

Inviscid Burgers equation with random kick forcing in noncompact setting

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Abstract

We develop ergodic theory of the inviscid Burgers equation with random kick forcing in noncompact setting. The results are parallel to those in our recent work on the Burgers equation with Poissonian forcing. However, the analysis based on the study of one-sided minimizers of the relevant action is different. In contrast with previous work, finite time coalescence of the minimizers does not hold, and hyperbolicity (exponential convergence of minimizers in reverse time) is not known. In order to establish a One Force — One Solution principle on each ergodic component, we use an extremely soft method to prove a weakened hyperbolicity property and to construct Busemann functions along appropriate subsequences.

Keywords: SPDE; Burgers equation; last passage percolation; invariant distributions; Busemann functions; One Force – One Solution Principle.

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1 Introduction

The Burgers equation was introduced by Burgers as a fluid dynamics model describing the evolving velocity profile of a 1D continuum of particles in the absence of pressure. Although it did not prove to be a good model for most interesting phenomena related to turbulence, it has re-emerged in various contexts including interface growth, traffic modeling, the large-scale structure of the Universe, etc., see [5] for a survey of the mathematics, physics, and numerics concerning the Burgers turbulence.

The inviscid Burgers equation is

$$\partial_t u(t, x) + \partial_x \left(\frac{u^2(t, x)}{2} \right) = \partial_x F(t, x). \quad (1.1)$$

In fluid dynamics terms, $u(t, x)$ is the velocity of the particle located at point $x \in \mathbb{R}$ at time $t \in \mathbb{R}$, and $f(t, x) = \partial_x F(t, x)$ is the external forcing term describing the acceleration of the particle at time t at point x .

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Even if the forcing is absent or smooth, solutions of this equation tend to develop discontinuities (shocks) in finite time. These shocks correspond to collisions between particles and provide archetypal examples of shock waves.

Although the classical solutions are well-defined only locally due to the inevitable singularity formation, one can work with generalized solutions. A natural global existence-uniqueness class of solutions is formed by so called entropy solutions, or viscosity solutions. They are globally well-defined and unique for a broad class of initial velocity profiles and forcing terms. Moreover, they have a natural interpretation in terms of physics and admit a variational characterization.

We are interested in the situation where the forcing is random, specifically, in the ergodic properties responsible for the long-term statistics of solutions. It is natural to expect the existence of invariant distributions balancing the influx of the energy pumped in by the external forcing and the dissipation of the energy at the shocks. This problem was considered first in compact settings, and a complete description of invariant distributions on a circle or, in the multi-dimensional version of the problem, on a torus, was obtained in [10], [20], [14]. The existence and uniqueness of an invariant distribution for the Burgers dynamics with random boundary conditions were proved in [2].

In these cases, the existence and uniqueness of an invariant distribution on the set of velocity profiles with a given average followed from the One Force — One Solution Principle (1F1S) that asserts that for any $v \in \mathbb{R}$ and for almost every realization of the forcing in the past, there is a unique velocity profile at the present that averages to v and is compatible with the history of the forcing. For any v , the collection of those velocity profiles indexed by all times $t \in \mathbb{R}$ forms a global solution that can be understood as a random one-point pullback attractor. Moreover, the distribution of this global solution at any fixed time is then a unique invariant distribution for the Markov semigroup generated by the Burgers equation on velocity profiles averaging to v .

In turn, to establish 1F1S, it is natural to employ a variational characterization of solutions called the Lax–Oleinik variational principle. To introduce it, we first represent the velocity profile as $u(t, x) = \partial_x U(t, x)$, where the potential $U(t, x)$ is a solution of the Hamilton–Jacobi–Bellman (HJB) equation

$$\partial_t U(t, x) + \frac{(\partial_x U(t, x))^2}{2} = F(t, x). \tag{1.2}$$

The viscosity solution of the Cauchy problem for this equation with initial data $U(t_0, \cdot) = U_0(\cdot)$ can be written as

$$U(t, x) = \inf_{\gamma: [t_0, t] \rightarrow \mathbb{R}} \left\{ U_0(\gamma(t_0)) + \frac{1}{2} \int_{t_0}^t \dot{\gamma}^2(s) ds + \int_{t_0}^t F(s, \gamma(s)) ds \right\}, \tag{1.3}$$

where the infimum is taken over all absolutely continuous curves γ satisfying $\gamma(t) = x$. Then the solution u of the Burgers equation can be found either by $u(t, x) = \partial_x U(t, x)$ or by using the slope of γ^* , the path on which the minimum in (1.3) is attained: $u(t, x) = \dot{\gamma}^*(t)$. The latter is related to the fact that in the HJB equation (1.2) the Hamiltonian is quadratic in $\partial_x U$. The sum of the last two terms on the right-hand side of (1.3) is often called the (Lagrangian) action of γ . The minimizing path γ^* can be identified with the trajectory of the particle that arrives to point x at time t . For most points (t, x) the minimizer is unique. However, there are exceptional locations corresponding to shocks where uniqueness does not hold.

This variational approach allows for an efficient analysis of the long term properties of the system via studying the behavior of minimizers over long time intervals. The Burgers equation preserves mean velocity and is invariant with respect to Galilean shear

space-time transformations, so without loss of generality let us confine ourselves to zero average velocity.

To establish 1F1S, one has to prove that minimizers over increasing time intervals in the past converge to limiting trajectories, so called one-sided infinite minimizers. These are paths with infinite history such that every finite restriction of such a path minimizes Lagrangian action over paths with the same endpoints. The entire space-time is foliated by these trajectories, and one can use their slopes to construct a global solution and prove that any global solution has to agree with this field of one-sided minimizers.

For this program to go through one needs an additional property of one-sided minimizers called hyperbolicity. It means that those paths approach each other in the past sufficiently fast. The reason to consider this property is that in order to use (1.3) in the proof that the velocity profile obtained from slopes of the minimizers is, in fact, a global solution, one has to keep track not only of the velocity, but also of the velocity potential. To find the increment of the velocity potential, we can consider two minimizers approaching each other in reverse time. Although the action corresponding to each of them is infinite, one still can make sense of the difference in action between these paths since there will be a diminishingly small contribution from times in a distant past. This analysis leads to an analogy between the global solution of the HJB equation (1.2) and Busemann functions in last passage percolation theory.

For 1D periodic (or circle) setting hyperbolicity was first established in [10] and recently a simpler proof exploiting the rigidity of 1D geometry was constructed in [6].

The first attempts of extending this program to noncompact settings, i.e., evolution of velocity profiles on the entire real line with no periodicity or other compactness assumption were [15], [27], and [3]. In [3], the program was carried out including the characterization of the global solution as a one-point random attractor and a description of the domain of attraction. However, the random forcing still was mostly concentrated in a compact set, so one can describe this kind of setting as quasi-compact.

In [4], the entire program was carried out for a fully noncompact space-time stationary setting on the real line. The forcing in [4] was assumed to be concentrated in Poissonian points in space-time. The similarity of this problem to last passage or first passage percolation and to Hammersley's process was exploited to adapt the methods of [21], [17], [18], [19], [29], [8], [7] to this setting, although several technical difficulties had to be overcome.

For every value $v \in \mathbb{R}$, the paper [4] constructs a unique family one-sided infinite minimizers with asymptotic slope v . The following strengthening of the hyperbolicity property was instrumental in constructing a global solution, proving its uniqueness and attraction property: with probability 1, any two of these one-sided minimizers coalesce, i.e., they meet at one of the Poissonian points and coincide from that point on. This strengthening of the hyperbolicity property is certainly an artefact of the model where the forcing is concentrated in a discrete set of space-time points.

The goal of this paper is to go through the same program in a noncompact setting where the forcing is still space-time stationary, but it is applied at discrete times, so that it is smooth in space and delta-type in time. From the fluid dynamics perspective it means that in our model, at every time $n \in \mathbb{Z}$ the velocity profile experiences a kick, i.e., the velocity of the particle at site $x \in \mathbb{R}$ is altered by a random amount $F_\omega(n, x)$, where $F_\omega(n, x)$ is a space-time stationary process. Between these kicks the velocity profile undergoes unforced Burgers evolution, i.e., each particle travels with a constant velocity at least before it collides with other particles. For simplicity of the analysis we will adopt a concrete "shot-noise" model for spatially smooth process $F_\omega(n, \cdot)$ such that these kicks at different times are i.i.d. We note that kick forcing models have been studied for

Burgers turbulence in compact setting in [20], [6] and for 2D Navier–Stokes system in [22], [23], [24].

Although some steps in the program for the kicked forcing model are mere adaptations of the methods of the previous paper [4], we have to tackle some new obstacles here. For example, some arguments that explicitly used the discrete nature of the Poisson point ensemble have to be enhanced or replaced with new ones.

One crucial difference between the old and new settings is that we can no longer rely on coalescence of one-sided minimizers. It is clear that distinct minimizers do not coalesce (all minimizers are solutions to the second-order Euler–Lagrange difference equation, so if two minimizers coincide with each other at two consecutive times, they are identical), but it is still not known if they are asymptotic to each other, i.e., if hyperbolicity holds true. This poses a serious difficulty. However, in this paper, we are able to replace hyperbolicity by a much weaker property. We prove that for any two one-sided minimizers γ^1, γ^2 with the same asymptotic slope,

$$\liminf_{m \rightarrow -\infty} \frac{|\gamma_m^1 - \gamma_m^2|}{|m|^{-1}} = 0. \quad (1.4)$$

In other words, we prove that one-sided minimizers approach each other rather closely *along a sequence* (m') of *pairing times*. It turns out that this weak hyperbolicity is sufficient to prove existence and uniqueness of a global solution of the Burgers equation in our model, and to study its domain of attraction. Of course, some limit transitions $m \rightarrow -\infty$ are replaced by limits along the pairing sequence of times m' . In particular, our definition of Busemann function is based on partial limits.

The new argument we use to prove (1.4) (we actually prove and use a slightly stronger statement) is quite soft and can be applied to other last passage percolation type models and random Lagrangian systems. However, we work only with the Burgers equation, since for this model system we can also complete all other steps of the program.

Let us also mention important recent work on geodesics and Busemann functions for discrete lattice models in [9], [12], [26], [13].

In the next two sections we explain the setting, main results, and the layout of the rest of the paper.

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2 The setting

We will consider the dynamics under which at time $n \in \mathbb{Z}$ the velocity profile receives an instantaneous random kick and then for time 1 solves the unforced Burgers equation before it receives the next kick, and so on. The forcing potential we want to consider can be informally written as

$$\sum_{n \in \mathbb{Z}} F_\omega(n, x) \delta(t - n)$$

where $\delta(\cdot)$ is Dirac's delta function and $F_\omega(n, \cdot)$, $n \in \mathbb{Z}$ is an i.i.d. sequence of stationary processes indexed by $x \in \mathbb{R}$ with finite range of dependence defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For simplicity, we work with a more concrete model where the random potential $(F_\omega(n, x))_{(n,x) \in \mathbb{Z} \times \mathbb{R}}$ is given by a shot-noise random field

$$F_\omega(n, x) = \sum_i \xi_{n,i} \phi \left(\frac{x - \eta_{n,i}}{\kappa_{n,i}} \right), \quad n \in \mathbb{Z}, x \in \mathbb{R},$$

where for each $n \in \mathbb{Z}$, $\{\eta_{n,i}\}_{i \in \mathbb{N}}$ is a Poisson point field on $\{n\} \times \mathbb{R}$ driven by the Lebesgue measure, $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function with bounded support, and the amplitudes $(\xi_{n,i})_{i \in \mathbb{N}}$ and scaling factors $(\kappa_{n,i})_{i \in \mathbb{N}}$ are two bounded i.i.d. sequences, jointly independent of each other and the point configuration $\{\eta_{n,i}\}$. There is no canonical enumeration of Poissonian points, so let us now give a more precise and enumeration-independent definition of this model of point influences using the approach of marked Poisson processes.

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable even function such that the set $\{x \in \mathbb{R} : \phi(x) \neq 0\}$ is non-empty and contained in $(-R_x, R_x)$ for some constant $R_x > 0$. Let $\{(\tau, \eta, \xi, \kappa)\}$ be a Poisson process on $\mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with driving measure

$$\mu(dn \times dx \times dv \times du) = P_\tau(dn) \times P_\eta(dx) \times P_\xi(dv) \times P_\kappa(du), \quad (2.1)$$

where P_τ is the counting measure on \mathbb{Z} , P_η is the Lebesgue measure on \mathbb{R} , P_ξ and P_κ are Borel probability measures on \mathbb{R} concentrated, respectively, on $[-R_\xi, R_\xi]$ and on $(0, R_\kappa]$ for some positive constants R_ξ, R_κ . To simplify the reasoning, we will assume that $R_x = R_\xi = R_\kappa = 1$, although extending all our results to the arbitrary values of these constants is straightforward. Some arguments will be easier if we assume that

$$P_\xi((0, R_\xi]) > 0, \quad \text{and} \quad P_\xi([-R_\xi, 0)) > 0, \quad (2.2)$$

although it is not necessary to make this assumption.

Let us recall the definition of a Poisson point field. It is convenient to assume that the probability space Ω_0 is the space of locally finite point configurations ω on $\mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{Z} \times \mathbb{R}^3$. This space is equipped with σ -algebra \mathcal{F}_0 generated by maps $\omega \mapsto \omega(B)$, for bounded Borel sets $B \subset \mathbb{Z} \times \mathbb{R}^3$, where $\omega(B)$ denotes the number of configuration points of ω in B . Then the probability measure P_0 on \mathcal{F}_0 is defined via the following two properties: (i) for any mutually disjoint bounded Borel sets B_1, \dots, B_k , $\omega(B_1), \dots, \omega(B_k)$ are independent random variables; (ii) for a bounded Borel set B , $\omega(B)$ has Poisson distribution with parameter $\mu(B)$, where μ is defined in (2.1).

The space-time projections $\{(\tau, \eta)\}$ of Poissonian $\{(\tau, \eta, \xi, \kappa)\}$ points form a Poisson point field in $\mathbb{Z} \times \mathbb{R}$ driven by $P_\tau(dn) \times P_\eta(dx)$. We will often refer to these points as space-time footprints of the original configuration points.

Now we can define the random kick forcing potential via

$$F_\omega(n, x) = \sum_{\eta, \xi, \kappa : (n, \eta, \xi, \kappa) \in \omega} \xi \phi \left(\frac{x - \eta}{\kappa} \right), \quad (n, x) \in \mathbb{Z} \times \mathbb{R}. \quad (2.3)$$

This sum is well-defined and differentiable for all (n, x) because nonzero contributions come only from finitely many Poissonian points with spatial component η satisfying $|\eta - x| \leq 1$. We will often omit the argument ω of $F(\cdot) = F_\omega(\cdot)$ for brevity.

It is immediate to see that $F(\cdot)$ is a space-time stationary process, i.e., the finite-dimensional distributions of $(F(n, x))_{(n,x) \in \mathbb{Z} \times \mathbb{R}}$ are the same as those of $(F(n + k, x + y))_{(n,x) \in \mathbb{Z} \times \mathbb{R}}$ for any choice of $(k, y) \in \mathbb{Z} \times \mathbb{R}$. Moreover, $(F_\omega(n, \cdot))_{n \in \mathbb{Z}}$, is an i.i.d. sequence of stationary processes. Each of these stationary processes has radius of dependence bounded by 2, i.e., σ -algebras generated by $(F(n, x))_{x \leq r}$ and $(F(n, x))_{x \geq r+2}$ are independent for any choice of $r \in \mathbb{R}$, since the former depends on Poissonian points with

spatial component η satisfying $\eta \leq r + 1$, the latter depends on Poissonian points with η satisfying $\eta \geq r + 1$, and the probability to have $\eta = r + 1$ is zero. Let us also note that all moments of

$$F_\omega^*(n, x) = \max\{|F_\omega^*(n, y)| : y \in [x, x + 1]\}, \tag{2.4}$$

are finite and can be easily estimated by the corresponding moments of the Poisson distribution. Moreover,

$$\varphi(\lambda) = \ln \mathbb{E} e^{\lambda |F_\omega^*(n, x)|} \in (0, \infty), \quad \lambda \in \mathbb{R}. \tag{2.5}$$

It will be convenient in this paper to work on a modified probability space

$$\Omega = \left\{ \omega \in \Omega_0 : \lim_{|x| \rightarrow \infty} \frac{F_\omega(n, x)}{|x|} = 0, \quad n \in \mathbb{Z} \right\} \in \mathcal{F}_0$$

instead of Ω_0 . The reason is that $P_0(\Omega) = 1$ due to the stationarity and moment assumptions on F_ω , and, as we will see later, on this set the Burgers dynamical system possesses some nice properties. Moreover, Ω is invariant under (i) Galilean space-time shear transformations $L^{a,v}$ moving each Poissonian point $(\tau, \eta, \xi, \kappa)$ to $(\tau, \eta + a + v\tau, \xi, \kappa)$, and (ii) space-time translations $\theta^{n,x}$ moving each Poissonian point $(\tau, \eta, \xi, \kappa)$ to $(\tau - n, \eta - x, \xi, \kappa)$. Here $n \in \mathbb{Z}, x \in \mathbb{R}, v \in \mathbb{R}$.

We denote the restrictions of \mathcal{F}_0 and P_0 onto Ω by \mathcal{F} and P . From now on for convenience we remove from Ω_0 the zero measure complement to Ω and work with the probability space (Ω, \mathcal{F}, P) . Under this transition, all the distributional properties are preserved.

Let us now define the Burgers dynamics associated with this random forcing potential and the corresponding dynamics for HJB equation. The latter may be called Hamilton – Jacobi – Bellman – Hopf – Lax – Oleinik (HJBHLO) dynamics.

The space of velocity potentials that we will consider will be \mathbb{H} , the space of all locally Lipschitz functions $W : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$\liminf_{x \rightarrow \pm\infty} \frac{W(x)}{|x|} > -\infty.$$

For a function $W \in \mathbb{H}$, a forcing realization F_ω determined by $\omega \in \Omega$, times $n_0, n_1 \in \mathbb{Z}$ satisfying $n_0 < n_1$, any a sequence of points $\gamma = (\gamma_{n_0}, \dots, \gamma_{n_1})$, we define the following action:

$$A_\omega^{n_0 n_1}(W, \gamma) = W(\gamma_{n_0}) + S^{n_0, n_1}(\gamma) + F_\omega^{n_0, n_1}(\gamma), \tag{2.6}$$

where

$$S^{n_0, n_1}(\gamma) = \frac{1}{2} \sum_{k=n_0}^{n_1-1} (\gamma_{k+1} - \gamma_k)^2$$

is the kinetic action associated to γ , and the potential action

$$F_\omega^{n_0, n_1}(\gamma) = \sum_{k=n_0}^{n_1-1} F_\omega(k, \gamma_k) \tag{2.7}$$

is responsible for the interaction with the external forcing potential F_ω . Sometimes, we will identify the sequence of points γ with the planar broken line consisting of segments connecting (n_0, γ_{n_0}) to (n_1, γ_{n_1}) , (n_1, γ_{n_1}) to (n_2, γ_{n_2}) , etc. Very often we will omit the argument ω of $A_\omega^{n_0 n_1}(W, \gamma)$, $F_\omega^{n_0, n_1}(\gamma)$ and other random variables.

Let us now consider the following minimization problem:

$$A_\omega^{n_0, n_1}(W, \gamma) \rightarrow \inf, \tag{2.8}$$

$$\gamma_{n_1} = x,$$

where the infimum is taken over all sequences $\gamma : \{n_0, \dots, n_1\} \rightarrow \mathbb{R}$. Let us denote the infimum value in (2.8) by $\Phi_\omega^{n_0, n_1} W(x)$.

If $n_1 = n_0 + 1$, then one can use the optimization problem (2.8) to find the viscosity solution of the unforced Burgers equation or the associated HJB equation with initial condition $W(\cdot) + F_\omega(n_0, \cdot)$, see, e.g., [11, Section 3.4]. Namely, $\Phi_\omega^{n_0, n_0+1} W(x)$ is well-defined and equal to the solution of the HJB at point $(n_0 + 1, x)$; and $\partial_x \Phi_\omega^{n_0, n_0+1} W(x)$ is the solution of the Burgers equation at point x . The latter can be also represented as $x - \gamma_{n_0}$, where $(\gamma_{n_0}, \gamma_{n_0+1}) = (\gamma_{n_0}, x)$ is the optimal two-point path solving (2.8).

Note that in [11], the assumption of global Lipschitzness of the initial data is used. Although that assumption does not hold for $W(\cdot) + F_\omega(n_0, \cdot)$, it can be easily replaced by local Lipschitzness and at most linear growth to $-\infty$. These properties hold for all $\omega \in \Omega$.

The general optimization problem (2.8) with an arbitrary gap between n_0 and n_1 corresponds to iterative applications of this one-step procedure, so the result matches the informal description of the dynamics given in the introduction: at time n_0 we start with a velocity profile W , alter it by $F(n_0, \cdot)$, then we solve the unforced Burgers–HJB equation for time 1, and then, at time $n_0 + 1$ we alter the solution by $F(n_0 + 1, \cdot)$ and then solve unforced Burgers equation for time 1, etc.

The family of random nonlinear operators $(\Phi_\omega^{n_0, n_1})_{n_0 \leq n_1}$ is the main object in this paper. Our main goal is to understand the asymptotics of $\Phi_\omega^{n_0, n_1}$ as $n_1 - n_0 \rightarrow \infty$. Let us now formulate the most important preliminary facts about the operators $\Phi_\omega^{n_0, n_1}$. We do not give proofs since they are just minor modifications of the textbook material in [11, Section 3.4].

Lemma 2.1. *For any $\omega \in \Omega$, $W \in \mathbb{H}$ and $n_0, n_1 \in \mathbb{Z}$ with $n_0 < n_1$, the following holds true:*

1. *For any $x \in \mathbb{R}$ there is a path $\gamma(x)$ that realizes the minimum in (2.8). In particular, the operator $\Phi_\omega^{n_0, n_1}$ is well-defined on \mathbb{H} .*
2. *The function $x \mapsto \Phi_\omega^{n_0, n_1} W(x)$ is locally Lipschitz.*
3. *The set O of points x such that the variational problem (2.8) admits a unique solution is open and dense in \mathbb{R} . The complement to O is at most countable.*
4. *If $x_0 \in O$, then $\Phi_\omega^{n_0, n_1} W(x)$ is differentiable at x_0 with respect to x and*

$$\partial_x \Phi_\omega^{n_0, n_1} W(x) \Big|_{x=x_0} = \gamma_{n_1}(x_0) - \gamma_{n_1-1}(x_0).$$

If $\gamma : [n_0, n_1] \rightarrow \mathbb{R}$ is the continuous curve linearly interpolating between points of the sequence $\gamma(x_0)$, then the right-hand side of this identity can be interpreted as $\dot{\gamma}(n_1)$, where the dot denotes (left) time derivative.

5. *If $x_0 \in \mathbb{R} \setminus O$, then the right and left derivatives of $\Phi_\omega^{n_0, n_1} W(x)$ w.r.t. x are well-defined at x_0 . They are equal to the slope of, respectively, the leftmost and rightmost minimizers realizing $\Phi_\omega^{n_0, n_1} W(x_0)$.*
6. *Let $\gamma(x)$ denote a minimizer with endpoint (n_1, x) . Then for every pair of points $x, y \in \mathbb{R}$ satisfying $x < y$, and every $n \in \{n_0, \dots, n_1\}$, $\gamma_n(x) < \gamma_n(y)$. Moreover, $\lim_{x \rightarrow \pm\infty} \gamma_n(x) = \pm\infty$.*

Optimal sequences $\gamma(x)$ and their continuous interpolations can be viewed as particle trajectories.

The following statement is the cocycle property for the operator family $(\Phi_\omega^{n_0, n_1})$. It is a direct consequence of Bellman's principle of dynamic programming.

Lemma 2.2. *If $\omega \in \Omega_1$, then for any $W \in \mathbb{H}$, any n_0, n_1, n_2 satisfying $n_0 < n_1 < n_2$, $\Phi_\omega^{n_1, n_2} \Phi_\omega^{n_0, n_1} W$ is well-defined and equals $\Phi_\omega^{n_0, n_2} W$. For any $x \in \mathbb{R}$, if γ is an optimal path realizing $\Phi_\omega^{n_0, n_2} W(x)$, then the restrictions of γ onto $\{n_0, \dots, n_1\}$ and $\{n_1, \dots, n_2\}$ are optimal paths realizing $\Phi_\omega^{n_0, n_1} W(\gamma_{n_1})$ and $\Phi_\omega^{n_1, n_2} (\Phi_\omega^{n_0, n_1} W)(x)$.*

Introducing $\Phi_\omega^n = \Phi_\omega^{0, n}$ we can rewrite the cocycle property as

$$\Phi_\omega^{n_1 + n_2} W = \Phi_{\theta^{n_1} \omega}^{n_2} \Phi_\omega^{n_1} W, \quad n_1, n_2 \in \mathbb{N}, \quad \omega \in \Omega,$$

where $\theta^n = \theta^{n, 0}$ denotes the time-shift on point configurations by n time units.

