get, for $b \in [k_n^{1/3}, k_n]$,

$$
\psi(b) \geq e^{-b} \mathbb{E}_b\left[e^{-S_m/\alpha_n} F\left(\mathbf{S}^{(m)} - \frac{b}{\sigma \sqrt{m}}\right) \mathbb{1}_{S_m - \frac{3}{2}\log n \in [\alpha_n^-, \alpha_n^+], \underline{S_m} \geq -L, \min_{\lfloor \lambda m \rfloor \leq j \leq m} S_j \geq \frac{3}{2}\log n} \right]
$$

$$
= e^{-b} \mathbb{E}\left[e^{-(S_m + b)/\alpha_n} F(\mathbf{S}^{(m)}) \mathbb{1}_{S_m - (\frac{3}{2}\log n - b) \in [\alpha_n^-, \alpha_n^+], \underline{S_m} \geq -(L+b), \min_{\lfloor \lambda m \rfloor \leq j \leq m} S_j \geq \frac{3}{2}\log n - b} \right].
$$

Then, we cut the segment $[\alpha_n^-, \alpha_n^+]$ in pieces of length $h > 0$, where h is any real number if S_1 is nonlattice and is the span of the lattice if S_1 is lattice, and we get that $\psi(b)$ is larger than

$$
e^{-b} \sum_{i = \lceil \alpha_n^-/h \rceil}^{\lfloor \alpha_n^+/h \rfloor - 1} \frac{e^{-(i+1)h/\alpha_n}}{n^{3/2\alpha_n}} \mathbb{E}\big[F(\mathbf{S}^{(m)}) \mathbb{1}_{S_m - (\frac{3}{2}\log n - b) \in [ih, (i+1)h), \underline{S_m} \geq -(L+b), \min_{\lfloor \lambda m \rfloor \leq j \leq m} S_j \geq \frac{3}{2}\log n - b} \big]
$$

$$
\geq \frac{e^{-b}}{n^{3/2\alpha_n}} \sum_{i = \lceil \alpha_n^-/h \rceil}^{\lfloor \alpha_n^+/h \rfloor - 1} e^{-(i+1)h/\alpha_n} \sqrt{\frac{\pi}{2}} \frac{\theta \theta^-}{\sigma} \frac{R(L+b)}{m^{3/2}} \big(\mathbb{E}[F(\mathfrak{e})] + o(1) \big) h R^-(ih),
$$

where the $o(1)$ is uniform in $b \in [k_n^{1/3}, k_n]$ and $i \in [\![\lceil \alpha_n^-/h \rceil, \lfloor \alpha_n^+/h \rfloor - 1]\!]$, by using Proposition 2.8, because we have $L + k_n \ll \sqrt{n}$ and $h(\lfloor \alpha_n^+/h \rfloor - 1) \ll \sqrt{n}$. Thus, we get, using that $R(L + b) \sim c_0(L + b)$ and $R^-(ih) \sim c_0^- ih$ uniformly in b and i,

$$\psi(b) \geq \frac{\mathrm{e}^{-b}}{n^{3/2\alpha_n}} \sqrt{\frac{\pi}{2}} \frac{\theta\theta^-}{\sigma} \frac{c_0(L+b)}{n^{3/2}} h\big(\mathbb{E}[F(\mathfrak{e})] + o(1)\big) \sum_{i = \lceil \alpha_n^-/h \rceil}^{\lfloor \alpha_n^+/h \rfloor - 1} \mathrm{e}^{-(i+1)h/\alpha_n} c_0^- ih$$

$$= (L + b)\mathrm{e}^{-b} \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} \frac{\alpha_n^2}{n^{3\beta_n/2}} \big(\mathbb{E}[F(\mathfrak{e})] + o(1)\big),$$

uniformly in $b \in [k_n^{1/3}, k_n]$, where we used (2.5) twice and also (5.10). Coming back to (5.15), we get that $\mathbb{E}[Y_n(F_n)]$ is larger than

$$\sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} \frac{\alpha_n^2}{n^{3\beta_n/2}} \big(\mathbb{E}[F(\mathfrak{e})] + o(1)\big) \mathbb{E}\bigg[\sum_{|x|=k_n} \mathbb{1}_{\underline{V}(x) \geq -L, V(x) \in [k_n^{1/3}, k_n]} \big(L + V(x)\big) \mathrm{e}^{-V(x)}\bigg]. \tag{5.16}$$

Using the many-to-one lemma, the expectation in (5.16) is equal to

$$\mathbb{E}\big[(S_{k_n} + L)\mathbb{1}_{\underline{S}_{k_n} \geq -L, S_{k_n} \in [k_n^{1/3}, k_n]}\big] \geq \mathbb{E}\big[(S_{k_n} + L)\mathbb{1}_{\underline{S}_{k_n} \geq -L, (S_{k_n}+L)/\sigma k_n^{1/2} \in [C^{-1}, C]}\big], \tag{5.17}$$

for all $C > 0$. We then choose a function $\chi : \mathbb{R}_+ \to \mathbb{R}$ continuous and bounded such that, for all $t \in \mathbb{R}_+$,

$$t\mathbb{1}_{t \in [2C^{-1}, C/2]} \leq \chi(t) \leq t\mathbb{1}_{t \in [C^{-1}, C]},$$

and (5.17) is larger than

$$\sigma k_n^{1/2} \mathbb{E}\bigg[\chi\bigg(\frac{S_{k_n} + L}{\sigma k_n^{1/2}}\bigg) \mathbb{1}_{\underline{S}_{k_n} \geq -L}\bigg] = \sigma k_n^{1/2}\big(1 + o(1)\big) \frac{\theta R(L)}{k_n^{1/2}} \int_0^\infty \chi(t) t \mathrm{e}^{-t^2/2} \, \mathrm{d}t$$

$$\geq \sigma\theta R(L)\big(1 + o(1)\big) \int_{2C^{-1}}^{C/2} t^2 \mathrm{e}^{-t^2/2} \, \mathrm{d}t,$$

by applying (2.6). Coming back to (5.16), we get

$$\liminf_{n \to \infty} \frac{n^{3\beta_n/2}}{\alpha_n^2} \mathbb{E}[Y_n(F_n)] \geq \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} \mathbb{E}[F(\mathfrak{e})] \sigma\theta R(L) \int_{2C^{-1}}^{C/2} t^2 \mathrm{e}^{-t^2/2} \, \mathrm{d}t \xrightarrow[C \to \infty]{} \frac{\theta}{\sigma^2} R(L)\mathbb{E}[F(\mathfrak{e})],$$

using that $\int_0^\infty t^2 \mathrm{e}^{-t^2/2} \, \mathrm{d}t = \sqrt{\frac{\pi}{2}}$. Since $\mathbb{E}_{\mathbb{Q}^{(L)}}[Y_n(F_n)/D_n^{(L)}] = \mathbb{E}[Y_n(F_n)]/R(L)$, it concludes the proof of Lemma 5.4. $\qquad\square$

5.4. *Addition of the second barrier*

In this section, we prove Lemma 5.2 with a method similar to the one used by Madaule [35] for his Lemma 4.9 (or Lemma 3.3 of Aïdékon [1]). The main difference is that we do not only consider particles that are at a distance of order 1 from $(3/2) \log n$, but we can nevertheless apply some of Madaule's results.

Proof of Lemma 5.2. We fix $L > 0$ and $\varepsilon, \eta > 0$. For all $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$ and $K > 0$, we have $|W_{n,\beta_n}^{(L)}(F) - W_{n,\beta_n}^{(L,K)}(F)| \leq 2\|F\| |W_{n,\beta_n}^{(L)} - W_{n,\beta_n}^{(L,K)}|$, where $W_{n,\beta_n}^{(L)} := W_{n,\beta_n}^{(L)}(1)$ and $W_{n,\beta_n}^{(L,K)} := W_{n,\beta_n}^{(L,K)}(1)$. Therefore, it is sufficient to show that we have

$$\limsup_{n \to \infty} \mathbb{Q}^{(L)}\bigg(\frac{n^{3\beta_n/2}}{\alpha_n^2} \frac{|W_{n,\beta_n}^{(L)} - W_{n,\beta_n}^{(L,K)}|}{D_n^{(L)}} > \varepsilon\bigg) \leq \eta, \tag{5.18}$$

for $K > 0$ large enough.

Using Proposition A.3 of Aïdékon [1], we know that $D_n^{(L)}$ converges in L^1 to $D_\infty^{(L)}$ under $\mathbb{P}$, so we can choose $M > 0$ large enough such that, for all $n \in \mathbb{N}$, $\mathbb{Q}^{(L)}(D_n^{(L)} > M) = \mathbb{E}[D_n^{(L)}\mathbb{1}_{D_n^{(L)} > M}]/R(L) \leq \eta/4$. Therefore, the probability in (5.18) is smaller than

$$\mathbb{Q}^{(L)}\left(n^{3\beta_n/2}\big|W_{n,\beta_n}^{(L)} - W_{n,\beta_n}^{(L,K)}\big| > \varepsilon\alpha_n^2 D_n^{(L)}, \frac{\eta}{4} \leq D_n^{(L)} \leq M\right) + \frac{\eta}{4} + \mathbb{Q}^{(L)}\left(D_n^{(L)} < \frac{\eta}{4}\right)$$

$$= \mathbb{E}\big[D_n^{(L)}\mathbb{1}_{n^{3\beta_n/2}|W_{n,\beta_n}^{(L)} - W_{n,\beta_n}^{(L,K)}| > \varepsilon\alpha_n^2 D_n^{(L)}, \frac{\eta}{4} \leq D_n^{(L)} \leq M}\big] + \frac{\eta}{4} + \frac{1}{R(L)}\mathbb{E}\big[D_n^{(L)}\mathbb{1}_{D_n^{(L)} < \frac{\eta}{4}}\big]$$

$$\leq M\mathbb{P}\left(n^{3\beta_n/2}\big|W_{n,\beta_n}^{(L)} - W_{n,\beta_n}^{(L,K)}\big| > \varepsilon\alpha_n^2\frac{\eta}{4}\right) + \frac{\eta}{4} + \frac{\eta}{4},$$

because $R(L) \geq 1$. Thus, we now want to prove that, for some $K > 0$ large enough, we have

$$\limsup_{n\to\infty}\mathbb{P}\big(n^{3\beta_n/2}\big|W_{n,\beta_n}^{(L)} - W_{n,\beta_n}^{(L,K)}\big| > \varepsilon'\alpha_n^2\big) \leq 2\eta',$$

with $\eta' := \eta/4M$ and $\varepsilon' := \varepsilon\eta/4$. Moreover, using (1.8), we can fix $K' \geq 0$ such that $\mathbb{P}(\min_{|z|=n} V(z) < \frac{3}{2}\log n - K') \leq \eta'$. Thus, our aim is now to show that we have

$$\limsup_{n\to\infty}\mathbb{P}\left(\big|W_{n,\beta_n}^{(L)} - W_{n,\beta_n}^{(L,K)}\big|\mathbb{1}_{\min_{|z|=n} V(z) \geq \frac{3}{2}\log n - K'} > \varepsilon'\frac{\alpha_n^2}{n^{3\beta_n/2}}\right) \leq \eta', \tag{5.19}$$

for some $K \geq K'$ large enough.

Now, following Madaule's [35] proof of his Lemma 4.9, we introduce the intervals $J_n(x) := [\frac{3}{2}\log n - x - 1, \frac{3}{2}\log n - x)$, for $x \in \mathbb{R}$, and the events, for $i, \ell \in \mathbb{N}$ and $\lfloor n/2 \rfloor \leq k \leq n$,

$$E_{i,k,\ell}(z) := \left\{\underline{V}(z) \geq -L, V(z_k) = \min_{\lfloor n/2 \rfloor \leq j \leq n} V(z_j) \in J_n(K) - \ell, V(z) \in J_n(K') + i\right\},$$

and, denoting $a_\ell := \lfloor e^{\nu(\ell+K)} \rfloor$ for some fixed $\nu \in (0,1)$,

$$F_\ell^1(z) := \bigcup_{i \geq 1, \lfloor n/2 \rfloor \leq k < n - a_\ell} E_{i,k,\ell}(z) \quad \text{and} \quad F_\ell^2(z) := \bigcup_{i \geq 1, n - a_\ell \leq k \leq n} E_{i,k,\ell}(z).$$

Then, we have

$$\big|W_{n,\beta_n}^{(L)} - W_{n,\beta_n}^{(L,K)}\big|\mathbb{1}_{\min_{|z|=n} V(z) \geq \frac{3}{2}\log n - K'} \leq \sum_{\ell \geq 0}\sum_{|z|=n} e^{-\beta_n V(z)}(\mathbb{1}_{F_\ell^1(z)} + \mathbb{1}_{F_\ell^2(z)}). \tag{5.20}$$

On the one hand, by Madaule's [35] proof of his Lemma 4.9, we have the inequality

$$\mathbb{P}\left(\sum_{|z|=n}\mathbb{1}_{F_\ell^2(z)} \geq 1\right) \leq c_{15}(1 + a_\ell)(1 + L)e^{-K-\ell}$$

and, therefore,

$$\mathbb{P}\left(\sum_{\ell \geq 0}\sum_{|z|=n} e^{-\beta_n V(z)}\mathbb{1}_{F_\ell^2(z)} > 0\right) \leq \sum_{\ell \geq 0}\mathbb{P}\left(\sum_{|z|=n}\mathbb{1}_{F_\ell^2(z)} \geq 1\right) \leq c_{16}(1 + L)e^{-(1-\nu)K}. \tag{5.21}$$

On the other hand, by using the many-to-one lemma, we get, for $i \geq 1$, $\lfloor n/2 \rfloor \leq k < n - a_\ell$ and $\ell \geq 0$,

$$\mathbb{E}\left[\sum_{|z|=n} e^{-\beta_n V(z)}\mathbb{1}_{E_{i,k,\ell}(z)}\right] = \mathbb{E}\big[e^{-S_n/\alpha_n}\mathbb{1}_{E_{i,k,\ell}}\big] \leq \frac{e^{-(i-1-K')/\alpha_n}}{n^{3/2\alpha_n}}\mathbb{P}(E_{i,k,\ell}), \tag{5.22}$$

where we set

$$E_{i,k,\ell} := \left\{ \underline{S}_n \geq -L,\, S_k = \min_{\lfloor n/2 \rfloor \leq j \leq n} S_j \in J_n(K) - \ell,\, S_n \in J_n(K') + i \right\}.$$

