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SHARP THRESHOLDS FOR CONTAGIOUS SETS IN RANDOM GRAPHS

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For fixed $r \geq 2$, we consider bootstrap percolation with threshold r on the Erdős–Rényi graph $\mathcal{G}_{n,p}$. We identify a threshold for p above which there is with high probability a set of size r that can infect the entire graph. This improves a result of Feige, Krivelevich and Reichman, which gives bounds for this threshold, up to multiplicative constants.

As an application of our results, we obtain an upper bound for the threshold for K_4-percolation on $\mathcal{G}_{n,p}$, as studied by Balogh, Bollobás and Morris. This bound is shown to be asymptotically sharp in subsequent work.

These thresholds are closely related to the survival probabilities of certain time-varying branching processes, and we derive asymptotic formulae for these survival probabilities, which are of interest in their own right.

1. Introduction.

1.1. *Bootstrap percolation.* The *r-neighbour bootstrap percolation process* on a graph $G = (V, E)$ evolves as follows. Initially, some set $V_0 \subset V$ is *infected*. Subsequently, any vertex that has at least r infected neighbours becomes infected, and remains infected. Formally the process is defined by

$$V_{t+1} = V_t \cup \{v : |N(v) \cap V_t| \geq r\},$$

where $N(v)$ is the set of neighbours of a vertex v. The sets V_t are increasing, and so converge to some set V_∞ of eventually infected vertices. We denote the infected set by $\langle V_0, G\rangle_r = V_\infty$. A *contagious set* for G is a set $I \subset V$ such that if we put $V_0 = I$ then we have that $\langle I, G\rangle_r = V$, that is, the infection of I results in the infection of all vertices of G.

Bootstrap percolation was introduced by Chalupa, Leath and Reich [20] (see also [17, 46, 48, 55, 58]), in the context of statistical physics, for the study of disordered magnetic systems. Since then it has been applied diversely in physics and in other areas, including computer science, neural networks and sociology, see [1, 3, 21, 22, 26–28, 30–32, 42, 47, 53, 57, 59, 60] and further references therein.

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Special cases of r-bootstrap percolation have been analyzed extensively on finite grids and infinite lattices; see, for instance [2, 9, 10, 12, 15, 18, 19, 33, 34, 36, 37, 50] (and references therein). Other special graphs of interest have also been studied, including hypercubes and trees; see [8, 11, 14, 29]. Recent work has focused on the case of random graphs (see, e.g., [3, 4, 16, 38]), and in particular, on the Erdős–Rényi random graph $\mathcal{G}_{n,p}$; see Janson, Łuczak, Turova and Vallier [39] (and also [6, 7, 23, 35, 40, 41, 49, 52, 56] for related results).

The main questions of interest in this field revolve around the size of the eventual infected set V_∞. In most works, the object of study is the probability that a *random* initial set is contagious, and its dependence on the size of V_0. For example, in [39], Theorem 3.1, for all $r \geq 2$ and $n^{-1} \ll p \ll n^{-1/r}$, the critical size for a random contagious set in $\mathcal{G}_{n,p}$ (selected independently of $\mathcal{G}_{n,p}$) is identified as $\frac{r-1}{r}((r-1)!/(np^r))^{1/(r-1)}$.

More recently, and in contrast with the results of [39], Feige, Krivelevich and Reichman [25] study the *existence* of small contagious sets in $\mathcal{G}_{n,p}$. We call a graph *susceptible* (or say that it *r-percolates*) if it contains a contagious set of the smallest possible size r. In [25], Theorem 1.2, the threshold for p above which $\mathcal{G}_{n,p}$ is likely to be susceptible is approximated, up to multiplicative constants.

Our main result is that susceptibility of $\mathcal{G}_{n,p}$ exhibits a sharp threshold. Let $p_c(n,r)$ denote the infimum over $p > 0$ so that $\mathcal{G}_{n,p}$ is susceptible with probability at least $1/2$. We identify the asymptotic p at which the probability transitions from $o(1)$ to $1 - o(1)$.

THEOREM 1.1. *Fix $r \geq 2$ and $\alpha > 0$. Let*

$$p = p(n) = \left(\frac{\alpha}{n \log^{r-1} n}\right)^{1/r}$$

and denote

$$\alpha_r = (r-1)!\left(\frac{r-1}{r}\right)^{2(r-1)}.$$

If $\alpha > \alpha_r$, then with high probability $\mathcal{G}_{n,p}$ is susceptible. If $\alpha < \alpha_r$, then there exists $\beta = \beta(\alpha, r)$ so that with high probability for every set I of size r we have that $|\langle I, \mathcal{G}_{n,p}\rangle_r| \leq \beta \log n$.

COROLLARY 1.2. *With the above notation, as $n \to \infty$*

$$p_c(n,r) = \left(\frac{\alpha_r}{n \log^{r-1} n}\right)^{1/r}\big(1 + o(1)\big).$$

Moreover the same asymptotic holds if the $1/2$ in the definition of p_c is replaced with any constant in $(0,1)$.

Thus, r-bootstrap percolation undergoes a sharp transition. For small p sets of size r infect at most $O(\log n)$ vertices, whereas for larger p there are contagious sets of size r. We remark that for $\alpha < \alpha_r$, with high probability $\mathcal{G}_{n,p}$ has susceptible subgraphs of size $\Theta(\log n)$. Moreover, our methods identify the largest β so that there are susceptible subgraphs of size $\beta \log n$ (see Proposition 2.1 below).

In closing, we compare Theorem 1.1 and the work of Janson, Łuczak, Turova and Vallier mentioned above, specifically, Theorem 3.1 of [39]. We find that if $p = (\alpha/(n \log^{r-1} n))^{1/r}$, where $\alpha = (1+\delta)\alpha_r$ for some small $\delta > 0$, then with high probability $\mathcal{G}_{n,p}$ has a contagious set of size r, however by [39], a fixed set, or a random set selected independently of $\mathcal{G}_{n,p}$, is likely to be contagious only if it is (roughly) of size at least $\frac{r}{r-1} \log n$.

1.2. *Graph bootstrap percolation and seeds.* Let H be some finite graph. The H*-bootstrap percolation* model, introduced by Bollobás [17], is a rule for adding edges to a graph G. An edge is added whenever its addition creates a copy of H within G. Eventually no further edges can be added, and the process terminates. Informally, the process completes all copies of H that are missing a single edge. Formally, we let $G_0 = G$, and then G_{i+1} is G_i together with every edge whose addition creates a subgraph which is isomorphic to H. Note that these are not necessarily induced subgraphs, so having more edges in G can only increase the final result. Note that the vertex set of G is fixed, and no vertices play any special role.

For a finite graph G, this procedure terminates once $G_{\tau+1} = G_\tau$, for some $\tau = \tau(G)$. We denote the resulting graph G_τ by $\langle G \rangle_H$. If $\langle G \rangle_H$ is the complete graph on the vertex set V, the graph G is said to H*-percolate* (or that it is H-percolating). The case $H = K_4$ is the minimal case of interest. Indeed, *all* graphs K_2-percolate, and a graph K_3-percolates if and only if it is connected. Hence, by a classical result of Erdős and Rényi [24], $\mathcal{G}_{n,p}$ will K_3-percolate with high probability precisely for $p > n^{-1} \log n + \Theta(n^{-1})$.

The main focus of [13] is H-bootstrap percolation in the case that $G = \mathcal{G}_{n,p}$ and $H = K_k$, for some $k \geq 4$. The critical thresholds are defined as

$$p_c(n, H) = \inf\{p > 0 : \mathbb{P}(\langle \mathcal{G}_{n,p} \rangle_H = K_n) \geq 1/2\}.$$

It is expected that this property has a sharp threshold for $H = K_k$ for any k, in the sense that for some $p_c = p_c(k)$ we have that $\mathcal{G}_{n,p}$ is K_k-percolating with high probability for $p > (1+\delta)p_c$ and is K_k-percolating with probability tending to 0 for $p = (1-\delta)p_c$.

Some bounds for $p_c(n, K_k)$, $k \geq 4$, are obtained in [13]. Among the main results of [13] is that $p_c(n, K_4) = \Theta(1/\sqrt{n \log n})$. However, neither the lower nor upper bound there achieves the correct constant. We improve the upper bound for $p_c(n, K_4)$ given in [13].

THEOREM 1.3. *Let* $p = \sqrt{\alpha/(n \log n)}$. *If* $\alpha > 1/3$, *then* $\mathcal{G}_{n,p}$ *is* K_4-*percolating with high probability. In particular as* $n \to \infty$, *we have that*

$$p_c(n, K_4) \le \frac{1 + o(1)}{\sqrt{3n \log n}}.$$

This upper bound is shown to be asymptotically sharp in subsequent work by Angel and Kolesnik [5] and Kolesnik [43], thereby identifying the asymptotic threshold for K_4-percolation.

One way for a graph G to K_{r+2}-percolate is if there is some ordering of the vertices so that vertices $1, \dots, r$ form a clique, and every other vertex is connected to at least r of the previous vertices according to the order. In this case, we call the clique formed by the first r vertices a *seed* for G. When $r = 2$, the seed is a clique of size 2, so we call it a *seed edge*.

LEMMA 1.4. *Fix* $r \ge 2$. *If* G *has a seed for* K_{r+2}-*bootstrap percolation, then* $\langle G \rangle_{K_{r+2}} = K_n$.

PROOF. We prove by induction that for $k \ge r$ the subgraph induced by the first k vertices percolates. For $k = r$, the definition of a seed implies that the subgraph is complete. Given that the first $k - 1$ vertices span a percolating graph, some number of steps will add all edges among them. Finally, vertex k has r neighbours among these, and so every edge between vertex k and a previous vertex can also be added by K_{r+2}-bootstrap percolation. □

In light of this, Theorem 1.3 above is a direct corollary of the following result.

THEOREM 1.5. *Let* $p = \sqrt{\alpha/(n \log n)}$. *As* $n \to \infty$, *the probability that* $\mathcal{G}_{n,p}$ *has a seed edge tends to* 1 *if* $\alpha > 1/3$ *and tends to* 0 *if* $\alpha < 1/3$.

The case of K_4-bootstrap percolation, corresponding to $r = 2$, appears to be special: It is reasonable to expect that the existence of a seed edge is the easiest way for a graph to K_4-percolate. This is similar to other situations where a threshold of interest on $\mathcal{G}_{n,p}$ coincides with that of a more fundamental event. For instance, with high probability, $\mathcal{G}_{n,p}$ is connected if and only if it has no isolated vertices (see [24]); $\mathcal{G}_{n,p}$ contains a Hamiltonian cycle if and only if the minimal degree is at least 2 (Komlós and Szemerédi [44]).

We do not expect the converse to Lemma 1.4 to hold. More precisely, if we increase p continuously and consider $\mathcal{G}_{n,p}$ with the natural coupling, there is a probability bounded from 0 that there is some time at which $\mathcal{G}_{n,p}$ will not have a seed edge, and yet will K_4-percolate. The reason for this is that there are other small structures that can take the place of a seed. We do believe that such structures are typically very small, since otherwise, there are at least two large structures within $\mathcal{G}_{n,p}$ that K_4-percolate independently. Since $p_c \to 0$, having multiple large percolating structures within $\mathcal{G}_{n,p}$ is less likely.

For $r > 2$, having a seed is no longer the easiest way for a graph to K_4-percolate. Indeed, by [13], the critical probability for K_{r+2}-bootstrap percolation is $n^{-(2r)/(r^2+3r-2)}$ up to (unknown) polylogarithmic factors (note that r in [13] is $r+2$ here). The threshold for having a seed is of order $n^{-1/r}(\log n)^{1/r-1}$, which is much larger (see Theorem 5.1 below).

1.3. *A nonhomogeneous branching process.* Given an edge $e = (x_0, x_1)$, we can explore the graph to determine if it is a seed edge. The number of vertices that are connected to both of its endpoints is roughly Poisson with mean np^2. In our context, the interesting p are $o(n^{-1/2})$, and therefore the number of such vertices has small mean, which we denote by $\varepsilon = np^2$. If there are any such vertices, denote them $x_2, \dots$. We then seek vertices connected to x_2 and at least one of x_0, x_1. The number of such vertices is roughly $\mathrm{Poi}(2\varepsilon)$. Indeed, the number of vertices connected to the kth vertex and at least one of the previous vertices is (approximately) $\mathrm{Poi}(k\varepsilon)$.

This leads us to the case $r = 2$ of the following nonhomogeneous branching process defined by parameters $r \in \mathbb{N}$ and $\varepsilon > 0$. The process starts with a single individual. The first $r-2$ individuals have precisely one child each. For $k \geq r-1$, the kth individual has a Poisson number of children with mean $\binom{k}{r-1}\varepsilon$, where here $\varepsilon = np^r$. Thus for $r = 2$ the kth individual has a mean of $k\varepsilon$ children. The process may die out (e.g., if individual $r-1$ has no children). However, if the process survives long enough the mean number of children exceeds one and the process becomes supercritical. Thus, the probability of survival is strictly between 0 and 1. Formally, this may be defined in terms of independent random variables $Z_k = \mathrm{Poi}(\binom{k}{r-1}\varepsilon)$ by $X_t = \sum_{k=r-1}^{t} Z_k - 1$. Survival is the event $\{X_t \geq 0, \forall t\}$.

THEOREM 1.6. *As $\varepsilon \to 0$, we have that*

$$\mathbb{P}(X_t > 0, \forall t) = \exp\left[-\frac{(r-1)^2}{r} k_r \big(1 + o(1)\big)\right],$$

where

$$k_r = k_r(\varepsilon) = \left(\frac{(r-1)!}{\varepsilon}\right)^{1/(r-1)}.$$

Note that $\varepsilon\binom{k_r}{r-1} \approx 1$. Hence, k_r is roughly the time at which the process becomes supercritical.

This process is closely related to the *binomial chain* representation of the r-bootstrap percolation dynamics, as utilized in [39], Chapter 10 (see also Scalia-Tomba [49] and Sellke [51]). In [39], Theorem 3.8, central limit theorems are established that describe the typical number of vertices eventually infected by small sets in $\mathcal{G}_{n,p}$ that are unlikely to be contagious. On the other hand, Theorem 1.6 is related to the event that $\mathcal{G}_{n,p}$ (where $\varepsilon = np^r$) has a contagious set of (the smallest possible) size r. In recent work [5], we identify the large deviations rate function corresponding to the event that a set of size ℓ in $\mathcal{G}_{n,p}$ eventually infects at least k vertices. These results, in the special case that $\ell/k_r \ll 1$ and $k \geq k_r$, imply Theorem 1.6.

1.4. *Outline of the proof.* In Section 2, we obtain a recurrence (2.1) for the number of graphs which r-percolate with the minimal number of edges. Using this, we estimate the asymptotics of such graphs, and thereby identify a quantity $\beta_*(\alpha)$, so that for $\alpha < \alpha_r$ (and p as in Theorem 1.1), with high probability no r-percolation (i.e., the r-bootstrap percolation process initialized by a set of r vertices) on $\mathcal{G}_{n,p}$ grows to size $\beta \log n$, for any $\beta \geq \beta_*(\alpha) + \delta$. Let $\beta_r(\alpha) = k_r(np^r)/\log n$, where $k_r = k_r(\varepsilon)$ is as defined in Section 1.3. Moreover, we find that $\beta_*(\alpha) < \beta_r(\alpha)$ for $\alpha < \alpha_r$, and that $\beta_*(\alpha) = \beta_r(\alpha)$ if $\alpha = \alpha_r$, suggesting that α_r is indeed the critical value of α.

In Section 3, we show by the second moment method that, if $\alpha > \alpha_r$, then $\mathcal{G}_{n,p}$ r-percolates with high probability. The main difficulty towards establishing this fact is that contagious sets are far from independent. One way to see (very roughly) that this is the case is as follows: For supercritical $\alpha > \alpha_r$, it is reasonable to presume that the expected number of contagious sets of size r is approximately n^{μ}, for some $\mu(\alpha) \downarrow 0$ as $\alpha \downarrow \alpha_r$. Let $r = 2$ (the cases $r > 2$ are similar), and suppose that some pair x, y infects a set V containing $\beta \log n$ vertices. Let x', y' be some other pair, such that $\{x, y\} \cap \{x', y'\} = \varnothing$. One way that x', y' can infect a set V' of size $\beta \log n$ is by first infecting some set V_1 where $|V \cap V_1| = 2$, and then infecting some $V_2 \subset V - V_1$ such that $|V_1 \cup V_2| = \beta \log n$. Note that this only implies the existence of at least three edges in $\mathcal{G}_{n,p}$ with at most one endpoint in V. To see this, observe that the first infected vertex $u \in V \cap V_1$ necessarily has at least two neighbours not in V, however the second vertex infected $v \neq u \in V \cap V_1$ may only have one such neighbour if $(u, v) \in E(\mathcal{G}_{n,p})$. As a result, it is perhaps not straightforward to obtain an upper bound for the conditional probability that x', y' infects $\beta \log n$ vertices, given that x, y infects $\beta \log n$ vertices, that is much smaller than p^3. Since there are $O(n^2)$ such pairs x', y', and since $p = \sqrt{\alpha/(n \log n)}$ (when $r = 2$ and p is close to p_c), it would appear that correlations are too high for a simple application of the second moment method.

To overcome this difficultly, we observe that if x', y' infects some set $V' = V_1 \cup V_2$ as above, and moreover $|V \cap V'| > 2$, then either (i) the second and third vertices $v, w \in V \cap V'$ that are infected (after the first vertex $u \in V \cap V'$, with two neighbours not in V, is infected) both have a neighbour not in V, thus giving a total of at least four edges in $\mathcal{G}_{n,p}$ with at least one endpoint not in V, or else (ii) the vertices u, v, w induce a triangle. For this reason, we instead consider contagious sets which infect *triangle-free* subgraphs of $\mathcal{G}_{n,p}$. To give some intuition for why this restriction should not effect the threshold (up to smaller order terms), note that the threshold p_c' for the existence of a contagious set of size r that induces a graph with at least one edge is much larger, $p_c' \gg p_c$. Therefore, although for p close to p_c there are many triangles in $\mathcal{G}_{n,p}$, we do not expect $\mathcal{G}_{n,p}$ to require a triangle in order to infect at large set of size $\beta \log n$.

