Regularity structures and renormalisation of FitzHugh–Nagumo SPDEs in three space dimensions

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Abstract

We prove local existence of solutions for a class of suitably renormalised coupled SPDE–ODE systems driven by space-time white noise, where the space dimension is equal to 2 or 3. This class includes in particular the FitzHugh–Nagumo system describing the evolution of action potentials of a large population of neurons, as well as models with multidimensional gating variables. The proof relies on the theory of regularity structures recently developed by M. Hairer, which is extended to include situations with semigroups that are not regularising in space. We also provide explicit expressions for the renormalisation constants, for a large class of cubic nonlinearities.

1 Introduction

Many spatially extended systems can be described by a reaction-diffusion equation coupled to an ordinary differential equation, of the form

$$\partial_t u = \mathfrak{D}\Delta_x u + F(u,v) \,,$$
$$\partial_t v = G(u,v) \,, \tag{1.1}$$

where $u = u(t,x)$ and $v = v(t,x)$ are functions of time $t \geqslant 0$ and space $x \in \mathbb{R}^d$ and take values in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, Δ_x denotes the usual Laplacian, $\mathfrak{D} \in \mathbb{R}^{m \times m}$ is a positive definite matrix of diffusion coefficients, and $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ are given sufficiently smooth functions. A well-studied situation of this kind arises in neuroscience, where $x \in \mathbb{R}$ measures the position along a neuron's

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axon, $u(t, x) \in \mathbb{R}$ is the membrane potential at time t and position x, and the so-called gating variables $v \in \mathbb{R}^n$ describe the number of open ion channels of n different types. Classical examples are the Hodgkin–Huxley model [25] and its simplifications such as the Morris–Lecar model [34] and the FitzHugh–Nagumo model [16, 37]. However, there are many more interesting situations outside neuroscience that can be described by coupled systems of the form (1.1), for instance in chemical kinetics [44], in population dynamics [8], and in pattern formation [2]. Furthermore, many other classical reaction-diffusion equations from mathematical biology, such as the Keller–Segel [26] or the Gray–Scott model [17] are of the form (1.1) if one regards the model parameters v not as constants via $\partial_t v = 0$ but as (slowly-)changing parameters via $\partial_t v = \tilde{\varepsilon}\widetilde{G}(u, v)$ for some small parameter $0 < \tilde{\varepsilon} \ll 1$ [29] and some smooth vector field $\widetilde{G}$. In fact, the number of applications of systems of the form (1.1) goes far beyond the areas we mentioned here.

However, in some cases, a deterministic model of the form (1.1) is not sufficient to describe the dynamics, and one has to add a stochastic term to capture the effect of thermal fluctuations and other unresolved dynamical processes affecting the system. In the case of neuron dynamics, see for instance the book and recent survey by Tuckwell [40, 42] and the recent reviews by Bressloff [6, 7]. In particular, stochastic versions of the FitzHugh-Nagumo PDE have attracted considerable recent attention [41, 43, 38] as well as related models of stochastic neural fields [30, 28]. In the derivation of these models and in related numerical studies, one frequently uses correlated [32] as well as space-time white noise [1] stochastic forcing.

Before studying properties of solutions to a stochastic partial differential equation (SPDE) with space-time white noise, one has to ensure that such an equation is well-defined, i.e., one has to attach a meaning of "solution" to a given SPDE. Depending on the space dimension d, this problem may in fact be extremely hard. Consider the case of a one-component system ($m = 1$, $n = 0$, $\mathfrak{D} \equiv 1$) of the form

$$\partial_t u = \Delta_x u + F(u) + \xi \,, \tag{1.2}$$

where ξ denotes space-time white noise (precise mathematical definitions will be given below). If $d = 1$, it is possible to define a notion of mild solution via the Duhamel principle in a quite general setting (see for instance [12]). In higher space dimension, the problem is much more difficult, owing to the fact that space-time white noise is extremely singular.

Consider for instance the case of the Allen–Cahn equation given by (1.2) with $F(u) = au - u^3$ (if $a = 0$ this is also known in quantum field theory as the dynamical Φ_d^4 model, cf [14]). In the case $d = 1$, the proof of existence of a unique solution goes back to Faris and Jona-Lasinio [13], and many quantitative properties of this solution are known (see for instance [9, 4, 3]). The two-dimensional case was solved by Da Prato and Debussche using a particular class of Besov spaces [11]. The case $d = 3$, however, was only solved very recently by Hairer, using his theory of regularity structures [19]. One of the difficulties is that a renormalisation procedure has to be used to properly define solutions. This is achieved by replacing space-time white noise ξ by a mollified version ξ^ε, solving the resulting regularised equation, and passing to the limit of vanishing regularisation (for an alternative approach, see [31]).

The theory of regularity structures provides a framework to study SPDEs with very singular noise, including but not limited to parabolic equations of the form (1.2). The basic idea is to construct an abstract space in which one can define algebraic operations on distributions (in the sense of generalised functions), such as multiplication, composition with a smooth function, and convolution with a kernel. The fixed-point equation obtained by applying Duhamel's principle to the SPDE with mollified noise is then lifted to the abstract space, solved in that space, and finally projected down to a distribution

in "physical" space. In addition, the theory allows to incorporate the renormalisation procedure that is needed in most cases; see for instance [20, 10] for introductions. Furthermore, there are already several recent applications of regularity structures, e.g., to large deviation theory of the Allen-Cahn equation [24], to the dynamical sine-Gordon model [23], to Wong-Zakai approximation of nonlinear parabolic SPDEs [22], to the parabolic Anderson model on bounded [19] as well as unbounded [21] domains, and to the KPZ equation [19, 18].

The purpose of the present work is to extend the theory of regularity structures to multicomponent systems of the form

$$
\begin{aligned}
\partial_t u &= \Delta_x u + F(u, v) + \xi \,, \\
\partial_t v &= G(u, v) \,.
\end{aligned}
\tag{1.3}
$$

The main result is the proof of existence, in space dimensions $d = 2$ and 3, of local solutions to the system (1.3) when F is a cubic polynomial and G is linear. This includes the case of the standard FitzHugh–Nagumo model, but also other equations such as the Koper model [27] with diffusion in the fast component, which features vectorial variables v. We will mostly focus on the case $d = 3$, first because this is the physically relevant case in many applications, but also because it is the technically more challenging case (the Allen–Cahn equation is known not to be renormalisable for $d \geqslant 4$, so the same will hold a *fortiori* for multicomponent systems with cubic nonlinearities).

The main technical difficulty that has to be overcome to obtain these results comes from the fact that the semigroup associated with the second equation in (1.3) is not at all regularising in space. Though one expects that this loss of regularisation is somehow compensated by the fact that no singular noise term acts on the equation for v, the general theory in [19] cannot be applied directly, because it uses in an essential way the assumption that the heat kernel is smooth everywhere except at the origin. Therefore, we have to extend the regularity structure constructed for the Allen–Cahn equation with new abstract symbols representing integration with respect to a singular kernel, and to prove suitable bounds involving these symbols. Furthermore, one has to analyse which role new symbols play in the renormalisation. We find that certain terms involving the singular kernel do not have to be renormalised, while on the other hand, general cubic nonlinearities yield renormalisation terms which do not appear in the Allen–Cahn case.

The remainder of this work is organised as follows. Section 2 contains all main local existence results. Section 3 gives a summary of the most important aspects of the theory of regularity structures contained in [19], illustrated in the case of the Allen–Cahn equation. In Section 4, we present our results allowing to represent the operation of integration with respect to a singular kernel. Section 5 contains the fixed-point argument proving local existence and uniqueness of solutions for the SPDE with mollified noise, and Section 6 deals with the renormalisation procedure which is necessary to pass to the limit of vanishing mollification. Finally, in Section 7 we complete the proofs of the main results.

Notations: We write $|x|$ to denote either the absolute value of $x \in \mathbb{R}$ or the ℓ^1-norm of $x \in \mathbb{R}^d$, while $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. If $a, b \in \mathbb{R}$ we use $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$, while $\lfloor a \rfloor$ denotes the largest integer less or equal a. If f, g are two real-valued functions depending on small parameters $\varepsilon, \delta, \ldots$ (which will be clear from the context), we write $f \lesssim g$ to indicate that there exists a constant $C > 0$ such that $f \leqslant Cg$ holds uniformly in the small parameters. We use the notation $f \asymp g$ to indicate that we have both $f \lesssim g$ and $g \lesssim f$.

2 Results

2.1 The standard FitzHugh–Nagumo equation

We start by considering the particular case of the standard FitzHugh–Nagumo SPDE, given by

$$\partial_t u = \Delta_x u + u - u^3 + v + \xi \,,$$
$$\partial_t v = a_1 u + a_2 v \,. \tag{2.1}$$

Here $\Delta_x = \sum_{j=1}^{d} \partial_{x_j}^2$ is the usual Laplacian, u and v are functions of time $t \geqslant 0$ and space $x \in \mathbb{T}^d$ (the torus in dimension $d = 2$ or $d = 3$), ξ stands for space-time white noise on $\mathbb{R} \times \mathbb{T}^d$, and a_1, a_2 are real parameters. As such, the system (2.1) is not well-posed, and a renormalisation procedure is required to define a notion of solution. To do this, we first choose a rescaled mollifier

$$\varrho_\varepsilon(t, x) = \frac{1}{\varepsilon^{d+2}} \varrho\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) \,, \tag{2.2}$$

where $\varrho : \mathbb{R}^{d+1} \to \mathbb{R}$ is a smooth compactly supported function of integral 1. We set $\xi^\varepsilon = \varrho_\varepsilon * \xi$, where the star stands for space-time convolution, and consider the sequence of equations

$$\partial_t u^\varepsilon = \Delta_x u^\varepsilon + \left[1 + C(\varepsilon)\right] u^\varepsilon - (u^\varepsilon)^3 + v^\varepsilon + \xi^\varepsilon \,,$$
$$\partial_t v^\varepsilon = a_1 u^\varepsilon + a_2 v^\varepsilon \,, \tag{2.3}$$

where $C(\varepsilon) \in \mathbb{R}$. Then our first result is the following, which is close in spirit to [19, Thm. 1.15].

Theorem 2.1 (Standard FitzHugh–Nagumo SPDE). *Assume u_0 belongs to the Hölder space $\mathcal{C}^\eta$ for some $\eta > -\frac{2}{3}$, and v_0 belongs to the Hölder space $\mathcal{C}^\gamma$ for some $\gamma > 1$. Then there exists a choice of the constant $C(\varepsilon)$ such that the system (2.3) with initial condition (u_0, v_0) admits a sequence of local solutions $(u^\varepsilon, v^\varepsilon)$, converging in probability to a limit (u, v) as $\varepsilon \to 0$. This limit is independent of the choice of mollifier ϱ.*

The proof of this theorem and the two next ones is given in Section 7. For the precise definition of Hölder spaces $\mathcal{C}^\eta$ with negative index, we refer to Definition 3.3 below. By *local solution* we mean that for any cut-off $L > 0$, the solution is defined up to the random time when the $\mathcal{C}^\eta$-norm of $(u^\varepsilon, v^\varepsilon)$ first reaches L. In other words, we cannot exclude the possibility of finite-time blow-up. Note, however, that in the deterministic case classical arguments involving positively invariant sets (see for instance [33]) show that the FitzHugh–Nagumo equations are globally well-posed. Since global well-posedness has also recently been proved for the stochastic Allen–Cahn equation [35, 36], one would not expect that the FitzHugh–Nagumo SPDE actually shows finite-time blow-up.

The constant $C(\varepsilon)$ admits an explicit expression in terms of the heat kernel[1]

$$G^{(d)}(t, x) = \frac{1}{|4\pi t|^{d/2}} \, e^{-\|x\|^2/4t} \, 1_{\{t>0\}} \,. \tag{2.4}$$

In dimension $d = 3$, $C(\varepsilon)$ is the same as for the dynamical Φ_3^4 model considered in [19]. Namely, setting $G_\varepsilon^{(d)} = G^{(d)} * \varrho_\varepsilon$, one has

$$C(\varepsilon) = 3 \int_{\mathbb{R}^4} G_\varepsilon^{(3)}(z)^2 \, dz - 18 \int_{\mathbb{R}^4} G^{(3)}(z) \mathcal{Q}_0^\varepsilon(z)^2 \, dz \,, \tag{2.5}$$

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[1] Below, the renormalisation constants will rather be defined in terms of a truncated version of the heat kernel. This does however not affect their singular parts (terms of order ε^{-1} and $\log(\varepsilon^{-1})$).

where $\mathcal{Q}_0^\varepsilon(z) = \int G_\varepsilon^{(3)}(z_1) G_\varepsilon^{(3)}(z_1 - z)\,\mathrm{d}z_1$. In particular, it has been known since the works of Feldman and Osterwalder [14, 15] that

$$C(\varepsilon) = \frac{C_1}{\varepsilon} + C_2 \log(\varepsilon^{-1}) + C_3 \tag{2.6}$$

for some constants $C_1, C_2, C_3 \in \mathbb{R}$. The constants C_1 and C_3 depend on the choice of mollifier ϱ, while C_2 is independent of ϱ (cf. [20, Rem. 6.2]).

In dimension $d = 2$, one can show (cf. [24, Rem. 2.14]) that the renormalisation constant is given by

$$C(\varepsilon) = 3 \int_{\mathbb{R}^3} G_\varepsilon^{(2)}(z)^2\,\mathrm{d}z = \frac{3}{4\pi} \log(\varepsilon^{-1}) + C_3 \,, \tag{2.7}$$

for some constant $C_3 \in \mathbb{R}$, depending again on the choice of ϱ.

The proof of Theorem 2.1 will also provide some information on the structure of the solutions. Indeed, we always have

$$u^\varepsilon(t, x) = \chi_\varepsilon(t, x) + \varphi^\varepsilon(t, x) \,, \tag{2.8}$$

where $\chi_\varepsilon = G^{(d)} * \xi^\varepsilon$ is the stochastic convolution of heat kernel and noise, and φ^ε converges to a function (as opposed to a distribution). In other words, the only singular term in the limit $\varepsilon \to 0$ is given by $\lim_{\varepsilon \to 0} \chi_\varepsilon = G^{(d)} * \xi$, which is independent of the nonlinear term in the equation. The function $v^\varepsilon(t, x)$ has the same structure, as it can be represented in terms of u^ε by solving a linear inhomogeneous equation.

2.2 More general cubic nonlinearities

We can now extend the above results to a more general class of systems, of the form

$$\begin{aligned}
\partial_t u &= \Delta_x u + F(u, v) + \xi \,, \\
\partial_t v &= a_1 u + a_2 v \,,
\end{aligned} \tag{2.9}$$

where F is a cubic polynomial of the form

$$F(u, v) = \alpha_1 u + \alpha_2 v + \beta_1 u^2 + \beta_2 uv + \beta_3 v^2 + \gamma_1 u^3 + \gamma_2 u^2 v + \gamma_3 uv^2 + \gamma_4 v^3 \,. \tag{2.10}$$

In that case, the renormalised equations take the form

$$\begin{aligned}
\partial_t u^\varepsilon &= \Delta_x u^\varepsilon + \left[F(u^\varepsilon, v^\varepsilon) + C_0(\varepsilon) + C_1(\varepsilon) u^\varepsilon + C_2(\varepsilon) v^\varepsilon \right] + \xi^\varepsilon \,, \\
\partial_t v^\varepsilon &= a_1 u^\varepsilon + a_2 v^\varepsilon \,,
\end{aligned} \tag{2.11}$$

where $C_0(\varepsilon), C_1(\varepsilon), C_2(\varepsilon) \in \mathbb{R}$, and we have the following result:

Theorem 2.2 (General cubic nonlinearities). *Assume u_0 and v_0 satisfy the same assumptions as in Theorem 2.1. Assume further that either $d = 2$, or $d = 3$ and $\gamma_2 = 0$. Then there exists a choice of constants $C_0(\varepsilon)$, $C_1(\varepsilon)$ and $C_2(\varepsilon)$ such that the system (2.11) with initial condition (u_0, v_0) admits a sequence of local solutions $(u^\varepsilon, v^\varepsilon)$, converging in probability to a limit (u, v) as $\varepsilon \to 0$. This limit is independent of the choice of mollifier ϱ.*

Note that in dimension $d = 3$, we assume that F contains no term of the form $\gamma_2 u^2 v$. This is because the method we use does not yield a simple form for the renormalised equation if such a term is present (see Section 6.3), which is an indirect consequence of the fact that the equation for v is not regularising in space (cf. Section 4.2). It is not clear at this stage whether this is just a technical artefact, or whether it has a deeper meaning.

The renormalisation constants can again be computed explicitly to leading order. They are given by

$$
\begin{aligned}
C_0(\varepsilon) &= -\frac{1}{3}\beta_1 C(\varepsilon) \,, \\
C_1(\varepsilon) &= -\gamma_1 C(\varepsilon) \,, \\
C_2(\varepsilon) &= -\frac{1}{3}\gamma_2 C(\varepsilon) \,,
\end{aligned}
\tag{2.12}
$$

where $C(\varepsilon)$ is given by

$$
C(\varepsilon) = 3\int_{\mathbb{R}^4} G^{(3)}_\varepsilon(z)^2 \,\mathrm{d}z + 18\gamma_1 \int_{\mathbb{R}^4} G^{(3)}(z)\mathcal{Q}_0^\varepsilon(z)^2 \,\mathrm{d}z
\tag{2.13}
$$

for $d = 3$ and by (2.7) for $d = 2$. Note in particular that these constants depend only on the coefficients of u^2, $u^2 v$ and u^3 in F.

2.3 Vectorial gating variables

Another generalisation of interest is to systems with vectorial variables v, of the form

$$
\begin{aligned}
\partial_t u &= \Delta_x u + F(u, v) + \xi \,, \\
\partial_t v &= u A_1 + A_2 v \,,
\end{aligned}
\tag{2.14}
$$

where $v(t, x)$ takes values in $\mathbb{R}^n$. Here $A_1 \in \mathbb{R}^n$ is a constant vector, and $A_2 \in \mathbb{R}^{n \times n}$ is a square matrix, while F is again a cubic polynomial in u and the v_i. This allows for instance to consider the Koper model [27] with spatial diffusion and space-time white noise in the fast variable; for the stochastic Koper model without the Laplacian representing spatial diffusion see [5]. The reaction terms of one version of the Koper model can be written in the form (2.14) with

$$
F(u, v) = 3u + v_1 - u^3 \,, \qquad A_1 = \begin{pmatrix} \epsilon_1 k \\ 0 \end{pmatrix} \,, \qquad A_2 = \begin{pmatrix} -2\epsilon_1 & \epsilon_1 \\ \epsilon_1 & -\epsilon_1 \end{pmatrix} \,,
\tag{2.15}
$$

where $k, \epsilon_1 \in \mathbb{R}$ are model parameters.

The natural candidate for the renormalised system associated to (2.14) is given by

$$
\begin{aligned}
\partial_t u^\varepsilon &= \Delta_x u^\varepsilon + \left[F(u^\varepsilon, v^\varepsilon) + C_0(\varepsilon) + C_1(\varepsilon)u^\varepsilon + \sum_{i=1}^n C_{2,i}(\varepsilon)v_i^\varepsilon \right] + \xi^\varepsilon \,, \\
\partial_t v^\varepsilon &= u^\varepsilon A_1 + A_2 v^\varepsilon \,,
\end{aligned}
\tag{2.16}
$$

for constants $C_0(\varepsilon), C_1(\varepsilon), C_{2,i}(\varepsilon) \in \mathbb{R}$. Indeed, we have the following result.

Theorem 2.3 (Vectorial variables v). *Assume u_0 and the components of v_0 satisfy the same assumptions as u_0 and v_0 in Theorem 2.1. Assume further that either $d = 2$, or $d = 3$ and $F(u, v)$ has no terms in $u^2 v_i$. Then there exists a choice of constants $C_0(\varepsilon)$, $C_1(\varepsilon)$, $C_{2,i}(\varepsilon)$ such that the system (2.16) with initial condition (u_0, v_0) admits a sequence of local solutions $(u^\varepsilon, v^\varepsilon)$, converging in probability to a limit (u, v) as $\varepsilon \to 0$. This limit is independent of the choice of mollifier ϱ.*

The renormalisation constants can again be expressed in terms of the coefficients of the initial equation (2.14). Writing

$$
F(u, v) = \alpha_1 u + \beta_1 u^2 + \gamma_1 u^3 + \sum_{i=1}^n \left[\alpha_{2,i} v_i + \beta_{2,i} u v_i + \gamma_{2,i} u^2 v_i \right] + R(u, v)
\tag{2.17}
$$

with $|R(u,v)| \leqslant C\|v\|^2$, we have

$$
\begin{aligned}
C_0(\varepsilon) &= -\frac{1}{3}\beta_1 C(\varepsilon) \,, \\
C_1(\varepsilon) &= -\gamma_1 C(\varepsilon) \,, \\
C_{2,i}(\varepsilon) &= -\frac{1}{3}\gamma_{2,i} C(\varepsilon) \,, \qquad i = 1,\ldots,n \,,
\end{aligned}
\tag{2.18}
$$

where $C(\varepsilon)$ is again the constant defined in (2.13) for $d = 3$ and in (2.7) for $d = 2$. Note that in the particular case of the Koper model, $C_1(\varepsilon)$ is equal to the constant $C(\varepsilon)$ obtained for the FitzHugh–Nagumo equation, while all other constants vanish. This is not surprising, since both models have the same nonlinearity.