We recall Equation (4.27) of Madaule [35]:

$$\mathbb{P}(E_{i,k,\ell}) \leq \begin{cases} c_{17} \frac{(1+L)\log n}{n^{3/2}(n-k+1)^{3/2}}(1+\ell+i) & \text{if } \lfloor n/2 \rfloor \leq k < \lfloor 3n/4 \rfloor, \\ c_{17} \frac{(1+L)}{n^{3/2}(n-k+1)^{3/2}}(1+\ell+i) & \text{if } \lfloor 3n/4 \rfloor \leq k \leq n. \end{cases} \tag{5.23}$$

Applying (5.23), we get

$$\sum_{k=\lfloor n/2 \rfloor}^{n-a_\ell-1} \mathbb{P}(E_{i,k,l}) \leq c_{18} \frac{(1+L)}{n^{3/2}}(1+\ell+i)\left(\frac{\log n}{\sqrt{n}} + a_\ell^{-1/2}\right). \tag{5.24}$$

Using (5.22), (5.24) and that $\mathbb{P}(E_{i,k,\ell}) = 0$ for $\ell \geq \frac{3}{2}\log n - K + L$, we have

$$\mathbb{E}\left[\sum_{\ell \geq 0} \sum_{|z|=n} e^{-\beta_n V(z)} \mathbb{1}_{F_\ell^1(z)}\right]$$

$$\leq \sum_{\ell \geq 0} \sum_{i \geq 1} \sum_{k=\lfloor n/2 \rfloor}^{n-a_\ell-1} \mathbb{E}\left[\sum_{|z|=n} e^{-\beta_n V(z)} \mathbb{1}_{E_{i,k,\ell}(z)}\right]$$

$$\leq c_{18} e^{(K'+1)/\alpha_n} \frac{(1+L)}{n^{3\beta_n/2}} \sum_{\ell=0}^{\lceil \frac{3}{2}\log n - K + L \rceil} \left(\frac{\log n}{\sqrt{n}} + a_\ell^{-1/2}\right) \sum_{i \geq 1} e^{-i/\alpha_n}(1+\ell+i). \tag{5.25}$$

We can bound the sum on i in (5.25) by $(1+\ell)\alpha_n^2(1+o(1))$ uniformly in ℓ. Moreover, taking K large enough such that $e^{-\nu K} < 1/2$, we have $a_\ell \geq e^{\nu(\ell+K)}/2$ for all $\ell \geq 0$. Thus, we get that (5.25) is smaller than

$$c_{19}(1+o(1)) \frac{(1+L)}{n^{3\beta_n/2}} \alpha_n^2 \left(\frac{(\log n)^3}{n^{1/2}} + e^{-\nu K/2}\right) = c_{19} e^{-\nu K/2}(1+L)(1+o(1)) \frac{\alpha_n^2}{n^{3\beta_n/2}},$$

and, thus, with the Markov inequality, we have

$$\mathbb{P}\left(\sum_{\ell \geq 0} \sum_{|z|=n} e^{-\beta_n V(z)} \mathbb{1}_{F_\ell^1(z)} > \varepsilon' \frac{\alpha_n^2}{n^{3\beta_n/2}}\right) \leq \frac{c_{19}}{\varepsilon'} e^{-\nu K/2}(1+L)(1+o(1)). \tag{5.26}$$

Finally, (5.19) follows from (5.20), (5.21) and (5.26) by taking K large enough and it concludes the proof of Lemma 5.2. $\qquad\square$

5.5. *From F to F_n*

We prove here that considering F_n instead of F does not change significantly the first moment.

Proof of Lemma 5.3. To control the first moment of $W_{n,\beta_n}^{(L,K)}(|F - F_n|)/D_n^{(L)}$ under $\mathbb{E}_{\mathbb{Q}^{(L)}}$, we follow the proof of Lemma 5.4 for the upper bound of the first moment of $W_{n,\beta_n}^{(L,K)}(F)/D_n^{(L)}$, but, instead of applying directly Proposition 2.8 as in (5.12), we use that

$$\mathbb{E}\left[|F - F_n|(\mathbf{S}^{(n)}) \mathbb{1}_{\underline{S}_n \geq -L, \min_{\lfloor n/2 \rfloor \leq j \leq n} S_j \geq \frac{3}{2}\log n - K, S_n - (\frac{3}{2}\log n - K) \in [ih, (i+1)h]}\right]$$

$$= o\left(\sqrt{\frac{\pi}{2}} \frac{\theta\theta^-}{\sigma} \frac{R(L)}{n^{3/2}} h R^-\big((i+1)h\big)\right),$$

uniformly in $i \in [0, h(\lceil(\alpha_n^+ + K)/h\rceil - 1)]$, by using Proposition 2.8 combined with Lemmas A.3 and A.5, because $F \in C_b^u(\mathcal{D}([0, 1]))$. The result follows with the same calculations as in the proof of Lemma 5.4. □

5.6. *The peeling lemma*

The aim of this subsection is to prove Lemma 5.5, which shows that introducing the event $\{z \in B_n\}$ does not change the first moment. This proof is based on the so-called peeling lemma (see Shi [40]), which controls that the spine, conditioned to have a specific trajectory, does not have too many and too low children. Such lemmas have been proved in the case where the spine ends up at a distance of constant order from $\frac{3}{2} \log n$ (see [1, Lemma C.1], [35, Lemma 7.1] and [40, Theorem 5.14]) and also when it ends up at a position of order $\sqrt{n}$ (see [3, Lemma 4.7]). Here we have to deal with the intermediate case where the spine end up far above $\frac{3}{2} \log n$ and far below $\sqrt{n}$.

In order to state the peeling lemma in a general setting, we introduce some notation. For $b, u, v \in \mathbb{R}$ and $n \in \mathbb{N}$, we set

$$A_n^{b,u,v} := \left\{ |z| = n : V(z) \in [v+b, v+b+1), \underline{V}(z) \geq -u, \min_{\lfloor n/2 \rfloor \leq j \leq n} V(z_j) \geq v \right\}.$$

We consider

$$a_i^{(n)} := \begin{cases} -u & \text{if } 0 \leq i < \lfloor n/2 \rfloor, \\ v & \text{if } \lfloor n/2 \rfloor \leq i \leq n, \end{cases} \quad \text{and} \quad \ell_i^{(n)} := \begin{cases} i^{1/7} & \text{if } 0 \leq i < \lfloor n/2 \rfloor, \\ (n-i)^{1/7} & \text{if } \lfloor n/2 \rfloor \leq i \leq n, \end{cases}$$

and the following set

$$B_n^\rho := \left\{ |z| = n : \forall j \in [\![0, n-1]\!], \sum_{y \in \Omega(z_{j+1})} \left(1 + [V(y) - a_j^{(n)}]_+\right) e^{-[V(y) - a_j^{(n)}]} \leq \rho e^{-\ell_j^{(n)}} \right\}.$$

We can now state our version of the peeling lemma, which covers the case where the spine ends up far below $\sqrt{n}$ and is therefore more general than the peeling lemmas in [1,35,40].

Lemma 5.7 (**Peeling lemma**). *For all $\varepsilon > 0$, there exist $\rho > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $b \in \mathbb{R}_+$ and $u, v \in [0, n^{1/8}]$,*

$$\mathbb{Q}\left(w_n \in A_n^{b,u,v} \cap \left(B_n^\rho\right)^c\right) \leq \varepsilon \frac{R(u)R^-(b)}{n^{3/2}}.$$

Remark 5.8. We present here the peeling lemma in terms of probability measure $\mathbb{Q}$, because it simplifies somehow the proof (for example, under $\mathbb{Q}$, the reproduction law along the spine does not depend on the position), but it is a direct consequence that, for all $\varepsilon > 0$, there exist $\rho > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $b \in \mathbb{R}_+$ and $u, v \in [0, n^{1/8}]$,

$$\mathbb{Q}^{(u)}\left(w_n^{(u)} \in A_n^{b,u,v} \cap \left(B_n^\rho\right)^c\right) \leq \varepsilon \frac{R(u+v+b)R^-(b)}{n^{3/2}}.$$

Before proving the peeling lemma, we first use it to show Lemma 5.5.

Proof of Lemma 5.5. We set $Y_n := Y_n(1)$ and $Y_n' := Y_n'(1)$ and, since F is bounded, it is sufficient to show that

$$\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n - Y_n'}{D_n^{(L)}}\right] = o\left(\frac{\alpha_n^2}{n^{3\beta_n/2}}\right).$$

We first change probabilities from $\mathbb{Q}^{(L)}$ to $\mathbb{Q}$: we have

$$\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n - Y_n'}{D_n^{(L)}}\right] = \frac{1}{R(L)} \mathbb{E}_{\mathbb{Q}}\left[\frac{Y_n - Y_n'}{W_n}\right] = \frac{1}{R(L)} \mathbb{E}_{\mathbb{Q}}\left[e^{-V(w_n)/\alpha_n} \mathbb{1}_{w_n \in A_n \cap B_n^c}\right]. \tag{5.27}$$

Then, setting $u = L$ and $v = \frac{3}{2} \log n$ and cutting the segment $[\alpha_n^-, \alpha_n^+]$ in pieces of length 1, we get that (5.27) is smaller than

$$\frac{1}{R(L)} \sum_{i=\lfloor \alpha_n^- \rfloor}^{\lceil \alpha_n^+ \rceil - 1} \mathrm{e}^{-i/\alpha_n} \mathbb{Q}\big(w_n \in A_n^{i,u,v} \cap B_n^c\big) \leq o\left(\frac{1}{n^{3/2}}\right) \sum_{i=\lfloor \alpha_n^- \rfloor}^{\lceil \alpha_n^+ \rceil - 1} \mathrm{e}^{-i/\alpha_n} R^-(i), \tag{5.28}$$

using Lemma 5.7 uniformly in i, noting that $B_n \subset B_n^{\rho_n}$ and $\rho_n \to \infty$. Using then (2.2) and (5.10), we get that (5.28) is a $o(\alpha_n^2/n^{3/2})$ and it concludes the proof of Lemma 5.5. □

Proof of Lemma 5.7. By Lemma 2.4 and (2.2), it exists $\mu > 0$ such that, for all $b \in \mathbb{R}_+$, $u, v \in [0, n^{1/7}]$ and $n \in \mathbb{N}^*$, we have

$$\mathbb{Q}\big(w_n \in A_n^{b,u,v}, \exists i \in [\![0, n]\!] : V(w_i) < a_i^{(n)} + 2\ell_i^{(n)} - \mu\big) \leq R(u) R^-(b) \left(\frac{\varepsilon}{3n^{3/2}} + c_7 \frac{9}{n^{2-3/7}}\right)$$

$$\leq \frac{\varepsilon}{2} \frac{R(u) R^-(b)}{n^{3/2}},$$

for n large enough (independent of u, v and b). Thus, it is sufficient to show that

$$\mathbb{Q}\big(w_n \in A_n^{b,u,v} \cap B_n^c, \forall j \in [\![0, n]\!], V(w_j) \geq a_j^{(n)} + 2\ell_j^{(n)} - \mu\big) \leq \frac{\varepsilon}{2} \frac{R(u) R^-(b)}{n^{3/2}}, \tag{5.29}$$

for n large enough, $b \in \mathbb{R}_+$ and $u, v \in [0, n^{1/8}]$. Therefore, we now prove (5.29). We first set, for $0 \leq i \leq n - 1$,

$$B_{n,i}^{\rho} := \left\{ |z| = n : \sum_{y \in \Omega(z_{i+1})} \big(1 + [V(y) - a_i^{(n)}]_+\big) \mathrm{e}^{-[V(y) - a_i^{(n)}]} \leq \rho \mathrm{e}^{-\ell_i^{(n)}} \right\}.$$

Since for all $u, v \in \mathbb{R}$, $1 + (u + v)_+ \leq (1 + u_+)(1 + v_+)$, we have

$$\sum_{y \in \Omega(z_{i+1})} \big(1 + [V(y) - a_i^{(n)}]_+\big) \mathrm{e}^{-[V(y) - a_i^{(n)}]} \leq \big(1 + [V(w_i) - a_i^{(n)}]_+\big) \mathrm{e}^{-[V(w_i) - a_i^{(n)}]} \Theta(w_{i+1}),$$

where we set, for $x \in \mathbb{T}$, by noting $\overleftarrow{x}$ the parent of x and $\Delta(x) := V(x) - V(\overleftarrow{x})$,

$$\Theta(x) := \sum_{y \in \Omega(x)} \big(1 + \Delta(y)_+\big) \mathrm{e}^{-\Delta(y)}.$$

Thus, we have, on event $\{w_n \notin B_{n,i}^{\rho}\} \cap \{V(w_i) \geq a_i^{(n)} + 2\ell_i^{(n)} - \mu\}$,

$$\big(1 + [V(w_i) - a_i^{(n)}]_+\big) \mathrm{e}^{-[V(w_i) - a_i^{(n)}]} \Theta(w_{i+1}) \geq \rho \mathrm{e}^{-\ell_i^{(n)}} \geq \rho \mathrm{e}^{-[V(w_i) - a_i^{(n)} + \mu]/2}$$

and, if we are moreover on event $\{w_n \in A_n^{b,u,v}\}$ so that $V(w_i) \geq a_i^{(n)}$, it implies that

$$\Theta(w_{i+1}) \geq \rho \mathrm{e}^{-\mu/2} \frac{\mathrm{e}^{[V(w_i) - a_i^{(n)}]/2}}{1 + [V(w_i) - a_i^{(n)}]_+} \geq c_{20} \rho \mathrm{e}^{-\mu/2} \mathrm{e}^{[V(w_i) - a_i^{(n)}]/3},$$

where $c_{20} := \inf_{u \geq 0} \mathrm{e}^{u/6}/(1 + u) > 0$. Therefore, we get

$$\mathbb{Q}\big(w_n \in A_n^{b,u,v} \cap \big(B_n^{\rho}\big)^c, \forall j \in [\![0, n]\!], V(w_j) \geq a_j^{(n)} + 2\ell_j^{(n)} - \mu\big) \leq \sum_{i=0}^{n-1} q_i, \tag{5.30}$$

where we set, for $0 \le i \le n - 1$,

$$q_i := \mathbb{Q}\big(w_n \in A_n^{b,u,v}, \forall j \in [\![0, n]\!], V(w_j) \ge a_j^{(n)} + 2\ell_j^{(n)} - \mu, \Theta(w_{i+1}) > \tilde{\rho}\mathrm{e}^{[V(w_i) - a_i^{(n)}]/3}\big)$$

and $\tilde{\rho} := c_{20}\rho\mathrm{e}^{-\mu/2}$. From now, we choose ρ such that $\tilde{\rho} \ge \mathrm{e}$.

