

ON THE UNIQUE CROSSING CONJECTURE OF DIACONIS AND PERLMAN ON CONVOLUTIONS OF GAMMA RANDOM VARIABLES

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Diaconis and Perlman [In *Topics in Statistical Dependence (Somerset, PA, 1987)* (1990) 147–166, IMS] conjecture that the distribution functions of two weighted sums of i.i.d. gamma random variables cross exactly once if one weight vector majorizes the other. We disprove this conjecture when the shape parameter of the gamma variates is $\alpha < 1$ and prove it when $\alpha \geq 1$.

1. Introduction. We say a real vector θ majorizes η , written as $\eta \prec \theta$, if there exists a doubly stochastic matrix D such that $\eta = \theta D$; see Marshall, Olkin and Arnold (2009) for basic properties of majorization. Let X_1, \dots, X_n be independent and identically distributed (i.i.d.) $\text{gamma}(\alpha, 1)$ random variables and denote the distribution function for $\sum_{i=1}^n \theta_i X_i$ by F_θ where $\theta \equiv (\theta_1, \dots, \theta_n)$ is a nonnegative weight vector. Diaconis and Perlman (1990) made the following.

CONJECTURE 1. *If $\eta \prec \theta$, but θ is not a permutation of η , then $F_\eta(x) - F_\theta(x)$ changes signs exactly once, from $-$ to $+$, as x increases from 0 to ∞ .*

Intuitively, $\sum_{i=1}^n \theta_i X_i$ is more dispersed when the weight vector θ becomes less uniform. This conjecture, known as the unique crossing conjecture (UCC), can be seen as a strong statement about the dispersion of these weighted sums in terms of tail probabilities. Diaconis and Perlman verified the UCC in the following cases: (a) $n = 2$; (b) $n = 3$ and $\alpha = 1$; (c) $n \geq 3$, $\alpha \geq 1$ and θ and η differ in only two components; (d) $n \geq 3$ and components of η are equal. Case (d) is useful for providing bounds on F_θ in terms of the distribution function of a single gamma variable. For this purpose, Diaconis and Perlman also carried out detailed analysis of the location of the crossing point between F_θ and F_η when all components of η are equal. Bounds based on a single gamma variable can also be found in the work of Roosta-Khorasani, Székely and Ascher (2015). Nevertheless, as remarked by several authors [Kochar and Xu (2012), Roosta-Khorasani and Székely (2015)] the UCC itself has remained an open problem.

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In this paper, we show that the UCC holds when $\alpha \geq 1$, but may fail when $\alpha < 1$, which is surprising as previous work has all supported the general validity of the UCC. This is relevant for understanding the behavior of tail probabilities for weighted sums of χ^2 random variables (corresponding to $\alpha = 1/2$) which arise naturally in statistical applications. For general α , weighted sums of gamma variables appear in diverse areas including reliability, actuarial science and statistics, and their properties have been extensively studied; see, for example, Bock et al. (1987), Diaconis and Perlman (1990), Székely and Bakirov (2003), Khaledi and Kochar (2004), Zhao and Balakrishnan (2009), Yu (2009, 2011), Kochar and Xu (2012) and Roosta-Khorasani and Székely (2015).

2. Special cases and a counterexample. Theorem 1 gives a positive answer to the UCC when $\alpha \geq 1$ and the weight vectors form a special configuration.

THEOREM 1. *Suppose $\alpha \geq 1$. Suppose $0 < \theta_1 \leq \dots \leq \theta_n$ and $\eta_1 \leq \dots \leq \eta_n$ and (a) there exists $2 \leq k \leq n$ such that $\theta_i < \eta_i$ for $i < k$ and $\theta_i > \eta_i$ for $i \geq k$; (b) $\prod_{i=1}^n \eta_i > \prod_{i=1}^n \theta_i$. Then there exists $x_0 \in (0, \infty)$ such that $F_\eta(x) < F_\theta(x)$ for $x \in (0, x_0)$ and the inequality is reversed for $x > x_0$.*

COROLLARY 1. *The UCC is valid if $n \geq 3$, $\alpha \geq 1$ and the weight vectors differ in at most three components.*

PROOF. When $n = 3$, conditions of Theorem 1 can be written as $\eta_{(1)} > \theta_{(1)} > 0$, $\eta_{(3)} < \theta_{(3)}$ and $\theta_1\theta_2\theta_3 < \eta_1\eta_2\eta_3$. It is easy to verify that, if $\eta \prec \theta$ and θ is all positive, and $\eta_i \neq \theta_j$ for all i, j , then these conditions are satisfied, and hence the UCC is valid.

If any $\eta_i = \theta_j$, then the problem reduces to the $n = 2$ case. As noted by Diaconis and Perlman, when $\alpha \geq 1$, one may extend the validity of the UCC for $n = 3$ to $n \geq 3$ provided that the weight vectors differ in at most three components.

Although Theorem 1 as stated requires all components of θ to be positive, the argument can be easily extended. Suppose η and θ differ in at most three components, $\eta \prec \theta$, and η is all positive but θ may have zero components. Then we can find $\eta \prec \tilde{\theta} \prec \theta$ such that $\tilde{\theta}$ is all positive, differs from η in the same (at most three) components and is infinitesimally close to θ . By the argument above, F_η and $F_{\tilde{\theta}}$ cross only once. If F_η and F_θ cross two or more times, then this implies that at least one crossing point has suddenly disappeared when a perturbation (no matter how slight) is made to θ , which is impossible. \square

To prove Theorem 1, we need some preliminary results. The following lemma is a special case of Theorem 1 of Yu (2011).

LEMMA 1. *For $n \geq 2$ and $\alpha > 0$, if $\log \eta \prec \log \theta$ then $F_\eta(x) \geq F_\theta(x)$ for all $x \in (0, \infty)$, that is, $F_\eta \leq_{\text{st}} F_\theta$.*

Based on Lemma 1, if there exists a vector τ such that $\eta \leq \tau$ coordinate-wise and $\log \tau < \log \theta$ (effectively $\log \eta <_w \log \theta$), then $F_\eta \leq_{st} F_\tau \leq_{st} F_\theta$. In this way, the conditions of Lemma 1 can be weakened.

Proposition 1 summarizes conditions for unique crossing for $n = 2$ and general $\alpha > 0$.

PROPOSITION 1. *Suppose $n = 2$. Then F_η crosses F_θ exactly once, and from below, as x increases from 0 to ∞ if and only if $\theta_1\theta_2 < \eta_1\eta_2$ and $\max(\theta_1, \theta_2) > \max(\eta_1, \eta_2)$.*

PROOF. Let us assume $\theta_1 \leq \theta_2$ and $\eta_1 \leq \eta_2$ without loss of generality.

Necessity. Suppose F_η crosses F_θ exactly once from below. Then $F_\eta(x) - F_\theta(x)$ is negative for sufficiently small $x > 0$ and is positive for sufficiently large x . But

$$\frac{1 - F_\theta(x)}{1 - F_\eta(x)} \leq \frac{\Pr(\theta_2(X_1 + X_2) > x)}{\Pr(\eta_2 X_2 > x)}.$$

The latter ratio is asymptotic to $g_{2\alpha}(x/\theta_2)/g_\alpha(x/\eta_2)$ as $x \rightarrow \infty$ where $g_\alpha(t) \equiv t^{\alpha-1}e^{-t}/\Gamma(\alpha)$. It is clear that if $\theta_2 < \eta_2$ then $(1 - F_\theta(x))/(1 - F_\eta(x)) \rightarrow 0$ as $x \rightarrow \infty$, and hence we must have $\theta_2 > \eta_2$. (It is easy to dismiss the boundary case $\theta_2 = \eta_2$.) On the other hand,

$$\lim_{x \downarrow 0} \frac{F_\theta(x)}{F_\eta(x)} = \lim_{x \downarrow 0} \frac{f_\theta(x)}{f_\eta(x)} = \left(\frac{\theta_1\theta_2}{\eta_1\eta_2}\right)^{-\alpha};$$

see, for example, Yu (2009), equation (13). Hence we must have $\theta_1\theta_2 \leq \eta_1\eta_2$. To rule out the boundary case, note that if $\theta_1\theta_2 = \eta_1\eta_2$, then $(\log \eta_1, \log \eta_2) < (\log \theta_1, \log \theta_2)$ and, by Lemma 1, there is no crossing.