Remark 2.4. The notion of *local subcriticality* given in [19, Assumption 8.3] suggests that the class of renormalisable SPDE–ODE models of the form (1.3) is larger than the one considered here. More precisely, in dimension $d = 3$, one would expect models with quartic F and linear G to be renormalisable, as well as models with F quartic in u and quadratic in v and G quadratic in u and linear in v. The reason we did not include them in our analysis is a technical one: quartic F would produce solutions whose singular component is not homogeneous in space and time, as in (2.8). The way in which we lift the singular kernel to the regularity structure is not able to deal with such situations (see Section 4.2 below). We plan to further investigate this issue in future work. ◊

3 Regularity structure for the Allen–Cahn equation

This section serves the double purpose of giving a very brief account of the theory of regularity structures contained in [19], and of describing a regularity structure for the Allen–Cahn equation

$$
\partial_t u = \Delta_x u + u - u^3 + \xi \,,
\tag{3.1}
$$

where $\Delta_x = \sum_{j=1}^d \partial_{x_j}^2$ is the usual Laplacian, $\xi = \xi(t,x)$ denotes space-time white noise on $\mathbb{R} \times \mathbb{T}^d$ where $\mathbb{T}^d$ denotes the d-dimensional torus with $d = 2$ or $d = 3$, and we seek a solution $u : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ for a given initial condition $u_0 = u(0,x)$ in a suitable function space. The regularity structure for (3.1) will serve as our starting point to build a larger structure allowing to represent the FitzHugh–Nagumo equation.

By Duhamel's principle, one possible solution concept for (3.1) is to consider the following integral equation

$$
u_t = \int_0^t S(t-s)\big[u_s - u_s^3 + \xi_s\big]\,\mathrm{d}s + S(t)u_0 \,,
\tag{3.2}
$$

where $S(t) = \mathrm{e}^{t\Delta_x}$ denotes the semigroup of the heat equation compatible with the boundary conditions and $u_s := u(s,\cdot)$, $\xi_s := \xi(s,\cdot)$. The purpose of a regularity structure is to provide an abstract space in which one can construct a fixed point of (3.2) when ξ is replaced by a mollified version ξ^ε. Then the idea is to take the limit $\varepsilon \to 0$, and to project the solution to a distribution on $[0,T] \times \mathbb{T}^d$, where "distribution" is understood in the sense of a "generalized function" representing a sample path.

3.1 Regularity structures

Definition 3.1 ([19, Def. 2.1]). *A regularity structure is a triple $(A, T, \mathcal{G})$ consisting of*

(R1) an index set $A \subset \mathbb{R}$, containing 0, which is bounded from below and locally finite;

(R2) a model space T, which is a graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$, where each T_α is a Banach space; the space T_0 is isomorphic to $\mathbb{R}$ and its unit is denoted $\mathbf{1}$;

(R3) a structure group $\mathcal{G}$ *of linear operators acting on* T*, such that*

$$\Gamma\tau - \tau \in \bigoplus_{\beta < \alpha} T_\beta =: T_\alpha^- \tag{3.3}$$

holds for every $\Gamma \in \mathcal{G}$*, every* $\alpha \in A$ *and every* $\tau \in T_\alpha$*; furthermore,* $\Gamma \mathbf{1} = \mathbf{1}$ *for every* $\Gamma \in \mathcal{G}$*.*

Example 3.2. A simple but important example of regularity structure is the polynomial regularity structure for d variables. In that case, $A = \mathbb{N}_0$ is the set of non-negative integers. For $\ell \in \mathbb{N}_0$, T_ℓ is the space of homogeneous polynomials in d variables of degree ℓ. It is spanned by the monomials $X^k = X_1^{k_1} \ldots X_d^{k_d}$ for which $|k| = k_1 + \cdots + k_d = \ell$. Finally, the structure group $\mathcal{G}$ is defined by

$$\Gamma_h(X^k) = (X - h)^k , \qquad h \in \mathbb{R}^d . \tag{3.4}$$

This group is isomorphic to $\mathbb{R}^d$, and one easily sees that Γ_h satisfies the requirement (3.3). The interpretation of $\mathcal{G}$ is that it allows to convert a Taylor expansion around a point $x \in \mathbb{R}^d$ into the expansion around another point $x + h$. ♦

We will henceforth denote by $\overline{T}$ the polynomial regularity structure with $d+1$ variables $X_0, \ldots, X_d$, where X_0 represents the time variable. Since the linear part of the Allen–Cahn equation is given by a parabolic operator, it turns out to be useful to make the time variable "count double". This is done by defining the *parabolic scaling*

$$\mathfrak{s} = (2, 1, \ldots, 1) \in \mathbb{N}^{d+1} , \tag{3.5}$$

acting on $\mathbb{R}^{d+1}$ and declaring that a monomial X^k has homogeneity $|X^k|_\mathfrak{s} = |k|_\mathfrak{s}$ where the *scaled degree* is defined by $|k|_\mathfrak{s} = 2k_0 + \sum_{i=1}^d k_i$.

The regularity structure of the Allen–Cahn equation (3.3) in $\mathbb{R}^d$ is built by enlarging $\overline{T}$, i.e., by adding new symbols other than the polynomial symbols X^k to $\overline{T}$ (see [24]). The noise is represented by a symbol Ξ. In order to account for the fact that space-time white noise has Hölder regularity α for any $\alpha < -(d+2)/2$, we set

$$\alpha_0 = -\frac{d+2}{2} - \kappa \tag{3.6}$$

where $\kappa > 0$ will be chosen sufficiently small in the sequel, and declare that Ξ has homogeneity $|\Xi|_\mathfrak{s} = \alpha_0$.

The set of symbols is equipped by a product, which by definition is commutative and associative with unit $\mathbf{1}$. The product of two elements $\tau, \sigma \in T$ has homogeneity $|\tau\sigma|_\mathfrak{s} := |\tau|_\mathfrak{s} + |\sigma|_\mathfrak{s}$. Furthermore, integration against the heat kernel is represented by a map $\mathcal{I} : T \to T$, which by definition satisfies $|\mathcal{I}(\tau)|_\mathfrak{s} := |\tau|_\mathfrak{s} + 2$, in order to account for the regularizing effect of the heat kernel.

Let $\mathcal{F}$ be the set containing all possible products of symbols $\mathbf{1}, \Xi, X_i$ and their images by $\mathcal{I}$, and denote by $\mathcal{H} = \text{span}(\mathcal{F})$ the vector space spanned by all these symbols. This is an infinite-dimensional space but in practice only a finite-dimensional subspace of $\mathcal{H}$ will be needed. Let $\mathcal{U}$ be the smallest set containing $\mathbf{1}, X_i$ and $\mathcal{I}(\Xi)$, and such that

$$\tau_1, \tau_2, \tau_3 \in \mathcal{U} \quad \Rightarrow \quad \mathcal{I}(\tau_1\tau_2\tau_3) \in \mathcal{U} . \tag{3.7}$$

We then set

$$\mathcal{F}_F = \{\Xi\} \cup \{\tau_1\tau_2\tau_3 : \tau_i \in \mathcal{U}\} , \tag{3.8}$$

and define the model space as being the vector space $T = \text{span}(\mathcal{F}_F)$ spanned by $\mathcal{F}_F$. The index set is then defined as

$$A = \left\{ |\tau|_\mathfrak{s} : \tau \in \mathcal{F}_F \right\} . \tag{3.9}$$

τ	Symb	$|\tau|_{\mathfrak{s}}$	$d=3$	$d=2$	$\Delta(\tau)$
Ξ	Ξ	α_0	$-\frac{5}{2}-\kappa$	$-2-\kappa$	$\Xi\otimes\mathbf{1}$
$\mathcal{I}(\Xi)^3$	⟨sym⟩	$3\alpha_0+6$	$-\frac{3}{2}-3\kappa$	$0-3\kappa$	⟨sym⟩$\otimes\mathbf{1}$
$\mathcal{I}(\Xi)^2$	⟨sym⟩	$2\alpha_0+4$	$-1-2\kappa$	$0-2\kappa$	⟨sym⟩$\otimes\mathbf{1}$
$\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)^2$	⟨sym⟩	$5\alpha_0+12$	$-\frac{1}{2}-5\kappa$	$2-5\kappa$	⟨sym⟩$\otimes\mathbf{1}+$⟨sym⟩$\otimes\mathcal{J}($⟨sym⟩$)$
$\mathcal{I}(\Xi)$	⟨sym⟩	α_0+2	$-\frac{1}{2}-\kappa$	$0-\kappa$	⟨sym⟩$\otimes\mathbf{1}$
$\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)$	⟨sym⟩	$4\alpha_0+10$	$0-4\kappa$	$2-4\kappa$	⟨sym⟩$\otimes\mathbf{1}+$⟨sym⟩$\otimes\mathcal{J}($⟨sym⟩$)$
$\mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)^2$	⟨sym⟩	$4\alpha_0+10$	$0-4\kappa$	$2-4\kappa$	⟨sym⟩$\otimes\mathbf{1}+$⟨sym⟩$\otimes\mathcal{J}($⟨sym⟩$)$
$\mathcal{I}(\Xi)^2 X_i$	⟨sym⟩X_i	$2\alpha_0+5$	$0-2\kappa$	$1-2\kappa$	⟨sym⟩$\otimes X_i+$⟨sym⟩$X_i\otimes\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	0	0	0	$\mathbf{1}\otimes\mathbf{1}$
$\mathcal{I}(\mathcal{I}(\Xi)^3)$	⟨sym⟩	$3\alpha_0+8$	$\frac{1}{2}-3\kappa$	$2-3\kappa$	⟨sym⟩$\otimes\mathbf{1}+\mathbf{1}\otimes\mathcal{J}($⟨sym⟩$)$
$\mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)$	⟨sym⟩	$3\alpha_0+8$	$\frac{1}{2}-3\kappa$	$2-3\kappa$	⟨sym⟩$\otimes\mathbf{1}+$⟨sym⟩$\otimes\mathcal{J}($⟨sym⟩$)$
$\mathcal{I}(\mathcal{I}(\Xi))\mathcal{I}(\Xi)^2$	⟨sym⟩	$3\alpha_0+8$	$\frac{1}{2}-3\kappa$	$2-3\kappa$	⟨sym⟩$\otimes\mathbf{1}+$⟨sym⟩$\otimes\mathcal{J}($⟨sym⟩$)+$ ⟨sym⟩$X_i\otimes\mathcal{J}_i($⟨sym⟩$)+$⟨sym⟩$\otimes X_i\mathcal{J}_i($⟨sym⟩$)$
$\mathcal{I}(\mathcal{I}(\Xi)^2)$	⟨sym⟩	$2\alpha_0+6$	$1-2\kappa$	$2-2\kappa$	⟨sym⟩$\otimes\mathbf{1}+\mathbf{1}\otimes\mathcal{J}($⟨sym⟩$)$
$\mathcal{I}(\mathcal{I}(\Xi))\mathcal{I}(\Xi)$	⟨sym⟩	$2\alpha_0+6$	$1-2\kappa$	$2-2\kappa$	⟨sym⟩$\otimes\mathbf{1}+$⟨sym⟩$\otimes\mathcal{J}($⟨sym⟩$)+$ ⟨sym⟩$X_i\otimes\mathcal{J}_i($⟨sym⟩$)+$⟨sym⟩$\otimes X_i\mathcal{J}_i($⟨sym⟩$)$
X_i	X_i	1	1	1	$X_i\otimes\mathbf{1}+\mathbf{1}\otimes X_i$
$\mathcal{I}(\mathcal{I}(\Xi))$	⟨sym⟩	α_0+4	$\frac{3}{2}-\kappa$	$2-\kappa$	⟨sym⟩$\otimes\mathbf{1}+\mathbf{1}\otimes\mathcal{J}($⟨sym⟩$)+$ $X_i\otimes\mathcal{J}_i($⟨sym⟩$)+\mathbf{1}\otimes X_i\mathcal{J}_i($⟨sym⟩$)$

Table 1: Elements of $\mathcal{F}_F$ of lowest homogeneity for the Allen–Cahn equation. They have been ordered by increasing homogeneity for the case $d=3$. The expressions for $\Delta(\tau)$ are shown for the case $d=3$. We have written $\mathcal{J}$ instead of $\mathcal{J}_0$ and $\mathcal{J}_i$ instead of $\mathcal{J}_{e_i}$, where the e_i are canonical basis vectors of $\mathbb{Z}_+^{d+1}$. Summation over the index i is understood.

The model space T admits a natural grading

$$T=\bigoplus_{\gamma\in A}T_\gamma \tag{3.10}$$

obtained by letting T_γ be the vector space spanned by elements of homogeneity γ. We equip each T_γ with a norm $\|\cdot\|_\gamma$; the choice of norm is irrelevant since [19, Lemma 8.10] shows the (nontrivial) fact that all T_γ are finite-dimensional.

Table 1 shows the elements in $\mathcal{F}_F$ of lowest homogeneity, using a graphical representation introduced by Hairer. Each symbol Ξ is denoted by a dot, and integration with respect to $\mathcal{I}$ is denoted by a vertical line pointing downwards. For instance, $\mathcal{I}(\Xi)=$ ⟨sym⟩. Multiplication of symbols is represented by joining them at the base, so for instance $\mathcal{I}(\Xi)^2=$ ⟨sym⟩.

Finally, we have to extend the definition of the structure group $\mathcal{G}$, which is the most involved part of the construction. The first step is to introduce a set $\mathcal{F}_+$, which contains all formal expressions of the type

$$X^k\prod_j\mathcal{J}_{k_j}\tau_j \ , \tag{3.11}$$

where $\tau_j \in \mathcal{F}$ and the multiindices k_j are such that $|\tau_j|_{\mathfrak{s}} + 2 - |k_j|_{\mathfrak{s}} > 0$. By definition, an expression of the form (3.11) has homogeneity $|k|_{\mathfrak{s}} + \sum_j (|\tau_j|_{\mathfrak{s}} + 2 - |k_j|_{\mathfrak{s}})$, which is always strictly positive, except for the element $\mathbf{1}$, which has homogeneity 0. The set $\mathcal{F}_F^+ \subset \mathcal{F}_+$ is defined similarly, but with $\tau_j \in \mathcal{F}_F$. We set $\mathcal{H}_+ = \operatorname{span}(\mathcal{F}_+)$ and $T^+ = \operatorname{span}(\mathcal{F}_F^+)$.

The last ingredient is given by a bilinear map $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_+$ defined by setting

$$\Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}\,, \qquad \Delta X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i\,, \qquad \Delta \Xi = \Xi \otimes \mathbf{1}\,, \tag{3.12}$$

and extended inductively to all of $\mathcal{H}$ by the rules

$$\Delta(\tau\sigma) = (\Delta\tau)(\Delta\sigma)\,,$$
$$\Delta(\mathcal{I}\tau) = (\mathcal{I} \otimes \operatorname{Id})\Delta(\tau) + \sum_{\ell,m} \frac{X^\ell}{\ell!} \otimes \frac{X^m}{m!} \mathcal{J}_{\ell+m}\tau\,, \tag{3.13}$$

where Id denotes the identity map and we impose that

$$\mathcal{J}_k \tau = 0 \qquad \text{if } |k|_{\mathfrak{s}} \geqslant |\tau|_{\mathfrak{s}} + 2\,, \tag{3.14}$$

while $\mathcal{J}_k \tau$ is a new formal symbol otherwise. Table 1 also shows $\Delta(\tau)$ for the first few elements of $\mathcal{F}_F$.

Denote by $\mathcal{H}_+^\star$ the dual of $\mathcal{H}_+$, that is, the set of linear maps $h : \mathcal{H}_+ \to \mathbb{R}$, that we will write as $\tau \mapsto \langle h, \tau \rangle$. Let $\mathcal{G}_\star$ be the set of grouplike elements $g \in \mathcal{H}_+^\star$, that is, those satisfying $\langle g, \tau\sigma \rangle = \langle g, \tau \rangle \langle g, \sigma \rangle$ for all $\tau, \sigma \in \mathcal{H}_+$. Then $\mathcal{G}$ is obtained by identifying elements of $\mathcal{G}_\star$ acting the same way on T^+. The action of $\mathcal{G}$ on T is defined by

$$(g, \tau) \mapsto \Gamma_g \tau = (\operatorname{Id} \otimes g)\Delta\tau\,. \tag{3.15}$$

The fact that $(A, T, \mathcal{G})$ constructed in this way is indeed a regularity structure is proved in [19, Thm. 8.24].

3.2 Models and reconstruction theorem

A regularity structure can be used to represent an SPDE as a fixed-point problem in an abstract space. One also has to provide a connection between the abstract space and the "physical" space the solution lives in. First, we need to introduce a version of the classical Hölder spaces $\mathcal{C}^\alpha$ on $\mathbb{R}^{d+1}$. In order to take the parabolic scaling into account, the Euclidean norm is replaced by

$$\|(t, x)\|_{\mathfrak{s}} = |t|^{1/2} + \sum_{i=1}^d |x_i|\,. \tag{3.16}$$

Denote by $\langle \mathbf{1}, \cdot \rangle$ the element of the dual of T defined by $\langle \mathbf{1}, \mathbf{1} \rangle = 1$ and $\langle \mathbf{1}, \tau \rangle = 0$ for all $\tau \in \bigoplus_{\gamma \neq 0} T_\gamma$. If $\alpha > 0$, the space $\mathcal{C}_{\mathfrak{s}}^\alpha$ is defined as the set of all functions $\varphi : \mathbb{R}^{d+1} \to \mathbb{R}$ such that there exists a function $\hat\varphi : \mathbb{R}^{d+1} \to T_\alpha^-$ such that $\langle \mathbf{1}, \hat\varphi(z) \rangle = \varphi(z)$ for every $z = (t, x) \in \mathbb{R}^{d+1}$ and such that for every compact set $\mathfrak{K} \subset \mathbb{R}^{d+1}$ the estimate

$$\|\hat\varphi(z + h) - \Gamma_h \hat\varphi(z)\|_\beta \lesssim \|h\|_{\mathfrak{s}}^{\alpha - \beta} \tag{3.17}$$

holds uniformly over $\beta < \alpha$, $\|h\|_{\mathfrak{s}} \leqslant 1$ and $z \in \mathfrak{K}$ (cf [19, Def. 2.14]). Lemma 2.12 in [19] shows that in the case of the Euclidean scaling $\mathfrak{s} = (1, \ldots, 1)$, this definition coincides with the usual definition of Hölder spaces.

If $\alpha < 0$, $\mathcal{C}_{\mathfrak{s}}^\alpha$ can be defined in a natural way as a subset of Schwarz distributions $\mathcal{S}'(\mathbb{R}^{d+1})$ as follows. Given $r \in \mathbb{N}$, denote by $\mathcal{B}_{\mathfrak{s},0}^r$ the space of functions of class $\mathcal{C}^r$ which are supported in the set $\{z \in \mathbb{R}^{d+1} : \|z\|_{\mathfrak{s}} \leqslant 1\}$.

Definition 3.3 ([19, Def. 3.7]). *Assume $\alpha < 0$ and let $r = -\lfloor \alpha \rfloor$. A Schwarz distribution $v \in \mathcal{S}'(\mathbb{R}^{d+1})$ belongs to $\mathcal{C}_\mathfrak{s}^\alpha$ if it belongs to the dual of $\mathcal{C}_0^r$ and for every compact set $\mathfrak{K} \subset \mathbb{R}^{d+1}$, there exists a constant C such that*

$$\langle v, \mathcal{S}_{\mathfrak{s},z}^\delta \eta \rangle \leqslant C\delta^\alpha \tag{3.18}$$

holds for all $\eta \in \mathcal{B}_{\mathfrak{s},0}^r$ with $\|\eta\|_{\mathcal{C}^r} \leqslant 1$, all $\delta \in (0,1]$ and all $z \in \mathfrak{K}$. Here

$$(\mathcal{S}_{\mathfrak{s},(t,x)}^\delta \eta)(\bar{t}, \bar{x}) := \delta^{-(d+2)} \eta\big(\delta^{-2}(\bar{t} - t), \delta^{-1}(\bar{x} - x)\big) . \tag{3.19}$$

Next we introduce the notion of a *model* associated with a regularity structure $(A, T, \mathcal{G})$.

Definition 3.4 ([19, Def. 2.17]). *A model for a regularity structure $(A, T, \mathcal{G})$ with scaling $\mathfrak{s}$ is a pair (Π, Γ), defined by a collection $\{\Pi_z : T \to \mathcal{S}'(\mathbb{R}^{d+1})\}_{z \in \mathbb{R}^{d+1}}$ of continuous linear maps and a map $\Gamma : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathcal{G}$ with the following properties.*

(M1) $\Gamma_{zz} = \mathrm{Id}$ is the identity of $\mathcal{G}$ and $\Gamma_{zz'}\Gamma_{z'z''} = \Gamma_{zz''}$ for all $z, z', z'' \in \mathbb{R}^{d+1}$.

(M2) $\Pi_{z'} = \Pi_z \Gamma_{zz'}$ for all $z, z' \in \mathbb{R}^{d+1}$.

(M3) For any $\gamma \in \mathbb{R}$ and any compact set $\mathfrak{K} \subset \mathbb{R}^{d+1}$, one has

$$\|\Pi\|_{\gamma;\mathfrak{K}} := \sup_{z \in \mathfrak{K}} \sup_{\alpha < \gamma} \sup_{\tau \in T_\alpha} \sup_{\eta \in \mathcal{B}_{\mathfrak{s},0}^r} \sup_{0 < \delta \leqslant 1} \frac{\left|\langle \Pi_z \tau, \mathcal{S}_{\mathfrak{s},z}^\delta \eta \rangle\right|}{\delta^\alpha \|\tau\|} < \infty . \tag{3.20}$$

(M4) For any $\gamma \in \mathbb{R}$ and any compact set $\mathfrak{K} \subset \mathbb{R}^{d+1}$, one has

$$\|\Gamma\|_{\gamma;\mathfrak{K}} := \sup_{z,z' \in \mathfrak{K}} \sup_{\alpha < \gamma} \sup_{\beta < \alpha} \sup_{\tau \in T_\alpha} \frac{\|\Gamma_{zz'}\tau\|_\beta}{\|\tau\| \|z - z'\|_\mathfrak{s}^{\alpha - \beta}} < \infty . \tag{3.21}$$

In the case of the Allen–Cahn equation with mollified noise $\xi^\varepsilon := \xi * \varrho_\varepsilon$, where $\varrho(t, x)$ is a mollifier and $\varrho_\varepsilon(t, x) = \varepsilon^{-(d+2)} \varrho(t/\varepsilon^2, x/\varepsilon)$, a canonical way of building a model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$ proceeds as follows. First, we can just define

$$(\Pi_z^\varepsilon \Xi)(\bar{z}) = \xi^\varepsilon(\bar{z}) ,$$
$$(\Pi_z^\varepsilon X^k)(\bar{z}) = (\bar{z} - z)^k ,$$
$$(\Pi_z^\varepsilon \tau \bar{\tau})(\bar{z}) = (\Pi_z^\varepsilon \tau)(\bar{z})(\Pi_z^\varepsilon \bar{\tau})(\bar{z}) \qquad \forall \tau, \bar{\tau} \in T . \tag{3.22}$$

To also include the integration map in the fixed point problem let $G = G(t, x)$ be the heat kernel defined as the fundamental solution associated to $\partial_t u = \Delta_x u$. We split the heat kernel G as

$$G = R + K , \tag{3.23}$$

where $K : (\mathbb{R}^{d+1} \setminus \{0\}) \to \mathbb{R}$ is a singular part satisfying specific algebraic properties, while $R : \mathbb{R}^{d+1} \to \mathbb{R}$ is a smooth part. The properties of K are:

- K is supported in $\{|x|^2 + |t| \leqslant 1\}$ where $|x| = \sum_{j=1}^d |x_j|$;
- $K(t, x) = 0$ for $t \leqslant 0$ and $K(t, -x) = K(t, x)$ for all (t, x);
- we have

$$K(t, x) = \frac{1}{|4\pi t|^{d/2}} \, \mathrm{e}^{-\|x\|^2/4t} \qquad \text{for } |x|^2 + |t| \leqslant \frac{1}{2} \tag{3.24}$$

  and $K(t, x)$ is smooth for $|x|^2 + |t| > \frac{1}{2}$;
- furthermore, one has vanishing moments

$$\int_{\mathbb{R}^{d+1}} K(t, x) P(t, x) \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{3.25}$$

for all polynomials P of parabolic degree less or equal some fixed $\zeta \geqslant 2$.