We first consider the case $0 \le i < \lfloor n/2 \rfloor$. Since $a_i^{(n)} = -u$, we have

$$q_i \le \mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{\underline{V}(w_i) \ge -u, \Theta(w_{i+1}) > \tilde{\rho}\mathrm{e}^{[V(w_i)+u]/3}} G_i\big(V(w_{i+1})\big)\big],$$

where we set, for $x \in \mathbb{R}$,

$$\begin{aligned}
G_i(x) &:= \mathbb{Q}_x\Big(\underline{V}(w_{n-i-1}) \ge -u, V(w_{n-i-1}) - v \in [b, b+1), \min_{\lfloor n/2 \rfloor - i - 1 \le j \le n-i-1} V(w_j) \ge v\Big) \\
&\le \mathbb{P}\Big(\underline{S}_{n-i-1} \ge -u - x, S_{n-i-1} - (v - x) \in [b, b+1), \min_{\lfloor (n-i-1)/2 \rfloor \le j \le n-i-1} S_j \ge v - x\Big) \\
&\le c_5 \frac{(1 + x + u)2(b+2)}{(n - i)^{3/2}} \\
&\le c_{21} \frac{(1 + x + u)R^-(b)}{n^{3/2}},
\end{aligned}$$

using successively Proposition 2.2(ii), (2.8) and (2.2). We thus get

$$\begin{aligned}
q_i &\le c_{21} \frac{R^-(b)}{n^{3/2}} \mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{\underline{V}(w_i) \ge -u, \Theta(w_{i+1}) > \tilde{\rho}\mathrm{e}^{[V(w_i)+u]/3}}\big(1 + V(w_{i+1}) + u\big)\big] \\
&\le c_{21} \frac{R^-(b)}{n^{3/2}} \mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{\underline{V}(w_i) \ge -u, V(w_i)+u < 3\log \frac{\Theta(w_{i+1})}{\tilde{\rho}}}\big(1 + V(w_i) + u + \Delta(w_{i+1})\big)\big] \\
&\le c_{21} \frac{R^-(b)}{n^{3/2}} \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\underline{V}(w_i) \ge -u, V(w_i)+u < 3\log \frac{\Theta(w_{i+1})}{\tilde{\rho}}}\Big[1 + 3\log_+ \frac{\Theta(w_{i+1})}{\tilde{\rho}} + \big(\Delta(w_{i+1})\big)_+\Big]\Big].
\end{aligned}$$

But, under $\mathbb{Q}$, $(\Theta(w_{i+1}), \Delta(w_{i+1}))$ is independent of $(V(w_j))_{0 \le j \le i}$ and has moreover the same law as $(X + \widetilde{X}, V(w_1))$, where X and $\widetilde{X}$ are defined in (1.4). Therefore, we get, by integrating first on $(V(w_j))_{0 \le j \le i}$,

$$q_i \le c_{21} \frac{R^-(b)}{n^{3/2}} \mathbb{E}_{\mathbb{Q}}\Big[F_i(X + \widetilde{X})\Big(1 + 3\log_+ \frac{X + \widetilde{X}}{\tilde{\rho}} + V(w_1)_+\Big)\Big], \tag{5.31}$$

where we set, for $x > 0$,

$$F_i(x) := \mathbb{Q}\Big(\underline{V}(w_i) \ge -u, V(w_i) + u < 3\log \frac{x}{\tilde{\rho}}\Big) = \mathbb{1}_{0 < \log \frac{x}{\tilde{\rho}}}\mathbb{P}(\underline{S}_i \ge -u, S_i < -u + 3\log_+ x),$$

by using Proposition 2.2(ii) and that $\tilde{\rho} \ge 1$. Then, applying (2.9), we have

$$\sum_{i=k_n}^{\lfloor n/2 \rfloor - 1} F_i(x) \le \mathbb{1}_{x > \tilde{\rho}} c_6 (1 + 3\log_+ x)(1 + u) \le 3\frac{c_6}{c_1} R(u)(1 + \log_+ x)\mathbb{1}_{x > \tilde{\rho}}$$

using also (2.2). Coming back to (5.31) and noting that, since $X + \widetilde{X} > \tilde{\rho} \ge \mathrm{e}$, we have $\log_+(X + \widetilde{X}) \ge 1$, this gives

$$\sum_{i=k_n}^{\lfloor n/2 \rfloor - 1} q_i \le c_{22} \frac{R(u)R^-(b)}{n^{3/2}} \mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{X+\widetilde{X} > \tilde{\rho}} \log_+(X + \widetilde{X})\big(\log_+(X + \widetilde{X}) + V(w_1)_+\big)\big] \tag{5.32}$$

and concludes the case $0 \le i < \lfloor n/2 \rfloor$.

We now consider the case $\lfloor n/2 \rfloor \leq i < n$, so $a_i^{(n)} = v$. On event $\{w_n \in A_n^{b,u,v}, \Theta(w_{i+1}) > \tilde{\rho} e^{[V(w_i)-v]/3}\}$, we have $V(w_i) = V(w_n) - (V(w_n) - V(w_i)) \in [v + b - (V(w_n) - V(w_i)), v + b + 1 - (V(w_n) - V(w_i))]$, $\Theta(w_{i+1}) > \tilde{\rho} \vee \tilde{\rho} e^{[V(w_i)-V(w_n)+b]/3}$ and, for all $i + 1 \leq j \leq n$, $V(w_n) - V(w_j) \leq b + 1$. Moreover, we set

$$\tilde{a}_j^{(n)} := \begin{cases} -u & \text{if } 0 \leq j < \lfloor n/4 \rfloor, \\ v & \text{if } \lfloor n/4 \rfloor \leq j \leq n, \end{cases}$$

and note that, on event $\{w_n \in A_n^{b,u,v}, \forall j \in [\![0,n]\!], V(w_j) \geq a_j^{(n)} + 2\ell_j^{(n)} - \mu\}$, for n large enough, we have $\forall j \in [\![\lfloor n/4 \rfloor, \lfloor n/2 \rfloor - 1]\!], V(w_j) \geq v = \tilde{a}_j^{(n)}$, because $\ell_j^{(n)} \geq \lfloor n/4 \rfloor^{1/7}$ and $u + v \leq 2n^{1/8}$. Thus, we get

$$q_i \leq \mathbb{Q}\big(\forall j \in [\![1,i]\!], V(w_j) \geq \tilde{a}_j^{(n)}, V(w_i) - \big(v - \big(V(w_n) - V(w_i)\big)\big) \in [b, b+1],$$
$$\Theta(w_{i+1}) > \tilde{\rho} \vee \tilde{\rho} e^{[V(w_i)-V(w_n)+b]/3}, \forall j \in [\![i+1,n]\!], V(w_n) - V(w_j) \leq b+1\big). \tag{5.33}$$

Under $\mathbb{Q}$, $(\Theta(w_{i+1}), (V(w_n) - V(w_j))_{i \leq j \leq n})$ is independent of $(V(w_j))_{0 \leq j \leq i}$, so the right-hand side of (5.33) is equal to

$$\mathbb{E}_{\mathbb{Q}}\big[H_i\big(V(w_n) - V(w_i)\big)\mathbb{1}_{\Theta(w_{i+1})>\tilde{\rho}\vee\tilde{\rho}e^{[V(w_i)-V(w_n)+b]/3}, \forall j \in [\![i+1,n]\!], V(w_n)-V(w_j)\leq b+1}\big], \tag{5.34}$$

where we set, for $x \in \mathbb{R}$,

$$H_i(x) := \mathbb{Q}\Big(\underline{V}(w_i) \geq -u, \min_{\lfloor n/4 \rfloor \leq j \leq i} V(w_j) \geq v, V(w_i) - (v - x) \in [b, b+1]\Big)$$

$$\leq \mathbb{P}\Big(\underline{S}_i \geq -u, \min_{\lfloor i/2 \rfloor \leq j \leq i} S_j \geq v, S_i - v \in [b - x, b - x + 1]\Big)$$

$$\leq c_{23} \frac{R(u)}{n^{3/2}}(1 + b - x),$$

using successively Proposition 2.2(ii), (2.8) and (2.2) as before. On the event $\{\Theta(w_{i+1}) > \tilde{\rho} e^{[V(w_i)-V(w_n)+b]/3}\}$, using that $\tilde{\rho} \geq 1$, we have $b - (V(w_n) - V(w_i)) \leq 3\log_+ \Theta(w_{i+1})$ and also $V(w_n) - V(w_{i+1}) \geq b - 3\log_+ \Theta(w_{i+1}) - \Delta(w_{i+1})_+$, therefore, (5.34) is smaller than

$$c_{23} \frac{R(u)}{n^{3/2}} \mathbb{E}_{\mathbb{Q}}\big[\big(1 + 3\log_+ \Theta(w_{i+1})\big)\mathbb{1}_{\Theta(w_{i+1})>\tilde{\rho}}$$
$$\times \mathbb{1}_{V(w_n)-V(w_{i+1})\geq b-3\log_+ \Theta(w_{i+1})-\Delta(w_{i+1})_+, \forall j \in [\![i+1,n]\!], V(w_n)-V(w_j)\leq b+1}\big]. \tag{5.35}$$

Note then that, under $\mathbb{Q}$, $(\Theta(w_{i+1}), \Delta(w_{i+1}))$ is independent of $(V(w_n) - V(w_j))_{i+1 \leq j \leq n}$ and has the same law as $(X + \tilde{X}, V(w_1))$, so (5.35) is equal to

$$c_{23} \frac{R(u)}{n^{3/2}} \mathbb{E}_{\mathbb{Q}}\big[\big(1 + 3\log_+(X + \tilde{X})\big)\mathbb{1}_{X+\tilde{X}>\tilde{\rho}}\Gamma_i\big(3\log_+(X + \tilde{X}) + V(w_1)_+\big)\big], \tag{5.36}$$

where we set, for $x \geq 0$,

$$\Gamma_i(x) := \mathbb{Q}\big(V(w_n) - V(w_{i+1}) \geq b - x, \forall j \in [\![i+1,n]\!], V(w_n) - V(w_j) \leq b+1\big)$$
$$= \mathbb{P}\big(S_{n-i-1}^- \leq (x+1) - (b+1), \underline{S}_{n-i-1}^- \geq -(b+1)\big),$$

by applying Proposition 2.2(ii) and then reversing time. Thus, using (2.9) and (2.2), we get

$$\sum_{i=\lfloor n/2 \rfloor}^{n-1} \Gamma_i(x) \leq c_6^-(1 + x + 1)(1 + b + 1) \leq 4\frac{c_6^-}{c_1^-}(1 + x)R^-(b).$$

On event $\{X + \widetilde{X} > \tilde{\rho}\}$, we have $\log_+(X + \widetilde{X}) \geq 1$ so, coming back to (5.36), we get

$$\sum_{i=\lfloor n/2 \rfloor}^{n-1} q_i \leq c_{24} \frac{R(u)R^-(b)}{n^{3/2}} \mathbb{E}_{\mathbb{Q}} \big[\log_+(X + \widetilde{X}) \mathbb{1}_{X+\widetilde{X}>\tilde{\rho}} \big(\log_+(X + \widetilde{X}) + V(w_1)_+ \big) \big]. \tag{5.37}$$

Finally, using (5.32), (5.37) and that $\mathbb{E}_{\mathbb{Q}}[V(w_1)_+ | \mathscr{F}_1] = \widetilde{X}/X$ by Proposition 2.2(i), we get

$$\sum_{i=0}^{n-1} q_i \leq (c_{22} + c_{24}) \frac{R(u)R^-(b)}{n^{3/2}} \mathbb{E}\big[\big(X \log_+^2(X + \widetilde{X}) + \widetilde{X} \log_+(X + \widetilde{X}) \big) \mathbb{1}_{X+\widetilde{X}>\tilde{\rho}} \big]. \tag{5.38}$$

Using (1.3), we can choose ρ large enough such that the expectation in (5.38) is smaller than $\varepsilon/2(c_{22} + c_{24})$ and, recalling (5.30), it proves (5.29) and concludes the proof of Lemma 5.7. $\qquad\square$

5.7. *Second moment of* $Y'_n(F_n)$

For $F \in \mathcal{C}(\mathcal{D}([0, 1]))$, by decomposing along the spine, $Y'_n(F)$ is equal to

$$\sum_{i=0}^{n-1} \sum_{y \in \Omega(w_{i+1}^{(L)})} \sum_{|z|=n, z \geq y} e^{-\beta_n V(z)} F\big(\mathbf{V}(z)\big) \mathbb{1}_{z \in A_n \cap B_n} + e^{-\beta_n V(w_n^{(L)})} F\big(\mathbf{V}(w_n^{(L)})\big) \mathbb{1}_{w_n^{(L)} \in A_n \cap B_n}.$$