Sufficiency. Assume $\theta_1\theta_2 < \eta_1\eta_2$ and $\theta_2 > \eta_2$. Retracing the proof of the necessity part, we can deduce that F_η crosses F_θ at least once, from below. To show that the crossing point is unique, we slightly extend the arguments of Diaconis and Perlman (1990). We have

$$F_\theta(x) - F_\eta(x) = xu^{-2} \int_0^\infty (H_\theta(u) - H_\eta(u))g_{2\alpha}(x/u) du,$$

where $H_\theta(u) = \Pr(\theta_1 W_1 + \theta_2 W_2 \leq u)$ and W_1 is a $\text{beta}(\alpha, \alpha)$ random variable with $W_1 = 1 - W_2 = X_1/(X_1 + X_2)$. The kernel $g_{2\alpha}(x/u)$ is strictly totally positive (STP) for $(x, u) \in (0, \infty)^2$. The claim would follow from variation-diminishing properties of STP kernels if we can show that $H_\theta(u) - H_\eta(u)$ changes signs only once, from + to -, as u increases on $(0, \infty)$. Note that

$$H_\theta(u) - H_\eta(u) = B\left(\frac{\eta_2 - u}{\eta_2 - \eta_1}\right) - B\left(\frac{\theta_2 - u}{\theta_2 - \theta_1}\right),$$

where $B(\cdot)$ denotes the $\text{beta}(\alpha, \alpha)$ distribution function. Let

$$u^* = \frac{\theta_2\eta_1 - \eta_2\theta_1}{\theta_2 - \theta_1 - \eta_2 + \eta_1}.$$

Under the assumptions, we have $\theta_1 < \eta_1 \leq \eta_2 < \theta_2$. It follows that $\eta_1 \leq u^* \leq \eta_2$ and

$$H_\theta(u) - H_\eta(u) \begin{cases} > 0 & \text{if } \theta_1 < u < u^*, \\ < 0 & \text{if } u^* < u < \theta_2, \\ = 0 & \text{otherwise.} \end{cases}$$

That is, $H_\theta(u) - H_\eta(u)$ has only one sign change at u^* , as needed. \square

REMARK 1. This proposition is closely related to Theorem 3.6 of [Kochar and Xu \(2011\)](#) and Proposition 3.1 of [Kochar and Xu \(2012\)](#) who compare F_θ and F_η according to the *star order* [[Shaked and Shanthikumar \(2007\)](#)]. $F_\eta \leq_* F_\theta$ means $F_\eta(x)$ crosses $F_\theta(cx)$ at most once, and always from below, for all $c > 0$. Proposition 1 can be used to recover a special case of Proposition 3.1 of [Kochar and Xu \(2012\)](#).

COROLLARY 2. If $\theta_2 > \theta_1, \eta_2 > \eta_1$ and $\theta_2/\theta_1 > \eta_2/\eta_1$, then $F_\eta \leq_* F_\theta$.

PROOF. In the stated parameter configuration, one can show that, for every $c > 0$, either θ and $c\eta$ satisfy the necessary conditions of Proposition 1 and F_θ and $F_{c\eta}$ cross exactly once, or they are ordered by the usual stochastic order, and there is no crossing. In other words, $F_\eta \leq_* F_\theta$. \square

PROOF OF THEOREM 1. We use induction on n . The case of $n = 2$ is given by Proposition 1. Suppose $n \geq 3$. The following argument works for $k < n$, and can be modified (with a different definition of τ) to handle the $k = n$ case. Let us consider $\tau \equiv (\theta_1, \dots, \theta_{k-2}, \delta, \eta_k, \theta_{k+1}, \dots, \theta_n)$, where

$$\delta_* \equiv \frac{\theta_{k-1}\theta_k}{\eta_k} \leq \delta \leq \min\left(\theta_k, \prod_{i \neq k} \eta_i / \prod_{i \neq k-1, k} \theta_i\right) \equiv \delta^*.$$

It is easy to see that $\delta_* < \delta^*$, and for $\delta \in (\delta_*, \delta^*)$ we have $\delta > \theta_{k-1}$ and $\prod_{i=1}^n \eta_i > \prod_{i=1}^n \tau_i > \prod_{i=1}^n \theta_i$. Also, $\tau_i < \eta_i$ for $i < k - 1$ and $\tau_i > \eta_i$ for $i \geq k + 1$ (including $i = n$ since $k < n$). Hence the sequence $\tau_{(i)} - \eta_i, i = 1, \dots, n$, has exactly one sign change, whether $\delta > \eta_{k-1}$ or not. (In the special case $k = 2$, we have $\delta < \eta_1$.) Here, we use $\tau_{(i)}$ rather than τ_i to account for possible switching between η_{k-1} and δ when we rearrange τ . As τ and θ differ in only two components, and τ and η have at least one (η_k) in common, by the induction hypothesis, F_τ crosses F_θ at most once, from below (say at x_*) and F_η crosses F_τ at most once, from below (say at x^*). When $\alpha \geq 1$, the gamma density is log-concave, which ensures that adding identical components does not create multiple crossings [see [Diaconis and Perlman \(1990\)](#), proof of their Proposition 2.3]. It is possible that the original single crossing is annihilated. If δ is large, then F_τ could stay entirely below F_θ

(effectively $x_* = \infty$). It is not possible, however, for F_η to stay entirely below F_τ , because $\tau_n > \eta_n$.

Note that F_τ stochastically increases in δ , which implies that x_* increases while x^* decreases in δ . This monotonicity can then be used to show that the crossing points (as long as they are finite) are continuous functions of δ . Specifically, fix $\delta_0 \in (\delta_*, \delta^*)$ such that $x_*(\delta_0)$ is finite. Then, by the continuity of F_θ and F_τ , and the monotonicity of x_* , we have

$$F_\theta(x_*(\delta_0+)) = \lim_{\delta \downarrow \delta_0} F_\theta(x_*(\delta)) = \lim_{\delta \downarrow \delta_0} F_\tau(x_*(\delta)) = F_\tau(x_*(\delta_0+)).$$

That is, F_θ and $F_\tau(\delta_0)$ crosses at $x_*(\delta_0+)$. By uniqueness of the crossing point, we have $x_*(\delta_0) = x_*(\delta_0+)$, and similarly $x_*(\delta_0) = x_*(\delta_0-)$, showing that x_* is continuous.

At $\delta = \delta_*$, we have $F_\tau \leq_{st} F_\theta$ by Lemma 1. So there is no crossing between F_τ and F_θ , that is, $x_* \downarrow 0$ as $\delta \downarrow \delta_*$ and $\lim_{\delta \downarrow \delta_*} x^* > 0$. At $\delta = \theta_k$, we have $F_\tau \geq_{st} F_\theta$. Thus $x_* \uparrow \infty$ as $\delta \uparrow \theta_k$ if $\delta^* = \theta_k$. The other possibility is $\delta^* = \prod_{i \neq k} \eta_i / \prod_{i \neq k, k-1} \theta_i$. In this case, at $\delta = \delta^*$ we have $F_\tau \geq_{st} F_\eta$ again by Lemma 1, because $\log \eta < \log \tau$. Indeed, because $\log(\eta_i/\tau_i)$ changes signs only once (after τ is arranged in increasing order), $\sum_{i=1}^l \log(\eta_i/\tau_i)$ first increases, and then decreases. At $l = 1$, we have $\eta_1 > \tau_1$. At $l = n$, we have $\sum_{i=1}^n \log(\eta_i/\tau_i) = 0$. Thus $\sum_{i=1}^l \log(\eta_i/\tau_i) \geq 0$ for all $l = 1, \dots, n$, that is, $\log \eta < \log \tau$. It follows that $x^* \downarrow 0$ as $\delta \uparrow \delta^*$.

Regardless of whether $\delta^* = \theta_k$, we have $x_* < x^*$ as $\delta \rightarrow \delta_*$ and $x_* > x^*$ as $\delta \rightarrow \delta^*$. By continuity, there exists some δ such that $x_* = x^*$ and

$$\begin{aligned} F_\theta(x) > F_\tau(x) > F_\eta(x), & \quad 0 < x < x^*; \\ F_\theta(x) < F_\tau(x) < F_\eta(x), & \quad x^* < x < \infty. \end{aligned}$$

It follows that F_η crosses F_θ exactly once, from below. \square

Theorem 2 gives a negative answer to the UCC for $n = 3$ and $\alpha < 1$. Note that counterexamples for $n > 3$ can be generated from a counterexample for $n = 3$ by appending small enough components to the weight vectors.