Lemma 5.5 in [19] shows that such a splitting indeed exists and also the vanishing moments condition holds for K; both parts K, R then satisfy a number of derivative bounds. Furthermore, it follows from Lemma 5.5 in [19] that there exists a decomposition

$$K(z - z') = \sum_{n \geqslant 0} K_n(z - z') \tag{3.26}$$

where $n \in \mathbb{N}_0$ and suitable derivative bounds and vanishing moment conditions hold for each of the kernels K_n. With this construction, the abstract integration map is represented by the Taylor-series-like expression

$$(\Pi_z^\varepsilon \mathcal{I}\tau)(\bar{z}) = \int K(\bar{z} - \bar{z}')(\Pi_z^\varepsilon \tau)(\mathrm{d}\bar{z}') + \sum_\ell \frac{(\bar{z} - z)^\ell}{\ell!} \langle f_z^\varepsilon, \mathcal{J}_\ell \tau \rangle , \tag{3.27}$$

where the linear forms $f_z^\varepsilon \in \mathcal{G} \subset \mathcal{H}_+^\star$ are constructed as follows:

$$\langle f_z^\varepsilon, \mathbf{1} \rangle = 1 ,$$
$$\langle f_z^\varepsilon, X_i \rangle = -z_i ,$$
$$\langle f_z^\varepsilon, \tau\bar{\tau} \rangle = \langle f_z^\varepsilon, \tau \rangle \langle f_z^\varepsilon, \bar{\tau} \rangle \qquad \forall \tau, \bar{\tau} \in T ,$$
$$\langle f_z^\varepsilon, \mathcal{J}_\ell \tau \rangle = - \int D^\ell K(z - \bar{z})(\Pi_z^\varepsilon \tau)(\mathrm{d}\bar{z}) . \tag{3.28}$$

Note that in the more general kernel case $K = K(z, \bar{z})$ one has to replace D^ℓ by D_1^ℓ and the subscript indicates derivative with respect to the first argument. The second term in the definition (3.27) of $\Pi_z^\varepsilon \mathcal{I}\tau$ may seem somewhat strange, and will be motivated in the next section. Note that it should really be interpreted by applying both sides to test functions, i.e., for all smooth compactly supported functions ψ and all $\tau \in T_\alpha$ one requires

$$\langle \Pi_z^\varepsilon \mathcal{I}\tau, \psi \rangle = \sum_{n \geqslant 0} \int_{\mathbb{R}^{d+1}} \psi(z') \langle \Pi_z^\varepsilon \tau, K_{n;zz'}^\alpha \rangle \, \mathrm{d}z' \tag{3.29}$$

where $K_{n;zz'}^\alpha$ is defined via

$$K_{n;zz'}^\alpha(z'') = K_n(z' - z'') - \sum_{|k|_{\mathfrak{s}} < \alpha+2} \frac{(z' - z)^k}{k!} D^k K_n(z - z'') . \tag{3.30}$$

The group elements $\Gamma_{z\bar{z}}^\varepsilon$ are then simply defined by

$$\Gamma_{z\bar{z}}^\varepsilon = (F_z^\varepsilon)^{-1} F_{\bar{z}}^\varepsilon \qquad \text{where } F_z^\varepsilon = \Gamma_{f_z^\varepsilon} \tag{3.31}$$

(recall that F_z^ε is invertible because f_z^ε is grouplike). In this way, the algebraic property (M1) of Definition 3.4 is automatically satisfied. Regarding the algebraic property (M2), we can note the following relations. Using Sweedler's notation [39], we write $\Delta\tau = \tau^{(1)} \otimes \tau^{(2)}$, although $\Delta\tau$ is usually a sum of such terms. Let further $\gamma_{z\bar{z}}^\varepsilon$ denote the element of the structure group such that $\Gamma_{z\bar{z}}^\varepsilon = \Gamma_{\gamma_{z\bar{z}}^\varepsilon}^\varepsilon$. Then we have by (3.15)

$$\Pi_z^\varepsilon \Gamma_{z\bar{z}}^\varepsilon \tau = \Pi_z^\varepsilon (\mathrm{Id} \otimes \gamma_{z\bar{z}}^\varepsilon)\Delta\tau = \Pi_z^\varepsilon \tau^{(1)} \langle \gamma_{z\bar{z}}^\varepsilon, \tau^{(2)} \rangle . \tag{3.32}$$

Property (M2) thus amounts to the relation

$$\Pi_{\bar{z}}^\varepsilon \tau = \Pi_z^\varepsilon \tau^{(1)} \langle \gamma_{z\bar{z}}^\varepsilon, \tau^{(2)} \rangle , \tag{3.33}$$

which provides some intuition for the meaning of Δ. It is easy to check that this relation holds for elements τ of the polynomial regularity structure. In the general case, the fact that $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$ is indeed a model is a nontrivial fact, proved in [19, Prop. 8.27].

Remark 3.5. It is very important to realise that the canonical model just built is not the only possible model for a given regularity structure. This freedom in the choice of model will be used when introducing the renormalisation procedure. All models, however, will share many properties with the canonical model. The only rule that will be modified is the product rule $\Pi_z^\varepsilon(\tau\bar\tau) = \Pi_z^\varepsilon(\tau)\Pi_z^\varepsilon(\bar\tau)$. $\lozenge$

We can now introduce the spaces $\mathcal{D}^\gamma$, which play an analogous role as the $\mathcal{C}_{\mathfrak{s}}^\alpha$ on the level of the regularity structure.

Definition 3.6 ([19, Def. 3.1])**.** *Let* $\gamma \in \mathbb{R}$. *Given a model* $Z = (\Pi, \Gamma)$, *the space* $\mathcal{D}^\gamma = \mathcal{D}^\gamma(Z)$ *consists of all functions* $f : \mathbb{R}^{d+1} \to T_\gamma^-$ *such that for every compact set* $\mathfrak{K} \subset \mathbb{R}^{d+1}$ *one has*

$$\|f\|_{\gamma;\mathfrak{K}} := \sup_{z \in \mathfrak{K}} \sup_{\beta < \gamma} \|f(z)\|_\beta + \sup_{\substack{z,\bar{z} \in \mathfrak{K} \\ \|z-\bar{z}\|_{\mathfrak{s}} \leqslant 1}} \sup_{\beta < \gamma} \frac{\|f(z) - \Gamma_{z\bar{z}}f(\bar{z})\|_\beta}{\|z - \bar{z}\|_{\mathfrak{s}}^{\gamma-\beta}} < \infty \,. \tag{3.34}$$

In the particular case of the polynomial regularity structure, it is again quite straightforward to check that the requirement (3.34) is equivalent to f being the Taylor expansion of an element of the Hölder space $\mathcal{C}_{\mathfrak{s}}^\gamma$. The spaces $\mathcal{D}^\gamma$ depend on the model via Γ, but not on Π, cf [19, Remark 3.4]. In order to compare elements of $\mathcal{D}^\gamma$ for different models, it is useful to introduce

$$\|f; \bar{f}\|_{\gamma;\mathfrak{K}} = \|f - \bar{f}\|_{\gamma;\mathfrak{K}} + \sup_{\substack{z,\bar{z} \in \mathfrak{K} \\ \|z-\bar{z}\|_{\mathfrak{s}} \leqslant 1}} \sup_{\beta < \gamma} \frac{\|f(z) - \bar{f}(z) - \Gamma_{z\bar{z}}f(\bar{z}) + \bar{\Gamma}_{z\bar{z}}\bar{f}(\bar{z})\|_\beta}{\|z - \bar{z}\|_{\mathfrak{s}}^{\gamma-\beta}} \tag{3.35}$$

(which is in general *not* a function of $f - \bar{f}$), where

$$\|f - \bar{f}\|_{\gamma;\mathfrak{K}} = \sup_{z \in \mathfrak{K}} \sup_{\beta < \gamma} \|f(z) - \bar{f}(z)\|_\beta \,. \tag{3.36}$$

The central result allowing to link elements in $\mathcal{D}^\gamma$ and in $\mathcal{C}_{\mathfrak{s}}^\alpha$ is the *reconstruction theorem* [19, Thm 3.10]. It states that if $\alpha_* = \min A$ and given $r > |\alpha_*|$ there exists, for any $\gamma \in \mathbb{R}$, a continuous linear map $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{C}_{\mathfrak{s}}^{\alpha_*}$ such that

$$\left|\langle \mathcal{R}f - \Pi_z f(z), \mathcal{S}_{\mathfrak{s},z}^\delta \eta \rangle\right| \leqslant C\delta^\gamma \|\Pi\|_{\gamma;\bar{\mathfrak{K}}} \|f\|_{\gamma;\bar{\mathfrak{K}}} \tag{3.37}$$

holds uniformly over all test functions $\eta \in \mathcal{B}_{\mathfrak{s},0}^r$, all $\delta \in (0,1]$, all $f \in \mathcal{D}^\gamma$ and all $z \in \mathfrak{K}$. The constant $C > 0$ depends only on γ and on the regularity structure, and the set $\bar{\mathfrak{K}}$ is the 1-fattening of $\mathfrak{K}$ (i.e., the points at distance at most 1 from $\mathfrak{K}$). If $\gamma > 0$, then (3.37) defines $\mathcal{R}$ uniquely. The reconstruction theorem also provides bounds on the dependence of $\mathcal{R}$ on the model (Π, Γ). Heuristically, (3.37) states that $\mathcal{R}f$ locally looks like $\Pi_z f(z)$ near any point $z \in \mathbb{R}^{d+1}$, up to terms of order γ.

3.3 Lifting the convolution maps

We can now lift the operation of convolution with the heat kernel to the space $\mathcal{D}^\gamma(Z)$. The requirement (3.25) is for compatibility with the condition that $\mathcal{I}$ should define an abstract integration map of order 2 in the sense of [19, Def. 5.7], namely

- $\mathcal{I} : T_\alpha \to T_{\alpha+2}$ for every $\alpha \in A$;
- $\mathcal{I}\tau = 0$ for every τ in $\overline{T}$;
- $\mathcal{I}\Gamma\tau - \Gamma\mathcal{I}\tau \in \overline{T}$ for every $\tau \in T$ and every $\Gamma \in \mathcal{G}$.

The central result is given by the so-called *multilevel Schauder estimates* [19, Thm 5.12], which state in particular that for all $\gamma \in \mathbb{R}$ such that $\gamma + 2 \notin \mathbb{N}$, there exists a map $\mathcal{K}_\gamma : \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+2}$ such that

$$\mathcal{R}\mathcal{K}_\gamma f = K * \mathcal{R}f \tag{3.38}$$

holds for all $f \in \mathcal{D}^\gamma$. In other words, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{D}^\gamma & \xrightarrow{\ \mathcal{K}_\gamma\ } & \mathcal{D}^{\gamma+2} \\
{\scriptstyle \mathcal{R}}\downarrow & & \downarrow{\scriptstyle \mathcal{R}} \\
\mathcal{C}_\mathfrak{s}^{\alpha_*} & \xrightarrow[\ K_*\ ]{} & \mathcal{C}_\mathfrak{s}^{\alpha_*}
\end{array}$$

The map $\mathcal{K}_\gamma$ has the following expression. For any $f \in \mathcal{D}^\gamma$,

$$(\mathcal{K}_\gamma f)(z) = \mathcal{I}f(z) + \mathcal{J}(z)f(z) + (\mathcal{N}_\gamma f)(z) \,, \tag{3.39}$$

where for each $\tau \in T_\alpha$,

$$\mathcal{J}(z)\tau = \sum_{|k|_\mathfrak{s} < \alpha+2} \frac{X^k}{k!} \int_{\mathbb{R}^{d+1}} D^k K(z - \bar{z})(\Pi_z \tau)(\mathrm{d}\bar{z}) \,, \tag{3.40}$$

$$(\mathcal{N}_\gamma f)(z) = \sum_{|k|_\mathfrak{s} < \gamma+2} \frac{X^k}{k!} \int_{\mathbb{R}^{d+1}} D^k K(z - \bar{z})(\mathcal{R}f - \Pi_z f(z))(\mathrm{d}\bar{z}) \,. \tag{3.41}$$

Note that the last two operators have values in $\overline{T}$, the polynomial part of the regularity structure, and that the only nonlocal operator is $\mathcal{N}_\gamma$.

The rôle of the operators $\mathcal{J}$ and $\mathcal{N}_\gamma$ is to ensure that $\mathcal{K}_\gamma f$ has the properties required to belong to $\mathcal{D}^{\gamma+2}$, which would not be the case if one simply sets $\mathcal{K}_\gamma f = \mathcal{I}f$. The maps $\mathcal{J}$ are related to the coefficients $\mathcal{J}_{k_j}$ appearing in (3.11); in fact, the $\mathcal{J}_{k_j}\tau_j$ play the rôle of placeholders for the $\mathcal{J}(z)\tau$.

As for the smooth part R of the heat kernel, it can be lifted as in [19, (7.7)]. Namely, with a smooth kernel R we associate maps $R_\gamma : \mathcal{C}_\mathfrak{s}^{\alpha_*} \to \mathcal{D}^\gamma$ given by

$$(R_\gamma v)(z) = \sum_{|k|_\mathfrak{s} < \gamma} \frac{X^k}{k!} \int_{\mathbb{R}^{d+1}} D^k R(z - \bar{z}) v(\bar{z}) \, \mathrm{d}\bar{z} := \sum_{|k|_\mathfrak{s} < \gamma} \frac{X^k}{k!} \langle v, D^k R(z - \cdot) \rangle \,. \tag{3.42}$$

It follows from [19, Prop 3.28] that for $v \in \mathcal{C}_\mathfrak{s}^{\alpha_*}$ one has

$$(\mathcal{R}R_\gamma v)(z) = \langle v, R(z - \cdot) \rangle \,,$$

and thus

$$\mathcal{R}R_\gamma \mathcal{R}f = R * \mathcal{R}f \,. \tag{3.43}$$

In other words, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{D}^\gamma & \xrightarrow{\ R_\gamma \mathcal{R}\ } & \mathcal{D}^\gamma \\
{\scriptstyle \mathcal{R}}\downarrow & {\scriptstyle R_\gamma}\nearrow & \downarrow{\scriptstyle \mathcal{R}} \\
\mathcal{C}_\mathfrak{s}^{\alpha_*} & \xrightarrow[\ R_*\ ]{} & \mathcal{C}_\mathfrak{s}^{\alpha_*}
\end{array}$$

Assume that for an appropriate choice of γ, one can find an element $U \in \mathcal{D}^\gamma(Z^\varepsilon)$ satisfying the fixed-point equation

$$U = (\mathcal{K}_{\bar{\gamma}} + R_\gamma \mathcal{R})\boldsymbol{R}^+(\Xi + U - U^3) + Gu_0 \,, \tag{3.44}$$

for some $\bar{\gamma} \geqslant \gamma - 2$, where $\boldsymbol{R}^+(t,x) = 1_{\{t>0\}}$ and Gu_0 denotes a suitable lift of the convolution in space of heat kernel and initial condition. Applying the reconstruction operator $\mathcal{R}$ to both sides of this equation, one can show that $u = \mathcal{R}U$ satisfies

$$u = (K + R) * (\boldsymbol{R}^+[u - u^3 + \xi^\varepsilon]) + Gu_0 \,, \tag{3.45}$$

which is equivalent to (3.2); in (3.45) we also use Gu_0 as a notation for the usual convolution in space of heat kernel and initial condition. This is basically the strategy implemented in Theorem 7.8 and Section 9.4 of [19], except that one has to deal with two additional technical difficulties. The first one is that due to the singular behaviour of the heat kernel as time goes to 0, the definition of the space $\mathcal{D}^\gamma$ has to be modified. We will apply this modification to our case in Section 4.3. The second difficulty is that although a fixed point exists for every mollification parameter $\varepsilon > 0$, it cannot converge as $\varepsilon \to 0$. This is why a renormalisation procedure is needed, which we will adapt to our case in Section 6.

4 Extension of the regularity structure

Our aim is now to extend the regularity structure built for the Allen–Cahn equation, in order to allow to represent a family of coupled SPDEs–ODEs of the form

$$
\begin{aligned}
\partial_t u &= \Delta_x u + F(u,v) + \xi^\varepsilon \ , \\
\partial_t v &= u A_1 + A_2 v \ ,
\end{aligned}
\tag{4.1}
$$

where $F(u,v)$ is a cubic polynomial, and $A_1 \in \mathbb{R}^n$, $A_2 \in \mathbb{R}^{n \times n}$ are either scalars as for the classical FitzHugh–Nagumo case with $n = 1$, or a vector and a matrix if v has multiple components. Duhamel's formula allows us to represent (mild) solutions of (4.1) as

$$
\begin{aligned}
u_t &= \int_0^t S(t-s)\big[\xi_s^\varepsilon + F(u_s, v_s)\big]\, \mathrm{d}s + S(t)u_0 \ , \\
v_t &= \int_0^t u_s Q(t-s)\, \mathrm{d}s + \mathrm{e}^{tA_2} v_0 \ ,
\end{aligned}
\tag{4.2}
$$

where $Q(t) := \mathrm{e}^{tA_2} A_1$. We thus have to lift to the regularity structure the operation of time-integration with respect to Q, which has no smoothing effect in space.

In the case where v has values in $\mathbb{R}^n$ with $n > 1$, the kernel $Q(t)$ is in fact a vector of dimension n. In what follows, we will mainly deal with the case of scalar Q, since the generalisation to the vectorial case is rather straightforward.

4.1 Extension theorem

Let us fix a finite time horizon T. Then we can always assume that the kernel Q satisfies the following properties:

- $Q(t)$ is supported on $[0, 2T]$ and smooth for $t > 0$;
- $Q(t) = \mathrm{e}^{tA_2} A_1$ for all $t \in [0, T]$.

The reason why this is allowed is that we will be interested in showing existence of solutions on a sufficiently small interval $[0, T]$, so that the behaviour of Q outside this interval is not going to matter. Also, since Q is bounded, it will not be necessary to decompose it as a sum of Q_n concentrated in sets of radius 2^{-n}, as in the case of the heat kernel in [19, Section 5], cf (3.26).

In order to be able to represent the time-integration map, we will have to extend our regularity structure and the associated model. We do this by adding to T new symbols denoted $\mathcal{E}(\tau)$, $\tau \in T \setminus \overline{T}$, which we represent by an open blue dot. Thus for instance we write $\mathcal{E}(\mathcal{I}(\Xi)) = \mathcal{E}(\uparrow) = \mathbf{\circ}$. In practice, we will only need to apply $\mathcal{E}$ to elements τ of homogeneity $|\tau|_{\mathfrak{s}} \in (-2, 0)$. We thus set $V = \{\tau \in T \colon -2 < |\tau|_{\mathfrak{s}} < 0\}$ (V is called a *sector* of T). If $\tau \notin V$, we simply set $\mathcal{E}(\tau) = 0$. We also postulate that for $\tau \in V$, $\mathcal{E}(\tau)$ has the same homogeneity as τ, and we choose the norm on the vector space generated by the

new symbols in such a way that

$$\|\mathcal{E}(\tau)\|_\alpha = \|\tau\|_\alpha \qquad \forall \alpha \in V .\tag{4.3}$$

The operator $\mathcal{E}$ defined in this way is an abstract integration map of order 0 on V in the sense of [19, Def. 5.7], i.e.,

- $\mathcal{E}: V \cap T_\alpha \to T_\alpha$ for every $\alpha \in A$;
- $\mathcal{E}\tau = 0$ for every $\tau \in V \cap \overline{T}$;
- $\mathcal{E}\Gamma\tau - \Gamma\mathcal{E}\tau \in \overline{T}$ for every $\tau \in V$ and every $\Gamma \in \mathcal{G}$.

The first two properties are obvious by definition of $\mathcal{E}\tau$, in particular we have $V \cap \overline{T} = \emptyset$. Regarding the third property, we first extend the structure group by setting

$$\Delta(\mathcal{E}\tau) = (\mathcal{E} \otimes \mathrm{Id})\Delta\tau .\tag{4.4}$$

The third required property of $\mathcal{E}$ is indeed satisfied since

$$\Gamma_g \mathcal{E}\tau = (\mathrm{Id} \otimes g)\Delta(\mathcal{E}\tau) = (\mathcal{E} \otimes g)\Delta\tau = \mathcal{E}\Gamma_g \tau ,\tag{4.5}$$

so that in fact $\Gamma_g \mathcal{E}\tau - \mathcal{E}\Gamma_g\tau = 0$ for all $\tau \in T$ and all $g \in \mathcal{G}$.