We cut this sum in two pieces, depending on whether the lineage of the considered particle z splits off from the spine's lineage before or after time k_n:

$$Y'^{[0,k_n)}_n(F) := \sum_{i=0}^{k_n-1} \sum_{y \in \Omega(w_{i+1}^{(L)})} \sum_{|z|=n, z \geq y} e^{-\beta_n V(z)} F\big(\mathbf{V}(z)\big) \mathbb{1}_{z \in A_n \cap B_n},$$

$$Y'^{[k_n,n]}_n(F) := Y'_n(F) - Y'^{[0,k_n)}_n(F).$$

We define in the same way $D_n^{(L),[0,k_n)}$ and $D_n^{(L),[k_n,n]}$. Then, by Aïdékon and Shi [3, Lemma 4.7], since $(\log n)^6 \ll k_n \ll \sqrt{n}$, we have

$$\inf_{u \in [k_n^{1/3}, k_n]} \mathbb{Q}^{(L)}\big(D_n^{(L),[k_n,n]} \leq n^{-2} | V(w_{k_n}^{(L)}) = u\big) \xrightarrow[n \to \infty]{} 1. \tag{5.39}$$

Proof of Lemma 5.6. First note that, using Proposition 2.3(i),

$$\frac{Y'_n(F_n)}{D_n^{(L)}} = \sum_{|z|=n} \mathbb{Q}^{(L)}\big(w_n^{(L)} = z | \mathscr{F}_n\big) \frac{e^{-V(z)/\alpha_n}}{R_L(V(z))} F_n\big(\mathbf{V}(z)\big) \mathbb{1}_{z \in A_n \cap B_n}$$

$$= \mathbb{E}_{\mathbb{Q}^{(L)}} \bigg[\frac{e^{-V(w_n^{(L)})/\alpha_n}}{R_L(V(w_n^{(L)}))} F_n\big(\mathbf{V}(w_n^{(L)})\big) \mathbb{1}_{w_n^{(L)} \in A_n \cap B_n} \bigg| \mathscr{F}_n \bigg]$$

and, thus, we have

$$\mathbb{E}_{\mathbb{Q}^{(L)}} \bigg[\bigg(\frac{Y'_n(F_n)}{D_n^{(L)}} \bigg)^2 \bigg] = \mathbb{E}_{\mathbb{Q}^{(L)}} \bigg[\frac{Y'_n(F_n)}{D_n^{(L)}} \frac{e^{-V(w_n^{(L)})/\alpha_n}}{R_L(V(w_n^{(L)}))} F_n\big(\mathbf{V}(w_n^{(L)})\big) \mathbb{1}_{w_n^{(L)} \in A_n \cap B_n} \bigg] =: \mathbb{E}_{\mathbb{Q}^{(L)}}^{[0,k_n)} + \mathbb{E}_{\mathbb{Q}^{(L)}}^{[k_n,n]},$$

by splitting $Y'_n(F_n) = Y'^{[0,k_n)}_n(F_n) + Y'^{[k_n,n]}_n(F_n)$. The first part will give the right order and constant and the second part will be negligible. Recall that F is assumed to be nonnegative.

We begin by bounding $\mathbb{E}_{\mathbb{Q}^{(L)}}^{[0,k_n)}$. Using $D_n^{(L)} \geq D_n^{(L),[0,k_n)}$ and $\mathbb{1}_{w_n^{(L)} \in B_n} \leq 1$, we get

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{(L)}}^{[0,k_n)} &\leq \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n'^{[0,k_n)}(F_n)}{D_n^{(L),[0,k_n)}} \frac{\mathrm{e}^{-V(w_n^{(L)})/\alpha_n}}{R_L(V(w_n^{(L)}))} F_n\left(\mathbf{V}\left(w_n^{(L)}\right)\right) \mathbb{1}_{w_n^{(L)} \in A_n} \right] \\
&= \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n'^{[0,k_n)}(F_n)}{D_n^{(L),[0,k_n)}} \varphi\left(V\left(w_{k_n}^{(L)}\right)\right) \mathbb{1}_{V(w_{k_n}^{(L)}) \in [k_n^{1/3}, k_n]} \right],
\end{aligned}
\tag{5.40}
$$

where we set, for $b \in [k_n^{1/3}, k_n]$ and with $m := n - k_n$,

$$
\begin{aligned}
\varphi(b) &:= \mathbb{E}_{\mathbb{Q}_b^{(L)}}\left[\frac{\mathrm{e}^{-V(w_m^{(L)})/\alpha_n}}{R_L(V(w_m^{(L)}))} F\left(\mathbf{V}\left(w_m^{(L)}\right) - \frac{b}{\sigma\sqrt{m}}\right) \mathbb{1}_{\forall 0 \leq i \leq n-k_n, V(w_i^{(L)}) \in I_{n,i+k_n}} \right] \\
&\leq \frac{1}{R_L(b)} \mathbb{E}\left[\mathrm{e}^{-(S_m+b)/\alpha_n} F\left(\mathbf{S}^{(m)}\right) \mathbb{1}_{\underline{S}_m \geq -(L+b), \min_{\lfloor \lambda m \rfloor \leq j \leq m} S_j \geq \frac{3}{2}\log n - b, S_m - (\frac{3}{2}\log n - b) \in [\alpha_n^-, \alpha_n^+]} \right],
\end{aligned}
$$

for some fixed $\lambda \in (1/2, 1)$ and n large enough, by applying Proposition 2.3(ii). Then, proceeding in the same way as for the lower bound of $\psi(b)$ in the proof of Lemma 5.4 (but with a sum on i from $\lfloor \alpha_n^-/h \rfloor$ to $\lceil \alpha_n^+/h \rceil - 1$), we get the upper bound

$$
\varphi(b) \leq \frac{1}{R_L(b)}(L+b)\sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} \frac{\alpha_n^2}{n^{3\beta_n/2}} \left(\mathbb{E}\left[F(\mathfrak{e})\right] + o(1)\right) = \frac{\theta}{\sigma^2} \frac{\alpha_n^2}{n^{3\beta_n/2}} \left(\mathbb{E}\left[F(\mathfrak{e})\right] + o(1)\right),
\tag{5.41}
$$

uniformly in $b \in [k_n^{1/3}, k_n]$, using (2.1) and (2.5) for the last equality. Coming back to (5.40), we showed that

$$
\mathbb{E}_{\mathbb{Q}^{(L)}}^{[0,k_n)} \leq \frac{\theta}{\sigma^2} \frac{\alpha_n^2}{n^{3\beta_n/2}} \left(\mathbb{E}\left[F(\mathfrak{e})\right] + o(1)\right) \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n'^{[0,k_n)}(F_n)}{D_n^{(L),[0,k_n)}} \mathbb{1}_{V(w_{k_n}^{(L)}) \in [k_n^{1/3}, k_n]} \right],
\tag{5.42}
$$

and thus we now want to bound the expectation in (5.42). We proceed in a way similar to the proof of Lemma 4.5 of Aïdekon and Shi [3], by introducing the event $\{D_n^{(L),[k_n,n]} \leq n^{-2}\}$:

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n'^{[0,k_n)}(F_n)}{D_n^{(L),[0,k_n)}} \mathbb{1}_{D_n^{(L),[k_n,n]} \leq n^{-2}} \right] \\
&= \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n'^{[0,k_n)}(F_n)}{D_n^{(L),[0,k_n)}} \mathbb{Q}^{(L)}\left(D_n^{(L),[k_n,n]} \leq n^{-2} | V\left(w_{k_n}^{(L)}\right) = u\right) \right] \\
&\geq \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n'^{[0,k_n)}(F_n)}{D_n^{(L),[0,k_n)}} \mathbb{1}_{V(w_{k_n}^{(L)}) \in [k_n^{1/3}, k_n]} \right] \inf_{u \in [k_n^{1/3}, k_n]} \mathbb{Q}^{(L)}\left(D_n^{(L),[k_n,n]} \leq n^{-2} | V\left(w_{k_n}^{(L)}\right) = u\right).
\end{aligned}
$$

Thus, applying (5.39), we get that the expectation in (5.42) is smaller than

$$
\begin{aligned}
&\left(1 + o(1)\right) \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n'^{[0,k_n)}(F_n)}{D_n^{(L),[0,k_n)}} \mathbb{1}_{D_n^{(L),[k_n,n]} \leq n^{-2}} \right] \\
&\leq \left(1 + o(1)\right)\left(\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n'^{[0,k_n)}(F_n)}{D_n^{(L),[0,k_n)}} \mathbb{1}_{D_n^{(L)} \geq n^{-7/4}, D_n^{(L),[k_n,n]} \leq n^{-2}} \right] + \|F\| \mathbb{Q}^{(L)}\left(D_n^{(L)} < n^{-7/4}\right) \right) \\
&\leq \left(1 + o(1)\right)\left(\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{Y_n'^{[0,k_n)}(F_n)}{(1 - n^{-1/4})D_n^{(L)}} \right] + \|F\| n^{-7/4} \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{1}{D_n^{(L)}} \right] \right),
\end{aligned}
\tag{5.43}
$$

using that $Y_n'^{[0,k_n)}(F_n)/D_n^{(L),[0,k_n)} \leq \|F\|$ and applying the Markov inequality for the second term. Using that $Y_n'^{[0,k_n)} \leq Y_n' \leq W_{n,\beta_n}^{(L,K)}$, and then Lemmas 5.4 and 5.3, we get that (5.43) is smaller than

$$(1 + o(1))\left(\left(\mathbb{E}[F(\mathfrak{e})] + o(1)\right)\frac{\theta}{\sigma^2}\frac{\alpha_n^2}{n^{3\beta_n/2}} + \|F\|\frac{n^{-7/4}}{R(L)}\right) = \left(\mathbb{E}[F(\mathfrak{e})] + o(1)\right)\frac{\theta}{\sigma^2}\frac{\alpha_n^2}{n^{3\beta_n/2}},$$

because $3\beta_n/2 \leq 7/4$ for n large enough. Thus, coming back to (5.42), we proved that

$$\mathbb{E}_{\mathbb{Q}^{(L)}}^{[0,k_n)} \leq \left(\left(\mathbb{E}[F(\mathfrak{e})] + o(1)\right)\frac{\theta}{\sigma^2}\frac{\alpha_n^2}{n^{3\beta_n/2}}\right)^2. \tag{5.44}$$

We now want to show that $\mathbb{E}_{\mathbb{Q}^{(L)}}^{[k_n,n]} = o((\alpha_n^2/n^{3\beta_n/2})^2)$ and, by bounding $F \leq \|F\|$, it is sufficient to deal with the case $F \equiv 1$. Using that $D_n^{(L),[0,k_n)} \leq D_n^{(L)}$, $\mathbb{1}_{z \in B_n} \leq 1$ and breaking down $Y_n'^{[k_n,n]}(1)$ along the spine, we first have that $\mathbb{E}_{\mathbb{Q}^{(L)}}^{[k_n,n]}$ is smaller than

$$\sum_{i=k_n}^{n-1} \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{1}{D_n^{(L),[0,k_n)}}\frac{e^{-V(w_n^{(L)})/\alpha_n}}{R_L(V(w_n^{(L)}))}\mathbb{1}_{w_n^{(L)} \in A_n \cap B_n}\sum_{y \in \Omega(w_{i+1}^{(L)})}\sum_{|z|=n, z \geq y}e^{-\beta_n V(z)}\mathbb{1}_{z \in A_n}\right]$$

$$+ \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{1}{D_n^{(L),[0,k_n)}}\frac{e^{-V(w_n^{(L)})/\alpha_n}}{R_L(V(w_n^{(L)}))}\mathbb{1}_{w_n^{(L)} \in A_n \cap B_n}e^{-\beta_n V(w_n^{(L)})}\right]. \tag{5.45}$$

Noting that, on the event $\{z \in A_n\}$, we have $e^{-\beta_n V(z)} \leq e^{-V(z)}n^{-3/2\alpha_n}$, the first term in (5.45) is smaller than

$$\sum_{i=k_n}^{n-1} \mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{1}{D_n^{(L),[0,k_n)}}\frac{e^{-V(w_n^{(L)})/\alpha_n}}{R_L(V(w_n^{(L)}))}\mathbb{1}_{w_n^{(L)} \in A_n \cap B_n}\sum_{y \in \Omega(w_{i+1}^{(L)})}\sum_{|z|=n, z \geq y}e^{-V(z)}n^{-3/2\alpha_n}\right]$$

$$= n^{-3/2\alpha_n}\sum_{i=k_n}^{n-1}\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{1}{D_n^{(L),[0,k_n)}}\frac{e^{-V(w_n^{(L)})/\alpha_n}}{R_L(V(w_n^{(L)}))}\mathbb{1}_{w_n^{(L)} \in A_n \cap B_n}\sum_{y \in \Omega(w_{i+1}^{(L)})}e^{-V(y)}\right], \tag{5.46}$$

by conditioning with respect to $\mathscr{G}_\infty := \sigma(V(w_i^{(L)}), V(y), y \in \Omega(w_{i+1}^{(L)}), i \in \mathbb{N})$ and noting that, given $\mathscr{G}_\infty$, $D_n^{(L),[0,k_n)}$ is independent of $(V(z), |z| = n, z \geq y, y \in \Omega(w_{i+1}^{(L)}))$. Noting that we are on the event $\{w_n^{(L)} \in B_n\}$, (5.46) is smaller than

$$n^{-3/2\alpha_n}\sum_{i=k_n}^{n-1}\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{1}{D_n^{(L),[0,k_n)}}\frac{e^{-V(w_n^{(L)})/\alpha_n}}{R_L(V(w_n^{(L)}))}\mathbb{1}_{w_n^{(L)} \in A_n \cap B_n}\rho_n e^{-a_i^{(n)}-\ell_i^{(n)}}\right]$$

$$\leq \frac{\rho_n}{n^{3/2\alpha_n}}\left(\sum_{i=k_n}^{\lfloor n/2\rfloor-1}e^{L-i^{1/7}} + \sum_{i=\lfloor n/2\rfloor}^{n-1}\frac{e^{-(n-i)^{1/7}}}{n^{3/2}}\right)\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{1}{D_n^{(L),[0,k_n)}}\frac{e^{-V(w_n^{(L)})/\alpha_n}}{R_L(V(w_n^{(L)}))}\mathbb{1}_{w_n^{(L)} \in A_n}\right]$$

$$\leq c_{25}e^L\frac{\rho_n}{n^{3\beta_n/2}}(1 + o(1))\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{1}{D_n^{(L),[0,k_n)}}\frac{e^{-V(w_n^{(L)})/\alpha_n}}{R_L(V(w_n^{(L)}))}\mathbb{1}_{w_n^{(L)} \in A_n}\right], \tag{5.47}$$

because $k_n^{1/7} \gg \log n$ and therefore the sum for $i \in [\![k_n, \lfloor n/2\rfloor - 1]\!]$ is a $o(n^{-3/2})$. For the second term in (5.45), we use that, on the event $\{w_n^{(L)} \in A_n\}$, $e^{-\beta_n V(w_n^{(L)})} \leq n^{-3\beta_n/2}$, and, thus, combining with (5.47), we get

$$\mathbb{E}_{\mathbb{Q}^{(L)}}^{[k_n,n]} \leq c_{25}e^L\frac{\rho_n}{n^{3\beta_n/2}}(1 + o(1))\mathbb{E}_{\mathbb{Q}^{(L)}}\left[\frac{1}{D_n^{(L),[0,k_n)}}\frac{e^{-V(w_n^{(L)})/\alpha_n}}{R_L(V(w_n^{(L)}))}\mathbb{1}_{w_n^{(L)} \in A_n}\right]$$

$$= c_{25} e^L \frac{\rho_n}{n^{3\beta_n/2}} \big(1 + o(1)\big) \mathbb{E}_{\mathbb{Q}^{(L)}} \left[\frac{1}{D_n^{(L),[0,k_n]}} \varphi\big(V\big(w_{k_n}^{(L)}\big)\big) \mathbb{1}_{V(w_{k_n}^{(L)}) \in [k_n^{1/3}, k_n]} \right],$$

where the function φ has been defined previously in the proof. Using (5.41) again, we get

$$\mathbb{E}_{\mathbb{Q}^{(L)}}^{[k_n,n]} \leq c_{26} e^L \frac{\rho_n}{n^{3\beta_n/2}} \frac{\alpha_n^2}{n^{3\beta_n/2}} \big(1 + o(1)\big) \mathbb{E}_{\mathbb{Q}^{(L)}} \left[\frac{1}{D_n^{(L),[0,k_n]}} \mathbb{1}_{V(w_{k_n}^{(L)}) \in [k_n^{1/3}, k_n]} \right]. \tag{5.48}$$