More specifically, we modify the recurrence (2.1) to obtain a recursive lower bound for graphs which r-percolate without using triangles, and show that this restriction does not significantly effect the asymptotics. Using Mantel's theorem [45] we establish the approximate independence of correspondingly restricted r-percolations, which we call $\hat{r}$-percolations, with relative ease.

A secondary obstacle is the need for a lower bound for the asymptotics of graphs which $\hat{r}$-percolate, with a significant proportion of vertices in the *top level* [i.e., vertices v of a triangle-free, r-percolating graph $G = (V, E)$ such that $v \in V_t - V_{t-1}$ where $V_t = V$]. Such bounds are required to estimate the growth of supercritical $\hat{r}$-percolations on $\mathcal{G}_{n,p}$, which have grown larger than the critical size $\beta_r(\alpha) \log n$ (discussed in the first paragraph of this section). Using a lower bound for the overall number of graphs which $\hat{r}$-percolate, we obtain a lower bound for the number of such graphs with $i = \Omega(k)$ vertices in the top level. This estimate, together with the approximate independence of $\hat{r}$-percolations, is sufficient to show that with high probability $\mathcal{G}_{n,p}$ has subgraphs of size $\beta \log n$ which r-percolate, for some $\beta \geq \beta_*(\alpha) + \delta$ [where $\beta_*(\alpha) > \beta_r(\alpha)$, for $\alpha > \alpha_r$].

Finally, to conclude, we show by the first moment method that for any given $A > 0$, with high probability an r-percolation which survives to size $(\beta_*(\alpha) + \delta) \log n$ will continue to survive to size $A \log n$. Having established the existence of an r-percolating subgraph of $\mathcal{G}_{n,p}$ of size $A \log n$, for a sufficiently large value of A (depending on the difference $\alpha - \alpha_r$), it is straightforward (by sprinkling) to show that with high probability $\mathcal{G}_{n,p}$ is susceptible.

2. Lower bound for $p_c(n, r)$.

In this section we prove the subcritical case of Theorem 1.1, by the first moment method. Throughout this section, we fix some $r \geq 2$. More precisely, we prove the following proposition.

PROPOSITION 2.1. *Let*

$$\alpha_r = (r-1)!\left(\frac{r-1}{r}\right)^{2(r-1)}, \qquad p = \vartheta_r(\alpha, n) = \left(\frac{\alpha}{n \log^{r-1} n}\right)^{1/r}.$$

Define $\beta_*(\alpha)$ to be the unique positive root of

$$r + \beta \log\left(\frac{\alpha \beta^{r-1}}{(r-1)!}\right) - \frac{\alpha \beta^r}{r!} - \beta(r-2).$$

For any $\alpha < \alpha_r$ and $\delta > 0$, with high probability, for every $I \subset [n]$ of size r, we have that $|\langle I, \mathcal{G}_{n,p} \rangle_r| \leq (\beta_*(\alpha) + \delta) \log n$.

The methods of Section 3 can be used to show that with high probability there are sets I of size r which infect $(\beta_* - \delta)\log n$ vertices. Hence, for $\alpha < \alpha_r$, with high probability the maximum of $|\langle I, \mathcal{G}_{n,p}\rangle_r|$ over sets I of size r is equal to $(\beta_* + o(1))\log n$.

For $\alpha < \alpha_r$, we have (see Lemma 2.11) the following upper bound:

$$\beta_*(\alpha) \le \left(\frac{(r-1)!}{\alpha}\right)^{1/(r-1)}.$$

[In fact, it can be shown by elementary calculus that α can be replaced with α_r on the right-hand side, resulting in the slightly improved upper bound of $\beta_*(\alpha) < (r/(r-1))^2$.] This is asymptotically optimal for $\alpha \sim \alpha_r$.

We note here that Proposition 2.1 can alternatively be established using the large deviation estimates developed in our subsequent work [5]. These two approaches are completely different, and so perhaps of independent interest: In this work, Proposition 2.1 is proved by analyzing the combinatorics of susceptible graphs directly, whereas the key result in [5], Theorem 3.2, (from which, in the special case of $\varepsilon = 0$, Proposition 2.1 follows) is proved using variational calculus to identify optimal trajectories of a branching process related to that discussed in Section 1.3.

2.1. *Small susceptible graphs.* As noted in the Introduction, a key idea is to study the number of subgraphs of size $k = \Theta(\log n)$ which are susceptible with the minimal number of edges. If none exist, then there can be no contagious set in G. Thus, an important step is developing estimates for the number of such susceptible graphs of size k.

For a graph G and initial infected set V_0, recall that $V_t = V_t(V_0, G)$ is the set of vertices infected up to and including step t. We let $\tau = \inf\{t : V_t = V_{t+1}\}$. We put $I_0 = V_0$ and $I_t = V_t - V_{t-1}$, for $t \ge 1$. We refer to I_t as the set of vertices infected in *level* t. In particular, the *top level* of a susceptible graph G (for a given V_0) is I_τ.

For a graph G, we let $V(G)$ and $E(G)$ denote its vertex and edge sets, and put $|G| = |V(G)|$.

We call a graph *minimally susceptible* if it is susceptible and has exactly $r(|G| - r)$ edges. If a graph G is susceptible, it has at least $r(|G| - r)$ edges, since each vertex in I_t, $t \ge 1$, is connected to r vertices in V_{t-1}.

For $k \in \mathbb{N}$, let $[k] = \{1, 2, \dots, k\}$.

DEFINITION 2.2. Let $m_r(k)$ denote the number of minimally susceptible graphs G with vertex set $[k]$ such that $[r]$ is a contagious set for G. Let $m_r(k,i)$ denote the number of such graphs with i vertices infected in the top level. Hence, $m_r(k) = \sum_{i=1}^{k-r} m_r(k,i)$.

We note that $m_r(k, k-r) = 1$, and claim that for $i < k - r$,

$$m_r(k,i) = \binom{k-r}{i} \sum_{j=1}^{k-r-i} a_r(k-i,j)^i m_r(k-i,j), \tag{2.1}$$

where

$$a_r(x, y) = \binom{x}{r} - \binom{x-y}{r}. \tag{2.2}$$

To see this, note that removing the top level from a minimally susceptible graph G of size k leaves a minimally susceptible graph G' of size $k-i$. If the top level of G' has size j, then all vertices in the top level of G are connected to r vertices of G', with at least one in the top level of G'. Thus, each vertex has $a_r(k-i, j)$ options for the connections. The $\binom{k-r}{i}$ term accounts for the set of possible labels of the top level of G.

To study asymptotics of m, it is convenient to define

$$\sigma_r(k, i) = \frac{m_r(k, i)}{(k-r)!}\left(\frac{(r-1)!}{k^{r-1}}\right)^k. \tag{2.3}$$

Substituting this in (2.1) gives

$$\sigma_r(k, i) = \sum_{j=1}^{k-r-i} A_r(k, i, j)\sigma_r(k-i, j) \qquad \text{for } i < k-r, \tag{2.4}$$

where

$$A_r(k, i, j) = \frac{j^i}{i!}\left(\frac{k-i}{k}\right)^{(r-1)k}\left(\frac{(r-1)!}{(k-i)^{r-1}}\frac{a_r(k-i, j)}{j}\right)^i. \tag{2.5}$$

We make the following observation.

LEMMA 2.3. *Let $A_r(k, i, j)$ be as in (2.5) and put $A_r(i, j) = j^i e^{-(r-1)i}/i!$. For any $i < k-r$ and $j \le k-r-i$, we have that $A_r(k, i, j)$ is increasing in k and converges to $A_r(i, j)$.*

PROOF. It is well known that for $m > 0$ we have $(1-m/k)^k$ is increasing and tends to e^{-m}. Thus,

$$\frac{j^i}{i!}\left(\frac{k-i}{k}\right)^{(r-1)k} \to A_r(i, j).$$

The lemma follows by (2.5) and the following claim, a formula which will also be of later use.

CLAIM 2.4. *For all integers $x \ge r$ and $1 \le y \le x-r$, we have that*

$$\frac{(r-1)!}{x^{r-1}}\frac{a_r(x, y)}{y} = \frac{1}{y}\sum_{\ell=1}^{y}\left(\frac{x-\ell}{x}\right)^{r-1}.$$

PROOF. For an integer $m \geq r$, let $(m)_r = m!/(m-r)!$ denote the rth falling factorial of the integer m. Since

$$(m)_r - (m-1)_r = r(m-1)^{r-1}$$

it follows that

$$\frac{(r-1)!}{x^{r-1}} \frac{a_r(x,y)}{y} = \frac{(x)_r - (x-y)_r}{ryx^{r-1}} = \frac{1}{y} \sum_{\ell=1}^{y} \left(\frac{x-\ell}{x} \right)^{r-1}$$

as required. $\square$

Since each term on the right-hand side of Claim 2.4 is increasing to 1, the same holds for their average. The proof is complete. $\square$

2.2. *Upper bounds for susceptible graphs*. Our first task is to derive estimates for the number of minimally susceptible graphs of size k with i vertices in the top level. This relies on the recurrence (2.1).

LEMMA 2.5. *Fix $r \geq 2$. For all $k > r$ and $i \leq k - r$, we have that*

$$m_r(k,i) \leq \frac{e^{-i-(r-2)k}}{\sqrt{i}} (k-r)! \left(\frac{k^{r-1}}{(r-1)!} \right)^k.$$

Equivalently, $\sigma_r(k,i) \leq i^{-1/2} e^{-i-(r-2)k}$.

PROOF. Since $m_r(k, k-r) = 1$, it is straightforward to verify that the claim holds in the case that $i = k - r$. For the remaining cases $i < k - r$, we prove the claim by induction on k. Applying the inductive hypothesis to the right-hand sum of (2.4), bounding $A_r(k,i,j)$ therein by $A_r(i,j)$ using Lemma 2.3, and extending the sum to all j, we find that

$$\sigma_r(k,i) \leq \sum_{j=1}^{\infty} A_r(i,j) j^{-1/2} e^{-j-(r-2)(k-i)}.$$

Thus it suffices to prove that this sum is at most $i^{-1/2} e^{-i-(r-2)k}$. Using the definition of $A_r(i,j)$ and cancelling the $e^{-(2-r)k}$ factors, we need the following claim.

CLAIM 2.6. *For any $i \geq 1$ we have*

$$\sum_{j=1}^{\infty} \frac{j^i e^{-i}}{i!} j^{-1/2} e^{-j} \leq i^{-1/2} e^{-i}.$$

This is proved in Appendix A.1.

We remark that Claim 2.6 is fundamentally a pointwise bound for the Perron eigenvector of the infinite operator A_2. (Other values of r follow since the influence of r cancels out.) This eigenvector decays roughly as e^{-i}, but with some lower order fluctuations. It appears that the $\sqrt{i}$ correction can be replaced by various other slowly growing functions of i. However, Claim 2.6 fails for certain i without the $\sqrt{j}$ term. $\square$

2.3. *Susceptible subgraphs of* $\mathcal{G}_{n,p}$. With Lemma 2.5 at hand, we obtain upper bounds for the growth probabilities of r-percolations on $\mathcal{G}_{n,p}$.

DEFINITION 2.7. A set I of size r is called *k-contagious* in the graph $\mathcal{G}_{n,p}$ if there is some t so that $|V_t(I, \mathcal{G}_{n,p})| = k$, that is, there is some time at which there are exactly k infected vertices. The set I is called (k, i)-contagious if in addition the number of vertices infected at step t is i, that is, $|I_t(I, \mathcal{G}_{n,p})| = i$.

DEFINITION 2.8. Let $P_r(k, i) = P_r(p, k, i)$ denote the probability that a given $I \subset [n]$, with $|I| = r$, is (k, i)-contagious. Let $P_r(k) = \sum_i P_r(k, i)$ denote the probability that such an I is k-contagious. Finally, let $E_r(k, i)$ and $E_r(k)$ denote the expected number of such subsets I.

We remark that $P_r(k)$ is not the same as the probability of *survival* to size k, which is given by $\sum_{\ell \ge k} \sum_{i > \ell - k} P_r(\ell, i)$.

LEMMA 2.9. *Let $\alpha > 0$, and let $p = \vartheta_r(\alpha, n)$ (as defined in Proposition 2.1) and $\varepsilon = np^r = \alpha / \log^{r-1} n$. For $i \le k - r$ and $k \le n^{1/(r(r+1))}$, we have that*

$$P_r(k, i) \le \bigl(1 + o(1)\bigr) \frac{e^{-\varepsilon \binom{k-i}{r}} \varepsilon^{k-r}}{(k-r)!} m_r(k, i),$$

where $o(1)$ depends on n, but not on i, k.

PROOF. Let $I \subset [n]$, with $|I| = r$, be given, and put

$$\ell_r(k, i) = \frac{e^{-\varepsilon \binom{k-i}{r}} \varepsilon^{k-r}}{(k-r)!} m_r(k, i)$$

so that the lemma states that $P_r(k, i) \le (1 + o(1)) \ell_r(k, i)$. This follows by a union bound: If I is (k, i)-contagious, then I is a contagious set for a minimally susceptible subgraph $G \subset \mathcal{G}_{n,p}$ (perhaps not induced) of size k with i vertices infected in the top level, and all vertices in $v \in V(G)^c$ are connected to at most $r - 1$ vertices below the top level of G [so that $V(G) = V_t(I, \mathcal{G}_{n,p})$, for some t]. There are $\binom{n}{k-r}$ choices for the vertices of G and $m_r(k, i)$ choices for its edges. For any such v

and G, the probability that v is connected to r vertices below the top level of G is bounded from below by

$$\binom{k-i}{r} p^r (1-p)^{k-i-r} > \binom{k-i}{r} p^r (1-p)^k.$$

Hence,

$$P_r(k,i) < \binom{n}{k-r} m_r(k,i) p^{r(k-r)} \left(1 - \binom{k-i}{r} p^r (1-p)^k\right)^{n-k}.$$

By the inequalities $\binom{n}{k} \le n^k/k!$ and $1 - x < e^{-x}$, it follows that

$$\log \frac{P_r(k,i)}{\ell_r(k,i)} < \varepsilon \binom{k-i}{r} \left(1 - (1-p)^k \left(1 - \frac{k}{n}\right)\right).$$

By the inequality $(1-x)^y \ge 1 - xy$, and since $k \le n^{1/(r(r+1))}$, the right-hand side is bounded by

$$\varepsilon k^{r+1}\bigl(p + (1-pk)/n\bigr) \le \varepsilon n^{1/r}(p + 1/n) \ll 1$$

as $n \to \infty$. Hence, $P_r(k,i) \le (1+o(1))\ell_r(k,i)$, as claimed. □

As a corollary, we obtain a bound for $E_r(k,i)$.

LEMMA 2.10. *Let $\alpha, \beta_0 > 0$. Put $p = \vartheta_r(\alpha, n)$. For all $k = \beta \log n$ and $i = \gamma k$, such that $\beta \le \beta_0$, we have that*

$$E_r(k,i) \lesssim n^{\mu} \log^{r(r-1)} n,$$

where

$$\mu = \mu_r(\alpha,\beta,\gamma) = r + \beta \log\left(\frac{\alpha\beta^{r-1}}{(r-1)!}\right) - \frac{\alpha\beta^r}{r!}(1-\gamma)^r - \beta(r-2+\gamma). \tag{2.6}$$

Here $\lesssim$ denotes inequality up to a constant depending on α, β_0, but not on β, γ.

PROOF. Let $r \ge 2$ and $\alpha, \beta_0 > 0$ be given. Put $\varepsilon = np^r$. By Lemmas 2.5 and 2.9, for all $k = \beta \log n$ and $i = \gamma k$, with $\beta \le \beta_0$, we have that $E_r(k,i)$ is bounded from above by

$$(1+o(1))\binom{n}{r}\left(\frac{\varepsilon k^{r-1}}{(r-1)!}\right)^k \varepsilon^{-r} e^{-i-(r-2)k-\varepsilon\binom{k-i}{r}} \lesssim n^{\mu} \log^{r(r-1)} n.$$

The $\sqrt{i}$ term from Lemma 2.5 is safely dropped for this upper bound. □

2.4. *Subcritical bounds*. In this section we prove Proposition 2.1.

The case of $\gamma = 0$ in Lemma 2.10 (corresponding to values of i such that $i/k \ll 1$) is of particular importance for the growth of subcritical r-percolations. For this reason, we set $\mu_r^*(\alpha, \beta) = \mu_r(\alpha, \beta, 0)$. The next result in particular shows that $\beta_*(\alpha)$, as in Proposition 2.1, is well defined.

LEMMA 2.11. *Let $\alpha > 0$. Let α_r be as defined in Proposition 2.1. Put*

$$\beta_r(\alpha) = \left(\frac{(r-1)!}{\alpha}\right)^{1/(r-1)}.$$

(i) *The function $\mu_r^*(\alpha, \beta)$ is decreasing in β, with a unique zero at $\beta_*(\alpha)$.*

(ii) *We have that*

$$\mu_r^*(\alpha, \beta_r(\alpha)) = r - \beta_r(\alpha)\frac{(r-1)^2}{r}$$

and hence $\beta_*(\alpha) = \beta_r(\alpha)$ (resp. $>$ or $<$) if $\alpha = \alpha_r$ (resp. $>$ or $<$).

The quantity $\beta_*(\alpha)$ also plays a crucial role in analyzing the growth of supercritical r-percolations on $\mathcal{G}_{n,p}$; see Section 3.5 below.