Since $\mathcal{E}$ is not regularity-increasing, there can be in principle infinitely many symbols of given homogeneity. The trick, however, will be to build the fixed-point map in such a way that only finitely many new symbols are needed. In practice, it will turn out that the only required new symbols are $\overset{\bullet}{\underset{\bullet}{\,}}$, and those obtained by applying $\mathcal{I}$ to existing symbols and/or multiplying them. However, since it is of independent interest, we are going to describe the extension procedure in a more abstract, inductive way. Given a subset $W \subset V$, on which the canonical model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$ is defined, we want to extend the model to a larger set $\widehat{W} = W \cup \{\mathcal{E}(\tau) \colon \tau \in W\}$. We can then apply the usual extension theorem [19, Thm 5.14] to extend the model to $\widehat{W} \cup \{\mathcal{I}(\tau) \colon \tau \in \widehat{W}\}$, and so on as often as needed.

The inductive step from W to $\widehat{W}$ goes as follows. Assume that Z^ε already satisfies Definition 3.4 on a regularity structure $(A, W, \mathcal{G})$. We define the extended model by setting

$$(\Pi^\varepsilon_{t,x}\mathcal{E}\tau)(\bar{t}, \bar{x}) = \int_{\bar{t}-2T}^{\bar{t}} Q(\bar{t} - s)(\Pi^\varepsilon_{t,x}\tau)(s, \bar{x})\,\mathrm{d}s \tag{4.6}$$

for all $\tau \in W$. Writing as before $F_z^\varepsilon = \Gamma_{f_z^\varepsilon}$, the new group elements are defined by setting

$$\Gamma^\varepsilon_{zz'}(\mathcal{E}\tau) = (F_z^\varepsilon)^{-1}F_{z'}^\varepsilon(\mathcal{E}\tau) \tag{4.7}$$

for all $z, z' \in \mathbb{R}^{d+1}$.

Remark 4.1. An alternative would be to define $\mathcal{E}$ on the sector $V = \{\tau \in T \colon -2 < |\tau|_\mathfrak{s} < 2\}$ by the expression

$$(\Pi^\varepsilon_{t,x}\mathcal{E}\tau)(\bar{t}, \bar{x}) = \begin{cases} \displaystyle\int_{\bar{t}-2T}^{\bar{t}} Q(\bar{t} - s)(\Pi^\varepsilon_{t,x}\tau)(s, \bar{x})\,\mathrm{d}s & \text{if } |\tau|_\mathfrak{s} < 0 , \\[4mm] \displaystyle\int_{\bar{t}-2T}^{\bar{t}} Q(\bar{t} - s)(\Pi^\varepsilon_{t,x}\tau)(s, \bar{x})\,\mathrm{d}s + \langle f_{t,x}^\varepsilon, \mathcal{J}^Q\tau\rangle & \text{if } 0 \leqslant |\tau|_\mathfrak{s} < 2 , \end{cases} \tag{4.8}$$

where

$$\langle f_{t,x}^\varepsilon, \mathcal{J}^Q\tau\rangle = -\int_{t-2T}^{t} Q(t - s)(\Pi^\varepsilon_{t,x}\tau)(s, x)\,\mathrm{d}s .\tag{4.9}$$

The new symbol $\mathcal{J}^Q$ is needed to ensure the property $(\Pi^\varepsilon_{t,x}\mathcal{E}\tau)(t,x) = 0$ when $|\tau|_{\mathfrak{s}} \geqslant 0$, which is necessary when lifting the fixed-point equation. While this defines an extended model with the required properties, the fact that Q acts by convolution in time only limits the regularity of its lift to the regularity structure (cf. Remark 4.7 below). We will see in the next subsection why it is sufficient to introduce new symbols $\mathcal{E}\tau$ only when $|\tau|_{\mathfrak{s}} < 0$. $\Diamond$

Remark 4.2. In cases where $v(t,x)$ takes values in $\mathbb{R}^n$ and $Q(t) \in \mathbb{R}^n$, we should in fact introduce n commuting symbols $\mathcal{E}_1, \ldots, \mathcal{E}_n$, and define the extended model by the relations

$$(\Pi^\varepsilon_{t,x}\mathcal{E}_i\tau)(\bar{t},\bar{x}) = \int_{\bar{t}-2T}^{\bar{t}} Q_i(\bar{t} - s)(\Pi^\varepsilon_{t,x}\tau)(s,\bar{x})\,\mathrm{d}s \tag{4.10}$$

for $i = 1, \ldots n$. The results that follow remain true when $\mathcal{E}$ and Q are replaced by $\mathcal{E}_i$ and Q_i. $\Diamond$

We now have to check that $(\Pi^\varepsilon, \Gamma^\varepsilon)$ indeed defines a model on $\widehat{W}$. The following lemma contains a technical estimate preparing the proof of that fact.

Lemma 4.3. *Assume that for any compact set $\mathfrak{K} \subset \mathbb{R}^{d+1}$ there exists a constant $C_{\mathfrak{K}}$ such that*

$$|(\Pi^\varepsilon_z\tau)(\bar{z})| \leqslant C_{\mathfrak{K}}\|z - \bar{z}\|_{\mathfrak{s}}^{|\tau|_{\mathfrak{s}}}\|\tau\| \tag{4.11}$$

for all $\tau \in W$ and all $z, \bar{z} \in \mathfrak{K}$. Then there exists a constant C_0, depending only on Q, such that

$$|(\Pi^\varepsilon_z\mathcal{E}\tau)(\bar{z})| \leqslant C_0 C_{\overline{\mathfrak{K}}}\|z - \bar{z}\|_{\mathfrak{s}}^{|\tau|_{\mathfrak{s}}}\|\tau\| \tag{4.12}$$

holds for all $\tau \in W$ and all $z, \bar{z} \in \mathfrak{K}$, where $\overline{\mathfrak{K}} = \{(t,x) \colon \exists \bar{t} \in \mathbb{R} \colon |\bar{t} - t| \leqslant 2T, (\bar{t},x) \in \mathfrak{K}\}$ (for brevity we shall call $\overline{\mathfrak{K}}$ the $2T$-fattening of $\mathfrak{K}$, although strictly speaking it is only a fattening in the time direction).

Proof. Using the definition (4.6) of $\Pi^\varepsilon_{t,x}\mathcal{E}\tau$ and the assumption (4.11), we obtain

$$
\begin{aligned}
|(\Pi^\varepsilon_{t,x}\mathcal{E}\tau)(\bar{t},\bar{x})| &\leqslant C_{\overline{\mathfrak{K}}}\int_{\bar{t}-2T}^{\bar{t}} |Q(\bar{t} - s)|\|(t,x) - (s,\bar{x})\|_{\mathfrak{s}}^\alpha\,\mathrm{d}s\,\|\tau\| \\
&\leqslant C_{\overline{\mathfrak{K}}}\int_0^{2T} |Q(\bar{s})||t - \bar{t} + \bar{s}|^{\alpha/2}\,\mathrm{d}\bar{s}\,\|\tau\| + C_{\overline{\mathfrak{K}}}\int_0^{2T} |Q(\bar{s})|\,\mathrm{d}\bar{s}\,|x - \bar{x}|^\alpha\|\tau\|\,,
\end{aligned}
$$

where $\alpha = |\tau|_{\mathfrak{s}}$. The required bound thus follows if we can show that

$$\int_0^{2T} |t - \bar{t} + \bar{s}|^{\alpha/2}\,\mathrm{d}\bar{s} \lesssim |t - \bar{t}|^{\alpha/2}$$

holds for $|t - \bar{t}| \lesssim 1$. By treating separately the cases $t > \bar{t}$ and $t < \bar{t}$, one sees that the left-hand side is always bounded above by a constant times $(|t - \bar{t}| + 2T)^{1+\alpha/2}$, which is bounded above for $\alpha > -2$. Since on the other hand, the right-hand side is bounded below by a positive constant for $\alpha < 0$, the result follows. $\square$

Note that [19, Prop. 8.27] shows in particular that the assumption (4.11) is satisfied by any canonical model for mollified noise Π^ε built as in Section 3.2.

We can now state the main result of this subsection, which is an adaptation of the extension theorem [19, Thm. 5.14] and of [19, Prop. 8.27] to our degenerate situation.

Proposition 4.4 (Extension theorem for $\mathcal{E}$)**.** *Let $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$ be a model for the regularity structure $(A, W, \mathcal{G})$, where $W \subset V$, and such that $\bar{z} \mapsto \Pi^\varepsilon_z\tau(\bar{z})$ is continuous and satisfies (4.11) for any $\varepsilon > 0$. Let $\widehat{W} = W \cup \{\mathcal{E}\tau \colon \tau \in W\}$. Then $\widehat{Z}^\varepsilon = (\widehat{\Pi}^\varepsilon, \widehat{\Gamma}^\varepsilon)$ obtained*

by extending Z^ε in the above way is a model for $(A, \widehat{W}, \mathcal{G})$, which satisfies (4.12) for any $\varepsilon > 0$. Furthermore,

$$\|\widehat{\Gamma}^\varepsilon\|_{\gamma;\mathfrak{K}} = \|\Gamma^\varepsilon\|_{\gamma;\mathfrak{K}} \tag{4.13}$$

holds for any $\gamma \in \mathbb{R}$ and any compact $\mathfrak{K} \subset \mathbb{R}^{d+1}$.

Proof. We have to prove that $\widehat{Z}^\varepsilon$ satisfies the assumptions (M1)–(M4) of Definition 3.4. Property (M1) is automatically satisfied owing to (4.7). Regarding property (M2), it is known [19, Sec. 8.3] that there exists a linear map $\mathbf{\Pi}^\varepsilon : T \to \mathcal{S}'(\mathbb{R}^{d+1})$ such that $\mathbf{\Pi}^\varepsilon\tau = \Pi_z^\varepsilon(F_z^\varepsilon)^{-1}\tau$ is independent of z for any $\tau \in T$. Property (M2) holds if we can find for any $\tau \in W$ a distribution $\mathbf{\Pi}^\varepsilon(\mathcal{E}\tau)$, independent of (t, x), such that

$$\Pi_{t,x}^\varepsilon(\mathcal{E}\tau) = \mathbf{\Pi}^\varepsilon F_{t,x}^\varepsilon(\mathcal{E}\tau) \qquad \forall z = (t, x) \in \mathbb{R}^{d+1} \ .$$

To achieve this, we simply define $\mathbf{\Pi}^\varepsilon$ on the extended structure by

$$(\mathbf{\Pi}^\varepsilon\mathcal{E}\tau)(\bar{t}, \bar{x}) = \int_{\bar{t}-2T}^{\bar{t}} Q(\bar{t} - s)(\mathbf{\Pi}^\varepsilon\tau)(s, \bar{x}) \, \mathrm{d}s \ .$$

Indeed, we have (writing as usual $\Delta\tau = \tau^{(1)} \otimes \tau^{(2)}$)

$$\begin{aligned}
(\mathbf{\Pi}^\varepsilon F_z^\varepsilon \mathcal{E}\tau)(\bar{t}, \bar{x}) &= \mathbf{\Pi}^\varepsilon(\mathcal{E} \otimes f_z^\varepsilon)\Delta\tau(\bar{t}, \bar{x}) \\
&= (\mathbf{\Pi}^\varepsilon\mathcal{E}\tau^{(1)})(\bar{t}, \bar{x})\langle f_z^\varepsilon, \tau^{(2)}\rangle \\
&= \int_{\bar{t}-2T}^{\bar{t}} Q(\bar{t} - s)(\mathbf{\Pi}^\varepsilon\tau^{(1)})(s, \bar{x}) \, \mathrm{d}s \, \langle f_z^\varepsilon, \tau^{(2)}\rangle \\
&= \int_{\bar{t}-2T}^{\bar{t}} Q(\bar{t} - s)(\Pi_z^\varepsilon\tau)(s, \bar{x}) \, \mathrm{d}s \\
&= (\Pi_z^\varepsilon\mathcal{E}\tau)(\bar{t}, \bar{x}) \ .
\end{aligned}$$

To obtain the fourth line, we have used the fact that $\Pi_z^\varepsilon = \mathbf{\Pi}^\varepsilon F_z^\varepsilon$ holds for the original model, and thus

$$(\Pi_z^\varepsilon\tau)(s, \bar{x}) = (\mathbf{\Pi}^\varepsilon F_z^\varepsilon\tau)(s, \bar{x}) = (\mathbf{\Pi}^\varepsilon(\mathrm{Id} \otimes f_z^\varepsilon)\Delta\tau)(s, \bar{x}) = (\mathbf{\Pi}^\varepsilon\tau^{(1)})(s, \bar{x})\langle f_z^\varepsilon, \tau^{(2)}\rangle \ .$$

Property (M3) is a direct consequence of (4.12). Indeed, this bound implies that for any localised test function η of integral 1,

$$\begin{aligned}
\left|\langle\Pi_{t,x}^\varepsilon\mathcal{E}\tau, \eta_{t,x}^\delta\rangle\right| &= \left|\iint (\Pi_{t,x}^\varepsilon\mathcal{E}\tau)(\bar{t}, \bar{x})\frac{1}{\delta^{d+2}}\eta\left(\frac{\bar{t} - t}{\delta^2}, \frac{\bar{x} - x}{\delta}\right) \mathrm{d}\bar{t} \, \mathrm{d}\bar{x}\right| \\
&\leqslant \iint \left|(\Pi_{t,x}^\varepsilon\mathcal{E}\tau)(t + \delta^2 s, x + \delta y)\right|\left|\eta(s, y)\right| \mathrm{d}s \, \mathrm{d}y \\
&\lesssim \delta^{|\tau|_\mathfrak{s}}\|\tau\|
\end{aligned}$$

as required by (3.20). Finally, in order to prove Property (M4), we recall from (4.5) that

$$\Gamma_{z\bar{z}}^\varepsilon\mathcal{E}\tau = \mathcal{E}\Gamma_{z\bar{z}}^\varepsilon\tau \ .$$

If $|\tau|_\mathfrak{s} \geqslant 0$, there is nothing to prove. If $\beta < \alpha = |\tau|_\mathfrak{s} < 0$, then

$$\|\Gamma_{z\bar{z}}^\varepsilon\mathcal{E}\tau\|_\beta = \|\mathcal{E}\Gamma_{z\bar{z}}^\varepsilon\tau\|_\beta = \|\Gamma_{z\bar{z}}^\varepsilon\tau\|_\beta \leqslant \|\Gamma^\varepsilon\|_{\gamma;\mathfrak{K}}\|\tau\|\|z - \bar{z}\|_\mathfrak{s}^{\alpha-\beta}$$

for all $z, \bar{z} \in \mathfrak{K}$. This completes the proof that $\widehat{Z}^\varepsilon$ is a model, and also proves (4.13). $\qquad\square$

4.2 Multilevel Schauder estimates

We now would like to construct an operator $\mathcal{K}_\gamma^Q : \mathcal{D}^\gamma \to \mathcal{D}^\gamma$ which lifts the operation of integration against Q to the regularity structure. Formally, this means that we should have

$$\mathcal{R}\mathcal{K}_\gamma^Q f = K_v * \mathcal{R}f \,, \qquad K_v(t,x,\bar t,\bar x) = Q(t - \bar t)\delta(x - \bar x) \tag{4.14}$$

for all $f \in \mathcal{D}^\gamma$. In fact, this should really be interpreted as

$$(\mathcal{R}\mathcal{K}_\gamma^Q f)(t,x) = \int_{t-2T}^{t} Q(t - \bar t)(\mathcal{R}f)(\bar t, x)\, \mathrm{d}\bar t \,, \tag{4.15}$$

or in terms of test functions

$$\langle \mathcal{R}\mathcal{K}_\gamma^Q f, \psi \rangle = \langle \mathcal{R}f, \hat\psi \rangle \,, \tag{4.16}$$

where $\hat\psi$ is defined by

$$\hat\psi(\bar t, y) := \int_\mathbb{R} Q(s - \bar t)\psi(s,y)\, \mathrm{d}s = \int_{\bar t}^{\bar t + 2T} Q(s - \bar t)\psi(s,y)\, \mathrm{d}s \,. \tag{4.17}$$

The problem with such a plan is that the kernel K_v being singular in space, we cannot apply an expansion as in (3.40) and (3.41). However, for the class of equations we are interested in, we do not actually need to define $\mathcal{K}_\gamma^Q$ on *all* of $\mathcal{D}^\gamma$. It will be quite sufficient to define it on a subset of $\mathcal{D}^\gamma$, which is given by the functions whose components with negative homogeneity do not depend on (t,x). This motivates the following definition.

Definition 4.5. *Let $Z = (\Pi, \Gamma)$ be a model. The space $\mathcal{D}_0^\gamma(Z) = \mathcal{D}_0^\gamma(\Gamma)$ is the space of functions $f \in \mathcal{D}^\gamma(\Gamma)$ of the form*

$$f(z) = \sum_{\tau \in T\,:\, -2 < |\tau|_\mathfrak{s} < 0} c_\tau \tau + \sum_{\tau \in T\,:\, |\tau|_\mathfrak{s} \geqslant 0} \hat c_\tau(z)\tau =: f_- + f_+(z) \,, \tag{4.18}$$

where the c_τ do not depend on z.

Remark 4.6. Not all constant functions belong to $\mathcal{D}_0^\gamma$, as they still have to satisfy the analytical bound (3.34), which includes the requirement $\|f(z) - \Gamma_{z\bar z}f(\bar z)\|_\beta = \mathcal{O}(\|z - \bar z\|^{\gamma - \beta})$ as $z \to \bar z$ for all $\beta < \gamma$. In fact, since $\Gamma_{z\bar z}\tau = (\mathrm{Id} \otimes \gamma_{z\bar z})\Delta\tau$, we see that a sufficient condition for having $\Gamma_{z\bar z}\tau = \tau$ is

$$\Delta\tau = \tau \otimes \mathbf{1} \,. \tag{4.19}$$

Thus the sum defining f_- should involve only terms $c_\tau \tau$ such that $\Delta\tau = \tau \otimes \mathbf{1}$. As can be seen in Table 1, in our case this property is satisfied by the symbols Ξ, $\uparrow$, $\vee$, $\mathbf{\vee}$, and will also hold e.g. for $\mathring{\vee}$, but it does *not* hold for $\mathbf{\vee}$ for instance. Note that owing to (3.15) and (3.33), (4.19) implies $\Pi_{\bar z}\tau = \Pi_z\tau$ for all $z, \bar z \in \mathbb{R}^{d+1}$, i.e., the model does not depend on the base point z for these τ. $\lozenge$

With this notation in place, we define the operator $\mathcal{K}_\gamma^Q$ on $\mathcal{D}_0^\gamma$ by

$$(\mathcal{K}_\gamma^Q f)(z) = \mathcal{E}f_- + \int_{t-2T}^{t} Q(t - s)f_+(s,x)\, \mathrm{d}s \,. \tag{4.20}$$

Note that this definition indeed requires $\mathcal{E}$ to be defined only on symbols τ of strictly negative homogeneity.

Remark 4.7. An alternative definition of $\mathcal{K}_\gamma^Q$, closer in spirit to (3.39), would be to set

$$(\mathcal{K}_\gamma^Q f)(z) = \mathcal{E}f(z) + \hat{\mathcal{J}}^Q(z)f_+(z) + (\mathcal{N}_\gamma^Q f_+)(z) \,,$$

where $\mathcal{E}$ is defined for positive-homogeneous terms as in Remark 4.1 and

$$\hat{\mathcal{J}}^Q(t,x)\tau = \int_{t-2T}^t Q(t-s)(\Pi_{t,x}^{(\varepsilon)}\tau)(s,x)\,\mathrm{d}s\,\mathbf{1}\,,$$

$$(\mathcal{N}_\gamma^Q f_+)(z) = \int_{t-2T}^t Q(t-s)(\mathcal{R}f_+ - \Pi_{t,x}^{(\varepsilon)}f_+(t,x))(s,x)\,\mathrm{d}s\,\mathbf{1}\,.$$

While it is easily checked that $\mathcal{K}_\gamma^Q$ defined in this way indeed satisfies (4.15), the problem is that it typically does not map $\mathcal{D}^\gamma$ into itself, but only into some $\mathcal{D}^{\gamma'}$ with $\gamma' < 1$. This is due to the fact that we cannot Taylor-expand the kernel K_v in space, preventing us from subtracting higher-order polynomial terms as in (3.41). $\Diamond$

The following lemma on translation invariance of the canonical model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$ will allow us to prove that $\mathcal{K}_\gamma^Q$ given by (4.20) satisfies the required properties. We give its proof in Appendix A.

Lemma 4.8. *Let $(\Pi^\varepsilon, \Gamma^\varepsilon)$ be the canonical model defined by (3.22), (3.27) and (4.6). For all $h, z, \bar{z} \in \mathbb{R}^{d+1}$, all $\tau \in T$, all $\bar{\tau} \in \mathcal{F}_+$ and all $\varepsilon > 0$, one has*

$$\Pi_{z+h}^\varepsilon \tau(\bar{z} + h) = \Pi_z^\varepsilon \tau(\bar{z})\,,$$
$$\Gamma_{z+h,\bar{z}+h}^\varepsilon \tau = \Gamma_{z\bar{z}}^\varepsilon \tau\,,$$
$$\langle \gamma_{z+h,\bar{z}+h}^\varepsilon, \bar{\tau} \rangle = \langle \gamma_{z\bar{z}}^\varepsilon, \bar{\tau} \rangle\,. \tag{4.21}$$

The following result shows that the multilevel Schauder estimates contained in [19, Thm. 5.12] also hold in the case of $\mathcal{E}$, in a similar form.