Let us denote

$$\mathbb{H}(v_-, v_+) = \left\{ W \in \mathbb{H} : \lim_{x \rightarrow \pm\infty} \frac{W(x)}{x} = v_\pm \right\}, \quad v_-, v_+ \in \mathbb{R}.$$

The following result shows that these spaces are invariant under HJBHLO dynamics. Along with Lemma 2.2 it allows to treat the dynamics as a random dynamical system with perfect cocycle property (see, e.g., [1, Section 1.1]). We give a proof of this lemma in Section 12.

Lemma 2.3. *For any $\omega \in \Omega$, for any n_0, n_1 with $n_0 < n_1$,*

1. *If $W \in \mathbb{H}$, then $\Phi_\omega^{n_0, n_1} W \in \mathbb{H}$.*
2. *If $W \in \mathbb{H}(v_-, v_+)$ for some v_-, v_+ , then $\Phi_\omega^{n_0, n_1} W \in \mathbb{H}(v_-, v_+)$.*

In nonrandom setting the family of operators (Φ^{n_0, n_1}) constructed via a variational problem of type (2.8) is called a HJBHLO evolution semigroup, see [28, Definition 7.33], but in our setting it would be more precise to call it a HJBHLO cocycle.

Potentials are naturally defined up to an additive constant. It is thus convenient to work with $\hat{\mathbb{H}}$, the space of equivalence classes of potentials from \mathbb{H} . Also, we can introduce spaces $\hat{\mathbb{H}}(v_-, v_+)$ as classes of potentials in $\mathbb{H}(v_-, v_+)$ coinciding up to an additive constant. The cocycle Φ can be projected on $\hat{\mathbb{H}}$ in a natural way. We denote the resulting cocycle on $\hat{\mathbb{H}}$ by $\hat{\Phi}$.

Since a velocity field determines its potential uniquely up to an additive constant, we can also introduce dynamics on velocity fields. We can introduce the Burgers dynamics on the space \mathbb{H}' of functions w (actually, classes of equivalence of functions since we do not distinguish two functions coinciding almost everywhere) such that for some function $W \in \mathbb{H}$ and almost every x , $w(x) = W'(x) = \partial_x W(x)$. For all $v_-, v_+ \in \mathbb{R}$, we can also introduce $\mathbb{H}'(v_-, v_+)$, the space consisting of functions w such that the potential W defined by $W(x) = \int_0^x w(y) dy$ belongs to $\mathbb{H}(v_-, v_+)$. One can interpret this space as the space of velocity profile with well-defined one-sided averages v_- and v_+ .

We will write $w_2 = \Psi_\omega^{n_0, n_1} w_1$ if $w_1 = W_1'$, $w_2 = W_2'$, and $W_2 = \Phi_\omega^{n_0, n_1} W_1$ for some $W_1, W_2 \in \mathbb{H}$. Of course, the maps belonging to the Burgers cocycle $(\Psi_\omega^{n_0, n_1})$ map \mathbb{H}' into itself and the spaces $\mathbb{H}'(v_-, v_+)$ are also invariant. However, these maps have additional regularity that we are about to exploit.

It follows from Lemma 2.1 that for all $\omega \in \Omega$ and every $w \in \mathbb{H}'$ there is a *cadlag* version of $\Psi_\omega^{n_0, n_1} w$ (i.e., it is right-continuous and has left limits at all points) such that the function

$$M(x) = x - \Psi_\omega^{n_0, n_1} w(x), \quad x \in \mathbb{R},$$

is strictly increasing and satisfies $\lim_{x \rightarrow \pm\infty} M(x) = \pm\infty$. In particular, M has at most countably many discontinuities, and so does $\Psi_\omega^{n_0, n_1} w$. In the fluid dynamics interpretation, the particle that arrives at x at time n_1 with velocity $\Psi_\omega^{n_0, n_1} w(x)$, is located at $M(x)$ at time $n_1 - 1$. Monotonicity means that the paths of those particles do not cross before n_1 . Each discontinuity of M corresponds to a pair of particles arriving at the same point

at time n_1 with different velocities and creating a shock. Let us also note that one can recover $\Psi_\omega^{n_0, n_1} w$ from M :

$$\Psi_\omega^{n_0, n_1} w(x) = x - M(x), \quad x \in \mathbb{R}.$$

This description of solutions of the Burgers equation in terms of monotone maps allows one to study the Burgers dynamics as evolution in spaces \mathbb{G} or $\mathbb{G}(v_-, v_+)$ consisting of cadlag functions $w \in \mathbb{H}'$ (or, respectively, $w \in \mathbb{H}'(v_-, v_+)$) such that $M_w : x \mapsto x - w(x)$ is an increasing function satisfying $\lim_{x \rightarrow \pm\infty} M(x) = \pm\infty$. Of course, M_w retains all the information about w since $w(x) = x - M_w(x)$.

There are some advantages of work with monotone functions. For example, we will utilize the following mode of convergence: a sequence of \mathbb{G} -functions $(w_n)_{n \in \mathbb{N}}$ converges to $w \in \mathbb{G}$ iff $\lim_{n \rightarrow \infty} M_{w_n}(x) = M_w(x)$ for all x in $\mathcal{C}(w)$, the set of continuity points of w . This is equivalent to $\lim_{n \rightarrow \infty} w_n(x) = w(x)$ for all $x \in \mathcal{C}(w)$. It is easy to define a metric on \mathbb{G} compatible with this mode of convergence. We discuss one such metric in Section 11.

3 Main results

We say that $u(n, x) = u_\omega(n, x)$, $(n, x) \in \mathbb{Z} \times \mathbb{R}$ is a global solution for the cocycle Ψ if there is a set $\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ such that for all $\omega \in \Omega'$, all m and n with $m < n$, we have $\Psi_\omega^{m, n} u_\omega(m, \cdot) = u_\omega(n, \cdot)$. We can also introduce the global solution as a skew-invariant function: $u_\omega(x)$, $x \in \mathbb{R}$ is called skew-invariant if there is a set $\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ such that for any $n \in \mathbb{Z}$, $\theta^n \Omega' = \Omega'$, and for any $n \in \mathbb{N}$ and $\omega \in \Omega'$, $\Psi_\omega^n u_\omega = u_{\theta^n \omega}$. Here and further on $\theta^n = \theta^{n, 0}$ is the time shift on Ω (we recall that space-time shifts $\theta^{n, x}$ were introduced in the previous section).

If $u_\omega(x)$ is a skew-invariant function, then $u_\omega(n, x) = u_{\theta^n \omega}(x)$ is a global solution. One can naturally view the potentials of $u_\omega(x)$ and $u_\omega(n, x)$ as a skew-invariant function and global solution for the cocycle $\hat{\Phi}$.

To state our first result, a description of global solutions, we need more notation. For a subset A of $\mathbb{Z} \times \mathbb{R}$, we denote by \mathcal{F}_A the σ -sub-algebra of \mathcal{F} generated by $\omega|_A$, the restriction of Poisson point configuration ω to $A \times \mathbb{R} \times \mathbb{R}$, i.e., by random variables $\omega(B)$ where B runs through Borel subsets of $A \times \mathbb{R} \times \mathbb{R}$. In other words, \mathcal{F}_A is generated by Poissonian points with space-time footprint in A .

Theorem 3.1. *For every $v \in \mathbb{R}$ there is a unique (up to zero-measure modifications) skew-invariant function $u_v : \Omega \rightarrow \mathbb{H}'$ such that for almost every $\omega \in \Omega$, $u_{v, \omega} \in \mathbb{H}'(v, v)$. The potential $U_{v, \omega}$ defined by $U_{v, \omega}(x) = \int^x u_{v, \omega}(y) dy$ is a unique skew-invariant potential in $\hat{\mathbb{H}}(v, v)$. The skew-invariant functions $U_{v, \omega}$ and $u_{v, \omega}$ are measurable w.r.t. $\mathcal{F}|_{(-\mathbb{N}) \times \mathbb{R}}$, i.e., they depend only on the history of the forcing. The spatial random process $(u_{v, \omega}(x))_{x \in \mathbb{R}}$ is stationary and ergodic with respect to space shifts.*

Notice that this theorem can be interpreted as a 1F1S Principle: for any velocity value v , the solution at time 0 with mean velocity v is uniquely determined by the history of the forcing: $u_{v, \omega} \stackrel{\text{a.s.}}{=} \chi_v(\omega|_{(-\mathbb{N}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}})$ for some deterministic functional χ_v of the point configurations in the past, i.e., in $(-\mathbb{N}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ (we actually describe χ_v in the proof, it is constructed via one-sided action minimizers). Since the forcing is stationary in time, we obtain that $u_{v, \theta^n \omega}$ is a stationary process in n , and the distribution of $u_{v, \omega}$ is an invariant distribution for the corresponding Markov semi-group, concentrated on $\mathbb{H}'(v, v)$.

The next result shows that each of the global solutions constructed in Theorem 3.1 plays the role of a one-point pullback attractor. To describe the domains of attraction we need to introduce several assumptions on initial potentials $W \in \mathbb{H}$. Namely, we

will assume that there is $v \in \mathbb{R}$ such that W and v satisfy one of the following sets of conditions:

$$\begin{aligned} v &= 0, \\ \liminf_{x \rightarrow +\infty} \frac{W(x)}{x} &\geq 0, \\ \limsup_{x \rightarrow -\infty} \frac{W(x)}{x} &\leq 0, \end{aligned} \tag{3.1}$$

or

$$\begin{aligned} v &> 0, \\ \lim_{x \rightarrow -\infty} \frac{W(x)}{x} &= v, \\ \liminf_{x \rightarrow +\infty} \frac{W(x)}{x} &> -v, \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} v &< 0, \\ \lim_{x \rightarrow +\infty} \frac{W(x)}{x} &= v, \\ \limsup_{x \rightarrow -\infty} \frac{W(x)}{x} &< -v. \end{aligned} \tag{3.3}$$

Condition (3.1) means that there is no macroscopic flux of particles from infinity toward the origin for the initial velocity profile W' . In particular, any $W \in \mathbb{H}(0, 0)$ or any $W \in \mathbb{H}(v_-, v_+)$ with $v_- \leq 0$ and $v_+ \geq 0$ satisfies (3.1). It is natural to call the arising phenomenon a rarefaction fan. We will see that in this case the long-term behavior is described by the global solution u_0 with mean velocity $v = 0$.

Condition (3.2) means that the initial velocity profile W' creates the influx of particles from $-\infty$ with effective velocity $v \geq 0$, and the influence of the particles at $+\infty$ is not as strong. In particular, any $W \in \mathbb{H}(v, v_+)$ with $v \geq 0$ and $v_+ > -v$ (e.g., $v_+ = v$) satisfies (3.2). We will see that in this case the long-term behavior is described by the global solution u_v .

Condition (3.3) describes a situation symmetric to (3.2), where in the long run the system is dominated by the flux of particles from $+\infty$.

The following precise statement supplements Theorem 3.1 and describes the basins of attraction of the global solutions u_v in terms of conditions (3.1)–(3.3).

Theorem 3.2. *For every $v \in \mathbb{R}$, there is a set $\tilde{\Omega} \in \mathcal{F}$ with $P(\tilde{\Omega}) = 1$ such that if $\omega \in \tilde{\Omega}$, $W \in \mathbb{H}$, and one of conditions (3.1),(3.2),(3.3) holds, then $w = W'$ belongs to the domain of pullback attraction of u_v in the sense of space \mathbb{G} : for any $n \in \mathbb{R}$ and any $x \in \mathcal{C}(u_v(n, \cdot))$,*

$$\lim_{m \rightarrow -\infty} \Psi_\omega^{m,n} w(x) = u_{v,\omega}(n, x).$$

The last statement of the theorem implies that for every $v \in \mathbb{R}$, the invariant measure on $\mathbb{H}'(v, v)$ described after Theorem 3.1 is unique and for any initial condition $w = W' \in \mathbb{H}'$ satisfying one of conditions (3.1),(3.2), and (3.3), the distribution of the random velocity profile at time n weakly converges to the unique stationary distribution on $\mathbb{H}'(v, v)$ as $n \rightarrow \infty$, in the topology of the space \mathbb{G} . However, our approach does not produce any convergence rate estimates.

The proofs of Theorems 3.1 and 3.2 are given in Sections 9 and 10, but most of the preparatory work is carried out in Sections 4–8.

The long-term behavior of the cocycles Φ and Ψ defined through the optimization problem (2.8) depends on the asymptotic behavior of the action minimizers over long time intervals. The natural notion that plays a crucial role in this paper is the notion of backward one-sided infinite minimizers or geodesics. A path $\gamma : \{\dots, n-1, n\} \rightarrow \mathbb{R}$ with $\gamma(n) = x$ is called a backward one-sided minimizer if its restriction onto any time interval $\{m, \dots, n\}$ provides the minimum to the action $A_\omega^{m,n}(W, \cdot)$ defined in (2.6) among paths connecting $\gamma(m)$ to x .

It can be shown (see Lemma 7.1) that any backward minimizer γ has an asymptotic slope $v = \lim_{n \rightarrow -\infty} (\gamma(n)/n)$. On the other hand, for every space-time point (n, x) and every $v \in \mathbb{R}$ there is a backward minimizer with slope v and endpoint (n, x) . The following theorem describes the most important properties of backward minimizers associated with the random potential F_ω .

Theorem 3.3. *For every $v \in \mathbb{R}$ there is a set of full measure Ω' such that for all $\omega \in \Omega'$ and all $(n, x) \in \mathbb{Z} \times \mathbb{R}$ except countably many there is a unique backward minimizer with asymptotic slope v . For $\omega \in \Omega'$, any two one-sided minimizers γ^1, γ^2 with asymptotic slope v , satisfy*

$$\liminf_{m \rightarrow -\infty} \frac{|\gamma_m^1 - \gamma_m^2|}{|m|^{-1}} = 0. \tag{3.4}$$

The proof of this core statement of this paper is spread over Sections 4 through 8. Sections 4 through 7 are technical extensions of the corresponding results for the Poissonian forcing from [4], although there are some new difficulties and some simplifications. In Section 4, we apply the sub-additive ergodic theorem to derive the linear growth of action. In Section 5, we prove quantitative estimates on deviations from the linear growth. We use these results in Section 6 to analyze deviations of optimal paths from straight lines. In Section 7, we prove the existence of infinite one-sided optimal paths and their properties. In Section 8, we prove a weak hyperbolicity property that is actually slightly stronger than (3.4). This is where new ideas are introduced and the exposition differs significantly from [4], where every two minimizers are proved to coalesce in finite time. It is this weak hyperbolicity that is used in Sections 9 and 10 to construct global solutions and study their properties. In Section 11, we discuss the convergence in space G . Section 12 contains some auxiliary lemmas.

4 Optimal action asymptotics and the shape function

Let us start with a note that the definition of the action has a slight time-reversal asymmetry: in the potential action, one of the endpoints of $\{n_0, \dots, n_1\}$ is included in summation in (2.7), and the other is not, whereas sometimes it is useful or convenient to work with the time-reversed version of the potential action and define it as

$$F_\omega^{n_0, n_1}(\gamma) = \sum_{k=n_0+1}^{n_1} F_\omega(k, \gamma_k). \tag{4.1}$$

Several estimates in this and forthcoming sections hold true for both definitions of the action, and to avoid separate discussion of both cases we will introduce an additional parameter $p \in [0, 1]$ and use the following generalized definition of the potential action:

$$\begin{aligned} F_\omega^{n_0, n_1}(\gamma) &= pF_\omega(n_0, \gamma_{n_0}) + \sum_{k=n_0+1}^{n_1-1} F_\omega(k, \gamma_k) + (1-p)F_\omega(n_1, \gamma_{n_1}) \\ &= \sum_{k=n_0}^{n_1-1} (pF_\omega(k, \gamma_k) + (1-p)F_\omega(k+1, \gamma_{k+1})). \end{aligned} \tag{4.2}$$

If $p = 1$ or 0 , then we recover definition (2.7) or, respectively, (4.1).

In this section we study the asymptotic behavior of the optimal action between space-time points (n, x) and (m, y) denoted by

$$\begin{aligned} A^{n,m}(x, y) &= A_{\omega}^{n,m}(x, y) = \min_{\gamma: \gamma_n=x, \gamma_m=y} A_{\omega}^{n,m}(0, \gamma) \\ &= \min_{\gamma: \gamma_n=x, \gamma_m=y} (S^{n,m}(\gamma) + F_{\omega}^{n,m}(\gamma)). \end{aligned} \tag{4.3}$$

We will use the generalized action that involves generalized potential action $F_{\omega}^{n_0, n_1}(\gamma)$ defined in (4.2) for an arbitrary value of parameter $p \in [0, 1]$.

Although to construct stationary solutions for the Burgers equation, we will need the asymptotic behavior as $n \rightarrow -\infty$, it is more convenient and equally useful to work with positive times and studying the limit $m \rightarrow -\infty$.

We begin with some simple observations on Galilean shear transformations of the point field.

Lemma 4.1. *Let $a, v \in \mathbb{R}$ and let $L = L^{a,v}$ be a transformation of space-time defined by $L(n, x) = (n, x + a + vn)$.*

1. *Suppose that γ is a path defined on a time interval $\{n_0, \dots, n_1\}$ and let $\bar{\gamma}$ be defined by $(n, \bar{\gamma}_n) = L(n, \gamma_n)$. Then*

$$S^{n_0, n_1}(\bar{\gamma}) = S^{n_0, n_1}(\gamma) + (\gamma_{n_1} - \gamma_{n_0})v + \frac{(n_1 - n_0)v^2}{2}.$$

2. *For any $\omega \in \Omega$, let us define $L\omega$ by pointwise application of L to the space-time footprint of configuration points: $(\tau, \eta, \xi, \kappa) \in \omega$ iff $(\tau, \eta + a + v\tau, \xi, \kappa) \in L\omega$. Then for any $\omega \in \Omega$, any time interval $\{n_0, \dots, n_1\}$, and any points $x_0, x_1, \bar{x}_0, \bar{x}_1 \in \mathbb{R}$ satisfying $L(n_0, x_0) = (n_0, \bar{x}_0)$ and $L(n_1, x_1) = (n_1, \bar{x}_1)$,*

$$A_{L\omega}^{n_0, n_1}(\bar{x}_0, \bar{x}_1) = A_{\omega}^{n_0, n_1}(x_0, x_1) + (x_1 - x_0)v + \frac{(n_1 - n_0)v^2}{2}.$$

Also, minimizers realizing $A_{\omega}^{n_0, n_1}(x_0, x_1)$ are mapped onto minimizers realizing $A_{L\omega}^{n_0, n_1}(\bar{x}_0, \bar{x}_1)$ under L .

3. *The measure \mathbb{P} is invariant under L .*
4. *For any points $x_0, x_1, \bar{x}_0, \bar{x}_1$ and any time interval $\{n_0, \dots, n_1\}$,*

$$A^{n_0, n_1}(\bar{x}_0, \bar{x}_1) \stackrel{distr}{=} A^{n_0, n_1}(x_0, x_1) + (x_1 - x_0)v + \frac{(n_1 - n_0)v^2}{2},$$

where

$$v = \frac{(\bar{x}_1 - x_1) - (\bar{x}_0 - x_0)}{n_1 - n_0}.$$

Proof: The first part of the Lemma is a simple computation:

$$\begin{aligned} S^{n_0, n_1}(\bar{\gamma}) &= \frac{1}{2} \sum_{n=n_0}^{n_1-1} (\gamma_{n+1} + v(n+1) - \gamma_n - vn)^2 = \frac{1}{2} \sum_{n=n_0}^{n_1-1} (\gamma_{n+1} - \gamma_n + v)^2 \\ &= \frac{1}{2} \sum_{n=n_0}^{n_1-1} (\gamma_{n+1} - \gamma_n)^2 + v \sum_{n=n_0}^{n_1-1} (\gamma_{n+1} - \gamma_n) + \frac{1}{2} \sum_{n=n_0}^{n_1-1} v^2. \end{aligned}$$

The second part follows from the first one. To prove the third part, it is sufficient to notice that the measure μ driving the Poisson process is invariant under L . The last

part is a consequence of the previous two parts, once one finds an appropriate Galilean transformation sending (n_0, x_0) to (n_0, \bar{x}_0) and (n_1, x_1) to (n_1, \bar{x}_1) . \square

The next useful property is the sub-additivity of action along any direction: for any velocity $v \in \mathbb{R}$, and any $n, m \geq 0$, we have

$$A^{0,n+m}(0, v(n+m)) \leq A^{0,n}(0, vn) + A^{n,n+m}(vn, v(n+m)).$$

This means that we can apply Kingman's sub-additive ergodic theorem to the function $n \mapsto A^{0,n}(0, vn)$ if we can show that $-EA^{0,n}(0, vn)$ grows at most linearly in n . We claim this linear bound in the following result:

Lemma 4.2. *Let $v \in \mathbb{R}$. There exist constants $C = C(v) > 0$ and $n_0 > 0$ such that for all $n \geq n_0$*

$$E|A^{0,n}(0, vn)| \leq Cn.$$

Proof: This statement and its proof are adapted from [4]. Lemma 4.1 implies that it is enough to prove this for $v = 0$. So in this proof we work with $A^n = A^n(0, 0)$.

