The Annals of Applied Probability
2018, Vol. 28, No. 1, 482–510
https://doi.org/10.1214/17-AAP1310

BSDES WITH MEAN REFLECTION

BY PHILIPPE BRIAND, ROMUALD ELIE[1] AND YING HU[2]

Université de Savoie, Université Paris-Est & Projet MathRisk ENPC-INRIA-UMLV and Université Rennes 1

In this paper, we study a new type of BSDE, where the distribution of the Y-component of the solution is required to satisfy an additional constraint, written in terms of the expectation of a loss function. This constraint is imposed at any deterministic time t and is typically weaker than the classical pointwise one associated to reflected BSDEs. Focusing on solutions (Y, Z, K) with deterministic K, we obtain the well-posedness of such equation, in the presence of a natural Skorokhod-type condition. Such condition indeed ensures the minimality of the enhanced solution, under an additional structural condition on the driver. Our results extend to the more general framework where the constraint is written in terms of a static risk measure on Y. In particular, we provide an application to the super-hedging of claims under running risk management constraint.

1. Introduction. Since the seminar work of Pardoux and Peng [14], strong connections between backward stochastic differential equations (BSDEs) and stochastic control problems have been extensively documented. The solution of a BSDE typically consists of an adapted pair of processes (Y, Z) with the following dynamics:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s \cdot dB_s, \qquad 0 \le t \le T.$$

In [14], Pardoux and Peng provided the existence of a unique solution (Y, Z) to above equation for a given square integrable terminal condition ξ and a Lipschitz random driver f. Since then, many extensions have been derived in several directions. For example, the regularity of the driver has been weakened. The underlying dynamics can contain jumps. These extensions allow to provide representation of solutions to a large class of stochastic control problems, and to tackle many problems in mathematical finance.

Received May 2016; revised January 2017.

[1]Supported in part by the ANR grant LIQUIRISK and the Chair Finance and Sustainable Development.

[2]Supported in part by Lebesgue center of mathematics ("Investissements d'avenir" program—ANR-11-LABX-0020-01 and by ANR-15-CE05-0024-02).

MSC2010 subject classifications. 60H10, 91G10.

Key words and phrases. Backward stochastic differential equations, mean reflection, Skorokhod type minimal condition, super-hedging, risk management constraint.

More interestingly, the consideration of additional conditions on the stochastic control problems of interest naturally led to the consideration of constrained BSDEs. In such a case, the solution of a constrained BSDE contains an additional adapted nondecreasing process K, such that (Y, Z, K) satisfies

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds - \int_t^T Z_s \cdot dB_s + K_T - K_t, \qquad 0 \le t \le T, \tag{1}$$

together with a chosen constraint on the solution. The process K interprets as the extra cost on the value process Y, due to the additional constraint. In such a framework, this equation admits an infinite number of solutions, as the roles of Y and K are too closely connected. The underlying stochastic control problem of interest typically indicates that one should look for the minimal solution (in terms of Y) of such equation. Motivated by optimal stopping or related obstacle problems, El Karoui et al. [10] introduced the notion of a reflected BSDE, where the constraint is of the form

$$Y_t \ge L_t, \qquad 0 \le t \le T.$$

The obstacle process L provides a lower bound on the solution Y and is interpreted as the reward payoff. It is worth noticing that the minimal solution (Y, Z, K) is fully characterized by the following so-called Skorokhod condition:

$$\int_0^T (Y_t - L_t)^+\,dK_t = 0.$$

This condition intuitively indicates that the process K is only allowed to push the value process Y whenever the constraint is binding.

The class of constrained BSDEs has been significantly enlarged in the recent literature. Cvitanic and Karatzas [8] utilized a doubly reflected BSDE, where the process Y lies in between two processes, to study zero sum Dynkin games. Considering super-hedging problems where the admissible portfolios are restricted to a convex set $\mathbf{C}$ (e.g., $\mathbf{C} = \mathbb{R}^+$ for no short sell constraints), Buckdahn and Hu [3, 4] and Cvitanic et al. [9] studied the well-posedness of BSDE (1) together with the constraint: $Z_t \in \mathbf{C}$, for $t \in [0, T]$. More generally, Peng and Xu [15] considered pointwise constraints of the form $\varphi(t, Y_t, Z_t) \ge 0$, where φ is nondecreasing in y. Furthermore, the study of optimal switching problems [6, 11–13] led to multidimensional systems of BSDEs with oblique reflections.

In contrast to the previously mentioned pointwise constraints on the solution, Bouchard et al. [2] introduced a type of BSDE with weak terminal condition, which emerges from quantile hedging or related controlled loss control problems. In their framework, the terminal condition is replaced by a constraint on the distribution of the random variable Y_T, more specifically, it is required that

$$\mathbb{E}\big[\ell(Y_T - \xi)\big] \ge 0,$$

for some loss function ℓ.

The purpose of this paper is to determine the impact of a dynamic version of such type of constraint, by studying the BSDE (1) together with a running constraint in expectation of the form

$$\mathbb{E}\big[\ell(t, Y_t)\big] \geq 0, \qquad 0 \leq t \leq T, \tag{2}$$

where $(\ell(t,\cdot))_{0\leq t\leq T}$ is a collection of nondecreasing (random) functions. It is worth noticing that the previous running constraint is only imposed on deterministic times $t \in [0, T]$. In the spirit of the above mentioned Skorokhod condition for reflected BSDEs, we look towards so-called flat solutions, that is, satisfying the extra condition

$$\int_0^T \mathbb{E}\big[\ell(t, Y_t)\big]\, dK_t = 0. \tag{3}$$

In terms of applications, it is worth noticing that the constraint (2) can be replaced by a more general version of the form

$$\rho(t, Y_t) \leq q_t, \qquad 0 \leq t \leq T, \tag{4}$$

where $(\rho(t,\cdot))_t$ is a time indexed collection of static risk measures, and $(q_t)_t$ are associated benchmark levels. This framework is in fact the main motivation of this paper, but we chose to present our main argumentation within the constraint (2) for sake of clarity and simplicity. We present in the last section of the paper an application to the super-replication of claims, when the investment portfolio Y satisfies a risk management constraint of the form (4).

Example and main result. In order to introduce the definition of solution that we use in the paper, let us start with the following example. We consider the following BSDE with mean reflexion:

$$\begin{aligned} &Y_t = \xi - \int_t^T \gamma\, ds - \int_t^T Z_s \cdot dB_s + K_T - K_t, \qquad 0 \leq t \leq T,\\ &\mathbb{E}[Y_t] \geq u, \qquad 0 \leq t \leq T, \qquad \int_0^T \big(\mathbb{E}[Y_t] - u\big)\, dK_t = 0, \end{aligned} \tag{5}$$

with $\gamma > 0$, and the terminal condition ξ such that $u < \mathbb{E}[\xi] < u + \gamma T$. Let $t^* \in (0, T)$ be given by $\mathbb{E}[\xi] - \gamma(T - t^*) = u$. It is straightforward to check that the triple (Y, Z, K) defined by

$$\begin{aligned} &Y_t = \mathbb{E}[\xi \mid \mathcal{F}_t] - \gamma(T-t) + \big(\mathbb{E}[\xi] - \gamma(T-t) - u\big)^-,\\ &K_t = \gamma\big(t \wedge t^*\big), \qquad \xi = \mathbb{E}[\xi] + \int_0^T Z_s\, dB_s, \end{aligned}$$

is a flat solution to the BSDE with mean reflexion (5).

For any real α, we set

$$M_t^\alpha := \exp\big(\alpha B_t - \alpha^2 t/2\big) \quad \text{and} \quad K_t^\alpha := \int_0^t M_s^\alpha\, dK_s, \qquad 0 \leq t \leq T.$$

Given K^α, let (Y^α, Z^α) be the solution to the classical BSDE

$$Y_t^\alpha = \xi - \int_t^T \gamma\, ds - \int_t^T Z_s^\alpha\, dB_s + K_T^\alpha - K_t^\alpha, \qquad 0 \le t \le T.$$

For all $0 \le t \le T$, since $\mathbb{E}[M_t^\alpha] = 1$ and K is deterministic, $\mathbb{E}[K_t^\alpha] = K_t$ and $\mathbb{E}[Y_t^\alpha] = \mathbb{E}[Y_t] \ge u$. Moreover, since $\mathbb{E}[Y_t] = u$ for $t \le t^*$, $\mathbb{E}[Y_t] - u = 0$ dK-a.e. and we have

$$\int_0^T (\mathbb{E}[Y_t^\alpha] - u_t)\, dK_t^\alpha = \int_0^T (\mathbb{E}[Y_t^0] - u_t) M_t^\alpha\, dK_t = 0.$$

Hence, for any real α, $(Y^\alpha, Z^\alpha, K^\alpha)$ is also a flat solution to (5). This example emphasizes that BSDEs with mean reflexion may have an infinite number of solutions if one allows K to be random.

A possible way to overcome this difficulty is to work with minimal solutions. Let us assume that our BSDE (5) as a minimal flat solution $(\bar{Y}, \bar{Z}, \bar{K})$. We have, for any real α, for $0 < t \le T$,

$$\begin{aligned}\bar{Y}_t \le Y_t^\alpha &= \mathbb{E}[\xi \mid \mathcal{F}_t] - \gamma(T-t) + \mathbb{E}\left[\int_t^T M_s^\alpha\, dK_s \,\Big|\, \mathcal{F}_t\right] \\ &= \mathbb{E}[\xi \mid \mathcal{F}_t] - \gamma(T-t) + M_t^\alpha (K_T - K_t).\end{aligned}$$

As a byproduct, sending α to $+\infty$, we deduce $\bar{Y}_t \le \mathbb{E}[\xi \mid \mathcal{F}_t] - \gamma(T-t)$ for $t > 0$, and in particular

$$\mathbb{E}[\bar{Y}_t] \le \mathbb{E}[\xi] - \gamma(T-t) \qquad \forall 0 < t \le T.$$

Since $\mathbb{E}[\xi] - \gamma T < u$, for $t > 0$ small enough, $\mathbb{E}[\bar{Y}_t] < u$. The constraint is not satisfied and we get a contradiction: BSDE (5) has no minimal solution.