THEOREM 2. *For every $0 < \alpha < 1$, there exist positive vectors θ and η with $n = 3$ such that $\eta < \theta$ and $F_\eta(x) - F_\theta(x)$ changes signs at least three times as x increases from 0 to ∞ .*

Theorem 2 is derived through a perturbation analysis rather than extensive numerical calculations. Our counterexamples have the feature that $\theta_1 \approx \theta_2 \ll \theta_3$, and η is a small perturbation of θ which changes all three components. We show that for a suitable choice of such θ and η , there exists a point at which $F_\eta(x)$ crosses $F_\theta(x)$ from above. Since $F_\eta(x)$ has to cross $F_\theta(x)$ from below for sufficiently

small $x > 0$ and for sufficiently large x , it follows that there are at least three crossing points. The rest of this section makes this precise. We need the following result, which is slightly modified from Lemma 1 of [Rinott, Scarsini and Yu \(2012\)](#); see also [Székely and Bakirov \(2003\)](#).

LEMMA 2. *Suppose X_i and $Z_i, i = 1, 2$ are independent random variables with $X_i \sim \text{gamma}(\alpha, 1)$ and $Z_i \sim \text{expo}(1), i = 1, 2$ and, independently, G is a weighted sum of i.i.d. gamma variates. Fix $\theta_i^* > 0, i = 1, 2$ and let $\theta_1 = \theta_1^* - \delta$ and $\theta_2 = \theta_2^* + \delta$. Then*

$$\begin{aligned} & \frac{\partial}{\partial \delta} \Pr(\theta_1 X_1 + \theta_2 X_2 + G \leq x) \\ &= \alpha(\theta_2 - \theta_1) \frac{\partial^2}{\partial x^2} \Pr(\theta_1(X_1 + Z_1) + \theta_2(X_2 + Z_2) + G \leq x). \end{aligned}$$

We also need a result concerning mixtures of gamma densities [$g_\alpha(t) \equiv t^{\alpha-1} e^{-t} / \Gamma(\alpha)$].

LEMMA 3. *If $\alpha \in (0, 1)$, then there exists $\lambda \in (0, \infty)$ such that the mixture density $(\lambda g_{1+\alpha}(x) + g_\alpha(x)) / (1 + \lambda)$ is bimodal. There exists no such λ if $\alpha \geq 1$.*

PROOF. For fixed $\lambda > 0$, let $h(x) \equiv \Gamma(1 + \alpha)(\lambda g_{1+\alpha}(x) + g_\alpha(x))$. Simple calculation yields

$$h'(x) = e^{-x} x^{\alpha-2} [-\lambda x^2 + \alpha(\lambda - 1)x + \alpha(\alpha - 1)].$$

If $\alpha > 1$, the quadratic in square brackets has only one positive root, and the sign pattern of $h'(x)$ is $+, -$ on $(0, \infty)$. Hence $h(x)$ is unimodal. We can similarly analyze the case of $\alpha = 1$.

If $\alpha \in (0, 1)$, then for large enough λ the quadratic in square brackets has two distinct positive roots, and the sign pattern of $h'(x)$ becomes $-, +, -$ on $(0, \infty)$, showing that $h(x)$ has precisely two modes (one at zero, the other positive) with a local minimum in between. \square

PROOF OF THEOREM 2. Since $\alpha \in (0, 1)$, by Lemma 3 one can choose $\lambda > 0$ such that $s(x) \equiv \lambda g_{1+\alpha}(x) + g_\alpha(x)$ is bimodal, with a local minimum at $x_0 > 0$. Choose $w > 0$ small enough so that $s'(x_0 - w) < 0$ and $s'(x_0 + w) > 0$.

Let $\theta_1 = \varepsilon - \delta, \theta_2 = \varepsilon + \delta - \lambda \delta^2, \theta_3 = 1 + \varepsilon + \lambda \delta^2$, where ε and δ are to be determined. We require $\varepsilon > \delta > 0$ and $\varepsilon < 1/\lambda$ so that the weight vector θ is positive. Let $X_i, i = 1, 2, \dots$ be independent $\text{gamma}(\alpha, 1)$ variates and $Z_i \sim \text{expo}(1)$ independently. Define $G_0 \equiv \sum_{i=1}^3 \theta_i X_i$. Denote the density of $G_1 \equiv G_0 + \sum_{i=1}^2 \theta_i Z_i$ by f_1 and that of $G_2 \equiv G_0 + \sum_{i=2}^3 \theta_i Z_i$ by f_2 . As $\varepsilon \downarrow 0$ and $\varepsilon > \delta$, we have $f_1(x) \rightarrow g_\alpha(x)$ and $f_2(x) \rightarrow g_{1+\alpha}(x)$ pointwise in $x \in (0, \infty)$. To show this, let

$\nu = 2 + 3\alpha$ and $T = \sum_{i=1}^3 X_i + \sum_{i=1}^2 Z_i$. Then $T \sim \text{gamma}(\nu, 1)$ and we may use the independence of T and $S \equiv G_1/T$ to obtain

$$(1) \quad f_1(x) = \frac{x^{\nu-1}}{\Gamma(\nu)} E[e^{-x/S} S^{-\nu}];$$

$$(2) \quad f_1'(x) = \frac{\nu - 1}{x} f_1(x) - \frac{x^{\nu-1}}{\Gamma(\nu)} E[e^{-x/S} S^{-\nu-1}].$$

For fixed $x > 0$, the function $e^{-x/s} s^{-\nu}$ vanishes as $s \downarrow 0$ and is bounded and continuous over $s \in (0, \infty)$, achieving its maximum at $s = x/\nu$. As $\varepsilon \downarrow 0$, we have $S \rightarrow \text{beta}(\alpha, 2 + 2\alpha)$ in distribution. Hence $E[e^{-x/S} S^{-\nu}]$ converges, and $f_1(x)$ converges to $g_\alpha(x)$. Similarly, convergence holds for the derivatives of $f_i, i = 1, 2$. In fact, from (1) we can regard $f_1(x)$ as an analytic function on the open right half of the complex plane, and the convergence just mentioned is uniform in compact subsets.

By Lemma 2 and the chain rule, with $F_\theta(x) = \Pr(G_0 \leq x)$, we have

$$\begin{aligned} \frac{\partial}{\partial \delta} F_\theta(x) &= \alpha [(\theta_2 - \theta_1) f_1'(x) + 2\lambda\delta(\theta_3 - \theta_2) f_2'(x)] \\ &= 2\alpha \delta s'_\delta(x), \end{aligned}$$

where $s'_\delta(x) \equiv \lambda(1 - \delta + 2\lambda\delta^2) f_2'(x) + (1 - \lambda\delta/2) f_1'(x)$. Since $s'_\delta(x) \rightarrow s'(x)$ as $\varepsilon \downarrow 0$, we may choose $\varepsilon < 1/\lambda$ small enough so that, as long as $\delta < \varepsilon$, we have $s'_\delta(x_0 - w) < 0$ and $s'_\delta(x_0 + w) > 0$. Let $\eta = (\varepsilon, \varepsilon, 1 + \varepsilon)$. Then $\eta < \theta$ and by the mean value theorem

$$F_\theta(x) - F_\eta(x) = 2\alpha \delta \delta^* s'_{\delta^*}(x), \quad \delta^* \in (0, \delta).$$

But the right-hand side is strictly negative at $x = x_0 - w$ and strictly positive at $x = x_0 + w$, indicating at least one sign change in $x \in (x_0 - w, x_0 + w)$. Since $F_\theta(x) - F_\eta(x) > 0$ for sufficiently small $x > 0$ and $F_\theta(x) - F_\eta(x) < 0$ for sufficiently large x , we have at least two additional sign changes, both from $+$ to $-$, in $x \in (0, x_0 - w]$ and $x \in [x_0 + w, \infty)$, respectively. \square

3. Main result and proof.

THEOREM 3. *Conjecture 1 is valid if $\alpha \geq 1$.*

The rest of this paper is devoted to a proof of the above result. We extend the techniques of the previous section. By analyzing the distribution function crossing patterns of gamma convolutions, we reduce the problem to a particular configuration of the weight vectors θ and η that are sufficiently close. For this local case, relationship between crossing points to modes of a mixture of gamma convolutions

is explored. We introduce a new stochastic order and derive monotonicity properties concerning densities of gamma convolutions. These tools further reduce the problem, leading to Theorem 3.

Lemma 4 shows that UCC holds locally for a particular configuration.