Let $(\gamma_0, \dots, \gamma_n)$ be a path realizing A^n . Let us denote

$$\Sigma(\gamma) = \sum_{j=0}^{n-1} k_j, \tag{4.4}$$

where $k_j = \lceil \gamma_{j+1} \rceil - \lceil \gamma_j \rceil + 1$, and set

$$E_{n,m} = \{\Sigma(\gamma) = m \text{ for some } \gamma \text{ realizing } A^n\}.$$

Lemma 4.3. *There are constants $C_1 > 0$ and $R \in \mathbb{N}$ such that if $m \geq Rn$, then*

$$P(E_{n,m}) \leq \exp(-C_1 m^2/n).$$

Proof: If a path γ realizes $E_{n,m}$, then

$$A^n \geq \frac{1}{2} \sum_{j=0}^{n-1} (k_j - 2)_+^2 - F_\omega^*([\gamma_0], \dots, [\gamma_n]), \tag{4.5}$$

where $a_+ = 0 \vee a$, and

$$F_\omega^*(i_0, \dots, i_n) = \sum_{j=0}^n F_\omega^*(j, i_j), \quad i_0, \dots, i_n \in \mathbb{Z}.$$

We recall that $F_\omega^*(j, k)$ was introduced in (2.4). Since A^n is optimal,

$$A^n \leq A_\omega^{0,n}(0, 0, \dots, 0) = pF_\omega(0, 0) + \sum_{j=1}^{n-1} F_\omega(j, 0) + (1-p)F_\omega(n, 0). \tag{4.6}$$

Since $a \mapsto (a - 2)_+^2$ is convex, we can use Jensen's inequality to see that

$$\frac{1}{2} \sum_{j=0}^{n-1} (k_j - 2)_+^2 \geq \frac{1}{2} n \left(\frac{m}{n} - 2 \right)_+^2.$$

Combining this with (4.5) and (4.6), we obtain

$$pF_\omega(0, 0) + \sum_{j=1}^{n-1} F_\omega(j, 0) + (1-p)F_\omega(n, 0) \geq \frac{1}{2} n \left(\frac{m}{n} - 2 \right)_+^2 - F_\omega^*([\gamma_0], \dots, [\gamma_n]).$$

We conclude that on $E_{n,m}$,

$$2F_{\omega,n,m}^* \geq \frac{1}{2}n \left(\frac{m}{n} - 2\right)_+^2, \tag{4.7}$$

where

$$F_{\omega,n,m}^* = \max \left\{ F_{\omega}^*(i_0, \dots, i_n) : \sum_{j=0}^{n-1} (|i_{j+1} - i_j| + 1) \leq m \right\}.$$

So we need a tail estimate for $F_{\omega,n,m}^*$. Since the distribution of $F_{\omega}^*(i_0, \dots, i_n)$ does not depend on the choice of (i_0, \dots, i_n) , we obtain that for any $r > 0$,

$$\mathbb{P}\{F_{\omega,n,m}^* > r\} \leq N_{n,m} \mathbb{P} \left\{ \sum_{j=0}^n F_{\omega}^*(0, j) > r \right\}, \tag{4.8}$$

where $N_{n,m}$ is the size of the set $\{(i_0, \dots, i_n) : \sum_{j=0}^{n-1} (|i_{j+1} - i_j| + 1) \leq m\}$, $n \leq m$. Let us estimate $N_{n,m}$ first. The number of ways to represent m as a sum of n ordered nonnegative terms is $\binom{m+n-1}{n-1} \leq 2^{m+n-1}$. Since we also may choose the sign of $i_{j+1} - i_j$, we obtain an additional factor of 2^n , so we obtain a crude estimate

$$N_{n,m} \leq e^{\rho m}, \quad m \geq n, \tag{4.9}$$

for some $\rho > 0$. We also have,

$$\mathbb{P} \left\{ \sum_{j=0}^n F_{\omega}^*(0, j) > r \right\} \leq e^{-\lambda r} \mathbb{E} e^{\lambda \sum_{j=0}^n F_{\omega}^*(0, j)} = e^{-\lambda r + \varphi(\lambda)(n+1)}, \quad r, \lambda > 0.$$

and, combining this with (4.8) and (4.9), we obtain

$$\mathbb{P}\{F_{\omega,n,m}^* > my\} \leq e^{\rho m - \lambda my + \varphi(\lambda)(n+1)} \leq e^{m(\rho - \lambda y + 2\varphi(\lambda))}, \quad m \geq n.$$

So choosing first any $\lambda > 0$, then any $y_0 > 0$ such that

$$\rho - \lambda y_0 + 2\varphi(\lambda) < -\lambda y_0/2,$$

we obtain

$$\mathbb{P}\{F_{\omega,n,m}^* > my\} \leq e^{-Kmy}, \quad y \geq y_0, \quad m \geq n, \tag{4.10}$$

where $K = \lambda/2$. If $R > 2$ and $m \geq Rn$, then using (4.7), denoting

$$y = \frac{1}{4} \frac{n}{m} \left(\frac{m}{n} - 2\right)^2 = \frac{1}{4} \frac{(m - 2n)^2}{nm} \geq \frac{m(1 - 2R^{-1})^2}{4n},$$

noticing that the right-hand side is bounded below by $R(1 - 2R^{-1})^2/4$ which exceeds y_0 for sufficiently large R , we obtain from (4.10):

$$\mathbb{P}(E_{m,n}) \leq e^{-Kmy} \leq e^{-Km \frac{m(1-2R^{-1})^2}{4n}},$$

and the lemma follows with $C_1 = K(1 - 2R^{-1})^2/4$. □

Lemma 4.4. For any $k \geq 1$, there is $c_k > 0$ such that for all $n, m \in \mathbb{N}$ with $m \geq n$,

$$\mathbb{E} F_{\omega,n,m}^{*k} \leq c_k m^k.$$

Proof: Clearly,

$$EF_{\omega,n,m}^{*k} = EF_{\omega,n,m}^{*k} \mathbf{1}_{\{F_{\omega,n,m}^* \leq y_0 m\}} + EF_{\omega,n,m}^{*k} \mathbf{1}_{\{F_{\omega,n,m}^* > y_0 m\}}.$$

We can bound the first term simply by $(y_0 m)^k$. For the second term we can use (4.10):

$$\begin{aligned} EF_{\omega,n,m}^{*k} \mathbf{1}_{\{F_{\omega,n,m}^* > y_0 m\}} &\leq \sum_{i=0}^{\infty} ((y_0 + i + 1)m)^k \mathbb{P}\{F_{\omega,n,m}^* \in ((y_0 + i)m, (y_0 + i + 1)m]\} \\ &\leq \sum_{i=0}^{\infty} (y_0 + i + 1)^k m^k \mathbb{P}\{F_{\omega,n,m}^* > (y_0 + i)m\} \\ &\leq m^k \sum_{i=0}^{\infty} (y_0 + i + 1)^k e^{-Km(y_0+i)}. \end{aligned}$$

This series is uniformly convergent in m , so the proof is completed. □

We can now prove Lemma 4.2. From (4.5) and (4.6) we know that $|A^n| \leq F_{\omega,n,m}^*$ on $E_{n,m}$. So, using Lemmas 4.3 and 4.4, and the fact that $F_{\omega,n,m}^*$ is nondecreasing in m , we obtain

$$\begin{aligned} \mathbb{E}|A^n| &= \mathbb{E} \sum_{n \leq m \leq Rn} |A^n| \mathbf{1}_{E_{n,m}} + \sum_{m > Rn} \mathbb{E}|A^n| \mathbf{1}_{E_{n,m}} \\ &\leq \mathbb{E} \sum_{n \leq m \leq Rn} F_{\omega,n,m}^* \mathbf{1}_{E_{n,m}} + \sum_{m > Rn} \mathbb{E} F_{\omega,n,m}^* \mathbf{1}_{E_{n,m}} \\ &\leq \mathbb{E} F_{\omega,n,Rn}^* + \sum_{m > Rn} \sqrt{\mathbb{E} F_{\omega,n,m}^{*2}} \sqrt{\mathbb{P}(E_{n,m})} \\ &\leq c_1 Rn + \sqrt{c_2} \sum_{m > Rn} m \exp(-C_1 m^2 / (2n)) \\ &\leq Cn, \end{aligned}$$

for C big enough. □

In fact, we can use the last calculation to obtain the following generalization of Lemma 4.2 for higher moments of A^n :

Lemma 4.5. *Let $k \in \mathbb{N}$. Then there is a constant $C(k) > 0$ such that*

$$\mathbb{E}|A^n|^k \leq C(k)n^k, \quad n \in \mathbb{N}.$$

Remark 4.6. In fact, the analysis of all the proofs in this section shows that all the results above are valid if A^n is replaced by

$$\tilde{A}^n = \min_{\gamma \in \Gamma_n} (S^{0,n}(\gamma) + F_{\omega}^{0,n}(\gamma)),$$

where Γ_n is any set satisfying two conditions: (i) all elements of Γ_n are paths $\gamma : \{0, \dots, n\} \rightarrow \mathbb{R}$ such that $\gamma_0 = 0, \gamma_n = 0$; (ii) Γ_n contains the path $(0, 0, \dots, 0)$. We will need this in the proof of Lemma 5.6.

Now we arrive to the main result of this section describing the *shape function* α for our model.

Lemma 4.7. *For each $v \in \mathbb{R}$, the number $\alpha(v) \in \mathbb{R}$ defined by*

$$\alpha(v) = \inf_n \frac{\mathbb{E} A^{0,n}(0, vn)}{n}, \tag{4.11}$$

satisfies

$$\frac{A^{0,n}(0, vn)}{n} \rightarrow \alpha(v), \quad \text{a.s. and in } L^1, \quad n \rightarrow \infty, \quad (4.12)$$

and does not depend on the choice of constant p in definition (4.2). Moreover,

$$\alpha(v) = \alpha(0) + \frac{v^2}{2}, \quad v \in \mathbb{R}. \quad (4.13)$$

Proof: The number $\alpha(v)$ is finite due to Lemma 4.2. The sub-additive ergodic theorem now implies (4.12). The independence on p follows from

$$\lim_{n \rightarrow \infty} \frac{F_\omega(n, vn)}{n} = \lim_{n \rightarrow \infty} \frac{F_\omega(0, 0)}{n} = 0, \quad \text{a.s. and in } L^1.$$

To prove (4.13) we notice that the Galilean shear map $(n, x) \mapsto (n, x + vn)$ transforms the paths connecting $(0, 0)$ to $(n, 0)$ into paths connecting $(0, 0)$ to (n, vn) . Lemma 4.1 implies that under this map the optimal action over these paths is altered by a deterministic correction $v^2n/2$, but the measure \mathbb{P} is invariant under the lift of this transformation onto Ω , so the lemma follows. \square

We know now from (4.12) that $A^{0,n}(0, vn) \sim \alpha(v)n$ as $n \rightarrow \infty$ with probability 1. However, this is not enough for our purposes since we need quantitative estimates on deviations of $A^{0,n}(0, vn)$ from $\alpha(v)n$. This is the material of the next section.

5 Concentration inequality for optimal action

The goal of this section is to prove a concentration inequality for $A^n(vn) = A^n(0, vn) = A_\omega^n(0, vn) = A_\omega^{0,n}(0, vn)$. The methods are similar to those of [4], except for some technical moments. In particular, we use the Azuma–Hoeffding inequality instead of the Kesten inequality. Throughout this section we work with the version of action defined in (4.2).

Theorem 5.1. *There are positive constants c_0, c_1, c_2, c_3, c_4 such that for any $v \in \mathbb{R}$, all $n > c_0$, and all $u \in (c_3n^{1/2} \ln^2 n, c_4n^{3/2} \ln n]$,*

$$\mathbb{P}\{|A^n(0, vn) - \alpha(v)n| > u\} \leq c_1 \exp\left\{-c_2 \frac{u}{n^{1/2} \ln n}\right\}.$$

Remark 5.2. In our setting one can also prove a similar bound for small values of n , but we will mostly need the theorem as it is stated.

Due to the invariance under shear transformations (Lemmas 4.1 and 4.7), it is sufficient to prove this theorem for $v = 0$. We will first derive a similar inequality with $\alpha(0)n$ replaced by EA^n , and then we will have to estimate the corresponding approximation error.

Lemma 5.3. *There are positive constants b_0, b_1, b_2, b_3, b_4 such that for all $n > b_0$ and all $u \in (b_3n^{1/2} \ln n, b_4n^{3/2} \ln n]$,*

$$\mathbb{P}\{|A^n - EA^n| > u\} \leq b_1 \exp\left\{-b_2 \frac{u}{n^{1/2} \ln n}\right\}.$$

We will need an estimate on probabilities of the following events:

$$B_n(u) = \left\{ \max_{0 \leq k \leq n} |\gamma_k| > u \text{ for some } \gamma \text{ realizing } A^n \right\}.$$

Lemma 5.4. *There is a constant $C_2 > 0$ such that if $n \in \mathbb{N}$ and $u \geq Rn$, then*

$$\mathbb{P}(B_n(u)) \leq C_2 \exp(-C_1 u^2/n),$$

where constants C_1 and R were introduced in Lemma 4.3.

Proof: If $\max_{0 \leq k \leq n} |\gamma_k| > u$, then $\Sigma(\gamma) \geq u$. Lemma 4.3 implies

$$\begin{aligned} \mathbb{P}(B_n(u)) &\leq \sum_{m \geq u} \exp(-C_1 m^2/n) \\ &\leq \exp(-C_1 u^2/n) \sum_{m \geq u} \exp(-C_1(m^2 - u^2)/n) \\ &\leq \exp(-C_1 u^2/n) \sum_{m \geq u} \exp(-C_1 R(m - u)) \\ &\leq C_2 \exp(-C_1 u^2/n), \end{aligned}$$

where $C_2 = \sum_{m=0}^{\infty} e^{-C_1 R m}$. □

Having Lemma 5.4 in mind, we define \tilde{A}^n to be the optimal action over all paths connecting $(0, 0)$ to $(0, n)$ and staying within $[-Rn, Rn]$.

Lemma 5.5. *Let constants R, C_1, C_2 be defined in Lemmas 4.3 and 5.4. For any $n \in \mathbb{N}$,*

$$\mathbb{P}\{A^n \neq \tilde{A}^n\} \leq C_2 \exp(-R^2 C_1 n).$$

Proof: It is sufficient to notice that

$$\mathbb{P}\{A^n \neq \tilde{A}^n\} \leq \mathbb{P}(B_n(Rn))$$

and apply Lemma 5.4. □

Lemma 5.6. *There is a constant D_1 such that for all $n \in \mathbb{N}$,*

$$0 \leq \mathbb{E}\tilde{A}^n - \mathbb{E}A^n \leq \mathbb{E}(|A^n| + |\tilde{A}^n|)\mathbf{1}_{B_n(Rn)} \leq D_1.$$

Proof: The first inequality is obvious, since $A^n \leq \tilde{A}^n$. We also have

$$\begin{aligned} \mathbb{E}\tilde{A}^n - \mathbb{E}A^n &\leq \mathbb{E}(\tilde{A}^n - A^n)\mathbf{1}_{B_n(Rn)} \\ &\leq \mathbb{E}|A^n|\mathbf{1}_{B_n(Rn)} + \mathbb{E}|\tilde{A}^n|\mathbf{1}_{B_n(Rn)}. \end{aligned} \tag{5.1}$$

Lemmas 4.5 and 5.4 give:

$$\begin{aligned} \mathbb{E}|A^n|\mathbf{1}_{B_n(Rn)} &\leq \sqrt{\mathbb{E}|A^n|^2} \sqrt{\mathbb{P}(B_n(Rn))} \\ &\leq \sqrt{C(2)n} \sqrt{C_2 \exp(-C_1 R^2 n)}. \end{aligned}$$

Remark 4.6 shows that the same estimate applies to \tilde{A}^n :

$$\mathbb{E}|\tilde{A}^n|\mathbf{1}_{B_n(Rn)} \leq \sqrt{C(2)n} \sqrt{C_2 \exp(-C_1 R^2 n)},$$

so the the right-hand side of (5.1) is uniformly bounded in n . □

As in [4], we could use Kesten’s concentration inequality to estimate the deviations of \tilde{A}^n from its mean. However, in our discrete time setting, we also can use a more basic tool, the following Azuma–Hoeffding inequality [16]:

Lemma 5.7. *Let $(\mathcal{F}_k)_{0 \leq k \leq N}$ be a filtration. Suppose $(M_k)_{0 \leq k \leq N}$ is a martingale with respect to $(\mathcal{F}_k)_{0 \leq k \leq N}$ such that for some constant $c > 0$ the increments $\Delta_k = M_k - M_{k-1}$ satisfy*

$$|\Delta_k| < c, \quad k = 1, \dots, N.$$

Then

$$\mathbb{P}\{M_N - M_0 \geq x\} \leq \exp\left(-\frac{x^2}{2Nc^2}\right).$$

To use this lemma in our framework, we must introduce an appropriate martingale. For a given natural $n \geq 2$, we will use $N = n - 1$. To define a filtration $(\mathcal{F}_k)_{0 \leq k \leq n-1}$, we introduce $Q_j = \{j\} \times [-Rn, Rn)$ and $Q_j^+ = \{j\} \times [-Rn - 1, Rn + 1)$, $j = 1, \dots, n - 1$.

Using the notation introduced before the statement of Theorem 3.1, we set $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and

$$\mathcal{F}_k = \sigma\left(\omega \Big|_{\bigcup_{j=1}^k Q_j^+}\right), \quad k = 1, \dots, n - 1.$$

We introduce a martingale $(M_k, \mathcal{F}_k)_{0 \leq k \leq n-1}$ by

$$M_k = \mathbb{E}(\tilde{A}^n | \mathcal{F}_k), \quad 0 \leq k \leq n - 1.$$

Note that $M_0 = \mathbb{E}\tilde{A}^n$ and $M_{n-1} = \tilde{A}^n$.

Let us denote by P_k the distribution of $\omega \Big|_{Q_k^+}$ on the sample space Ω_k of finite point configurations in $Q_k^+ \times \mathbb{R} \times \mathbb{R}$. For $\omega, \sigma \in \prod_{k=1}^{n-1} \Omega_k$ we write

$$[\omega, \sigma]_k = (\omega_1, \dots, \omega_k, \sigma_{k+1}, \dots, \sigma_N) \in \prod_{k=1}^{n-1} \Omega_k.$$

Then, for $k = 1, \dots, n - 1$,

$$\begin{aligned} \Delta_k(\omega_1, \dots, \omega_k) &:= M_k - M_{k-1} \\ &= \int \tilde{A}_{[\omega, \sigma]_k}^n \prod_{j=k+1}^{n-1} dP_j(\sigma_j) - \int \tilde{A}_{[\omega, \sigma]_{k-1}}^n \prod_{j=k}^{n-1} dP_j(\sigma_j) \\ &= \int \left(\tilde{A}_{[\omega, \sigma]_k}^n - \tilde{A}_{[\omega, \sigma]_{k-1}}^n \right) \prod_{j=k}^{n-1} dP_j(\sigma_j). \end{aligned}$$

For any set $B \in \mathbb{Z} \times \mathbb{R}$, we will denote

$$F_\omega^*(B) = \sup\{|F_\omega(n, x)| : (n, x) \in B\}. \tag{5.2}$$

Lemma 5.8. *Let $k \in \{1, \dots, n - 1\}$. Then*

$$|\tilde{A}_{[\omega, \sigma]_k}^n - \tilde{A}_{[\omega, \sigma]_{k-1}}^n| \leq F_{[\omega, \sigma]_k}^*(Q_k) + F_{[\omega, \sigma]_{k-1}}^*(Q_k).$$

Proof: Changing ω_k to σ_k we change the action of any path passing through Q_k by at most $F_{[\omega, \sigma]_k}^*(Q_k) + F_{[\omega, \sigma]_{k-1}}^*(Q_k)$, and the statement follows. \square

The next step is to introduce a truncation of configuration ω . For $j \in \{1, \dots, n - 1\}$ and $i \in \{-Rn - 1, \dots, Rn\}$, we denote $Q_{ji} = \{j\} \times [i, i + 1)$. We define $\bar{\omega}$ by erasing all configuration points of ω in each block $Q_{ji} \times \mathbb{R} \times \mathbb{R}$ with $\omega(Q_{ji} \times \mathbb{R} \times \mathbb{R}) > b \ln n$. The value $b > 0$ will be chosen later. The restrictions of $\bar{\omega}$ to blocks $Q_{ji} \times \mathbb{R} \times \mathbb{R}$ are mutually independent. Lemma 5.8 applies to truncated configurations as well. Since for any segment $\{k\} \times [a, b]$ and any point configuration ω ,

$$F_\omega^*(\{k\} \times [a, b]) \leq \omega(\{k\} \times [a - 1, b + 1] \times \mathbb{R} \times \mathbb{R}),$$

Lemma 5.8 gives:

$$|\tilde{A}_{[\bar{\omega}, \bar{\sigma}]_k}^n - \tilde{A}_{[\bar{\omega}, \bar{\sigma}]_{k-1}}^n| \leq 6b \ln n, \quad k = 1, \dots, n - 1,$$

where $\bar{\sigma}$ is the truncation of σ . Therefore,

$$|\Delta_k(\bar{\omega}_1, \dots, \bar{\omega}_k)| \leq \int 6b \ln n \prod_{j=k}^N dP_j(\sigma_j) \leq 6b \ln n.$$

Now Lemma 5.7 directly implies the following estimate:

Lemma 5.9. For all $x > 0$,

$$P \left\{ |\tilde{A}^n(\bar{\omega}) - E\tilde{A}_n(\bar{\omega})| > x \right\} \leq 2 \exp \left(-\frac{x^2}{72(n-1)b^2 \ln^2 n} \right).$$

Let us now estimate the discrepancy between $\tilde{A}^n(\bar{\omega})$ and $\tilde{A}^n(\omega)$:

Lemma 5.10. If $b > 2$, then there is n_0 such that for any $x > 0$ and any $n > n_0$,

$$P\{|\tilde{A}^n(\bar{\omega}) - \tilde{A}^n(\omega)| > x\} \leq 2e^{-x}.$$

Proof: Let us define $\xi_{ji} = \omega(Q_{ji} \times \mathbb{R} \times \mathbb{R})$. Then

$$|\tilde{A}^n(\bar{\omega}) - \tilde{A}^n(\omega)| \leq \sum_{j,i} \xi_{ji} \mathbf{1}_{\{\xi_{ji} > b \ln n\}}.$$

By Markov's inequality and mutual independence of ξ_{ji} ,

$$P \left\{ \sum_{j,i} \xi_{ji} \mathbf{1}_{\{\xi_{ji} > b \ln n\}} > x \right\} \leq e^{-x} [Ee^{\xi \mathbf{1}_{\{\xi > b \ln n\}}}]^{2R(n+1)(n-1)},$$

where ξ is a r.v. with the same distribution as any of ξ_{ji} . The lemma will follow from

$$\lim_{n \rightarrow \infty} [Ee^{\xi \mathbf{1}_{\{\xi > b \ln n\}}}]^{2R(n+1)(n-1)} = 1,$$

which is implied by

$$Ee^{\xi \mathbf{1}_{\{\xi > b \ln n\}}} \leq 1 + \frac{Ee^{2\xi}}{e^{b \ln n}} \leq 1 + \frac{Ee^{2\xi}}{n^b},$$

and $b > 2$. □

We also need an estimate on the discrepancy between $E\tilde{A}^n(\bar{\omega})$ and $E\tilde{A}^n(\omega)$. It is a direct consequence of Lemma 5.10:

Lemma 5.11. There is a constant D_2 such that for all $n \in \mathbb{N}$,

$$|E\tilde{A}^n(\bar{\omega}) - E\tilde{A}^n(\omega)| < D_2.$$

Proof of Lemma 5.3: Lemmas 5.6 and 5.11 imply that for $u > D_1 + D_2$

$$\begin{aligned} P\{|A^n(\omega) - EA^n(\omega)| > u\} &\leq P\{|A^n(\omega) - \tilde{A}^n(\omega)| > (u - D_1 - D_2)/3\} \\ &\quad + P\{|\tilde{A}^n(\omega) - \tilde{A}^n(\bar{\omega})| > (u - D_1 - D_2)/3\} \\ &\quad + P\{|\tilde{A}^n(\bar{\omega}) - E\tilde{A}^n(\bar{\omega})| > (u - D_1 - D_2)/3\}. \end{aligned}$$

The lemma follows from the estimates of the three terms provided by Lemmas 5.5, 5.9, and 5.10. □

The following lemma gives an estimate on how EA^n changes under argument doubling. We will use this lemma estimate on $EA^n - \alpha(0)n$ and bridge the gap between Lemma 5.3 and Theorem 5.1.

Lemma 5.12. There is a number $b_0 > 0$ such that for any $n > n_0$,

$$0 \leq 2EA^n - EA^{2n} \leq b_0 n^{1/2} \ln^2 n.$$

Proof: The first inequality follows from $A^{0,2n}(0,0) \leq A^{0,n}(0,0) + A^{n,2n}(0,0)$. Let us prove the second one.

Let γ be the (rightmost, for definiteness) minimizer from $(0, 0)$ to $(2n, 0)$. Then

$$A^{2n} \geq \min_{|x| \leq 2Rn} A^{0,n}(0, x) + \min_{|x| \leq 2Rn} A^{n,2n}(x, 0) + A^{2n} \mathbf{1}_{\{\max_{0 \leq k \leq 2n} |\gamma_k| > 2Rn\}}.$$

Therefore, by Lemma 5.6,

$$\mathbb{E} A^{2n} \geq \mathbb{E} \min_{|x| \leq 2Rn} A^n(0, x) + \mathbb{E} \min_{|x| \leq 2Rn} A^{n,2n}(x, 0) - D_1. \tag{5.3}$$

We will estimate the first term of the right-hand side. To that end we define $I_n = \{-2Rn, \dots, 2Rn - 2, 2Rn - 1\}$ and $\bar{I}_n = I_n \cup \{2Rn\}$. Let now γ be the minimizer from $(0, 0)$ to (x, n) , with $x \in [k, k + 1]$ for some $k \in I_n$. Let us introduce γ^+ and γ^- satisfying $\gamma_j^+ = \gamma_j^- = \gamma_j$ for all $j < n$ and $\gamma_n^+ = k + 1$, $\gamma_n^- = k$. Comparing γ to γ_+ if $\gamma_{n-1} \geq k + 1/2$ and to γ_- if $\gamma_{n-1} < k + 1/2$, we see that

$$S^{0,n}(\gamma_{\pm}) \leq S^{0,n-1}(\gamma) + \frac{(1/2)^2}{2} = S^{0,n-1}(\gamma) + \frac{1}{8},$$

so

$$A^n(x) \geq \min\{A^n(k), A^n(k + 1)\} - 2F^*(n, k) - \frac{1}{8}.$$

But

$$\mathbb{E} \max_{k \in \bar{I}_n} F_{\omega}^*(n, k) \leq 3\mathbb{E} \max\{\omega(\{n\} \times [k, k + 1]) : k = -2Rn - 1, \dots, k = 2Rn + 1\},$$

and so there is a constant $c > 0$ such that

$$\mathbb{E} \min_{|x| \leq 2Rn} A^n(0, x) \geq \mathbb{E} \min_{k \in \bar{I}_n} A^n(0, k) - c(\ln n + 1).$$

Lemma 4.1 implies $\min_x \mathbb{E} A^n(x) = \mathbb{E} A^n(0)$. Therefore, denoting

$$X_n = \max_{k \in \bar{I}_n} \{(\mathbb{E} A^n(k) - A^n(k))_+\},$$

we obtain

$$\begin{aligned} \mathbb{E} \min_{|x| \leq 2Rn} A^n(0, x) &\geq \min_{k \in \bar{I}_n} \mathbb{E} A^n(k) - \mathbb{E} X_n - c(\ln n + 1) \\ &\geq \mathbb{E} A^n - \mathbb{E} X_n - c(\ln n + 1), \end{aligned} \tag{5.4}$$

Similarly, we obtain for the second term in (5.3)

$$\mathbb{E} \min_{|x| \leq 2Rn} A^{n,2n}(x, 0) \geq \mathbb{E} A^{n,2n}(0, 0) - \mathbb{E} Y_n - c(\ln n + 1), \tag{5.5}$$

where

$$Y_n = \max_{k \in \bar{I}_n} \{(\mathbb{E} A^{n,2n}(k, 0) - A^{n,2n}(k, 0))_+\}.$$

For a constant r to be determined later, we introduce the event

$$E = \{X_n + Y_n \leq r(\ln^2 n)\sqrt{n}\}.$$

Then

$$X_n + Y_n \leq r(\ln^2 n)\sqrt{n} \mathbf{1}_E + (X_n + Y_n) \mathbf{1}_{E^c}.$$

Therefore,

$$\mathbb{E}(X_n + Y_n) \leq r(\ln^2 n)\sqrt{n} + \sqrt{\mathbb{E} X_n^2 \mathbb{P}(E^c)} + \sqrt{\mathbb{E} Y_n^2 \mathbb{P}(E^c)}. \tag{5.6}$$

Let us estimate the second term in (5.6). According to Lemma 4.1, the random variables $A^n(k) - EA^n(k)$, $k \in I_n$ have the same distribution, so replacing the maximum in the definition of X_n^2 with summation we obtain

$$EX_n^2 \leq (4Rn + 1)E(A^n - EA^n)_+^2 \leq (4Rn + 1)E(A^n)^2 \leq Cn^3, \tag{5.7}$$

for some $C > 0$ and all $n \in \mathbb{N}$, where we used Lemma 4.5 in the last inequality.