Taking into account the outputs of this very simple example, in order to get existence and uniqueness of solutions to mean reflected BSDEs, we consider only deterministic flat solutions meaning that the process K is required to be a deterministic function. More precisely, we will show that mean reflected BSDEs of type (1)–(2)–(3) have a unique deterministic flat solution as soon as the driver is Lipschitz continuous and the function ℓ is bi-Lipschitz: there exist $0 < c \le C$ such that

$$c|x-y| \le \left|\ell(t,x) - \ell(t,y)\right| \le C|x-y|.$$

The rest of the paper is organized as follows: Section 2 presents the problem setting, clarifies the assumptions and discusses the main results of the paper. In Section 3, we construct the unique solution to the system (1)–(2)–(3) whenever ℓ is linear and deterministic. The general case is treated in Section 4, where we derive the well-posedness of the system (1)–(2)–(3). In Section 5, we obtain, in a special case, the minimality of the enhanced solution, whereas the mathematical finance application is provided in Section 6.

Notation. Throughout this paper, we are given a finite horizon T and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a d-dimensional standard Brownian motion $B = (B_t)_{0\geq t\leq T}$. We will work with the usual augmented filtration of B, $\{\mathcal{F}_t\}_{0\leq t\leq T}$. Any element $x \in \mathbb{R}^d$ will be identified to a column vector with ith component x^i and Euclidian norm $|x|$. $\mathcal{C}_T$ denotes the set $C([0,T],\mathbb{R})$ of continuous functions from $[0,T]$ to $\mathbb{R}$. For a given set of parameters α, $C(\alpha)$ will denote a constant only depending on these parameters, and which may change from line to line. Finally, we classically denote by:

- $L^2(\mathcal{F}_t)$ the set of real valued $\mathcal{F}_t$-measurable square integrable random variables, for any $t \in [0,T]$;
- $\mathcal{S}^2$ the set of real valued $\mathcal{F}$-adapted continuous processes Y on $[0,T]$ such that $\|Y\|_{\mathcal{S}^2} := \mathbb{E}[\sup_{0\leq r\leq T} |Y_r|^2]^{\frac{1}{2}} < \infty$;
- $\mathcal{H}^2$ the set of predictable $\mathbb{R}^d$-valued processes Z s.t. $\|Z\|_{\mathcal{H}^2} := \mathbb{E}[\int_0^T | Z_r|^2\,dr]^{\frac{1}{2}} < \infty$;
- $\mathcal{A}^2$ is the closed subset of $\mathcal{S}^2$ consisting of nondecreasing processes $K = (K_t)_{0\leq t\leq T}$ with $K_0 = 0$;
- $\mathcal{A}^2_D$ the subset of deterministic elements of $\mathcal{A}^2$.

2. Problem setup.

2.1. *BSDEs with mean reflexion*. The main purpose of this paper is to construct solutions (Y, Z, K) to the following BSDE:

$$(6)\quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds - \int_t^T Z_s \cdot dB_s + K_T - K_t, \qquad 0 \leq t \leq T,$$

$$(7)\quad \mathbb{E}\big[\ell(t, Y_t)\big] \geq 0, \qquad 0 \leq t \leq T,$$

where the second equation is a running constraint in expectation on the component Y of the solution. In opposition to classical reflected BSDE where (7) would typically be a pointwise constraint, the constraint considered here depends on the distribution of Y. We call this new type of constrained equations as *BSDEs with mean reflexion*.

The nondecreasing function ℓ can be interpreted as a loss function and typical examples of interest are:

- $\ell(t, x) = x - u_t$ where u is a deterministic continuous benchmark, that the process Y is required to beat in expectation;
- $\ell(t, x) = \mathbf{1}_{x\geq u_t} - v_t$ (or any smoothed equivalent), so that the process Y is now required to beat deterministic continuous benchmark u with a probability greater than v_t, for any time t;
- $\ell(t, x) = U(x, \xi_t) - u_t$ where U is a concave utility function, $(\xi_t)_t$ is a running random benchmark of interest and $(u_t)_t$ a given deterministic confidence level.

Whenever ℓ is a strictly increasing function, the corresponding classical reflected BSDE is characterized by the dynamics (6) together with the stronger pointwise constraint

$$\ell(t, Y_t) \geq 0, \qquad 0 \leq t \leq T.$$

In such a case, the Y-component of the solution to the BSDE is reflected on the boundary process $([\ell(t, \cdot)]^{-1}(0))$. Observe that our constrained BSDE of interest weakens the condition imposed on Y, from pointwise constraint to a constraint on expectation.

REMARK 1. Observe that condition (7) is required for deterministic time in $[0, T]$, rather than all the possible stopping times smaller than T. In our framework, considering a constraint on all stopping times would strongly strengthen the constraint of interest. On the contrary, both type of pointwise conditions are by construction equivalent for classical reflected BSDEs.

2.2. *Assumptions on coefficients*. The parameters of BSDE with mean reflection are the terminal condition ξ, the driver f as well as the loss function ℓ. These parameters are supposed to satisfy the following assumptions:

(H_f) The driver $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is a measurable map with respect to $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R})$, $\mathcal{P}$ being the sigma algebra of progressive sets of $\Omega \times [0, T]$, and there exists $\lambda \geq 0$ such that, $\mathbb{P}$-a.s., for all $t \in [0, T]$,

$$\forall y, p, z, q, \qquad \left| f(t, y, z) - f(t, p, q) \right| \leq \lambda \big(|y - p| + |z - q| \big)$$

and

$$\mathbb{E}\left[\int_0^T \left| f(t, 0, 0) \right|^2 dt \right] < +\infty.$$

(H_ξ) The terminal condition ξ is a square-integrable $\mathcal{F}_T$-measurable random variable such that

$$\mathbb{E}\big[\ell(T, \xi)\big] \geq 0.$$

(H_ℓ) The loss function $\ell : \Omega \times [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable map with respect to $\mathcal{F}_T \times \mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{R})$ and there exists $C \geq 0$ such that, $\mathbb{P}$-a.s.:

1. $(t, y) \longmapsto \ell(t, y)$ is continuous,
2. $\forall t \in [0, T]$, $y \longmapsto \ell(t, y)$ is strictly increasing,
3. $\forall t \in [0, T]$, $\mathbb{E}[\ell(t, \infty)] > 0$,
4. $\forall t \in [0, T]$, $\forall y \in \mathbb{R}$, $|\ell(t, y)| \leq C(1 + |y|)$.

REMARK 2. We choose to work with Lipschitz and square integrability assumptions on the driver and terminal condition. We restrict to this simple framework, in order to decrease the amount of technical details and emphasize the novelty induced by the additional constraint (7).

REMARK 3. Observe that Condition (H_ξ) ensures that the constraint is automatically satisfied at maturity. This condition implies that no a priori facelift procedure is required on the terminal payoff ξ.

2.3. *Definition of solution, main results and discussion*. We now turn to the definition of a solution to the BSDE (6) with mean reflexion (7).

DEFINITION. A square integrable solution to the BSDE (6) with mean reflection (7) is a triple of processes $(Y, Z, K) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{A}^2$ satisfying (6) together with the constraint (7). A solution is said to be *flat* if moreover K increases only when necessary, that is, when we have

$$\int_0^T \mathbb{E}[\ell(t, Y_t)]\, dK_t = 0. \tag{8}$$

When K is deterministic, that is, $K \in \mathcal{A}^2_D$, we call the solution deterministic.

As discussed in the Introduction, we observe that allowing K to be random leads to the existence of multiple flat solutions. We have also seen that it may induce nonexistence of minimal solution for the BSDE (6) with mean reflection (7). This is why we chose here to restrict to the consideration of so-called *deterministic solutions*, that is, solutions (Y, Z, K) with deterministic compensator K.

In particular, focusing on deterministic solutions, we verify, in some special cases, that the flatness condition (8) can directly imply the minimality property of the solution beyond all the deterministic ones. This is in particular the case for drivers with deterministic linear dependence in y; see Condition (24).

The main result of this paper is the existence and uniqueness of deterministic flat solution to the BSDE (6) with mean reflection (7). This is first achieved for the particular case of linear loss function ℓ; see Proposition 4 and Theorem 5 in Section 3. The proof follows a constructive approach when the driver does not depend on Y and Z, together with a contraction property in order to tackle any Lipschitz driver function. An alternative approach via penalization is also provided in a particular case. When the driver is not linear, the well-posedness of the system (6)–(7)–(8) is also established, under an additional assumption on the loss function, denoted (H_L) below; see Proposition 7 and Theorem 9 in Section 4.

In a similar fashion, we explain in Section 6 below how the constraint in expectation (7) can be replaced by a constraint of the form $\rho(\cdot, Y_\cdot) \le q_\cdot$, where $(\rho(t, \cdot))_t$ is a collection of static risk measures computed at time 0, and q is a collection of time-indexed benchmarks. In particular, solving this equation allows, for example, to represent the super-hedging price of a claim ξ, whenever any admissible portfolios require to satisfy at any date t a running risk management constraint written in terms of risk measures.

REMARK 4. Since the constraint (7) concerns the distribution of the solution to the BSDE, it is tempting to understand the possible connection between such type of BSDE and corresponding constrained McKean Vlasov BSDEs. This topic seems promising in particular for the mean field game literature and is left for further research.

REMARK 5. We wish to point out that the constraint $\mathbb{E}[\ell(t, Y_t)] \geq 0$ is imposed on each $t \in [0, T]$ but not on all stopping times smaller than T. As we will see in the example below, there is no reason to have $\mathbb{E}[\ell(\tau, Y_\tau)] \geq 0$ for all stopping times. Indeed, let us choose $\xi = 1$. For any λ, the deterministic flat solution to

$$Y_t = \xi - \lambda \int_t^T B_s \, ds - \int_t^T Z_s \, dB_s + (K_T - K_t), \qquad \mathbb{E}[Y_t] \geq 0, \qquad 0 \leq t \leq T,$$

is given by

$$Y_t = 1 - \lambda(T - t) B_t, \qquad Z_t = -\lambda(T - t), \qquad K_t = 0,$$

since the unconstrained solution already satisfies $\mathbb{E}[Y_t] = 1 \geq 0$. However, choosing the bounded stopping time $\tau = \inf\{s \geq 0 : B_s \geq 1\} \wedge T$, we have

$$Y_\tau = 1 - \lambda(T - \tau) B_\tau = 1 - \lambda(T - \tau)\mathbf{1}_{\tau < T}, \qquad \mathbb{E}[Y_\tau] = 1 - \lambda \mathbb{E}\big[(T - \tau)\mathbf{1}_{\tau < T}\big].$$

Thus, if λ is large enough, $\mathbb{E}[Y_\tau] < 0$ and the constraint is not satisfied for some bounded stopping time.