Proceeding in the same way as before by using (5.39) to introduce the event $\{D_n^{(L),[k_n,n]} \leq n^{-2}\}$, the expectation in (5.48) is smaller than

$$\big(1 + o(1)\big) \mathbb{E}_{\mathbb{Q}^{(L)}} \left[\frac{1}{D_n^{(L),[0,k_n]}} \mathbb{1}_{D_n^{(L),[k_n,n]} \leq n^{-2}} \right]$$

$$\leq \big(1 + o(1)\big) \left(\mathbb{E}_{\mathbb{Q}^{(L)}} \left[\frac{1}{D_n^{(L),[0,k_n]}} \mathbb{1}_{D_n^{(L)} \geq n^{-7/4}, D_n^{(L),[k_n,n]} \leq n^{-2}} \right] + \mathbb{Q}^{(L)}\big(D_n^{(L)} < n^{-7/4}\big) \right). \tag{5.49}$$

As for (5.43), (5.49) is equal to $(1 + o(1))/R(L)$ and, coming back to (5.48), we get

$$\mathbb{E}_{\mathbb{Q}^{(L)}}^{[k_n,n]} \leq c_{27} \frac{e^L}{R(L)} \frac{\rho_n}{n^{3\beta_n/2}} \frac{\alpha_n^2}{n^{3\beta_n/2}} \big(1 + o(1)\big) = o\left(\left(\frac{\alpha_n^2}{n^{3\beta_n/2}} \right)^2 \right),$$

because $\rho_n \ll \alpha_n^2$. This concludes the proof of Lemma 5.6. $\qquad\square$

Appendix A: Convergence of random measures

In this section, we present some results concerning convergence of random or deterministic probability measures on a polish space S or more specifically on $\mathcal{D}([0, 1])$. Some of these results are classical, but we state them here with uniformity in some parameter $\theta \in \Theta$.

A.1. *General space*

Let (S, d), (S_1, d_1) and (S_2, d_2) be Polish spaces. We consider some set Θ. In the sequel, for each $\theta \in \Theta$, $(\mu_n^\theta)_{n \in \mathbb{N}}$ will denote a sequence of random probability measures on S and $(\xi_n^\theta)_{n \in \mathbb{N}}$ a sequence of deterministic probability measures on S. Moreover, μ and ξ will be probability measures on S, that are deterministic and do not depend on θ.

Lemma A.1. *Assume that for all Lipschitz $F \in \mathcal{C}_b(S)$ and $\varepsilon > 0$, $\mathbb{P}(|\mu_n^\theta(F) - \mu(F)| \geq \varepsilon) \to 0$ as $n \to \infty$ uniformly in $\theta \in \Theta$. Then, the same convergence holds for all $F \in \mathcal{C}_b(S)$.*

Note that in the case where we consider a deterministic sequence $(\xi_n^\theta)_{n \in \mathbb{N}}$, it simply means that $\xi_n^\theta(F) \to \xi(F)$ uniformly in $\theta \in \Theta$. It is necessary that the limit does not depend on θ.

Proof. We follow the proof of Portmanteau Theorem in Billingsley [13, Theorem 2.1]. Thus, we first consider a closed set A and $\varepsilon > 0$ and we want to show that

$$\lim_{n \to \infty} \sup_{\theta \in \Theta} \mathbb{P}\big(\mu_n^\theta(A) \geq \mu(A) + \varepsilon\big) = 0. \tag{A.1}$$

We consider, for each $\eta > 0$, the function $F^\eta \colon x \mapsto 1 - (1 \wedge \eta^{-1} d(x, A)) \in [0, 1]$ that is Lipschitz and such that $F^\eta \downarrow \mathbb{1}_A$. Thus, by dominated convergence, we have $\mu(F^\eta) \to \mu(A)$ as $\eta \to 0$. We fix η such that $\mu(F^\eta) \leq \mu(A) + (\varepsilon/2)$. Since $\mu_n^\theta(F^\eta) \geq \mu_n^\theta(A)$, we get

$$\sup_{\theta \in \Theta} \mathbb{P}\big(\mu_n^\theta(A) \geq \mu(A) + \varepsilon\big) \leq \sup_{\theta \in \Theta} \mathbb{P}\left(\mu_n^\theta\big(F^\eta\big) \geq \mu\big(F^\eta\big) + \frac{\varepsilon}{2} \right) \xrightarrow[n \to \infty]{} 0,$$

by using the assumption of the lemma. From (A.1), we get that, for all set A such that $\mu(\partial A) = 0$ $\mathbb{P}$-a.s. and all $\varepsilon > 0$,

$$\lim_{n \to \infty} \sup_{\theta \in \Theta} \mathbb{P}\big(\big|\mu_n^\theta(A) - \mu(A)\big| \ge \varepsilon\big) = 0. \tag{A.2}$$

We now consider $F \in \mathcal{C}_b(S)$ and we can assume that F is nonnegative. We fix $\varepsilon > 0$ and set $M := \|F\|$. Firstly, we have

$$\int_0^M \mathbb{E}\big[\mu\big(\partial\{F > t\}\big)\big]\,dt \le \int_0^M \mathbb{E}\big[\mu\big(\{F = t\}\big)\big]\,dt = \mathbb{E}\bigg[\int_S \int_0^M \mathbb{1}_{F(x)=t}\,dt\,d\mu(x)\bigg] = 0,$$

and, therefore, for almost every $t \in [0, M]$ (in the sense of the Lebesgue measure), $\mathbb{P}$-a.s., $\mu(\partial\{F > t\}) = 0$. Thus, for all $N \in \mathbb{N}$, we can fix a subdivision $0 = t_0 < t_1 < \cdots < t_N = M$ such that for all $0 \le k \le N-1$, $t_{k+1} - t_k \le 2M/N$ and for all $1 \le k \le N-1$, $\mu(\partial\{F > t_k\}) = 0$ $\mathbb{P}$-a.s. Since $\mu(F) = \int_0^M \mu(F > t)\,dt$ and $t \mapsto \mu(F > t)$ is nonincreasing, we have

$$\sum_{k=1}^N (t_k - t_{k-1})\mu(F > t_k) \le \mu(F) \le \sum_{k=0}^{N-1} (t_{k+1} - t_k)\mu(F > t_k) \tag{A.3}$$

and the same holds for μ_n^θ instead of μ, for all $n \in \mathbb{N}$ and $\theta \in \Theta$. Since, in (A.3), the left-hand side and right-hand side of (A.3) tend to $\mu(F)$ as $N \to \infty$, we can choose N large enough such that they are at most at distance $\varepsilon/2$ from $\mu(F)$. Then, using (A.3) for μ_n^θ, it follows that

$$\sup_{\theta \in \Theta} \mathbb{P}\big(\big|\mu_n^\theta(F) - \mu(F)\big| \ge \varepsilon\big) \le \sup_{\theta \in \Theta} \Bigg(\mathbb{P}\bigg(\sum_{k=1}^{N-1} (t_{k+1} - t_k)\big(\mu_n^\theta(F > t_k) - \mu(F > t_k)\big) \ge \frac{\varepsilon}{4} \bigg)$$

$$+ \mathbb{P}\bigg(\sum_{k=1}^{N-1} (t_{k+1} - t_k)\big(\mu_n^\theta(F > t_k) - \mu(F > t_k)\big) \le -\frac{\varepsilon}{4} \bigg) \Bigg),$$

by noting that, in the sums, the term for $k = N$ is zero, since $t_N = M = \|F\|$, and the term for $k = 0$ is smaller than $2t_1 \le 4M/N \le \varepsilon/4$, if we choose N large enough. Then, we have

$$\sup_{\theta \in \Theta} \mathbb{P}\big(\big|\mu_n^\theta(F) - \mu(F)\big| \ge \varepsilon\big) \le \sup_{\theta \in \Theta} \sum_{k=1}^{N-1} \mathbb{P}\bigg(\big|\mu_n^\theta(F > t_k) - \mu(F > t_k)\big| \ge \frac{\varepsilon N}{8M(N-1)} \bigg) \xrightarrow[n \to \infty]{} 0,$$

by using (A.2), because for all $1 \le k \le N-1$, $\mu(\partial\{F > t_k\}) = 0$ $\mathbb{P}$-a.s. $\qquad\square$

Lemma A.2. *We consider the product space $S := S_1 \times S_2$. Assume that, for all $G_1 \in \mathcal{C}_b^u(S_1)$ and $G_2 \in \mathcal{C}_b^u(S_2)$, $\xi_n^\theta(G_1 \star G_2) \to \xi(G_1 \star G_2)$ as $n \to \infty$ uniformly in $\theta \in \Theta$, where $G_1 \star G_2 \colon (x, y) \in S \mapsto G_1(x)G_2(y)$. Then, for all $F \in \mathcal{C}_b(S)$, $\xi_n^\theta(F) \to \xi(F)$ uniformly in $\theta \in \Theta$.*

Proof. Using Lemma A.1, it is sufficient to consider $F \in \mathcal{C}_b^u(S)$. Let $\varepsilon > 0$. Since F is uniformly continuous, it exists $\eta > 0$ such that for any $x, y \in S$ that verify $d(x, y) \le \eta$, we have $|F(x) - F(y)| \le \varepsilon$. Since S_2 is separable for the metric d, it exists $(y_i)_{i \in \mathbb{N}} \in S_2^\mathbb{N}$ such that $\bigcup_{i \in \mathbb{N}} B_2(y_i, \eta/2) = S_2$, where $B_2(y, r)$ denotes the open ball of radius r centred at y in S_2. Now we consider a compact set $K \subset S$ such that $\xi(K) \ge 1 - \varepsilon$ and K' the image of K under the canonical projection $S \to S_2$. Since K' is a compact set of S_2, we can extract a finite cover $K' \subset \bigcup_{i=0}^N B(y_i, \eta/2)$. Using again the compacity of K', there exist nonnegative Lipschitz functions $\chi_0, \ldots, \chi_N \in \mathcal{C}_b(S_2)$ such that, for all $0 \le i \le N$, $\operatorname{supp} \chi_i \subset B(y_i, \eta/2)$, $\chi_0 + \cdots + \chi_N \le 1$ and $\chi_0 + \cdots + \chi_N = 1$ on K'. Finally, we set $\chi := 1 \star (\chi_0 + \cdots + \chi_N)$, so that χ is a Lipschitz continuous function from $S \to [0, 1]$ and $\chi = 1$ on K.

We can now construct some functions of the form $G_1 \star G_2$ to approach F. For $0 \le i \le N$, we set $G_1^i \colon x \in S_1 \mapsto F(x, y_i)$ and $G_2^i \colon y \in S_2 \mapsto \chi_i(y)$. By the triangle inequality, we have that $|\xi_n^\theta(F) - \xi(F)|$ is smaller than

$$\left| \xi_n^\theta \left(F - \sum_{i=0}^N G_1^i \star G_2^i \right) \right| + \sum_{i=0}^N |\xi_n^\theta (G_1^i \star G_2^i) - \xi(G_1^i \star G_2^i)| + \left| \xi \left(F - \sum_{i=0}^N G_1^i \star G_2^i \right) \right|. \tag{A.4}$$

The second term in (A.4) tends to 0 as $n \to \infty$ uniformly in $\theta \in \Theta$ by the assumption of the lemma. On the other hand, we have

$$\left| F - \sum_{i=0}^N G_1^i \star G_2^i \right| \le \|F\|(1 - \chi) + \left| \chi F - \sum_{i=0}^N G_1^i \star G_2^i \right| \le \|F\|(1 - \chi) + \varepsilon \chi,$$

because of the choice of η. Thus, we get that the first and third terms of (A.4) are smaller than $2\varepsilon + \|F\|(\xi_n^\theta(1 - \chi) + \xi(1 - \chi))$. Since $1 - \chi = 1 \star (1 - \chi_0 - \cdots - \chi_N)$, we can use again the assumption of the lemma to get that $\xi_n^\theta(1 - \chi) \to \xi(1 - \chi)$ as $n \to \infty$ uniformly in $\theta \in \Theta$. Noting that $1 - \chi \le \mathbb{1}_{K^c}$ and $\xi(K^c) \le \varepsilon$, it proves Lemma A.2. $\qquad\square$

A.2. Weak convergence in $\mathcal{D}([0, 1])$

We keep here the notation of the previous subsection, but we take $S = \mathcal{D}([0, 1])$. Recall the definition of the Skorokhod distance d on $\mathcal{D}([0, 1])$: for $x, y \in \mathcal{D}([0, 1])$,

$$d(x, y) := \inf_{\lambda \in \Lambda} \left(\|\lambda - \mathrm{id}\|_\infty \vee \|x - y \circ \lambda\|_\infty \right),$$

where we set $\Lambda := \{\lambda \colon [0, 1] \to [0, 1] | \lambda(0) = 0, \lambda(1) = 1, \lambda \text{ continuous and increasing}\}$, and that, equipped with this distance, $\mathcal{D}([0, 1])$ is a polish space (see Billingsley [13]).