Also, Lemma 5.3 shows that

$$\begin{aligned} P(E^c) &\leq \sum_{k \in I_n} P \left\{ A^n(k) - EA^n(k) > \frac{r}{2}(\ln^2 n)\sqrt{n} \right\} \\ &\quad + \sum_{k \in I_n} P \left\{ A^{n,2n}(k, 0) - EA^{n,2n}(k, 0) > \frac{r}{2}(\ln^2 n)\sqrt{n} \right\} \\ &\leq 2(4Rn + 1)b_1 \exp \left(-b_2 \frac{r \ln n}{2} \right). \end{aligned} \tag{5.8}$$

The same estimates apply to the third term in (5.6), and we can now finish the proof by choosing r to be large enough and combining estimates (5.3)–(5.8). \square

With this lemma at hand we can now use a discrete version of Lemma 4.2 from [19]. Its proof literally repeats the proof of the original lemma where the argument of a and g was assumed to be continuous.

Lemma 5.13. *Suppose the functions $a : \mathbb{N} \rightarrow \mathbb{R}$ and $g : \mathbb{N} \rightarrow \mathbb{R}_+$ satisfy the following conditions: $a(n)/n \rightarrow \nu \in \mathbb{R}$ and $g(n)/n \rightarrow 0$ as $n \rightarrow \infty$, $a(2n) \geq 2a(n) - g(n)$ and $\psi \equiv \limsup_{n \rightarrow \infty} g(2n)/g(n) < 2$. Then, for any $c > 1/(2 - \psi)$, and for all large n ,*

$$a(n) \leq \nu n + cg(n).$$

Taking $a(n) = EA^n$, $\nu = \alpha(0)$, $g(n) = b_0 n^{1/2} \ln^2 n$, $\psi = \sqrt{2}$, $c = 2$, we conclude that for $b'_0 = 2b_0$ and large n ,

$$0 \leq EA^n - \alpha(0)n \leq b'_0 n^{1/2} \ln^2 n,$$

and Theorem 5.1 follows from this estimate, Lemma 5.3, and the shear invariance established in Lemma 4.1. \square

6 Straightness estimates

As in [4],[8], and [7] we keep following the ideas from [19] and [29], adapting the program to our specific situation. The step that we make in this section is to estimate deviations of minimizers from straight lines.

We will need a curvature estimate for the shape function constructed in Section 4. Recalling that the shape function $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ was introduced in Lemma 4.7, we define $\alpha_0 = \alpha(0)$ and extend α to a function on $\mathbb{N} \times \mathbb{R}$:

$$\alpha(n, x) = n\alpha\left(\frac{x}{n}\right) = n\left(\alpha_0 + \frac{1}{2}\left(\frac{x}{n}\right)^2\right) = \alpha_0 n + \frac{x^2}{2n}, \quad (n, x) \in \mathbb{N} \times \mathbb{R}.$$

We remark that, in contrast with the situation in [4], we do not know the sign of α_0 .

Lemma 4.7 implies that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\lim_{m \rightarrow \infty} \frac{A^{0,mn}(0, mx)}{m} \stackrel{\text{a.s.}}{=} \alpha(n, x).$$

We need a convexity estimate of this function α . For $(n, x) \in \mathbb{N} \times \mathbb{R}$, $L > 0$, we define

$$\mathcal{C}(n, x, L) := \left\{ (m, y) \in \mathbb{Z} \times \mathbb{R} : m \in \{n + 1, \dots, 2n\} \text{ and } \left| y - \frac{m}{n} x \right| \leq L \right\}.$$

So $\mathcal{C}(n, x, L)$ is a parallelogram of width $2L$ with one pair of sides parallel to the x -coordinate axis and the other one parallel to $[(n, x), (2n, 2x)]$ (for any two points p, q on the plane $\mathbb{R} \times \mathbb{R} \supset \mathbb{Z} \times \mathbb{R}$, $[p, q]$ denotes the straight line segment connecting these two points):

$$\partial_S^\pm \mathcal{C}(n, x, L) := \left\{ (m, y) \in \mathbb{Z} \times \mathbb{R} : m \in \{n + 1, \dots, 2n\} \text{ and } y - \frac{m}{n}x = \pm L \right\}.$$

The union of $\partial_S^+ \mathcal{C}(n, x, L)$ and $\partial_S^- \mathcal{C}(n, x, L)$ is

$$\partial_S \mathcal{C}(n, x, L) := \left\{ (m, y) \in \mathbb{Z} \times \mathbb{R} : m \in \{n + 1, \dots, 2n\} \text{ and } \left| y - \frac{m}{n}x \right| = L \right\}.$$

The following lemma will play the role of Lemma 2.1 in [29].

Lemma 6.1. *For all $(n, x), (m, y) \in \mathbb{N} \times \mathbb{R}$, such that $m > n$, we have*

$$\alpha(m - n, y - x) + \alpha(n, x) = \alpha(m, y) + \frac{n}{2m(m - n)} \left(y - \frac{m}{n}x \right)^2. \tag{6.1}$$

Proof:

$$\begin{aligned} \alpha(m - n, y - x) + \alpha(n, x) &= \alpha_0(m - n) + \frac{(y - x)^2}{2(m - n)} + \alpha_0 n + \frac{x^2}{2n} \\ &= \alpha_0 m + \frac{(y - x)^2}{2(m - n)} + \frac{x^2}{2n} \\ &= \alpha(m, y) - \frac{y^2}{2m} + \frac{(y - x)^2}{2(m - n)} + \frac{x^2}{2n}. \end{aligned}$$

Identity (6.1) now follows from a straightforward comparison of the algebraic expressions involved.

One may also argue that for fixed m , $\alpha(m - n, y - x) + \alpha(n, x) - \alpha(m, y)$ is a quadratic function in y . The minimum of this function equals 0, and is attained at \tilde{y} such that (m, \tilde{y}) is a multiple of (n, x) , i.e., $\tilde{y} = mx/n$. The lemma follows by computing the coefficient in front of y^2 . \square

This deterministic convexity lemma, together with the concentration bound of Section 5, will help us to show that minimizers cannot deviate from a straight line too much.

We will need one more auxiliary estimate:

Lemma 6.2. *Let $c_1, c_2 > 0$, $0 < n_1 < n_2 < n_1 + c_1$, $|x| < c_2 n_1$. Then*

$$\alpha(n_1, x) - \alpha(n_2, x) < |\alpha_0|c_1 + c_2^2 c_1.$$

Proof: A straightforward computation gives:

$$\begin{aligned} \alpha(n_1, x) - \alpha(n_2, x) &= \alpha_0 n_1 + \frac{x^2}{2n_1} - \alpha_0 n_2 + \frac{x^2}{2n_2} \\ &\leq |\alpha_0| |n_1 - n_2| + \frac{x^2}{2} \frac{n_2 - n_1}{n_1^2}, \end{aligned}$$

and the lemma follows. \square

For $(n, x), (m, y) \in \mathbb{Z} \times \mathbb{R}$ satisfying $n < m$, we denote by $\gamma^{(n,x),(m,y)}$ the rightmost point-to-point minimizer connecting (n, x) to (m, y) and by $A((n, x), (m, y)) = A^{n,m}(x, y)$ the associated optimal action. For $(n, x), (m, y) \in \mathbb{Z} \times \mathbb{R}$ with $n < m$, we define the events

$$\begin{aligned} G^+((n, x), (m, y)) &= \left\{ \exists \tilde{0} \in [0, 1], \exists \tilde{y} > y : \gamma_n^{(0,\tilde{0}), (m,\tilde{y})} \in [x, x + 1] \right\}, \\ G^-((n, x), (m, y)) &= \left\{ \exists \tilde{0} \in [0, 1], \exists \tilde{y} < y : \gamma_n^{(0,\tilde{0}), (m,\tilde{y})} \in [x, x + 1] \right\}. \end{aligned}$$

These events say that there is a minimizer starting close to 0 at time 0, ending at time m to the right (respectively, left) of y , and passing close to x at time n . To estimate the probability of this event, we first have to control the action to and from points close to (n, x) .

Lemma 6.3. *Suppose $(n, x), (m, y) \in \mathbb{Z} \times \mathbb{R}$ and $n < m$. Let $\tilde{x} \in [x, x + 1]$, $\tilde{y} \in [y, y + 1]$. Then*

$$A^{n,m}(\tilde{x}, \tilde{y}) \geq A^{n-1,m+1}(x, y) - F^*(n-1, x) - F^*(n, x) - F^*(m, y) - F^*(m+1, y) - 1.$$

If, additionally, $n + 3 \leq m$, then

$$A^{n,m}(\tilde{x}, \tilde{y}) \leq A^{n+1,m-1}(x, y) + F^*(n, x) + F^*(n+1, x) + F^*(m-1, y) + F^*(m, y) + 1.$$

Proof: We recall that we work with action defined by (4.2). The first inequality is a result of a direct comparison of the action of $\gamma^{(n-1,x),(m+1,y)}$ to that of

$$\gamma_k = \begin{cases} x, & k = n - 1, \\ \gamma_k^{(n,x),(m,y)}, & k \in \{n, \dots, m\}, \\ y, & k = m + 1. \end{cases}$$

The second inequality is a result of a direct comparison of the action of

$$\gamma_k = \begin{cases} \tilde{x}, & k = n, \\ \gamma_k^{(n+1,x),(m-1,y)}, & k \in \{n+1, \dots, m-1\}, \\ \tilde{y}, & k = m, \end{cases}$$

to the action of $\gamma^{(n,\tilde{x}),(m,\tilde{y})}$. □

Lemma 6.4. *Fix $\delta \in (0, 1/4)$ and $v > 0$. There exist constants $c_1, c_2, M > 0$ such that for all (n, x) with $n > M$ and $(m, y) \in \partial_S^\pm \mathcal{C}(n, x, n^{1-\delta})$, with $m > n$, we have*

$$P(G^\pm((n, x), (m, y))) \leq c_1 \exp\left(-c_2 n^{1/2-2\delta} / \ln n\right).$$

Proof: Let us consider only the case of $\partial_S^+ \mathcal{C}(n, x, n^{1-\delta})$ and $G^+((n, x), (m, y))$. The shear invariance implies that it is sufficient to consider $x = -1$, so $[x, x + 1] = [-1, 0]$.

On $G^+((n, x), (m, y))$, there are numbers $\tilde{0} \in [0, 1]$, $\tilde{x} \in [-1, 0]$ and $\tilde{y} > y$ such that

$$A^{0,m}(\tilde{0}, \tilde{y}) = A^{0,n}(\tilde{0}, \tilde{x}) + A^{n,m}(\tilde{x}, \tilde{y}) \tag{6.2}$$

Let $i = [y]$. Let us first consider the case where $i \geq 4Rn$. Then

$$\frac{n}{m} i > Rm. \tag{6.3}$$

By monotonicity of dependence of point-to-point minimizers on the endpoints, we have $\gamma_n^{(0,0),(m,i)} < 0$. It means that on $[0, m]$, $\gamma^{(0,0),(m,i)}$ deviates from a straight line by ni/m . Denoting the latter event by $B(n, m, i)$, we can use Lemma 5.4 to write:

$$P(B(n, m, i)) \leq C_2 \exp(-C_1(ni/m)^2/m) \leq C_2 \exp(-C_1 i^2 n^2 / m^3).$$

For $B(n, m, 4Rn+) = \bigcup_{i \geq 4Rn} B(n, m, i)$, we obtain

$$P(B(n, m, 4Rn+)) \leq C_2 \sum_{i \geq 4Rn} \exp(-C_1 i^2 n^2 / m^3).$$

Since the first term in this series is bounded by $C_2 \exp(-2C_1 R^2 n)$, and the ratio of two consecutive terms is bounded by $\exp(-C_1 R)$, we conclude that

$$P(B(n, m, 4Rn+)) \leq C_3 \exp(-2C_1 R^2 n). \tag{6.4}$$

Let us now consider the case where $[\tilde{y}] = i \in (n^{1-\delta} - 1, 4Rn)$. Lemma 6.3 implies that

$$\begin{aligned} A^{0,m}(\tilde{0}, \tilde{y}) &\leq A^{1,m-1}(0, i) + F^*(0, 0) + F^*(0, 1) + F^*(m-1, i) + F^*(m, i) + 1, \\ A^{0,n}(\tilde{0}, \tilde{x}) &\geq A^{-1,n+1}(0, 0) - F^*(-1, 0) - F^*(0, 0) - F^*(n, 0) - F^*(n+1, 0) - 1, \\ A^{n,m}(\tilde{x}, \tilde{y}) &\geq A^{n-1,m+1}(0, i) - F^*(n-1, 0) - F^*(n, 0) - F^*(m, i) - F^*(m+1, i) - 1. \end{aligned}$$

Along with (6.2), this implies

$$A^{1,m-1}(0, i) - A^{-1,n+1}(0, 0) - A^{n-1,m+1}(0, i) \geq -3 - X(n, m, i), \tag{6.5}$$

where

$$\begin{aligned} X(n, m, i) &= F^*(0, 0) + F^*(0, 1) + F^*(m-1, i) + F^*(m, i) \\ &\quad + F^*(-1, 0) + F^*(0, 0) + F^*(n, 0) + F^*(n+1, 0) \\ &\quad + F^*(n-1, 0) + F^*(n, 0) + F^*(m, i) + F^*(m+1, i). \end{aligned}$$

Let us now approximate the left-hand side of (6.5) using the extension of shape function α introduced in the beginning of this section. Lemma 6.2 implies that there is a constant L such that under the constraints we have imposed on n, m , and i ,

$$\alpha(m-2, i) < \alpha(m+2, i) + L \tag{6.6}$$

and

$$\alpha(n+2, 0) > \alpha(n, 0) - L. \tag{6.7}$$

Since $m+2 \leq 3n$ and $m+2-n \leq 2n$, Lemma 6.1 implies that

$$\alpha(m-n+2, i) + \alpha(n, 0) \geq \alpha(m+2, i) + \frac{i^2}{12n}. \tag{6.8}$$

Combining (6.6),(6.7),(6.8) with (6.5), we obtain

$$\begin{aligned} &(A^{1,m-1}(0, i) - \alpha(m-2, i)) - (A^{-1,n+1}(0, 0) - \alpha(n+2, 0)) \\ &\quad - (A^{n-1,m+1}(0, i) - \alpha(m-n+2, i)) + X(n, m, i) \\ &\geq -3 - 2L + \frac{i^2}{12n} \geq \frac{i^2}{15n} \end{aligned}$$

if $n > M$ for sufficiently large M .

Let us define events

$$\begin{aligned} E_1(n, m, i) &= \left\{ A^{1,m-1}(0, i) - \alpha(m-2, i) \geq \frac{i^2}{60n} \right\}, \\ E_2(n, m, i) &= \left\{ A^{-1,n+1}(0, 0) - \alpha(n+2, 0) \leq -\frac{i^2}{60n} \right\}, \\ E_3(n, m, i) &= \left\{ A^{n-1,m+1}(0, i) - \alpha(m-n+2, i) \leq -\frac{i^2}{60n} \right\}, \\ E_4(n, m, i) &= \left\{ X(n, m, i) \geq \frac{i^2}{60n} \right\}. \end{aligned}$$

We would like to show that for some constants c'_1, c'_2 ,

$$P(E_1(n, m, i)) + P(E_2(n, m, i)) + P(E_3(n, m, i)) + P(E_4(n, m, i)) \leq c'_1 \exp\left(-c'_2 \frac{i^2}{n^{3/2} \ln n}\right). \quad (6.9)$$

Sufficient estimates for first two terms follow from Theorem 5.1, and for the last one from (2.5). Let us estimate $P(E_3(n, m, i))$. By the shear invariance,

$$P(E_3(n, m, i)) = P\left\{A^{m-n+2} - \alpha_0 \cdot (m - n + 2) \leq -\frac{i^2}{60n}\right\}.$$

Since $m - n + 2$ may be small, Theorem 5.1 does not apply directly. However, to estimate the right-hand side, we notice that $A^{m-n+2} \geq A^m - A^{m-n+2,m}$ and, therefore,

$$P(E_3(n, m, i)) \leq P\left\{|A^m - \alpha_0 m| \geq \frac{i^2}{120n}\right\} + P\left\{|A^{n-2} - \alpha_0(n-2)| \geq \frac{i^2}{120n}\right\},$$

and the desired estimate follows from an application of Theorem 5.1 to both terms on the right-hand side. So, (6.9) follows, and we obtain

$$\sum_{n^{1-\delta}-1 < i < 4Rn} (P(E_1(n, m, i)) + P(E_2(n, m, i)) + P(E_3(n, m, i)) + P(E_4(n, m, i))) \leq c'_1 \exp\left(-c'_2 \frac{n^{2-2\delta}}{4n^{3/2} \ln n}\right) \sum_{i > n^{1-\delta}/2} \exp\left(-c'_2 \frac{i^2 - n^{2-2\delta}/4}{n^{3/2} \ln n}\right),$$

where the series factor in the right-hand side is uniformly bounded in n . This, along with (6.4) implies the theorem. \square

The above Lemma can be used to show that a minimal path starting close to the origin and passing close to (n, x) , with high probability will not exit the parallelogram $\mathcal{C}(n, x, n^{1-\delta})$ through the lateral sides.

Lemma 6.5. Fix $\delta \in (0, 1/4)$ and $v > 0$. There exist constants $c_1, c_2, M, \kappa > 0$ such that for all (n, x) with $n > M$,

$$P(H(n, x)) \leq c_1 \exp(-c_2 n^\kappa),$$

where $H(n, x) = H^+(n, x) \cup H^-(n, x)$ and

$$H^\pm(n, x) = \bigcup_{(m,y) \in \partial_S^\pm \mathcal{C}(n,x,n^{1-\delta})} G^\pm((n, x), (m, y)).$$

Proof: There are n admissible values of m , so we can apply Lemma 6.4 and choose any $\kappa \in (0, 1/2 - 2\delta)$. \square

Let us now prove δ -straightness of minimizers, as was introduced by Newman in [25]. For a path γ and $n \in \mathbb{Z}$, we define

$$\gamma^{\text{out}}(n) = \{(m, \gamma_m) : m > n\}.$$

For $v > 0$ we define

$$\text{Co}(v) = \{(n, x) \in \mathbb{N} \times \mathbb{R} : |x| \leq nv\}.$$

For $(n, x) \in \mathbb{N} \times \mathbb{R}$ and $\eta > 0$ we define:

$$\text{Co}(n, x, \eta) = \{(m, y) \in \mathbb{N} \times \mathbb{R} : |y/m - x/n| \leq \eta\}. \quad (6.10)$$

Lemma 6.6 (δ -straightness). For $\delta \in (0, 1/4)$ and $v > 0$ we have with probability one that there exists $M = M(\omega) > 0$ (depending on v and δ) and nonrandom $Q > 0$ (depending only on δ), such that for all $\tilde{0} \in [0, 1]$, for all $(m, y) \in \mathbb{N} \times \mathbb{R}^+$ and for all $(k, z) \in \gamma^{(0, \tilde{0}), (m, y)} \cap \text{Co}(v)$ with $k > M$, we have

$$\gamma^{\text{out}}(k) \subset \text{Co}(k, z, Qk^{-\delta}),$$

for $\gamma = \gamma^{(0, \tilde{0}), (m, y)}$.

This lemma states that if a minimizer starting near $(0, 0)$ passes through a remote point (k, z) , it has to stay in a narrow cone around the ray $\mathbb{N} \cdot (k, z)$.

Proof: Using Lemma 6.5 and the Borel–Cantelli Lemma, we see that for any $v' > v$, there is a (random) $M > 0$ such that if $j > M$ and $(j, i) \in \text{Co}(v')$ then $H^c(j, i)$ holds.

We conclude that any minimizer γ passing through $\{0\} \times [0, 1]$ and $\{j\} \times [i, i+1] \subset \text{Co}(v')$ with $j > M$ satisfies

$$\left| \gamma_n - \frac{ni}{j} \right| < j^{1-\delta}, \quad n \in \{j+1, \dots, 2j\}. \tag{6.11}$$

So for any such minimizer and point (k, z) satisfying the conditions of the lemma and M chosen as above, we can choose numbers $i_0 = [z], i_1, i_2, \dots$ such that for each $l, \gamma_{2^l k} \in [i_l, i_l + 1]$, and

$$\left| \gamma_n - \frac{ni_l}{2^l k} \right| < (2^l k)^{1-\delta}, \quad n \in \{2^l k + 1, 2^l k + 2, \dots, 2^{l+1} k\}.$$

In particular, for all l

$$|i_{l+1} - 2i_l| < (2^l k)^{1-\delta} + 1 \leq 2(2^l k)^{1-\delta}.$$

Therefore, for all l ,

$$|i_l - 2^l i_0| \leq \sum_{j=0}^{l-1} 2^{l-j-1} |i_{j+1} - 2i_j| \leq \sum_{j=0}^{l-1} 2^{l-j-1} 2(2^j k)^{1-\delta} \leq 2^l k^{1-\delta} \sum_{i=0}^{l-1} 2^{-\delta j},$$

so

$$\left| \frac{i_l}{2^l k} - \frac{i_0}{k} \right| \leq ck^{-\delta} \tag{6.12}$$

for some constant c .

For $n \in [2^l k, 2^{l+1} k], l \geq 1$, (6.11) implies

$$\left| \gamma_n - \frac{ni_l}{2^l k} \right| < (2^l k)^{1-\delta} \leq n^{1-\delta} k^{1-\delta}. \tag{6.13}$$

Now we obtain the lemma by combining (6.12) and (6.13). □

As a side product of the proof of Lemma 6.6 we obtain the following statement:

Lemma 6.7. For $\delta \in (0, 1/4)$ and $v > 0$ there are nonrandom numbers $M, Q, C_1, C_2, \kappa > 0$ such that if

$$G_n = \left\{ \exists \tilde{0} \in [0, 1], (m, y) \in \mathbb{Z} \times \mathbb{R}, (k, z) \in \gamma^{(0, \tilde{0}), (m, y)} \cap \text{Co}(v) : \right. \\ \left. k > n, \gamma^{\text{out}}(k) \not\subset \text{Co}(k, z, Qk^{-\delta}) \right\},$$

then for $n \geq M$,

$$P(G_n) \leq C_1 e^{-C_2 n^\kappa}.$$

Proof: Arguing as in the proof of Lemma 6.6, we obtain that if G_n holds, then at least one event $H(j, i)$ is violated for $j \geq n$, and the lemma follows now from Lemma 6.5. □

7 Existence and uniqueness of semi-infinite minimizers

7.1 Existence

With δ -straightness at hand, we can prove some important properties of minimizing paths. A semi-infinite minimizer starting at $(n, x) \in \mathbb{Z} \times \mathbb{R}$ is a path $\gamma : \{n, n + 1, \dots\} \rightarrow \mathbb{R}$ such that $\gamma_n = x$ and the restriction of γ to any finite time interval is a point-to-point minimizer. We call (n, x) the endpoint of γ .