LEMMA 4. *Suppose $\alpha \geq 1$. Given an index $1 < k < n - 1$ let $0 < \theta_1 \leq \dots \leq \theta_{k-1} < \theta_k \leq \theta_{k+1} < \theta_{k+2} \leq \dots \leq \theta_n$ and $\delta_i > 0, i = 1, \dots, n$, and let η be defined as follows:*

$$\eta_i = \begin{cases} \theta_i + \delta_i & i = 1, \dots, k - 1; \\ \theta_i - \sum_{j=1}^{k-1} \delta_j & i = k; \\ \theta_i + \sum_{j=k+2}^n \delta_j & i = k + 1; \\ \theta_i - \delta_i & i = k + 2, \dots, n. \end{cases}$$

Then F_η crosses F_θ exactly once from below if $\sum_{i \neq k, k+1} \delta_i$ is small enough.

To deduce Theorem 3 from Lemma 4, we build on our proof of Theorems 1 and 2. Let us introduce a majorization-type ordering which may be of independent interest. As usual, we write $\theta_{(1)}, \theta_{(2)}, \dots, \theta_{(n)}$ as θ rearranged in increasing order.

DEFINITION 1. We say a real vector θ *V-majorizes* η , written as $\eta <_V \theta$, if there exists $\tilde{\theta}$ such that $\eta < \tilde{\theta}$ and indices $1 \leq k_1, k_2 \leq n$ such that

$$(3) \quad \theta_{(i)} \begin{cases} \leq \tilde{\theta}_{(i)} \leq \eta_{(i)} & 1 \leq i \leq k_1; \\ = \tilde{\theta}_{(i)} & k_1 < i < k_2; \\ \geq \tilde{\theta}_{(i)} \geq \eta_{(i)} & k_2 \leq i \leq n. \end{cases}$$

Simply put, θ V-majorizes η if θ is obtained from a vector $\tilde{\theta}$ that majorizes η by increasing (and decreasing) the largest (smallest) few components of $\tilde{\theta}$ which are already larger (smaller) than those of η . Let us record some useful properties of $<_V$.

PROPOSITION 2. *Let η, θ be positive vectors such that $\eta <_V \theta$. (a) If $\prod_{i=1}^n (\eta_i / \theta_i) \geq 1$ then $\log \eta <^w \log \theta$. (b) If $\prod_{i=1}^n (\eta_i / \theta_i) \leq 1$ then $F_\eta \leq_{st} F_\theta$.*

PROOF. Assume the coordinates of θ (resp., η) are arranged in increasing order, and assume $\theta \neq \eta$. Let $\tilde{\theta}$ (also arranged in increasing order) be such that $\eta < \tilde{\theta}$ and (3) is satisfied.

Part (a). Note that $\tilde{\theta}_i \geq \theta_i$ for $1 \leq i < k_2$. From $\eta < \tilde{\theta}$, we get $\log \eta <^w \log \tilde{\theta}$ and

$$\prod_{i=1}^l \eta_i \geq \prod_{i=1}^l \tilde{\theta}_i \geq \prod_{i=1}^l \theta_i, \quad 1 \leq l < k_2.$$

For $i \geq k_2$, we have $\eta_i \leq \theta_i$ which means $\prod_{i=1}^l (\eta_i / \theta_i)$ between $l = k_2 - 1$ and $l = n$ is minimized at $l = n$. Thus, to ensure $\log \eta <^w \log \theta$ we only need $\prod_{i=1}^n \eta_i \geq \prod_{i=1}^n \theta_i$.

Part (b). We may define

$$\theta_i^* \equiv \begin{cases} \theta_i, & i < k_2, \\ \theta_i^{1-\lambda} \eta_i^\lambda, & i \geq k_2; \end{cases} \quad \lambda = \frac{\sum_{i=1}^n \log(\theta_i / \eta_i)}{\sum_{i=k_2}^n \log(\theta_i / \eta_i)}.$$

Then $\lambda \in [0, 1]$, $\theta^* \leq \theta$ (coordinate-wise), and $\prod_{i=1}^n (\eta_i / \theta_i^*) = 1$. The reasoning of part (a) yields $\log \eta < \log \theta^*$. By Lemma 1, $F_\eta \leq_{st} F_{\theta^*} \leq_{st} F_\theta$. \square

With the notion of $<_V$, we can suitably generalize Conjecture 1 and prove it, building on a special case, Lemma 4, which we will establish later.

THEOREM 4. *Suppose $\alpha \geq 1$, and η, θ are positive weight vectors such that $\eta <_V \theta$, $\theta_{(n)} > \eta_{(n)}$ and $\prod_{i=1}^n (\eta_i / \theta_i) > 1$. Then there exists $x_0 \in (0, \infty)$ such that $F_\eta(x) < F_\theta(x)$ for $x \in (0, x_0)$ and $F_\eta(x) > F_\theta(x)$ for $x > x_0$.*

It is easy to verify that if $\eta < \theta$ and (i) θ is all positive and (ii) $\eta_i \neq \theta_j$ for all i, j , then the conditions of Theorem 4 are satisfied (assuming $\alpha \geq 1$ of course). When deducing Theorem 3 from Theorem 4, both restrictions (i) and (ii) can be relaxed in the same way as explained in the proof of Corollary 1.

PROOF OF THEOREM 4. We will use induction on n . The case of $n = 3$ is covered by Theorem 1. For $n \geq 4$, assume $\theta_1 \leq \dots \leq \theta_n$, $\eta_1 \leq \dots \leq \eta_n$ and $\theta_i \neq \eta_i$ for all i . Let us define

$$j \equiv \min\{i : \theta_i > \eta_i, 1 \leq i \leq n\}; \quad k \equiv \max\{i : \theta_i < \eta_i, 1 \leq i \leq n\}.$$

These indices must exist because $\theta_n > \eta_n$ and $\theta_1 < \eta_1$. Moreover, we have $j \geq 2, k \leq n - 1, j \neq k$ and $k \geq j - 1$. Consider two cases:

(a) $k = j - 1$. Then $\log(\eta_i / \theta_i)$ has only one sign change and the claim follows from Theorem 1.

(b) $k > j$. Since $\eta <_V \theta$, there exists $\tilde{\theta}$ such that $\eta < \tilde{\theta}$ and (3) holds. In (3), we necessarily have $k_1 \leq j - 1$ and $k_2 \geq k + 1$, and hence $\tilde{\theta}_i = \theta_i, i = j, \dots, k$. Let $\delta = \eta - \tilde{\theta}$. Then $\delta_i \geq 0$ for $i < j$ and $\delta_i \leq 0$ for $i > k$. Define a weight vector τ

parameterized by t_1, t_2 as follows:

$$(4) \quad \tau_i \equiv \begin{cases} \theta_i + t_1 \delta_i & i < j, \\ \theta_i - t_1 \sum_{l=1}^{j-1} \delta_l & i = j, \\ \theta_i & j < i < k; \\ \theta_i - t_2 \sum_{l=k+1}^n \delta_l & i = k, \\ \theta_i + t_2 \delta_i & i > k. \end{cases}$$

We require $0 \leq t_1 \leq c_1$ and $0 \leq t_2 \leq c_2$ where

$$c_1 = \frac{\theta_j - \eta_j}{\sum_{i=1}^{j-1} \delta_i}; \quad c_2 = \frac{\theta_k - \eta_k}{\sum_{i=k+1}^n \delta_i}.$$

Because $\eta < \tilde{\theta}$, we have $\sum_{i=1}^{j-1} \delta_i \geq -\delta_j > 0, \sum_{i=k+1}^n \delta_i \leq -\delta_k < 0$ and $c_1, c_2 \in (0, 1)$. Define $\tilde{\tau} = \tau + \tilde{\theta} - \theta$. Using $t_i \leq c_i, i = 1, 2$, we can show that components of $\tilde{\tau}$ are in increasing order, and that $\eta < \tilde{\tau}$. Moreover, $\tau_i \leq \tilde{\tau}_i \leq \eta_i$ for $i < j$ and $\tau_i \geq \tilde{\tau}_i \geq \eta_i$ for $i > k$. It follows that $\eta <_{\vee} \tau$. Also, from (4) and (i) $\tau_i \leq \eta_i \leq \eta_j \leq \tau_j$ for $i < j$ and (ii) $\tau_i \geq \eta_i \geq \eta_k \geq \tau_k$ for $i > k$, we can deduce that $\tau < \theta$, which yields $\prod_{i=1}^n (\tau_i / \theta_i) \geq 1$. In fact, if we have t, \tilde{t} such that $0 \leq t_i \leq \tilde{t}_i \leq c_i, i = 1, 2$, then $\eta <_{\vee} \tau(\tilde{t}) < \tau(t)$.