Proposition 4.9 (Multilevel Schauder estimates on $\mathcal{K}_\gamma^Q$). *If $\gamma > 0$, then the operator $\mathcal{K}_\gamma^Q$ maps $\mathcal{D}_0^\gamma$ into $\mathcal{D}_0^\gamma$, and satisfies*

$$\|\mathcal{K}_\gamma^Q f\|_{\gamma;\mathfrak{K}} \leqslant (1 \vee \|Q\|_{L^1})\|f\|_{\gamma;\overline{\mathfrak{K}}}\,, \tag{4.22}$$

where $\|Q\|_{L^1} = \int_{\mathbb{R}} |Q(t)|\,\mathrm{d}t = \int_0^{2T} |Q(t)|\,\mathrm{d}t$, and $\overline{\mathfrak{K}}$ is the $2T$-fattening of $\mathfrak{K}$. The identity

$$(\mathcal{R}\mathcal{K}_\gamma^Q f)(t,x) = \int_{t-2T}^t Q(t-s)\mathcal{R}f(s,x)\,\mathrm{d}s \tag{4.23}$$

holds for all $f \in \mathcal{D}_0^\gamma$. Furthermore, if $(\overline{\Pi}, \overline{\Gamma})$ is a second translation-invariant model satisfying (3.27) and (4.6), and $\overline{\mathcal{K}}_\gamma^Q$ is the associated lift, then

$$\|\mathcal{K}_\gamma^Q f\,;\overline{\mathcal{K}}_\gamma^Q \bar{f}\|_{\gamma;\mathfrak{K}} \leqslant (1 \vee \|Q\|_{L^1})\|f\,;\bar{f}\|_{\gamma;\overline{\mathfrak{K}}} \tag{4.24}$$

holds for all $f \in \mathcal{D}_0^\gamma(\Gamma)$ and $\bar{f} \in \mathcal{D}_0^\gamma(\overline{\Gamma})$.

Proof. We start by showing that relation (4.23) holds, assuming $f, \mathcal{K}_\gamma^Q f \in \mathcal{D}_0^\gamma$. It follows from the reconstruction theorem and its corollary [19, Prop. 3.28] (see also (4.11)) that

$$\mathcal{R}f_+(z) = \langle \mathbf{1}, f_+(z) \rangle = (\Pi_z^\varepsilon f_+(z))(z)$$

for all $z \in \mathbb{R}^{d+1}$. Furthermore, let

$$(\widehat{\mathcal{R}}f_-)(\bar{z}) := \Pi_z^\varepsilon f_-(\bar{z}) = \sum_{-2 < |\tau|_\mathfrak{s} < 0} c_\tau (\Pi_z^\varepsilon \tau)(\bar{z})$$

(where the right-hand side actually does not depend on z, cf. Remark 4.6). Then we see that $\widehat{\mathcal{R}}f_-$ trivially satisfies the reconstruction theorem, so that by uniqueness of the reconstruction operator if $\gamma > 0$ we have

$$\mathcal{R}f_- = \widehat{\mathcal{R}}f_- = \Pi_z^\varepsilon f_-$$

whenever $f_- \in \mathcal{D}_0^\gamma$. Thus in fact

$$\mathcal{R}f(z) = \big(\Pi_z^\varepsilon f(z)\big)(z)$$

holds for all $z \in \mathbb{R}^{d+1}$. It follows by (4.5) and the same argument as the one yielding $\mathcal{R}f_- = \Pi_z^\varepsilon f_-$ that

$$
\begin{aligned}
(\mathcal{R}\mathcal{E}f_-)(t,x) &= (\Pi_{t,x}^\varepsilon \mathcal{E}f_-)(t,x) \\
&= \int_{t-2T}^t Q(t-s)(\Pi_{t,x}^\varepsilon f_-)(s,x)\,\mathrm{d}s \\
&= \int_{t-2T}^t Q(t-s)\mathcal{R}f_-(s,x)\,\mathrm{d}s \ ,
\end{aligned}
$$

which implies that we indeed have

$$
\begin{aligned}
(\mathcal{R}\mathcal{K}_\gamma^Q f)(t,x) &= (\mathcal{R}\mathcal{E}f_-)(t,x) + \int_{t-2T}^t Q(t-s)\mathcal{R}f_+(s,x)\,\mathrm{d}x \\
&= \int_{t-2T}^t Q(t-s)\mathcal{R}f(s,x)\,\mathrm{d}x \ .
\end{aligned}
$$

Next we check that $\mathcal{K}_\gamma^Q$ indeed maps $\mathcal{D}_0^\gamma$ into itself. Since $\mathcal{E}$ preserves homogeneity, it is in fact sufficient to show that $\mathcal{K}_\gamma^Q$ maps $\mathcal{D}_0^\gamma$ into $\mathcal{D}^\gamma$. First we note that for any $\beta < \gamma$,

$$
\begin{aligned}
\|\mathcal{K}_\gamma^Q f(z)\|_\beta &\leqslant \|\mathcal{E}f_-(z)\|_\beta + \int_{t-2T}^t |Q(t-s)|\|f_+(s,x)\|_\beta\,\mathrm{d}s \\
&\leqslant \|f_-(z)\|_\beta + \|Q\|_{L^1} \sup_{s\in[t-2T,t]} \|f_+(s,x)\|_\beta \\
&\leqslant (1 \vee \|Q\|_{L^1})\|f\|_{\gamma;\overline{\mathfrak{K}}} \ ,
\end{aligned}
$$

(note that depending on the sign of β, either one or the other term in the sums contributes, but never both). Regarding terms evaluated in different locations, note that (4.5) implies that

$$
\begin{aligned}
\|\mathcal{E}f_- - \Gamma_{z\bar{z}}^\varepsilon \mathcal{E}f_-\|_\beta &= \|\mathcal{E}(f_- - \Gamma_{z\bar{z}}^\varepsilon f_-)\|_\beta \\
&= \|f_- - \Gamma_{z\bar{z}}^\varepsilon f_-\|_\beta \\
&\leqslant \|f_-\|_{\gamma;\mathfrak{K}} \|z - \bar{z}\|_{\mathfrak{s}}^{\gamma-\beta} \ .
\end{aligned}
$$

As for the positive-homogeneous part $g = \mathcal{K}_\gamma^Q f_+$, using Lemma 4.8 we get

$$
\begin{aligned}
g(t,x) - \Gamma_{t,x;\bar{t},\bar{x}}^\varepsilon g(\bar{t},\bar{x}) &= \int_0^{2T} Q(s)\big[f_+(t-s,x) - \Gamma_{t,x;\bar{t},\bar{x}}^\varepsilon f_+(\bar{t}-s,\bar{x})\big]\,\mathrm{d}s \\
&= \int_0^{2T} Q(s)\big[f_+(t-s,x) - \Gamma_{t-s,x;\bar{t}-s,\bar{x}}^\varepsilon f_+(\bar{t}-s,\bar{x})\big]\,\mathrm{d}s \ ,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\|g(t,x) - \Gamma_{t,x;\bar{t},\bar{x}}^\varepsilon g(\bar{t},\bar{x})\|_\beta &\leqslant \int_0^{2T} |Q(s)|\|f_+(t-s,x) - \Gamma_{t-s,x;\bar{t}-s,\bar{x}}^\varepsilon f_+(\bar{t}-s,\bar{x})\|_\beta\,\mathrm{d}s \\
&\leqslant \|Q\|_{L^1}\|f_+\|_{\gamma;\overline{\mathfrak{K}}} \|(t,x) - (\bar{t},\bar{x})\|_{\mathfrak{s}}^{\gamma-\beta} \ .
\end{aligned}
$$

Collecting the estimates obtained and using the definition (3.34) of $\|\cdot\|$, we see that $\mathcal{K}_\gamma^Q f \in \mathcal{D}^\gamma$ and that it satisfies (4.22).

Consider finally the case where we have a second modelled distribution $\bar{f} \in \mathcal{D}_0^\gamma(\overline{\Gamma})$. Using again (4.5), it is immediate to see that

$$\|\mathcal{E}f_- \,; \mathcal{E}\bar{f}_-\|_{\gamma;\mathfrak{K}} \leqslant \|f_- \,; \bar{f}_-\|_{\gamma;\overline{\mathfrak{K}}} \,.$$

Writing $\bar{g}(z) = \overline{\mathcal{K}}_\gamma^Q \bar{f}_+(z)$, it follows in the same way as above that

$$\|g - \bar{g}\|_{\gamma;\mathfrak{K}} \leqslant \|Q\|_{L^1} \|f_+ - \bar{f}_+\|_{\gamma,\overline{\mathfrak{K}}} \,.$$

Finally, we have

$$g(t,x) - \bar{g}(t,x) - \Gamma_{t,x;\bar{t},\bar{x}}^\varepsilon g(\bar{t},\bar{x}) + \overline{\Gamma}_{t,x;\bar{t},\bar{x}}\bar{g}(\bar{t},\bar{x})$$
$$= \int_0^{2T} Q(s)\big[f_+(t-s,x) - \bar{f}_+(t-s,x) - \Gamma_{t,x;\bar{t},\bar{x}}^\varepsilon f_+(\bar{t}-s,\bar{x}) + \overline{\Gamma}_{t,x;\bar{t},\bar{x}}\bar{f}_+(\bar{t}-s,\bar{x})\big]\,\mathrm{d}s \,.$$

Using translation invariance of Γ^ε and $\overline{\Gamma}$, we obtain for all $\beta < \gamma$

$$\|g(t,x) - \bar{g}(t,x) - \Gamma_{t,x;\bar{t},\bar{x}}^\varepsilon g(\bar{t},\bar{x}) + \overline{\Gamma}_{t,x;\bar{t},\bar{x}}\bar{g}(\bar{t},\bar{x})\|_\beta$$
$$\leqslant \|Q\|_{L^1} \sup_{s \in [0,2T]} \|(t-s,x) - (\bar{t}-s,\bar{x})\|_{\mathfrak{s}}^{\gamma-\beta} \|f_+ \,; \bar{f}_+\|_{\gamma;\overline{\mathfrak{K}}}$$
$$= \|Q\|_{L^1} \|(t,x) - (\bar{t},\bar{x})\|_{\mathfrak{s}}^{\gamma-\beta} \|f_+ \,; \bar{f}_+\|_{\gamma;\overline{\mathfrak{K}}} \,,$$

from which (4.24) follows easily. $\qquad\square$

4.3 Extension to the space $\mathcal{D}^{\gamma,\eta}$

As already mentioned at the end of Section 3.3, the definition of the spaces $\mathcal{D}^\gamma$ has to be modified in order to be able to deal with the time-zero singularity of the heat kernel. In particular, we have to check that the estimates on $\mathcal{K}_\gamma^Q$ established in Proposition 4.9 still hold in these modified spaces.

We recall a few notations from [19, Section 6]. Let $P = \{(t,x) \in \mathbb{R}^{d+1} : t = 0\}$ denote the time-zero hyperplane, and introduce the scaled norms

$$\|z\|_P = 1 \wedge |t|^{1/2} \,, \qquad \|z,\bar{z}\|_P = \|z\|_P \wedge \|\bar{z}\|_P = 1 \wedge |t|^{1/2} \wedge |\bar{t}|^{1/2} \,. \qquad (4.25)$$

For a compact $\mathfrak{K} \subset \mathbb{R}^{d+1}$, let

$$\mathfrak{K}_P = \big\{(z,\bar{z}) \in (\mathfrak{K} \setminus P)^2 : z \neq \bar{z}, \|z - \bar{z}\|_{\mathfrak{s}} \leqslant \|z,\bar{z}\|_P\big\} \,. \qquad (4.26)$$

Definition 4.10 ([19, Def. 6.2])**.** *Given a model (Π,Γ) and constants $\gamma > 0$ and $\eta \in \mathbb{R}$, set*

$$\|f\|_{\gamma,\eta;\mathfrak{K}} := \sup_{z \in \mathfrak{K} \setminus P} \sup_{\beta < \gamma} \frac{\|f(z)\|_\beta}{\|z\|_P^{(\eta-\beta) \wedge 0}} \,, \qquad [\![f]\!]_{\gamma,\eta;\mathfrak{K}} := \sup_{z \in \mathfrak{K} \setminus P} \sup_{\beta < \gamma} \frac{\|f(z)\|_\beta}{\|z\|_P^{\eta-\beta}} \,. \qquad (4.27)$$

The space $\mathcal{D}^{\gamma,\eta}$ consists of all functions $f : \mathbb{R}^{d+1} \setminus P \to T_\gamma^-$ such that

$$\|\!|\!|f|\!|\!|_{\gamma,\eta;\mathfrak{K}} := \|f\|_{\gamma,\eta;\mathfrak{K}} + \sup_{(z,\bar{z}) \in \mathfrak{K}_P} \sup_{\beta < \gamma} \frac{\|f(z) - \Gamma_{z\bar{z}}f(\bar{z})\|_\beta}{\|z - \bar{z}\|_{\mathfrak{s}}^{\gamma-\beta}\|z,\bar{z}\|_P^{\eta-\gamma}} < \infty \,. \qquad (4.28)$$

Given a second model $(\overline{\Pi},\overline{\Gamma})$ and $\bar{f} \in \mathcal{D}^{\gamma,\eta}(\overline{\Gamma})$, we also set

$$\|\!|\!|f \,; \bar{f}|\!|\!|_{\gamma,\eta;\mathfrak{K}} := \|f - \bar{f}\|_{\gamma,\eta;\mathfrak{K}} + \sup_{(z,\bar{z}) \in \mathfrak{K}_P} \sup_{\beta < \gamma} \frac{\|f(z) - \bar{f}(z) - \Gamma_{z\bar{z}}f(\bar{z}) + \overline{\Gamma}_{z\bar{z}}\bar{f}(\bar{z})\|_\beta}{\|z - \bar{z}\|_{\mathfrak{s}}^{\gamma-\beta}\|z,\bar{z}\|_P^{\eta-\gamma}} \,. \qquad (4.29)$$

Finally, similarly to Definition 4.5, we define $\mathcal{D}_0^{\gamma,\eta}$ to be the set of $f \in \mathcal{D}^{\gamma,\eta}$ whose components with negative homogeneity do not depend on z.

Most of the results in [19, Section 6] are directly applicable to the present setting. The only result that has to be adapted is Proposition 6.16, which takes here the following form.

Proposition 4.11 (Multilevel Schauder estimates on $\mathcal{D}_0^{\gamma,\eta}$)**.** *Let $f \in \mathcal{D}_0^{\gamma,\eta}(\Gamma^\varepsilon)$ with*

$$-2 < \eta < 0 < \gamma < \eta + 2 , \tag{4.30}$$

and assume that $f_+(t, x) = 0$ whenever $t < 0$. Then $\mathcal{K}_\gamma^Q f \in \mathcal{D}_0^{\gamma,\eta}(\Gamma^\varepsilon)$, and there exists a constant C_1, depending only on Q, such that

$$\|\mathcal{K}_\gamma^Q f\|_{\gamma,\eta;\mathfrak{K}} \leqslant C_1 \big[1 + \|\Gamma^\varepsilon\|_{\gamma;\overline{\mathfrak{K}}}\big] \|f\|_{\gamma,\eta;\overline{\mathfrak{K}}} , \tag{4.31}$$

where $\overline{\mathfrak{K}}$ is the $2T$-fattening of $\mathfrak{K}$. Furthermore, let $\bar{f} \in \mathcal{D}_0^{\gamma,\eta}(\overline{\Gamma})$, where $(\overline{\Pi}, \overline{\Gamma})$ is another translation-invariant model satisfying (3.27) and (4.6), and assume that $\bar{f}_+(t, x) = 0$ whenever $t < 0$. Then there exists a constant C_2, depending only on Q, such that

$$\|\mathcal{K}_\gamma^Q f ; \overline{\mathcal{K}}_\gamma^Q \bar{f}\|_{\gamma,\eta;\mathfrak{K}} \leqslant C_1 \big[1 + \|\Gamma^\varepsilon\|_{\gamma;\overline{\mathfrak{K}}}\big] \|f ; \bar{f}\|_{\gamma,\eta;\overline{\mathfrak{K}}} + C_2 \|f\|_{\gamma,\eta;\overline{\mathfrak{K}}} \|\overline{\Gamma} - \Gamma^\varepsilon\|_{\gamma;\overline{\mathfrak{K}}} . \tag{4.32}$$

Proof. It follows as in the proof of Proposition 4.9 that

$$\|\mathcal{E}f_-\|_{\gamma,\eta;\mathfrak{K}} \leqslant \|f_-\|_{\gamma,\eta;\overline{\mathfrak{K}}} , \qquad \|\mathcal{E}f_- ; \mathcal{E}\bar{f}_-\|_{\gamma,\eta;\mathfrak{K}} \leqslant \|f_- ; \bar{f}_-\|_{\gamma,\eta;\overline{\mathfrak{K}}} ,$$

so that we only have to consider the positive-homogeneous part f_+ of f. For convenience, we again write $g(z)$ instead of $(\mathcal{K}_\gamma^Q f_+)(z)$. Note that

$$\|f_+\|_{\gamma,\eta;\mathfrak{K}} = \sup_{z \in \mathfrak{K} \setminus P} \sup_{\beta < \gamma} \frac{\|f_+(z)\|_\beta}{\|z\|_P^{(\eta-\beta)\wedge 0}} = \sup_{z=(t,x) \in \mathfrak{K} \setminus P} \sup_{0 \leqslant \beta < \gamma} \Big[(1 \wedge |t|)^{(\beta-\eta)/2} \|f_+(z)\|_\beta\Big] ,$$

since we may always assume $\beta \geqslant 0$ and thus $(\eta - \beta) \wedge 0 = \eta - \beta$. Since $f_+(t, x) = 0$ for $t < 0$ we have

$$\|g(t,x)\|_\beta \leqslant \int_{(t-2T)\vee 0}^t |Q(t-s)| \|f_+(s,x)\|_\beta \, \mathrm{d}s \leqslant \int_0^t \frac{|Q(t-s)|}{(1 \wedge s)^{(\beta-\eta)/2}} \, \mathrm{d}s \, \|f_+\|_{\gamma,\eta;\overline{\mathfrak{K}}} .$$

Since $\beta < \gamma < \eta + 2$, the integral is convergent and we have

$$\|g\|_{\gamma,\eta;\mathfrak{K}} \leqslant C_Q \|f_+\|_{\gamma,\eta;\overline{\mathfrak{K}}} ,$$

where C_Q depends only on Q. In order to estimate $\|g\|_{\gamma,\eta;\mathfrak{K}}$, we also need to bound differences of the form $g(z) - \Gamma_{z\bar{z}}^\varepsilon g(\bar{z})$. From now on we assume that $z = (t,x)$ and $\bar{z} = (\bar{t}, \bar{x})$ with $t \leqslant \bar{t}$, since the case $t > \bar{t}$ follows by symmetry. Using the fact that f_+ is supported on $\{t \geqslant 0\}$ and translation invariance, we get

$$\|g(z) - \Gamma_{z\bar{z}}^\varepsilon g(\bar{z})\|_\beta \leqslant \int_0^{2T \wedge \bar{t}} |Q(s)| \|f_+(t-s,x) - \Gamma_{t-s,x;\bar{t}-s,\bar{x}}^\varepsilon f_+(\bar{t}-s,\bar{x})\|_\beta \, \mathrm{d}s .$$

However, there is a difficulty because $(z, \bar{z}) \in \mathfrak{K}_P$ does not automatically imply that one has $((t-s, x), (\bar{t}-s, \bar{x})) \in \mathfrak{K}_P$. In fact,

$$((t-s, x), (\bar{t}-s, \bar{x})) \in \mathfrak{K}_P \Leftrightarrow (1 \wedge |t-s| \wedge |\bar{t}-s|)^{1/2} \geqslant \|(t-s,x) - (\bar{t}-s,\bar{x})\|_{\mathfrak{s}}$$
$$\Leftrightarrow 1 \wedge |t-s| \wedge |\bar{t}-s| \geqslant \|z - \bar{z}\|_{\mathfrak{s}}^2 .$$

By definition of $\mathfrak{K}_P$, we can assume that $\|z - \bar{z}\|_{\mathfrak{s}} \leqslant 1$, so that $s \leqslant t - \|z - \bar{z}\|_{\mathfrak{s}}^2$ is a sufficient condition for having $((t-s, x), (\bar{t}-s, \bar{x})) \in \mathfrak{K}_P$. We split the integration interval into

$[0, t - \|z - \bar{z}\|_{\mathfrak{s}}^2]$ and $[t - \|z - \bar{z}\|_{\mathfrak{s}}^2, 2T \wedge \bar{t}]$, and treat each interval separately. For the first interval, we use the definition of $\|f_+\|$ to get

$$
\int_0^{t - \|z - \bar{z}\|_{\mathfrak{s}}^2} |Q(s)| \| f_+(t - s, x) - \Gamma_{t-s,x;\bar{t}-s,\bar{x}}^\varepsilon f_+(\bar{t} - s, \bar{x}) \|_\beta \, \mathrm{d}s
$$

$$
\leqslant \int_0^{t - \|z - \bar{z}\|_{\mathfrak{s}}^2} \frac{|Q(s)|}{(1 \wedge (t - s))^{(\gamma - \eta)/2}} \, \mathrm{d}s \, \|z - \bar{z}\|_{\mathfrak{s}}^{\gamma - \beta} \| f_+ \|_{\gamma, \eta; \overline{\mathfrak{K}}}
$$

$$
\leqslant \frac{C_Q}{(1 \wedge t)^{(\gamma - \eta)/2}} \|z - \bar{z}\|_{\mathfrak{s}}^{\gamma - \beta} \| f_+ \|_{\gamma, \eta; \overline{\mathfrak{K}}} \, .
$$

For the second part of the integral, we bound $\| f_+(t - s, x) - \Gamma_{z\bar{z}}^\varepsilon f_+(\bar{t} - s, \bar{x}) \|_\beta$ above by the sum $\| f_+(t - s, x) \|_\beta + \| \Gamma_{z\bar{z}}^\varepsilon f_+(\bar{t} - s, \bar{x}) \|_\beta$, and treat each term separately. Using the assumption $f_+(t, x) = 0$ for $t < 0$ and performing the change of variables $s = t - \bar{s}$, the first term becomes

$$
\int_0^{\|z - \bar{z}\|_{\mathfrak{s}}^2} |Q(t - \bar{s})| \| f_+(\bar{s}, x) \|_\beta \, \mathrm{d}\bar{s} \leqslant \int_0^{\|z - \bar{z}\|_{\mathfrak{s}}^2} \frac{|Q(t - \bar{s})|}{(1 \wedge \bar{s})^{(\beta - \eta)/2}} \, \mathrm{d}\bar{s} \, \| f_+ \|_{\gamma, \eta; \overline{\mathfrak{K}}}
$$

$$
\lesssim \|z - \bar{z}\|_{\mathfrak{s}}^{2 - \beta + \eta} \| f_+ \|_{\gamma, \eta; \overline{\mathfrak{K}}} \, .
$$

For the second term, we use the definition of $\| \Gamma^\varepsilon \|$ and the decomposition of f_+ as a linear combination of τ to obtain

$$
\| \Gamma_{z\bar{z}}^\varepsilon f_+(\bar{s}, \bar{x}) \|_\beta \leqslant \sum_\delta \| f_+(\bar{s}, \bar{x}) \|_\delta \|z - \bar{z}\|_{\mathfrak{s}}^{\delta - \beta} \| \Gamma^\varepsilon \|_{\gamma; \overline{\mathfrak{K}}}
$$

$$
\leqslant \sum_\delta \frac{\| f_+ \|_{\gamma, \eta; \overline{\mathfrak{K}}}}{(1 \wedge \bar{s})^{(\delta - \eta)/2}} \|z - \bar{z}\|_{\mathfrak{s}}^{\delta - \beta} \| \Gamma^\varepsilon \|_{\gamma; \overline{\mathfrak{K}}} \, ,
$$

where the sum runs over finitely many values of $\delta \in [0, \gamma]$. Setting $s = \bar{t} - \bar{s}$, this yields a contribution

$$
\int_0^{\|z - \bar{z}\|_{\mathfrak{s}}^2 + \bar{t} - t} |Q(\bar{t} - \bar{s})| \| \Gamma_{z\bar{z}}^\varepsilon f_+(\bar{s}, x) \|_\beta \, \mathrm{d}\bar{s}
$$

$$
\leqslant \sum_\delta \int_0^{\|z - \bar{z}\|_{\mathfrak{s}}^2 + \bar{t} - t} \frac{|Q(\bar{t} - \bar{s})|}{(1 \wedge \bar{s})^{(\delta - \eta)/2}} \, \mathrm{d}\bar{s} \, \|z - \bar{z}\|_{\mathfrak{s}}^{\delta - \beta} \| f_+ \|_{\gamma, \eta; \overline{\mathfrak{K}}} \| \Gamma^\varepsilon \|_{\gamma; \overline{\mathfrak{K}}}
$$

$$
\lesssim \sum_\delta \|z - \bar{z}\|_{\mathfrak{s}}^{2 - \delta + \eta} \|z - \bar{z}\|_{\mathfrak{s}}^{\delta - \beta} \| f_+ \|_{\gamma, \eta; \overline{\mathfrak{K}}} \| \Gamma^\varepsilon \|_{\gamma; \overline{\mathfrak{K}}}
$$

$$
\lesssim \|z - \bar{z}\|_{\mathfrak{s}}^{2 + \eta - \beta} \| f_+ \|_{\gamma, \eta; \overline{\mathfrak{K}}} \| \Gamma^\varepsilon \|_{\gamma; \overline{\mathfrak{K}}} \, .
$$

Combining the different bounds, we obtain (4.31).