Lemma 7.1. *With probability one, all semi-infinite minimizers have an asymptotic slope (velocity, direction): for every minimizer γ , there is $v \in \mathbb{R} \cup \{\pm\infty\}$ depending on γ such that*

$$\lim_{n \rightarrow \infty} \frac{\gamma_n}{n} = v.$$

Proof: Let us fix a sequence $v_n \rightarrow \infty$. Using the translation invariance of the forcing potential F , with probability one, for any $(j, i) \in \mathbb{Z}^2$ we can choose a corresponding sequence of constants $M_n(j, i) > 0$ such that the statement in Lemma 6.6 holds for the entire sequence, for paths starting in $\{j\} \times [i, i + 1]$.

Let us take some one-sided minimizer γ . If $\gamma_k/k \rightarrow +\infty$ or $-\infty$, then the desired statement is automatically true. In the opposite case we have

$$\liminf_{k \rightarrow \infty} \frac{|\gamma_k|}{k} < \infty.$$

This implies that there exist $n \geq 1$ and a sequence $k_m \rightarrow \infty$ such that $|\gamma_{k_m}|/k_m \leq v_n$. Let us and choose $(j, i) \in \mathbb{Z}^2$ such that $(k_1, \gamma_{k_1}) \in j \times [i, i + 1]$. For m large enough, we will have that $k_m > M_n(j, i)$ and, therefore,

$$\gamma^{\text{out}}(k_m) \subset (j, i) + \text{Co}(k_m - j, \gamma_{k_m} - i, Q|k_m - j|^{-\delta}),$$

for a constant $Q > 0$ and m large enough. Therefore, $\bar{v} = \limsup_{k \rightarrow \infty} \gamma_k/k$ and $\underline{v} = \liminf_{k \rightarrow \infty} \gamma_k/k$ are well-defined and satisfy

$$\bar{v} - \underline{v} < Q|k_m - j|^{-\delta}.$$

Since the right-hand side converges to 0 as $m \rightarrow \infty$, we obtain $\bar{v} = \underline{v}$. □

Lemma 7.2. *Let γ be a point-to-point minimizer between (n, x) and (m, y) . Then for all $k \in \{n + 1, \dots, m - 1\}$,*

$$\gamma_{k+1} - \gamma_k = \gamma_k - \gamma_{k-1} + f(k, \gamma_k), \tag{7.1}$$

or, equivalently,

$$\gamma_{k+1} = 2\gamma_k - \gamma_{k-1} + f(k, \gamma_k), \tag{7.2}$$

where $f(k, x) = \partial_x F(k, x)$ for all $(k, x) \in \mathbb{Z} \times \mathbb{R}$.

Proof: For all $\omega \in \Omega$, F_ω is a sum of finitely many smooth functions on every interval $\{k\} \times (x - 1, x + 1)$. Therefore, action is a smooth function of paths. The lemma follows from equating the partial derivative of the action with respect to γ_k to 0. □

Equations 7.1 and (7.2) are discrete time versions of Euler–Lagrange equations. Their meaning is that path γ instantaneously changes its velocity by $f(k, \gamma_k) = \partial_x F(k, \gamma_k)$ at time k .

The following is the last technical lemma we need to prove existence of one-sided minimizers.

Lemma 7.3. *Let $n_i \uparrow \infty$ and x_i, z_i satisfy $|x_i| < |z_i|$ and $|x_i - z_i| > 2Rn_i$ for all $i \in \mathbb{N}$. Then with probability 1, there is $N(\omega)$ such that if $n_i > N(\omega)$, then*

$$|\gamma_{n_i}^{(0,0),(n_i+1,x_i)}| < z_i.$$

Proof: This Lemma is a consequence of Lemma 5.4 and the Borel–Cantelli Lemma. \square

The next lemma states existence of one-sided minimizers with a given asymptotic slope and provides a way to construct them.

Lemma 7.4. *With probability one, for every $v \in \mathbb{R}$ and for every sequence $(m_i, y_i) \in \mathbb{Z} \times \mathbb{R}$ with $m_i \rightarrow \infty$ and*

$$\lim_{i \rightarrow \infty} \frac{y_i}{m_i} = v,$$

and for every $(n, x) \in \mathbb{Z} \times \mathbb{R}$, there exists a subsequence (i_k) such that the minimizing paths $\gamma^{(n,x),(m_{i_k},y_{i_k})}$ converge pointwise to a semi-infinite minimizer starting at (n, x) and with asymptotic slope equal to v .

Proof: Without loss of generality, we can assume that $(n, x) \in \{0\} \times [0, 1]$ since $\mathbb{Z} \times \mathbb{R}$ can be represented as a countable union of shifts of this set. We fix $\delta \in (0, 1/4)$ and a sequence $v_l \rightarrow \infty$, and then choose $Q > 0$ and $M_l = M_l(\omega) \rightarrow \infty$ such that the statement of Lemma 6.6 holds for every triplet (v_l, M_l, Q) . From these triplets we choose a triplet (v_0, M, Q) such that $v_0 > v + 2QM^{-\delta} + 2R$.

Passing to a subsequence if needed, we can make sure that (m_i) is an increasing sequence satisfying $m_i \geq M$ for all i , and $(m_j, y_j) \in \text{Co}(1, v, Qm_i^{-\delta})$ for all $j > i$.

Consider the paths $\gamma^j = \gamma^{(0,x),(m_j,y_j)}$. We claim that there is $t \in \mathbb{N}$ and $C > 0$ such that infinitely many paths γ^j satisfy

$$|\gamma_t^j|, |\gamma_{t+1}^j| < C. \tag{7.3}$$

To see that, let us analyse the restrictions on paths γ_j that we have.

The situation where γ^j visits a point $(p, z) \in \text{Co}(v_0) \setminus \text{Co}(1, v, 2Qm_i^{-\delta})$ satisfying $p > M$ is impossible since this path would violate the δ -straightness condition: the relevant cone through (p, z) will not overlap with $\text{Co}(1, v, Qm_i^{-\delta})$, and therefore it cannot contain (m_j, y_j) .

Suppose now that γ^j does not visit $\text{Co}(v_0) \setminus \text{Co}(1, v, 2Qm_i^{-\delta})$ but visits $\text{Co}(v_0)^c$. Then there are points $(p, z_1) \notin \text{Co}(v_0)$ and $(p+1, z_2) \in \text{Co}(1, v, 2Qm_i^{-\delta})$ such that $\gamma_p^{(0,x),(p+1,z_2)} = z_1$. Let us consider the case $z_1 > v_0p$ (the case $z_1 < -v_0p$ is treated similarly). Due to monotonicity of dependence of minimizers on endpoints, $\gamma_p^{(0,1),(p+1,(v+2Qm_i^{-\delta})(p+1))} > v_0p$. Now Lemma 7.3 implies that this can happen only for finitely many values of p . So there exists N such that no path γ^j visits $\text{Co}(v_0)^c$ after time N .

Combining the claims of last two paragraphs, we obtain that for each $i \geq 1$ and $j > i$, the path γ^j lies in the cone $\text{Co}(1, v, 2Qm_i^{-\delta})$ for times larger than $m_i \vee N$. In particular, our claim about (7.3) holds true.

Using compactness of $[-C, C]$, we can find a subsequence of minimizers γ_j converging to a limit at times t and $t + 1$. Let us recall that minimizers solve the Euler–Lagrange equation (7.2). The values of a solution of (7.2) at t and $t + 1$ determine the entire solution uniquely. Also, due to continuous dependence of solutions on the initial data, the convergence of the solution at times $t, t + 1$ implies the pointwise convergence at all times. Due to Lemma 12.1, the resulting limiting infinite trajectory is a one-sided minimizer. Since all the pre-limiting finite minimizers lie within the cones $\text{Co}(1, v, 2Qm_i^{-\delta})$ for sufficiently large times, so does the limiting one-sided minimizer. Therefore, its asymptotic slope equals v . \square

7.2 Uniqueness

Let us now prove uniqueness of one-sided minimizers with given asymptotic slope.

Lemma 7.5. *Let $\omega \in \Omega$ and $(n, x) \in \mathbb{Z} \times \mathbb{R}$. If γ^1 and γ^2 are two distinct one-sided minimizers with endpoint (n, x) , then they do not intersect as curves in $\mathbb{R} \times \mathbb{R}$.*

Proof: This follows directly from Lemma 12.2. \square

Lemma 7.6. *Let $(n, x) \in \mathbb{R} \times \mathbb{Z}$ and $v \in \mathbb{R}$. With probability 1, there is a unique one-sided minimizer with slope v and endpoint (n, x) .*

Proof: We fix (n, x) . First, we know that with probability 1, for each $v \in \mathbb{R}$, there is a one-sided minimizer with asymptotic slope v . To each direction $v \in \mathbb{R}$ we assign an interval $I_v = (a_v, b_v)$, where $a_v = \inf \gamma_1$ and $b_v = \sup \gamma_1$ with infimum and supremum taken over all one-sided minimizers γ with slope v . If there is a unique one-sided minimizer with slope v , we set $I_v = \emptyset$. Lemma 7.5 implies that if $v_1 < v_2$, then every minimizer γ^1 with slope v_1 and every minimizer γ^2 with slope v_2 satisfy $\gamma_k^1 < \gamma_k^2$, for all times $k > n$. In particular $I_{v_1} \cap I_{v_2} = \emptyset$. Since one can place at most countably many disjoint open intervals in \mathbb{R} , we have that there are at most countably many values of v such that $I_v \neq \emptyset$, i.e., $a_v \neq b_v$. However, $\mathbb{P}\{I_v \neq \emptyset\} = p$ does not depend on v due to shear invariance. Therefore, we can take any probability density f on \mathbb{R} and write

$$p = \int_{\mathbb{R}} \mathbb{P}\{I_v \neq \emptyset\} f(v) dv = \int_{\mathbb{R}} \mathbb{E} \mathbf{1}_{\{I_v \neq \emptyset\}} f(v) dv = \mathbb{E} \int_{\mathbb{R}} \mathbf{1}_{\{I_v \neq \emptyset\}} f(v) dv = 0,$$

since $I_v \neq \emptyset$ can be true at most for countably many v . So, for any $v \in \mathbb{R}$, $\mathbb{P}\{I_v \neq \emptyset\} = 0$, and the lemma follows. \square

Lemma 7.7. *Under the conditions of the previous lemma, let γ denote the (a.s.-unique) minimizer with endpoint (n, x) and slope v . Then there is an event of probability 1 such that on that event, for any sequence of points $(m_i, y_i) \in \mathbb{Z} \times \mathbb{R}$ such that $m_i \rightarrow \infty$ and $y_i/m_i \rightarrow v$ and all $k > n$, we have $\gamma_k^{(n,x),(m_i,y_i)} \rightarrow \gamma_k$.*

Proof: Let us check that for any sequence (i') we can choose a subsequence (i'') such that the corresponding minimizers $\gamma^{(n,x),(m_{i''},y_{i''})}$ converge to γ . In fact, Lemma 7.4 allows to find a subsubsequence (i''') such that the corresponding minimizers $\gamma^{(n,x),(m_{i'''},y_{i'''})}$ converge pointwise to a limiting infinite one-sided minimizer. However, there is a unique one-sided minimizer γ , and the desired convergence follows. \square

Lemma 7.8. *Let $(n, x) \in \mathbb{Z} \times \mathbb{R}$ and $v \in \mathbb{R}$. With probability 1, the following holds true: there is a unique minimizer γ with endpoint $(n, x) \in \mathbb{Z} \times \mathbb{R}$ and slope $v \in \mathbb{R}$; if $y \neq x$, then no minimizer $\tilde{\gamma}$ with endpoint (n, y) and slope v can intersect γ .*

Proof: Without loss of generality let us assume that $y > x$. Minimizers γ and $\tilde{\gamma}$ have at most one intersection point (the argument for this claim is similar to the proof of Lemma 12.2). Suppose they have exactly one intersection point. Then there is an integer $m > n$ such that $\tilde{\gamma}$ and γ intersect strictly between times $m - 1$ and $m + 1$.

Let us consider the sequence of minimizers $\beta^k = \gamma^{(n,x),(k,\tilde{\gamma}_k)}$. Lemma 7.7 implies that these minimizers converge pointwise to γ . In particular, $\beta_j^k \rightarrow \gamma_j$, $j \in \{m - 1, m, m + 1\}$. Therefore, for sufficiently large k the point-to-point minimizers β^k and $\tilde{\gamma}$ coincide at time k and intersect between times $m - 1$ and $m + 1$. This is a contradiction with Lemma 12.2, and the proof is completed. \square

Lemma 7.9. *Let $v \in \mathbb{R}$. With probability 1 the following holds true: for all $(n, x) \in \mathbb{Z} \times \mathbb{R}$ there is a one-sided minimizer with endpoint (n, x) and slope v ; the minimizers are unique for all (n, x) except countably many.*

Proof: Let us fix v . Lemma 7.7 implies that with probability 1, unique minimizers $\gamma(n, q)$ with slope v exist simultaneously for all points $(n, q) \in \mathbb{Z} \times \mathbb{Q}$. Let us take a point $(n, x) \in \mathbb{Z} \times \mathbb{R}$ with $x \notin \mathbb{Q}$. Due to Lemma 7.8, none of $\gamma(n, q)$ with $(n, q) \in (\mathbb{Z} \times (\mathbb{Q} \cap [[x], [x] + 1]))$ intersect each other. In particular, they all are squeezed between $\gamma(n, [x])$ and $\gamma(n, [x] + 1)$. So we can take a sequence of rational points $q_i \downarrow x$ and use Lemma 12.1 to conclude that the pointwise limit of $\gamma(n, q_i)$ is a minimizer with endpoint (n, x) . Since this minimizer $\gamma^+(n, x)$ lies between $\gamma(n, [x])$ and $\gamma(n, [x] + 1)$, its asymptotic slope is also equal to v .

We can also construct another minimizer $\gamma^-(n, x)$ for any point (n, x) taking left limits $q_i \uparrow x$. The set of all points (n, x) such that these two minimizers do not coincide is at most countable since each discrepancy defines an interval and these intervals are disjoint. Lemma 7.8 implies that if the limits on the right and on the left coincide, then the constructed minimizer is unique, so there is at most countable set of shock points where the minimizer is not unique. \square

A useful point of view at the field of minimizers is via families of monotone maps associated with them. Namely, for each $n \in \mathbb{Z}$ we can consider a monotone right-continuous map defined by $x \mapsto \gamma_{n-1}^+(n, x)$. This map has at most countably many discontinuities and, in terms of fluid dynamics, takes the particle that arrives to point x at time n and outputs its position at time $n - 1$. Of course, one can also consider the inverses of those maps. They are also monotone maps that have intervals of constancy corresponding to particles absorbed into shocks. Compositions of maps from both families correspond to particle dynamics over longer time intervals.

8 Weak hyperbolicity

Here we switch back to the point of view where the action is given by (2.6) and one-sided minimizers are “backward” ones. For an endpoint (n, x) a one-sided minimizer γ is defined on $\{\dots, n - 1, n\}$. Let us recall that the guiding idea is to construct global solutions of the Burgers equation using these minimizers collecting information about the history of the forcing before time n .

In contrast with the existing work on the Burgers equation and last-passage type percolation models, for our model, we are not able to prove hyperbolicity, i.e., the asymptotic closeness of one-sided minimizers in reverse time. However, in this section we prove a weakened hyperbolicity property that is still useful for our model and hopefully will be useful for other related models. The most important new ideas of this paper are introduced in this section and the following ones.

From the previous section we know that for a fixed $v \in \mathbb{R}$, with probability 1, to each point $(n, x) \in \mathbb{Z} \times \mathbb{R}$, we can assign the rightmost minimizer $\gamma^+(n, x)$ and leftmost minimizer $\gamma^-(n, x)$ with slope v . Throughout this section we will be working with $v = 0$ (although the results are valid for all $v \in \mathbb{R}$ due to the Galilean invariance of the system), so we suppress the dependence on v in the notation. For all but countably many (n, x) , $\gamma^+(n, x) = \gamma^-(n, x)$.

Let us define for $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$ satisfying $x \leq y$,

$$W_k(n, x, y) = \sum_{i=0}^2 (\gamma_{k-i}^+(n, y) - \gamma_{k-i}^-(n, x)), \quad k \leq n.$$

We also define $W_k(n, x) = W_k(n, x, x)$ for $(n, x) \in \mathbb{Z} \times \mathbb{R}$. Let us state the main result of this section.

Theorem 8.1. *With probability 1, for all $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$ satisfying $x \leq y$,*

$$\liminf_{k \rightarrow -\infty} \frac{W_k(n, x, y)}{(n - k)^{-1}} = 0. \tag{8.1}$$

This theorem is a corollary of the following fact:

Lemma 8.2. *With probability 1, for all $(n, x) \in \mathbb{Z} \times \mathbb{R}$,*

$$\liminf_{k \rightarrow -\infty} \frac{W_k(n, x)}{(n - k)^{-1}} = 0. \tag{8.2}$$

Derivation of Theorem 8.1 from Lemma 8.2: Suppose that with positive probability there are n, x, y such that (8.1) fails. Our goal is to prove that then with positive probability there is (n', x') such that (8.2) fails for (n, x) replaced with (n', x') .

Due to space-time stationarity we obtain that there is $r > 0$ with the following property: with positive probability there are $x, y \in [-r, r]$ such that $x < y$, (8.1) fails, and the unique one-sided minimizers γ^x for $(0, x)$ and γ^y for $(0, y)$ satisfy $-r < \gamma_{-1}^x < \gamma_{-1}^y < r$. We denote this event by A .

Let B be the following event: $F_\omega(0, w) = 0$ for all $w \in [-r, r]$ and $F_\omega(0, -r - 1) \vee F_\omega(0, r + 1) < -3(r + 1)^2$. Then $P(B) > 0$ due to (2.2) and the fact that Poisson random variables are unbounded. Also, events A and B are independent since A depends only on the realization of ω for negative times. Therefore, $P(A \cap B) > 0$.

Let us prove that if A and B hold, then no minimizer with endpoint $(1, z)$ for some $z \in \mathbb{R}$ can pass between x and y .

Suppose that the opposite holds, namely, there is $z \in \mathbb{R}$ such that $p = \gamma_0(1, z)$ satisfies $x < p < y$. Then $\gamma(1, z)$ has to pass between γ^x and γ^y , so, denoting $q = \gamma_{-1}(1, z)$, we obtain $-r < q < r$. We claim that, contrary to our assumption, the path $(\gamma_{-1}(1, z), \gamma_0(1, z), \gamma_1(1, z)) = ((-1, q), (0, p), (1, z))$ is not a minimizing path between $(-1, q)$ and $(1, z)$. Namely, we claim that one of the paths $\gamma^+ = ((-1, q), (0, r + 1), (1, z))$ and $\gamma^- = ((-1, q), (0, -r - 1), (1, z))$ has smaller action on time interval $\{-1, 0, 1\}$ than $\gamma(1, z)$. Suppose $z > 0$, then, noticing that the contribution of $(-1, q)$ is common for all paths under consideration and using the definition of B , we obtain

$$\begin{aligned} A^{-1,1}(\gamma(1, z)) - A^{-1,1}(\gamma^+) &= \frac{1}{2}(p - q)^2 + \frac{1}{2}(z - p)^2 + F_\omega(0, p) \\ &\quad - \frac{1}{2}(r + 1 - q)^2 - \frac{1}{2}(z - r - 1)^2 - F_\omega(0, r + 1) \\ &> (r + 1 - p)z \geq 0, \end{aligned}$$

and, similarly, for $z < 0$,

$$A^{-1,1}(\gamma(1, z)) - A^{-1,1}(\gamma^-) > 0.$$

In both cases we obtain a contradiction that proves our claim that on $A \cap B$ no one-sided minimizer with endpoint after time 0, can pass between γ^x and γ^y .

Due to stationarity, it is impossible for all minimizers with endpoint after time 0 to pass on the left of x at time 0, or for all of these minimizers to pass on the right of y at time 0. Therefore, due to the monotonicity of the minimizers with respect to the endpoint, there is $z \in \mathbb{R}$ such that for all $z' < z$, $\gamma_0(1, z') \leq x$ and for all $z' > z$, $\gamma_0(1, z') \geq y$. In particular $(1, z)$ is a shock point, i.e., there are two one-sided minimizers with endpoint $(1, z)$, one passing on the right of $(0, y)$ and another on the left of $(0, x)$. Since (8.1) fails and $W_k(1, z) > W_k(0, x, y)$ for all $k < 0$, we conclude that on the event $A \cap B$ of positive probability, there is a point $(1, z)$ such that (8.2) fails with (n, x) replaced by $(1, z)$. This contradicts Lemma 8.2, and the proof is completed. \square

For the proof of Lemma 8.2, we need some notation and auxiliary results.

For every shock (n, x) we define the area absorbed in this shock by

$$\Lambda(n, x) = \{(m, y) : m \leq n, \gamma_m^-(n, x) \leq y \leq \gamma_m^+(n, x)\}.$$

If $(m, y) \in \Lambda(n, x)$, we write $(n, x) \prec (m, y)$ and say that (n, x) absorbs (m, y) or inherits from (m, y) . In that case (n, x) is called a successor of (m, y) , and (m, y) is called a predecessor of (n, x) . We have chosen the notation to ensure that every shock counts as its own predecessor.

Lemma 8.3. *Every shock (n, x) has a unique successor at time $n + 1$, i.e., there is a unique shock $(n + 1, z)$ such that $(n + 1, z) \prec (n, x)$.*

Proof: As in the derivation of Theorem 8.1 from Lemma 8.2, it is not possible for all minimizers corresponding to points $(n+1, z'), z \in \mathbb{R}$ to pass on one side of (n, x) . So there is a point z such that for all $z' < z, \gamma_n(n+1, z') \leq x$ and for all $z' > z, \gamma_n(n+1, z') \geq x$. Moreover, $\gamma_n(n+1, z')$ cannot be continuous at $z' = z$, since otherwise, due to the Euler-Lagrange equation (7.2), $\gamma_{n-1}(n+1, z')$ would also be continuous at $z' = z$, in contradiction with the presence of shock at (n, x) . So, $(n+1, z)$ is a shock point and $(n+1, z) \prec (n, x)$. \square

Two shocks (n, x) and (n, y) are said to *merge* or *coalesce* if they have a common successor at time $n+1$, i.e., there is $z \in \mathbb{R}$ such that $(n+1, z) \prec (n, x)$ and $(n+1, z) \prec (n, y)$.

Lemma 8.4. *Let $N, M \in \mathbb{N}, \delta > 0$. Then $P(B_\delta) = 0$, where*

$$B_\delta = \left\{ \text{there is } r_0 > 0 : \text{for all } r > r_0, \max_{1 \leq i \leq M} \max_{-N \leq k \leq i} \gamma_k^+(i, r) > (1 + \delta)r \right\}.$$

Proof: The lemma follows directly from the fact that

$$X(r) = \max_{1 \leq i \leq M} \max_{-N \leq k \leq i} \gamma_k^+(i, r) - r, \quad r \in \mathbb{R},$$

is a stationary process. \square

Proof of Lemma 8.2: Suppose that with positive probability there is a point (n, x) such that

$$\liminf_{k \rightarrow -\infty} \frac{W_k(n, x)}{(n - k)^{-1}} > 0. \tag{8.3}$$

Then there is a number $q > 0$ such that with positive probability there is a point (n, x) satisfying

$$W_k(n, x) > \frac{q}{n - k}, \quad k < n. \tag{8.4}$$

Our goal is to show that the systematic presence of such shocks all over the space-time leads to a contradiction. The strategy is to show how to construct a family of shocks with disjoint absorbed areas and such that the union of those areas is too large to fit into the space-time.

For any set $A \subset \mathbb{R}$ we introduce its density

$$\rho(A) = \lim_{r \rightarrow +\infty} \frac{|A \cap [0, r]|}{r} \in [0, +\infty],$$

if the limit is well-defined. Here bars mean the number of elements in the set.

Let $A_n, n \in \mathbb{N}$ be the (random) set of points (n, x) satisfying (8.4). By ergodic theorem, there is a deterministic number $a > 0$ such that $\rho(A_n) = a$ almost surely for all $n \in \mathbb{Z}$.