Let us denote $\tau(t = (c_1, 0))$ by v , which has the feature that $v_j = \eta_j$. Define

$$\gamma_i \equiv \begin{cases} \theta_i^{1-c_3} v_i^{c_3} & i < j, \\ v_i & i \geq j; \end{cases} \quad c_3 = \frac{\log(\theta_j / v_j)}{\sum_{i=1}^{j-1} \log(v_i / \theta_i)}.$$

Then $c_3 \in (0, 1)$, and $\log \gamma < \log \theta$. Moreover, since $v_i \geq \gamma_i$ for $i < j$ and $\eta <_{\vee} v$, we have $\eta <_{\vee} \gamma$.

Also, denote $\tilde{v} \equiv \tau(t = (0, c_2))$, and define $\tilde{\gamma}_i = \theta_i, i \neq k$ and $\tilde{\gamma}_k = \eta_k$. We have $\tilde{v}_k = \eta_k, \tilde{\gamma} \geq \tilde{v}, \tilde{\gamma} \geq \theta$ and $\eta <_{\vee} \tilde{\gamma}$.

In view of Lemma 4, let $\varepsilon > 0$ be small enough so that as long as $0 < t_i \leq \varepsilon, i = 1, 2$, we have $F_{\tau(t_1, t_2)}$ crosses F_{θ} exactly once from below. Construct a continuum of weight vectors $\rho(s)$ such that $\rho(-3) = \gamma, \rho(-2) = v, \rho(-1) = \tau(\varepsilon, 0), \rho(1) = \tau(0, \varepsilon), \rho(2) = \tilde{v}, \rho(3) = \tilde{\gamma}$ and values of $\rho(s)$ in between are defined through linear interpolation. In particular, for $s \in (-1, 1)$ we have $\rho(s) = \tau(t_1, t_2)$ with $t_1 = (1 - s)\varepsilon/2$ and $t_2 = (1 + s)\varepsilon/2$. By the choice of ε , we know $F_{\rho(s)}$ crosses F_{θ} exactly once, from below, for $s \in (-1, 1)$. The same holds for $s \in (-3, -1] \cup [1, 3)$ by Theorem 1, with the possible exception of some s in the upper portion of the interval $(2, 3)$. When $s \in (2, 3)$ it is possible that $F_{\rho(s)}$ stays entirely below F_{θ} . This is not possible for $s \in [-2, 2]$ because $\rho(s) < \theta$ and the means of the two distributions are equal. It is not possible for $s \in (-3, -2)$ because the mean of $F_{\rho(s)}$ is even smaller than that of F_{θ} . To verify the conditions of Theorem 1, we

examine the subintervals of $(-3, -1] \cup [1, 3)$ and note that $\rho_{(i)}(s) - \theta_i$ has only one sign change as a function of $i = 1, \dots, n$.

Let x_* denote a crossing point between F_η and F_θ (the conditions ensure at least one crossing). Note that $F_{\rho(s)}$ stochastically decreases as $s \downarrow -3$, with $F_{\rho(-3)} \leq_{st} F_\theta$, and $F_{\rho(s)}$ stochastically increases as $s \uparrow 3$, with $F_{\rho(3)} \geq_{st} F_\theta$. Although monotonicity need not hold when s moves away from the boundary, by continuity, $F_{\rho(s)}$ must cross F_θ at precisely x_* for some $s_* \in (-3, 3)$.

Suppose $s_* \in [2, 3)$. Then it is easy to verify $\eta \prec_V \rho(s_*)$. Note that $\rho(s_*) \neq \eta$ but $F_\eta(x_*) = F_{\rho(s_*)}(x_*)$. By Proposition 2, we have $\prod_{i=1}^n (\eta_i / \rho_i(s_*)) > 1$. Since η and $\rho(s_*)$ have η_k in common, by the induction hypothesis, F_η crosses $F_{\rho(s_*)}$ exactly once, from below, at the same crossing point x_* between $F_{\rho(s_*)}$ and F_θ . As in the proof of Theorem 1, we conclude that x_* is the only crossing point between F_η and F_θ .

The case of $s_* \in (-3, -2]$ is similar.

Suppose $s_* \in (-2, 2)$. Regardless of which subinterval s_* falls into, we have some $t^* \equiv (t_1^*, t_2^*)$ not identically zero such that

$$(5) \quad F_{\tau(t^*)}(x) < F_\theta(x), \quad x \in (0, x_*); \quad F_{\tau(t^*)}(x) > F_\theta(x), \quad x > x_*.$$

By Proposition 2, we must have $\prod_{i=1}^n \tau_i(t^*) \leq \prod_{i=1}^n \eta_i$, that is, $t^* \in \Omega$, with $\Omega \equiv \{t = (t_1, t_2) : t_i \in [0, c_i], i = 1, 2; \prod_{i=1}^n \eta_i \geq \prod_{i=1}^n \tau_i(t)\}$. If t^* lies in the interior of Ω , then repeating the entire argument with $\tau(t^*)$ in place of θ [which corresponds to $t = (0, 0)$] we conclude that, either the claim does hold, or there exists $t_i^{**} \geq t_i^*$, with strict inequality for at least one $i = 1, 2$, such that $F_{\tau(t^{**})}$ crosses $F_{\tau(t^*)}$ (and hence F_θ) exactly once, from below, at x_* . And $t^{**} \in \Omega$. Let Ω_0 be the set of $t \in \Omega$ such that (i) $t \geq t^*$, and (ii) $F_{\tau(t)}(x) \leq F_{\tau(t^*)}(x)$ for $x \in (0, x_*)$ and the inequality is reversed for $x > x_*$. By continuity, Ω_0 is a closed set. Let ω^* be an element of Ω_0 with maximal value of $\omega_1 + \omega_2$. The above discussion shows that, either the claim holds, or ω^* does not belong to the interior of Ω , that is, $\omega_i^* = c_i$ for at least one $i = 1, 2$. We can rule out the other boundary situation $\prod_{i=1}^n \eta_i = \prod_{i=1}^n \tau_i(\omega^*)$ in view of Proposition 2, unless $\tau(\omega^*)$ is a permutation of η , in which case the claim follows from the definition of Ω_0 and the strict inequalities (5). In other cases, by the induction hypothesis, F_η crosses $F_{\tau(\omega^*)}$ exactly once from below, at x_* ; the claim follows from this, the definition of Ω_0 , and (5). \square

To treat the local case of Lemma 4, a key tool is the following Lemma 5, which connects whether there are multiple crossing points when the weight vector is perturbed locally to whether mixtures of several gamma convolutions are always unimodal.

LEMMA 5. For a fixed positive weight vector θ , let η be defined by $\eta = \theta + \sum_{k=1}^K \tau^{(k)}$ where, associated with each k , we have a pair of indices $i_k \neq j_k$ and a

real number $\delta_k > 0$ such that $\theta_{i_k} < \theta_{j_k}$ and

$$\tau_i^{(k)} = \begin{cases} \delta_k & i = i_k, \\ -\delta_k & i = j_k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f_k(x|\delta)$, $\delta \equiv (\delta_1, \dots, \delta_K)$, denote the density of $\sum_{i=1}^n \eta_i X_i + \eta_{i_k} Z_{i_k} + \eta_{j_k} Z_{j_k}$, where $X_i \sim \text{gamma}(\alpha, 1)$, $Z_i \sim \text{expo}(1)$ are mutually independent. Suppose, for arbitrary constants $\lambda_k \geq 0$ such that $\sum_k \lambda_k = 1$, we have $\sum_k \lambda_k f_k(x|0)$ is unimodal, with a strictly negative second derivative at the mode, and no saddle points. Then for small enough $\sum_{k=1}^K \delta_k$, F_η crosses F_θ exactly once, from below.