PROOF OF LEMMA 2.11. For the first claim, we note that by setting $\gamma = 0$ in (2.6) we obtain

$$\mu_r^*(\alpha, \beta) = r + \beta \log\left(\frac{\alpha\beta^{r-1}}{(r-1)!}\right) - \frac{\alpha\beta^r}{r!} - \beta(r-2). \tag{2.7}$$

Therefore,

$$\frac{\partial}{\partial\beta}\mu_r^*(\alpha, \beta) = 1 + \log\left(\frac{\alpha\beta^{r-1}}{(r-1)!}\right) - \frac{\alpha\beta^{r-1}}{(r-1)!}.$$

Since $\alpha\beta_r(\alpha)^{r-1}/(r-1)! = 1$, the above expression is equal to 0 at $\beta = \beta_r(\alpha)$ and negative for all other $\beta > 0$. Hence, $\mu_*(\alpha, \beta)$ is decreasing in β, as claimed. Moreover, since $\lim_{\beta\to 0^+} \mu_r^*(\alpha, \beta) = r$ and $\lim_{\beta\to\infty} \mu_r^*(\alpha, \beta) = -\infty$, $\beta_*(\alpha)$ is well defined.

We obtain the expression for $\mu_r^*(\alpha, \beta_r(\alpha))$ in the second claim by (2.7) and the equality $\alpha\beta_r(\alpha)^{r-1}/(r-1)! = 1$. The conclusion of the claim thus follows by the first claim, noting that $\beta_r(\alpha)$ is decreasing in α and $\mu_r^*(\alpha_r, \beta_r(\alpha_r)) = 0$ since $\beta_r(\alpha_r) = (r/(r-1))^2$. □

We are ready to prove the main result of this section.

PROOF OF PROPOSITION 2.1. Let $\alpha < \alpha_r$ and $\delta > 0$ be given. First, we show that with high probability $\mathcal{G}_{n,p}$ contains no m-contagious set, for $m = \beta \log n$ with $\beta \in [\beta_*(\alpha) + \delta, \beta_r(\alpha)]$. To this end, we make the following claim.

CLAIM 2.12. *For all $\beta \le \beta_r(\alpha)$, we have that $\mu_r(\alpha, \beta, \gamma) \le \mu_r^*(\alpha, \beta)$.*

PROOF. By (2.6), we have that

$$\frac{\partial}{\partial \gamma}\mu_r(\alpha, \beta, \gamma) = -\beta\left(1 - \frac{\alpha\beta^{r-1}}{(r-1)!}(1-\gamma)^{r-1}\right).$$

Therefore, for any fixed $\beta \le \beta_r(\alpha)$, we find that $\mu_r(\alpha, \beta, \gamma)$ is decreasing in $\gamma < 1$, and the claim follows, recalling that $\mu_r^*(\alpha, \beta) = \mu_r(\alpha, \beta, 0)$. □

By Lemmas 2.10, 2.11 and 2.12, we find by summing over all $O(\log^2 n)$ relevant values of k and i that the probability that such a m-contagious set exists is bounded (up to a multiplicative constant) by

$$n^{\mu_*(\alpha, \beta_*(\alpha)+\delta)} \log^{r(r-1)+2} n \ll 1.$$

It remains to show that with high probability $\mathcal{G}_{n,p}$ has no m-contagious set I, for some $m \ge \beta_r \log n$. Towards this, note that if such a set I exists, then there is some t so that

$$|V_t(I, \mathcal{G}_{n,p})| < \beta_r \log n \le |V_{t+1}(I, \mathcal{G}_{n,p})|.$$

Letting $k = |V_t(I, \mathcal{G}_{n,p})|$, we find that for some $k < \beta_r \log n$ there is a k-contagious set I, and $m - k$ further vertices with r neighbours in $V_t(I, \mathcal{G}_{n,p})$.

The expected number of k-contagious sets with i vertices infected in the top level is $E_r(k, i)$. Let $p_r(k, i)$ be the probability that for a given set of size k with i vertices identified as the top level, there are at least $\beta_r \log n - k$ vertices with at least r neighbours in the set, with at least one neighbour in the top level. Hence, the probability that $\mathcal{G}_{n,p}$ has a m-contagious set I, for some $m \ge \beta_r \log n$, is at most

$$\sum_{i<k<\beta_r(\alpha)\log n} E_r(k, i) p_r(k, i).$$

The proposition now follows by the following claim, proved in Appendix A.2.

CLAIM 2.13. *For all $k < \beta_r(\alpha) \log n$ and $i \le k - r$, we have that*

$$E_r(k, i) p_r(k, i) \lesssim n^{\mu_r^*(\alpha, \beta_r(\alpha))} \log^{r(r-1)} n,$$

where $\lesssim$ denotes inequality up to a constant, independent of i, k.

Indeed, by Claim 2.13, it follows by summing over all $O(\log^2 n)$ relevant i, k that the probability that $\mathcal{G}_{n,p}$ has an m-contagious set for some $m \ge \beta_r(\alpha) \log n$ is bounded (up to a constant) by

$$n^{\mu_r^*(\alpha, \beta_r)} \log^{r(r-1)+2} n \ll 1,$$

where the last inequality follows by Lemma 2.11, since $\alpha < \alpha_r$ and hence $\mu_r^*(\alpha, \beta_r(\alpha)) < 0$. □

3. Upper bound for $p_c(n,r)$. In this section we prove Theorem 1.1. In light of Proposition 2.1, it remains to prove that for $\alpha > \alpha_r$, with high probability $\mathcal{G}_{n,p}$ is susceptible. Fundamentally this is done using the second moment method. As discussed in Section 1.4, the main obstacle is that contagious sets are not sufficiently independent for a straightforward application of the second moment method. To this end, we restrict to a special type of contagious sets, which infect k vertices with no triangles.

As in the previous section, we fix $r \geq 2$ throughout.

3.1. *Triangle-free susceptible graphs.* Recall that a graph is *triangle-free* if it contains no subgraph which is isomorphic to K_3.

DEFINITION 3.1. Let $\hat{m}_r(k,i)$ denote the number of triangle-free graphs that contribute to $m_r(k,i)$ (see Section 2.1). Put $\hat{m}_r(k) = \sum_{i=1}^{k-r} \hat{m}_r(k,i)$.

Following Section 2.1, we obtain a recursive lower bound for $\hat{m}_r(k,i)$. We note that $\hat{m}_r(k,k-r) = m_r(k,k-r) = 1$. For $i < k-r$, we claim that

$$\hat{m}_r(k,i) \geq \binom{k-r}{i} \sum_{j=1}^{k-r-i} \hat{a}_r(k-i,j)^i \hat{m}_r(k-i,j), \tag{3.1}$$

where

$$\hat{a}_r(x,y) = \max\{0, a_r(x,y) - 2ryx^{r-2}\}. \tag{3.2}$$

Note that [in contrast to the recursion for $m(k,i)$], this is only a lower bound. To see (3.1), we argue that of the $a_r(k-i,j)$ ways to connect a vertex in the top level to lower levels, at most $2rj(k-i)^{r-2}$ create a triangle. This is so since the number of ways of choosing r vertices from $k-i$, including at least one of the top j and including at least one edge, is at most

$$jr\binom{k-i-2}{r-2} + jr(k-i-r)\binom{k-i-3}{r-3} < 2jr(k-i)^{r-2},$$

where the first (resp. second) term accounts for the case that an edge selected contains (resp. does not contain) a vertex among the top j.

Setting

$$\hat{\sigma}_r(k,i) = \frac{\hat{m}_r(k,i)}{(k-r)!}\left(\frac{(r-1)!}{k^{r-1}}\right)^k,$$

(3.1) reduces to

$$\hat{\sigma}_r(k,i) \geq \sum_{j=1}^{k-r-i} \hat{A}_r(k,i,j)\hat{\sigma}_r(k-i,j), \tag{3.3}$$

where

$$\hat{A}_r(k,i,j) = \frac{j^i}{i!}\left(\frac{k-i}{k}\right)^{(r-1)k}\left(\frac{(r-1)!}{(k-i)^{r-1}}\frac{\hat{a}_r(k-i,j)}{j}\right)^i. \tag{3.4}$$

The following observation indicates that restricting to susceptible graphs which are triangle-free does not have a significant effect on the asymptotics.

LEMMA 3.2. *Let $\hat{A}_r(k,i,j)$ be as in (3.4) and let $A_r(i,j)$ be as defined in Lemma 2.3. For any fixed $i, j \geq 1$, we have that $\hat{A}_r(k,i,j) \to A_r(i,j)$, as $k \to \infty$.*

PROOF. Fix $i, j \geq 1$. From their definitions we have that

$$\frac{\hat{A}_r(k,i,j)}{A_r(k,i,j)} = \left(\frac{\hat{a}_r(k-i,j)}{a_r(k-i,j)}\right)^i.$$

Since $a_r(k-i,j)$ is of order k^{r-1} and $\hat{a}_r(k-i,j) - a_r(k-i,j)$ is of order k^{r-2} (for fixed i, j), we have that $\hat{a}_r(k-i,j)/a_r(k-i,j) \to 1$, as $k \to \infty$. Since i is fixed, it follows by Lemma 2.3 that

$$\lim_{k\to\infty} \hat{A}_r(k,i,j) = \lim_{k\to\infty} A_r(k,i,j) = A_r(i,j). \qquad \square$$

In order to obtain asymptotic lower bounds for $\hat{m}_r(k,i)$ it is useful to further restrict to graphs with bounded level sizes.

DEFINITION 3.3. For $\ell \geq r$, let $\hat{m}_{r,\ell}(k) \leq \hat{m}_r(k)$ be the number of graphs that contribute to $\hat{m}_r(k)$ which have level sizes bounded by ℓ (i.e., $|I_i| \leq \ell$ for all i). Let $\hat{m}_{r,\ell}(k,i)$ be the number of such graphs with exactly $i \leq \ell$ vertices in the top level. Hence, $\hat{m}_{r,\ell}(k) = \sum_{i=1}^{\ell} \hat{m}_{r,\ell}(k,i)$.

Observe that for fixed k, $\hat{m}_{r,\ell}(k)$ is increasing in ℓ, and equals $m_r(k)$ for $\ell \geq k-r$.

Lemma 3.2 will be used to prove asymptotic lower bounds for $\hat{m}_{r,\ell}(k,i)$. When i is small, the resulting bounds are not sufficiently strong. Thus we also make use of the following lower bound for $\hat{m}_{r,\ell}(k,i)$ for values of i which are small compared with k. This is also used as a base case for an inductive proof of lower bounds for $\hat{m}_r(k,i)$ (see Lemma 3.6 below).

LEMMA 3.4. *For all relevant i, k and $\ell \geq r$ such that $k > r(r^2+1)+i+2$, we have that*

$$\hat{m}_{r,\ell}(k,i) \geq \binom{k-r}{i}\hat{b}_r(k,i)^i \hat{m}_{r,\ell}(k-i),$$

where

$$\hat{b}_r(k,i) = \binom{k-i-r-1}{r-1}\left(1 - \frac{r^3}{k-i-r-2}\right).$$

In particular $\hat{m}_{r,\ell}(k,i) > 0$ for such k.

PROOF. Let i, k, ℓ as in the lemma be given. We establish the lemma by considering the subset $\mathcal{H}$ of graphs contributing to $\hat{m}_{r,\ell}(k,i)$, constructed as follows. To obtain a graph $H \in \mathcal{H}$, select a subset $U \subset [k] - [r]$ of size i for the vertices in the top level of H, and a minimally susceptible, triangle-free graph H' on $[k] - U$ so that $[r]$ is a contagious set for H' with all level sizes bounded by ℓ and j vertices in the top level, for some $1 \le j \le \min\{k - r - i, \ell\}$. Let v denote the vertex in the top level of H' of largest index. For each $u \in U$, select a subset $V_u \subset [k] - U$ of size r which contains v, and so that no $v', v'' \in V_u$ are neighbours in H'. Finally, let H be the minimally susceptible graph on $[k]$ with subgraph H' such that each vertex $u \in U$ is joined by an edge to all vertices in V_u. By the choice of H' and the sets V_u, we have that H contributes to $\hat{m}_{r,\ell}(k,i)$. Moreover, by the choice of v, for any choice of U, H' and V_u, a unique graph H is obtained. Hence, $|\mathcal{H}| \le \hat{m}_{r,\ell}(k,i)$.

To conclude, we claim that, for each $u \in U$, the number of possibilities for V_u is bounded from below by

$$\binom{k-i-r-1}{r-1} - r(k-i-r-1)\binom{k-i-r-3}{r-3} \ge \hat{b}_r(k,i).$$

To see this, simply note that there are $r(k-i-r-1)$ edges in H' which do not join v to one of its r neighbours. Therefore,

$$\hat{m}_{r,\ell}(k,i) \ge \binom{k-r}{i} \hat{b}_r(k,i)^i \sum_j \hat{m}_{r,\ell}(k-i,j) = \binom{k-r}{i} \hat{b}_r(k,i)^i \hat{m}_{r,\ell}(k-i)$$

(where the sum is over $1 \le j \le \min\{k-r-i, \ell\}$) as claimed.

By the choice of i, k, $\hat{b}_r(k,i) > 0$. Hence $\hat{m}_{r,\ell}(k,i) > 0$, since $\hat{m}_{r,\ell}(k) > 0$ for all relevant k, ℓ, as is easily seen (e.g., by considering minimally susceptible, triangle-free graphs of size $k = nr + m$, for some $n \ge 1$ and $m \le r$, which have m vertices in the top level and r vertices in all levels below, and all vertices in level $i \ge 1$ are connected to all r vertices in level $i - 1$). $\square$

LEMMA 3.5. *As $k \to \infty$, we have that*

$$m_r(k) \ge \hat{m}_r(k) \ge e^{-o(k)} e^{-(r-2)k} (k-r)! \left(\frac{k^{r-1}}{(r-1)!} \right)^k.$$

Comparing this with Lemma 2.5, we see that the number of triangle-free susceptible graphs of size k is not much smaller than the number of susceptible graphs (up to an error of $e^{o(k)}$).

PROOF OF LEMMA 3.5. Let $\ell \ge r$. The idea is to use spectral analysis of the linear recursion (3.3), restricted to level sizes bounded by ℓ, and then take $\ell \to \infty$. However, some work is needed to write the recursion in a usable form.

Put

$$\hat{\sigma}_{r,\ell}(k,i) = \frac{\hat{m}_{r,\ell}(k,i)}{(k-r)!}\left(\frac{(r-1)!}{k^{r-1}}\right)^k.$$

Restricting (3.3) to $j \le \ell$, it follows that

$$\hat{\sigma}_{r,\ell}(k,i) \ge \sum_{j=1}^{\ell} \hat{A}_r(k,i,j)\hat{\sigma}_{r,\ell}(k-i,j) \qquad \text{for } i \le \ell. \tag{3.5}$$

In order to express (3.5) in matrix form, we introduce the following notation. For an $\ell \times \ell$ matrix M, let M_j, be the $\ell \times \ell$ matrix whose jth row is that of M and all other entries are 0. Let

$$\psi(M) = \begin{bmatrix} M_1 & M_2 & \cdots & M_{\ell-1} & M_\ell \\ I_\ell & & & & \\ & I_\ell & & & \\ & & \ddots & & \\ & & & I_\ell & \end{bmatrix},$$

where I_ℓ is the $\ell \times \ell$ identity matrix and all empty blocks are filled with 0's. For all relevant k, put

$$\hat{\Sigma}_k = \hat{\Sigma}_k(r,\ell) = \begin{bmatrix} \hat{\sigma}_k \\ \hat{\sigma}_{k-1} \\ \vdots \\ \hat{\sigma}_{k-\ell+1} \end{bmatrix},$$

where $\hat{\sigma}_k = \hat{\sigma}_k(r,\ell)$ is the $1 \times \ell$ vector with entries $(\hat{\sigma}_k)_j = \hat{\sigma}_{r,\ell}(k,j)$.

Using this notation, (3.5) can be written as

$$\hat{\Sigma}_k \ge \psi(\hat{A}_k)\hat{\Sigma}_{k-1}, \tag{3.6}$$

where $\hat{A}_k = \hat{A}_k(r,\ell)$ is the $\ell \times \ell$ matrix with entries $(\hat{A}_k)_{i,j} = \hat{A}_r(k,i,j)$.

By Lemma 3.4, we have that all coordinates of $\hat{\Sigma}_k$ are positive for all k large enough. Let $A = A(r,\ell)$ denote the $\ell \times \ell$ matrix with entries $A_{i,j} = A_r(i,j)$ (as defined in Lemma 2.3). For $\varepsilon > 0$, let $A_\varepsilon = A_\varepsilon(r,\ell)$, be the $\ell \times \ell$ matrix with entries $(A_\varepsilon)_{i,j} = A_{i,j} - \varepsilon$. By Lemma 3.2, for k large enough each entry of $\hat{A}_k$ is greater than the same entry of A_ε. Since $A > 0$, we have that $A_\varepsilon > 0$ for all sufficiently small $\varepsilon > 0$. Hence, by Lemma 3.2 and (3.6), for any such ε, there is a k_ε so that

$$\hat{\Sigma}_{k_\varepsilon+k} \ge \psi(A_\varepsilon)^k \hat{\Sigma}_{k_\varepsilon} > 0 \qquad \text{for } k \ge 0,$$

with entries of Σ_{k_ε} positive. Therefore, up to a factor of $e^{-o(k)}$, the growth rate of $\hat{\sigma}_{r,\ell}(k) = \sum_i \hat{\sigma}_{r,\ell}(k,i)$ is given by the Perron eigenvalue $\lambda = \lambda(r,\ell)$ of $\psi(A)$.

Let $D_\lambda = \mathrm{diag}(\lambda^{-i} : 1 \le i \le \ell)$. We claim that the Perron eigenvalue of $\psi(A)$ is characterized by the property that the Perron eigenvalue of $D_\lambda A$ is 1. To see this, one simply verifies that if $D_\lambda A v = v$, then

$$v_\lambda = \begin{bmatrix} \lambda^{\ell-1} v \\ \lambda^{\ell-2} v \\ \vdots \\ v \end{bmatrix}$$

satisfies $\psi(A) v_\lambda = \lambda v_\lambda$. If v has nonnegative entries, then 1 is the Perron eigenvalue of $D_\lambda A$ and λ the Perron eigenvalue of $\psi(A)$.