It remains to obtain a similar bound on $\| g \, ; \bar{g} \|$, where $\bar{g} = \overline{\mathcal{K}}_\gamma^Q \bar{f}_+$. The same argument as before yields

$$
\| g - \bar{g} \|_{\gamma, \eta; \mathfrak{K}} \leqslant C_Q \| f_+ - \bar{f}_+ \|_{\gamma, \eta; \overline{\mathfrak{K}}} \, .
$$

Regarding the second term in the definition of $\| g \, ; \bar{g} \|$, we again split the integral defining it into two parts. The first one is

$$
\int_0^{t - \|z - \bar{z}\|_{\mathfrak{s}}^2} |Q(s)| \| f_+(t - s, x) - \bar{f}_+(t - s, x) - \Gamma_{z\bar{z}}^\varepsilon f_+(\bar{t} - s, \bar{x}) + \overline{\Gamma}_{z\bar{z}} \bar{f}_+(\bar{t} - s, \bar{x}) \|_\beta \, \mathrm{d}s
$$

$$
\leqslant \frac{C_Q}{(1 \wedge t)^{(\gamma - \eta)/2}} \|z - \bar{z}\|_{\mathfrak{s}}^{\gamma - \beta} \| f_+ \, ; \bar{f}_+ \|_{\gamma, \eta; \overline{\mathfrak{K}}} \, .
$$

For the second part, we use the decomposition

$$\|f_+(t-s,x) - \bar{f}_+(t-s,x) - \Gamma^\varepsilon_{z\bar{z}}f_+(\bar{t}-s,\bar{x}) + \overline{\Gamma}_{z\bar{z}}\bar{f}_+(\bar{t}-s,\bar{x})\|_\beta$$
$$\leqslant \|f_+(t-s,x) - \bar{f}_+(t-s,x)\|_\beta$$
$$+ \|\Gamma^\varepsilon_{z\bar{z}}(f_+(\bar{t}-s,\bar{x}) - \bar{f}_+(\bar{t}-s,\bar{x}))\|_\beta$$
$$+ \|(\Gamma^\varepsilon_{z\bar{z}} - \overline{\Gamma}_{z\bar{z}})\bar{f}_+(\bar{t}-s,\bar{x})\|_\beta \ ,$$

and estimate the integral of each term separately, in the same way as above. Combining the different bounds, we obtain (4.32). □

Remark 4.12. Note that the condition (4.30) on η and γ excludes integer values for these parameters, except that we may have either $\eta = -1$ or $\gamma = 1$. The reason while these integer values are allowed is that owing to the parabolic scaling, the exponents such as $(\gamma - \eta)/2$ occurring in integrals over time can never take integer values that could make them divergent. ◊

5 The fixed-point equation

In this section, we construct a fixed-point equation lifting (4.2) to an appropriate space $\mathcal{D}^{\gamma,\eta}$, and prove that it admits a unique fixed point, depending Lipschitz-continuously on the lifted initial data and noise. The procedure is very similar to the one for the Allen–Cahn equation or the Φ^4_3 model carried out in [19, Section 9], with only a few adjustments required by the additional variable v.

5.1 The set-up

From now on, we assume that the space dimension is $d = 3$, since this is the most difficult case which is still renormalisable. Consider the fixed-point equation on $\mathcal{D}^{\gamma,\eta} \times \mathcal{D}^{\gamma,\eta}$ given by

$$U = (\mathcal{K}_{\bar{\gamma}} + R_\gamma \mathcal{R})\boldsymbol{R}^+ \big[\Xi + F(U,V)\big] + G u_0 \ ,$$
$$V = \mathcal{K}^Q_\gamma \boldsymbol{R}^+ U + \widehat{Q} v_0 \ , \tag{5.1}$$

where $\boldsymbol{R}^+(t,x) = 1_{\{t>0\}}$ and $\bar{\gamma} \geqslant \gamma - 2$ as before, and by definition, $\mathcal{R}\boldsymbol{R}^+\Xi$ is given by the distribution $\xi^\varepsilon 1_{\{t>0\}}$. Furthermore,

$$(Gu_0)(t,x) = (S(t)u_0)(x) = \int_{\mathbb{R}^3} G(t, x-y)u_0(y)\,\mathrm{d}y \ ,$$
$$(\widehat{Q}v_0)(t,x) = \mathrm{e}^{tA_2}\, v_0(x)1_{\{t>0\}} \ . \tag{5.2}$$

In fact, these distributions have to be lifted to $\mathcal{D}^{\gamma,\eta}$, which is always possible by construction, cf [19, Def. 2.14]. In order to understand the structure of the solution of the fixed-point equation (5.1), let us rewrite it in the form

$$U = \mathcal{I}\boldsymbol{R}^+\big[\Xi + F(U,V)\big] + \widehat{R}(U,V) + Gu_0 \ ,$$
$$V = \mathcal{E}\boldsymbol{R}^+U_- + \mathcal{Q}\boldsymbol{R}^+U_+ + \widehat{Q}v_0 \ , \tag{5.3}$$

where U_- and U_+ denote respectively the strictly negative homogeneous and positive homogeneous parts of U (cf. (4.18)), $\widehat{R}(U,V)$ contains the polynomial part of $\mathcal{K}_{\bar{\gamma}}\big[\Xi + F(U,V)\big]$ as well as the term $R_\gamma \mathcal{R}\big[\Xi + F(U,V)\big]$ (which always belongs to $\overline{T}$), and

$$\mathcal{Q}f_+(t,x) = \int_{t-2T}^t Q(t-s)f_+(s,x)\,\mathrm{d}s \ . \tag{5.4}$$

Assume F is a cubic polynomial of the form

$$F(u,v) = \alpha_1 u + \alpha_2 v + \beta_1 u^2 + \beta_2 uv + \beta_3 v^2 + \gamma_1 u^3 + \gamma_2 u^2 v + \gamma_3 uv^2 + \gamma_4 v^3 \ . \qquad (5.5)$$

Further assume that the initial conditions u_0 and v_0 are such that Gu_0 and $\widehat{Q}v_0$ are sufficiently regular. Iterating the map (5.3), starting for instance with identically zero functions, it is not difficult to see that any fixed point of (5.1) must have the form

$$U = \uparrow + \varphi\mathbf{1} + \left[a_1 \, \curlyvee + a_2 \, \curlyvee + a_3 \, \curlyvee + a_4 \, \curlyvee\right] + \left[b_1 \, \mathsf{Y} + b_2 \, \mathsf{Y} + b_3 \, \mathsf{Y}\right] + \langle \nabla\varphi, X \rangle + \varrho_U \ ,$$
$$V = \uparrow + \psi\mathbf{1} + \left[\hat{a}_1 \, \curlyvee + \hat{a}_2 \, \curlyvee + \hat{a}_3 \, \curlyvee + \hat{a}_4 \, \curlyvee\right] + \left[\hat{b}_1 \, \mathsf{Y} + \hat{b}_2 \, \mathsf{Y} + \hat{b}_3 \, \mathsf{Y}\right] + \langle \nabla\psi, X \rangle + \varrho_V \ ,$$
$$(5.6)$$

where the coefficients satisfy $a_i = \gamma_i$ for $i = 1, 2, 3, 4$,

$$\begin{aligned} b_1 &= \beta_1 + 3\varphi\gamma_1 + \psi\gamma_2 \ , \\ b_2 &= \beta_2 + 2\varphi\gamma_2 + 2\psi\gamma_3 \ , \\ b_3 &= \beta_3 + \varphi\gamma_3 + 3\psi\gamma_4 \ , \end{aligned} \qquad (5.7)$$

and $\hat{a}_i\tau = \mathcal{Q}(a_i\tau)$, $\hat{b}_i\tau = \mathcal{Q}(b_i\tau)$. In (5.6), ϱ_U, ϱ_V denote terms of homogeneity at least $\frac{3}{2} - \kappa$, and $\varphi(z)$, $\psi(z)$, $\nabla\varphi(z)$, $\nabla\psi(z)$ are independent functions — the notation is not supposed to suggest that φ and ψ are differentiable. The scalar product notation $\langle \nabla\varphi, X \rangle$ is a shorthand for $\sum_{i=0}^{3}(\nabla\varphi)_i X_i$.

Remark 5.1. Note that the only symbols of strictly negative homogeneity in (5.6) are the stochastic convolutions $\uparrow$ and $\uparrow$. Owing to translation invariance, they both satisfy the relation $\Delta\tau = \tau \otimes \mathbf{1}$ (cf. Remark 4.6), showing that U and V indeed belong to a $\mathcal{D}_0^{\gamma,\eta}$ space. The situation would be different if $F(u,v)$ contained a quartic term such as u^4. Indeed, this would result in the occurrence of a symbol $\curlyvee$ in the expansion (5.6) of U, which has strictly negative homogeneity in dimension $d = 3$. Since this symbol is not translation invariant (compare the case of $\curlyvee$ in Table 1), we cannot expect in this case to find a fixed point in $\mathcal{D}_0^{\gamma,\eta}$. $\Diamond$

Proposition 5.2 (Equivalence of fixed-point equations). *Assume that $\varepsilon > 0$ and that (5.1) admits a fixed point (U, V) where $U, V \in \mathcal{D}^{\gamma,\eta}$ for some $\gamma, \eta \in \mathbb{R}$. Then $(u, v) = (\mathcal{R}U, \mathcal{R}V)$ satisfies the fixed-point equation (4.2).*

Proof. The first step is to note that (3.27) and (4.6) imply

$$(\Pi_{\bar{z}}^{\varepsilon} \uparrow)(z) = \int K(z - z_1)\xi^{\varepsilon}(z_1)\,\mathrm{d}z_1 =: \chi_{\varepsilon}(z) \ ,$$
$$(\Pi_{\bar{z}}^{\varepsilon} \uparrow)(z) = \int K^Q(z - z_1)\xi^{\varepsilon}(z_1)\,\mathrm{d}z_1 =: \chi_{\varepsilon}^Q(z) \ ,$$

independently of $\bar{z}$, where we have used the notation

$$K^Q(t, x) = \int_0^{2T} K(t - s, x)Q(s)\,\mathrm{d}s \ .$$

Note that both in dimensions $d = 3$ and $d = 2$, the sum in (3.27) is empty, because in both cases $\uparrow$ has strictly negative homogeneity, cf. Table 1. Applying the reconstruction operator $\mathcal{R}$ to (5.6), since all functions are smooth for $\varepsilon > 0$, we obtain

$$\begin{aligned} u(z) &= \chi_{\varepsilon}(z) + \varphi(z) \ , \\ v(z) &= \chi_{\varepsilon}^Q(z) + \psi(z) \ . \end{aligned}$$

Here we have used the fact that if $f_+ \in \mathcal{D}^{\gamma,\eta}$ has only components of strictly positive homogeneity, one has $\mathcal{R}f_+(z) = (\Pi_z^\varepsilon f_+(z))(z) = 0$ (cf. (4.11) as well as [19, Prop. 3.28]). On the other hand, using (3.38) and (3.43), we obtain for all $f \in \mathcal{D}^{\gamma,\eta}$

$$\big(\mathcal{R}(\mathcal{K}_{\bar\gamma} + R_\gamma \mathcal{R})f\big)(z) = (G * \mathcal{R}f)(z) .$$

Applying $\mathcal{R}$ to the equations (5.1) and using (4.23), we thus obtain

$$u(t,x) = G * \mathcal{R}\boldsymbol{R}^+\big[\Xi + F(U,V)\big](t,x) + Gu_0(t,x) ,$$

$$v(t,x) = \int_0^t (\mathcal{R}\boldsymbol{R}^+ U)(s,x)Q(t-s)\,\mathrm{d}s + \widehat{Q}v_0(t,x) .$$

It is not difficult to check that $\mathcal{R}$ and $\boldsymbol{R}^+$ commute. Thus the result follows if we are able to show that

$$\mathcal{R}F(U,V) = F(\mathcal{R}U, \mathcal{R}V) .$$

Computing the part of nonpositive homogeneity of the difference $\mathcal{R}F(U,V) - F(\mathcal{R}U, \mathcal{R}V)$, we see that it is a linear combination of terms of the form

$$(\Pi_z^\varepsilon \,\text{⋎}\!\!\bullet)(z) , \quad (\Pi_z^\varepsilon \,\text{⋎}\!\!\bullet)(z) , \quad (\Pi_z^\varepsilon \,\text{⋎}\!\!\bullet)(z) \quad \text{and} \quad (\Pi_z^\varepsilon\,\uparrow)(z)^2 \langle \nabla\varphi, (\Pi_z^\varepsilon X)(z)\rangle ,$$

and all possible terms obtained from these by substitutions of the form $\uparrow \mapsto \,\text{⚲}$. It follows directly from (3.22) that $(\Pi_z^\varepsilon X)(z) = 0$. It thus remains to show that all terms of the three other types vanish as well. Applying (3.27), (3.28), and recalling (3.14) we obtain

$$(\Pi_{\bar z}^\varepsilon \,\text{Y})(z) = \int \big[K(z - z_1) - K(\bar z - z_1)\big](\Pi_{\bar z}^\varepsilon \,\text{⋁})(z_1)\,\mathrm{d}z_1 ,$$

which vanishes in $z = \bar z$. It follows that $(\Pi_z^\varepsilon \,\text{⋎}\!\!\bullet)(z) = (\Pi_z^\varepsilon \,\text{Y})(z)(\Pi_z^\varepsilon \,\text{⋁})(z) = 0$. All other terms can be treated similarly, and the result follows. $\square$

Remark 5.3. In dimension $d = 2$, the symbols $\text{⋎}\!\!\bullet$, $\text{⋎}\!\!\bullet$, $\text{⋎}\!\!\bullet$ have strictly positive homogeneity, so that the last part of the proof is not needed. $\diamond$

The expansion (5.6) shows in particular that both U and V have regularity $-\frac{1}{2} - \kappa$, and that their components of negative homogeneity do not depend on (t,x). We will thus look for a fixed point of (5.6) in a space $\mathcal{D}_0^{\gamma,\eta}$ for appropriate values of γ and η.

5.2 Local existence and uniqueness of the fixed point

In order to describe the effect of F, it will be useful to introduce the following slight generalisation of Definition 4.5. Given a model (Π, Γ), the space $\mathcal{D}_{\alpha,\beta}^{\gamma,\eta}(\Gamma)$ is the space of functions $f \in \mathcal{D}^{\gamma,\eta}(\Gamma)$ of the form

$$f(z) = \sum_{\tau \in T \,:\, \alpha \leqslant |\tau|_\mathfrak{s} < \beta} c_\tau \tau + \sum_{\tau \in T \,:\, |\tau|_\mathfrak{s} \geqslant \beta} \hat{c}_\tau(z)\tau , \tag{5.8}$$

where the c_τ do not depend on z.

The following result establishes the strong local Lipschitz continuity of F in the sense of [19, Section 7.3].

Proposition 5.4 (Strong local Lipschitz continuity). *Fix $\alpha < 0 < \gamma$ and set $\bar\gamma = \gamma + 2\alpha$ and $\bar\eta = 3\eta \wedge (\eta + 2\alpha)$. Let $\mathfrak{K} \subset \mathbb{R}^{d+1}$ be compact, and assume $U, V \in \mathcal{D}_{\alpha,0}^{\gamma,\eta}(\Gamma)$ are such that*

$$\|U\|_{\gamma,\eta;\mathfrak{K}} + \|V\|_{\gamma,\eta;\mathfrak{K}} \leqslant R . \tag{5.9}$$

Then $F(U,V) \in \mathcal{D}_{3\alpha,2\alpha}^{\bar\gamma,\bar\eta}(\Gamma)$, and there exists a constant $C(R)$ such that

$$\|F(U,V)\|_{\bar\gamma,\bar\eta;\mathfrak{K}} \leqslant C(R) . \tag{5.10}$$

Furthermore, if $(\overline{\Pi}, \overline{\Gamma})$ is a possibly different model and $\overline{U}, \overline{V} \in \mathcal{D}_{\alpha,0}^{\gamma,\eta}(\overline{\Gamma})$ satisfy the bound $\|\overline{U}\|_{\gamma,\eta;\mathfrak{K}} + \|\overline{V}\|_{\gamma,\eta;\mathfrak{K}} \leqslant R$, then

$$\|F(U,V)\,;F(\overline{U},\overline{V})\|_{\bar{\gamma},\bar{\eta};\mathfrak{K}} \leqslant C(R)\Big[\|U\,;\overline{U}\|_{\gamma,\eta;\mathfrak{K}} + \|V\,;\overline{V}\|_{\gamma,\eta;\mathfrak{K}} + \|\Gamma - \overline{\Gamma}\|_{2\gamma+\alpha;\mathfrak{K}}\Big]. \quad (5.11)$$

Proof. Writing $U(z) = U_- + U_+(z)$ where U_- contains all terms of negative homogeneity and applying [19, Prop. 6.12], we see that

$$\begin{aligned}
U(z)^2 &= U_-^2 + 2U_- U_+(z) + U_+(z)^2 & &\in \mathcal{D}_{2\alpha,\alpha}^{\gamma+\alpha,2\eta\wedge(\eta+\alpha)}, \\
U(z)^3 &= U_-^3 + 3U_-^2 U_+(z) + 3U_- U(z)^2 + U_+(z)^3 &\in \mathcal{D}_{3\alpha,2\alpha}^{\gamma+2\alpha,3\eta\wedge(\eta+2\alpha)},
\end{aligned}$$

where the multiplication of elements in $\mathcal{D}_{\cdot,\cdot}^{\gamma;\eta}$ is well-defined due to the results in [19, Sec. 4]. Similar relations hold for powers of V and cross-terms, proving (5.10). The proof of (5.11) is similar, using the bound on $\|f\,;g\|$ given in [19, Prop. 6.12]. $\qquad\square$

Remark 5.5. In cases where the two models (Π, Γ) and $(\overline{\Pi}, \overline{\Gamma})$ are identical, (5.11) automatically provides the bound

$$\|F(U,V) - F(\overline{U},\overline{V})\|_{\bar{\gamma},\bar{\eta};\mathfrak{K}} \leqslant C(R)\Big[\|U - \overline{U}\|_{\gamma,\eta;\mathfrak{K}} + \|V - \overline{V}\|_{\gamma,\eta;\mathfrak{K}}\Big]. \quad (5.12)$$

In order to proceed, it is convenient to rewrite the fixed-point equation (5.1) in a slightly different form. First we set

$$W := (\mathcal{K}_{\bar{\gamma}} + R_\gamma \mathcal{R})\boldsymbol{R}^+ \Xi. \quad (5.13)$$

For the time being, let us *assume* that $W \in \mathcal{D}_{\alpha_0+2,0}^{\gamma,\eta}$ for some $\gamma \in \mathbb{R}$ and all $\eta < \alpha_0 + 2$ (recall that $\alpha_0 = -\frac{5}{2} - \kappa$ in dimension $d = 3$); we come back to this point in Remark 5.7 below. Then we define the map

$$\mathcal{M}: U \mapsto (\mathcal{K}_{\bar{\gamma}} + R_\gamma \mathcal{R})\boldsymbol{R}^+ F\big(U, \mathcal{K}_\gamma^Q \boldsymbol{R}^+ U + \widehat{Q}v_0\big) + W + Gu_0. \quad (5.14)$$

If U is a fixed point of $\mathcal{M}$ and $V = \mathcal{K}_\gamma^Q \boldsymbol{R}^+ U + \widehat{Q}v_0$ then (U,V) is a fixed point of (5.1).