Since the shock areas (areas between leftmost and rightmost one-sided minimizers) are disjoint for different points in A_n , we see that

$$\sum_{i=0}^2 (\gamma_{n-1-i}^+(n, r) - \gamma_{n-1-i}^-(n, 0)) \geq \sum_{x \in A_n \cap [0, r]} W_{n-1}(n, x) \geq q|A_n \cap [0, r]|.$$

Since $|A_n \cap [0, r]|/r \rightarrow a$, we conclude that

$$\liminf_{r \rightarrow \infty} \frac{\sum_{i=0}^2 \gamma_{n-1-i}^+(n, r)}{r} \geq qa.$$

Due to Lemma 8.4, this is impossible unless $a \leq q^{-1} < \infty$.

For $i \in \mathbb{N}$, let B_i consist of all shocks in A_i that are successors of some shocks in A_{i-1} . Let C_i consist of those shocks in A_i that are not successors of any shocks in A_{i-1} .

Let D_i consist of those shocks in A_{i-1} whose successors are not in A_i . Let E_i consist of all shocks (i, x) in A_{i-1} such that there is $y < x$ satisfying $(i, y) \in A_{i-1}$ and both (i, x) and (i, y) have the same successor belonging to A_i . Let F_i be the set of all shocks $(i-1, x)$ in A_{i-1} such that no other point $(i-1, y) \in A_{i-1}$ has the same successor in A_i as $(i-1, x)$ and satisfies $y < x$. We then have

$$\rho(A_i) = \rho(B_i) + \rho(C_i), \tag{8.5}$$

and

$$\rho(A_{i-1}) = \rho(D_i) + \rho(E_i) + \rho(F_i). \tag{8.6}$$

To see this, first notice that all of the densities involved are well-defined deterministic numbers due to ergodic theorem, since they are defined via skew-invariant (w.r.t. spatial shifts) functionals of the ergodic environment. Then identities (8.5) and (8.6) follow from the fact that $A_i = B_i \cup C_i$ and $A_{i-1} = D_i \cup E_i \cup F_i$.

We already know that $\rho(A_i) = \rho(A_{i-1})$. Also, $\rho(B_i) = \rho(F_i)$, since the relation " \prec " defines a one-to-one monotone map between B_i and F_i and it preserves the density due to Lemma 8.4. So, (8.5) and (8.6) imply $\rho(C_i) = \rho(D_i) + \rho(E_i)$. Noticing that $\rho(E_i) > 0$ does not depend on i and denoting the common value by $\rho_E > 0$, we obtain

$$\rho(C_i) - \rho(D_i) = \rho_E > 0. \tag{8.7}$$

Let us recall that we are proving Lemma 8.2. We made an assumption that there are shocks with absorbed areas that are not too thin, i.e., that satisfy (8.3). Identity (8.7) computes the (positive) balance of densities of such shocks emerging and disappearing at time i . Our goal now is to exploit this identity to see that there is a systematic addition of area due to newly emerging shocks satisfying (8.3).

Let $n \in \mathbb{N}$. Let $\mathcal{A}^n = \bigcup_{i=1}^n A_i$ and let \mathcal{B}^n be the set of all shocks (i, x) such that $i \in \{1, \dots, n\}$ and (i, x) is a successor of some shock from \mathcal{A}^n . Since according to our definition each shock is its own successor, we have $\mathcal{A}^n \subset \mathcal{B}^n$. We want to partition \mathcal{B}^n into sequences of successive shocks and estimate the area absorbed by each of those sequences at times $-2, -1, -0$. We must be careful to avoid overlaps of these areas.

We say that $(i, x) \in \mathcal{B}^n$ is the *main predecessor* of $(i+1, z) \in \mathcal{B}^n$ if

1. $(i+1, z) \prec (i, x)$;
2. there is no $y < x$ such that $(i+1, z) \prec (i, y)$ and $(i, y) \in \mathcal{A}^n$;
3. if $(i, x) \in \mathcal{B}^n \setminus \mathcal{A}^n$ then (i) there is no $y < x$ such that $(i+1, z) \prec (i, y)$ and $(i, y) \in \mathcal{B}^n$;
(ii) there is no y such that $(i+1, z) \prec (i, y)$ and $(i, y) \in \mathcal{A}^n$.

In other words: if at time i there are predecessors of $(i+1, z)$ among \mathcal{A}^n , we choose the leftmost of them; if not, we choose the leftmost predecessor among \mathcal{B}^n . Clearly, if $(i+1, z) \in \mathcal{B}^n$ has some predecessors in \mathcal{B}^n at time i , then exactly one of them is the main one.

For any $k, m \in \mathbb{N}$ satisfying $k \leq m \leq n$ we define $Q_{k,m}$ as the set of sequences $\zeta : \{k, \dots, m\} \rightarrow \mathbb{R}$ satisfying the following conditions:

- (i) (i, ζ_i) is a shock for all $i \in \{k, \dots, m\}$;
- (ii) $(k, \zeta_k) \in A_k$;
- (iii) for all $i \in \{1, \dots, k-1\}$ and all $(i, x) \in A_i$, $(k, \zeta_k) \not\prec (i, x)$;
- (iv) for each $i \in \{k, \dots, m-1\}$, (i, ζ_i) is the main predecessor of $(i+1, \zeta_{i+1})$;

(v) if $m < n$, then (m, ζ_m) is not the main predecessor of its successor.

Sequences in $Q = \bigcup_{1 \leq k \leq m \leq n} Q_{k,m}$ viewed as sets of space-time points generate a partition of \mathcal{B}^n , and each shock sequence in Q is uniquely defined by its birth place, i.e., its first entry. Since the birth places of two distinct sequences cannot be successors of each other, the areas absorbed by them are mutually disjoint.

For $k, m \in \mathbb{N}$ such that $k \leq m \leq n$, we define

$$R_{k,m} = \{a : \{k, \dots, m\} \rightarrow \{0, 1\} : a_k = 1\}$$

and $a : Q_{k,m} \rightarrow R_{k,m}$ by

$$a_i(\zeta) = \begin{cases} 1, & (i, \zeta_i) \in A_i, \\ 0, & (i, \zeta_i) \notin A_i. \end{cases}$$

For $1 \leq k \leq m \leq n$ and $a \in R_{k,m}$, we introduce

$$Q_{k,m}^a = \{\zeta \in Q_{k,m} : a(\zeta) = a\}$$

and note that all sets $Q_{k,m}^a$ with all possible values of k, m , and a are mutually disjoint, and there are finitely many of them.

If $1 \leq k \leq m \leq n$, $a \in R_{k,m}$, and $i \in \{1, \dots, n\}$, then we define $Q_{k,m}^a(i)$ to be the section of $Q_{k,m}^a$ at level i , i.e., the set consisting of points $x \in \mathbb{R}$ such that (i, x) belongs to a path from $Q_{k,m}^a(i)$.

Lemma 8.5. For all $i, j \in \{k, \dots, m\}$, $\rho(Q_{k,m}^a(i)) = \rho(Q_{k,m}^a(j))$.

Proof: This is a corollary of Lemma 8.4. □

If $a \in R_{k,m}$ we write $k(a) = k$, $m(a) = m$. For $a \in S_n = \bigcup_{1 \leq k \leq m \leq n} R_{k,m}$, we introduce $\bar{a} \in \{0, 1\}^{\{0, \dots, n\}}$ by

$$\bar{a}_i = \begin{cases} a_i, & k(a) \leq i \leq m(a), \\ 0, & \text{otherwise.} \end{cases}$$

and $\Delta(a) \in \{-1, 0, 1\}^{\{1, \dots, n\}}$ by

$$\Delta_k(a) = \bar{a}_k - \bar{a}_{k-1}, \quad k = 1, \dots, n,$$

so $\Delta(a)$ is an alternating sequence of +1's and -1's with possibly some 0's between and around them. We define also

$$v(a) = \sum_{k=1}^n \frac{1}{k} \Delta_k(a), \quad a \in S_n,$$

and

$$w(a) = \frac{1}{k(a)}, \quad a \in S_n.$$

Clearly, $w(a) \geq v(a) > 0$ for all $a \in S_n$, due to the alternating character of the sequence $\Delta(a)$.

Let us define $\rho(Q_{k,m}^a) = \rho(Q_{k,m}^a(k))$ and $\rho(a) = \rho(Q_{k(a),m(a)}^a)$. Then

$$\sum_{a \in S_n} w(a) \rho(a) \geq \sum_{a \in S_n} v(a) \rho(a) = \sum_{a \in S_n} \sum_{k=1}^n \frac{1}{k} \Delta_k(a) \rho(a) = \sum_{k=1}^n \frac{c_k - d_k}{k}, \quad (8.8)$$

where for $k = 1, \dots, n$

$$c_k = \sum_{a \in S_n : \Delta_k(a) = 1} \rho(a),$$

$$d_k = \sum_{a \in S_n : \Delta_k(a) = -1} \rho(a).$$

Since

$$C_i = \bigcup_{a \in S_n: \Delta_i(a)=1} Q_{k(a),m(a)}^a(i),$$

$$D_i = \bigcup_{a \in S_n: \Delta_i(a)=-1} Q_{k(a),m(a)}^a(i-1),$$

we obtain from Lemma 8.5:

$$c_i = \sum_{a \in S_n: \Delta_i(a)=1} \rho(Q_{k(a),m(a)}^a) = \sum_{a \in S_n: \Delta_i(a)=1} \rho(Q_{k(a),m(a)}^a(i)) = \rho(C_i),$$

$$d_i = \sum_{a \in S_n: \Delta_i(a)=-1} \rho(Q_{k(a),m(a)}^a) = \sum_{a \in S_n: \Delta_i(a)=-1} \rho(Q_{k(a),m(a)}^a(i-1)) = \rho(D_i).$$

So, (8.8) and (8.7) imply

$$\sum_{a \in S_n} w(a)\rho(a) \geq \sum_{i=1}^n \frac{\rho(C_i) - \rho(D_i)}{i} \geq \rho_E \sum_{i=1}^n \frac{1}{i}. \tag{8.9}$$

By construction and by the definition of sets A_n , to each a and each sequence $\zeta \in Q_{k(a),m(a)}^a$ we can associate a triplet of intervals $J_l(\zeta) \subset \{l\} \times \mathbb{R}$, $l \in \{0, -1, -2\}$ with the following properties:

$$\sum_{l=-2}^0 |J_l(\zeta)| \geq qw(a), \quad \bigcup_{l=-2}^0 J_l(\zeta) \subset \Lambda(k(a), \zeta_{k(a)}). \tag{8.10}$$

In particular, intervals $J_l(\zeta^1)$ and $J_l(\zeta^2)$ for any $l \in \{0, -1, -2\}$ and any two distinct sequences ζ^1, ζ^2 do not overlap. Let us denote by

$$T_n(r) = \sum_{a \in S_n} \sum_{\gamma \in Q_{k(a),m(a)}^a: \zeta_{k(a)} \in [0,r]} \sum_{l=-2}^0 |J_l(\zeta)|,$$

the total trace at times $-2, -1, 0$ of non-overlapping shocks with birth locations in $\{1, \dots, n\} \times [0, r]$. Due to (8.9) and (8.10),

$$\liminf_{r \rightarrow \infty} \frac{T_n(r)}{r} \geq L_n,$$

where

$$L_n = q\rho_E \sum_{i=1}^n \frac{1}{i}.$$

Since $\lim_{n \rightarrow \infty} L_n = \infty$, we can choose n large enough to ensure $L_n > 8$. Then, for all sufficiently large r , we are guaranteed to have $T_n(r) > 7r$. Therefore, for all sufficiently large r we will have points in $\{1, \dots, n\} \times [0, r]$ with associated one-sided minimizers containing points on the right of $\{-2, -1, 0\} \times [0, 2r]$ (the total length of the segments this set consists of is only $6r$). Applying Lemma 8.4, we see that this happens with probability 0. Therefore, our assumption that with positive probability there exists a point satisfying (8.3) was wrong, so the proof of the lemma is completed. \square

9 Constructing stationary solutions

Let us begin with the following auxiliary result:

Lemma 9.1. *Let $K > 0$. With probability 1 there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the Lipschitz constant of $F_\omega(n, x)$ with respect to $x \in [-Kn, Kn]$ is bounded by $\ln n$.*

Proof: It is sufficient to prove that for sufficiently large $n \in \mathbb{N}$ and any $r \in \mathbb{N}$ with $|r| < Kn + 1$, the Lipschitz constant of $F_\omega(n, x)$ w.r.t. $x \in [r, r + 2]$ is bounded by $\ln n$. So let us take arbitrary n and r satisfying the conditions above. For any $x, y \in [r, r + 2]$, the definition (2.3) and our assumptions on the functions involved in it, imply

$$|F_\omega(n, x) - F_\omega(n, y)| \leq N_\omega(n, r)L|x - y|,$$

where

$$N_\omega(n, r) = \omega(\{n\} \times [r - 1, r + 3] \times \mathbb{R} \times \mathbb{R})$$

and L is the Lipschitz constant of ϕ . So the Lipschitz constant of $F_\omega(n, x)$ w.r.t. $x \in [r, r + 2]$ is bounded by $N_\omega(n, r)L$. Since for any $\lambda > 0$ there is $C(\lambda) > 0$ such that

$$\mathbb{P}\{N_\omega(n, r)L > \ln n\} \leq \frac{\mathbb{E}e^{\lambda N_\omega(n, r)L}}{e^{\lambda \ln n}} \leq \frac{C(\lambda)}{n^\lambda},$$

we can choose $\lambda = 3$ to see that

$$\sum_{n \in \mathbb{N}} \sum_{r: |r| < Kn + 1} \mathbb{P}\{N_\omega(n, r)L > \ln n\} < \infty,$$

so, the Borel–Cantelli Lemma implies that with probability 1 only finitely many events $\{N_\omega(n, r)L > \ln n\}$ happen, and the lemma follows. \square

Let us recall that for a fixed $v \in \mathbb{R}$, with probability 1, to each point $(n, x) \in \mathbb{Z} \times \mathbb{R}$, we can assign the rightmost minimizer $\gamma^+(n, x)$ and leftmost minimizer $\gamma^-(n, x)$ with slope v . For all but countably many (n, x) , $\gamma^+(n, x) = \gamma^-(n, x)$. In this section, we will denote $\gamma(n, x) = \gamma^+(n, x)$ for brevity. We also assume without loss of generality that $v = 0$ and suppress the dependence on v unless specifically stated otherwise.

Lemma 9.2. *Let $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$ with $x < y$. Let $(k_j)_{j \in \mathbb{N}}$ satisfy $k_j \rightarrow -\infty$ and*

$$W_{k_j}(n, x, y) < \frac{1}{n - k_j}, \quad j \in \mathbb{N} \tag{9.1}$$

(the existence of such a pairing sequence follows from Theorem 8.1). Then the limit

$$B(n, x, y) = \lim_{j \rightarrow \infty} \Delta_j$$

is well-defined and finite, where

$$\Delta_j = A^{k_j - 1, n}(\gamma(n, y)) - A^{k_j - 1, n}(\gamma(n, x)), \quad j \in \mathbb{N}.$$

Proof: It is sufficient to prove that Δ_j is a Cauchy sequence. Let us estimate $|\Delta_j - \Delta_m|$ for $m > j$. To simplify the notation, we denote

$$\begin{aligned} \gamma^1 &= \gamma(n, x), & \gamma^2 &= \gamma(n, y), \\ x_4 &= \gamma_{k_m - 1}^1, & y_4 &= \gamma_{k_m - 1}^2, \\ x_3 &= \gamma_{k_m - 2}^1, & y_3 &= \gamma_{k_m - 2}^2, \\ x_2 &= \gamma_{k_j}^1, & y_2 &= \gamma_{k_j}^2, \\ x_1 &= \gamma_{k_j - 1}^1, & y_1 &= \gamma_{k_j - 1}^2. \end{aligned}$$

Since γ^2 is an optimal path between $(k_j - 1, y_1)$ and $(k_m - 1, y_4)$, we have

$$\begin{aligned} A^{k_j - 1, k_m - 1}(\gamma^2) &\leq F_\omega(k_j - 1, y_1) + \frac{(x_2 - y_1)^2}{2} + A^{k_j, k_m - 2}(\gamma^1) \\ &\quad + F_\omega(k_m - 2, x_3) + \frac{(y_4 - x_3)^2}{2}. \end{aligned}$$

Combining this with

$$F_\omega(k_j - 1, x_1) + \frac{(x_2 - x_1)^2}{2} + A^{k_j, k_m - 2}(\gamma^1) + F_\omega(k_m - 2, x_3) + \frac{(x_4 - x_3)^2}{2} = A^{k_j - 1, k_m - 1}(\gamma^1),$$

we obtain

$$\Delta_j - \Delta_m \leq \delta_{j,m},$$

where

$$\begin{aligned} \delta_{j,m} &= \frac{(x_2 - y_1)^2}{2} - \frac{(x_2 - x_1)^2}{2} + \frac{(y_4 - x_3)^2}{2} - \frac{(x_4 - x_3)^2}{2} \\ &\quad + F_\omega(k_j - 1, y_1) - F_\omega(k_j - 1, x_1) \\ &= (x_1 - y_1) \left(x_2 - \frac{x_1 + y_1}{2} \right) + (x_4 - y_4) \left(x_3 - \frac{x_4 + y_4}{2} \right) \\ &\quad + F_\omega(k_j - 1, y_1) - F_\omega(k_j - 1, x_1). \end{aligned}$$

Therefore, using (9.1), the fact that $|\gamma_k^1| \vee |\gamma_k^2| = o(n - k)$ as $k \rightarrow -\infty$, and Lemma 9.1, we obtain that there is a function $\beta_1(j) \downarrow 0$ and a number J_1 such that if $j > J_1$ and $m > j$, then $\Delta_j - \Delta_m < \beta_1(j)$.

Interchanging the roles of γ^1 and γ^2 we also obtain that there is a function $\beta_2(j) \downarrow 0$ and a number J_2 such that if $j > J_2$ and $m > j$, then $-(\Delta_j - \Delta_m) < \beta_2(j)$.

Combining these last two statements we conclude that (Δ_j) is a Cauchy sequence. \square

The following lemma is an immediate extension of the previous ones. We can extend definition of W to nonsimultaneous points $(n_1, x_1), (n_2, x_2) \in \mathbb{Z} \times \mathbb{R}$:

$$W_k((n_1, x_1), (n_2, x_2)) = \sum_{i=0}^2 \left(\gamma_{k-i}^+(n_1, x_1) \vee \gamma_{k-i}^+(n_2, x_2) - \gamma_{k-i}^-(n_2, x_2) \wedge \gamma_{k-i}^-(n_1, x_1) \right), \quad k \leq n_1 \wedge n_2.$$

Lemma 9.3. *Let $(n_1, x_1), (n_2, x_2) \in \mathbb{Z} \times \mathbb{R}$. Then*

1. *There is a sequence $k_j \downarrow -\infty$ such that*

$$W_{k_j}((n_1, x_1), (n_2, x_2)) \leq \frac{1}{n_1 \wedge n_2 - k_j}, \quad j \in \mathbb{N}.$$

2. *For every pairing sequence (k_j) satisfying conditions of part 1, the following finite limit exists:*

$$B((n_1, x_1), (n_2, x_2)) = \lim_{j \rightarrow \infty} (A^{k_j - 1, n_2}(\gamma(n_2, x_2)) - A^{k_j - 1, n_1}(\gamma(n_1, x_1))).$$

3. *The limit in part 2 does not depend on the concrete choice of (k_j) .*

Proof: The first part of the lemma follows since we can apply Theorem 8.1 to points $(n_1 \wedge n_2, \gamma_{n_1 \wedge n_2}(n_1, x_1))$ and $(n_1 \wedge n_2, \gamma_{n_1 \wedge n_2}(n_2, x_2))$ that share the time coordinate. The second part holds since

$$A^{k_j - 1, n_1}(\gamma(n_1, x_1)) = A^{k_j - 1, n_1 \wedge n_2}(\gamma(n_1 \wedge n_2, \gamma_{n_1 \wedge n_2}(n_1, x_1))) + A^{n_1 \wedge n_2, n_1}(\gamma(n_1, x_1))$$

and

$$A^{k_j-1, n_2}(\gamma(n_2, x_2)) = A^{k_j-1, n_1 \wedge n_2}(\gamma(n_1 \wedge n_2, \gamma_{n_1 \wedge n_2}(n_2, x_2))) + A^{n_1 \wedge n_2, n_1}(\gamma(n_2, x_2)),$$

so one can apply Lemma 9.2 to points $(n_1 \wedge n_2, \gamma_{n_1 \wedge n_2}(n_1, x_1))$ and $(n_1 \wedge n_2, \gamma_{n_1 \wedge n_2}(n_2, x_2))$. The last part follows from the standard trick of interlacing the two sequences. \square

The function $B((n_1, x_1), (n_2, x_2)) = B_\omega((n_1, x_1), (n_2, x_2))$ may be called the Busemann function in analogy to the Busemann functions used in [4] and previous work on last-passage percolation, although we stress that in our setting we are currently able to prove convergence only along appropriate subsequences k_j . This function has several standard properties. Some of them are summarized in the following lemma:

Lemma 9.4. *Let B be defined as above.*

1. *The distribution of B is translation invariant: for any $\Delta \in \mathbb{Z} \times \mathbb{R}$,*

$$B(\cdot + \Delta, \cdot + \Delta) \stackrel{distr}{=} B(\cdot, \cdot).$$

2. *B is antisymmetric:*

$$B((n_1, x_1), (n_2, x_2)) = -B((n_2, x_2), (n_1, x_1)), \quad (n_1, x_1), (n_2, x_2) \in \mathbb{Z} \times \mathbb{R},$$

in particular $B(n, x), (n, x) = 0$ for any $(n, x) \in \mathbb{Z} \times \mathbb{R}$.

3. *B is additive: for any $(n_1, x_1), (n_2, x_2), (n_3, x_3) \in \mathbb{Z} \times \mathbb{R}$,*

$$B((n_1, x_1), (n_3, x_3)) = B((n_1, x_1), (n_2, x_2)) + B((n_2, x_2), (n_3, x_3)).$$

4. *For any $(n_1, x_1), (n_2, x_2) \in \mathbb{Z} \times \mathbb{R}$ satisfying $n_1 < n_2$,*

$$B((n_1, x_1), (n_2, x_2)) \leq A^{n_1, n_2}(x_1, x_2). \tag{9.2}$$

5. *For any $(n_1, x_1), (n_2, x_2) \in \mathbb{Z} \times \mathbb{R}$, $E|B(n_1, x_1), (n_2, x_2)| < \infty$.*

Proof: The first two parts are obvious. For the third part we need to ensure that the convergence in the definition of B for all the values of arguments involved holds along the same sequence k_j . That is true, since all one-sided minimizers are ordered and the gap between the two minimizers on the sides dominates the smaller gaps between the middle one and the side ones.

Let us prove part 4. We denote $\gamma^1 = \gamma(n_1, x_1)$ and $\gamma^2 = \gamma(n_2, x_2)$. Let us find a pairing sequence $(k_j)_{j \in \mathbb{N}}$ for γ^1, γ^2 . For any j let us create a path $\gamma(j)$ that starts at $(k_j - 1, \gamma_{k_j-1}^2)$, makes a step to $(k_j, \gamma_{k_j}^1)$, coincides with γ^1 between k_j and n_1 , and on $\{n_1, \dots, n_2\}$ coincides with the optimal path between (n_1, x_1) and (n_2, x_2) . Then

$$\begin{aligned} A^{k_j-1, n_2}(\gamma^2) &\leq A^{k_j-1, n_2}(\gamma(j)) \\ &\leq F(k_j - 1, \gamma_{k_j-1}^2) + \frac{(\gamma_{k_j}^1 - \gamma_{k_j-1}^2)^2}{2} + A^{k_j, n_1}(\gamma^1) + A^{n_1, n_2}(x_1, x_2) \\ &\leq F(k_j - 1, \gamma_{k_j-1}^2) - F(k_j - 1, \gamma_{k_j-1}^1) + \frac{(\gamma_{k_j}^1 - \gamma_{k_j-1}^2)^2 - (\gamma_{k_j}^1 - \gamma_{k_j-1}^1)^2}{2} \\ &\quad + A^{k_j-1, n_1}(\gamma^1) + A^{n_1, n_2}(x_1, x_2). \end{aligned}$$

Moving $A^{k_j-1, n_1}(\gamma^1)$ to the left-hand side, taking limit $j \rightarrow \infty$, using the pairing property of k_j , the sublinear growth of $|\gamma^1|, |\gamma^2|$, and Lemma 9.1, we finish the proof of part 4.