Lemma A.3. *Assume that, for all $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$, we have $\xi_n^\theta(F) \to \xi(F)$ as $n \to \infty$ uniformly in $\theta \in \Theta$. We consider $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$, $(u_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$, $(v_n)_{n \in \mathbb{N}} \in \mathcal{D}([0, 1])^{\mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}} \in \Lambda^{\mathbb{N}}$ such that $u_n \to 1$, $\|v_n\|_\infty \to 0$ and $\|\lambda_n - \mathrm{id}\|_\infty \to 0$. We set $F_n \colon x \in \mathcal{D}([0, 1]) \mapsto F(v_n + u_n(x \circ \lambda_n))$. Then, we have $\xi_n^\theta(|F_n - F|) \to 0$ uniformly in $\theta \in \Theta$.*

Proof. For $x \in \mathcal{D}([0, 1])$, we first have

$$d(x, v_n + u_n(x \circ \lambda_n)) \le \|v_n\|_\infty + d(x, x \circ \lambda_n) + d(x \circ \lambda_n, u_n(x \circ \lambda_n))$$
$$\le \|v_n\|_\infty + \|\lambda_n - \mathrm{id}\|_\infty + |u_n - 1|\|x\|_\infty. \tag{A.5}$$

Now, we consider $\varepsilon > 0$ and we fix $K > 0$ such that $\xi(\{\|x\|_\infty \ge K\}) \le \varepsilon$ and some Lipschitz function $\chi \colon \mathcal{D}([0, 1]) \to [0, 1]$ such that $\mathbb{1}_{\|x\|_\infty < K} \le \chi(x) \le \mathbb{1}_{\|x\|_\infty < K+1}$ for all $x \in \mathcal{D}([0, 1])$ (this is possible since $\|-\|_\infty$ is Lipschitz on $\mathcal{D}([0, 1])$). Thus, we have

$$\xi_n^\theta(|F_n - F|) \le \|F_n\|\xi_n^\theta(1 - \chi) + \xi_n^\theta(\chi|F_n - F|) + \|F\|\xi_n^\theta(1 - \chi). \tag{A.6}$$

On the one hand, for $x \in \mathcal{D}([0, 1])$, we have, using $\chi(x) \le \mathbb{1}_{\|x\|_\infty < K+1}$ and (A.5),

$$\chi(x)|F_n(x) - F(x)| \le \omega_F \left(\|v_n\|_\infty + \|\lambda_n - \mathrm{id}\|_\infty + |u_n - 1|(K + 1) \right),$$

so we get $|\xi_n^\theta(\chi(F_n - F))| \le \|\chi(F_n - F)\| \to 0$ as $n \to \infty$ uniformly in $\theta \in \Theta$. On the other hand, by using the assumption of the lemma with the function $1 - \chi$, the first and third terms in the right-hand side of (A.6) tend towards $2\|F\|\xi(1 - \chi)$ uniformly in $\theta \in \Theta$. Since $\xi(1 - \chi) \le \varepsilon$, it concludes the proof of Lemma A.3. $\qquad\square$

Remark A.4. In Lemma A.3, if the limit measure ξ^θ depends on $\theta \in \Theta$, then the result is still true under the additional assumption that $\sup_{\theta \in \Theta} \xi(\{\|x\|_\infty \geq K\}) \to 0$ as $K \to \infty$, so that in the proof K and χ could be chosen independently of θ.

Lemma A.5. *Assume that* $\xi(\{x(0) = 0\}) = 1$ *and that, for all* $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$, *we have* $\xi_n^\theta(F) \to \xi(F)$ *as* $n \to \infty$ *uniformly in* $\theta \in \Theta$. *We consider* $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$ *and* $(\kappa_n)_{n \in \mathbb{N}}$ *such that* $\kappa_n \to 0$. *We set* $F_n \colon x \in \mathcal{D}([0, 1]) \mapsto F(x_{\kappa_n + (1 - \kappa_n)t} - x_{\kappa_n}, t \in [0, 1])$. *Then, we have* $\xi_n^\theta(|F_n - F|) \to 0$ *uniformly in* $\theta \in \Theta$.

Proof. The function $\varphi_n \colon t \in [0, 1] \mapsto \kappa_n + (1 - \kappa_n)t$ is not bijective from $[0, 1]$ to $[0, 1]$, so we consider a function $\lambda_n \in \Lambda$ such that $\lambda_n(t) = \kappa_n + (1 - \kappa_n)t$ for $t \geq \kappa_n$ and that is linear on $[0, \kappa_n]$. Then, for $x \in \mathcal{D}([0, 1])$, we have

$$d(x, x \circ \varphi_n - x_{\kappa_n}) \leq \|\lambda_n - \mathrm{id}\|_\infty \vee \|x \circ \lambda_n - x \circ \varphi_n - x_{\kappa_n}\|_\infty \leq \kappa_n \vee 3 \max_{[0,\kappa_n]} |x|. \tag{A.7}$$

Let $\varepsilon > 0$. Since $\xi(\{x(0) = 0\}) = 1$, it exists $\delta > 0$ such that $\xi(\{\max_{[0,\delta]} |x| > \varepsilon\}) \leq \varepsilon$. Let $\chi \colon \mathcal{D}([0, 1]) \to [0, 1]$ be a Lipschitz function such that $\mathbb{1}_{\max_{[0,\delta]} |x| < \varepsilon} \leq \chi(x) \leq \mathbb{1}_{\max_{[0,\delta]} |x| < 2\varepsilon}$ for all $x \in \mathcal{D}([0, 1])$. Thus, we have, using the triangle inequality and then (A.7),

$$\xi_n^\theta(|F_n - F|) \leq \|F_n\| \xi_n^\theta(1 - \chi) + \xi_n^\theta(\chi |F_n - F|) + \|F\| \xi_n^\theta(1 - \chi)$$

$$\leq 2\|F\| \xi_n^\theta(1 - \chi) + \omega_F(\kappa_n \vee 6\varepsilon),$$

for n large enough such that $\kappa_n \leq \delta$ (so independent of θ). Finally, using the assumption of the lemma with the function $1 - \chi$ and recalling that $\xi(1 - \chi) \leq \varepsilon$, it concludes the proof. $\qquad\square$

Appendix B: Proofs of the preliminary results concerning random walk

In this section, we prove the results stated in Section 2. Section B.1 is devoted to the proof of Proposition 2.6 and Corollary 2.7. Sections B.2 and B.3 contain preliminary results for the proof of Proposition 2.8 in Section B.4.

B.1. *Convergence towards the 3-dimensional Bessel process and the Brownian meander*

We first recall a known invariance principle for the random walk conditioned to stay nonnegative for all time. For all $n \in \mathbb{N}$, $u \in \mathbb{R}_+$ and $B \in \mathscr{F}_n$, we set

$$\mathbb{P}_u^+(B) := \frac{1}{R(u)} \mathbb{E}_u \big[R(S_n) \mathbb{1}_B \mathbb{1}_{\underline{S}_n \geq 0} \big]. \tag{B.1}$$

It defines a probability measure $\mathbb{P}_u^+$, that is called the law of the random walk started at $u \in \mathbb{R}_+$ and conditioned to stay nonnegative for all time. Then we have the following invariance principle, by Theorem 1.1 of Caravenna and Chaumont [15]: for any $b \in \mathbb{R}_+$ and $(b_n)_{n \in \mathbb{N}}$ such that $b_n / \sigma \sqrt{n} \to b$ as $n \to \infty$ and for any $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$,

$$\mathbb{E}_{b_n}^+ \big[F(\mathbf{S}^{(n)}) \big] \xrightarrow[n \to \infty]{} \mathbb{E}_b \big[F(\mathcal{R}) \big], \tag{B.2}$$

where $\mathcal{R}$ denotes the 3-dimensional Bessel process on $[0, 1]$.

Proposition 2.6 follows from (B.2) and from the following link between the 3-dimensional Bessel process and the Brownian meander (see Imhof [29]): for all $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$, we have

$$\mathbb{E}\big[F(\mathcal{M}) \big] = \sqrt{\frac{\pi}{2}} \mathbb{E}\left[\frac{1}{\mathcal{R}(1)} F(\mathcal{R}) \right]. \tag{B.3}$$

Proof of Proposition 2.6. We can assume that F is nonnegative. For $K > 0$, we consider $\chi : \mathbb{R}_+ \to [0, 1]$ continuous such that $\mathbb{1}_{[K^{-1}, K]} \leq \chi \leq \mathbb{1}_{[(2K)^{-1}, 2K]}$. On the one hand, we have

$$\mathbb{E}_u\left[F\big(\mathbf{S}^{(n)}\big)\left(1 - \chi\left(\frac{S_n}{\sigma\sqrt{n}}\right)\right)\mathbb{1}_{\underline{S}_n \geq 0}\right] \leq \|F\|\frac{\theta R(u)}{\sqrt{n}}\left(\int_0^\infty \big(1 - \chi(t)\big)te^{-t^2/2}\,\mathrm{d}t + o(1)\right), \tag{B.4}$$

uniformly in $u \in [0, \gamma_n]$, using (2.6). On the other hand, we have

$$\mathbb{E}_u\left[F\big(\mathbf{S}^{(n)}\big)\chi\left(\frac{S_n}{\sigma\sqrt{n}}\right)\mathbb{1}_{\underline{S}_n \geq 0}\right] = \frac{R(u)}{\sqrt{n}}\mathbb{E}_u^+\left[h_n\left(\frac{S_n}{\sigma\sqrt{n}}\right)F\big(\mathbf{S}^{(n)}\big)\right], \tag{B.5}$$

where we set, for $z \in \mathbb{R}_+$, $h_n(z) := \sqrt{n}\chi(z)/R(z\sigma\sqrt{n})$. Using (2.1) and $\chi \leq \mathbb{1}_{[(2K)^{-1}, 2K]}$, we have $\limsup_{n\to\infty}\sup_{z\in\mathbb{R}_+}|h_n(z) - h(z)| = 0$, with $h(z) := \chi(z)/c_0\sigma z$. Thus, we get, uniformly in u because h_n and h do not depend on u,

$$\limsup_{n\to\infty}\left|\mathbb{E}_u^+\left[h_n\left(\frac{S_n}{\sigma\sqrt{n}}\right)F\big(\mathbf{S}^{(n)}\big)\right] - \mathbb{E}_u^+\left[h\left(\frac{S_n}{\sigma\sqrt{n}}\right)F\big(\mathbf{S}^{(n)}\big)\right]\right| = 0.$$

Moreover, using (B.2), (B.3) and (2.5), we have

$$\mathbb{E}_u^+\left[h\left(\frac{S_n}{\sigma\sqrt{n}}\right)F\big(\mathbf{S}^{(n)}\big)\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[\frac{\chi(\mathcal{R}(1))}{c_0\sigma\mathcal{R}(1)}F\big(\mathcal{R}(t), t \in [0, 1)\big)\right] = \theta\mathbb{E}\big[\chi\big(\mathcal{M}(1)\big)F(\mathcal{M})\big],$$

uniformly in $u \in [0, \gamma_n]$. Coming back to (B.4) and (B.5) and using that the density of $\mathcal{M}(1)$ is $t \mapsto te^{-t^2/2}\mathbb{1}_{t>0}$, we showed that

$$\limsup_{n\to\infty}\sup_{u\in[0,\gamma_n]}\left|\frac{\sqrt{n}}{\theta R(u)}\mathbb{E}_u\big[F\big(\mathbf{S}^{(n)}\big)\mathbb{1}_{\underline{S}_n \geq 0}\big] - \mathbb{E}\big[F(\mathcal{M})\big]\right| \leq 2\|F\|\int_0^\infty \big(1 - \chi(t)\big)te^{-t^2/2}\,\mathrm{d}t,$$

which tends to 0 as $K \to \infty$, so it concludes the proof. $\qquad\square$

Proof of Corollary 2.7. By Lemma A.2 of Madaule [34] (that holds under the assumption of this corollary, see Remark 3.1), it exists $c_{28}(L, C) > 0$ such that for all n large enough and $K \geq 0$,

$$\mathbb{E}\big[e^{CS_n/\sqrt{n}}\mathbb{1}_{\underline{S}_n \geq -L, S_n \geq K\sqrt{n}}\big] \leq \frac{c_{28}(L, C)}{\sqrt{n}}e^{-CK/2}. \tag{B.6}$$

We consider some $\varepsilon > 0$ and fix K large enough such that $c_{28}(L, C)e^{-CK/2} \leq \varepsilon\theta R(L)/\|F\|$ and also $\mathbb{E}[e^{C\sigma\mathcal{M}(1)}\mathbb{1}_{\mathcal{M}(1)\geq K}] \leq \varepsilon/\|F\|$. Considering a continuous function $\chi : \mathbb{R} \to [0, 1]$ such that $\mathbb{1}_{x\leq K} \leq \chi(x) \leq \mathbb{1}_{x\leq K+1}$ and using the triangle inequality, we get

$$\left|\frac{\sqrt{n}}{\theta R(L)}\mathbb{E}\big[e^{CS_n/\sqrt{n}}F\big(\mathbf{S}^{(n)}\big)\mathbb{1}_{\underline{S}_n \geq -L}\big] - \mathbb{E}\big[e^{C\sigma\mathcal{M}(1)}F(\mathcal{M})\big]\right|$$

$$\leq 2\varepsilon + \left|\frac{\sqrt{n}}{\theta R(L)}\mathbb{E}\left[e^{CS_n/\sqrt{n}}F\big(\mathbf{S}^{(n)}\big)\chi\left(\frac{S_n}{\sqrt{n}}\right)\mathbb{1}_{\underline{S}_n \geq -L}\right] - \mathbb{E}\big[e^{C\sigma\mathcal{M}(1)}F(\mathcal{M})\chi\big(\sigma\mathcal{M}(1)\big)\big]\right|. \tag{B.7}$$