Following [19, Section 7.1], we write $O = [-1,2] \times \mathbb{R}^3$ and $O_T = (-\infty, T] \times \mathbb{R}^3$, and introduce the shorthand $\|\cdot\|_{\gamma,\eta;T}$ for $\|\cdot\|_{\gamma,\eta;O_T}$, and define $\|\cdot\,;\cdot\|_{\gamma,\eta;T}$ similarly. Finally, we use the notation $\|Z\,;\bar{Z}\|_{\gamma;\mathfrak{K}} = \|\Pi - \overline{\Pi}\|_{\gamma;\mathfrak{K}} + \|\Gamma - \overline{\Gamma}\|_{\gamma;\mathfrak{K}}$ to quantify the difference between two models. The following result is an adaptation of [19, Thm. 7.8].

Proposition 5.6 (Existence and uniqueness of the fixed point). *Assume $-\frac{2}{3} < \eta < -\frac{1}{2}$ and $\alpha > -1$ are such that $\eta + 2\alpha > -2$, and assume $\gamma > -2\alpha$. For any u_0, v_0, W such that $W + Gu_0, \widehat{Q}v_0 \in \mathcal{D}_{\alpha,0}^{\gamma,\eta}$, there exists a time $T > 0$ such that $\mathcal{M}$ admits a unique fixed point $U^* \in \mathcal{D}^{\gamma,\eta}$ on $(0,T)$. Furthermore, the solution map $\mathcal{S}_T : (u_0, v_0, W, Z) \mapsto U^*$ is jointly Lipschitz continuous in the sense that if Z and $\bar{Z}$ are two models and (u_0, v_0, W) and $(\bar{u}_0, \bar{v}_0, \overline{W})$ are two sets of initial conditions and forcing terms such that*

$$\|Gu_0\,;G\bar{u}_0\|_{\gamma,\eta;T} + \|\widehat{Q}v_0\,;\widehat{Q}\bar{v}_0\|_{\gamma,\eta;T} + \|W\,;\overline{W}\|_{\gamma,\eta;T} + \|Z\,;\bar{Z}\|_{2\gamma+\alpha;O} \leqslant \delta, \quad (5.15)$$

then one has

$$\|U^*\,;\overline{U}^*\|_{\gamma,\eta;T} \leqslant C_0 \delta \quad (5.16)$$

for some constant $C_0 > 0$.

Proof. Given $R > 0$ let $\mathcal{B}(R) = \{f \in \mathcal{D}_{\alpha,0}^{\gamma,\eta} \colon \|f\|_{\gamma,\eta;T} \leqslant R\}$ and pick $U \in \mathcal{B}(R)$. By Proposition 4.11, we have

$$V := \mathcal{K}_\gamma^Q U + \widehat{Q}v_0 \in \mathcal{B}(C_1' R + \|\widehat{Q}v_0\|_{\gamma,\eta;T})$$

for a constant C_1' depending only on Q and on the model. Applying Proposition 5.4, we see that $F(U, V) \in \mathcal{D}_{3\alpha, 2\alpha}^{\bar{\gamma}, \bar{\eta}}$ and satisfies

$$\|F(U, V)\|_{\bar{\gamma}, \bar{\eta}; T} \leqslant C\left(R + C_1' R + \|\widehat{Q} v_0\|_{\gamma, \eta; T}\right) .$$

The assumptions on γ, η and α guarantee that $\bar{\gamma} > 0$ and $\bar{\eta} > -2$. We can thus apply Theorem 7.1 and Lemma 7.3 in [19] to obtain the existence of constants $C_2, \kappa > 0$ such that

$$\|\mathcal{M}(U)\|_{\gamma, \eta; T} \leqslant C_2 T^{\kappa/2} C\left(R + C_1' R + \|\widehat{Q} v_0\|_{\gamma, \eta; T}\right) + \|W\|_{\gamma, \eta; T} + \|G u_0\|_{\gamma, \eta; T}$$

for every $T \in (0, 1]$. Choosing

$$R = 1 + \left[\|W\|_{\gamma, \eta; T} + \|G u_0\|_{\gamma, \eta; T}\right]$$

and T sufficiently small, we see that $\mathcal{M}$ maps the set $\mathcal{B}(R)$ into itself.

Next we show that $\mathcal{M}$ is a contraction in $\mathcal{B}(R)$. For $U, \overline{U} \in \mathcal{B}(R)$, V as above and $\overline{V} = \mathcal{K}_\gamma^Q \overline{U} + \widehat{Q} v_0$, we have

$$\mathcal{M}(U) - \mathcal{M}(\overline{U}) = (\mathcal{K}_{\bar{\gamma}} + R_\gamma \mathcal{R}) \boldsymbol{R}^+ \left[F(U, V) - F(\overline{U}, \overline{V})\right] .$$

Using again Theorem 7.1 and Lemma 7.3 in [19] and applying (5.12) to bound the term $\left[F(U, V) - F(\overline{U}, \overline{V})\right]$, we get

$$\|\mathcal{M}(U) - \mathcal{M}(\overline{U})\|_{\gamma, \eta; T} \leqslant C_2 T^{\kappa/2} C\left(R + C_1' R + \|\widehat{Q} v_0\|_{\gamma, \eta; T}\right) \left[\|U - \overline{U}\|_{\gamma, \eta; T}\right] .$$

Taking again T small enough, we obtain that $\mathcal{M}$ is indeed a contraction, and the existence of a unique fixed point in a ball in $\mathcal{D}^{\gamma, \eta}$ follows from Banach's fixed-point theorem.

Finally, in order to prove the bound (5.16), we set $U_1 = (\mathcal{K}_{\bar{\gamma}} + R_\gamma \mathcal{R}) \boldsymbol{R}^+ F(U, V)$ and $\overline{U}_1 = (\mathcal{K}_{\bar{\gamma}} + R_\gamma \mathcal{R}) \boldsymbol{R}^+ F(\overline{U}, \overline{V})$. Using the inequality

$$\|f + g \,;\, \bar{f} + \bar{g}\|_{\gamma, \eta; T} \leqslant \|f \,;\, \bar{f}\|_{\gamma, \eta; T} + \|g \,;\, \bar{g}\|_{\gamma, \eta; T}$$

we obtain

$$\|\mathcal{M}(U) \,;\, \mathcal{M}(\overline{U})\|_{\gamma, \eta; T} \leqslant \|U_1 \,;\, \overline{U}_1\|_{\gamma, \eta; T} + \|W \,;\, \overline{W}\|_{\gamma, \eta; T} + \|G u_0 \,;\, G \bar{u}_0\|_{\gamma, \eta; T} \,,$$

where the first term on the right-hand side can be estimated with the help of (5.11). The required bound thus follows from [19, Thm. 7.1 and Lemma 7.3]. □

Remark 5.7. In fact, Proposition 5.6 can be applied as soon as we choose $\gamma > 1$. Indeed, let $\alpha = -\frac{1}{2} - \kappa$ and assume $\gamma = 1 + \bar{\kappa}$ for some $\bar{\kappa} > 0$.

1. The conditions on γ and η are satisfied provided $\kappa < \bar{\kappa}$ and $\kappa \leqslant \frac{1}{6}$.
2. Lemma 7.5 in [19] shows that if $u_0 \in \mathcal{C}^\eta$ (with the Euclidean scaling and recalling that we take $-\frac{2}{3} < \eta < -\frac{1}{2}$) then $G u_0 \in \mathcal{D}^{\gamma, \eta}$ for any $\gamma > (\eta \vee 0)$.
3. Definition 2.14 in [19] shows that if $v_0 \in \mathcal{C}^\gamma$, then it can be lifted to a an element $v_0 \in \mathcal{D}^\gamma$, and then $\widehat{Q} v_0$ will belong to $\mathcal{D}^{\gamma, \eta}$ as required.
4. As discussed in Section 9.4 of [19] (see Equation (9.18) and below), the assumption $W \in \mathcal{D}_{\alpha_0 + 2, 0}^{\gamma, \eta}$ is indeed satisfied for γ, η as in Proposition 5.6, provided we define $\mathcal{R} \boldsymbol{R}^+ \Xi$ as the distribution $\xi^\varepsilon 1_{\{t \geqslant 0\}}$. ◇

Let $Z^\varepsilon = \Psi(\xi^\varepsilon)$ denote the canonical model for mollified noise ξ^ε constructed in Sections 3.2 and 4.1. In exactly the same way as in [19, Prop. 9.8 and 9.10], we can now extend the solution map $\mathcal{S}_T$ to a maximal solution map $\mathcal{S}^L$, defined until the first time $\|(\mathcal{R} U)(t, \cdot)\|_\eta + \|(\mathcal{R} V)(t, \cdot)\|_\eta$ reaches a (large) cut-off value L. This map satisfies

$$\mathcal{R} \mathcal{S}^L(u_0, v_0, Z^\varepsilon) = \bar{\mathcal{S}}^L(u_0, v_0, \xi^\varepsilon) \,, \tag{5.17}$$

where $\bar{\mathcal{S}}^L$ is the solution map of the original SPDE–ODE system (4.1) with mollified noise ξ^ε. In other words, the following diagram commutes:

$$(u_0, v_0, Z^\varepsilon) \xmapsto{\quad \mathcal{S}^L \quad} (U^*, V^*)$$

$$\Psi \Big\uparrow \qquad\qquad \Big\downarrow \mathcal{R}$$

$$(u_0, v_0, \xi^\varepsilon) \xmapsto[\quad \bar{\mathcal{S}}^L \quad]{} (u^\varepsilon, v^\varepsilon)$$

6 Renormalisation of the equation

6.1 The renormalisation group

Consider a general model $Z = (\Pi, \Gamma)$ for our regularity structure, which is admissible in the sense that is satisfies (3.27), the last identity in (3.28) and (4.6). The model may also be specified by the pair $(\boldsymbol{\Pi}, f)$ such that for all $z, \bar{z} \in \mathbb{R}^{d+1}$, one has

$$\Pi_z = \boldsymbol{\Pi}\, \Gamma_{f_z} \qquad \text{and} \qquad \Gamma_{z\bar{z}} = (\Gamma_{f_z})^{-1}\Gamma_{f_{\bar{z}}} =: \Gamma_{\gamma_{z\bar{z}}} \;. \tag{6.1}$$

To build a renormalisation transformation, one first needs to define sets $\mathcal{F}_\star \subset \mathcal{F}_0 \subset \mathcal{F}_F$ such that

- $\mathcal{F}_0$ contains all $\tau \in \mathcal{F}_F$ of strictly negative homogeneity;
- for all $\tau \in \mathcal{F}_0$, one has $\Delta\tau \in \mathrm{span}(\mathcal{F}_0) \otimes \mathrm{span}(\mathcal{F}_0^+)$, where $\mathcal{F}_0^+ \subset \mathcal{F}_F^+$ contains all symbols of the form (3.11) with $\tau_j \in \mathcal{F}_\star$ and $|\mathcal{J}_{k_j}\tau_j|_\mathfrak{s} > 0$ (including the empty product).

See [19, Sect. 8.3], in particular Remark 8.37, for a proof that it is always possible to find sets $\mathcal{F}_\star$ and $\mathcal{F}_0$ satisfying these two properties. For the Allen–Cahn model, a possible choice is

$$\mathcal{F}_\star = \{\,\mathbin{\text{🜨}}, \vee, \uparrow\,\}\,,$$
$$\mathcal{F}_0 = \{\Xi, \mathbin{\text{🜨}}, \vee, \uparrow, \mathbin{\text{⚓}}, \vee X_i, \mathbin{\text{🜨}}, \mathbin{\text{🜨}}, \mathbf{1}, \mathbin{\text{🜨}}, \curlyvee, \mathbin{\text{🜨}}, X_i\}\,. \tag{6.2}$$

For FitzHugh–Nagumo type equations, we choose the same sets, enriched by all symbols obtained by substitutions $\uparrow \to \mathbin{\text{🜨}}$.

Let $\mathcal{H}_0 = \mathrm{span}(\mathcal{F}_0)$ and $\mathcal{H}_0^+ = \mathrm{span}(\mathcal{F}_0^+)$. By definition, a *renormalisation transformation* is a linear map $M : \mathcal{H}_0 \to \mathcal{H}_0$ such that $MX^k = X^k$, $M\mathcal{I}\tau = \mathcal{I}M\tau$ for all $\tau \in \mathcal{F}_0$ such that $\mathcal{I}\tau \in \mathcal{F}_0$ and $M\mathcal{E}\tau = \mathcal{E}M\tau$ for all $\tau \in \mathcal{F}_0$. With M, we would like to associate another admissible model $(\boldsymbol{\Pi}^M, f^M)$ by setting

$$\boldsymbol{\Pi}^M\tau = \boldsymbol{\Pi}M\tau \qquad \forall \tau \in \mathcal{F}_0 \;. \tag{6.3}$$

As shown in [19, Section 8.3], this is indeed possible if one is able to find linear maps $\Delta^M : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0^+$, and $\hat{M} : \mathcal{H}_0^+ \to \mathcal{H}_0^+$ such that

$$\Pi_z^M\tau = (\Pi_z \otimes f_z)\Delta^M\tau \;,$$
$$\langle f_z^M, \bar{\tau}\rangle = \langle f_z, \hat{M}\bar{\tau}\rangle \tag{6.4}$$

holds for all $\tau \in \mathcal{F}_0$ and all $\bar{\tau} \in \mathcal{F}_0^+$, where Π_z^M and f_z^M satisfy relations analogous to (6.1). The model $(\boldsymbol{\Pi}^M, f^M)$ is admissible provided the conditions

$$\hat{M}\mathcal{J}_k = \mathcal{M}(\mathcal{J}_k \otimes \mathrm{Id})\Delta^M \;,$$
$$(\mathrm{Id} \otimes \mathcal{M})(\Delta \otimes \mathrm{Id})\Delta^M = (M \otimes \hat{M})\Delta \tag{6.5}$$

hold, where $\mathcal{M}$ is the multiplication map, defined by $\mathcal{M}(\tau \otimes \sigma) = \tau\sigma$. The map $\hat{M}$ should also be a multiplicative morphism leaving the X^k invariant, that is,

$$\hat{M}(\tau_1 \tau_2) = (\hat{M}\tau_1)(\hat{M}\tau_2) \,, \qquad \hat{M}(X^k) = X^k \,. \tag{6.6}$$

It is shown in [19, Prop. 8.36] that given a linear map M, there is a unique choice of maps Δ^M and $\hat{M}$ such that (6.5) and (6.6) hold, making $(\mathbf{\Pi}^M, f^M)$ an admissible model. In addition, the elements $\Gamma_{z\bar{z}}^M$ can be built using a map $\hat{\Delta}^M : \mathcal{H}_0^+ \to \mathcal{H}_0^+ \otimes \mathcal{H}_0^+$ such that

$$\gamma_{z\bar{z}}^M = (\gamma_{z\bar{z}} \otimes f_{\bar{z}})\hat{\Delta}^M \,. \tag{6.7}$$

Definition 6.1 ([19, Def. 8.41]). *The* renormalisation group $\mathfrak{R}$ *is the set of linear maps* $M : \mathcal{H}_0 \to \mathcal{H}_0$ *satisfying* $M\mathcal{I} = \mathcal{I}M$, $M\mathcal{E} = \mathcal{E}M$, $M(X^k) = X^k$, *and such that for all* $\tau \in \mathcal{F}_0$ *and all* $\bar{\tau} \in \mathcal{F}_0^+$ *one has*

$$\Delta^M \tau - \tau \otimes \mathbf{1} \in \bigoplus_{\beta > \alpha} (T_\beta \cap \operatorname{span}(\mathcal{F}_0)) \otimes T^+ \,,$$
$$\hat{\Delta}^M \bar{\tau} - \bar{\tau} \otimes \mathbf{1} \in \bigoplus_{\beta > \alpha} (T_\beta \cap \operatorname{span}(\mathcal{F}_0^+)) \otimes T^+ \,, \tag{6.8}$$

for all $\tau \in T_\alpha$ *and all* $\bar{\tau} \in T_\alpha \cap \operatorname{span}(\mathcal{F}_0^+)$.

It is shown in [19, Lemma 8.43] that $\mathfrak{R}$ is indeed a group. The main result [19, Thm. 8.44] states that given $M \in \mathfrak{R}$, and an admissible model $(\mathbf{\Pi}, f)$, the model $(\mathbf{\Pi}^M, f^M)$ constructed as above is indeed an admissible model. In particular, it satisfies the required analytic bounds and behaves well under the extension theorem [19, Thm. 5.14].

Let us now introduce a particular subgroup of $\mathfrak{R}$ suited to our problem. In the case of the Allen–Cahn equation (cf [19, Sections 9.2 and 10.5]), a two-parameter group of transformations $M = \exp\{-C_1 L_1 - C_2 L_2\}$ is sufficient, where the generators L_1 and L_2 are defined by the substitution rules

$$L_1 : \vee\!\!\!\bullet \mapsto \mathbf{1} \,, \qquad L_2 : \vcenter{\hbox{$\mathcal{\psi}$}} \mapsto \mathbf{1} \,. \tag{6.9}$$

More precisely, $L_1\tau$ is in general a sum of terms, where each term corresponds to one occurrence of $\vee\!\!\!\bullet = \mathcal{I}(\Xi)^2$ in τ. Thus for instance $L_1(\vcenter{\hbox{$\psi$}}) = 3\!\uparrow$, because there are 3 ways of picking a pair of $\uparrow$ in $\vcenter{\hbox{$\psi$}}$, and replacing any of these pairs by $\mathbf{1}$ yields $\uparrow$.

In our case, we may *a priori* have to apply similar substitutions for all elements obtained from $\vee\!\!\!\bullet$ and $\vcenter{\hbox{$\psi$}}$ by replacing one or several $\mathcal{I}(\Xi)$ by $\mathcal{E}\mathcal{I}(\Xi)$. We thus introduce the sets of symbols

$$\mathcal{F}_1 = \{\vee\!\!\!\bullet, \vee\!\!\!\bullet, \vee\!\!\!\bullet\} \,,$$
$$\mathcal{F}_2 = \{\mathcal{I}(\tau_1)\tau_2 : \tau_1, \tau_2 \in \mathcal{F}_1\} \,. \tag{6.10}$$

Note that $\mathcal{F}_2$ has 9 elements. This leads us to define a 12-parameter subgroup of $\mathfrak{R}$ given by

$$M = \exp\Big\{ -\sum_{\tau \in \mathcal{F}_1} C_1(\tau)L_\tau - \sum_{\tau \in \mathcal{F}_2} C_2(\tau)L_\tau \Big\} \,, \tag{6.11}$$

where L_τ denotes the substitution $\tau \mapsto \mathbf{1}$, applied as often as possible. The exponential is then defined by its formal series, which is always finite because any application of L_τ strictly decreases the number of occurrences of $\mathcal{I}(\Xi)$ or $\mathcal{E}(\mathcal{I}(\Xi))$. In practice, however, we will see that only 2 of these parameters are really needed to renormalise the model.

We introduce the shorthands $C_1 = C_1(\vee\!\!\!\bullet)$, $C_1' = C_1(\vee\!\!\!\bullet)$ and $C_1'' = C_1(\vee\!\!\!\bullet)$. The action of M on a few elements that will occur in the computations below is given in the

following list:

$$M(\uparrow) = \uparrow \,, \tag{6.12a}$$

$$M(\vee) = \vee - C_1 \mathbf{1} \,, \tag{6.12b}$$

$$M(\Psi) = \Psi - 3C_1\uparrow \,, \tag{6.12c}$$

$$M(\bullet\!\vee) = \bullet\!\vee - C_2(\bullet\!\vee)\mathbf{1} - C_1 \Upsilon \,, \tag{6.12d}$$

$$M(\bullet\!\Psi) = \bullet\!\Psi - 3C_2(\bullet\!\vee)\uparrow - 3C_1 \bullet\!\vee - C_1 \Upsilon\!\Psi + 3C_1^2 \curlywedge \,. \tag{6.12e}$$

The action of M on elements such as $\vee$ is obtained by obvious substitutions, e.g. we have $M(\Psi) = \Psi - C_1''\uparrow - 2C_1' \curlyvee$. As in [19, Section 9.2], one can determine the map Δ^M and thus, via (6.4), the renormalised model Π^M. Again, we just list a few expressions that will play a rôle in the computations below:

$$\Pi_z^M(\uparrow) = \Pi_z(\uparrow) \,, \tag{6.13a}$$

$$\Pi_z^M(\vee) = \Pi_z(\uparrow)^2 - C_1 \,, \tag{6.13b}$$

$$\Pi_z^M(\Psi) = \Pi_z(\uparrow)^3 - 3C_1\Pi_z(\uparrow) \,, \tag{6.13c}$$

$$\Pi_z^M(\Upsilon) = \Pi_z(\Upsilon) \,, \tag{6.13d}$$

$$\Pi_z^M(\bullet\!\vee) = \Pi_z(\Upsilon)\Pi_z^M(\vee) - C_2(\bullet\!\vee) \,, \tag{6.13e}$$

$$\Pi_z^M(\bullet\!\Psi) = \Pi_z^M(\Upsilon\!\Psi)\Pi_z^M(\vee) - 3C_2(\bullet\!\vee)\Pi_z^M(\uparrow) + 3C_1\Pi_z(\vee X_i)\langle f_z, \mathcal{J}_{e_i}(\uparrow)\rangle \,. \tag{6.13f}$$

Similar relations with obvious substitutions hold for the elements obtained by having $\mathcal{E}$ act on some $\mathcal{I}(\Xi)$.