We leave the problem of mean reflected BSDEs with constraint on stopping times for further research.

2.4. *A priori estimate.* Let us conclude this section by providing a useful a priori estimate on any solution to the BSDE (6)–(7).

LEMMA 1. *Let* (Y, Z, K) *be a square integrable solution to the BSDE* (6) *with mean reflection* (7). *Then* Y *satisfies the following*:

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} |Y_t|^2\Big] \leq C(\lambda, T) \mathbb{E}\Big[|Y_0|^2 + K_T^2 + \int_0^T \big|f(s, 0, 0)\big|^2 \, ds + \int_0^T |Z_s|^2 \, ds\Big].$$

PROOF. By construction, we have

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s) \, ds + \int_0^t Z_s \cdot dB_s - K_t, \qquad 0 \leq t \leq T.$$

Because K is nondecreasing, Assumption (H_f) leads to

$$\begin{aligned} |Y_t| \leq |Y_0| + K_T + \int_0^T \big|f(s, 0, 0)\big| \, ds + \lambda \int_0^T |Z_s| \, ds \\ + \sup_{0 \leq t \leq T} \left| \int_0^t Z_s \cdot dB_s \right| + \lambda \int_0^t |Y_s| \, ds, \end{aligned}$$

for $t \in [0,T]$. Since Y has continuous paths, Gronwall's lemma gives

$$\sup_{0\le t\le T}|Y_t| \le e^{\lambda T}\biggl(|Y_0| + K_T + \int_0^T |f(s,0,0)|\,ds + \lambda\int_0^T |Z_s|\,ds + \sup_{0\le t\le T}\biggl|\int_0^t Z_s\cdot dB_s\biggr|\biggr),$$

and the result follows from the Burkholder–Davis–Gundy inequality. $\square$

REMARK 6. We deduce from the previous lemma that, when the generator has linear growth, the process Y belongs to $\mathcal{S}^2$ as soon as Z and K are square integrable.

3. The particular case of linear mean reflection. In this section, we consider a special case where the mean reflection is linear. Namely, $\ell : (t,y) \mapsto y - u_t$ so that the condition (7) is replaced by

$$\mathbb{E}[Y_t] \ge u_t, \qquad 0 \le t \le T, \tag{9}$$

where u is a deterministic continuous map from $[0,T]$ to $\mathbb{R}$. Hereby, we impose a running deterministic lower bound u on the expected value of the Y-component of the solution. Besides, we recall that Assumption (H_ξ) ensures that this constraint is already satisfied at maturity so that we have

$$\mathbb{E}[\xi] \ge u_T. \tag{10}$$

In this linear framework, we construct in Proposition 4 an explicit deterministic flat solution (Y,Z,K) to a BSDE with linear mean reflexion (9), when the driver does not depend on Y nor Z. On the other hand, Proposition 3 indicates that uniqueness holds within the class of deterministic flat solutions to (6)–(9).

Hereafter, we first derive an a priori estimate on the solution, and then tackle respectively the uniqueness and existence issues. In order to handle general drivers, the enhanced demonstration relies on a contraction argument, but an alternative approach via penalization is also provided in a particular case.

3.1. *A priori estimate*. The main mathematical advantage of considering a linear loss function ℓ is that it allows to use some of the computational tricks associated to classical reflected BSDEs, when the compensator K is deterministic. As detailed in the proof below, this enables us to derive the following a priori estimate on the solution to the BSDE with linear mean reflexion.

LEMMA 2. *Let (Y,Z,K) be a deterministic square integrable flat solution to the BSDE* (6) *with linear mean reflexion* (9). *Then*

$$\mathbb{E}\biggl[\sup_{0\le t\le T}|Y_t|^2 + \int_0^T |Z_s|^2\,ds\biggr] + K_T^2 \le C(\lambda,T)\biggl(\mathbb{E}\biggl[|\xi|^2 + \int_0^T |f(s,0,0)|^2\,ds\biggr] + \|u\|_\infty^2\biggr).$$

PROOF. Let us recall that the Lipschitz property of f implies

$$2y \cdot f(t,y,z) \le \big|f(t,0,0)\big|^2 + \frac{1}{2}|z|^2 + \big(1+2\lambda+2\lambda^2\big)|y|^2, \qquad \forall (y,z) \in \mathbb{R} \times \mathbb{R}^d.$$

Setting $\beta := 1 + 2\lambda + 2\lambda^2$, Itô's formula together with the previous inequality provides

$$e^{\beta t}|Y_t|^2 + \frac{1}{2}\int_t^T e^{\beta s}|Z_s|^2\,ds \le e^{\beta T}|\xi|^2 + \int_0^T e^{\beta s}\big|f(s,0,0)\big|^2\,ds + 2\int_t^T e^{\beta s} Y_s\,dK_s - 2\int_t^T e^{\beta s} Y_s Z_s \cdot dB_s,$$

for all $t \in [0,T]$. Since K is deterministic and ℓ is linear, we compute

$$2\mathbb{E}\bigg[\int_t^T e^{\beta s} Y_s\,dK_s\bigg] = 2\int_t^T e^{\beta s}\mathbb{E}[Y_s]\,dK_s = 2\int_t^T e^{\beta s}\big(\mathbb{E}[Y_s] - u_s\big)\,dK_s + 2\int_t^T e^{\beta s} u_s\,dK_s.$$

Besides the solution is flat so that condition (8) directly implies

$$2\mathbb{E}\bigg[\int_t^T e^{\beta s} Y_s\,dK_s\bigg] = 2\int_t^T e^{\beta s} u_s\,dK_s \le 2e^{\beta T}\|u\|_\infty K_T.$$

We deduce that

$$\sup_{0\le t\le T}\mathbb{E}\big[e^{\beta t}|Y_t|^2\big] + \mathbb{E}\bigg[\int_0^T e^{\beta s}|Z_s|^2\,ds\bigg] \le 3\bigg(\mathbb{E}\bigg[e^{\beta T}|\xi|^2 + \int_0^T e^{\beta s}\big|f(s,0,0)\big|^2\,ds\bigg] + 2e^{\beta T}\|u\|_\infty K_T\bigg),$$

from which we get, for any $\varepsilon > 0$,

$$\sup_{0\le t\le T}\mathbb{E}\big[|Y_t|^2\big] + \mathbb{E}\bigg[\int_0^T |Z_s|^2\,ds\bigg] \le C(\lambda,T,\varepsilon)\bigg(\mathbb{E}\bigg[|\xi|^2 + \int_0^T \big|f(s,0,0)\big|^2\,ds\bigg] + \|u\|_\infty^2\bigg) + \varepsilon K_T^2. \tag{11}$$

On the other hand, since K is deterministic, we have

$$K_T = \mathbb{E}[K_T] = Y_0 - \mathbb{E}[\xi] - \mathbb{E}\bigg[\int_0^T f(s,Y_s,Z_s)\,ds\bigg],$$

from which we deduce the inequality

$$K_T^2 \le C(\lambda,T)\bigg(\mathbb{E}\bigg[\int_0^T \big|f(s,0,0)\big|^2\,ds\bigg] + \sup_{0\le t\le T}\mathbb{E}\big[|Y_t|^2\big] + \mathbb{E}\bigg[\int_0^T |Z_s|^2\,ds\bigg]\bigg). \tag{12}$$

Combining this estimate with (11) and ε small enough, we get

$$\sup_{0\le t\le T}\mathbb{E}[|Y_t|^2]+\mathbb{E}\left[\int_0^T |Z_s|^2\,ds\right]+|K_T|^2 \\ \le C(\lambda,T)\left(\mathbb{E}\left[|\xi|^2+\int_0^T |f(s,0,0)|^2\,ds\right]+\|u\|_\infty^2\right)$$

and the result follows from Lemma 1. □

3.2. *Uniqueness of the deterministic flat solution.* The uniqueness of flat deterministic solution for a BSDE with linear mean reflection follows mainly from a similar argumentation for a classical reflected BSDE.

PROPOSITION 3. *The BSDE* (6) *with linear mean reflexion* (9) *has at most one square integrable deterministic flat solution.*

PROOF. Let us consider two such solutions (Y^1,Z^1,K^1) and (Y^2,Z^2,K^2) and denote

$$\delta Y := Y^1-Y^2,\qquad \delta Z := Z^1-Z^2\quad\text{and}\quad \delta K := K^1-K^2.$$

Setting $a := 2\lambda+2\lambda^2$ and arguing as in Lemma 2, Itô's formula gives easily

$$e^{at}|\delta Y_t|^2+\frac{1}{2}\int_t^T e^{as}|\delta Z_s|^2\,ds \le -2\int_t^T e^{as}\delta Y_s\delta Z_s\cdot dB_s+2\int_t^T e^{as}\delta Y_s\,d\delta K_s,$$

for $t\in[0,T]$. Let us observe that since both solutions are flat and deterministic and ℓ is linear, we nicely have

$$\begin{aligned}\mathbb{E}\left[\int_t^T e^{as}\delta Y_t\,d\delta K_s\right] &= \int_t^T e^{as}\big[(\mathbb{E}[Y_s^1]-u_s)-(\mathbb{E}[Y_s^2]-u_s)\big]\,dK_s^1 \\ &\quad -\int_t^T e^{as}\big[(\mathbb{E}[Y_s^1]-u_s)-(\mathbb{E}[Y_s^2]-u_s)\big]\,dK_s^2 \\ &= -\int_t^T e^{as}(\mathbb{E}[Y_s^2]-u_s)\,dK_s^1-\int_t^T e^{as}(\mathbb{E}[Y_s^1]-u_s)\,dK_s^2 \\ &\le 0,\end{aligned}$$

for any $t\in[0,T]$. Thus the result follows by taking expectations in the previous inequality. □

As detailed in the Introduction, considering deterministic K processes is a key for the obtention of a unique solution to the BSDE of interest. We now turn to the existence property.