If $\lambda < e^{-(r-2)}(e\ell)^{-1/\ell}$, we claim that every row sum of $D_\lambda A$ is greater than 1. Indeed, for all such λ, the sum of row $i \le \ell$ is [using the bound $i! \le ei(i/e)^i$]

$$\left(e^{r-1}\lambda\right)^{-i} \sum_{j=1}^{\ell} \frac{j^i}{i!} > \left(e^{r-1}\lambda\right)^{-i} \frac{\ell^i}{i!} > \frac{1}{ei}\left((e\ell)^{1/\ell} \frac{\ell}{i}\right)^i.$$

Twice differentiating the log of the right-hand side with respect to i, we obtain $-(i-1)/i^2$. Therefore, noting that for $i = \ell$ the right hand side above equals to 1, and for $i = 1$ it equals $(\ell/e)(e\ell)^{1/\ell} \ge 1$ for all relevant ℓ, the claim follows.

Since the spectral radius of a matrix is bounded below by its minimum row sum, it follows that for such λ, the spectral radius of $D_\lambda A$ is greater than 1. Since the spectral radius of $D_\lambda A$ is decreasing in λ, the Perron eigenvalue $\lambda(r, \ell)$ of $\psi(A)$ is at least $e^{-(r-2)}(e\ell)^{-1/\ell}$, and hence $\liminf_{\ell \to \infty} \lambda(r, \ell) \ge e^{-(r-2)}$. Taking $\ell \to \infty$, we find that

$$\hat{m}_r(k) \ge e^{-o(k)} e^{-(r-2)k} (k-r)! \left(\frac{k^{r-1}}{(r-1)!}\right)^k$$

as required. $\square$

We require a lower bound for the number of minimally susceptible graphs of size k with $i = \Omega(k)$ vertices in the top level in order to estimate the growth of supercritical r-percolations on $\mathcal{G}_{n,p}$.

LEMMA 3.6. *Let $\varepsilon \in (0, 1/(r+1))$. For all sufficiently large k and $i \le (\varepsilon/r)^2 k$, we have that*

$$\hat{m}_r(k, i) \ge e^{-i\varepsilon - (r-2)k - o(k)} (k-r)! \left(\frac{(k-i)k^{r-2}}{(r-1)!}\right)^k,$$

where $o(k)$ depends on k, ε, but not on i.

Although the proof is somewhat involved, the general scheme is straightforward. We use Lemmas 3.4 and 3.5 to obtain a sufficient bound for i, k in a range for which $i/k \ll 1$. Then, for all other relevant i, k we proceed by induction, using (3.1). The inductive step (Claim 3.7 below) of the proof appears in Appendix A.3.

PROOF OF LEMMA 3.6. Fix some k_r so that

$$k_r > \max\left\{e^{r/\varepsilon}, \frac{r(r^2+1)+2}{1-(\varepsilon/r)^2}\right\}.$$

Note that, for all $k > k_r$ and $i \le (\varepsilon/r)^2 k$, we have that $k/\log^2 k < (\varepsilon/r)^2 k$ and that Lemma 3.4 applies to $\hat{m}_r(k,i)$ [setting $\ell = k - r$, so that $\hat{m}_{r,\ell}(k,i) = \hat{m}_r(k,i)$].

For all relevant i, k, let

$$\hat{\rho}_r(k,i) = \frac{\hat{m}_r(k,i)}{(k-r)!}\left(\frac{(r-1)!}{(k-i)k^{r-2}}\right)^k. \tag{3.7}$$

By Lemma 3.5 there is some $f_r(k) \ll k$ such that

$$\hat{m}_r(k) \ge e^{-(r-2)k - f_r(k)}(k-r)!\left(\frac{k^{r-1}}{(r-1)!}\right)^k.$$

Without loss of generality, we assume f_r is nondecreasing.

By Lemma 3.4, we find that for all $k > k_r$ and relevant i, $\hat{\rho}_r(k,i)$ is bounded from below by

$$\frac{e^{-(r-2)(k-i)-f_r(k-i)}}{i!}\hat{b}_r(k,i)^i\left(\frac{(k-i)^{r-1}}{(r-1)!}\right)^{k-i}\left(\frac{(r-1)!}{(k-i)k^{r-2}}\right)^k.$$

By the bound $\binom{n}{k} \ge (n-k)^k/k!$,

$$\hat{b}_r(k,i) \ge \frac{(k-i-2r)^{r-1}}{(r-1)!}\left(1 - \frac{r^3}{k-i-r-2}\right).$$

Therefore the lower bound for $\hat{\rho}_r(k,i)$ above is bounded from below by (using the inequality $i! < i^i$)

$$C_r(k,i)g_r(k,i)e^{-(r-2)k - f_r(k-i) - i\log i},$$

where

$$C_r(k,i) = \left(1 - \frac{2r}{k-i}\right)^{(r-1)i}\left(1 - \frac{r^3}{k-i-r-2}\right)^i$$

and

$$g_r(k,i) = e^{(r-2)i}\left(\frac{k-i}{k}\right)^{(r-2)k}.$$

If $r = 2$, then $g_r \equiv 1$. We note that, for $r > 2$,

$$\frac{\partial}{\partial i}g_r(k,i) = -\frac{(r-2)i}{k-i}g_r(k,i) < 0$$

and so, for any such r and relevant k, $g_r(k,i)$ is decreasing in i. By the inequality $(1-x)^y > 1-xy$, for any $k > k_r$ and $i \le (\varepsilon/r)^2 k$, we have that

$$\begin{aligned}
C_r(k,i) &> 1 - \frac{2r^2 i}{k-i} - \frac{r^3 i}{k-i-r-2} \\
&> 1 - \frac{2\varepsilon^2}{1-(\varepsilon/r)^2} - \frac{r\varepsilon^2}{1-(\varepsilon/r)^2-(r+2)/k} \\
&> 1 - \frac{2/(r+1)^2}{1-1/r^4} - \frac{1/r}{1-1/r^4-(r+2)/k_r} > 0
\end{aligned}$$

since $\varepsilon < 1/(r+1)$, $k_r > e^{r/\varepsilon} > e^{r(r+1)}$ and $r \ge 2$. Altogether, for some constant $\xi' = \xi'(r) > 0$, we have that

$$\hat{\rho}_r(k,i) \ge \xi' e^{-(r-2)k - h_r(k)} \qquad \text{for } k > k_r \text{ and } i \le k/\log^2 k, \tag{3.8}$$

where

$$h_r(k) = f_r(k) - \log g_r\left(k, \frac{k}{\log^2 k}\right) + \frac{k}{\log^2 k}\log\left(\frac{k}{\log^2 k}\right). \tag{3.9}$$

We note that $h(k) \ll k$ as $k \to \infty$.

CLAIM 3.7. *For some $\xi = \xi(r,\varepsilon) > 0$, for all $k > k_r$ and $i \le (\varepsilon/r)^2 k$, we have that $\hat{\rho}_r(k,i) \ge \xi e^{-i\varepsilon-(r-2)k-h_r(k)}$.*

Claim 3.7 is proved in Appendix A.3.

Since $h_r(k) \ll k$ and ξ depends only on r, ε, the lemma follows by Claim 3.7 and (3.7). □

3.2. *$\hat{r}$-bootstrap percolation on $\mathcal{G}_{n,p}$.* We define *$\hat{r}$-percolation*, a restriction of r-percolation, which informally halts upon requiring a triangle. Formally, recall the definitions of $I_t(I,G)$ and $V_t(I,G)$ given in Section 2.1. Let $\hat{I}_t = I_t$ if G contains a triangle-free subgraph H such that $V_t(I,H) = V_t(I,G)$, and otherwise put $\hat{I}_t = \varnothing$. Put $\hat{V}_t = \bigcup_{s \le t} \hat{I}_s$.

DEFINITION 3.8. Let $\hat{P}_r(k,i) = \hat{P}_r(p,k,i)$ denote the probability that for a given $I \subset [n]$, with $|I| = r$, we have that $|\hat{V}_t(I,\mathcal{G}_{n,p})| = k$ and $|\hat{I}_t(I,\mathcal{G}_{n,p})| = i$, for some t. Let $\hat{E}_r(k,i)$ denote the expected number of such subsets I. We put $\hat{P}_r(k) = \sum_i \hat{P}_r(k,i)$ and $\hat{E}_r(k) = \sum_i \hat{E}_r(k,i)$.

Using Lemma 3.6 we obtain lower bounds for the growth probabilities of $\hat{r}$-percolations on $\mathcal{G}_{n,p}$.

LEMMA 3.9. *Let* $\alpha > 0$. *Put* $p = \vartheta_r(\alpha, n)$ *and* $\varepsilon = np^r = \alpha/\log^{r-1} n$. *For* $i \le k - r$ *and* $k \le n^{1/(r(r+1))}$, *we have that*

$$\hat{P}_r(k,i) \ge (1 - o(1)) \frac{e^{-\varepsilon\binom{k-i}{r}}\varepsilon^{k-r}}{(k-r)!}\hat{m}_r(k,i),$$

where $o(1)$ *depends on* n, *but not on* i, k.

PROOF. Let $I \subset [n]$, with $|I| = r$, be given. Put

$$\hat{\ell}_r(k,i) = \frac{e^{-\varepsilon\binom{k-i}{r}}\varepsilon^{k-r}}{(k-r)!}\hat{m}_r(k,i).$$

If for some $V \subset [n]$, with $|V| = k$ and $I \subset V$, we have that the subgraph $\mathcal{G}_V \subset \mathcal{G}_{n,p}$ induced by V is minimally susceptible and triangle-free, I is a contagious set for $\mathcal{G}_V$ with i vertices in the top level, and all vertices in V^c are connected to at most $r - 1$ vertices below the top level of $\mathcal{G}_V$, then it follows that $|\hat{V}_t(I, \mathcal{G}_{n,p})| = k$ and $|\hat{I}_t(I, \mathcal{G}_{n,p})| = i$ for some t. Hence,

$$\hat{P}_r(k,i) > \binom{n-r}{k-r}\hat{m}_r(k,i)p^{r(k-r)}(1-p)^{k^2}\left(1 - \binom{k-i}{r}p^r\right)^n.$$

By the inequalities $\binom{n}{k} \ge (n-k)^k/k!$ and $(1 - x/n)^n \ge e^{-x}(1 - x^2/n)$, it follows that

$$\frac{\hat{P}_r(k,i)}{\hat{\ell}_r(k,i)} > \left(1 - \frac{k}{n}\right)^k (1-p)^{k^2}\left(1 - \binom{k-i}{r}^2\frac{\varepsilon^2}{n}\right).$$

For all large n, the right-hand side is bounded from below by

$$\left(1 - \frac{k}{n}\right)^k \left(1 - \frac{1}{n^{1/r}}\right)^{k^2}\left(1 - \frac{k^{2r}}{n}\right) \sim 1$$

since $k \le n^{1/(r(r+1))} \ll n^{1/(2r)}$ and $r \ge 2$. It follows that $\hat{P}_r(k,i) \ge (1 - o(1))\hat{\ell}_r(k,i)$, where $o(1)$ depends on n, but not on i, k, as required. $\square$

3.3. *Supercritical bounds.* In this section we show that, for $\alpha > \alpha_r$, the expected number of supercritical $\hat{r}$-percolations on $\mathcal{G}_{n,p}$ which grow larger than a critical size $\beta_*(\alpha)\log n > \beta_r(\alpha)\log n$ (see Lemma 2.11) is large. The importance of $\beta_*(\alpha)$ is established in Section 3.5 below. Subsequent sections establish the existence of sets I of size r so that $\hat{r}$-percolation initialized at I grows larger than $\beta_*(\alpha)\log n$.

LEMMA 3.10. *Let* $\alpha, \beta_0 > 0$ *and* $\varepsilon \in (0, 1/(r+1))$. *Put* $p = \vartheta_r(\alpha, n)$. *For all sufficiently large* $k = \beta\log n$ *and* $i = \gamma k$, *with* $\beta \le \beta_0$ *and* $\gamma \le (\varepsilon/r)^2$, *we have that*

$$\hat{E}_r(k,i) \ge n^{\mu_\varepsilon - o(1)},$$

where

$$\mu_\varepsilon = \mu_{r,\varepsilon}(\alpha,\beta,\gamma) = r + \beta\log\left(\frac{\alpha\beta^{r-1}(1-\gamma)}{(r-1)!}\right) - \frac{\alpha\beta^r}{r!}(1-\gamma)^r - \beta(r-2+\varepsilon\gamma)$$

and $o(1)$ *depends on* $\alpha, \varepsilon, \beta_0$, *but not on* β, γ.

PROOF. Put $\delta = np^r$. By Lemmas 3.6 and 3.9, for large $k = \beta \log n$ and $i = \gamma k$, with $\beta \le \beta_0$ and $\gamma \le (\varepsilon/r)^2$,

$$\hat{E}_r(k,i) \ge \xi(n)\binom{n}{r}\left(\frac{\delta(k-i)k^{r-2}}{(r-1)!}\right)^k \delta^{-r} e^{-i\varepsilon-(r-2)k-\delta\binom{k-i}{r}-o(k)} = n^{\mu_\varepsilon - o(1)},$$

where $\xi(n) \sim 1$ depends only on n, and $o(k)$ depends only on r, ε, β_0. $\square$

We note that, for any $\alpha, \varepsilon > 0$,

$$\mu_{r,\varepsilon}(\alpha,\beta,0) = \mu_r^*(\alpha,\beta) \tag{3.10}$$

(where μ^* is as defined in Section 2.4).

We now state the main result of this section.

LEMMA 3.11. *Let* $\varepsilon < 1/(r+1)$. *Put* $\alpha_{r,\varepsilon} = (1+\varepsilon)\alpha_r$ *and* $p = \vartheta_r(\alpha_{r,\varepsilon}, n)$. *For some* $\delta(r,\varepsilon) > 0$ *and* $\zeta(r,\varepsilon) > 0$, *if* $k_n/\log n \in [\beta_*(\alpha_{r,\varepsilon}), \beta_*(\alpha_{r,\varepsilon}) + \delta]$ *for all large* n, *then* $\hat{E}_r(k_n) \gg n^\zeta$ *as* $n \to \infty$.

The proof appears in Appendix A.4. Although the argument is technical, the basic idea is straightforward: we show that, for some $\zeta > 0$, for all relevant k there is some i so that $\hat{E}_r(k,i) > n^\zeta$. For $k > \beta_* \log n$, values of i with this property are on the order of k, and so the proof makes use of Lemma 3.6.

3.4. *$\hat{r}$-percolations are almost independent.* For a set $I \subset [n]$, with $|I| = r$, let $\hat{\mathcal{E}}_k(I)$ denote the event that $\hat{r}$-percolation on $\mathcal{G}_{n,p}$ initialized by I grows to size k, that is, we have that $|\hat{V}_t(I)| = k$ for some t. Hence, $\hat{P}_r(k) = \mathbb{P}(\hat{\mathcal{E}}_k(I))$. In this section we show that for sets $I \ne I'$ of size r, and suitable values of k, p, the events $\hat{\mathcal{E}}_k(I)$ and $\hat{\mathcal{E}}_k(I')$ are approximately independent. Specifically, we establish the following lemma.

LEMMA 3.12. *Let* $\alpha, \beta > 0$ *and put* $p = \vartheta_r(\alpha, n)$. *Fix sets* $I \ne I'$ *such that* $|I| = |I'| = r$ *and* $|I \cap I'| = m$. *For* $\beta \log n \le k \le n^{1/(r(r+1))}$, *we have that* $\mathbb{P}(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}_k(I))$ *is bounded from above by*

$$(rk/n)^{r-m} + O\big(k^{2r}(kp)^{r(r-m)}\big) + \begin{cases} \big(1+o(1)\big)\hat{P}_r(k) & \text{if } m = 0, \\ o\big((n/k)^m\big)\hat{P}_r(k) & \text{if } 1 \le m < r, \end{cases}$$

where $o(1)$ *depends only on* n.

For sets $I \subset V$ of sizes r and k, let $\hat{\mathcal{E}}(I, V)$ be the event that for some t we have $\hat{V}_t(I) = V$. By symmetry these events all have the same probability. Since for a fixed I and different sets V these events are disjoint, we have that $\hat{P}_r(k) = \binom{n-r}{k-r}\mathbb{P}(\hat{\mathcal{E}}(I, V))$.

LEMMA 3.13. *Fix sets $I \subset V$, with $|I| = r$ and $|V| = k$.*

(i) *For any set of edges $E \subset [n]^2 - V^2$, the conditional probability that $E \subset E(\mathcal{G}_{n,p})$, given $\hat{\mathcal{E}}(I, V)$, is at most $p^{|E|}$.*

(ii) *For any $u \notin V$ and set of vertices $W \subset [n]$ such that $|W| = r$ and $|V \cap W| < r$, the conditional probability that $(u, w) \in \mathcal{G}_{n,p}$ for all $w \in W$, given $\hat{\mathcal{E}}(I, V)$, is at least $p^r(1 - p)^k$.*

PROOF. Let $\mathcal{G}_V$ denote the subgraph of $\mathcal{G}_{n,p}$ induced by V. The event $\hat{\mathcal{E}}(I, V)$ occurs if and only if for some t and triangle-free subgraph $H \subset \mathcal{G}_V$, we have that $V_t(I, H) = V_t(I, \mathcal{G}_V) = V$ and all vertices in V^c are connected to at most $r - 1$ vertices below the top level of H [i.e., $V - I_t(I, H)$]. This event is increasing in the set of edges of $\mathcal{G}_V$, and decreasing in edges outside V. By the FKG inequality,

$$\mathbb{P}\big(E \subset E(\mathcal{G}_{n,p})|\hat{\mathcal{E}}(I, V)\big) \le \mathbb{P}\big(E \subset E(\mathcal{G}_{n,p})\big) = p^{|E|}.$$

For claim (ii), let G be a possible value for $\mathcal{G}_V$ on $\hat{\mathcal{E}}(I, V)$, with a subgraph H as above, and $i \le k - r$ vertices infected in the top level [i.e., $I_t(I, H) = i$]. The conditional probability that u is connected to all vertices in W, given $\hat{\mathcal{E}}(I, V)$ and $\mathcal{G}_V = G$, is equal to

$$\frac{p^r \sum_{\ell=0}^{r-1-\ell_0} \binom{k-i-\ell_0}{\ell} p^\ell (1 - p)^{k-i-\ell_0-\ell}}{\sum_{\ell=0}^{r-1} \binom{k-i}{\ell} p^\ell (1 - p)^{k-i-\ell}},$$

where $\ell_0 < r$ is the number of vertices in W below the top level of H. Bounding the numerator by the $\ell = 0$ term and the denominator by 1, the above expression is at least $p^r(1 - p)^{k-i-\ell_0} \ge p^r(1 - p)^k$. Hence, summing over the possibilities for G, we obtain the second claim. □

The following result, a special case of Turán's theorem [54], plays an important role in establishing the approximate independence of $\hat{r}$-percolations.