PROOF. Note that $\eta < \theta$ if δ is small enough, and hence F_η crosses F_θ at least once, from below. By Lemma 2, we have

$$\frac{\partial F_\eta(x)}{\partial \delta_k} = \alpha(\eta_{i_k} - \eta_{j_k}) f'_k(x|\delta), \quad k = 1, \dots, K.$$

Then

$$(6) \quad F_\eta(x) - F_\theta(x) = \int_0^1 \sum_{k=1}^K \alpha(\theta_{i_k} - \theta_{j_k} + 2t\delta_k) \delta_k f'_k(x|t\delta) dt.$$

Suppose the claim does not hold, and there exists a sequence $\eta(l)$ corresponding to $\delta(l) \equiv (\delta_{1l}, \dots, \delta_{Kl})$ such that $\delta(l) \rightarrow 0$ and $F_{\eta(l)}(x) - F_\theta(x) = 0$ has at least two roots in $x \in (0, \infty)$, for each $l = 1, 2, \dots$. Denote $L(\delta) = \sum_{k=1}^K \delta_k (\theta_{i_k} - \theta_{j_k})$. By taking subsequences if necessary as $l \rightarrow \infty$, we may assume $\delta_{kl}(\theta_{i_k} - \theta_{j_k})/L(\delta(l)) \rightarrow \lambda_k$ for some nonnegative λ_k such that $\sum_{k=1}^K \lambda_k = 1$. We benefit from the fact that f_k can be regarded as analytic functions on the open right half of the complex plane, and as δ tends to zero they converge uniformly on compact subsets. It follows that

$$(7) \quad \lim_{l \rightarrow \infty} \frac{F_{\eta(l)}(x) - F_\theta(x)}{\alpha L(\delta(l))} = \sum_{k=1}^K \lambda_k f'_k(x|0),$$

and the convergence is uniform on compact subsets. Let D denote a finite horizontal strip within the open right half plane such that $D \cap \mathbb{R}$ encloses all possible roots $x \in (0, \infty)$ of $F_{\eta(l)}(x) - F_\theta(x) = 0$ for sufficiently large l . This is possible from bounds on location of the crossing points [Bock et al. (1987), Roosta-Khorasani and Székely (2015)]. In fact, when δ is sufficiently small and $t \in (0, 1)$, letting $M = \max(\theta)$ and $m = \min(\theta)$ we obtain $\text{gamma}(n\alpha + 2, m) \leq_{lr} f_k(\cdot|t\delta) \leq_{lr} \text{gamma}(n\alpha + 2, M)$, which implies $f'_k(x|t\delta) < 0$ for $x > M/(n\alpha + 1)$ and $f'_k(x|t\delta) > 0$ for $x < m/(n\alpha + 1)$. It follows from (6) that all positive real roots of $F_\eta(x) - F_\theta(x) = 0$ must be between $m/(n\alpha + 1)$ and $M/(n\alpha + 1)$. We can make D thin enough so that there are no other roots of $\sum_{k=1}^K \lambda_k f'_k(x|0)$ within

D except for the unique mode of the real function $\sum_{k=1}^K \lambda_k f_k(x|0)$ which, by assumption, must be a simple root. By (7), for large enough l , the number of roots of $F_{\eta(l)} - F_\theta$ within D , counting multiplicity, must be equal to one, which contradicts the assumption of multiple real roots. \square

Lemma 4 is a consequence of Lemma 6 and Lemma 5.

LEMMA 6. *In the setting of Lemma 4, let $X_i \sim \text{gamma}(\alpha, 1)$ and $Z_i \sim \text{expo}(1)$ be mutually independent. Let f_j denote the density of $\sum_{i=1}^n \theta_i X_i + \theta_j Z_j + \theta_k Z_k$ for $j = 1, \dots, k - 1$ and that of $\sum_{i=1}^n \theta_i X_i + \theta_j Z_j + \theta_{k+1} Z_{k+1}$ for $j = k + 2, \dots, n$. Then, for any $\lambda = (\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+2}, \dots, \lambda_n)$ such that $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, the mixture density $\sum_{i \neq k, k+1} \lambda_i f_i$ is unimodal with a strictly negative second derivative at the mode, and no saddle points.*

Lemma 6 requires detailed analysis. As a starting point, we prove some monotonicity properties concerning the densities of gamma convolutions in a simple case.

LEMMA 7. *For $\theta \in (0, 1)$, let $X \sim \text{gamma}(\alpha, 1)$ and $Z \sim \text{expo}(1)$ independently. Let $g_\alpha(x) = x^{\alpha-1} e^{-x} / \Gamma(\alpha)$ and denote the density of $X + \theta Z$ by $h(x)$. Then (a) if $\alpha \geq 1$ then $h'(x)/g'_\alpha(x)$ strictly increases in $x \in (\alpha - 1, \infty)$; (b) if $\alpha \geq 2$ then $h'(x)/g'_\alpha(x)$ also strictly increases in $x \in (0, \alpha - 1)$; (c) if $\alpha \geq 1$ then $h'(x)/g'_{\alpha+1}(x)$ strictly decreases in each of $x \in (0, \alpha)$ and $x \in (\alpha, \infty)$; (d) parts (a)–(c) still hold when the distribution of Z is replaced by a finite mixture of exponentials with rates > 1 .*

PROOF. In the $\alpha = 1$ case, the densities are amenable to direct calculations. Let us assume $\alpha > 1$. Denote $g \equiv g_\alpha$. We have

$$(8) \quad \theta h'(x) + h(x) = g(x), \quad x > 0,$$

which can be verified by comparing the Laplace transform of both sides. To prove (a), we will show $h''(x)g'(x) > h'(x)g''(x)$ for $x > \alpha - 1$. Differentiating (8) to eliminate $h''(x)$, and noting that $\theta g''(x) + g'(x) < 0$ for $x > \alpha - 1$, we equivalently need to show

$$(9) \quad u(x) \equiv e^{x/\theta} \left[\frac{g'^2(x)}{\theta g''(x) + g'(x)} - h'(x) \right] < 0, \quad x > \alpha - 1.$$

This holds for $x = \alpha - 1$ because $g'(\alpha - 1) = 0$ and h dominates g in the likelihood ratio order. By direct calculation, we have

$$(10) \quad u'(x) = \frac{\theta e^{x/\theta} g'(x)(g''^2(x) - g'(x)g'''(x))}{(\theta g''(x) + g'(x))^2} < 0, \quad x > \alpha - 1.$$

Thus $u(x) < 0$ for all $x > \alpha - 1$, as required.

To prove (b), we only need to show that $u(x) > 0$ for $x \in (0, x_*)$ and $u(x) < 0$ for $x \in (x_*, \alpha - 1)$ where $x_* < \alpha - 1$ is the unique positive root of $\theta g''(x) + g'(x) = 0$. When $\alpha \geq 2$, we have $u(x) \rightarrow 0$ as $x \downarrow 0$ and $u'(x) > 0$ for $x \in (0, x_*) \cup (x_*, \alpha - 1)$. We obtain the desired sign pattern of u as a consequence.

To prove (c), we similarly will show $h''(x)g'_{\alpha+1}(x) < h'(x)g''_{\alpha+1}(x)$ for $x > 0$. This is equivalent to $\tilde{u}(x) < 0$ for $x \in (0, x^*)$ and $\tilde{u}(x) > 0$ for $x \in (x^*, \infty)$, where $x^* \in (\alpha - 1, \alpha)$ is the unique positive root of $\theta g''_{\alpha+1}(x) + g'_{\alpha+1}(x) = 0$ and

$$\tilde{u}(x) \equiv e^{x/\theta} \left[\frac{g'(x)g'_{\alpha+1}(x)}{\theta g''_{\alpha+1}(x) + g'_{\alpha+1}(x)} - h'(x) \right].$$

This sign pattern can be proved by arguments parallel to the previous parts.

Part (d) is obvious. \square

The usefulness of these monotonicity properties is more apparent after we define the following stochastic order.

DEFINITION 2. Suppose f and g are twice continuously differentiable densities supported on an interval $I \subset (0, \infty)$. We say f is dominated by g in the *supplemented likelihood ratio ordering*, written as $f \leq_{\text{slr}} g$, if (a) $f'(x)g(x) \leq f(x)g'(x)$ for all $x \in I$ and (b) $f'(x)/g'(x)$ decreases in each of the sets $\{x : f'(x) > 0\}$ and $\{x : g'(x) < 0\}$.

Some properties of \leq_{slr} are summarized as follows.

PROPOSITION 3. (a) If $f \leq_{\text{slr}} g$ and $g \leq_{\text{slr}} h$, then $f \leq_{\text{slr}} h$. (b) If $\alpha \geq 2$ and $\theta \in (0, 1)$, then $g_\alpha \leq_{\text{slr}} g_\alpha * \text{expo}(\theta)$ where $*$ denotes convolution. (c) Suppose $\alpha \geq 1$, and $\theta \in (0, 1)$, then $g_\alpha * \text{expo}(\theta) \leq_{\text{slr}} g_{\alpha+1}$. (d) Parts (b) and (c) still hold if $\text{expo}(\theta)$ is replaced by a mixture of exponentials with rates ≥ 1 .