3.3. *Existence of a deterministic flat solution*. We first focus on the particular case where the driver f does not depend on Y nor Z. In this simple case, we construct explicitly the unique solution to a BSDE with linear mean reflection.

PROPOSITION 4. *Let C be a square integrable progressively measurable stochastic process or more generally in the space* $\mathrm{L}^2(\Omega; \mathrm{L}^1(0,T))$. *The BSDE with linear mean reflection*

$$
\begin{aligned}
&Y_t = \xi + \int_t^T C_s\,ds - \int_t^T Z_s \cdot dB_s + K_T - K_t, \\
&\mathbb{E}[Y_t] \ge u_t, \qquad 0 \le t \le T,
\end{aligned} \tag{13}
$$

has a unique square integrable deterministic flat solution.

PROOF. Let us set $x_t = \mathbb{E}[\xi + \int_t^T C_s\,ds]$. By Skorokhod's lemma, there exists a unique pair of deterministic functions $(y, K) : [0,T] \to \mathbb{R}$ such that K is nondecreasing and $K_0 = 0$ and we have

$$
y_t = x_t + K_T - K_t, \qquad y_t \ge u_t, \qquad \int_0^T (y_t - u_t)\,dK_t = 0. \tag{14}
$$

By construction, observe that K is continuous and $K_t = \sup_{0\le s\le T}(x_s - u_s)_- - \sup_{t\le s\le T}(x_s - u_s)_-$. Now, K being given, we know that the BSDE

$$
Y_t = \xi + \int_t^T C_s\,ds - \int_t^T Z_s \cdot dB_s + K_T - K_t, \qquad 0 \le t \le T,
$$

has a unique square integrable solution (Y, Z). Moreover, we have by construction $y_t = \mathbb{E}[Y_t]$. It follows from (14) that (Y, Z, K) is a deterministic flat solution of the BSDE (13). The uniqueness follows from Proposition 3. $\square$

We now turn to the general driver case and we will derive the well-posedness of the BSDE using a well-chosen contraction property.

THEOREM 5. *The BSDE* (6) *with linear mean reflexion* (9) *has a unique deterministic square integrable flat solution*.

PROOF. For given processes $U \in \mathcal{S}^2$ and $V \in \mathcal{H}^2$, let (Y, Z, K) be the deterministic square integrable flat solution to the BSDE

$$
\begin{aligned}
&Y_t = \xi + \int_t^T f(s, U_s, V_s)\,ds - \int_t^T Z_s \cdot dB_s + K_T - K_t, \\
&\mathbb{E}[Y_t] \ge u_t, \qquad 0 \le t \le T,
\end{aligned}
$$

as provided by Proposition 4. Let us show that the mapping $\Phi : (U, V) \longmapsto (Y, Z)$, from $\mathcal{S}^2 \times \mathcal{H}^2$ into itself, has a unique fixed point.

For this purpose, let us denote (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2) the two deterministic square integrable flat solutions to the above BSDE with given (U^1, V^1) and (U^2, V^2), respectively. Set

$$\delta Y := Y^1 - Y^2, \qquad \delta Z := Z^1 - Z^2, \qquad \delta K := K^1 - K^2,$$
$$\delta U := U^1 - U^2, \qquad \delta V := V^1 - V^2.$$

For $a = 4\lambda^2 + 1$, Itô's formula leads to

$$\begin{aligned}|\delta Y_0|^2 &+ \int_0^T e^{as}\left(|\delta Y_s|^2 + |\delta Z_s|^2\right) ds \\ &\leq \frac{1}{2}\int_0^T e^{as}\left(|\delta U_s|^2 + |\delta V_s|^2\right) ds - 2\int_0^T e^{as}\delta Y_s \delta Z_s \cdot dB_s \\ &\quad + 2\int_0^T e^{as}\delta Y_s \, d\delta K_s.\end{aligned}$$

As observed in the proof of Proposition 4, we compute

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{as}\delta Y_s \, d\delta K_s\right] &= -\int_0^T e^{as}\left(\mathbb{E}[Y_s^2] - u_s\right) dK_s^1 - \int_0^T e^{as}\left(\mathbb{E}[Y_s^1] - u_s\right) dK_s^2 \\ &\leq 0.\end{aligned}$$

It follows directly that

$$\mathbb{E}\left[\int_0^T e^{as}\left(|\delta Y_s|^2 + |\delta Z_s|^2\right) ds\right] \leq \frac{1}{2}\mathbb{E}\left[\int_0^T e^{as}\left(|\delta U_s|^2 + |\delta V_s|^2\right) ds\right]. \tag{15}$$

Since we have

$$\begin{aligned}\delta Y_t &= \mathbb{E}\left[\int_t^T \left(f(s, U_s^1, V_s^1) - f(s, U_s^2, V_s^2)\right) ds \,\Big|\, \mathcal{F}_t\right] \\ &\quad + (K_T^1 - K_t^1) - (K_T^2 - K_t^2), \\ K_T^i - K_t^i &= \sup_{t \leq s \leq T}\left(\mathbb{E}\left[\xi + \int_s^T f(r, U_r^i, V_r^i)\, dr\right] - u_s\right)_-,\end{aligned}$$

we get immediately

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |\delta Y_t|^2\right] \leq C\mathbb{E}\left[\int_0^T \left(|\delta U_s|^2 + |\delta V_s|^2\right) ds\right]. \tag{16}$$

As a by-product, Φ is continuous from $\mathcal{S}^2 \times \mathcal{H}^2$ into itself.

Moreover, starting from $(Y^0, Z^0) = (0, 0)$ and setting for $n \geq 1$, $(Y^n, Z^n) = \Phi(Y^{n-1}, Z^{n-1})$, we deduce from (15) that

$$\mathbb{E}\left[\int_0^T |Y_t^{n+1} - Y_t^n|^2 + \int_0^T |Z_t^{n+1} - Z_t^n|^2 \, dt\right] \leq C2^{-n},$$

and that the sequence $\{(Y^n, Z^n)\}_{n\geq 0}$ converges in $\mathcal{H}^2 \times \mathcal{H}^2$ to the unique fixed point of Φ. Finally, from (16), $\{(Y^n, Z^n)\}_{n\geq 0}$ is a Cauchy sequence in $\mathcal{S}^2 \times \mathcal{H}^2$. Hence Φ has a unique fixed point in $\mathcal{S}^2 \times \mathcal{H}^2$. $\square$

In order to handle classical reflected BSDE, a very helpful feature is the characterization of the solution as a limit of corresponding penalized classical BSDEs. The idea simply relies on the addition of a strong penalization on the driver of a classical BSDE, which is only active whenever the constraint is not satisfied. As the penalization strength increases, the Y component of the penalized solution also increases and converges at the limit to the minimal solution of the reflected BSDE. In our framework, the constraint only integrates the distribution of Y, and not the pointwise values of the process Y. For this reason, no comparison argument can ensure that a sequence of penalized BSDEs will be nondecreasing and the classical line of proof falls down. Nevertheless, whenever the benchmark function u is constant, we are able to identify the unique deterministic flat solution of a BSDE with linear mean reflexion as the limit of corresponding penalized BSDEs of the McKean–Vlasov type. This is the purpose of the next proposition proved in a particular case.

PROPOSITION 6. *Suppose that the benchmark $(u_t)_t$ is constant and also denoted u. For any positive integer n, let us consider (Y^n, Z^n) solution to the BSDE of McKean–Vlasov type*

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n)\, ds + \int_t^T n\bigl(u - \mathbb{E}[Y_s^n]\bigr)_+\, ds - \int_t^T Z_s^n \cdot dB_s, \qquad 0 \leq t \leq T,$$

and denote $K^n := \int_0^{\cdot} n(u - \mathbb{E}[Y_s^n])_+\, ds$. As n goes to infinity, (Y^n, Z^n, K^n) converges to the unique flat deterministic solution of the BSDE (6) with linear mean reflexion (9).

PROOF. The proof is postponed to the Appendix. $\square$

4. BSDE with general mean reflection. We now turn to the general case where $x \longmapsto \ell(t, \omega, x)$ is not necessarily linear. We recall that we still work under Assumptions (H_ξ)–(H_f)–(H_ℓ) presented in Section 2. In the same spirit as the approach presented in the previous section, we first construct explicitly a solution whenever the driver does not depend on Y nor Z, and then tackle the general case via a Picard contraction argument. The construction of an explicit solution in the nonlinear case is less natural and relies a lot on the use of the following operator:

$$L_t : \mathrm{L}^2(\mathcal{F}_T) \to [0, \infty), \qquad X \longmapsto \inf\bigl\{x \geq 0 : \mathbb{E}\bigl[\ell(t, x + X)\bigr] \geq 0\bigr\}, \tag{17}$$

defined for any $t \in [0, T]$. Since ℓ is of linear growth at infinity and $\mathbb{E}[\ell(t, \infty)] > 0$, L_t is well defined. Namely, $L_t(X)$ represents the minimal deterministic strength with which the random variable X must be pushed upward in order to satisfy the constraint of interest at time t. In the previous linear case where $\ell : (t, x) \mapsto x - u_t$, we simply explicitly have $L_t : X \mapsto (\mathbb{E}[X] - u_t)_-$.

We first focus on the constant driver case and we then are able to tackle the general case. For this last framework, a Lipschitz property for the operator L will be required.

4.1. *The constant driver case.* In this section, we demonstrate the well-posedness of the BSDE of interest in the constant driver case. As explained above, the operator L plays an important role in order to build a solution to such BSDE.

PROPOSITION 7. *Let C be a square integrable progressively measurable stochastic process or more generally in the space* $\mathrm{L}^2(\Omega; \mathrm{L}^1(0, T))$.

Then the BSDE with mean reflection

$$(18)\qquad \begin{aligned} &Y_t = \xi + \int_t^T C_s\, ds - \int_t^T Z_s \cdot dB_s + K_T - K_t, \\ &\mathbb{E}\big[\ell(t, Y_t)\big] \geq 0, \qquad 0 \leq t \leq T, \end{aligned}$$

has a unique square integrable deterministic flat solution.

PROOF. We derive the existence and uniqueness properties separately.