LEMMA 3.14 (Mantel's theorem [45]). *If a graph G is triangle-free, then we have that $e(G) \le \lfloor v(G)^2/4\rfloor$.*

In other words, a triangle-free graph has edge-density at most $1/2$. The number $2r - 1$ is key, since $\lfloor (2r - 1)^2/4\rfloor = r(r - 1)$, and thus

$$r(2r - 1) - \lfloor (2r - 1)^2/4\rfloor = r^2. \tag{3.11}$$

LEMMA 3.15. *Let $\alpha > 0$ and $k \le n^{1/(r(r+1))}$. Put $p = \vartheta_r(\alpha, n)$. Fix sets $I \subset V$ and I' such that $|I| = |I'| = r$, $|V| = k$ and $\ell = |V \cap I'| < r$. Let $\hat{\mathcal{E}}_{k,q}(I')$ denote the event that for some t we have that $\hat{V}_t(I') = V'$ for some $V' \supset I'$ such that $|V'| = k$ and $|V \cap V'| = q$. Then*

$$\mathbb{P}(\hat{\mathcal{E}}_{k,q}(I')|\hat{\mathcal{E}}(I,V)) \le \begin{cases} (1+o(1))\hat{P}_r(k) & q = 0, \\ o((n/k)^{\ell})\hat{P}_r(k) & 1 \le q < 2r-1, \\ k^{2r-1}(kp)^{r(r-\ell)} & q \ge 2r-1, \end{cases}$$

where $o(1)$ depends only on n.

PROOF. *Case* i ($q < 2r - 1$). We claim that

$$\mathbb{P}(\hat{\mathcal{E}}_{k,q}(I')|\hat{\mathcal{E}}(I,V)) \le \left(\left(\frac{n}{k}\right)^{\ell}\left(\frac{k^2}{np^{q/4}}\right)^{q}\right)\sum_{i=1}^{k-r}\hat{Q}_r(k,i), \tag{3.12}$$

where $\hat{Q}_r(k,i)$ is equal to

$$\binom{n}{k-r}\hat{m}_r(k,i)p^{r(k-r)}\left(1-\left(\binom{k-i}{r}-\binom{q}{r}\right)p^r(1-p)^{2k}\right)^{n-2k}.$$

To see this, note that if $\hat{\mathcal{E}}_{k,q}(I)$ occurs then for some V' such that $|V'| = k$, $I' \subset V'$, and $|V \cap V'| = q$, we have that I' is a contagious set for a triangle-free subgraph $H' \subset \mathcal{G}_{n,p}$ on V' with i vertices in the top level, for some $i \le k - r$, and all vertices in $(V \cup V')^c$ are connected to at most $r - 1$ vertices below the top level of H'. There are at most

$$\binom{k}{q-\ell}\binom{n-(q-\ell)}{k-r-(q-\ell)} \le \left(\frac{n}{k}\right)^{\ell}\left(\frac{k^2}{n}\right)^{q}\binom{n}{k-r}$$

such subsets V'. By Lemmas 3.13 and 3.14, for any such V' and i as above, the conditional probability that such a subgraph H' exists, given $\hat{\mathcal{E}}(I,V)$, is bounded by $\hat{m}(k,i)p^{r(k-r)-q^2/4}$, since at most $q^2/4$ edges of H' join vertices in $V \cap V'$. By Lemma 3.13, for any $u \in (V \cup V')^c$ and set V'' of r vertices below the top level of H' with at most $r - 1$ vertices in $V \cap V'$, the conditional probability that u is connected to all vertices in V'' is at least $p^r(1-p)^k$. Hence, again by Lemma 3.13, any such u is connected to all vertices in such a set V'' with conditional probability at least $(\binom{k-i}{r} - \binom{q}{r})p^r(1-p)^{2k}$. The claim follows.

To conclude, let $\hat{\ell}_r(k,i)$ be as in the proof of Lemma 3.9, which recall shows that $\hat{P}_r(k,i) \ge (1-o(1))\hat{\ell}_r(k,i)$ as $k \to \infty$, where $o(1)$ depends only on n. We have, by the inequalities $\binom{n}{k} \le n^k/k!$ and $1 - x < e^{-x}$, that

$$\log\frac{\hat{Q}_r(k,i)}{\hat{\ell}_r(k,i)} < \varepsilon\binom{k-i}{r}\left(1-(1-p)^{2k}\left(1-\frac{2k}{n}\right)\right) + \varepsilon q^r.$$

By the inequality $(1-x)^y \ge 1-xy$, and since $k \le n^{1/(r(r+1))}$, it follows that the right-hand side is at most $\varepsilon n^{1/r}(p+1/n)+\varepsilon q^r \ll 1$, and so

$$\hat{Q}_r(k,i) \le \big(1+o(1)\big)\hat{\ell}_r(k,i) \le \big(1+o(1)\big)\hat{P}_r(k,i),$$

where $o(1)$ depends only on n. Hence,

$$\sum_{i=1}^{k-r} \hat{Q}_r(k,i) \le \big(1+o(1)\big)\sum_{i=1}^{k-r} \hat{P}_r(k,i) = \big(1+o(1)\big)\hat{P}_r(k).$$

Finally, case (i) follows by (3.12) and noting that

$$\frac{np^{q/4}}{k^2} \gg \frac{np^{r/2}}{k^2} \gg n^{1/2-2/(r(r+1))} \gg 1$$

since $q<2r$, $k \le n^{1/(r(r+1))}$ and $r \ge 2$.

Case (ii) ($q \ge 2r-1$). Put $s = 2r-1-\ell$. If $\hat{\mathcal{E}}_{k,q}(I')$ occurs, then for some $\{v_j\}_{j=1}^s \subset V - I'$ and nondecreasing sequence $\{t_j\}_{j=1}^s$, we have that $v_j \in \hat{I}_{t_j}(I')$ and $\hat{V}_{t_j-1} \cap (V-I') = \{v_i : t_i < t_j\}$. Informally, t_j is the jth time that $\hat{r}$-percolation initialized by I' infects a vertex in $V-I'$. It follows that I' is a contagious set for a triangle-free subgraph $H \subset \mathcal{G}_{n,p}$ on $\hat{V}_{t_s-1} \cup \{v_i : t_i = t_s\}$. Since $v_j \in \hat{I}_{t_j}(I')$, note that v_j is r-connected to $\hat{V}_{t_j-1} \subset \hat{V}_{t_s-1}$ in H. Hence, by Lemma 3.14 and (3.11), there are at least

$$rs - \lfloor (2r-1)^2/4 \rfloor = r(r-\ell)$$

edges between $\{v_j\}_{j=1}^s$ and $\hat{V}_{t_s-1} - V$. Therefore, by Lemma 3.13, the conditional probability of $\hat{\mathcal{E}}_{k,q}(I')$, given $\hat{\mathcal{E}}(I,V)$, is bounded by $k^s(kp)^{r(r-\ell)} \le k^{2r-1}(kp)^{r(r-\ell)}$, as claimed. $\square$

Using Lemma 3.15 we establish the main result of this section.

PROOF OF LEMMA 3.12. For each $\ell \in [m,r]$, fix a set V_ℓ such that $I \subset V_\ell$, $|V_\ell| = k$ and $\ell = |V_\ell \cap I'|$. By symmetry, we have that

$$\begin{aligned}
\mathbb{P}\big(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}_k(I)\big) &= \sum_{\ell=m}^{r} \frac{\binom{r-m}{\ell-m}\binom{n-(2r-m)}{k-r-(\ell-m)}}{\binom{n-r}{k-r}} \mathbb{P}\big(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}(I,V_\ell)\big) \\
&\le r^{\ell-m}\binom{n-r}{k-r}^{-1} \sum_{\ell=m}^{r} \binom{n-r-(\ell-m)}{k-r-(\ell-m)} \mathbb{P}\big(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}(I,V_\ell)\big) \\
&\le \sum_{\ell=m}^{r} (rk/n)^{\ell-m} \mathbb{P}\big(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}(I,V_\ell)\big).
\end{aligned}$$

If $\ell = m$, then by Lemma 3.15, summing over $q \in [\ell, k]$, we get

$$\mathbb{P}\big(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}(I, V_m)\big) \le \begin{cases} \big(1+o(1)\big)\hat{P}_r(k) + k^{2r}(kp)^{r^2} & m = 0, \\ o\big((n/k)^m\big)\hat{P}_r(k) + k^{2r}(kp)^{r(r-m)} & 1 \le m < r. \end{cases}$$

Likewise, for any $m < \ell < r$,

$$\begin{aligned}(k/n)^{\ell-m}\mathbb{P}\big(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}(I, V_\ell)\big) &\le (rk/n)^{\ell-m}\big(o\big((n/k)^\ell\big)\hat{P}_r(k) + k^{2r}(kp)^{r(r-\ell)}\big) \\ &\le o\big((n/k)^m\big)\hat{P}_r(k) + k^{2r}(kp)^{r(r-m)}\big(np^r k^{r-1}\big)^{m-\ell} \\ &\le o\big((n/k)^m\big)\hat{P}_r(k) + k^{2r}(kp)^{r(r-m)}\big(\alpha\beta^{r-1}\big)^{m-\ell} \\ &= o\big((n/k)^m\big)\hat{P}_r(k) + O\big(k^{2r}(kp)^{r(r-m)}\big).\end{aligned}$$

Finally, for $\ell = r$ we bound $\mathbb{P}(\hat{\mathcal{E}}_k(I')|\hat{\mathcal{E}}(I, V_r)) \le 1$. Summing over $\ell \in [m, r]$ we obtain the result. $\square$

3.5. *Terminal r-percolations*. In this section we establish the importance of $\beta_*(\alpha)$ for the growth of supercritical r-percolations. Essentially, we find that an r-percolation on $\mathcal{G}_{n,p}$, having grown larger than $\beta_*(\alpha)\log n$, with high probability continues to grow.

DEFINITION 3.16. We say that $I \subset [n]$ is a *terminal* (k, i)*-contagious set* for $\mathcal{G}_{n,p}$ if $|V_\tau(I, \mathcal{G}_{n,p}, r)| = k$ and $|I_\tau(I, \mathcal{G}_{n,p}, r)| = i$.

LEMMA 3.17. *Let $\alpha > \alpha_r$ and $\beta_r^*(\alpha) < \beta_1 < \beta_2$. Put $p = \vartheta_r(\alpha, n)$. With high probability, $\mathcal{G}_{n,p}$ has no terminal k-contagious set, with $k = \beta\log n$, for all $\beta \in [\beta_1, \beta_2]$.*

PROOF. If I is a terminal (k, i)-contagious set for $\mathcal{G}_{n,p}$, then I is a contagious set for some subgraph $H \subset \mathcal{G}_{n,p}$ of size k with i vertices in the top level, and all vertices in $V(H)^c$ are connected to at most $r-1$ vertices in $V(H)$. Hence, the probability that a given I is as such is bounded from above by

$$\binom{n}{k-r} m_r(k,i) p^{r(k-r)} \left(1 - \binom{k}{r} p^r (1-p)^k\right)^{n-k}.$$

For all relevant $k \le \beta_2 \log n$, we find [bounding $(1-p)^k \ge 1 - pk$] that

$$1 - \binom{k}{r} p^r (1-p)^k \le 1 - \binom{k}{r} p^r + (kp)^{r+1} \ll 1 - \binom{k}{r} p^r + 1/n.$$

Put $\varepsilon = np^r$. By Lemma 2.5 [and the bounds $\binom{n}{k} \le n^k/k!$ and $1 - x < e^{-x}$], it follows that the expected number of terminal (k, i)-contagious sets, with $k = \beta\log n$ and $i = \gamma k$, for some $\beta \le \beta_2$, is bounded (for all large n) by

$$\binom{n}{r}\left(\frac{\varepsilon k^{r-1}}{(r-1)!}\right)^k \varepsilon^{-r} e^{-i-(r-2)k-\varepsilon\binom{k}{r}} \lesssim n^{\mu_r^*(\alpha,\beta)-\beta\gamma} \log^{r(r-1)} n,$$

where $\lesssim$ denotes inequality up to a constant depending on α, β_2, but not on β, γ.

By Lemma 2.11, we have that $\mu_r^*(\alpha, \beta) \le \mu_r^*(\alpha, \beta_1) < 0$ for all $\beta \in [\beta_1, \beta_2]$. Hence, summing over the $O(\log^2 n)$ relevant values of i, k, we find that the probability that $\mathcal{G}_{n,p}$ contains a terminal k-contagious set for some $k = \beta \log n$, with $\beta \in [\beta_1, \beta_2]$, is bounded (for all large n, and up to a constant) by

$$n^{\mu_r^*(\alpha,\beta_1)} \log^{r(r-1)+2} n \ll 1$$

giving the result. $\square$

3.6. *Almost sure susceptibility.* Finally, we complete the proof of Theorem 1.1. Using Lemmas 3.11, 3.12 and 3.17, we argue that if $\alpha > \alpha_r$, then with high probability $\mathcal{G}_{n,p}$ contains a large susceptible subgraph. By adding independent random graphs with small edge probabilities, we deduce that percolation occurs with high probability.

PROOF OF THEOREM 1.1. Proposition 2.1 gives the subcritical case $\alpha < \alpha_r$. Assume therefore that $\alpha > \alpha_r$. Let $\mathcal{G}_*, \mathcal{G}_i$, for $i \ge 0$, be independent random graphs with edge probabilities $p_* = \vartheta_r(\alpha_r + \varepsilon, n)$ and $p_i = 2^{-i(r-1)/r} p_\varepsilon$, where $p_\varepsilon = \vartheta_r(\varepsilon, n)$. Moreover, let $\varepsilon > 0$ be sufficiently small so that $\mathcal{G} = \mathcal{G}_* \cup \bigcup_{i\ge 0} \mathcal{G}_i$ is a random graph with edge probabilities at most $p = \vartheta_r(\alpha, n)$. Thus, to show that $\mathcal{G}_{n,p}$ is susceptible, it suffices to show that with high probability $\mathcal{G}$ is susceptible.

CLAIM 3.18. *Let $A > 0$. With high probability the graph $\mathcal{G}_*$ contains a susceptible subgraph on some set $U_0 \subset [n]$ of size $|U_0| \ge A \log n$.*

PROOF. Using Lemmas 3.11 and 3.12 we show by the second moment method that, with high probability, $\mathcal{G}_*$ contains a susceptible subgraph of size at least $(\beta_r^*(\alpha) + \delta_0) \log n$, for some $\delta_0 > 0$. By Lemma 3.17, this gives the claim.

Recall that Lemma 3.11 provides $\delta, \zeta > 0$ so that if $k_n/\log n \in [\beta_*(\alpha) + \delta/2, \beta_*(\alpha) + \delta]$, then $\hat{E}_r(k_n) \gg n^\zeta$. Fix such a sequence k_n. For each n, fix $I_n \subset [n]$ with $|I_n| = r$. Applying Lemma 3.12, it follows that

$$\begin{aligned}
\sum_I \frac{\mathbb{P}(\hat{\mathcal{E}}_{k_n}(I)|\hat{\mathcal{E}}_{k_n}(I_n))}{\hat{E}_r(k_n)} &\le 1 + o(1) + \binom{n}{r}^{-1} \sum_{m=1}^{r-1} \binom{n-r}{r-m} o\big((n/k_n)^m\big) \\
&\quad + n^{-\zeta} \sum_{m=0}^{r-1} n^{r-m}\big((k_n/n)^{r-m} + k_n^{2r}(k_n p_*)^{r(r-m)}\big) \\
&\le 1 + o(1) + n^{-\zeta} \sum_{m=0}^{r-1} \big(k_n^{r-m} + k_n^{2r}\big((k_n p_*)^r n\big)^{r-m}\big) \\
&= 1 + o(1) + O\big(n^{-\zeta} \log^{3r} n\big) \sim 1,
\end{aligned}$$

where the sum is over sets $I \subset [n]$ such that $|I| = r$ and $|I \cap I_n| = m$ for some $0 \le m < r$. Therefore with high probability some $\hat{r}$-percolation on $\mathcal{G}_*$ grows to size k_n. As discussed, the claim follows by the choice of k_n and Lemma 3.17. $\square$

We remark here that the result now follows immediately by Claim 3.18 and [39], Theorem 3.1(iii). (To see this, select some $A > \frac{r-1}{r}((r-1)!/\varepsilon)^{1/(r-1)}$ and apply Claim 3.18. Next, add to $\mathcal{G}_*$ an independent graph $\mathcal{G}_\varepsilon$ with edge probabilities p_ε, to obtain $\mathcal{G}_* \cup \mathcal{G}_\varepsilon$ with edge probabilities bounded by $\vartheta_r(\alpha, n)$. By [39], Theorem 3.1(iii), with high probability $\mathcal{G}_* \cup \mathcal{G}_\varepsilon$ is susceptible.) That being said, for the sake of completeness, we conclude by a simple sprinkling argument.

CLAIM 3.19. *There is some $A = A(\varepsilon)$ so that if U_0 is a set of size $|U_0| \ge A \log n$, then with high probability r-percolation on $\bigcup_{i\ge1} \mathcal{G}_i$ initialized at U_0 infects a set of vertices of order $n/\log n$.*

PROOF. Let $A = 2r(16r/\varepsilon)^{1/(r-1)}$. Moreover, assume that n is sufficiently large and ε is sufficiently small so that $A \ge 2$ and $A(2^{1-r}\varepsilon/\log n)^{1/r} \le 1/2$.