Let us prove the last part. Using additivity and translation invariance we see that it is sufficient to consider points $(0, 0)$ and (n, x) with $n < 0$. Parts 2 and 4 of this lemma along with Lemma 4.2 imply that

$$EB((0, 0), (n, x)) \geq -EA^{n,0}(x, 0) > -\infty.$$

So it remains to prove an upper bound. Furthermore,

$$B((0, 0), (n, x)) = \sum_{k=1}^{|n|} B\left(\left(-k, x \frac{k-1}{|n|}\right), \left(-k, x \frac{k}{|n|}\right)\right),$$

and all the terms on the right-hand side have the same distribution. Expectation of each of them belongs to $(-\infty, \infty]$ Therefore, $EB((0, 0)(n, x))$ is finite if and only if $EB((0, 0)(-1, x/|n|))$ is finite. So it is sufficient to prove $EB((0, 0), (-1, x)) < \infty$ for all $x \in \mathbb{R}$. Furthermore, we have

$$B((0, 0), (-2, 0)) = B((0, 0), (-1, x)) + B((-1, x), (-2, 0)),$$

where, due to the symmetry of the Poissonian process and the action, the distributions of two terms on the right-hand side coincide. Therefore, $EB((0, 0), (-1, x)) < \infty$ iff $EB((0, 0), (-2, 0)) < \infty$. Applying this once again, we see that it is sufficient to establish $EB((0, 0), (-1, 0)) < \infty$.

We denote $\gamma^0 = \gamma(0, 0)$ and $\gamma^1 = \gamma(0, -1)$ for brevity.

Let $L_1 > 0$ (we will later impose some conditions on L_1). $H = \{(n, x) \in \mathbb{Z} \times \mathbb{R} : |x| \leq L_1(-n - 1)\}$. Since γ^0 has asymptotic slope 0, the time τ defined by

$$\tau = \min\{n \leq 0 : \gamma_k^0 \notin H\} - 1$$

is finite. We define $z = \gamma^0(\tau)$ Let (k_j) be a pairing sequence for γ^0 and γ^1 . For $k_j < \tau$, we have

$$A^{k_j-1,-1}(\gamma^1) - A^{k_j-1,0}(\gamma^0) \leq Q_{k_j}(\gamma^0, \gamma^1) + A^{\tau,-1}(z, 0) - A^{\tau,0}(\gamma^0),$$

where

$$\begin{aligned} Q_n(\gamma^0, \gamma^1) &= F(n-1, \gamma_{n-1}^1) + \frac{(\gamma_n^0 - \gamma_{n-1}^1)^2}{2} \\ &\quad - F(n-1, \gamma_{n-1}^0) - \frac{(\gamma_n^0 - \gamma_{n-1}^0)^2}{2} \\ &= F(n-1, \gamma_{n-1}^1) - F(n-1, \gamma_{n-1}^0) \\ &\quad + (\gamma_{n-1}^1 - \gamma_{n-1}^0) \left(\frac{\gamma_{n-1}^0 + \gamma_{n-1}^1}{2} - \gamma_n^0 \right) \end{aligned}$$

Since $Q_{k_j}(\gamma^0, \gamma^1) \rightarrow 0$ as $j \rightarrow \infty$, we obtain

$$B((0, 0), (-1, 0)) \leq A^{\tau,-1}(z, 0) - A^{\tau,0}(\gamma^0) = A^{\tau,-1}(z, 0) - A^{\tau,0}(z, 0),$$

and it is sufficient to prove

$$EA^{\tau,-1}(z, 0) < \infty \tag{9.3}$$

and

$$EA^{\tau,0}(z, 0) > -\infty. \tag{9.4}$$

Let us estimate the tail of the distribution of τ . First, we choose $L_2 > 0$ so that $L_1 - L_2 > 2R$, where R is chosen according to Lemma 4.3. If $\tau = n$, $\gamma_{\tau+1}^0 > 0$, and $\gamma_{\tau}^0 -$

$\gamma_{\tau+1}^0 < -(L_1 - L_2)|n|$, then the optimal path γ' connecting $(n, L_1(|n| - 2) - (L_1 - L_2)|n|) = (n, L_2|n| - 2L_1)$ to $(0, 0)$ satisfies $\gamma'_{n+1} > L_1(|n| - 2)$, and thus, if $|n|$ is sufficiently large, say, greater than some n_1 , it deviates from the straight line connecting $(n, L_2|n| - 2L_1)$ to $(0, 0)$ by at least $L_1(|n| - 2) - (L_2|n| - 2L_1)(|n| - 1)/|n| > Rn$. Applying Lemma 5.4, we obtain that

$$P\{\tau = n, \gamma_{n+1}^0 > 0, \gamma_n^0 - \gamma_{n+1}^0 < -(L_1 - L_2)|n|\} \leq c_1 \exp(-c_2n), \quad n \geq n_1,$$

for some $c_1, c_2 > 0$, and, similarly,

$$P\{\tau = n, \gamma_{n+1}^0 < 0, \gamma_n^0 - \gamma_{n+1}^0 > (L_1 - L_2)|n|\} \leq c_1 \exp(-c_2n), \quad n \geq n_1.$$

Combining these two inequalities we obtain

$$P\{\tau = n, \gamma_n^0 \notin I_n \cup (-I_n)\} \leq 2c_1 \exp(-c_2n), \quad n \geq n_1, \tag{9.5}$$

where $I_n = [L_2|n| - 2L_1, L_1(|n| - 1)]$.

Let us now take any $n \in -\mathbb{N}$, any point $y \in I_n \cup (-I_n)$ and suppose that $\tau = n$ and $|\gamma_n^0 - y| \leq 1$. Since the asymptotic slope of γ^0 equals 0, for sufficiently large values of $|m|$ we will have $|\gamma_m^0| < L_2|m|/2$. Lemma 6.7 implies that probability of such an event is bounded by $C_1 e^{-C_2|m|^\kappa}$ for sufficiently large $|n|$, and since one can find $3(L_1 - L_2)|n|$ points y such that the union of segments $[y - 1, y + 1]$ covers the entire set $I_n \cup (-I_n)$, we obtain that there is $n_2 > 0$ such that

$$P\{\tau = n, \gamma_n^0 \in I_n \cup (-I_n)\} \leq 2(L_1 - L_2)|n|C_1 e^{-C_2|n|^\kappa}, \quad |n| \geq n_2. \tag{9.6}$$

Combining (9.5) and (9.6), we obtain that there are constants $\bar{C}_1, \bar{C}_2 > 0$ such that

$$P\{\tau = n\} \leq \bar{C}_1 e^{-\bar{C}_2|n|^\kappa}, \quad n \in \mathbb{N}. \tag{9.7}$$

Let us now prove (9.3). Notice that on $\{\tau = n\}$,

$$A^{\tau,-1}(z, 0) \leq F_\omega(n, z) + \frac{(L_1n)^2}{2} + A^{n+1,-1}(0, 0), \tag{9.8}$$

and

$$A^{n-1,-1}(0, 0) \leq F_\omega(n - 1, 0) + \frac{(L_1n)^2}{2} + A^{\tau,-1}(z, 0),$$

so

$$A^{\tau,-1}(z, 0) \geq -F_\omega(n - 1, 0) - \frac{(L_1n)^2}{2} + A^{n-1,-1}(0, 0). \tag{9.9}$$

Combining (9.8) and (9.9), we obtain that on $\{\tau = n\}$,

$$|A^{\tau,-1}(z, 0)| \leq F_\omega^*(\{n\} \times [-L_1|n|, L_1|n|]) + |F_\omega(n - 1, 0)| + (L_1n)^2 + |A^{n+1,-1}(0, 0)| + |A^{n-1,-1}(0, 0)|,$$

where F_ω^* was defined in (5.2). The Cauchy-Schwarz inequality implies

$$\begin{aligned} E|A^{\tau,-1}(z, 0)|\mathbf{1}_{\{\tau=n\}} &\leq \sqrt{P\{\tau = n\}} \left(E F_\omega^{*2}(\{n\} \times [-L_1|n|, L_1|n|]) \right. \\ &\quad + E|F_\omega(n - 1, 0)| + (L_1n)^2 \\ &\quad \left. + E(A^{n+1,-1}(0, 0))^2 + E(A^{n-1,-1}(0, 0))^2 \right). \end{aligned}$$

The first term in the parentheses grows logarithmically in $|n|$, the second term does not depend on n , and the remaining terms grow quadratically due to Lemma 4.5. Combining

this with (9.7), we finish the proof of (9.3). Inequality (9.4) can be proved in exactly the same way. \square

Let us now define

$$U_\omega(n, x) = U(n, x) = B((0, 0), (n, x)), \quad (n, x) \in \mathbb{Z} \times \mathbb{R}.$$

The main claim of this section is that thus defined U is skew invariant under of the HJBHLO cocycle, and its space derivative is a global solution of the Burgers equation.

Let us recall that the HJBHLO evolution is given by

$$\Phi^{m,n}W(y) = \inf_{x \in \mathbb{R}} \{W(x) + A^{m,n}(x, y)\}, \quad m \leq n, \quad y \in \mathbb{R}, \quad (9.10)$$

where $A^{m,n}(x, y)$ has been defined in (4.3).

Theorem 9.5. *The random function U is a global solution of the HJBHLO, i.e., for almost all $\omega \in \Omega$,*

$$\Phi_\omega^{n_1 n_2} U_\omega(n_1, \cdot) = U_\omega(n_2, \cdot), \quad n_1 < n_2.$$

Proof: Let γ be a minimizer with endpoint (n_2, x) . Then

$$\begin{aligned} U(n_2, x) &= U(n_1, \gamma_{n_1}) + (U(n_2, x) - U(n_1, \gamma_{n_1})) \\ &= U(n_1, \gamma_{n_1}) + A^{n_1, n_2}(\gamma_{n_1}, x). \end{aligned}$$

We need to show that the right-hand side is the infimum of $U(n_1, y) + A^{n_1, n_2}(y, x)$ over all $y \in \mathbb{R}$. Suppose that for some $y \in \mathbb{R}$,

$$U(n_1, y) + A^{n_1, n_2}(y, x) < U(n_1, \gamma_{n_1}) + A^{n_1, n_2}(\gamma_{n_1}, x). \quad (9.11)$$

Let us take any minimizer $\bar{\gamma}$ originating at (n_1, y) and use Lemma 9.3 to find a pairing sequence (k_j) for (n_1, y) and (n_1, γ_{n_1}) . Then (9.11) implies

$$\begin{aligned} \lim_{j \rightarrow \infty} (A^{k_j-1, n_1}(\bar{\gamma}) - A^{k_j-1, n_1}(\gamma)) &= U(n_1, y) - U(n_1, \gamma_{n_1}) \\ &< A^{n_1, n_2}(\gamma_{n_1}, x) - A^{n_1, n_2}(y, x). \end{aligned}$$

Denoting the right-hand side by δ , we conclude that for sufficiently large j ,

$$\begin{aligned} A^{k_j-1, n_1}(\bar{\gamma}) + A^{n_1, n_2}(y, x) &\leq A^{k_j-1, n_1}(\gamma) + A^{n_1, n_2}(\gamma_{n_1}, x) - \delta/2 \\ &\leq A^{k_j-1, n_2}(\gamma) - \delta/2. \end{aligned}$$

Let now $\gamma'(j)$ be the path starting at $(k_j - 1, \gamma_{k_j-1})$, making an immediate step to $(k_j, \bar{\gamma}_{k_j})$, coinciding with $\bar{\gamma}$ between times k_j and n_1 , and coinciding with the optimal path connecting y to x between n_1 and n_2 . We have

$$\begin{aligned} A^{k_j-1, n_2}(\gamma'(j)) &\leq A^{k_j-1, n_1}(\bar{\gamma}) + A^{n_1, n_2}(y, x) \\ &\quad + \frac{(\gamma_{k_j-1} - \bar{\gamma}_{k_j})^2 - (\bar{\gamma}_{k_j-1} - \bar{\gamma}_{k_j})^2}{2} \\ &\quad + F_\omega(k_j - 1, \gamma_{k_j-1}) - F_\omega(k_j - 1, \bar{\gamma}_{k_j-1}). \end{aligned}$$

Combining the last two inequalities, we obtain

$$A^{k_j-1, n_2}(\gamma'(j)) \leq A^{k_j-1, n_2}(\gamma) + r(j),$$

where

$$\begin{aligned} r(j) &= -\frac{\delta}{2} + \left(\frac{\gamma_{k_j-1} + \bar{\gamma}_{k_j-1}}{2} - \bar{\gamma}_{k_j} \right) (\gamma_{k_j-1} - \bar{\gamma}_{k_j-1}) \\ &\quad + F_\omega(k_j - 1, \gamma_{k_j-1}) - F_\omega(k_j - 1, \bar{\gamma}_{k_j-1}). \end{aligned}$$

For sufficiently large j , $r(j) < 0$. This is implied by sublinear growth of $\gamma_{k_j-1}, \bar{\gamma}_{k_j-1}, \bar{\gamma}_{k_j}$ in k_j , the rate of decay of $|\gamma_{k_j-1} - \bar{\gamma}_{k_j-1}|$ guaranteed by Lemma 9.3, and Lemma 9.1.

So, $A^{k_j-1, n_2}(\gamma'(j)) < A^{k_j-1, n_2}(\gamma)$ which contradicts the fact that γ is an optimal path between points $((k_j - 1), \gamma_{k_j-1})$ and (n_2, x) . Therefore, our assumption on existence of y satisfying (9.11) was wrong, and the proof is complete. \square

To prove $U(n, \cdot) \in \mathbb{H}$ for all $n \in \mathbb{Z}$, we begin with the Lipschitz property.

Lemma 9.6. For all $n \in \mathbb{Z}$, $U_v(n, \cdot)$ is locally Lipschitz.

Proof: Let us fix $n \in \mathbb{N}$ and any points $z_1, z_2 \in \mathbb{R}$ satisfying $z_1 < z_2$. Let $\gamma^- = \gamma^-(n, z_1)$ and $\gamma^+ = \gamma^+(n, z_2)$. Then $\gamma_k^- < \gamma_k^+$ for all $k \leq n$ and there is a pairing sequence $k_j \downarrow -\infty$ such that

$$|\gamma_{k_j}^+ - \gamma_{k_j}^-| + |\gamma_{k_j-1}^+ - \gamma_{k_j-1}^-| < \frac{1}{n - k_j}, \quad j \in \mathbb{N}.$$

Let us now take any points $x, y \in (z_1, z_2)$ and denote $\gamma^1 = \gamma(n, x)$, $\gamma^2 = \gamma(n, y)$. These minimizers pass between γ^- and γ^+ , and, in particular, the same sequence (k_j) is pairing for γ^1 and γ^2 , so

$$|\gamma_{k_j}^2 - \gamma_{k_j}^1| + |\gamma_{k_j-1}^2 - \gamma_{k_j-1}^1| < \frac{1}{n - k_j}, \quad j \in \mathbb{N}.$$

As in the proof of Lemma 9.3, we obtain

$$\begin{aligned} A^{k_j-1, n}(\gamma^2) - A^{k_j-1, n}(\gamma^1) &\leq (\gamma_{k_j-1}^1 - \gamma_{k_j-1}^2) \left(\gamma_{k_j}^1 - \frac{\gamma_{k_j-1}^1 + \gamma_{k_j-1}^2}{2} \right) \\ &\quad + (x - y) \left(\gamma_{n-1}^1 - \frac{x + y}{2} \right) \\ &\quad + F_\omega(k_j - 1, \gamma_{k_j-1}^2) - F_\omega(k_j - 1, \gamma_{k_j-1}^1). \end{aligned}$$

Taking $j \rightarrow \infty$, we obtain

$$U_n(y) - U_n(x) \leq (x - y) \left(\gamma_{n-1}^1 - \frac{x + y}{2} \right).$$

Since $\gamma_{n-1}^1 \in (\gamma_{n-1}^-, \gamma_{n-1}^+)$ irrespective of the choice of $x, y \in (z_1, z_2)$, we conclude that there is $C_1(z_1, z_2)$ such that

$$U(n, y) - U(n, x) \leq C_1(z_1, z_2)|x - y|, \quad z_1 < x < y < z_2.$$

Similarly, for some $C_2(z_1, z_2)$, we obtain

$$U(n, x) - U(n, y) \leq C_2(z_1, z_2)|x - y|, \quad z_1 < x < y < z_2,$$

which completes the proof. \square

Although we have always assumed that $v = 0$ in this section, all the definitions, constructions and results hold true for other values v as well, due to the Galilean shear invariance. Let us denote the corresponding Busemann function and global HJBHLO solution by B_v and U_v .

To prove that $U_v(n, \cdot) \in \mathbb{H}(v, v)$ for all n , we will compute the expectation of its spatial increments (we already know that it is well defined due to part 5 of Lemma 9.4), and prove that $u_v(n, \cdot)$ is ergodic with respect to the spatial variable.

Lemma 9.7. For any $(n, x) \in \mathbb{Z} \times \mathbb{R}$,

$$\mathbb{E}(U_v(n, x + 1) - U_v(n, x)) = \mathbb{E}B_v((n, x), (n, x + 1)) = v.$$

Proof: First, we consider the case $v = 0$. Due to the distributional invariance of the potential process F under reflections,

$$EB_0((n, x), (n, x + 1)) = EB_0((n, x + 1), (n, x)).$$

Combining this with the anti-symmetry of B_0 , we obtain $EB_0((n, x + 1), (n, x)) = 0$, as required.

In the general case, we can apply the shear transformation L of $\mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ defined by

$$L : (m, y, q, r) \mapsto (m, y + (n - m)v, q, r).$$

Due to Lemma 4.1, the one-sided minimizers with slope v will be mapped onto one-sided minimizers of slope 0 for the new potential $F_{L(\omega)}$. We already know that

$$EB_0((n, x + 1), (n, x))_{L(\omega)} = 0.$$

A direct computation based on Lemma 4.1 gives

$$B_{0,L(\omega)}((n, x), (n, x + 1)) = B_{v,\omega}((n, x), (n, x + 1)) - v,$$

and our statement follows since L preserves the driving measure μ and hence the distribution of the Poisson process and the potential F . \square

So far we have worked with solutions of the Hamilton–Jacobi equation. One can obtain the corresponding solutions of the Burgers equation by

$$u_v(n, x) = \gamma_{v,n}(n, x) - \gamma_{v,n}(n - 1, x).$$

Then $U_v(n, x) - U_v(n, 0) = \int_0^x u_v(n, y) dy$. We recall that $\Psi^{m,n}w$ denotes the solution at time n of the Burgers equation with initial condition w imposed at time m .

Lemma 9.8. *The function u_v defined above is a global solution of the kicked Burgers equation. If $m \leq n$, then*

$$\Psi^{m,n}u_v(m, \cdot) = u_v(n, \cdot), \quad m \leq n.$$

Proof: This statement is a direct consequence of Lemma 2.1, Theorem 9.5, and the definition of the Burgers cocycle Ψ . \square

Lemma 9.9. *For any v and any n , the process $u_v(n, \cdot)$ is stationary and ergodic with respect to spatial translations.*

Proof: Denoting $u_v(n, 0)(\omega) = \xi(\omega)$, we see that due to space-time invariance of the procedure of constructing of one-sided minimizers, $u_v(n, x) = \xi(\tau_x\omega)$, where τ_x denotes the space shift of the Poisson process by distance x . Since the measure \mathbb{P} is invariant and ergodic with respect to spatial translations, we conclude that so is $u_v(n, \cdot)$. \square

Theorem 9.10. *For any $v \in \mathbb{R}$ and any $n \in \mathbb{N}$, we have $U_{v,\omega}(n, \cdot) \in \mathbb{H}(v, v)$. The sequence $u_{v,\omega}(n, \cdot), n \in \mathbb{N}$ is a stationary process with values in $\mathbb{G}(v, v)$.*

Proof: The first claim is a direct consequence of Lemmas 9.7 and 9.9, and Birkhoff’s ergodic theorem. The second claim follows from the first one and the space-time invariance of the construction of minimizers. \square

10 Stationary solutions: uniqueness and basins of attraction

In this section we prove Theorem 3.2 and the uniqueness part in Theorem 3.1. The key step is the following observation.

Lemma 10.1. *Let $n \in \mathbb{Z}$ and suppose that an initial condition W satisfies one of the conditions (3.1),(3.2),(3.3). With probability one, the following holds true for every $y \in \mathbb{R}$. Let $y^*(m)$ be a solution of the optimization problem (9.10) for each $m \leq n$. Then*

$$\lim_{m \rightarrow -\infty} \frac{y^*(m)}{m} = v.$$

Proof of Theorem 3.2: Since u_v is a solution of the Burgers equation over any finite time interval, $u_v(n, \cdot)$ is continuous at y iff there is a unique one-sided minimizer $\gamma(n, y)$ with endpoint (n, y) . Moreover, in this case, $u_v(n, y) = \gamma_n(n, y) - \gamma_{n-1}(n, y)$.

So let us take such a point y . Lemma 10.1 and Lemma 7.7 guarantee then that solutions $y^*(m)$ for optimization problem (9.10) and the corresponding optimal paths $\gamma^{(m, y^*(m)), (n, y)}$ realizing $A^{m, n}(y^*(m), y)$ converge pointwise to the infinite one-sided minimizer $\gamma(n, x)$. In particular,

$$\begin{aligned} \Psi^{m, n} w(y) &= \gamma_n^{(m, y^*(m)), (n, y)} - \gamma_{n-1}^{(m, y^*(m)), (n, y)} \\ &\rightarrow \gamma_n(n, y) - \gamma_{n-1}(n, y) = u_v(n, y), \quad m \rightarrow -\infty, \end{aligned}$$

which completes the proof. □

Proof of Lemma 10.1: We will only prove the sufficiency of condition (3.1). The proof of sufficiency of conditions (3.2) and (3.3) follows the same lines and we omit it.

Let us also restrict ourselves to $n = 0$ for simplicity. The proof does not change for other values of n .

Since y^* is increasing in y , it is sufficient to show that the conclusion of the lemma holds with probability 1 for fixed y . The stationarity of the forcing potential implies that we can assume $y = 0$.

We must show that for any $\varepsilon > 0$ it is extremely unlikely for a path γ with $\gamma_0 = 0$ and $|\gamma_m| > \varepsilon|m|$ to provide a solution to (9.10) if $|m|$ is large. For definiteness, let us work with paths satisfying $0^*(m) = \gamma_m > \varepsilon|m|$.

For any $\delta > 0$ and for sufficiently large $|m|$,

$$W(0) + A^{m, 0}(0, 0) < (\alpha(0) + \delta)|m|.$$

If additionally $x = 0^*(m) > \varepsilon|m|$, then for $i = \lfloor x - \varepsilon|m| \rfloor$,

$$\inf_{z \in [\varepsilon|m| + i, \varepsilon|m| + i + 1]} W(z) + A^{m-1, 0}(\varepsilon|m| + i, 0) < (\alpha(0) + \delta)|m| + \frac{1}{2} + F(m-1, \varepsilon|m| + i).$$

Condition (3.1) at $+\infty$ implies that there is r_1 such that for $m < -r_1$ and all $i \in \mathbb{N}$,

$$\inf_{z \in [\varepsilon|m| + i, \varepsilon|m| + i + 1]} W(z) > -(|m| + i)\delta,$$

so there is $r_2 > 0$ such that if $m < -r_2$ and $i \in \mathbb{N}$, then

$$\begin{aligned} A^{m-1, 0}(\varepsilon|m| + i, 0) - F(m-1, \varepsilon|m| + i) &< (\alpha(0) + 2\delta)|m| + \delta i + \frac{1}{2} \\ &< |m| \left(\alpha(0) + 3\delta + \delta \frac{i}{|m|} \right). \end{aligned}$$

Let us denote by B_{mi} the event defined by this inequality. Then $B_{mi} \subset C_{mi} \cup D_{mi}$, where

$$\begin{aligned} C_{mi} &= \left\{ -F(m-1, \varepsilon|m| + i) < -\delta(|m| + i) \right\} \\ D_{mi} &= \left\{ A^{m-1, 0}(\varepsilon|m| + i, 0) < |m| \left(\alpha(0) + 4\delta + 2\delta \frac{i}{|m|} \right) \right\} \end{aligned}$$

Due to the Borel–Cantelli lemma, to show that with probability 1, events B_{mi} can happen only for finitely many values of m , it suffices to show that for some $\beta > 0$ and $c > 0$,

$$\sum_{m \leq -c} \sum_i P(C_{mi}) < \infty, \tag{10.1}$$

$$\sum_{m \leq -c} \sum_{i \leq \beta|m|} P(D_{mi}) < \infty, \tag{10.2}$$

and

$$\sum_{m \leq -c} \sum_{i > \beta|m|} P(D_{mi}) < \infty. \tag{10.3}$$

Inequality (10.1) does not depend on β and holds for any c since

$$P(C_{mi}) \leq e^{-\delta(|m|+i)} E e^{F(0,0)}.$$

Denoting $\alpha_{mi} = \alpha\left(\frac{\varepsilon|m|+i}{|m|+1}\right)$, using shear and translation invariance, we obtain

$$P(D_{mi}) = P\left\{A^{|m|+1} - \alpha(0)(|m| + 1) < |m|\left(\alpha(0) + 4\delta + \frac{2\delta i}{|m|} - \frac{|m| + 1}{|m|} \alpha_{mi}\right)\right\}.$$

If δ is sufficiently small, then, using Lemma 4.7, we can find r_2 such that for all $m < -r_2$ and all i ,

$$|m|\left(\alpha(0) + 4\delta + \frac{2\delta i}{|m|} - \frac{|m| + 1}{|m|} \alpha_{mi}\right) < -(|m| + 1)\left(\frac{\varepsilon^2}{2} + \frac{i^2}{2|m|^2}\right),$$

so

$$P(D_{mi}) \leq P\left\{A^{|m|+1} - \alpha(0)(|m| + 1) < -(|m| + 1)\left(\frac{\varepsilon^2}{2} + \frac{i^2}{2|m|^2}\right)\right\}. \tag{10.4}$$

Now (10.2) follows (with any $c \geq r_2$ and with arbitrary choice of β) from Theorem 5.1.