Step 1. Existence. In order to solve (18), let us define

$$\Psi_t := L_t(X_t) \qquad \text{where } X_t = \mathbb{E}\bigg[\xi + \int_t^T C_s\, ds \,\Big|\, \mathcal{F}_t\bigg], 0 \leq t \leq T.$$

Since ℓ is continuous in space, observe that

$$(19)\qquad \mathbb{E}\big[\ell(t, X_t + \Psi_t)\big] \geq 0, \qquad 0 \leq t \leq T.$$

Let us now show that Ψ is moreover continuous. Observe first that the map $x \longmapsto \mathbb{E}[\ell(t, x + X)]$ is continuous and strictly increasing. If $\mathbb{E}[\ell(t, X_t)] \leq 0$, since ℓ is continuous and has linear growth, for any $x < L_t(X_t) < y$, one has

$$\begin{aligned} \lim_{s \to t} \mathbb{E}\big[\ell(s, x + X_s)\big] &= \mathbb{E}\big[\ell(t, x + X_t)\big] < 0 = \mathbb{E}\big[\ell\big(t, L_t(X_t) + X_t\big)\big] \\ &< \mathbb{E}\big[\ell(t, y + X_t)\big] = \lim_{s \to t} \mathbb{E}\big[\ell(s, y + X_s)\big]. \end{aligned}$$

Then, if $|s - t|$ is small enough, $\mathbb{E}[\ell(s, x + X_s)] < 0$, $\mathbb{E}[\ell(s, y + X_s)] > 0$ and $x \leq L_s(X_s) \leq y$.

If $\mathbb{E}[\ell(t, X_t)] > 0$, $L_t(X_t) = 0$, and $\lim_{s \to t} \mathbb{E}[\ell(s, X_s)] = \mathbb{E}[\ell(t, X_t)] > 0$. If $|s - t|$ is small enough, $\mathbb{E}[\ell(s, X_s)] > 0$ and $L_s(X_s) = 0$.

We are now in position to define the continuous process K by

$$K_t := \sup_{0\le s\le T} \Psi_s - \sup_{t\le s\le T} \Psi_s \qquad \text{so that } K_T - K_t = \sup_{t\le s\le T} \Psi_s, \qquad 0 \le t \le T.$$

Observe that K is deterministic, nondecreasing with $K_0 = 0$. Given this process K, let (Y, Z) be the unique solution to the classical BSDE with the dynamics of (18). Then, since $x \longmapsto \ell(t, x)$ is nondecreasing, we deduce from (19) that

$$\begin{aligned} \mathbb{E}\big[\ell(t, Y_t)\big] &= \mathbb{E}\big[\ell(t, X_t + K_T - K_t)\big] = \mathbb{E}\Big[\ell\Big(t, X_t + \sup_{t\le s\le T} \Psi_s\Big)\Big] \\ &\ge \mathbb{E}\big[\ell(t, X_t + \Psi_t)\big] \ge 0. \end{aligned} \tag{20}$$

Hence, (Y, Z, K) is a deterministic solution to the BSDE with weak reflexion (18).

Let us now verify that it is also flat. By definition of K, observe that $\sup_{t\le s\le T} \Psi_s = \Psi_t\ dK_t$-a.e. and $\mathbf{1}_{\Psi_t=0} = 0\ dK_t$-a.e. Thus, by (20) we compute

$$\int_0^T \mathbb{E}\big[\ell(t, Y_t)\big]\, dK_t = \int_0^T \mathbb{E}\big[\ell(t, X_t + \Psi_t)\big]\, dK_t = \int_0^T \mathbb{E}\big[\ell(t, X_t + \Psi_t)\big]\mathbf{1}_{\Psi_t>0}\, dK_t.$$

Besides, since ℓ is continuous in space, we have $\mathbb{E}[\ell(t, X_t + \Psi_t)] = 0$ as soon as $\Psi_t > 0$, so that

$$\int_0^T \mathbb{E}\big[\ell(t, X_t + \Psi_t)\big]\mathbf{1}_{\Psi_t>0}\, dK_t = 0,$$

and (Y, Z, K) is a flat solution.

Step 2. Uniqueness. Let (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2) be two deterministic flat solutions to the BSDE with mean reflexion (18). We work towards a contradiction and suppose that there exists $t_1 < T$ such that

$$K^1_T - K^1_{t_1} > K^2_T - K^2_{t_1}.$$

Setting t_2 as the first time t after t_1 such that $K^1_T - K^1_t = K^2_T - K^2_t$, we observe that

$$K^1_T - K^1_t > K^2_T - K^2_t, \qquad t_1 \le t < t_2.$$

Since ℓ is strictly increasing, this implies that

$$\mathbb{E}\big[\ell\big(t, X_t + K^1_T - K^1_t\big)\big] > \mathbb{E}\big[\ell\big(t, X_t + K^2_T - K^2_t\big)\big] \ge 0, \qquad t_1 \le t < t_2.$$

But (Y^1, Z^1, K^1) is a flat solution and hereby

$$\int_{t_1}^{t_2} \mathbb{E}\big[\ell\big(t, X_t + K^1_T - K^1_t\big)\big]\, dK^1_t = 0,$$

so that we must have $dK^1 = 0$ on the interval $[t_1, t_2]$. We deduce that

$$K^1_T - K^1_{t_2} = K^1_T - K^1_{t_1} > K^2_T - K^2_{t_1} \ge K^2_T - K^2_{t_2},$$

which contradicts the definition of t_2. Hence $K^1 = K^2$ and the uniqueness of solution to classical BSDEs directly implies that (Y^1, Z^1, K^1) coincides with (Y^2, Z^2, K^2). $\square$

4.2. *Existence and uniqueness for the general case*. Now that the well-posedness for constant driver is established, we can focus on the BSDE (6) with mean reflexion (7) in full generality. In order for the solution to be well defined, we will require a Lipschitz property of the operator L defined in (17), that we present in the following an additional assumption:

(H_L) The operator L_t is Lipschitz continuous for the L^1-norm, uniformly in time: namely there exists a constant $C \geq 0$ such that

$$\left|L_t(X) - L_t(Y)\right| \leq C\mathbb{E}\big[|X - Y|\big], \qquad 0 \leq t \leq T, X, Y \in \mathrm{L}^2(\mathcal{F}_t),$$

where the operator L_t is defined in (17).

It is worth noticing that the previous assumption (H_L) is automatically satisfied as soon as the loss function ℓ is a bi-Lipschitz function in x. More precisely, we consider the following alternative assumption on ℓ:

$(H_{b\ell})$ The loss function $\ell : \Omega \times [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable map with respect to $\mathcal{F}_T \times \mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{R})$ and there exists $0 < c_l \leq C_l$ such that, $\mathbb{P}$-a.s.:

1. $\forall y \in \mathbb{R}$, $t \longmapsto \ell(t, y)$ is continuous,
2. $\forall t \in [0, T]$, $y \longmapsto \ell(t, y)$ is strictly increasing,
3. $\forall t \in [0, T]$, $\forall y \in \mathbb{R}$, $|\ell(t, y)| \leq C_l(1 + |y|)$,
4. $\forall t \in [0, T]$,

$$c_\ell|x - y| \leq \left|\ell(t, x) - \ell(t, y)\right| \leq C_\ell|x - y|, \qquad x, y \in \mathbb{R}. \tag{21}$$

LEMMA 8. *Assume $(H_{b\ell})$. Then both Assumptions (H_ℓ) and (H_L) hold.*

PROOF. Observe first that $(H_{b\ell})$ implies directly that (H_ℓ) holds. Fix now $t \in [0, T]$ and let X and Y be two random variables in $\mathrm{L}^2(\mathcal{F}_T)$.

Since ℓ is nondecreasing, the lower bound of (21) gives

$$\ell\left(t, L_t(X) + \frac{C_\ell}{c_\ell}\mathbb{E}\big[|X - Y|\big] + Y\right) \geq c_\ell\frac{C_\ell}{c_\ell}\mathbb{E}\big[|X - Y|\big] + \ell\big(t, L_t(X) + Y\big),$$

and using the upper bound we get

$$\ell\big(t, L_t(X) + Y\big) \geq \ell\big(t, L_t(X) + X\big) - C_l|X - Y|,$$

from which it follows

$$\ell\left(t, L_t(X) + \frac{C_\ell}{c_\ell}\mathbb{E}\big[|X - Y|\big] + Y\right) \geq \ell\big(t, L_t(X) + X\big) - C_l|X - Y| + C_l\mathbb{E}\big[|X - Y|\big].$$

Since $\mathbb{E}[\ell(t, X + L_t(X))] \geq 0$, we obtain by taking the expectation of the previous inequality

$$\mathbb{E}\left[\ell\left(t, L_t(X) + \frac{C_\ell}{c_\ell}\mathbb{E}\big[|X - Y|\big] + Y\right)\right] \geq 0.$$

By definition of $L_t(Y)$, this directly implies that

$$L_t(Y) \le L_t(X) + \frac{C_\ell}{c_\ell}\mathbb{E}\big[|X - Y|\big].$$

By symmetry of X and Y, we conclude that

$$\big|L_t(X) - L_t(Y)\big| \le \frac{C_\ell}{c_\ell}\mathbb{E}\big[|X - Y|\big]. \qquad \square$$

We are now in position to state the main result of the paper, providing the well-posedness of BSDEs with mean reflexion.

THEOREM 9. *In addition to the running assumptions* (H_ξ)–(H_f)–(H_ℓ), *let us assume moreover that* (H_L) *is satisfied. Then there exists a unique deterministic flat solution* $(Y, Z, K) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{A}^2_D$ *to the BSDE* (6) *with mean reflexion* (7).