We define a sequence of disjoint sets U_i as follows. Given U_i, we consider all vertices not in $U_0, \dots, U_i$, and add to U_{i+1} some $2^{i+1} A \log n$ vertices that are r-connected in $\mathcal{G}_{i+1}$ to U_i (say, those of lowest index).

We first argue that, as long as at most $n/2$ vertices are included in $\bigcup_{j=1}^i U_j$ and $2^i \le n/\log^2 n$, the probability that we can find $2^{i+1} A \log n$ vertices to populate U_{i+1} is at least $1 - n^{-1}$. Indeed, a vertex not in $\bigcup_{j=1}^i U_j$ is at least r-connected in $\mathcal{G}_{i+1}$ to U_i with probability bounded from below [using the bounds $(1-x)^r \ge 1 - xr$ and $\binom{n}{k} \ge (n/k)^k$] by

$$\binom{|U_i|}{r} p_{i+1}^r (1 - p_{i+1})^{|U_i| - r} \ge \left(\frac{|U_i| p_{i+1}}{r}\right)^r (1 - |U_i| p_{i+1}) \ge \frac{1}{2}\left(\frac{|U_i| p_{i+1}}{r}\right)^r,$$

since, for all large n,

$$|U_i| p_{i+1} = 2^{-(r-1)/r}(A \log n)\left(\frac{2^i \varepsilon}{n \log^{r-1} n}\right)^{1/r} \le A\left(\frac{2^{1-r}\varepsilon}{\log n}\right)^{1/r} \le \frac{1}{2}.$$

Hence the expected number of such vertices is at least

$$\frac{n}{2}\frac{1}{2}\left(\frac{|U_i| p_{i+1}}{r}\right)^r = \frac{\varepsilon}{4r}\left(\frac{A}{2r}\right)^{r-1}(2^i A \log n) = 2^{i+2} A \log n$$

by the choice of A. Therefore by Chernoff's bound, such a set U_{i+1} of size $2^{i+1} A \log n$ can be selected with probability at least $1 - \exp(-2^{i-1} A \log n) \ge 1 - n^{-1}$ (since $A \ge 2$ and $i \ge 0$), as required. Since the number of levels before reaching $n/2$ vertices is $O(\log n)$, the claim follows. $\square$

By Claims 3.18 and 3.19, with high probability $\mathcal{G}_* \cup \bigcup_{i\ge1} \mathcal{G}_i$ contains a susceptible subgraph on some $U \subset [n]$ of order $n/\log n$. To conclude, we observe that given this, by adding $\mathcal{G}_0$ we have that $\mathcal{G} = \mathcal{G}_* \cup \bigcup_{i\geq 0} \mathcal{G}_i$ is susceptible with high (conditional) probability. Indeed, the expected number of vertices in U^c which are connected in $\mathcal{G}_0$ to at most $r-1$ vertices in U is bounded from above by

$$n\sum_{j=0}^{r-1}\binom{|U|}{j}p_0^j(1-p_0)^{|U|-j} \ll n\big(|U|p_0\big)^r e^{-p_0(|U|-r)} \ll n^r e^{-n^{(1-1/r)/2}} \ll 1.$$

Hence, $\mathcal{G}$ is susceptible with high probability, as required. $\square$

4. Time dependent branching processes. In this section we prove Theorem 1.6, giving estimates for the survival probabilities for a family of nonhomogenous branching process which are closely related to contagious sets in $\mathcal{G}_{n,p}$.

Recall that in our branching process, the nth individual has a Poisson number of children with mean $\binom{n}{r-1}\varepsilon$. This does not specify the order of the individuals (i.e., which of these children is next). While the order would affect the resulting tree, the choice of order clearly does not affect the probability of survival. In light of this, we can use the breadth first order: Define generation 0 to be the first $r-1$ individuals, and let generation k be all children of individuals from generation $k-1$. All individuals in a generation appear in the order before any individual of a later generation. Let Y_t be the size of generation t, and $S_t = \sum_{i\leq t} Y_i$.

Let $\Psi_r(k,i)$ be the probability that for some t we have $S_t = k$ and $Y_t = i$.

LEMMA 4.1. *We have that*

$$\Psi_r(k,i) = \frac{e^{-\varepsilon\binom{k-i}{r}}\varepsilon^{k-r}}{(k-r)!}m_r(k,i).$$

PROOF. We first give an equivalent branching process. Instead of each individual having a number of children, children will have r parents. We start with r individuals (indexed $0,\dots,r-1$), and every subset of size r of the population gives rise to an independent $\mathrm{Poi}(\varepsilon)$ additional individuals. Thus, the initial set of r individuals produces $\mathrm{Poi}(\varepsilon)$ further individuals, indexed $r,\dots$. Individual k together with each subset of $r-1$ of the previous individuals has $\mathrm{Poi}(\varepsilon)$ children, so overall individual k has $\mathrm{Poi}(\binom{k}{r-1}\varepsilon)$ children where k is the maximal parent.

Let X_S be the number of children of a set S of individuals. A graph contributing to $m_r(k,i)$ requires $\mathrm{Poi}(\varepsilon)$ variables to equal X_S, so the probability is $\prod e^{-\varepsilon}\varepsilon^{X_s}/X_S!$. Up to generation t this considers $\binom{k-i}{r}$ sets, and $\sum X_S = k-r$, giving the terms involving ε in the claim. The combinatorial terms $\prod X_S!$ and $(k-r)!$ come from possible labelings of the graph. $\square$

PROOF OF THEOREM 1.6. Up to the $o(1)$ term appearing in the statement of the theorem, the survival of (X_t) is equivalent to the probability p_S that for some t

we have that $S_t \geq k_r$, where $(S_t)_{t\geq 0}$ is as defined above Lemma 4.1 and $k_r = k_r(\varepsilon)$ is as in the theorem. By Lemma 4.1,

$$p_S \geq \sum_i \Psi_r(k_r, i) \geq \frac{e^{-\varepsilon\binom{k_r}{r}}\varepsilon^{k_r-r}}{(k_r-r)!}\sum_i m_r(k_r, i) \geq \frac{e^{-\varepsilon\binom{k_r}{r}}\varepsilon^{k_r-r}}{(k_r-r)!} m_r(k_r).$$

By Lemma 3.5, as $\varepsilon \to 0$, the right-hand side is bounded from below by

$$e^{-o(k_r)}e^{-(r-2)k_r-\varepsilon\binom{k_r}{r}}\left(\varepsilon\frac{k_r^{r-1}}{(r-1)!}\right)^{k_r}\varepsilon^{-r} = e^{-\frac{(r-1)^2}{r}k_r(1+o(1))}.$$

On the other hand, we note that the formula for $\Psi_r(k, i)$ in Lemma 4.1 agrees with the upper bound for $P_r(k, i)$ in Lemma 2.9 [up to the $1 + o(1)$ factor]. Hence, using the bounds in Lemma 2.5 and slightly modifying of the proof of Proposition 2.1 (since here we have Poisson random variables instead of Binomial random variables), it can be shown that

$$p_S \leq e^{o(k_r)}\frac{e^{-\varepsilon\binom{k_r}{r}}\varepsilon^{k_r-r}}{(k_r-r)!} m_r(k_r) = e^{-\frac{(r-1)^2}{r}k_r(1+o(1))}$$

completing the proof. $\square$

As already mentioned in Section 1.3, these branching processes are treated in greater generality in our subsequent work [5].

5. Graph bootstrap percolation. Fix $r \geq 2$ and a graph H. We say that a graph G is (H, r)-*susceptible* if for some $H' \subset G$ we have that H' is isomorphic to H and $V(H)$ is a contagious set for G. We call such a subgraph H' a *contagious copy of* H. Hence, a seed for K_{r+2}-bootstrap percolation, as discussed in Section 1.2, is a contagious copy of K_r. Let $p_c(n, H, r)$ denote the infimum over $p > 0$ such that $\mathcal{G}_{n,p}$ is (H, r)-susceptible with probability at least $1/2$.

By the arguments in Sections 2 and 3, with only minor changes, we obtain the following result. We omit the proof.

THEOREM 5.1. *Fix $r \geq 2$ and $H \subset K_r$ with $v(H) = r$ and $e(H) = e$. Put*

$$\alpha_{r,e} = (r-1)!\left(\frac{(r-1)^2}{r^2-e}\right)^{r-1}.$$

As $n \to \infty$,

$$p_c(n, H, r) = \left(\frac{\alpha_{r,e}}{n\log^{r-1} n}\right)^{1/r}\bigl(1 + o(1)\bigr).$$

We obtain Theorem 1.5, from which Theorem 1.3 follows, as a special case.

PROOF OF THEOREM 1.5. The result follows by Theorem 5.1, taking $r = 2$ and $e = 1$, in which case $\alpha_{2,1} = 1/3$. $\square$

APPENDIX: TECHNICAL LEMMAS

We collect in this appendix several technical results used above.

A.1. Proof of Claim 2.6.

PROOF OF CLAIM 2.6. By the bound $i! > \sqrt{2\pi i}(i/e)^i$, it suffices to verify that

$$\frac{(e/i)^i}{\sqrt{2\pi}}\Lambda(i) \le 1 \qquad \text{for } i \ge 1, \tag{A.1}$$

where $\Lambda(i) = \mathrm{Li}(-i + 1/2, 1/e)$ and $\mathrm{Li}(s, z) = \sum_{j=1}^{\infty} z^j j^{-s}$ is the polylogarithm function.

Let Γ denote the gamma function. From the relationship between Li and the Herwitz zeta function, it can be shown that $\Lambda(i)/\Gamma(i + 1/2) \sim 1$, as $i \to \infty$, and hence $(e/i)^i\Lambda(i) \to \sqrt{2\pi}$, as $i \to \infty$. It appears (numerically) that $(e/i)^i\Lambda(i)$ increases monotonically to $\sqrt{2\pi}$, however this is perhaps not simple to verify (or true). Instead, we find a suitable upper bound for $\Lambda(i)$.

CLAIM A.2. *For all $i \ge 1$, we have that*

$$\Lambda(i) < \Gamma(i + 1/2)\bigl(1 + ab^i\bigr),$$

where $a = \zeta(3/2)$ and $b = e/(2\pi)$, and ζ is the Riemann zeta function.

PROOF. For all $|u| < 2\pi$ and $s \notin \mathbb{N}$, we have the series representation

$$\mathrm{Li}\bigl(s, e^u\bigr) = \Gamma(1 - s)(-u)^{s-1} + \sum_{\ell=0}^{\infty} \frac{\zeta(s - \ell)}{\ell!}u^\ell.$$

Hence

$$\Lambda(i) = \Gamma(i + 1/2) + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!}\zeta(1/2 - i - \ell). \tag{A.2}$$

Recall the functional equation for ζ:

$$\zeta(x) = 2^x\pi^{x-1}\sin(\pi x/2)\Gamma(1 - x)\zeta(1 - x).$$

Therefore, since $\zeta(1/2+x) > 0$ is decreasing in $x \ge 1$ we have that, for all relevant i, ℓ,

$$\bigl|\zeta(1/2 - i - \ell)\bigr| \le a\sqrt{\frac{2}{\pi}}\frac{\Gamma(\ell + i + 1/2)}{(2\pi)^{\ell+i}} < a\frac{\Gamma(\ell + i + 1/2)}{(2\pi)^{\ell+i}}. \tag{A.3}$$

Applying (A.2), (A.3) [and the inequalities $\Gamma(x+\ell) < (x+\ell-1)^\ell\Gamma(x)$, $\ell! > \sqrt{2\pi\ell}(\ell/e)^\ell$, and $(1+x/\ell)^\ell < e^\ell$], we find that, for all $i \geq 1$,

$$\begin{aligned}\frac{\Lambda(i)}{\Gamma(i+1/2)} - 1 &< \frac{a}{(2\pi)^i}\sum_{\ell=0}^{\infty}\frac{(\ell+i-1/2)^\ell}{(2\pi)^\ell \ell!}\\ &< \frac{ab^i}{e^i}\left(1+\sum_{\ell=1}^{\infty}\frac{1}{\sqrt{2\pi\ell}}\left(\frac{e}{2\pi}\left(1+\frac{i-1/2}{\ell}\right)\right)^\ell\right)\\ &< ab^i\left(\frac{1}{e}+\frac{1}{\sqrt{2e\pi}}\sum_{\ell=1}^{\infty}\left(\frac{e}{2\pi}\right)^\ell\right)\\ &< ab^i\end{aligned}$$

establishing the claim. □

By Claim A.2, the formula

$$\Gamma(i+1/2) = \sqrt{\pi}\frac{i!}{4^i}\binom{2i}{i},$$

and the bounds

$$\binom{2i}{i} < \frac{4^i}{\sqrt{\pi i}}\left(1-\frac{1}{9i}\right)$$

and

$$i! < \sqrt{2\pi i}\left(\frac{i}{e}\right)^i\left(1-\frac{1}{12i}\right)^{-1}$$

(valid for all $i \geq 1$), we find that

$$\frac{(e/i)^i}{\sqrt{2\pi}}\Lambda(i) < \frac{4}{3}\frac{9i-1}{12i-1}\left(1+ab^i\right) \qquad \text{for } i \geq 1. \tag{A.4}$$

Differentiating the right-hand side of (A.4), and dividing by the positive term $4/(3(12i-1)^2)$, we obtain

$$3 + ab^i\left(3+\log(b)\left(108i^2-12i+1\right)\right)$$

which, for $i \geq 11$, is bounded from below by

$$3 + 108ab^i\log(b)i^2 > 3 - 237b^i i^2 > 0.$$

Hence, for $i \geq 11$, the right-hand side of (A.4) increases monotonically to 1 as $i \to \infty$. It follows that (A.1) holds for all $i \geq 11$. Inequality (A.1), for $i \leq 10$, can be verified numerically (e.g., by interval arithmetic), completing the proof of Claim 2.6. □

A.2. Proof of Claim 2.13.

PROOF OF CLAIM 2.13. By Lemma 2.10, for all $k = \beta \log n$ and $i = \gamma k$ as in the lemma, we have that

$$E_r(k,i) \lesssim n^{\mu_r(\alpha,\beta,\gamma)} \log^{r(r-1)} n. \tag{A.5}$$

We find a suitable upper bound for $p_r(k,i)$ as follows. For $\beta < \beta_r(\alpha)$, put $\ell_\beta = \xi_\beta \log n$, where $\xi_\beta = \beta_r(\alpha) - \beta$. For a given set V of size k with i vertices identified as the top level, there are $a_r(k,i)$ ways to select r vertices in V with at least one in the top level. Hence, for $k = \beta \log n$ with $\beta < \beta_r(\alpha)$, it follows that

$$p_r(k,i) \le \binom{n}{\ell_\beta} \big(a_r(k,i) p^r\big)^{\ell_\beta}.$$

By Claim 2.4, we have that $a_r(k,i) < ik^{r-1}/(r-1)!$. Hence, applying the bound $\binom{n}{\ell} \le (ne/\ell)^\ell$, we find that

$$p_r(k,i) \le \left(\frac{e\alpha\beta^r\gamma}{\xi_\beta (r-1)!}\right)^{\ell_\beta}.$$

Hence, by Lemma 2.10,

$$E_r(k,i) p_r(k,i) \lesssim n^{\bar{\mu}_r(\alpha,\beta,\gamma)} \log^{r(r-1)} n, \tag{A.6}$$

where

$$\bar{\mu}_r(\alpha,\beta,\gamma) = \mu_r(\alpha,\beta,\gamma) + \xi_\beta \log\left(\frac{e\alpha\beta^r\gamma}{\xi_\beta(r-1)!}\right). \tag{A.7}$$

Therefore, by (A.5), (A.6), we obtain Claim 2.13 by the following fact.

CLAIM A.3. *For any $\gamma \in (0,1)$, we have that*

$$\min\{\mu_r(\alpha,\beta,\gamma), \bar{\mu}_r(\alpha,\beta,\gamma)\} \le \mu_r^*(\alpha,\beta_r)$$

for all $\beta \in (0, \beta_r(\alpha)]$.

PROOF. For convenience, we simplify notation as follows: Put $\beta_r = \beta_r(\alpha)$. We parametrize β using a variable δ: for $\delta \in (0,1]$, let $\beta_\delta = \delta\beta_r$. For $\gamma \in (0,1)$, let $\mu_r(\delta,\gamma) = \mu_r(\alpha,\beta_\delta,\gamma)$, $\bar{\mu}_r(\delta,\gamma) = \bar{\mu}_r(\alpha,\beta_\delta,\gamma)$, and $\delta_\gamma = \delta_\gamma(r) = 1 - \sqrt{\gamma/r}$. Finally, put $\mu_r^* = \mu_r(1,0) = \mu_r^*(\alpha,\beta_r)$. In this notation, Claim A.3 states that

$$\min\{\mu_r(\delta,\gamma), \bar{\mu}_r(\delta,\gamma)\} \le \mu_r^* \qquad \text{for } \delta \in (0,1].$$

Since $\alpha\beta_r^{r-1}/(r-1)! = 1$, it follows that $\alpha\beta_\delta^{r-1}/(r-1)! = \delta^{r-1}$. Therefore, by (2.6), (A.7), we have that

$$\mu_r(\delta,\gamma) = r - \beta_r\left(\frac{\delta^r}{r}(1-\gamma)^r + \delta(r-2+\gamma) - (r-1)\delta\log\delta\right) \tag{A.8}$$

and

$$\bar{\mu}_r(\delta,\gamma) = \mu_r(\delta,\gamma) + \beta_r(1-\delta)\log\left(\frac{e\gamma\delta^r}{1-\delta}\right). \tag{A.9}$$

We obtain Claim A.3 by the following subclaims (as we explain below the statements).

SUBCLAIM A.4. *For any fixed $\gamma \in (0,1)$, we have that $\mu_r(\delta,\gamma)$ and $\bar{\mu}_r(\delta,\gamma)$ are convex and concave in $\delta \in (0,1)$, respectively.*

SUBCLAIM A.5. *For $\gamma \in (0,1)$, we have that*:

(i) $\mu_r(1,\gamma) < \mu_r^*$,
(ii) $\mu_r(\delta_\gamma,\gamma) < \mu_r^*$, *and*
(iii) $e\gamma\delta_\gamma^r/(1-\delta_\gamma) < 1$.