Then, note that in Proposition 2.6, we can replace $\mathbb{E}_u[F(\mathbf{S}^{(n)})\mathbb{1}_{\underline{S}_n \geq 0}]$ by $\mathbb{E}[F(\mathbf{S}^{(n)})\mathbb{1}_{\underline{S}_n \geq -u}]$: indeed, it works when $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$ and we extend to $F \in \mathcal{C}_b(\mathcal{D}([0, 1]))$ by Lemma A.1. Thus, applying Proposition 2.6 with the function $x \in \mathcal{D}([0, 1]) \mapsto e^{C\sigma x_1}F(x)\chi(\sigma x_1)$ which belongs to $\mathcal{C}_b(\mathcal{D}([0, 1]))$ we get that the right-hand of (B.7) tends to 2ε as $n \to \infty$ and it concludes the proof of Corollary 2.7. $\qquad\square$

B.2. *Local limit theorems*

We first recall the classical Stone's [41] local limit theorem: letting $h > 0$ be any real number if S_1 is nonlattice and be the span of the lattice if S_1 is lattice, we have

$$\mathbb{P}\big(S_n \in [b, b + h)\big) = \frac{h}{\sigma\sqrt{2\pi n}} e^{-b^2/2\sigma^2 n} + o\bigg(\frac{1}{\sqrt{n}}\bigg), \tag{B.8}$$

as $n \to \infty$, uniformly in $b \in \mathbb{R}$. Thus we have the following uniform bound: it exists $c_{29} > 0$ such that, for all $n \geq 1$ and $b \in \mathbb{R}$,

$$\mathbb{P}\big(S_n \in [b, b + 1)\big) \leq \frac{c_{29}}{\sqrt{n}}. \tag{B.9}$$

Now we state a local limit theorem for the random walk staying above a barrier, in the case where the starting point is at distance of order $\sqrt{n}$ from the barrier and the endpoint at distance $o(\sqrt{n})$.

Lemma B.1. *Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\gamma_n \ll \sqrt{n}$ as $n \to \infty$. We set $f : t \in \mathbb{R} \mapsto$
$te^{-t^2/2}\mathbb{1}_{t\geq 0}$.*

(i) *If the law of S_1 is nonlattice, then, for all $h > 0$,*

$$\mathbb{P}\big(S_n \in [u - b, u - b + h), \underline{S}_n \geq -b\big) = \frac{\theta^-}{\sigma n} f\bigg(\frac{b}{\sigma\sqrt{n}}\bigg) \int_u^{u+h} R^-(t)\,\mathrm{d}t + o\bigg(\frac{R^-(u)}{n}\bigg),$$

 as $n \to \infty$, uniformly in $b \in \mathbb{R}$ and in $u \in [0, \gamma_n]$.

(ii) *If the law of S_1 is (h, a)-lattice, then,*

$$\mathbb{P}(S_n = u - b, \underline{S}_n \geq -b) = \frac{\theta^-}{\sigma n} f\bigg(\frac{b}{\sigma\sqrt{n}}\bigg) h R^-(u) + o\bigg(\frac{R^-(u)}{n}\bigg),$$

 as $n \to \infty$, uniformly in $b \in \mathbb{R}$ and $u \in [0, \gamma_n] \cap (b + an + h\mathbb{Z})$.

Proof. First note that, for each $D > 0$, by Propositions 11, 18 and 24 of Doney [26], both estimates of Lemma B.1 holds uniformly in $b \in [D^{-1}\sqrt{n}, D\sqrt{n}]$ and $u \in [0, \gamma_n]$.[4] Noting also that f tends to 0 at 0 and at infinity, it is sufficient to prove that, for each $h > 0$ and $\varepsilon > 0$, there exist D and n large enough such that $\mathbb{P}(S_n \in [u - b, u - b + h), \underline{S}_n \geq -b) \leq \varepsilon R^-(u)/n$ for all $b \notin [D^{-1}\sqrt{n}, D\sqrt{n}]$ and $u \in \mathbb{R}_+$. Reversing time, we can equivalently prove that $\mathbb{P}(S_n \in [b - u, b - u + h), \underline{S}_n \geq -u) \leq \varepsilon R(u)/n$.

For $b \in [0, D^{-1}\sqrt{n}]$, using (2.7) and (2.2), we get $\mathbb{P}(S_n \in [b - u, b - u + h), \underline{S}_n \geq -u) \leq c_{30}R(u)(1 + D^{-1}\sqrt{n})/n^{3/2}$. For D and n large enough and independent of u, this is smaller than $\varepsilon R(u)/n$.

For $b > D\sqrt{n}$, we cancel the lower barrier between times $\lfloor n/2 \rfloor + 1$ and n so that we get

$$\mathbb{P}\big(S_n \in [b - u, b - u + h), \underline{S}_n \geq -u\big) \leq \mathbb{E}\bigg[\mathbb{1}_{\underline{S}_{\lfloor n/2\rfloor} \geq -u} h\bigg(\frac{S_{\lfloor n/2\rfloor} + u}{\sigma\lfloor n/2\rfloor^{1/2}}\bigg)\bigg], \tag{B.10}$$

where we set for $x \geq 0$, $h(x) := \mathbb{P}(S_{n-\lfloor n/2\rfloor} + x\sigma\lfloor n/2\rfloor^{1/2} \in [b, b+h))$. Using (B.8), we have, uniformly in $b > D\sqrt{n}$ and $x \in \mathbb{R}_+$,

$$h(x) \leq \frac{c_{31}}{\sqrt{n}} \exp\bigg(-\frac{(x\sigma\lfloor n/2\rfloor^{1/2} - b)^2}{2\sigma^2(n - \lfloor n/2\rfloor)}\bigg) + o\bigg(\frac{1}{\sqrt{n}}\bigg) \leq \frac{c_{31}}{\sqrt{n}} g_D(x) + o\bigg(\frac{1}{\sqrt{n}}\bigg),$$

---

[4]Doney states his results in terms of the renewal function for the first *weak* increasing ladder height process of S and of $\mathbb{P}(\min_{1\leq k\leq n} S_n > 0)$. Our formulation follows from Remark 4.6 of Caravenna and Chaumont [16] (although they work only in the lattice and absolutely continuous cases, this remark does not rely on these assumptions).

where $g_D \colon \mathbb{R}_+ \to [0, 1]$, obtained by taking the supremum on $b > D\sqrt{n}$, is continuous and converges simply to 0 as $D \to \infty$. Thus, using (2.6) and (2.4), we get that (B.10) is smaller than

$$\frac{c_{31}}{\sqrt{n}} \frac{\theta R(u)}{\sqrt{n}} \left(\int_0^\infty g_D(t) t \mathrm{e}^{-t^2/2} \, \mathrm{d}t + o(1) \right) + \frac{c_3 R(u)}{\sqrt{n}} o\!\left(\frac{1}{\sqrt{n}} \right), \tag{B.11}$$

uniformly in $b \geq D\sqrt{n}$ and $u \in [0, \gamma_n]$. By the dominated convergence theorem, the integral in (B.11) tends to 0 as $D \to \infty$ and, thus, (B.10) is smaller than $\varepsilon R(u)/n$ for D and n large enough and independent of u. $\qquad\square$

B.3. *Convergence towards the Bessel bridge*

In this subsection, we are going to prove the following generalization of Lemma 2.4 of Chen, Madaule and Mallein [22]. It proves that conditioned by the event of Lemma B.1 the trajectory $\mathbf{S}^{(n)}$ converges to the Bessel bridge. This is a first step in the proof of Proposition 2.8.

Lemma B.2. *Let* $(\gamma_n)_{n \in \mathbb{N}}$ *be a sequence of positive numbers such that* $\gamma_n \ll \sqrt{n}$ *as* $n \to \infty$. *We set* $f \colon t \in \mathbb{R} \mapsto t\mathrm{e}^{-t^2/2}\mathbb{1}_{t \geq 0}$ *and we denote by* $\rho_{b,0}^1$ *the 3-dimensional Bessel bridge of length* 1 *from* $b \in \mathbb{R}_+$ *to* 0.

(i) *If the law of* S_1 *is nonlattice, then, for all* $h > 0$ *and* $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$,

$$\mathbb{E}_{b\sigma\sqrt{n}}\big[F(\mathbf{S}^{(n)})\mathbb{1}_{S_n \in [u, u+h), \underline{S}_n \geq 0}\big] = \frac{\theta^-}{\sigma n} \int_u^{u+h} R^-(t) \, \mathrm{d}t f(b) \mathbb{E}\big[F(\rho_{b,0}^1)\big] + o\!\left(\frac{R^-(u)}{n} \right),$$

as $n \to \infty$, *uniformly in* $b \in \mathbb{R}$ *and in* $u \in [0, \gamma_n]$.

(ii) *If the law of* S_1 *is* (h, a)-*lattice, then, for all* $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$,

$$\mathbb{E}_{b\sigma\sqrt{n}}\big[F(\mathbf{S}^{(n)})\mathbb{1}_{S_n = u, \underline{S}_n \geq 0}\big] = \frac{\theta^-}{\sigma n} h R^-(u) f(b) \mathbb{E}\big[F(\rho_{b,0}^1)\big] + o\!\left(\frac{R^-(u)}{n} \right),$$

as $n \to \infty$, *uniformly in* $b \in \mathbb{R}$ *and* $u \in [0, \gamma_n] \cap (b\sigma\sqrt{n} + an + h\mathbb{Z})$.

Proof. We will treat only the nonlattice case, because the proof in the lattice case is exactly the same, with $hR^-(u)$ instead of $\int_u^{u+h} R^-(t) \, \mathrm{d}t$. Moreover, since F is bounded, $f(b) \to 0$ as $b \to \infty$ and Lemma B.1 deals with the case $F \equiv 1$, it is sufficient to show that, for each $K > 0$, the estimate holds uniformly in $b \in [0, K]$ instead of $b \in \mathbb{R}_+$.

We first assume that $\forall x \in \mathcal{D}([0, 1])$, $F(x) = F'(x_t, t \in [0, 1-\varepsilon])$ for some $F' \in \mathcal{C}_b^u(\mathcal{D}([0, 1-\varepsilon]))$. Thus, $F(\mathbf{S}^{(n)})$ is $\mathscr{F}_m$-measurable with $m := \lceil(1-\varepsilon)n\rceil$ and we have

$$\mathbb{E}_{b\sigma\sqrt{n}}\big[F(\mathbf{S}^{(n)})\mathbb{1}_{S_n \in [u, u+h), \underline{S}_n \geq 0}\big] = \mathbb{E}_{b\sigma\sqrt{n}}\left[F(\mathbf{S}^{(n)})g\!\left(\frac{S_m}{\sigma\sqrt{n}} \right)\mathbb{1}_{\underline{S}_m \geq 0}\right], \tag{B.12}$$

where we set, for $z \in \mathbb{R}_+$,

$$g(z) := \mathbb{P}_{z\sigma\sqrt{n}}\big(\underline{S}_{n-m} \geq 0, S_{n-m} \in [u, u+h)\big) = \frac{\theta^-}{\sigma \varepsilon n} f\!\left(\frac{z}{\sqrt{\varepsilon}} \right) \int_u^{u+h} R^-(t) \, \mathrm{d}t + o\!\left(\frac{R^-(u)}{n} \right),$$

uniformly in $u \in [0, \gamma_n]$ and $z \in \mathbb{R}_+$, using Lemma B.1. Therefore, (B.12) is equal to

$$\frac{\theta^-}{\sigma \varepsilon n} \int_u^{u+h} R^-(t) \, \mathrm{d}t \mathbb{E}_{b\sigma\sqrt{n}}\left[F(\mathbf{S}^{(n)}) f\!\left(\frac{S_m}{\sigma\sqrt{\varepsilon n}} \right)\mathbb{1}_{\underline{S}_m \geq 0}\right] + \mathbb{E}_{b\sigma\sqrt{n}}\big[F(\mathbf{S}^{(n)})\mathbb{1}_{\underline{S}_m \geq 0}\big] o\!\left(\frac{R^-(u)}{n} \right),$$

uniformly in $u \in [0, \gamma_n]$ and $b \in [0, K]$. Thus, it is now sufficient to prove that

$$\mathbb{E}_{b\sigma\sqrt{n}}\left[F(\mathbf{S}^{(n)}) f\!\left(\frac{S_m}{\sigma\sqrt{\varepsilon n}} \right)\mathbb{1}_{\underline{S}_m \geq 0}\right] \xrightarrow[n \to \infty]{} \varepsilon f(b) \mathbb{E}\big[F(\rho_{b,0}^1)\big], \tag{B.13}$$

uniformly in $b \in [0, K]$. On the one hand, by Equation (2.30) of Chen, Madaule and Mallein [22], we have

$$\varepsilon f(b)\mathbb{E}\big[F\big(\rho_{b,0}^1\big)\big] = \frac{b}{\sqrt{\varepsilon}}\mathbb{E}_b\big[F'\big(\mathcal{R}(s), s \in [0, 1-\varepsilon])\mathrm{e}^{-\mathcal{R}(1-\varepsilon)^2/2\varepsilon}\big]. \tag{B.14}$$

On the other hand, recalling (B.1), we get

$$\mathbb{E}_{b\sigma\sqrt{n}}\bigg[F\big(\mathbf{S}^{(n)}\big)f\bigg(\frac{S_m}{\sigma\sqrt{\varepsilon n}}\bigg)\mathbb{1}_{\underline{S}_m \geq 0}\bigg] = \mathbb{E}_{b\sigma\sqrt{n}}^+\bigg[F\big(\mathbf{S}^{(n)}\big)h_{\varepsilon,b}^n\bigg(\frac{S_m}{\sigma\sqrt{n}}\bigg)\bigg], \tag{B.15}$$

where we set, for $z \in \mathbb{R}_+$,

$$h_{\varepsilon,b}^n(z) := f\bigg(\frac{z}{\sqrt{\varepsilon}}\bigg)\frac{R(b\sigma\sqrt{n})}{R(z\sigma\sqrt{n})} = \frac{z}{\sqrt{\varepsilon}}\mathrm{e}^{-z^2/2\varepsilon}\frac{R(b\sigma\sqrt{n})}{R(z\sigma\sqrt{n})}.$$