To prove (10.3) we need an auxiliary lemma. In its statement and proof we use the notation introduced in Section 4.

Lemma 10.2. *There are constants $c_1, c_2, X_0, N_0 > 0$ such that for $n > N_0, x > X_0$,*

$$P\{A^n \leq -xn\} \leq c_1 e^{-c_2 xn}.$$

Now (10.3) is a consequence of this lemma and (10.4) if we choose $c > N_0$ and β satisfying

$$\frac{\varepsilon^2}{2} + \frac{\beta^2}{2} - \alpha(0) > X_0.$$

It remains to prove Lemma 10.2.

Proof of Lemma 10.2: Let us take $c_3 > 0$ and write

$$P\{A^n \leq -xn\} \leq P\{\Sigma \leq c_3 xn, A^n \leq -xn\} + P\{\Sigma > c_3 xn\}, \tag{10.5}$$

where $\Sigma = \Sigma(\gamma)$ has been defined in (4.4) for a path γ realizing A^n .

To estimate the first term on the right-hand side, we choose $c_3 > 0$ and $x_0 > 0$ so that for all $x > x_0$ and all $n \in \mathbb{N}$, $\lceil c_3 xn \rceil > n$ and $xn > y_0 \lceil c_3 xn \rceil$, where y_0 was introduced before inequality (4.10). We can now apply that inequality to conclude that for some constants $c_4 > 0$,

$$P\{\Sigma \leq c_3 xn, A^n \leq -xn\} \leq P\{F_{\omega, n, \lceil c_3 xn \rceil}^* \geq xn\} \leq e^{-c_4 xn}.$$

The second term on the right-hand side of (10.5) can be estimated using Lemma 4.3. If $c_3x \geq R$ and n is sufficiently large, then

$$\begin{aligned} \mathbb{P}\{\Sigma > c_3xn\} &\leq \sum_{m \geq c_3xn} \mathbb{P}(E_{n,m}) \\ &\leq \sum_{m \geq c_3xn} C_1 \exp(-C_2m^2/n) \\ &\leq C'_1 \exp(-C'_2x^2n), \end{aligned}$$

for some constants $C'_1, C'_2 > 0$, which completes the proof. □

Proof of uniqueness in Theorem 3.1: Let $w_\omega = W'_\omega$ be a global solution of the Burgers equation such that for each $n \in \mathbb{Z}$, $w(n, \cdot) \in \mathbb{H}'(v, v)$ with probability 1. Then, for any $n \leq N$, $w(0, \cdot) = \Psi_\omega^{-n,0}w(-n, \cdot)$. The cadlag version of w belongs to $\mathbb{G}(v, v)$. For any $x \in \mathbb{R}$, the trajectory solving the Euler–Lagrange equation and terminating at x with velocity $w(x)$ is a minimizer on every finite interval. Therefore it is a one-sided minimizer and must have an asymptotic slope $\tilde{v}(x)$. Notice that $\tilde{v}(x)$ is monotone in x , since on any finite interval the minimizers cannot intersect. Due to spatial translation invariance, $\tilde{v}(x)$ is a stationary process in x , so $\tilde{v}(x) = \tilde{v}$ has to be a constant. Hence, $w_\omega(0, \cdot)$ almost surely coincides with $u_{\tilde{v}}(0, \cdot)$. Since the latter belongs to $\mathbb{G}(\tilde{v}, \tilde{v})$ almost surely, we see that $\tilde{v} = v$, so $w_\omega(0, \cdot)$ almost surely coincides with $u_v(0, \cdot)$, which completes the proof. □

11 Metric on \mathbb{G}

Let us recall that \mathbb{G} consists of all cadlag functions $w : \mathbb{R} \rightarrow \mathbb{R}$ such that $M_w : \mathbb{R} \rightarrow \mathbb{R}$ defined by $M_w(x) = x - w(x)$ is a strictly increasing function satisfying $\lim_{x \rightarrow \pm\infty} M_w(x) = \pm\infty$. The goal of this section is to introduce a metric d on \mathbb{G} such that $\lim_{n \rightarrow \infty} d(w_n, w) = 0$ is equivalent to $\lim_{n \rightarrow \infty} w_n(x) = w(x)$ for all $x \in \mathcal{C}(w)$. First we note that for every w , the inverse M_w^{-1} of M_w defined by

$$M_w^{-1}(y) = \inf\{x : M_w(x) \geq y\}, \quad x \in \mathbb{R},$$

is a continuous function retaining all the information about M_w and w . Let us define d as the metric of locally uniform convergence on continuous functions on \mathbb{R} :

$$d(u, w) = \sum_{N=1}^{\infty} 2^{-N} d_N(u, w), \quad u, w \in \mathbb{G},$$

where

$$d_N(u, w) = \sup_{x \in [-N, N]} |M_u^{-1}(x) - M_w^{-1}(x)| \wedge 1, \quad u, w \in \mathbb{G}.$$

Lemma 11.1. *Let $(w_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{G} and $w \in \mathbb{G}$. Then $d(w_n, w) \rightarrow 0$ as $n \rightarrow \infty$, iff $\lim_{n \rightarrow \infty} w_n(x) = w(x)$ for all $x \in \mathcal{C}(w)$.*

Proof: Suppose $d(w_n, w) \rightarrow 0$ as $n \rightarrow \infty$. We need to prove $M_{w_n}(x) \rightarrow M_w(x)$ for each $x \in \mathcal{C}(w) = \mathcal{C}(M_w)$. So let us take such an x and any $\varepsilon > 0$. We can find $x^- < x$ and $x^+ > x$ such that

$$M_w(x) - \varepsilon < M_w(x^-) < M_w(x) < M_w(x^+) < M_w(x) + \varepsilon.$$

Let us denote $y^- = M_w(x^-)$, $y = M_w(x)$, $y^+ = M_w(x^+)$. Since $d(w_n, w) \rightarrow 0$, we see that there is n_0 such that for $n > n_0$, $M_{w_n}^{-1}(y^-) < (x^- + x)/2$ and $M_{w_n}^{-1}(y^+) > (x + x^+)/2$. Therefore, for $n > n_0$, $y^- < M_{w_n}(x) < y^+$, so $M_w(x) - \varepsilon < M_{w_n}(x) < M_w(x) + \varepsilon$, and our claim follows.

Now let us assume that $\lim_{n \rightarrow \infty} w_n(x) = w(x)$ for all $x \in \mathcal{C}(w)$. We need to show that for any $N \in \mathbb{N}$, $d_N(w_n, w) \rightarrow 0$. First, we can find points $z^-, z^+ \in \mathcal{C}(w)$ such that $M_w(z^-) < -N$ and $M_w(z^+) > N$. There is n_0 such that $[-N, N] \subset (M_{w_n}(z^-), M_{w_n}(z^+))$ for all $n > n_0$.

For any $\varepsilon > 0$ let us find a finite collection of points $x_0, x_1, \dots, x_m \in \mathcal{C}(w)$ such that $z^- = x_0 < x_1 < \dots < x_m = z^+$ and $x_k - x_{k-1} < \varepsilon/2$ for all $k = 1, \dots, m$. Let us denote $y_k = M_w(x_k)$, $k = 0, 1, \dots, m$. Points $y_k = M_w(x_k)$, $k = 0, 1, \dots, m$ form a strictly increasing sequence. Let us denote

$$\Delta = \min_{k=1, \dots, m} (y_k - y_{k-1}) \wedge (-N - y_0) \wedge (y_m - N) > 0.$$

Since $x_k \in \mathcal{C}(w)$, we can find $n_1 \geq n_0$ such that for all $n > n_1$ and all $k = 0, \dots, m$, $|M_{w_n}(x_k) - y_k| < \Delta/2$. So, if $y \in [y_{k-1}, y_k] \cap [-N, N]$, then $M_{w_n}^{-1}(y) \in [x_{(k-2) \vee 0}, x_{(k+1) \wedge m}]$ and $M_w^{-1}(y) \in [x_{k-1}, x_k]$. Hence, $|M_{w_n}^{-1}(y) - M_w^{-1}(y)| \leq \varepsilon$. Since $[-N, N] \subset \bigcup_{k=1}^m [y_{k-1}, y_k]$, we obtain $d_N(w_n, w) \leq \varepsilon$ for $n \geq n_1$. This is the desired uniform estimate. \square

12 Auxiliary lemmas

Proof of Lemma 2.3: The local Lipschitzness of $\Phi_\omega^{n_0, n_1} W$ is part 2 of Lemma 2.1, so let us establish the behavior as $x \rightarrow \infty$. Let us begin with the second part of the Lemma.

Due to the cocycle property it is sufficient to consider the situation where $n_1 = n_0 + 1$. Let us take $W \in \mathbb{H}(v_-, v_+)$. We have

$$\Phi_\omega^{n_0, n_0+1} W(x) \leq W(x) + F_\omega(n_0, x), \quad x \in \mathbb{R}.$$

Since $\lim_{x \rightarrow \infty} (F_\omega(n_0, x)/x) = 0$ on Ω_1 , we see that

$$\limsup_{x \rightarrow +\infty} \frac{\Phi_\omega^{n_0, n_0+1} W(x)}{x} \leq v_+.$$

Let us prove that

$$\liminf_{x \rightarrow +\infty} \frac{\Phi_\omega^{n_0, n_0+1} W(x)}{x} \geq v_+.$$

If this inequality is violated, then there is $\varepsilon > 0$ and two increasing sequences $(x_k)_{k \in \mathbb{N}}$, $(y_k)_{k \in \mathbb{N}}$ such that $x_k \rightarrow +\infty$ and

$$V(y_k) + \frac{1}{2}(x_k - y_k)^2 < (v_+ - \varepsilon)x_k, \quad k \in \mathbb{N}, \tag{12.1}$$

where $V(y) = W(y) + F(y)$, $y \in \mathbb{R}$. Inequality (12.1) implies that y_k cannot be bounded, so we obtain $y_k \rightarrow +\infty$. Moreover, it follows from (12.1) and from $\lim_{y \rightarrow \infty} V(y)/y = v_+$ that for sufficiently large k ,

$$v_+ y_k < (v_+ - \varepsilon/2)x_k,$$

so y_k/x_k cannot converge to 1. Therefore, there is a subsequence (k') and a constant $c > 0$ such that $|y_{k'} - x_{k'}| > c(y_{k'} + x_{k'})$, so

$$\begin{aligned} V(y_{k'}) &\leq -\frac{1}{2}(x_{k'} - y_{k'})^2 + (v_+ - \varepsilon)x_{k'} \\ &\leq -\frac{c^2 y_{k'}^2}{2} - \frac{c^2 x_{k'}^2}{2} + (v_+ - \varepsilon)x_{k'} \\ &\leq -\frac{c^2 y_{k'}^2}{2}, \end{aligned}$$

for sufficiently large k' which contradicts the linear growth of V . A similar analysis applies to the behavior at $-\infty$, and the second part of the Lemma is proved completely.

To prove the first part, assume that $W \in \mathbb{H}$, but $\Phi_\omega^{n_0, n_0+1} W \notin \mathbb{H}$ due to the behavior, say, at $+\infty$. Then there are two increasing sequences $(x_k)_{k \in \mathbb{N}}$, $(y_k)_{k \in \mathbb{N}}$ such that $x_k \rightarrow +\infty$ and

$$\frac{V(y_k) + \frac{1}{2}(x_k - y_k)^2}{x_k} \rightarrow -\infty, \quad k \rightarrow \infty. \tag{12.2}$$

This means that $V(y_k)/x_k \rightarrow -\infty$. Since $V(y_k)/y_k$ is bounded below for large k , we conclude that $x_k/y_k \rightarrow 0$. Now (12.2) implies that $V(y_k) \leq -y_k^2/4$ for large k , which contradicts $W \in \mathbb{H}$. Similar reasoning applies to the behavior near $-\infty$. \square

Lemma 12.1. *A pointwise limit of a sequence of point-to-point minimizers is a point-to-point minimizer.*

Proof: Suppose $(\gamma^k)_{k \in \mathbb{N}}$ is a sequence of point-to-point minimizers on a time interval $\{n_1, \dots, n_2\}$. Suppose that γ is a path such that $\gamma_j^k \rightarrow \gamma_j$ as $k \rightarrow \infty$ for $j \in \{n_1, \dots, n_2\}$. If γ is not a point-to-point minimizer, then there is a path β satisfying $\beta_{n_1} = \gamma_{n_1}$, $\beta_{n_2} = \gamma_{n_2}$, $A^{n_1, n_2}(\beta) < A^{n_1, n_2}(\gamma)$. Let us introduce paths $\beta^k = (\gamma_{n_1}^k, \beta_{n_1+1}, \dots, \beta_{n_2-1}, \gamma_{n_2}^k)$. Since $A^{n_1, n_2}(\gamma^k) \rightarrow A^{n_1, n_2}(\gamma)$ and $A^{n_1, n_2}(\beta^k) \rightarrow A^{n_1, n_2}(\beta)$, we obtain $A^{n_1, n_2}(\beta^k) < A^{n_1, n_2}(\gamma^k)$ for sufficiently large k which contradicts the minimizing property of γ^k . \square

Lemma 12.2. *Let $\omega \in \Omega$ and $(n, x) \in \mathbb{Z} \times \mathbb{R}$. If γ^1 and γ^2 are two distinct point-to-point minimizers on $\{n, \dots, n'\}$ satisfying $\gamma_n^1 = \gamma_n^2 = x$, then, as curves in $\mathbb{R} \times \mathbb{R}$, they do not intersect on time interval (n, n') .*

Proof of Lemma 12.2: If γ^1 and γ^2 have two consecutive points in common, they coincide due to the Euler-Lagrange equation.

Suppose $\gamma_m^1 = \gamma_m^2 = y$ and $\gamma_{m-1}^1 < \gamma_{m-1}^2$ for some $m \in \{n+1, \dots, n'-1\}$. By the Euler-Lagrange equation, $\gamma_{m+1}^1 > \gamma_{m+1}^2$. For $\delta > 0$, we define

$$\begin{aligned} \tilde{\gamma}^1 &= (x, \gamma_{n+1}^1, \dots, \gamma_{m-1}^1, y - \delta, \gamma_{m+1}^2), \\ \tilde{\gamma}^2 &= (x, \gamma_{n+1}^2, \dots, \gamma_{m-1}^2, y + \delta, \gamma_{m+1}^1). \end{aligned}$$

Then

$$\begin{aligned} &A^{n, m+1}(\tilde{\gamma}^1) + A^{n, m+1}(\tilde{\gamma}^2) - (A^{n, m+1}(\gamma^1) + A^{n, m+1}(\gamma^2)) \\ &= F(m, y + \delta) + F(m, y - \delta) - 2F(m, y) \\ &\quad + \frac{1}{2} \left((\gamma_{m+1}^1 - y - \delta)^2 + (\gamma_{m+1}^2 - y + \delta)^2 + (y + \delta - \gamma_{m-1}^2)^2 + (y - \delta - \gamma_{m-1}^1)^2 \right. \\ &\quad \left. - (\gamma_{m+1}^1 - y)^2 - (\gamma_{m+1}^2 - y)^2 - (y - \gamma_{m-1}^2)^2 - (y - \gamma_{m-1}^1)^2 \right) \\ &= F(m, y + \delta) + F(m, y - \delta) - 2F(m, y) + 2\delta^2 + \delta(\gamma_{m+1}^2 - \gamma_{m+1}^1 + \gamma_{m-1}^1 - \gamma_{m-1}^2). \end{aligned}$$

Since $\delta(\gamma_{m+1}^2 - \gamma_{m+1}^1 + \gamma_{m-1}^1 - \gamma_{m-1}^2) < 0$ and the sum of the remaining terms is $o(\delta)$, $\delta \rightarrow 0$, due to differentiability of F , we obtain that for sufficiently small δ ,

$$A^{n, m+1}(\tilde{\gamma}^1) + A^{n, m+1}(\tilde{\gamma}^2) < A^{n, m+1}(\gamma^1) + A^{n, m+1}(\gamma^2),$$

so either $A^{n, m+1}(\tilde{\gamma}^1) < A^{n, m+1}(\gamma^2)$ or $A^{n, m+1}(\tilde{\gamma}^2) < A^{n, m+1}(\gamma^1)$ which contradicts the minimizing property of γ^1, γ^2 .

It remains to exclude the case where $\gamma_m^1 < \gamma_m^2$ and $\gamma_{m+1}^1 > \gamma_{m+1}^2$ for some $m \in \{n+1, \dots, n-1\}$. In this case, we define

$$\begin{aligned} \tilde{\gamma}^1 &= (x, \gamma_{n+1}^1, \dots, \gamma_m^1, \gamma_{m+1}^2), \\ \tilde{\gamma}^2 &= (x, \gamma_{n+1}^2, \dots, \gamma_m^2, \gamma_{m+1}^1). \end{aligned}$$

Then

$$\begin{aligned} & A^{n,m+1}(\tilde{\gamma}^1) + A^{n,m+1}(\tilde{\gamma}^2) - (A^{n,m+1}(\gamma^1) + A^{n,m+1}(\gamma^2)) \\ &= \frac{1}{2} \left((\gamma_{m+1}^2 - \gamma_m^1)^2 + (\gamma_{m+1}^1 - \gamma_m^2)^2 - (\gamma_{m+1}^1 - \gamma_m^1)^2 - (\gamma_{m+1}^2 - \gamma_m^2)^2 \right) \\ &= (\gamma_m^2 - \gamma_m^1)(\gamma_{m+1}^2 - \gamma_{m+1}^1) < 0, \end{aligned}$$

so, as above, at least one of the paths γ^1, γ^2 is not optimal. This contradiction completes the proof of the lemma. \square

References

- [1] Ludwig Arnold, *Random dynamical systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR-1723992
- [2] Yuri Bakhtin, *Burgers equation with random boundary conditions*, Proc. Amer. Math. Soc. **135** (2007), no. 7, 2257–2262 (electronic). MR-2299503
- [3] Yuri Bakhtin, *The Burgers equation with Poisson random forcing*, Ann. Probab. **41** (2013), no. 4, 2961–2989. MR-3112935
- [4] Yuri Bakhtin, Eric Cator, and Konstantin Khanin, *Space-time stationary solutions for the Burgers equation*, J. Amer. Math. Soc. **27** (2014), no. 1, 193–238. MR-3110798
- [5] Jérémie Bec and Konstantin Khanin, *Burgers turbulence*, Phys. Rep. **447** (2007), no. 1-2, 1–66. MR-2318582
- [6] Alexandre Boritchev and Konstantin Khanin, *On the hyperbolicity of minimizers for 1D random Lagrangian systems*, Nonlinearity **26** (2013), no. 1, 65–80. MR-3001762
- [7] Eric Cator and Leandro P. R. Pimentel, *Busemann functions and equilibrium measures in last passage percolation models*, Probab. Theory Related Fields **154** (2012), no. 1-2, 89–125. MR-2981418
- [8] Eric Cator and Leandro P.R. Pimentel, *A shape theorem and semi-infinite geodesics for the Hammersley model with random weights*, ALEA **8** (2011), 163–175. MR-2783936
- [9] Michael Damron and Jack Hanson, *Busemann functions and infinite geodesics in two-dimensional first-passage percolation*, Comm. Math. Phys. **325** (2014), no. 3, 917–963. MR-3152744
- [10] Weinan E, K. Khanin, A. Mazel, and Ya. Sinai, *Invariant measures for Burgers equation with stochastic forcing*, Ann. of Math. (2) **151** (2000), no. 3, 877–960. MR-1779561
- [11] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998. MR-1625845
- [12] N. Georgiou, F. Rassoul-Agha, and T. Seppäläinen, *Variational formulas and cocycle solutions for directed polymer and percolation models*, ArXiv e-prints, arXiv:1311.3016 (2013).
- [13] N. Georgiou, F. Rassoul-Agha, and T. Seppäläinen, *Stationary cocycles and Busemann functions for the corner growth model*, ArXiv e-prints, arXiv:1510.00859 (2015).
- [14] Diogo Gomes, Renato Iturriaga, Konstantin Khanin, and Pablo Padilla, *Viscosity limit of stationary distributions for the random forced Burgers equation*, Mosc. Math. J. **5** (2005), no. 3, 613–631, 743. MR-2241814
- [15] Viet Ha Hoang and Konstantin Khanin, *Random Burgers equation and Lagrangian systems in non-compact domains*, Nonlinearity **16** (2003), no. 3, 819–842. MR-1975784
- [16] Wassily Hoeffding, *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. **58** (1963), 13–30. MR-0144363
- [17] C. Douglas Howard and Charles M. Newman, *Euclidean models of first-passage percolation*, Probability Theory and Related Fields **108** (1997), 153–170, 10.1007/s004400050105. MR-1452554
- [18] C. Douglas Howard and Charles M. Newman, *From greedy lattice animals to Euclidean first-passage percolation*, Perplexing problems in probability, Progr. Probab., vol. 44, Birkhäuser Boston, Boston, MA, 1999, pp. 107–119. MR-1703127

- [19] C. Douglas Howard and Charles M. Newman, *Geodesics and spanning trees for Euclidean first-passage percolation*, Ann. Probab. **29** (2001), no. 2, 577–623. MR-1849171
- [20] R. Iturriaga and K. Khanin, *Burgers turbulence and random Lagrangian systems*, Comm. Math. Phys. **232** (2003), no. 3, 377–428. MR-1952472
- [21] Harry Kesten, *On the speed of convergence in first-passage percolation*, Ann. Appl. Probab. **3** (1993), no. 2, 296–338. MR-1221154
- [22] Sergei Kuksin, Andrey Piatnitski, and Armen Shirikyan, *A coupling approach to randomly forced nonlinear PDEs. II*, Comm. Math. Phys. **230** (2002), no. 1, 81–85. MR-1927233
- [23] Sergei Kuksin and Armen Shirikyan, *A coupling approach to randomly forced nonlinear PDE's. I*, Comm. Math. Phys. **221** (2001), no. 2, 351–366. MR-1845328
- [24] Sergei B. Kuksin, *Randomly forced nonlinear PDEs and statistical hydrodynamics in 2 space dimensions*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2006. MR-2225710
- [25] Charles M. Newman, *A surface view of first-passage percolation*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 1017–1023. MR-1404001
- [26] F. Rassoul-Agha, T. Seppäläinen, and A. Yilmaz, *Variational formulas and disorder regimes of random walks in random potentials*, ArXiv e-prints, arXiv:1410.4474 (2014). MR-3176363
- [27] Toufic M. Suidan, *Stationary measures for a randomly forced Burgers equation*, Comm. Pure Appl. Math. **58** (2005), no. 5, 620–638. MR-2141893
- [28] Cédric Villani, *Optimal transport. old and new*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009. MR-2459454
- [29] Mario V. Wüthrich, *Asymptotic behaviour of semi-infinite geodesics for maximal increasing subsequences in the plane*, In and out of equilibrium (Mambucaba, 2000), Progr. Probab., vol. 51, Birkhäuser Boston, Boston, MA, 2002, pp. 205–226. MR-1901954