PROOF. Let us consider σ and τ in the time interval $[0, T]$ with $\sigma \le \tau$. Given $Y_\tau \in \mathrm{L}^2(\mathcal{F}_\tau)$, $\{U_t\}_{\sigma \le t \le \tau} \in \mathcal{S}^2$ and $\{V_t\}_{\sigma \le t \le \tau} \in \mathcal{H}^2$, Proposition 7 ensures the existence of a triple of processes $\{(Y_t, Z_t, R_t)\}_{\sigma \le t \le \tau}$ solution to the BSDE with mean reflexion

$$Y_t = Y_\tau + \int_t^\tau f(s, U_s, V_s)\,ds - \int_t^\tau Z_s \cdot dB_s + R_t, \qquad \sigma \le t \le \tau,$$

$$\mathbb{E}\big[\ell(t, Y_t)\big] \ge 0, \qquad \sigma \le t \le \tau, \qquad \int_\sigma^\tau \mathbb{E}\big[\ell(t, Y_t)\big]\,dR_t = 0,$$

where we conveniently denoted $R_\cdot = K_\tau - K_\cdot$. In this setting, R is nonincreasing with $R_\tau = 0$ and, for $\sigma \le t \le \tau$,

$$R_t = \sup_{t \le s \le \tau} L_s(X_s) \qquad \text{with } X_t = \mathbb{E}\bigg[Y_\tau + \int_t^\tau f(s, U_s, V_s)\,ds \,\Big|\, \mathcal{F}_t\bigg]. \tag{22}$$

Let (Y', Z', R') be the solution associated to (U', V') and the same Y_τ.

We have, with usual notation,

$$\delta Y_t = \mathbb{E}\bigg[\int_t^\tau \big[f(s, U_s, V_s) - f\big(s, U'_s, V'_s\big)\big]\,ds \,\Big|\, \mathcal{F}_t\bigg] + \delta R_t, \qquad \sigma \le t \le \tau,$$

from which we deduce immediately, since f is assumed to be Lipschitz that

$$\mathbb{E}\Big[\sup_{\sigma \le t \le \tau} |\delta Y_t|^2\Big] \le C(\lambda)\mathbb{E}\bigg[\bigg(\int_\sigma^\tau \big(|\delta U_s| + |\delta V_s|\big)\,ds\bigg)^2\bigg] + \sup_{\sigma \le t \le \tau} |\delta R_t|^2.$$

Besides, since (H_L) holds, we deduce from the representation (22) together with $\delta Y_\tau = 0$ and the Lipschitz property of f that, for $\sigma \le t \le \tau$,

$$\begin{aligned}|\delta R_t| &\le \Big|\sup_{t \le s \le \tau} L_s(X_s) - \sup_{t \le s \le \tau} L_s\big(X'_s\big)\Big| \le \sup_{t \le s \le \tau} \big|L_s(X_s) - L_s\big(X'_s\big)\big| \\ &\le \sup_{t \le s \le \tau} \mathbb{E}\big[|\delta X_s|\big] \le C(\lambda)\mathbb{E}\bigg[\int_\sigma^\tau \big(|\delta U_s| + |\delta V_s|\big)\,ds\bigg].\end{aligned}$$

Combining the previous estimates together with the Cauchy–Schwarz inequality, we deduce

$$\mathbb{E}\Big[\sup_{\sigma\le t\le\tau}|\delta Y_t|^2\Big]\le C(\lambda)\mathbb{E}\Big[\Big(\int_\sigma^\tau\big(|\delta U_s|+|\delta V_s|\big)\,ds\Big)^2\Big],$$

and writing

$$\int_\sigma^\tau \delta Z_s\cdot dB_s=\delta Y_\tau-\delta Y_\sigma+\delta R_\tau-\delta R_\sigma+\int_\sigma^\tau\big[f(s,U_s,V_s)-f\big(s,U'_s,V'_s\big)\big]\,ds$$

we finally have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{\sigma\le t\le\tau}|\delta Y_t|^2+\int_\sigma^\tau|\delta Z_s|^2\,ds\Big]\\
&\quad\le C(\lambda)\mathbb{E}\Big[\Big(\int_\sigma^\tau\big(|\delta U_s|+|\delta V_s|\big)\,ds\Big)^2\Big]\\
&\quad\le C(\lambda)(\tau-\sigma)\max(1,\tau-\sigma)\mathbb{E}\Big[\sup_{\sigma\le t\le\tau}|\delta U_t|^2+\int_\sigma^\tau|\delta V_s|^2\,ds\Big].
\end{aligned}\tag{23}$$

Of course, this inequality shows that the BSDE (6) with mean reflexion (7) has a unique solution whenever T is small enough.

To cover the general case, let us pick $n\ge 1$ such that $C(\lambda)\min(T,T^2)/n^2<1$. For $i=0,\dots,n$, let us set $T_i:=iT/n$. Starting from the interval $[T_{n-1},T_n]$ and $Y_{T_n}=\xi$, let for $i=n,\dots,1$, (Y^i,Z^i,R^i) the unique solution to the BSDE with mean reflexion

$$\begin{aligned}
&Y_t^i=Y_{T_i}^{i+1}+\int_t^{T_i}f\big(s,Y_s^i,Z_s^i\big)\,ds-\int_t^{T_i}Z_s^i\cdot dB_s+R_t^i,\\
&\mathbb{E}\big[\ell\big(t,Y_t^i\big)\big]\ge 0,\qquad T_{i-1}\le t\le T_i,\\
&\int_{T_{i-1}}^{T_i}\mathbb{E}\big[\ell\big(t,Y_t^i\big)\big]\,dR_t^i=0,\\
&R^i \text{ continuous and nonincreasing on } [T_{i-1},T_i] \text{ with } R_{T_i}^i=0.
\end{aligned}$$

Let us define (Y,Z,R) on $[0,T]$ by setting

$$Y_t=Y_0^1\mathbf{1}_0(t)+\sum_{i=1}^n Y_t^i\mathbf{1}_{]T_{i-1},T_i]}(t),\qquad Z_t=\sum_{i=1}^n Z_t^i\mathbf{1}_{]T_{i-1},T_i[}(t),$$

and $R_t=R_t^n$ on $[T_{n-1},T_n]$ and, for $i=n-1,\dots,1$, $R_t=R_t^i+R_{T_i}$ on $[T_{i-1},T_i]$. Since $R_{T_i}^i=0$, R is continuous and nonincreasing. Finally, let us define $K_t=R_0-R_t$ to get a nondecreasing continuous function with $K_0=0$. Since $R_T=0$, $K_T=R_0$ and $R_t=K_T-K_t$.

It is straightforward to check that (Y,Z,K) is a solution to the BSDE (6) with mean reflexion (7). Uniqueness follows from the uniqueness on each small interval. □

As a by-product, taking into account Lemma 8, we have the following result.

COROLLARY 10. *Let* (H_ξ), (H_f) *and* $(H_{b\ell})$ *hold.*
Then there exists a unique deterministic flat solution $(Y, Z, K) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{A}_D^2$ *to the BSDE* (6) *with mean reflexion* (7).

5. Minimality of the deterministic flat solution. Let us recall that for classical reflected BSDE, the Skorokhod condition ensures the minimality of the enhanced solution in the class of all supersolutions to the reflected BSDE. By minimality, we refer to minimality in terms of the Y-component of the solution. The Skorokhod condition indicates that the compensator K only pushes the solution when the condition is binding, that is, only when it is really necessary. This motivates us to consider BSDEs with mean reflection which satisfy the corresponding flatness condition (8).

Now, that the existence of a unique deterministic flat solution to the BSDE (6) with mean reflexion (7) has been established, it is natural to consider if this flatness condition (8) also implies the minimality among all the deterministic solutions. Since the constraint is given in expectation instead of pointwisely, it is not obvious that only the condition at time t determines the minimal upward kick to apply on the solution at time t. Under additional assumption on the structure of the driver function f, we are able to verify that such minimality property is indeed satisfied.

THEOREM 11. *Suppose that the driver function f is of the form*

$$f : (t, y, z) \mapsto a_t y + h(t, z), \tag{24}$$

where a is a deterministic and bounded measurable function. If ℓ is strictly increasing, a deterministic flat solution (Y, Z, K) is minimal among all the deterministic solutions.

PROOF. Let (Y, Z, K) be a deterministic flat solution, and (Y', Z', K') be any deterministic solution. We want to prove that $Y \le Y'$. We first focus on the particular case where the driver does not depend on y and then tackle the general case where f is given by (24).

Step 1. Driver of the form $f(t, z)$. Since the driver function f does not depend on y, the processes $(Y - (K_T - K), Z)$ and $(Y' - (K'_T - K'), Z')$ are both solutions of the same classical BSDE, and we deduce that

$$Y_t - (K_T - K_t) = Y'_t - \bigl(K'_T - K'_t\bigr), \qquad 0 \le t \le T. \tag{25}$$

Hereby, proving that $Y \le Y'$ boils down to showing that $K_T - K \le K'_T - K'$. We work towards a contradiction and suppose the existence of $t_1 < T$ such that

$$K_T - K_{t_1} > K'_T - K'_{t_1}.$$

Let t_2 be the first time such that $K_T - K_. \geq K'_T - K'_.$. Obviously, t_2 is a deterministic time smaller than T and by continuity of K and K', we get $K_T - K_{t_2} = K'_T - K'_{t_2}$ and

$$K_T - K_t > K'_T - K'_t, \qquad t_1 \leq t < t_2. \tag{26}$$

We deduce from (25) that $Y > Y'$, on $[t_1, t_2)$, and the strict monotonicity of ℓ implies

$$\mathbb{E}[\ell(t, Y_t)] > \mathbb{E}[\ell(t, Y'_t)] \geq 0, \qquad t_1 \leq t \leq t_2.$$

Since Y is a flat solution, we have $\int_0^T \mathbb{E}[\ell(Y_s)]\, dK_s = 0$ and we deduce that $dK_t = 0$, for $t \in [t_1, t_2)$. Therefore,

$$K'_T - K'_{t_1} < K_T - K_{t_1} = K_T - K_{t_2} = K'_T - K'_{t_2},$$

which is a contradiction since K' must be nondecreasing.

Step 2. Driver of the form (24). Let us denote $A_t := \int_0^t a_s\, ds$ for $0 \leq t \leq T$. Making the following transformation,

$$\tilde{Y}_t = e^{A_t} Y_t, \qquad \tilde{Z}_t = e^{A_t} Z_t, \qquad \tilde{K}_t = e^{A_t} K_t,$$

we verify easily that $(\tilde{Y}, \tilde{Z}, \tilde{K})$ is a flat deterministic solution to the BSDE with mean reflection associated to the parameters

$$\tilde{\xi} = e^{A_T} \xi, \qquad \tilde{f}(t, z) = e^{A_t} f\big(t, e^{-A_t} z\big) \quad \text{and} \quad \tilde{\ell}(t, y) = \ell\big(t, e^{-A_t} y\big).$$

According to the previous step $\tilde{Y}$ is minimal within the class of deterministic solutions, and Y inherits this property by a straightforward argument. □

REMARK 7. As a by-product, this proof provides an alternative argument in order to derive the uniqueness of the flat deterministic solution of BSDEs with mean reflexion and driver of the form (24). It is in fact a generalization of the proof presented in Proposition 7 for the constant driver case.