Indeed, by Subclaim A.5(ii), (iii), we have that $\bar{\mu}_r(\delta_\gamma,\gamma) < \mu_r(\delta_\gamma,\gamma) < \mu_r^*$. Therefore, noting that $\lim_{\delta\to 1^-}\bar{\mu}_r(\delta,\gamma) = \mu_r(1,\gamma)$, $\lim_{\delta\to 0^+}\mu_r(\delta,\gamma) = r$, and $\lim_{\delta\to 0^+}\bar{\mu}_r(\delta,\gamma) = -\infty$ [see (A.8), (A.9)], we then obtain Claim A.3 by applying Subclaims A.4 and A.5(i).

PROOF OF SUBCLAIM A.4. By (A.8), for any $\gamma \in (0,1)$, we have that

$$\frac{\partial^2}{\partial\delta^2}\mu_r(\delta,\gamma) = \frac{(r-1)\beta_r}{\delta}\big(1-\delta^{r-1}(1-\gamma)^r\big) > 0$$

for all $\delta \in (0,1)$. Moreover, by (A.8), (A.9), the above expression, and noting that

$$\frac{\partial^2}{\partial\delta^2}(1-\delta)\log\left(\frac{e\gamma\delta^r}{1-\delta}\right) = -\frac{r-(r-1)\delta^2}{\delta^2(1-\delta)},$$

it follows that, for any $\gamma \in (0,1)$,

$$\begin{aligned}
\frac{\partial^2}{\partial\delta^2}\bar{\mu}_r(\delta,\gamma) &= -\frac{\beta_r}{\delta^2(1-\delta)}\big(r-(r-1)\delta^2-\delta(1-\delta)\big(1-\delta^{r-1}(1-\gamma)^r\big)\big) \\
&= -\frac{\beta_r}{\delta^2(1-\delta)}\big(1+(r-1)(1-\delta)\big(1+\delta^r(1-\gamma)^r\big)\big) \\
&< 0
\end{aligned}$$

for all $\delta \in (0,1)$. The claim follows. $\square$

PROOF OF SUBCLAIM A.5. Note that $\mu_r^* = r - \beta_r(r-1)^2/r$. Since, by (A.8),

$$\mu_r(1,\gamma) = r - \beta_r\left(\frac{(1-\gamma)^r}{r} + r - 2 + \gamma\right)$$

claim (i) follows immediately by the inequality $(1-\gamma)^r > 1 - r\gamma$.

Next, we note that by (A.8), to establish claim (ii) we need to show that $f_r(\delta_\gamma, \gamma) > (r-1)^2/r$, where

$$f_r(\delta, \gamma) = \frac{\delta^r}{r}(1-\gamma)^r + \delta(r-2+\gamma) - (r-1)\delta\log\delta.$$

We treat the cases $\gamma \in (0, 1/r)$ and $\gamma \in [1/r, 1)$ separately. By the inequality $\log\delta \le 1-\delta$, we have that

$$f_r(\delta, \gamma) > \delta(r-2+\gamma) - (r-1)\delta(1-\delta).$$

The right-hand side is equal to $(r-1)^2/r$ when $\delta = \delta_\gamma$ and $\gamma = 1/r$ or $\gamma = 1$. Setting $\delta = \delta_\gamma$ in the right-hand side, and differentiating twice with respect to γ, we obtain $-(1+3\gamma)/(4\sqrt{\gamma^3 r}) < 0$. It follows that $f_r(\delta, \gamma) > (r-1)^2/r$ for all $\gamma \in [1/r, 1)$. For $\gamma \in (0, 1/r)$, we note that by the bound $(1-\gamma)^r > 1-\gamma r$,

$$f_r(\delta, \gamma) > \frac{\delta^r}{r}(1-\gamma r) + \delta(r-2+\gamma) - (r-1)\delta\log\delta.$$

Setting $\zeta = \sqrt{\gamma/r}$, $f_r(\delta_\gamma, \gamma)$ is bounded from below by

$$\frac{(1-\zeta)^r}{r}\bigl(1-(r\zeta)^2\bigr) + (1-\zeta)\bigl(r-2+r\zeta^2\bigr) - (r-1)(1-\zeta)\log(1-\zeta).$$

Hence, it suffices to show that this expression is bounded from below by $(r-1)^2/r$ for all $\zeta \in (0, 1/r)$. To this end, we note that it is equal to $(r-1)^2/r$ when $\zeta = 0$, and claim that it is increasing in $\zeta \le 1/r$. Indeed, differentiating with respect to ζ, we obtain

$$(1-\zeta)^{r-1}\bigl(r(r+2)\zeta^2 - 2r\zeta - 1\bigr) - 3r\zeta^2 + 2r\zeta + 1 + (r-1)\log(1-\zeta).$$

Note that $r(r+2)\zeta^2 - 2r\zeta - 1 < 0$ for all $\zeta \in [0, 1/r]$. Hence, since $(1-\zeta)^{r-1} \le (1+(r-1)\zeta)^{-1}$ and $\log(1-\zeta) \ge -\zeta(1+\zeta)$ for all relevant $\zeta \le 1/2$, the above expression is bounded from below by

$$\frac{(r-1)\zeta^2(2(1-2\zeta)r+\zeta)}{1+(r-1)\zeta} > 0.$$

It follows that $f_r(\delta_\gamma, \gamma) > (r-1)^2/r$ for all $\gamma \in (0, 1/r)$. Altogether, claim (ii) is proved.

Finally, for claim (iii), let $g_r(\delta, \gamma) = e\gamma\delta^r/(1-\delta)$. In this notation, claim (iii) states that $g_r(\delta_\gamma, \gamma) < 1$. To verify this inequality, we note that

$$\frac{\partial}{\partial\delta} g_r(\delta, \gamma) = \frac{e\gamma\delta^{r-1}}{(1-\delta)^2}\bigl(r-(r-1)\delta\bigr)$$

and hence

$$\frac{\partial}{\partial\delta} g_r(\delta_\gamma, \gamma) = e\delta_\gamma^{r-1}\bigl(r+(r-1)\sqrt{\gamma r}\bigr).$$

Therefore, noting that

$$\frac{\partial}{\partial\gamma} g_r(\delta,\gamma)\bigg|_{\delta=\delta_\gamma} = \frac{e\delta_\gamma^r}{1-\delta_\gamma} = e\delta_\gamma^{r-1}\left(\sqrt{\frac{r}{\gamma}}-1\right)$$

and

$$\frac{\partial}{\partial\gamma}\delta_\gamma = -\frac{1}{2\sqrt{\gamma r}},$$

it follows that

$$\frac{\partial}{\partial\gamma} g_r(\delta_\gamma,\gamma) = \frac{e\delta_\gamma^{r-1}}{2}\left(\sqrt{\frac{r}{\gamma}}-(r+1)\right).$$

Therefore, for any $r \geq 2$, $g_r(\delta_\gamma,\gamma)$ is maximized at $\gamma = r/(r+1)^2$. Therefore we find that, for all relevant γ,

$$g_r(\delta_\gamma,\gamma) \leq g_r\big(r/(r+1), r/(r+1)^2\big) = e\left(\frac{r}{r+1}\right)^{r+1} < 1$$

giving the claim. $\square$

As discussed, Subclaims A.4 and A.5 imply Claim A.3. $\square$

To conclude, we recall that Claim A.3 implies Claim 2.13. $\square$

A.3. Proof of Claim 3.7.

PROOF OF CLAIM 3.7. We recall the relevant quantities defined in the proof of Lemma 3.6; see (3.7), (3.8), (3.9). We have that

$$\hat{\rho}_r(k,i) \geq \xi' e^{-(r-2)k-h_r(k)} \qquad \text{for } k > k_r \text{ and } i \leq k/\log^2 k,$$

where

$$h_r(k) = f_r(k) - \log g_r\left(k, \frac{k}{\log^2 k}\right) + \frac{k}{\log^2 k}\log\left(\frac{k}{\log^2 k}\right),$$

$f_r(k)$ is nondecreasing and $f_r(k) \ll k$, and $g_r(k,i) = e^{(r-2)i}(\frac{k-i}{k})^{(r-2)k}$. Claim 3.7 states that for some $\xi > 0$, for all large k and $i \leq (\varepsilon/r)^2 k$, we have that $\hat{\rho}_r(k,i) \geq \xi e^{-i\varepsilon-(r-2)k-h_r(k)}$.

SUBCLAIM A.6. *For all $k > k_r$, we have that $h_r(k)$ is increasing in k.*

PROOF. Since $f_r(k)$ is nondecreasing and $k/\log^2 k$ is increasing, it suffices by (3.9) to show that $g_r(k, k/\log^2 k)$ is decreasing for $k > k_r$ (and assuming $r > 2$, as else $g_r \equiv 1$ and so there is nothing to prove). To this end, we note that

$$\frac{\partial}{\partial i} g_r(k,i) = -\frac{(r-2)i}{k-i} g_r(k,i),$$

$$\frac{\partial}{\partial k}\frac{k}{\log^2 k}=\frac{\log k-2}{\log^3 k},$$

and

$$\frac{\partial}{\partial k}g_r(k,i)=\frac{r-2}{k-i}\biggl((k-i)\log\biggl(\frac{k-i}{k}\biggr)+i\biggr)g_r(k,i).$$

Hence, differentiating $g_r(k, k/\log^2 k)$ with respect to k, and dividing by

$$-\frac{(r-2)k}{k(1-\log^{-2}k)\log^3 k}g_r\bigl(k,k/\log^2 k\bigr)<0$$

we obtain

$$\bigl(\log^3 k\bigr)\bigl(1-\log^{-2}k\bigr)\log\biggl(\frac{\log^2 k}{\log^2 k-1}\biggr)-\frac{\log^3 k-\log k+2}{\log^2 k}.$$

By the inequality $\log x>2(x-1)/(x+1)$ (valid for $x>1$), the above expression is bounded from below by

$$\frac{\log^3 k-4\log^2 k-\log k+2}{(\log^2 k)(2\log^2 k-1)}>\frac{\log k-5}{2\log^2 k-1}>0$$

for all $k>k_r$, since $k_r>e^{r/\varepsilon}>e^{r(r+1)}$ and $r>2$. The claim follows. $\square$

Fix some $k_*=k_*(r,\varepsilon)>k_r/(1-(\varepsilon/r)^2)$ so that $k/\log^2 k$ is larger than $9(r/\varepsilon)^4$ and $(r+2)!/(1-\varepsilon)$ for all $k\ge k_*$. Note that, for all $k\ge k_*$ and $i\le(\varepsilon/r)^2k$, we have that $k-i>k_r$. By (3.8), select some $\xi(r,\varepsilon)\le\xi'$ so that the claim holds for all $k>k_r$ and relevant i, provided that either $i\le k/\log^2 k$ or $k\le k_*$.

We establish the remaining cases, $k>k_*$ and $k/\log^2 k<i\le(\varepsilon/r)^2k$, by induction. To this end, let $k>k_*$ be given, and assume that the claim holds for all $k'<k$ and relevant i. By (3.1) it follows that

$$\hat{\rho}_r(k,i)\ge\sum_{j=1}^{k-r-i}\hat{B}_r(k,i,j)\hat{\rho}_r(k-i,j)\qquad(i<k-r),\tag{A.10}$$

where

$$\hat{B}_r(k,i,j)=\frac{j^i}{i!}\biggl(\frac{k-i}{k}\biggr)^{(r-2)k}\biggl(\frac{k-i-j}{k-i}\biggr)^{k-i}\biggl(\frac{(r-1)!}{(k-i)^{r-1}}\frac{\hat{a}_r(k-i,j)}{j}\biggr)^i.$$

SUBCLAIM A.7. *For all $(r+2)!\le i, j\le k/r^2$, we have that*

$$\hat{B}_r(k,i,j)\ge\frac{j^i}{i!}\biggl(\frac{k-i}{k}\biggr)^{(r-2)k}\biggl(\frac{k-i-j}{k-i}\biggr)^{k+(r-2)i}.$$

PROOF. By the formula for $\hat{B}_r(k,i,j)$ above, it suffices to show that

$$\frac{(r-1)!}{(k-i)^{r-1}}\frac{\hat{a}_r(k-i,j)}{j} > \left(\frac{k-i-j}{k-i}\right)^{r-1}.$$

To this end, we note that by (3.2) and Claim 2.4 the left-hand side is bounded from below by

$$\frac{1}{j}\sum_{\ell=1}^{j}\left(\frac{k-i-\ell}{k-i}\right)^{r-1} - \frac{2r!}{k-i}.$$

Since, for any integer m, $(1-y/x)^m - (1-(y+1/2)/x)^m$ is decreasing in y, for $y<x$, it follows that

$$\frac{1}{j}\sum_{\ell=1}^{j}\left(\frac{k-i-\ell}{k-i}\right)^{r-1} \geq \left(\frac{k-i-(j+1)/2}{k-i}\right)^{r-1}.$$

Thus, applying the inequalities $1-xy \leq (1-x)^y \leq 1/(1+xy)$, we find that

$$\frac{(r-1)!}{(k-i)^{r-1}}\frac{\hat{a}_r(k-i,j)}{j} - \left(\frac{k-i-j}{k-i}\right)^{r-1}$$

is bounded from below by

$$1 - \frac{(j+1)(r-1)}{2(k-i)} - \frac{2r!}{k-i} - \frac{1}{1+j(r-1)/(k-i)}$$

which equals

$$\frac{((r-1)j-(r+4r!-1))(k-i) - ((r-1)j+(r+4r!-1))(r-1)j}{2(k-i)(k-i+(r-1)j)}.$$

It thus remains to show that the numerator in the above expression is nonnegative, for all i, j as in the claim. To see this, we observe that $r+4r!-1 < (r-1)(r+2)!$ for all $r \geq 2$. Hence, for $(r+2)! \leq i, j \leq k/r^2$ and $r \geq 2$, the numerator divided by $(r-1)k > 0$ is bounded from below by

$$\big(j-(r+2)!\big)\left(1-\frac{1}{r^2}\right) - \big(j+(r+2)!\big)\frac{1}{r^2} = \left(1-\frac{2}{r^2}\right)\big(j-(r+2)!\big) \geq 0$$

as required. The claim follows. □

Applying Subclaim A.7, the inductive hypothesis (recalling that $k-i > k_r$ by the choice of k_*), and the bound $i! < 3\sqrt{i}(i/e)^i$ to (A.10), it follows that

$$\hat{\rho}_r(k,i) > \xi \frac{e^{-(r-2)k+(r-1)i-h_r(k-i)}}{3\sqrt{i}}\left(\frac{k-i}{k}\right)^{(r-2)k}\sum_{j\in\mathcal{J}_{r,\varepsilon}}\psi_{r,\varepsilon}(i/k, j/i)^k, \tag{A.11}$$

where $\mathcal{J}_{r,\varepsilon}(k,i)$ is the set of j satisfying $(r+2)! \leq j \leq (\varepsilon/r)^2(k-i)$, and

$$\psi_{r,\varepsilon}(\gamma,\delta) = \delta^{\gamma} e^{-\delta\gamma\varepsilon}\left(1-\frac{\delta\gamma}{1-\gamma}\right)^{1+\gamma(r-2)}.$$

SUBCLAIM A.8. *Put $\delta_\varepsilon = 1 - \varepsilon$ and $\delta_{r,\varepsilon} = \delta_\varepsilon + (\varepsilon/r)^2$. For any fixed $\gamma \le (\varepsilon/r)^2$, we have that $\psi_{r,\varepsilon}(\gamma, \delta)$ is increasing in δ, for $\delta \in [\delta_\varepsilon, \delta_{r,\varepsilon}]$.*

PROOF. Differentiating $\psi_{r,\varepsilon}(\gamma, \delta)$ with respect to δ, we obtain

$$\frac{\psi_{r,\varepsilon}(\gamma, \delta)\gamma}{\delta(1-\gamma-\delta\gamma)}\bigl(\varepsilon\gamma\delta^2 - \bigl(1+\varepsilon+\gamma(r-1-\varepsilon)\bigr)\delta + 1 - \gamma\bigr).$$

Hence, to establish the claim, it suffices to show that

$$\varepsilon\gamma\delta_{r,\varepsilon}^2 - \bigl(1+\varepsilon+\gamma(r-1-\varepsilon)\bigr)\delta_{r,\varepsilon} + 1 - \gamma$$

is positive for relevant γ. Moreover, since the above expression is decreasing in γ, we need only verify the case $\gamma = (\varepsilon/r)^2$. Setting γ as such in the above expression, and then dividing by ε^2/r^6, we obtain

$$r^6 - (1-\varepsilon)r^5 - \bigl(1+3\varepsilon^2-\varepsilon^3\bigr)r^4 - r^3\varepsilon^2 + \varepsilon^2\bigl(1+3\varepsilon-2\varepsilon^2\bigr)r^2 + \varepsilon^5.$$

For $\varepsilon < 1/r$ and $r \ge 2$, this expression is bounded from below by

$$r\bigl(r^5 - r^4 - \bigl(1+3/r^2\bigr)r^3 - 1\bigr) \ge r > 0$$

as required, giving the claim. $\square$

By the choice of k_* and since $k > k_*$, for all relevant $k/\log^2 k \le i \le (\varepsilon/r)^2 k$, we have that $\delta_\varepsilon i \ge (r+2)!$ and

$$\frac{\delta_{r,\varepsilon} i}{k-i} \le (\varepsilon/r)^2 \frac{1-\varepsilon+(\varepsilon/r)^2}{1-(\varepsilon/r)^2} \le (\varepsilon/r)^2,$$

where the second inequality follows since

$$\frac{\partial}{\partial\varepsilon}\frac{1-\varepsilon+(\varepsilon/r)^2}{1-(\varepsilon/r)^2} = -r^2\frac{(r^2+\varepsilon^2-4\varepsilon)}{(r-\varepsilon)^2(r+\varepsilon)^2} < 0$$

for all $r \ge 2$. Hence, for all such i, k, we have that $j \in \mathcal{J}_{r,\varepsilon}(k,i)$ for all $j \in [\delta_\varepsilon, \delta_{r,\varepsilon}]$. Therefore, for any such i, k, by (A.11) and Subclaim A.8, we have that

$$\begin{aligned}
\hat{\rho}_r(k,i) &> \xi \frac{e^{-(r-2)k+(r-1)i-h_r(k-i)}}{3\sqrt{i}}\left(\frac{k-i}{k}\right)^{(r-2)k} \sum_{\delta_\varepsilon i \le j \le \delta_{r,\varepsilon} i} \psi_{r,\varepsilon}(i/k, j/i)^k \\
&> \xi \frac{(\delta_{r,\varepsilon}-\delta_\varepsilon)\sqrt{i}}{3} e^{-(r-2)k+(r-1)i-h_r(k-i)}\left(\frac{k-i}{k}\right)^{(r-2)k} \psi_{r,\varepsilon}(i/k, \delta_\varepsilon)^k \\
&> \xi e^{-(r-2)k+(r-1)i-h_r(k-i)}\left(\frac{k-i}{k}\right)^{(r-2)k} \psi_{r,\varepsilon}(i/k, \delta_\varepsilon)^k,
\end{aligned}$$

where the last inequality follows since $k > k_*$ and $i \ge k/\log^2 k$ and by the choice of k_*, we have that $\delta_{r,\varepsilon} - \delta_\varepsilon = (\varepsilon/r)^2 > 3/\sqrt{i}$.