Since $\mathrm{e}^{-z^2/2\varepsilon} \to 0$ as $z \to \infty$ and using (2.1), it is clear that

$$\sup_{b\in[0,K],z\in\mathbb{R}_+}\bigg|h_{\varepsilon,b}^n(z) - \frac{b}{\sqrt{\varepsilon}}\mathrm{e}^{-z^2/2\varepsilon}\bigg| \xrightarrow[n\to\infty]{} 0, \tag{B.16}$$

so we get, combining (B.14), (B.15) and (B.16),

$$\limsup_{n\to\infty}\sup_{b\in[0,K]}\bigg|\mathbb{E}_{b\sigma\sqrt{n}}\bigg[F\big(\mathbf{S}^{(n)}\big)f\bigg(\frac{S_m}{\sigma\sqrt{\varepsilon n}}\bigg)\mathbb{1}_{\underline{S}_m\geq 0}\bigg] - \varepsilon f(b)\mathbb{E}\big[F\big(\rho_{b,0}^1\big)\big]\bigg|$$

$$\leq \limsup_{n\to\infty}\sup_{b\in[0,K]}\frac{b}{\sqrt{\varepsilon}}\big|\mathbb{E}_{b\sigma\sqrt{n}}^+\big[F\big(\mathbf{S}^{(n)}\big)\mathrm{e}^{-S_m^2/2\sigma^2\varepsilon n}\big] - \mathbb{E}_b\big[F'\big(\mathcal{R}(s), s \in [0, 1-\varepsilon])\mathrm{e}^{-\mathcal{R}(1-\varepsilon)^2/2\varepsilon}\big]\big|,$$

which is equal to 0 by applying (B.2). It proves (B.13) and so it concludes the case where $F(x) = F'(x_t, t \in [0, 1-\varepsilon])$ for some $F' \in \mathcal{C}_b^u(\mathcal{D}([0, 1-\varepsilon]))$.

We now want to extend the result to the case $F \in \mathcal{C}_b^u(\mathcal{D}([0, 1]))$. For $\varepsilon > 0$ and $x \in \mathcal{D}([0, 1])$, we define $\varphi_\varepsilon(x) \in \mathcal{D}([0, 1])$ by $\varphi_\varepsilon(x)|_{[0,1-\varepsilon)} = x|_{[0,1-\varepsilon)}$ and $\varphi_\varepsilon(x)|_{[1-\varepsilon,1]} \equiv 0$, so that $F \circ \varphi_\varepsilon$ satisfies the assumption of the particular case that is already proved. Thus, it is now sufficient to show that, for each $\eta > 0$, it exists $\varepsilon > 0$ such that

$$\limsup_{n\to\infty}\sup_{u\in[0,\gamma_n],b\in[0,K]}\mathbb{E}_{b\sigma\sqrt{n}}\big[\big|F\big(\mathbf{S}^{(n)}\big) - F \circ \varphi_\varepsilon\big(\mathbf{S}^{(n)}\big)\big|\mathbb{1}_{S_n\in[u,u+h],\underline{S}_n\geq 0}\big] \leq \eta\frac{R^-(u)}{n}, \tag{B.17}$$

$$\sup_{b\in[0,K]}f(b)\mathbb{E}\big[\big|F\big(\rho_{b,0}^1\big) - F \circ \varphi_\varepsilon\big(\rho_{b,0}^1\big)\big|\big] \leq \eta. \tag{B.18}$$

We first prove (B.18): we have

$$\mathbb{E}\big[\big|F\big(\rho_{b,0}^1\big) - F \circ \varphi_\varepsilon\big(\rho_{b,0}^1\big)\big|\big] \leq \mathbb{E}\big[\omega_F\big(\big\|\rho_{b,0}^1 - \varphi_\varepsilon\big(\rho_{b,0}^1\big)\big\|_\infty\big)\big]$$

$$\leq \mathbb{E}\Big[\omega_F\Big(\sup_{t\in[1-\varepsilon,1]}\rho_{b,0}^1(t)\Big)\Big] =: \mathbb{E}_{b,\varepsilon}.$$

Since the function $b \mapsto \mathbb{E}_{b,\varepsilon}$ is nondecreasing and $f \leq 1$, the left-hand side of (B.18) is smaller than $\mathbb{E}_{K,\varepsilon}$ and, thus, it tends to 0 as $\varepsilon \to 0$ by dominated convergence, because ω_F is bounded. Now, we prove (B.17): using that $d(\mathbf{S}^{(n)}, \varphi_\varepsilon(\mathbf{S}^{(n)})) \leq \max_{0\leq k\leq\varepsilon n} S_{n-k}/\sigma\sqrt{n}$ and reversing time, we get that the expectation in (B.17) is smaller than

$$\mathbb{E}\bigg[\omega_F\bigg(\sup_{0\leq k\leq\varepsilon n}\frac{S_k^- + u + h}{\sigma\sqrt{n}}\bigg)\mathbb{1}_{S_n^- - b\sigma\sqrt{n}\in(-u-h,-u],\underline{S}_n^-\geq-u-h}\bigg]. \tag{B.19}$$

Conditioning with respect to $\mathscr{F}_{\lfloor n/2 \rfloor}$ and applying (B.9), we get that (B.19) is smaller than

$$
\frac{c_{29}(1+h)}{(n-\lfloor n/2 \rfloor)^{1/2}} \mathbb{E}\left[\omega_F\left(\sup_{0 \le k \le \varepsilon n} \frac{S_k^- + u + h}{\sigma\sqrt{n}} \right) \mathbb{1}_{\underline{S}_{\lfloor n/2 \rfloor}^- \ge -u-h} \right]
$$

$$
= \frac{c_{29}(1+h)}{\sqrt{n/2}} \frac{\theta^- R^-(u+h)}{\sqrt{n/2}} \left(\mathbb{E}\left[\omega_F\left(\frac{1}{\sqrt{2}} \sup_{t \in [0,2\varepsilon]} \mathcal{M}_t \right) \right] + o(1) \right),
\tag{B.20}
$$

uniformly in $u \in [0, \gamma_n]$ and $b \in [0, K]$, by using Lemma 2.6 to get the last equality. The expectation in the right-hand side of (B.20) does not depend on n, b and u and tends to 0 as $\varepsilon \to 0$ by dominated convergence, so it shows (B.17) and concludes the proof. □

B.4. *Convergence towards the Brownian excursion*

We prove here Proposition 2.8, in a similar way as Lemma 2.5 of Chen, Madaule and Mallein [22], but directly with a first barrier that can be different of 0. Following [22], we fix some $\lambda \in (0, 1)$ and, for $G_1 \colon \mathcal{D}([0, \lambda]) \to \mathbb{R}$, $G_2 \colon \mathcal{D}([0, 1-\lambda]) \to \mathbb{R}$ and $x \in \mathcal{D}([0, 1])$, we set

$$
G_1 \star G_2(x) := G_1\big(x_s, s \in [0, \lambda]\big) G_2\big(x_{\lambda+s}, s \in [0, 1-\lambda]\big).
$$

Then, by [22, Lemma 2.3], we have, for each $G_1 \in \mathcal{C}_b(\mathcal{D}([0, \lambda]))$ and $G_2 \in \mathcal{C}_b(\mathcal{D}([0, 1-\lambda]))$,

$$
\mathbb{E}\big[G_1 \star G_2(\mathfrak{e})\big] = \sqrt{\frac{2}{\pi}} \frac{1}{\lambda^{1/2}(1-\lambda)^{3/2}} \mathbb{E}\big[\mathcal{M}(\lambda)e^{-\mathcal{M}(\lambda)^2/2(1-\lambda)} G_1(\mathcal{M}) G_2\big(\rho_{\mathcal{M}(\lambda),0}^{1-\lambda}\big)\big],
\tag{B.21}
$$

where $\mathfrak{e}$ is the normalized Brownian excursion, $\mathcal{M}$ is the Brownian meander of length λ and $(\rho_{z,0}^{1-\lambda})_{z \in \mathbb{R}_+}$ is a family of Bessel bridges of length $1 - \lambda$ from z to 0, independent of $\mathcal{M}$.

Proof of Proposition 2.8. Using Lemma A.2, it is sufficient to consider the case where $F = G_1 \star G_2$ for some $G_1 \in \mathcal{C}_b^u(\mathcal{D}([0, \lambda]))$ and $G_2 \in \mathcal{C}_b^u(\mathcal{D}([0, 1-\lambda]))$. The proof will be only treated in the lattice case: in the nonlattice case, the reasoning is exactly the same with $\int_b^{b+h} R^-(t)\,dt$ instead of $hR^-(b)$. We set $m := \lfloor \lambda n \rfloor$ and take the conditional expectation according to $\mathscr{F}_m$ to get that

$$
\mathbb{E}\big[F\big(\mathbf{S}^{(n)}\big)\mathbb{1}_{\underline{S}_{\lfloor \lambda n \rfloor} \ge -u, \min_{\lfloor \lambda n \rfloor \le i \le n} S_i \ge v, S_n = v+b}\big] = \mathbb{E}\left[G_1\left(\frac{S_{\lfloor sn \rfloor}}{\sigma\sqrt{n}}, s \le \lambda \right) \varphi\left(\frac{S_m}{\sigma\sqrt{n}} \right) \mathbb{1}_{\underline{S}_m \ge -u} \right],
\tag{B.22}
$$

where we set $\varphi(z) := \mathbb{E}_{z\sigma\sqrt{n}}[G_2(S_{\lfloor (s+\lambda)n \rfloor -m}/\sigma\sqrt{n}, s \le 1-\lambda)\mathbb{1}_{\underline{S}_{n-m} \ge v, S_n = v+b}]$, for each $z \in \mathbb{R}$ such that $v + b + z\sigma\sqrt{n} \in a(n-m) + h\mathbb{Z}$. Then, since G_2 is uniformly continuous, using Lemma B.2 combined with Lemma A.3,[5] we get that $\varphi(z)$ is equal to

$$
\frac{\theta^-}{\sigma(1-\lambda)n} h R^-(b) f\left(\frac{z}{\sqrt{1-\lambda}} \right) \mathbb{E}\big[G_2\big(\sqrt{1-\lambda}\,\rho_{z/\sqrt{1-\lambda},0}^1(s/(1-\lambda)), s \le 1-\lambda\big)\big] + o\left(\frac{R^-(b)}{n} \right),
$$

uniformly in $v \in [-\gamma_n, \gamma_n]$, $b \in [0, \gamma_n] \cap (-v + an + h\mathbb{Z})$ and $z \in \mathbb{R}$ such that $v + b + z\sigma\sqrt{n} \in a(n-m) + h\mathbb{Z}$ (note that if $z < 0$, the fact that the Bessel bridge is not well-defined is not a problem, because $f(z/\sqrt{1-\lambda}) = 0$ so the first term is zero). Since the last expectation is equal to $\mathbb{E}[G_2(\rho_{z,0}^{1-\lambda})]$ by scaling properties of the Bessel bridge, (B.22) is

---

[5]Here the limit measure is the law of $\rho_{z,0}^{1-\lambda}$ that depends on the parameter (b, v, z). Thus, using Remark A.4, we should prove that $\mathbb{P}(\|\rho_{z,0}^{1-\lambda}\|_\infty > K) \to 0$ as $K \to \infty$ uniformly in $z \in \mathbb{R}_+$, but it is obviously false. However, since $f(z/\sqrt{1-\lambda}) \to 0$ as $z \to \infty$, it is sufficient to have, for each $M > 0$, $\mathbb{P}(\|\rho_{z,0}^{1-\lambda}\|_\infty > K) \to 0$ as $K \to \infty$ uniformly in $z \in [0, M]$ and this is clearly true.

equal to

$$\frac{\theta^-}{\sigma(1-\lambda)n}hR^-(b)\mathbb{E}\left[G_1\left(\frac{S_{\lfloor sn\rfloor}}{\sigma\sqrt{n}}, s\leq\lambda\right)f\left(\frac{S_m}{\sqrt{1-\lambda}\sigma\sqrt{n}}\right)G_2\left(\rho^{1-\lambda}_{S_m/\sigma\sqrt{n},0}\right)\mathbb{1}_{\underline{S}_m\geq-u}\right]$$

$$+\mathbb{E}\left[G_1\left(\frac{S_{\lfloor sn\rfloor}}{\sigma\sqrt{n}}, s\leq\lambda\right)\mathbb{1}_{\underline{S}_m\geq-u}\right]o\left(\frac{R^-(b)}{n}\right), \tag{B.23}$$

where $(\rho^{1-\lambda}_{z,0})_{z\in\mathbb{R}_+}$ is independent of $(S_n)_{n\in\mathbb{N}}$. Then, since the function $z\in\mathbb{R}\mapsto\mathbb{E}[G_2(\rho^{1-\lambda}_{z,0})]$ is continuous, using Lemma 2.6 combined with Lemma A.3, we get that (B.23) is equal to

$$\frac{\theta^-}{\sigma(1-\lambda)n}hR^-(b)\frac{\theta R(u)}{\sqrt{m}}\left(\mathbb{E}\left[G_1(\mathcal{M})f\left(\frac{\mathcal{M}(\lambda)}{\sqrt{1-\lambda}}\right)G_2\left(\rho^{1-\lambda}_{\mathcal{M}(\lambda),0}\right)\right]+o(1)\right)+o\left(\frac{R^-(b)R(u)}{n^{3/2}}\right),$$

where $\mathcal{M}$ is the Brownian meander of length λ, independent of $(\rho^{1-\lambda}_{z,0})_{z\in\mathbb{R}_+}$. Finally, recalling the definition of f and using (B.21), it concludes the proof of Proposition 2.8. □

Acknowledgements

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