REMARK 8. For general drivers, the minimality of the solution under the Skorokhod condition is a difficult question. In the previous example, which is really particular ($Y - Y'$ is deterministic), we obtain the minimality of Y in the usual sense. We do not believe that this result is true in general even though we can not exhibit any counterexample. An alternative viewpoint may be to look towards a minimality "in mean", but our attempts in that direction remained unfruitful, and we leave this open point for further research.

6. Extension and application. Interpreting Y as the value of a portfolio, the constraint (7) imposes at any date t a constraint on the distribution of Y_t, seen from time 0. The form of constraint that we considered so far is the expectation of a loss function. From a financial point of view, an investor may be required to control the risk of any admissible portfolio. In order to measure the underlying risk of a portfolio, the natural tools in the mathematical finance literature are the so-called risk measures; see, for example, [1]. We emphasize in this section how our framework of study allows to encompass such type of running static risk measure constraint. Then we present an application for the problem of super-hedging a claim under a given running risk measure constraint.

6.1. *BSDE with risk measure reflection.* For a fixed t, a static risk measure is a map $\rho(t,\cdot): L^2(\mathcal{F}_t) \longrightarrow \mathbb{R}$ satisfying $\rho(t,0)=0$ together with:

- Monotonicity: $X \le Y \Longrightarrow \rho(t,X) \ge \rho(t,Y)$, for $X, Y \in L^2(\mathcal{F}_t)$.
- Translation invariance: $\rho(t, X+m) = \rho(t,X) - m$, for $X \in L^2(\mathcal{F}_t)$ and $m \in \mathbb{R}$.

Hereby, for a given $t \in [0,T]$, $\rho(t,X)$ is a real number which measures the risk associated to the wealth random variable X. Risk measures can similarly be characterized by their so-called acceptance set, which defines as

$$\mathcal{A}_\rho^t = \{X \in L^2(\mathcal{F}_t) : \rho(t,X) \le 0\}.$$

Similarly, given a set $\mathcal{A}^t$, one can define a static risk measure by setting

$$\rho(t,X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}^t\},$$

so that the acceptance set $\mathcal{A}^t$ and the risk measure $\rho(t,\cdot)$ share a one to one correspondence. For a given collection of static risk measures $(\rho(t,\cdot))_t$, a wealth process Y will be considered admissible in our framework as soon as it satisfies

$$\rho(t, Y_t) \le q_t, \qquad 0 \le t \le T, \tag{27}$$

where q is a given time indexed deterministic benchmark. For example, the risk measuring tool of ρ could simply not depend on time, but be compared to the deterministic benchmark q, which evolves with time, by either tightening or relaxing the constraint. We now look towards solutions of BSDEs subject to the additional constraint (27). In the same spirit as above, a flat solution to such type of BSDE will be required to satisfy

$$\int_0^T [q_t - \rho(t, Y_t)]\, dK_t = 0. \tag{28}$$

The next theorem indicates that we are able to consider BSDEs under risk measure constraint of the form (27), in a similar fashion as the one developed in the previous sections.

THEOREM 12. *Let $\rho(t,.) : [0,T] \times \mathrm{L}^2 \longrightarrow \mathbb{R}$ be a collection of monotonic and translation invariant risk measures, which are continuous with time and Lipschitz in space, that is,*

$$\left|\rho(t,X) - \rho(t,Y)\right| \le C\mathbb{E}\big[|X-Y|\big], \qquad 0 \le t \le T, X, Y \in \mathrm{L}^2(\mathcal{F}_t).$$

If we are moreover given a continuous deterministic benchmark q and ξ satisfies $\rho(T,\xi) \le q_T$, then the "BSDE with risk measure reflection"

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,ds - \int_t^T Z_s \cdot dB_s + K_T - K_t, \qquad 0 \le t \le T,$$

$$\rho(t, Y_t) \le q_t, \qquad 0 \le t \le T, \qquad \int_0^T \big[q_t - \rho(t, Y_t)\big]\,dK_t = 0,$$

admits a unique deterministic flat solution.

Besides, if f satisfies (24), *the deterministic flat solution is minimal among all deterministic solutions.*

PROOF. The reasoning simply follows the arguments of Proposition 7, Theorem 9 and Theorem 11. The main distinction is that the map L_t is replaced by the risk measure $\rho(t,\cdot) - q_t$, for any $t \in [0,T]$. Besides, the translation invariance property conveniently replaces the strict monotonicity of ℓ in the proofs. □

Typical examples considered in the literature are coherent risk measures of the form

$$\rho(t,X) = \sup\{\mathbb{E}^{\mathbb{Q}}[-X] : \mathbb{Q} \in \mathcal{Q}_t\},$$

where $\mathcal{Q}_t$ is a set of probabilities absolutely continuous w.r.t. $\mathbb{P}$. As soon as the set of probability change densities is bounded, $\rho(t,\cdot)$ is Lipschitz. This is particular the case for the classical expected shortfall risk measure, defined as

$$\rho_\alpha^{\mathrm{ES}}(t,X) := \frac{1}{\alpha_t}\int_0^{\alpha_t} \mathrm{VaR}_s(X)\,ds,$$

where $\alpha_t \in (0,1)$ denotes a given precision level and VaR_s is the Value at Risk of level s. Indeed, the expected shortfall (or AVaR) rewrites also this way

$$\rho_\alpha^{\mathrm{ES}}(t,X) = \sup\left\{\mathbb{E}^{\mathbb{Q}}[-X] : \frac{d\mathbb{Q}}{d\mathbb{P}} \le \frac{1}{\alpha_t}\right\}.$$

6.2. *Application to super-hedging under risk constraint.* We now turn to an application in mathematical finance and consider a stock market endowed with a bond with deterministic interest rate r and a vector of d stocks with dynamics

$$dS_t = S_t(\mu_t\,dt + \sigma_t\,dB_t), \qquad 0 \le t \le T,$$

where the drift μ and the volatility σ are square integrable predictable processes. We assume that $\sigma_t\sigma_t' - \varepsilon I \succeq 0$ for some $\varepsilon > 0$, in order to ensure the completeness of the market. For a given initial capital x, we consider portfolios $X^{x,\pi,K}$ driven by a consumption-investment strategy (π, K), and whose dynamics are given by

$$\begin{aligned} dX_t^{x,\pi,K} &= X_t^{x,\pi,K}\left(r_t\,dt + (\mu_t - r_t\mathbf{1})'\pi_t \frac{dS_t}{S_t}\right) - dK_t \\ &= r_t X_t^{x,\pi,K}\,dt + (\mu_t - r_t\mathbf{1})'\pi_t\,dt + \pi_t'\sigma_t\,dB_t - dK_t, \qquad 0 \le t \le T. \end{aligned}$$

Using such portfolios, a financial engineer is willing to hedge a possibly non-Markovian claim $\xi \in L^2(\mathcal{F}_T)$. For regulatory purposes, the risk management department of his financial institution imposes his restrictions on the class of admissible investment strategies. Namely, a portfolio wealth process $X^{x,\pi,K}$ is considered admissible if and only if it satisfies the following constraint:

$$\rho_\alpha^{\mathrm{ES}}\big(t, X_t^{x,\pi,K}\big) \le q_t, \qquad 0 \le t \le T,$$

where (α, q) are a time indexed collection of deterministic quantile and level benchmarks. These benchmarks can, for example, be chosen in such a way that the constraint becomes either tighter or weaker, as we approach the maturity T. In such a case, the careful investor is looking for the super-hedging price

$$Y_0 = \inf\big\{x \in \mathbb{R}, \exists(\pi, K) \in \mathcal{A}, \text{s.t. } X_T^{x,\pi,K} \ge \xi \text{ and } \rho_\alpha^{\mathrm{ES}}(t, X_t) \le q_t, \forall t \in [0, T]\big\},$$

and associated consumption-investment strategy. Applying the results of this paper, we deduce that, if the investor restricts to deterministic consumption strategies, Y_0 is well defined as the starting point of the unique deterministic flat solution to the following BSDE with risk measure reflection:

$$\begin{aligned} Y_t = \xi &+ \int_t^T \big(-r_s Y_s - (\mu_s - r_s\mathbf{1})'\sigma_s^{-1} Z_s\big)\,ds \\ &- \int_t^T Z_s \cdot dB_s + K_T - K_t, \qquad 0 \le t \le T, \end{aligned}$$

$$\rho_\alpha^{\mathrm{ES}}(t, Y_t) \le q_t, \qquad 0 \le t \le T, \qquad \int_0^T \big[q_t - \rho_\alpha^{\mathrm{ES}}(t, Y_t)\big]\,dK_t = 0.$$

Indeed, the driver function satisfies (24), so that the flat solution is minimal among all deterministic ones.

APPENDIX

Observe first that the solution (Y^n, Z^n) is well and uniquely defined, according to the results of [5] up to slight modifications discussed for example in [7].