SUBCLAIM A.9. *Fix $k/\log^2 k \le i \le (\varepsilon/r)^2 k$, and define $\zeta_r(k,i)$ such that*

$$\hat{\rho}_r(k,i) = \xi e^{-\zeta_r(k,i)\varepsilon i-(r-2)k-h_r(k)}.$$

We have that $\zeta_r(k,i) < 1$.

PROOF. Letting $\gamma = i/k$, it follows by the bound for $\hat{\rho}_r(k,i)$ above, and since $k > k_*$ and hence $h_r(k-i) < h_r(k)$ by Subclaim A.6 and the choice of k_*, that $\zeta_r(k,i)$ is bounded from above by

$$\delta_\varepsilon - \frac{r-1}{\varepsilon} - \frac{r-2}{\varepsilon\gamma}\log(1-\gamma) - \frac{1}{\varepsilon}\log\delta_\varepsilon - \frac{1+\gamma(r-2)}{\varepsilon\gamma}\log\left(1-\frac{\delta_\varepsilon\gamma}{1-\gamma}\right).$$

Recall that $\delta_\varepsilon = 1-\varepsilon$. Applying the bound $-\log(1-x) \le x/(1-x)$ for $x=\gamma$ and $x = \delta_\varepsilon\gamma/(1-\gamma)$, and the bound $-\log(1-x) \le x + (1+x)x^2/2$ for $x=\varepsilon$ [valid for any $x < 1/3$, and so for all relevant $\varepsilon < 1/(r+1)$ with $r \ge 2$], we find that the expression above is bounded from above by

$$\nu(\varepsilon,\gamma) = 2 - \frac{\varepsilon(1-\varepsilon)}{2} - \frac{1-(r-1)\gamma}{\varepsilon(1-\gamma)} + \frac{(1-\varepsilon)(1+(r-2)\gamma)}{\varepsilon(1-(2-\varepsilon)\gamma)}.$$

Therefore, noting that

$$\frac{\partial}{\partial\gamma}\nu(\varepsilon,\gamma) = \frac{r-2}{\varepsilon(1-\gamma)^2} + \frac{(1-\varepsilon)(r-\varepsilon)}{\varepsilon(1-(2-\varepsilon)\gamma)^2} > 0,$$

to establish the subclaim, it suffices to verify that $\nu(\varepsilon,(\varepsilon/r)^2) < 1$ for all $r \ge 2$ and $\varepsilon < 1/(r+1)$. Furthermore, since

$$\nu\big(\varepsilon,(\varepsilon/r)^2\big) = 2 - \frac{\varepsilon(1-\varepsilon)}{2} - \frac{r^2-\varepsilon^2(r-1)}{\varepsilon(r^2-\varepsilon^2)} + \frac{(1-\varepsilon)(r^2+\varepsilon^2(r-2))}{\varepsilon(r^2-2\varepsilon^2+\varepsilon^3)}$$

and hence

$$\frac{\partial}{\partial r}\nu\big(\varepsilon,(\varepsilon/r)^2\big) = -\frac{\varepsilon(r(r-4)+\varepsilon^2)}{(r^2-\varepsilon^2)^2} - \frac{\varepsilon(1-\varepsilon)(r(r-2\varepsilon)+\varepsilon^2(2-\varepsilon))}{(r^2-2\varepsilon^2+\varepsilon^3)^2} < 0$$

for all $k \ge 4$ and $\varepsilon < 1$, we need only verify the cases $r \le 4$.

To this end, let $\eta(r,\varepsilon)$ denote the difference of the numerator and denominator of $\nu(\varepsilon,(\varepsilon/r)^2)$ (in its factorized form), namely

$$-\varepsilon^7 + 3\varepsilon^6 + (r^2-4)\varepsilon^5 - 2(2r^2-2r+1)\varepsilon^4 + (5r^2-6r+8)\varepsilon^3 + r^2(r^2-2r-2)\varepsilon^2 - r^2(r-2)^2\varepsilon.$$

For all $\varepsilon < 1/3$, we have that

$$\eta(2,\varepsilon) = -\varepsilon^2(1-\varepsilon)(2-\varepsilon)(2+\varepsilon)\big(2-2\varepsilon+\varepsilon^2\big) < -\varepsilon^2 < 0.$$

Similarly,

$$\eta(3,\varepsilon) = -\varepsilon\big(9-9\varepsilon-35\varepsilon^2+26\varepsilon^3-5\varepsilon^4-3\varepsilon^5+\varepsilon^6\big) < -\varepsilon < 0$$

and

$$\eta(4,\varepsilon) = -\varepsilon\bigl(64 - 96\varepsilon - 64\varepsilon^2 + 50\varepsilon^3 - 12\varepsilon^4 - 3\varepsilon^5 + \varepsilon^6\bigr) < -\varepsilon < 0.$$

It follows that $\nu(\varepsilon, (\varepsilon/r)^2) < 1$ for all $\varepsilon < 1/3$ and $k \leq 4$, and hence for all $k \geq 2$, giving the subclaim. $\square$

By Subclaim A.9, we find that $\hat{\rho}_r(k,i) = \xi e^{-\varepsilon i-(r-2)k-h_r(k)}$ for all i,k such that $k/\log^2 k \leq i \leq (\varepsilon/r)^2 k$, completing the induction, and thus giving Claim 3.7. $\square$

A.4. Proof of Lemma 3.11.

PROOF OF LEMMA 3.11. Put $\alpha_{r,\varepsilon} = (1+\varepsilon)\alpha_r$. Let $\beta_r = \beta_r(\alpha_{r,\varepsilon})$ and $\beta_* = \beta_*(\alpha_{r,\varepsilon})$. For $\beta > 0$ and $\gamma \in [0,1)$, let $\mu_{r,\varepsilon}(\beta,\gamma) = \mu_{r,\varepsilon}(\alpha_{r,\varepsilon},\beta,\gamma)$ and $\mu_r^*(\beta) = \mu_r^*(\alpha_{r,\varepsilon},\beta)$. Let $\gamma_{r,\varepsilon}^*(\beta)$ denote the maximizer of $\mu_{r,\varepsilon}(\beta,\gamma)$ over $\gamma \in [0,1)$, which is well defined, since for all $\gamma \in (0,1)$,

$$\text{(A.12)} \qquad \frac{\partial^2}{\partial\gamma^2}\mu_{r,\varepsilon}(\beta,\gamma) - \frac{\beta}{(1-\gamma)^2} - \frac{\alpha_{r,\varepsilon}\beta^r}{(r-2)!}(1-\gamma)^{r-2} < 0$$

and $\lim_{\gamma\to 1^-}\mu_{r,\varepsilon}(\beta,\gamma) = -\infty$. Finally, put $\gamma_{r,\varepsilon}(\beta) = \min\{\gamma_{r,\varepsilon}^*(\beta), (\varepsilon/r)^2\}$.

We show that $\mu_{r,\varepsilon}(\beta,\gamma_{r,\varepsilon}(\beta))$ is bounded away from 0 for $\beta \in [\beta_r^*, \beta_r^*+\delta]$, for some $\delta > 0$. By Lemma 3.10, the result follows.

CLAIM A.10. *For $\gamma \in (0,1)$, let*

$$\beta_{r,\varepsilon}(\gamma) = \frac{(1/(1-\gamma)+\varepsilon)^{1/(r-1)}}{1-\gamma}\beta_r$$

and put

$$\beta_{r,\varepsilon} = \lim_{\gamma\to 0^+}\beta_{r,\varepsilon}(\gamma) = (1+\varepsilon)^{1/(r-1)}\beta_r.$$

We have that:

(i) $\gamma_{r,\varepsilon}^*(\beta) = 0$, *for all* $\beta \leq \beta_{r,\varepsilon}$,
(ii) *for* $\beta > \beta_{r,\varepsilon}$, $\gamma = \gamma_{r,\varepsilon}^*(\beta)$ *if and only if* $\beta = \beta_{r,\varepsilon}(\gamma)$, *and*
(iii) $\gamma_{r,\varepsilon}^*(\beta)$ *is increasing in* β, *for* $\beta \geq \beta_{r,\varepsilon}$.

PROOF. By (A.12), we have that $\mu_{r,\varepsilon}(\beta,\gamma)$ is concave in γ. Therefore, since

$$\frac{\partial}{\partial\gamma}\mu_{r,\varepsilon}(\beta,\gamma) - \beta\biggl(\frac{1}{1-\gamma} + \varepsilon - \frac{\alpha_{r,\varepsilon}\beta^{r-1}}{(r-1)!}(1-\gamma)^{r-1}\biggr)$$

and hence, for any $\xi > 0$,

$$\frac{\partial}{\partial\gamma}\mu_{r,\varepsilon}(\xi\beta_r,\gamma) = -\xi\beta_r\biggl(\frac{1}{1-\gamma} + \varepsilon - \xi^{r-1}(1-\gamma)^{r-1}\biggr),$$

the first two claims follow. The third claim is a consequence of the second claim and the fact that $\beta_{r,\varepsilon}(\gamma)$ is increasing in γ. □

By the following claims, we obtain the lemma (as we discuss below the statements).

CLAIM A.11. *For $\beta > 0$ and $\gamma \in [0, 1)$, let*

$$\omega_{r,\varepsilon}(\beta,\gamma) = \mu_{r,\varepsilon}(\beta,\gamma) - \mu_r^*(\beta).$$

We have that:

(i) $\omega_{r,\varepsilon}(\beta, \gamma_{r,\varepsilon}(\beta)) = 0$, *for all $\beta \le \beta_{r,\varepsilon}$, and*
(ii) $\omega_{r,\varepsilon}(\beta, \gamma_{r,\varepsilon}(\beta))$ *is increasing in β, for $\beta \ge \beta_{r,\varepsilon}$.*

CLAIM A.12. *We have that $\beta_{r,\varepsilon} < \beta_*$.*

Indeed, the claims together imply that $\omega_{r,\varepsilon}(\beta_*, \gamma_{r,\varepsilon}(\beta_*)) > 0$. Therefore, since $\mu_r^*(\beta_*) = 0$, we thus have that $\mu_{r,\varepsilon}(\beta_*, \gamma_{r,\varepsilon}(\beta_*)) > 0$. Therefore, by the continuity of $\mu_{r,\varepsilon}(\beta, \gamma_{r,\varepsilon}(\beta))$ in β, it follows that $\mu_{r,\varepsilon}(\beta, \gamma_{r,\varepsilon}(\beta)) > 0$ for all $\beta \in [\beta_*, \beta_* + \delta]$, for some $\delta > 0$. As discussed the lemma follows, applying Lemma 3.10.

PROOF OF CLAIM A.11. We note that the first claim follows by (3.10) and Claim A.10(i).

For the second claim, we show that (a) $\omega_{r,\varepsilon}(\beta, \gamma^*_{r,\varepsilon}(\beta))$ is increasing in β, for $\beta \ge \beta_{r,\varepsilon}$ such that $\gamma^*_{r,\varepsilon}(\beta) \le (\varepsilon/r)^2$, and (b) $\omega_{r,\varepsilon}(\beta, (\varepsilon/r)^2)$ is increasing in β, for $\beta \ge \beta_{r,\varepsilon}$. By Claim A.10(iii), this implies the claim.

Since $\gamma^*_{r,\varepsilon}(\beta)$ maximizes $\mu_{r,\varepsilon}(\beta,\gamma)$, and so $\partial\omega_{r,\varepsilon}(\beta, \gamma^*_{r,\varepsilon}(\beta))/\partial\gamma = 0$, it follows that

$$\frac{\partial}{\partial\beta}\omega_{r,\varepsilon}\big(\beta, \gamma^*_{r,\varepsilon}(\beta)\big) = \frac{\partial}{\partial\beta}\omega_{r,\varepsilon}(\beta,\gamma)\bigg|_{\gamma=\gamma^*_{r,\varepsilon}(\beta)}.$$

Hence, by Claim A.10(ii), to establish (a) we show that for all $\gamma \le (\varepsilon/r)^2$, $\partial\omega_{r,\varepsilon}(\beta_{r,\varepsilon}(\gamma),\gamma)/\partial\beta > 0$. To this end, we observe that

$$\frac{\partial}{\partial\beta}\omega_{r,\varepsilon}(\beta,\gamma) = \log(1-\gamma) - \varepsilon\gamma + \frac{\alpha_{r,\varepsilon}\beta^{r-1}}{(r-1)!}\big(1-(1-\gamma)^r\big). \tag{A.13}$$

Setting $\beta = \beta_{r,\varepsilon}(\gamma)$, the above expression simplifies as

$$\log(1-\gamma) - \varepsilon\gamma + \frac{1/(1-\gamma)+\varepsilon}{(1-\gamma)^{r-1}}\big(1-(1-\gamma)^r\big).$$

By the inequalities $(1-x)^y \le 1/(1+xy)$ and $\log(1-x) \ge -x/(1-x)$, this expression is bounded from below by

$$-\frac{\gamma}{1-\gamma} - \varepsilon\gamma + \big(1+(r-1)\gamma\big)\left(\frac{1}{1-\gamma}+\varepsilon\right)\left(1-\frac{1}{1+\gamma r}\right)$$

which factors as

$$\frac{\gamma(1+\varepsilon(1-\gamma))}{(1-\gamma)(1+\gamma r)}\big(r-1+\gamma r(r-2)\big)>0$$

and (a) follows.

Similarly, we note that by (A.13), for any $\beta \geq \beta_{r,\varepsilon}$ and $\gamma > 0$,

$$\begin{aligned}\frac{\partial}{\partial\beta}\omega_{r,\varepsilon}(\beta,\gamma) &\geq \log(1-\gamma)-\varepsilon\gamma+\frac{\alpha_{r,\varepsilon}\beta_{r,\varepsilon}^{r-1}}{(r-1)!}\big(1-(1-\gamma)^r\big)\\ &= \log(1-\gamma)-\varepsilon\gamma+(1+\varepsilon)\big(1-(1-\gamma)^r\big).\end{aligned}$$

Hence, using the same bounds for $(1-x)^y$ and $\log(1-x)$ as above, we find that for all such $\beta \geq \beta_{r,\varepsilon}$, $\partial\omega_{r,\varepsilon}(\beta,(\varepsilon/r)^2)/\partial\beta$ is bounded from below by

$$\frac{\varepsilon^2(r^3(r-1)(1+\varepsilon)-2r^2\varepsilon^2-r(2r-1)\varepsilon^3+\varepsilon^5)}{(r-\varepsilon)(r+\varepsilon)(r+\varepsilon^2)r^2}.$$

For $\varepsilon < 1/r$, the numerator is bounded from below by

$$\varepsilon^2\left(r^3(r-1)-2-\frac{2r-1}{r^2}\right)=\frac{\varepsilon^2}{r}\big(r^6-r^5-2r^2-2r+1\big)>0$$

since $r \geq 2$. Hence $\partial\omega_{r,\varepsilon}(\beta,(\varepsilon/r)^2)/\partial\beta > 0$, giving (b), and thus completing the proof of the second claim. $\square$

PROOF OF CLAIM A.12. By Lemma 2.11, the claim is equivalent to the inequality $\mu_r^*(\beta_{r,\varepsilon}) > 0$. To see this, we note that

$$\beta_r=\left(\frac{(r-1)!}{\alpha_{r,\varepsilon}}\right)^{1/(r-1)}=\left(\frac{1}{1+\varepsilon}\right)^{1/(r-1)}\left(\frac{r-1}{r}\right)^2,$$

and hence by (2.7), for any $\xi > 0$, we have that

$$\begin{aligned}\mu_r^*(\xi^{1/(r-1)}\beta_r) &= r-\xi^{1/(r-1)}\beta_r\left(r-2+\frac{\xi}{r}-\log\xi\right)\\ &= r-\left(\frac{r}{r-1}\right)^2\left(\frac{\xi}{1+\varepsilon}\right)^{1/(r-1)}\left(r-2+\frac{\xi}{r}-\log\xi\right).\end{aligned}$$

In particular,

$$\mu_r^*(\beta_{r,\varepsilon})=r-\left(\frac{r}{r-1}\right)^2\left(r-2+\frac{1+\varepsilon}{r}-\log(1+\varepsilon)\right).$$

Therefore, by the bound $\log(1+x) > x/(1+x)$, we find that

$$\mu_r^*(\beta_{r,\varepsilon})>\frac{\varepsilon r(r-1-\varepsilon)}{(1+\varepsilon)(r-1)^2}>0$$

as required. $\square$

As discussed, Lemma 3.11 follows by Claims A.11 and A.12. $\square$

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