PROOF. Part (a) is obvious. Parts (b)–(d) are restating Lemma 7. \square

PROPOSITION 4. Suppose f and g are unimodal (see Remark 2), $f \leq_{\text{slr}} g$ and h is Polya frequency order 3. Assume we can take the derivatives inside the absolutely convergent integrals and obtain $(f * h)' = f' * h$ and $(g * h)' = g' * h$. Then $f * h \leq_{\text{slr}} g * h$.

PROOF. Let us denote $\tilde{f} = f * h$ and $\tilde{g} = g * h$. Since h is PF3, the likelihood ratio ordering is preserved, that is, $\tilde{f} \leq_{\text{lr}} \tilde{g}$. Moreover, \tilde{f} and \tilde{g} are unimodal. Let $\lambda > 0$ and consider the function $f'(x) - \lambda g'(x)$. By assumption, $f'(x)/g'(x)$ decreases on each of $I_+ \equiv \{x : f'(x) > 0\}$ and $I_- \equiv \{x : g'(x) < 0\}$. Assume these are nonempty, otherwise the argument can be suitably modified. Note that by $f \leq_{\text{lr}} g$ we have $g'(x) > 0$ for $x \in I_+$ and $f'(x) < 0$ for $x \in I_-$. On the set $I_0 \equiv \{x :$

$g'(x) \geq 0, f'(x) \leq 0$ we have $f'(x) - \lambda g'(x) \leq 0$. Overall $f' - \lambda g'$ changes signs at most twice, and the sign sequence is $+, -, +$ in the case of two changes. By the variation-diminishing properties of totally positive kernels [Karlin (1968)], the same is true for $(f' - \lambda g') * h = \tilde{f}' - \lambda \tilde{g}'$. We need to show \tilde{f}'/\tilde{g}' decreases on each of $\tilde{I}_+ \equiv \{x : \tilde{f}'(x) > 0\}$ and $\tilde{I}_- \equiv \{x : \tilde{g}'(x) < 0\}$. Denote the upper end point of \tilde{I}_+ by x_0 . For $0 < \lambda < \tilde{f}'(x_0)/\tilde{g}'(x_0)$, if $\tilde{f}'(x) - \lambda \tilde{g}'(x)$ ever crosses zero from below in $x \in I_+$, then it must be nonnegative for $x \geq x_0$, in order not to violate the sign pattern of $+, -, +$. Thus

$$\tilde{f}(x_0) = - \int_{x_0}^{\infty} \tilde{f}'(x) dx \leq -\lambda \int_{x_0}^{\infty} \tilde{g}'(x) dx = \lambda \tilde{g}(x_0)$$

which contradicts $\lambda < \tilde{f}'(x_0)/\tilde{g}'(x_0)$. With a small perturbation this still applies when $\lambda = \tilde{f}'(x_0)/\tilde{g}'(x_0)$. For $\lambda > \tilde{f}'(x_0)/\tilde{g}'(x_0)$, we have $\tilde{f}'(x)/\tilde{g}'(x) < \lambda$ for $x \in I_+$ sufficiently close to x_0 , because of the likelihood ratio ordering. In order not to violate the sign pattern, $\tilde{f}'(x) - \lambda \tilde{g}'(x)$ cannot cross zero from below in $x \in I_+$ in this case either. Because λ is arbitrary, $\tilde{f}'(x)/\tilde{g}'(x)$ must decrease for $x \in I_+$. The case of $x \in \tilde{I}_-$ is similar. \square

REMARK 2. We impose a restricted form of unimodality, which is satisfied by the gamma convolutions. For the above proof to be valid, we need the set I_0 to be situated between I_+ and I_- . This will be satisfied if we assume the closures of I_+ and I_- are intervals. So, an isolated saddle point is allowed, but not a flat ridge. We will note down such restrictions when needed.

Proposition 5 reveals the intimate relation between \leq_{slr} and the unimodality of the mixture of two densities with arbitrary mixing proportions. It allows us to reduce the problem of unimodality needed in Lemma 6 to manageable special cases.

PROPOSITION 5. *Let f_i and $h_i, i = 1, 2$, be twice continuously differentiable and unimodal densities supported on $(0, \infty)$ such that*

$$h_1 \leq_{\text{slr}} f_1 \leq_{\text{lr}} f_2 \leq_{\text{slr}} h_2.$$

Suppose the mixture density $ph_1 + (1 - p)h_2$ is unimodal for all $p \in [0, 1]$. Then so is $pf_1 + (1 - p)f_2$, assuming f'_1 and f'_2 do not vanish simultaneously in between the modes of f_1 and f_2 .

PROOF. Let x_* and x^* denote the modes of f_1 and f_2 , respectively. In the case of a possible plateau, x_* (resp., x^*) denotes the leftmost (resp., rightmost) mode of f_1 (resp., f_2). Obviously, all modes of the mixture $pf_1 + (1 - p)f_2$ are in the interval $[x_*, x^*]$. Moreover, for each $x_0 \in (x_*, x^*)$ such that $f'_i(x_0) \neq 0, i = 1, 2$, we may set $\lambda = -f'_1(x_0)/f'_2(x_0)$ to obtain a stationary point of this mixture density.

By the likelihood ratio ordering, we necessarily have $f'_1(x_0) < 0 < f'_2(x_0)$ and $\lambda > 0$. To show that the mixture is unimodal, suppose f'_2 does not vanish on (x_*, x^*) . Then we can show that $-f'_1(x)/f'_2(x)$ increases on (x_*, x^*) , which is equivalent to

$$(11) \quad f''_1(x_0)f'_2(x_0) \leq f''_2(x_0)f'_1(x_0), \quad x_0 \in (x_*, x^*).$$

Condition (11) is necessary because, if the mixture is unimodal, then a stationary point can never be a local minimum, and hence $f''_1(x_0) + \lambda f''_2(x_0) \leq 0$. On the other hand, if $\lambda = -f'_1(x_0)/f'_2(x_0)$ is an increasing function of $x_0 \in (x_*, x^*)$, then stationary points of the mixture corresponding to the same λ form a connected interval, showing that the mixture is unimodal. A close inspection shows that (11) is sufficient as long as the saddle points of f_1 and f_2 on (x_*, x^*) do not coincide.

Applying this criterion to the mixture $ph_1 + (1 - p)h_2$, we have

$$h''_1(x_0)h'_2(x_0) \leq h''_2(x_0)h'_1(x_0), \quad x_0 \in (x_*, x^*),$$

which yields, as long as $f'_1(x_0) \neq 0 \neq f'_2(x_0)$,

$$\frac{f''_2(x_0)}{f'_2(x_0)} \leq \frac{h''_2(x_0)}{h'_2(x_0)} \leq \frac{h''_1(x_0)}{h'_1(x_0)} \leq \frac{f''_1(x_0)}{f'_1(x_0)}, \quad x_0 \in (x_*, x^*),$$

in view of $h_1 \leq_{\text{slr}} f_1$ and $f_2 \leq_{\text{slr}} h_2$, and (11) is established. \square

Next, we present two log-concavity results needed in the proof of Lemma 6. See also Székely and Bakirov (2003), Theorem 4, for general results related to Lemma 8 below.

LEMMA 8. *Suppose $X_i \sim \text{gamma}(\alpha, 1)$ and $Z_i \sim \text{expo}(1)$, $i = 1, 2$, are mutually independent where $\alpha \geq 1$. Let $\delta_1, \delta_2 > 0$. Then arbitrary mixtures of $\delta_1 X_1 + \delta_2 X_2$ and $\delta_1(X_1 + Z_1) + \delta_2(X_2 + Z_2)$ are unimodal.*

PROOF. We show that when $\alpha = 1$, such mixtures are log-concave. If $\alpha > 1$, then we can write $\sum_{i=1}^2 \delta_i X_i = \sum_{i=1}^2 \delta_i (X_i^* + Y_i)$ where $X_i^* \sim \text{expo}(1)$ and $Y_i \sim \text{gamma}(\alpha - 1, 1)$ independently. We can similarly “split off” $\delta_1 Y_1 + \delta_2 Y_2$ from $\sum_{i=1}^2 \delta_i (X_i + Z_i)$. Because $\delta_1 Y_1 + \delta_2 Y_2$ is unimodal, the result follows from the log-concave result in the $\alpha = 1$ case.