Step 1. Uniform a priori estimate on the sequence $(Y^n, Z^n, K^n)_n$. Since K^n is deterministic, we have

$$\begin{aligned} 2\mathbb{E}\left[\int_t^T e^{as} Y_s^n\,dK_s^n\right] &= 2\int_t^T e^{as}\mathbb{E}[Y_s^n]\,dK_s^n \\ &= 2\int_t^T e^{as}\big(\mathbb{E}[Y_s^n] - u\big)\,dK_s^n + 2\int_t^T e^{as} u\,dK_s^n \end{aligned}$$

$$
\begin{aligned}
&= -2n \int_t^T e^{as} (u - \mathbb{E}[Y_s^n])_+^2 \, ds + 2 \int_t^T e^{as} u \, dK_s^n \\
&\le 2u \int_t^T e^{as} \, dK_s^n,
\end{aligned}
$$

for any constant a and $t \in [0, T]$. Thus, arguing as in the proof of Lemma 2, we get the following estimate on the solution (Y^n, Z^n):

$$
\begin{aligned}
&\sup_{n\ge 1} \biggl(\mathbb{E}\biggl[\sup_{0\le t\le T} |Y_t^n|^2 + \int_0^T |Z_s^n|^2 \, ds \biggr] + |K_T^n|^2 \biggr) \\
&\qquad \le C(\lambda, T) \biggl(\mathbb{E}\biggl[|\xi|^2 + \int_0^T |f(s, 0, 0)|^2 \, ds \biggr] + u^2 \biggr).
\end{aligned}
$$

Step 2. Convergence of the sequence $(Y^n, Z^n, K^n)_n$. Since the constraint is satisfied at maturity, observe also that $((u - \mathbb{E}[Y_0^n])_+)^2$ rewrites

$$
\begin{aligned}
&|(u - \mathbb{E}[Y_0^n])_+|^2 + 2n \int_0^T |(u - \mathbb{E}[Y_s^n])_+|^2 \, ds \\
&\qquad = -2 \int_0^T \mathbb{E}[f(s, Y_s^n, Z_s^n)] (u - \mathbb{E}[Y_s^n])_+ \, ds \\
&\qquad \le n \int_0^T |(u - \mathbb{E}[Y_s^n])_+|^2 \, ds + \frac{C(\lambda, T)}{n},
\end{aligned}
$$

according to the previous estimate. Hence, we deduce for later use that

$$
n^2 \int_0^T |(u - \mathbb{E}[Y_s^n])_+|^2 \, ds \le C(\lambda, T). \tag{29}
$$

We now look towards a contracting property of the sequence (Y^n, Z^n) and denote $\delta X := X^{n+1} - X^n$ for $X = Y, Z$ or K. Setting $a := \frac{1}{2} + 2\lambda + 2\lambda^2$, a standard computation based on Itô's formula provides

$$
\begin{aligned}
&e^{at} |\delta Y_t|^2 + \frac{1}{2} \int_t^T e^{as} (|\delta Y_s|^2 + |\delta Z_s|^2) \, ds \\
&\qquad \le 2 \int_t^T e^{as} \delta Y_s \, d\delta K_s - 2 \int_t^T e^{as} \delta Y_s \delta Z_s \cdot dB_s, \qquad 0 \le t \le T,
\end{aligned}
$$

from which we deduce that

$$
\sup_{0\le t\le T} \mathbb{E}\biggl[|\delta Y_t|^2 + \int_0^T (|\delta Y_s|^2 + |\delta Z_s|^2) \, ds \biggr] \le 2 \sup_{0\le t\le T} \mathbb{E}\biggl[\int_t^T e^{as} \delta Y_s \, d\delta K_s \biggr]. \tag{30}
$$

For any $s \in [0, T]$, denoting $v_s^n := (u - y_s^n)_+$ where y_s^n stands for $\mathbb{E}[Y_s^n]$, we have $dK_s^n = n v_s^n \, ds$ and

$$\begin{aligned}\mathbb{E}\Big[\int_t^T e^{as} \delta Y_s \, d\delta K_s\Big] \\ = \int_t^T e^{as} \big[y_s^{n+1} - y_s^n\big]\big[(n+1)v_s^{n+1} - n v_s^n\big] \, ds, \qquad 0 \le t \le T.\end{aligned}$$

Moreover, we compute

$$\begin{aligned}&\big[y^{n+1} - y^n\big]\big[(n+1)v^{n+1} - n v^n\big] \\ &\quad = \big[\big(u - y^n\big) - \big(u - y^{n+1}\big)\big]\big[(n+1)v^{n+1} - n v^n\big] \\ &\quad \le -n\big|v^n\big|^2 + (2n+1)v^n v^{n+1} - (n+1)\big|v^{n+1}\big|^2.\end{aligned}$$

But we have

$$-nx^2 + (2n+1)xy - (n+1)y^2 = -n\bigg(x - \Big(1 + \frac{1}{2n}\Big)y\bigg)^2 + \frac{y^2}{4n}, \qquad x, y \in \mathbb{R},$$

so that combining the previous estimates with (29), we deduce

$$\mathbb{E}\Big[\int_0^T e^{as} \delta Y_s \, d\delta K_s\Big] \le \frac{1}{4n}\int_0^T |v_s^{n+1}|^2 \, ds \le \frac{C(\lambda, T)}{n^3}.$$

Plugging this estimate in (30), it follows that

$$\sup_{0 \le t \le T} \mathbb{E}\big[|\delta Y_t|^2\big] + \mathbb{E}\Big[\int_0^T \big(|\delta Y_s|^2 + |\delta Z_s|^2\big) \, ds\Big] \le \frac{C(\lambda, T)}{n^3}.$$

Setting $\Delta_t K^n = K_T^n - K_t^n$ and reminding that K^n is deterministic, observe that

$$\Delta_t K^{n+1} - \Delta_t K^n = \mathbb{E}[\delta Y_t] - \mathbb{E}\Big[\int_t^T \big(f\big(s, Y_s^{n+1}, Z_s^{n+1}\big) - f\big(s, Y_s^n, Z_s^n\big)\big) \, ds\Big],$$

from which we deduce

$$\sup_{0 \le t \le T} \big|\Delta_t K^{n+1} - \Delta_t K^n\big| \le \frac{C(\lambda, T)}{n^3}.$$

Since we have

$$\delta Y_t = \mathbb{E}\bigg(\int_t^T \big(f\big(s, Y_s^{n+1}, Z_s^{n+1}\big) - f\big(s, Y_s^n, Z_s^n\big)\big) \, ds \,\Big|\, \mathcal{F}_t\bigg) + \Delta_t K^{n+1} - \Delta_t K^n,$$

combining the above and Burkholder–Davis–Gundy inequality, we conclude that $(Y^n, Z^n, K^n)_n$ converges strongly to a limit (Y, Z, K), namely

$$\mathbb{E}\Big[\sup_{0 \le t \le T} |Y_t^n - Y_t|^2 + \int_0^T |Z^n - Z_s|^2 \, ds\Big] + \sup_{0 \le t \le T} |K_t^n - K_t|^2 \longrightarrow_{n \to \infty} 0.$$

Step 3. Properties of the limit (Y, Z, K). Passing to the limit the dynamics of $(Y^n, Z^n, K^n)_n$, remark that (Y, Z, K) satisfies (6). Observe also that, by construction, K is deterministic, nondecreasing with $K_0 = 0$. Besides, the estimate (29) directly implies that

$$\int_0^T \left|(u - \mathbb{E}[Y_t])_+\right|^2 dt = \lim_{n\to\infty} \int_0^T \left|(u - \mathbb{E}[Y_t^n])_+\right|^2 dt = 0,$$

so that $\mathbb{E}[Y_t] \geq u$, for any $t \in [0, T]$. Finally, from Lemma 13 below, since $(\mathbb{E}[Y^n], K^n)$ converges to $(\mathbb{E}[Y], K)$ in $\mathcal{C}([0, T])$, we have

$$\lim_{n\to\infty} \int_0^T (\mathbb{E}[Y_t^n] - u)_+ \, dK_t^n = \int_0^T (\mathbb{E}[Y_t] - u)_+ \, dK_t$$

and, on the other hand,

$$\int_0^T (\mathbb{E}[Y_t^n] - u)_+ \, dK_t^n = n \int_0^T (\mathbb{E}[Y_t^n] - u)_+ (u - \mathbb{E}[Y_t^n])_+ \, dt = 0.$$

It follows that (Y, Z, K) is the unique flat deterministic solution to the BSDE (6) with linear mean reflection (9).

We now complete the argumentation by proving a rather elementary lemma, that we just used in the previous proof.

LEMMA 13. *Let $(u^n)_{n\geq 1}$ and $(K^n)_{n\geq 1}$ be two convergent sequences of $(\mathcal{C}_T, |\cdot|_\infty)$. We assume that, for each $n \geq 1$, K^n is nondecreasing and we denote by u and K the corresponding limits of $(u^n)_n$ and $(K^n)_n$. We have*

$$\lim_{n\to\infty} \int_0^T u_t^n \, dK_t^n = \int_0^T u_t \, dK_t.$$

PROOF. For any piecewise constant function h, we have

$$\begin{aligned}\int_0^T u_s^n \, dK_s^n - \int_0^T u_s \, dK_s &= \int_0^T [u_s^n - u_s] \, dK_s^n + \int_0^T [u_s - h_s] \, dK_s^n \\ &\quad + \int_0^T h_s \, dK_s^n - \int_0^T h_s \, dK_s + \int_0^T [h_s - u_s] \, dK_s,\end{aligned}$$

from which we deduce that

$$\begin{aligned}\left|\int_0^T u_s^n \, dK_s^n - \int_0^T u_s \, dK_s\right| &\leq |u^n - u|_\infty |K^n|_\infty + |u - h|_\infty (|K^n|_\infty + |K|_\infty) \\ &\quad + \left|\int_0^T h_s \, dK_s^n - \int_0^T h_s \, dK_s\right|.\end{aligned}$$

Since h is piecewise constant, we have

$$\lim_{n\to\infty} \int_0^T h_s \, dK_s^n = \int_0^T h_s \, dK_s$$

and

$$\limsup \left| \int_0^T u_s^n \, dK_s^n - \int_0^T u_s \, dK_s \right| \leq 2 \, |u - h|_\infty \, |K|_\infty,$$

from which we get the result since piecewise constant functions on $[0, T]$ are dense in $(\mathcal{C}_T, |\cdot|_\infty)$. $\square$

Acknowledgement. The authors would like to thank the referees for their careful reading and fruitful comments.

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P. BRIAND
LABORATOIRE DE MATHÉMATIQUES
CNRS UMR 5127
UNIVERSITÉ DE SAVOIE
CAMPUS SCIENTIFIQUE
73376 LE BOURGET DU LAC
FRANCE
E-MAIL: philippe.briand@univ-savoie.fr

R. ELIE
LAMA UMR CNRS 8050
UNIVERSITÉ PARIS-EST &
PROJET MATHRISK
ENPC-INRIA-UMLV
CHAMPS-SUR-MARNE
FRANCE
E-MAIL: romuald.elie@univ-mlv.fr

Y. HU
IRMAR UMR CNRS 6625
UNIVERSITÉ RENNES 1
CAMPUS DE BEAULIEU
35042 RENNES CEDEX
FRANCE
E-MAIL: ying.hu@univ-rennes1.fr