Let us assume $\delta_2 = 1$ and $\delta \equiv \delta_1 \in (0, 1)$. When $\alpha = 1$, the densities of $\delta X_1 + X_2$ and $\delta(X_1 + Z_1) + X_2 + Z_2$ are, respectively,

$$h_1(x) = \frac{e^{-x} - e^{-x/\delta}}{1 - \delta}; \quad h_2(x) = \frac{x(e^{-x} + e^{-x/\delta}) - 2\delta h_1(x)}{(1 - \delta)^2}.$$

For $\lambda > -2\delta$ and $\varepsilon \equiv \delta^{-1} - 1$, let

$$q(x) \equiv x(e^{\varepsilon x} + 1) + \lambda(e^{\varepsilon x} - 1).$$

We only need to show that $q(x)$ is log-concave. A quick calculation yields

$$e^{-\varepsilon x} [q'^2(x) - q''(x)q(x)] = e^{\varepsilon x} + e^{-\varepsilon x} - 2 - (\varepsilon x)^2 + (\lambda\varepsilon + 2)^2$$

which is positive for all $x > 0$. \square

LEMMA 9. *Let Y be an arbitrary mixture of $k \geq 1$ exponentials with means $\delta_i, i = 1, \dots, k$ such that $\max(\delta_i) \leq \delta$. Let $Z \sim \text{expo}(\delta)$ independently of Y . Then $Y + Z$ is strictly log-concave.*

PROOF. Suppose $\max(\delta_i) < \delta$. The density of $U \equiv Y + Z$ can be written as $h(u) = \sum_{i=1}^k \lambda_i (e^{-u/\delta} - e^{-u/\delta_i})$ for some constants $\lambda_i > 0$. We know $h(u)$ is strictly log-concave on $(0, \infty)$ because $e^{u/\delta} h(u)$ is strictly concave. A slight modification works when $\max(\delta_i) = \delta$. \square

We are now ready to present the proof of Lemma 6, which concludes the derivation of our main result.

PROOF OF LEMMA 6. We shall use the notation \leq_{slr} with the random variables as well as the densities. Let Y_1 be an arbitrary mixture of $\theta_j Z_j$ for $j = 1, \dots, k - 1$; let Y_2 be an arbitrary mixture of $\theta_j Z_j$ for $j = k + 2, \dots, n$. Then $Y_1 \leq_{\text{lr}} \theta_k Z_k \leq_{\text{lr}} Y_2$, and these have strictly decreasing densities. Define

$$\tilde{X} \equiv \theta_n X_n + \theta_{k+1} X_{k+1}, \quad W_1 \equiv \tilde{X} + \theta_k Z_k + Y_1, \quad W_2 \equiv \tilde{X} + \theta_{k+1} Z_{k+1} + Y_2.$$

We have $W_1 \leq_{\text{lr}} W_2$, and W_1, W_2 are unimodal (since $\alpha \geq 1$); W_1 is in fact log-concave by Lemma 9. If we can show that arbitrary mixtures of W_1 and W_2 are unimodal, then so are those of $f_j, j = 1, \dots, k - 1, k + 2, \dots, n$, by adding $\sum_{i \neq k+1, n} \theta_i X_i$, which is log-concave.

Lemma 7 yields $\theta_n X_n + Y_2 \leq_{\text{slr}} \theta_n (X_n + Z_n)$. Convolving both sides with $\theta_{k+1}(X_{k+1} + Z_{k+1})$, which is PF3 [Karlin (1968)], we obtain

$$W_2 \leq_{\text{slr}} \theta_{k+1}(X_{k+1} + Z_{k+1}) + \theta_n(X_n + Z_n).$$

Lemma 7 also yields

$$(12) \quad \theta_n(X_n + Z_n) \leq_{\text{slr}} \theta_n(X_n + Z_n) + \theta_k Z_k \leq_{\text{slr}} \theta_n(X_n + Z_n) + \theta_k Z_k + Y_1,$$

where the second \leq_{slr} is obtained by convolving $\theta_n(X_n + Z_n) \leq_{\text{slr}} \theta_n(X_n + Z_n) + Y_1$ with $\theta_k Z_k$. Convolving (12) with $\theta_{k+1}(X_{k+1} + Z_{k+1})$ yields

$$W_2 \leq_{\text{slr}} W_3 \equiv \theta_{k+1}(X_{k+1} + Z_{k+1}) + \theta_n(X_n + Z_n) + \theta_k Z_k + Y_1.$$

By Proposition 5, we only need to show that arbitrary mixtures of W_1 and W_3 are unimodal. But this is a consequence of Lemma 8, which shows that arbitrary mixtures of \tilde{X} and $\theta_{k+1}(X_{k+1} + Z_{k+1}) + \theta_n(X_n + Z_n)$ are unimodal, and Lemma 9, which shows that $\theta_k Z_k + Y_1$ is log-concave.

Strict unimodality, in the sense of a strictly negative second derivative at the mode, and no saddle points, can be established by a careful examination of the above steps. For example, in Proposition 5, the claim still holds if we use strict unimodality in both the condition on h_i and the conclusion on f_i . Also, in addition to being unimodal, the density of a mixture of \tilde{X} and $\theta_{k+1}(X_{k+1} + Z_{k+1}) + \theta_n(X_n + Z_n)$ is analytic on $(0, \infty)$, vanishes at $0+$, and has a bounded first derivative. One can then verify that the step of adding $\theta_k Z_k + Y_1$, which is strictly log-concave, yields a strictly unimodal density. \square

REFERENCES

- BOCK, M. E., DIACONIS, P., HUFFER, F. W. and PERLMAN, M. D. (1987). Inequalities for linear combinations of gamma random variables. *Canad. J. Statist.* **15** 387–395. [MR0939218](#)
- DIACONIS, P. and PERLMAN, M. D. (1990). Bounds for tail probabilities of weighted sums of independent gamma random variables. In *Topics in Statistical Dependence (Somerset, PA, 1987)*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **16** 147–166. IMS, Hayward, CA. [MR1193975](#)
- KARLIN, S. (1968). *Total Positivity*. Stanford Univ. Press, Stanford, CA.
- KHALEDI, B.-E. and KOCHAR, S. C. (2004). Ordering convolutions of gamma random variables. *Sankhyā* **66** 466–473. [MR2108202](#)
- KOCHAR, S. and XU, M. (2011). The tail behavior of the convolutions of gamma random variables. *J. Statist. Plann. Inference* **141** 418–428. [MR2719506](#)
- KOCHAR, S. and XU, M. (2012). Some unified results on comparing linear combinations of independent gamma random variables. *Probab. Engrg. Inform. Sci.* **26** 393–404. [MR2943335](#)
- MARSHALL, A. W., OLKIN, I. and ARNOLD, B. (2009). *Inequalities: Theory of Majorization and Its Applications*, 2nd ed. Springer, New York.
- RINOTT, Y., SCARSINI, M. and YU, Y. (2012). A Colonel Blotto gladiator game. *Math. Oper. Res.* **37** 574–590. [MR2997892](#)
- ROOSTA-KHORASANI, F. and SZÉKELY, G. J. (2015). Schur properties of convolutions of gamma random variables. *Metrika* **78** 997–1014. [MR3407592](#)
- ROOSTA-KHORASANI, F., SZÉKELY, G. J. and ASCHER, U. M. (2015). Assessing stochastic algorithms for large scale nonlinear least squares problems using extremal probabilities of linear combinations of gamma random variables. *SIAM/ASA J. Uncertain. Quantif.* **3** 61–90. [MR3300412](#)
- SHAKED, M. and SHANTHIKUMAR, J. G. (2007). *Stochastic Orders*. Springer, New York. [MR2265633](#)
- SZÉKELY, G. J. and BAKIROV, N. K. (2003). Extremal probabilities for Gaussian quadratic forms. *Probab. Theory Related Fields* **126** 184–202. [MR1990053](#)
- YU, Y. (2009). Stochastic ordering of exponential family distributions and their mixtures. *J. Appl. Probab.* **46** 244–254. [MR2508516](#)
- YU, Y. (2011). Some stochastic inequalities for weighted sums. *Bernoulli* **17** 1044–1053. [MR2817616](#)
- ZHAO, P. and BALAKRISHNAN, N. (2009). Likelihood ratio ordering of convolutions of heterogeneous exponential and geometric random variables. *Statist. Probab. Lett.* **79** 1717–1723. [MR2547942](#)

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