6.2 Convergence of the renormalised models

From now on, we assume that the driving noise ξ is Gaussian white noise on $D = \mathbb{R} \times \mathbb{T}^d$, where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the d-dimensional torus . This means that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the Hilbert space $H = L^2(D)$, and a collection of centred jointly Gaussian random variables $\{W_h\}_{h\in H}$ which are assumed to satisfy

$$\mathbb{E}\{W_h W_{\bar{h}}\} = \langle h, \bar{h}\rangle \tag{6.14}$$

for all $h, \bar{h} \in H$. Then ξ is the distribution defined by $\langle \xi, \varphi\rangle = W_{\pi\varphi}$ for every test function φ, where π is the canonical projection from $L^2(\mathbb{R}^{d+1})$ to $L^2(D)$.

Let $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$ be the canonical model built for mollified noise ξ^ε. Our aim is now to find a specific sequence of renormalisation maps M_ε in $\mathfrak{R}$ such that the sequence of models $\widehat{Z}^\varepsilon = M_\varepsilon Z^\varepsilon$ defined by

$$\widehat{\Pi}_z^{(\varepsilon)} = (\Pi_z^\varepsilon)^{M_\varepsilon} \tag{6.15}$$

converges in a suitable sense to a limiting model $\widehat{Z}$.

The relevant result to achieve this is Theorem 10.7 in [19]. It states that if for all localised scaled test functions $\eta_z^\delta = \mathcal{S}_{\mathfrak{s},z}^\delta \eta$ and all $\tau \in \mathcal{F}_- = \{\tau \in \mathcal{F}_F \colon |\tau|_{\mathfrak{s}} < 0\}$, one can find random variables $\widehat{\Pi}_z\tau$ such that

$$\mathbb{E}\big|\langle \widehat{\Pi}_z\tau, \eta_z^\delta\rangle\big|^2 \lesssim \delta^{2|\tau|_{\mathfrak{s}}+\kappa} \,,$$
$$\mathbb{E}\big|\langle \widehat{\Pi}_z\tau - \widehat{\Pi}_z^{(\varepsilon)}\tau, \eta_z^\delta\rangle\big|^2 \lesssim \varepsilon^{2\theta}\delta^{2|\tau|_{\mathfrak{s}}+\kappa} \tag{6.16}$$

holds for some $\kappa, \theta > 0$, then there exists a unique admissible model $\widehat{Z}$ such that for all compact sets $\mathfrak{K} \subset \mathbb{R}^{d+1}$ and all $p \geqslant 1$ one has

$$\mathbb{E}\|\widehat{Z}\|_{\gamma;\mathfrak{K}}^p \lesssim 1 \,, \qquad \mathbb{E}\|\widehat{Z} \,; \widehat{Z}^\varepsilon\|_{\gamma;\mathfrak{K}}^p \lesssim \varepsilon^{\theta p} \,. \tag{6.17}$$

Here $\|Z\|_{\gamma;\mathfrak{K}} = \|\Pi\|_{\gamma;\mathfrak{K}} + \|\Gamma\|_{\gamma;\mathfrak{K}}$, and $\|Z;\overline{Z}\|_{\gamma;\mathfrak{K}}$ is defined in Section 5.2. In (6.16), each random variable $\widehat{\Pi}_z\tau$ should in addition belong to the inhomogeneous Wiener chaos of order $\|\tau\|$, where $\|\tau\|$ is the number of occurrences of Ξ in τ (cf [19, Section 10.1]). More precisely, Wiener's isometry allows one to define linear maps $I_k : H^{\otimes k} \to L^2$, as explained in [19, Section 10.1]. Then the Wiener chaos expansion of $\widehat{\Pi}_0^{(\varepsilon)}\tau$ is obtained by finding distributions $(\widehat{\mathcal{W}}^{(\varepsilon;k)}\tau)(z) \in H^{\otimes k}$ such that for every test function φ,

$$\langle \widehat{\Pi}_0^{(\varepsilon)}\tau, \varphi \rangle = \sum_{k \leqslant \|\tau\|} I_k \left(\int_{\mathbb{R}^{d+1}} \varphi(z)(\widehat{\mathcal{W}}^{(\varepsilon;k)}\tau)(z)\,\mathrm{d}z \right). \tag{6.18}$$

Let further $\mathcal{W}^{(\varepsilon;k)}$ be the distributions defined in a similar way from the bare model Π^ε. As usual, we will express (6.18) by the slightly more informal notation

$$(\widehat{\Pi}_0^{(\varepsilon)}\tau)(z) = \sum_{k \leqslant \|\tau\|} I_k\big((\widehat{\mathcal{W}}^{(\varepsilon;k)}\tau)(z)\big). \tag{6.19}$$

Proposition 10.11 in [19] provides conditions on the $\widehat{\mathcal{W}}^{(\varepsilon;k)}$ from which the required conditions (6.16) follow. Namely, for each $\tau \in \mathcal{F}_-$, one should find functions $\widehat{\mathcal{W}}^{(k)}$ such that

$$\left| \langle (\widehat{\mathcal{W}}^{(k)}\tau)(z), (\widehat{\mathcal{W}}^{(k)}\tau)(\bar{z}) \rangle \right| \leqslant C \sum_{\lambda \in B_{k,\tau}} \big(\|z\|_{\mathfrak{s}} + \|\bar{z}\|_{\mathfrak{s}} \big)^\lambda \|z - \bar{z}\|_{\mathfrak{s}}^{\bar{\kappa}+2|\tau|_{\mathfrak{s}}-\lambda},$$

$$\left| \langle (\delta\widehat{\mathcal{W}}^{(\varepsilon;k)}\tau)(z), (\delta\widehat{\mathcal{W}}^{(\varepsilon;k)}\tau)(\bar{z}) \rangle \right| \leqslant C\varepsilon^{2\theta} \sum_{\lambda \in B_{k,\tau}} \big(\|z\|_{\mathfrak{s}} + \|\bar{z}\|_{\mathfrak{s}} \big)^\lambda \|z - \bar{z}\|_{\mathfrak{s}}^{\bar{\kappa}+2|\tau|_{\mathfrak{s}}-\lambda} \tag{6.20}$$

for some $\bar{\kappa}, \theta > 0$, where $\delta\widehat{\mathcal{W}}^{(\varepsilon;k)} = \widehat{\mathcal{W}}^{(\varepsilon;k)} - \widehat{\mathcal{W}}^{(k)}$ and each $B_{k,\tau}$ is a finite set of indices $\lambda \in [0, 2|\tau|_{\mathfrak{s}} + \kappa + |\mathfrak{s}|)$, where $|\mathfrak{s}| = d + 2 = 5$. Furthermore, the scalar product in (6.20) is the canonical scalar product on $H^{\otimes k}$ induced by the scalar product on H.

In order to obtain such bounds for our system, we can to a large extent adapt the proof of [19, Thm. 10.22]. As a matter of fact, it is immediate to see that this theorem can be extended to our case, because the involved kernels have singularities which are not worse than in the Allen–Cahn case. The aim of the discussion that follows is thus only to determine which parameters of the renormalisation group introduced above are really needed to ensure convergence.

The first step is to observe that if ϱ_ε is the mollifyer, i.e., if $\xi^\varepsilon = \varrho_\varepsilon * \xi$, noting $K_\varepsilon = K * \varrho_\varepsilon$ we have

$$(\Pi_{\bar{z}}^\varepsilon \,{}^\uparrow)(z) = (K_\varepsilon * \xi)(z) = \int K_\varepsilon(z - z_1)\xi(z_1)\,\mathrm{d}z_1 =: \chi_\varepsilon(z), \tag{6.21}$$

which is independent of $\bar{z}$ and belongs to the first Wiener chaos with

$$\big((\widehat{\mathcal{W}}^{(\varepsilon;1)}\,{}^\uparrow)(z)\big)(z_1) = K_\varepsilon(z - z_1). \tag{6.22}$$

As a consequence, we simply set

$$(\widehat{\Pi}_{\bar{z}}^{(\varepsilon)}\,{}^\uparrow)(z) = \chi_\varepsilon(z), \qquad (\widehat{\Pi}_{\bar{z}}\,{}^\uparrow)(z) = \chi_0(z) := (K * \xi)(z). \tag{6.23}$$

Due to translation invariance, it will henceforth be sufficient to evaluate models in $\bar{z} = 0$. As already seen in the proof of Proposition 5.2, we have

$$(\Pi_0^\varepsilon \,{}^\curlyvee)(z) = (K^Q * \xi^\varepsilon)(z) = \int K^Q(z - z_1)\xi^\varepsilon(z_1)\,\mathrm{d}z_1 =: \chi_\varepsilon^Q(z). \tag{6.24}$$

This can be rewritten in the form

$$\chi_\varepsilon^Q(z) = (K_\varepsilon^Q * \xi)(z) = \int K_\varepsilon^Q(z - z_1)\xi(z_1)\,\mathrm{d}z_1 \,, \tag{6.25}$$

where we have set

$$K_\varepsilon^Q(t, x) = \int_0^{2T} Q(s)K_\varepsilon(t - s, x)\,\mathrm{d}s \,. \tag{6.26}$$

It follows that we can set

$$(\widehat{\Pi}_0^{(\varepsilon)}\,{\scriptstyle\text{\char"019}})(z) = \chi_\varepsilon^Q(z) \,, \qquad (\widehat{\Pi}_0\,{\scriptstyle\text{\char"019}})(z) = \chi_0^Q(z) := (K^Q * \xi)(z) \,. \tag{6.27}$$

The second step is to consider products of terms $\mathord{\text{\char"019}}$ and $\mathord{\text{\char"019}}$. To this end, we observe that by definition of the canonical model (see (3.22)),

$$(\Pi_0^\varepsilon\,\mathord{\text{\char"2228}})(z) = \chi_\varepsilon(z)^2 = \iint K_\varepsilon(z - z_1)K_\varepsilon(z - z_2)\xi(z_1)\xi(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 \,, \tag{6.28}$$

which belongs to the second inhomogeneous Wiener chaos. Renormalisation is needed because the right-hand side is known to diverge in the limit $\varepsilon \to 0$. However, if one introduces the Wick product

$$\xi(z_1) \diamond \xi(z_2) = \xi(z_1)\xi(z_2) - \delta(z_1 - z_2) \,, \tag{6.29}$$

then one obtains

$$(\Pi_0^\varepsilon\,\mathord{\text{\char"2228}})(z) = \iint K_\varepsilon(z - z_1)K_\varepsilon(z - z_2)\xi(z_1)\diamond\xi(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 + \int K_\varepsilon(z - z_1)^2\,\mathrm{d}z_1 \,. \tag{6.30}$$

The first term on the right-hand side belongs to the second homogeneous Wiener chaos, with

$$\big((\widehat{\mathcal{W}}^{(\varepsilon;2)}\,\mathord{\text{\char"2228}})(z)\big)(z_1, z_2) = K_\varepsilon(z - z_1)K_\varepsilon(z - z_2) \,, \tag{6.31}$$

and is known to converge as $\varepsilon \to 0$. We can thus define

$$(\widehat{\Pi}_0^{(\varepsilon)}\,\mathord{\text{\char"2228}})(z) = \iint K_\varepsilon(z - z_1)K_\varepsilon(z - z_2)\xi(z_1)\diamond\xi(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 =: \chi_\varepsilon^{\diamond 2}(z) \,, \tag{6.32}$$

which is indeed compatible with (6.13b), provided we set

$$C_1(\varepsilon) = \int K_\varepsilon(z - z_1)^2\,\mathrm{d}z_1 = \int K_\varepsilon(-z_1)^2\,\mathrm{d}z_1 \,. \tag{6.33}$$

The limiting model $(\widehat{\Pi}_0\,\mathord{\text{\char"2228}})(z)$ is then naturally defined by the same expression as in (6.32), but with K_ε replaced by K.

In order to deal with the term $\mathord{\text{\char"2228}}$, the relevant Wick product formula is

$$\begin{aligned}\xi(z_1)\diamond\xi(z_2)\diamond\xi(z_3) = {}&\xi(z_1)\xi(z_2)\xi(z_3)\\ &- \xi(z_1)\delta(z_2 - z_3) - \xi(z_2)\delta(z_3 - z_1) - \xi(z_3)\delta(z_1 - z_2) \,.\end{aligned} \tag{6.34}$$

In view of (6.13c), this is indeed compatible with the definition

$$(\widehat{\Pi}_0^{(\varepsilon)}\,\mathord{\text{\char"2228}})(z) = \iiint K_\varepsilon(z - z_1)K_\varepsilon(z - z_2)K_\varepsilon(z - z_3)\xi(z_1)\diamond\xi(z_2)\diamond\xi(z_3)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3 \,, \tag{6.35}$$

without the need to introduce a further renormalisation constant.

Before being able to deal with terms such as , we have to compute the model associated with $\curlyvee$. Applying (3.27) (in which the sum only contains the term $\ell = 0$) and the last relation in (3.28), we obtain

$$(\Pi_0^\varepsilon \curlyvee)(z) = \int \big[K(z - z_1) - K(-z_1)\big](\Pi_0^\varepsilon \vee)(z_1)\,\mathrm{d}z_1$$

$$= \int \big[K(z - z_1) - K(-z_1)\big]\chi_\varepsilon^{\diamond 2}(z_1)\,\mathrm{d}z_1$$

$$=: (K * \chi_\varepsilon^{\diamond 2})(z) - (K * \chi_\varepsilon^{\diamond 2})(0)\,, \tag{6.36}$$

where we have used the fact that the integrals of $K(z - \cdot)$ and K are equal to cancel out the term C_1. It thus follows from (6.13e) that

$$(\widehat{\Pi}_0^{(\varepsilon)} \tau)(z) = \big[(K * \chi_\varepsilon^{\diamond 2})(z) - (K * \chi_\varepsilon^{\diamond 2})(0)\big]\chi_\varepsilon^{\diamond 2}(z) - C_2(\tau)\,. \tag{6.37}$$

Similar expressions hold for the other terms of $\mathcal{F}_2$. In order to compute the Wiener chaos decomposition of expressions such as (6.37), we will need the identity

$$\big(\xi(z_1) \diamond \xi(z_2)\big)\big(\xi(z_3) \diamond \xi(z_4)\big) = \xi(z_1) \diamond \xi(z_2) \diamond \xi(z_3) \diamond \xi_4$$
$$+ \xi(z_1) \diamond \xi(z_3)\delta(z_2 - z_4) + \xi(z_1) \diamond \xi(z_4)\delta(z_2 - z_3)$$
$$+ \xi(z_2) \diamond \xi(z_3)\delta(z_1 - z_4) + \xi(z_2) \diamond \xi(z_4)\delta(z_1 - z_3)$$
$$+ \delta(z_1 - z_3)\delta(z_2 - z_4) + \delta(z_1 - z_4)\delta(z_2 - z_3)\,, \tag{6.38}$$

which follows from [19, Lemma 10.3] and the Wick rule $I_k(f) \diamond I_\ell(g) = I_{k+\ell}(f \otimes g)$. It turns out that the important term regarding renormalisation is the contribution to the zeroth Wiener chaos, resulting from the last line in (6.38), which is given in the case $\tau = \tau$ by

$$(\widehat{\mathcal{W}}^{(\varepsilon;0)} \tau)(z) = 2 \iiint K(z - \bar{z})K_\varepsilon(\bar{z} - z_1)K_\varepsilon(\bar{z} - z_2)K_\varepsilon(z - z_1)K_\varepsilon(z - z_2)\,\mathrm{d}\bar{z}\,\mathrm{d}z_1\,\mathrm{d}z_2$$

$$+ 2 \iiint K(-\bar{z})K_\varepsilon(\bar{z} - z_1)K_\varepsilon(\bar{z} - z_2)K_\varepsilon(z - z_1)K_\varepsilon(z - z_2)\,\mathrm{d}\bar{z}\,\mathrm{d}z_1\,\mathrm{d}z_2$$

$$- C_2(\tau)\,. \tag{6.39}$$

As shown in [19, Thm. 10.22], only the first integral on the right-hand side (which is independent of z by translation invariance) diverges as $\varepsilon \to 0$, which imposes a choice for $C_2(\tau)$. In order to represent this quantity and related ones appearing when dealing with the other terms, we introduce the notations

$$\mathcal{Q}_0^\varepsilon(z) = \int K_\varepsilon(z_1)K_\varepsilon(z_1 - z)\,\mathrm{d}z_1\,,$$

$$\mathcal{Q}_1^\varepsilon(z) = \int K_\varepsilon(z_1)K_\varepsilon^Q(z_1 - z)\,\mathrm{d}z_1\,,$$

$$\mathcal{Q}_2^\varepsilon(z) = \int K_\varepsilon^Q(z_1)K_\varepsilon^Q(z_1 - z)\,\mathrm{d}z_1\,. \tag{6.40}$$

In particular, we have

$$C_2(\tau) = 2 \int K(z)\mathcal{Q}_0^\varepsilon(z)^2\,\mathrm{d}z\,. \tag{6.41}$$

Before proceeding to the convergence result, we collect a number of bounds on integrals involving the kernels K_ε and K_ε^Q. In what follows we will sometimes write K_0 in place of K as well as K_0^Q in place of K^Q, while $|x| = |x_1| + |x_2| + |x_3|$ denotes the ℓ^1-norm of $x \in \mathbb{R}^3$, so that $\|(t,x)\|_\mathfrak{s} = |t|^{1/2} + |x|$, cf (3.16). Note however that $|x|$ and the Euclidean norm $\|x\|$ are equivalent.

Lemma 6.2. *The following bounds hold for all $z = (t, x) \in \mathbb{R}^4$ and all $\varepsilon \in [0, 1]$:*

$$|K_\varepsilon(z)| \lesssim \frac{1}{(\|z\|_{\mathfrak{s}} + \varepsilon)^3} \, , \qquad |K_\varepsilon^Q(z)| \lesssim \frac{1}{|x| + \varepsilon} \, , \tag{6.42}$$

and

$$\int K_\varepsilon(z)^2 \, \mathrm{d}z \lesssim \frac{1}{\varepsilon} \, , \qquad \int K_\varepsilon(z) K_\varepsilon^Q(z) \, \mathrm{d}z \lesssim 1 \, , \qquad \int K_\varepsilon^Q(z)^2 \, \mathrm{d}z \lesssim 1 \, . \tag{6.43}$$

Furthermore, one has

$$|\mathcal{Q}_0^\varepsilon(z)| \lesssim \frac{1}{\|z\|_{\mathfrak{s}} + \varepsilon} \, , \qquad |\mathcal{Q}_1^\varepsilon(z)| \lesssim 1 \, , \qquad |\mathcal{Q}_2^\varepsilon(z)| \lesssim 1 \, . \tag{6.44}$$

Finally, for $i, j \in \{0, 1, 2\}$ let

$$I_{ij}(\varepsilon) = \int K(z) \mathcal{Q}_i^\varepsilon(z) \mathcal{Q}_j^\varepsilon(z) \, \mathrm{d}z \, . \tag{6.45}$$

Then $I_{00}(\varepsilon) \asymp |\log \varepsilon|$, while all other $I_{ij}(\varepsilon)$ are bounded uniformly in ε.

The proof is given in Appendix B. It is important to emphasize that Lemma 6.2 is the key result to show that certain terms do not have to be renormalised. For example, consider the difference between the first bound in (6.43) and the second and third bounds. This explains why certain symbols involving the $\mathcal{E}$-operator do not have to be renormalised while the same symbol without an additional $\mathcal{E}$-operator has to be renormalised; see also Appendix B for a detailed computation.

Lemma 6.3. *For any $\varepsilon, \theta \in (0, 1]$ and any $(t, x) \in \mathbb{R}^4$, one has the bound*

$$\left| K^Q(t, x) - K_\varepsilon^Q(t, x) \right| \lesssim \frac{\varepsilon^\theta}{|x|^{1+\theta}} \, . \tag{6.46}$$

As a consequence,

$$\int \left(K_\varepsilon^Q(z - z_1) - K^Q(z - z_1) \right) \left(K_\varepsilon^Q(\bar{z} - z_1) - K^Q(\bar{z} - z_1) \right) \mathrm{d}z_1 \lesssim \varepsilon^{2\theta} \tag{6.47}$$

holds for all $\theta \in (0, \frac{1}{2})$ and all $z, \bar{z} \in \mathbb{R}^4$.

We give the proof in Appendix C.

We can now state the main result of this section, which is an adaptation of [19, Thm. 10.22].

Proposition 6.4 (Convergence of the renormalised models)**.** *There exists a random model $\widehat{Z}$, independent of the choice of mollifier ϱ, and a family of renormalisation transformations $M_\varepsilon \in \mathfrak{R}$ such that for any $\theta < -\frac{5}{2} - \alpha_0 = \kappa$ (recall from (3.6) the definition of the noise regularity α_0), any compact set $\mathfrak{K}$ and any $\gamma < \zeta$, one has*

$$\mathbb{E} \| M_\varepsilon Z^\varepsilon \, ; \, \widehat{Z} \|_{\gamma; \mathfrak{K}} \lesssim \varepsilon^\theta \tag{6.48}$$

uniformly over $\varepsilon \in (0, 1]$. Here ζ is such that (3.25) holds for all polynomials P of parabolic degree less or equal ζ.

Proof. As already stated, the proof follows along the lines of [19, Thm. 10.22], with some changes due to the presence of the additional integration operator $\mathcal{E}$. Thus we will only comment on those parts of the proof that need to be adapted.

The result will follow if we are able to choose the constants $C_1(\tau)$ and $C_2(\tau)$ in (6.11) for each $\varepsilon \in (0, 1]$ in such a way that the bounds (6.16) hold for all $\tau \in \mathcal{F}_-$; equivalently,

the bounds (6.20) should hold for all these τ. The limiting model $\widehat{Z}$ will be constructed step by step during this process.

The first step is to set

$$(\widehat{\Pi}_0 \Xi)(z) = \xi(z) \,,$$

and the fact that the required bounds are satisfied is a consequence of Proposition 9.5 in [19] (see also Lemma 10.2).

In the case of $\tau = \uparrow$, the limiting model is defined by (6.23), and the conclusion follows as shown in [19, Thm. 10.22] by applying the bounds on convolutions in [19, Lemmas 10.14 and 10.17]. We thus turn to the case $\tau = \overset{\bullet}{\underset{\circ}{\text{}}}$. By (6.27), we have

$$\big((\widehat{\mathcal{W}}^{(\varepsilon;1)} \overset{\bullet}{\underset{\circ}{\text{}}})(z)\big)(z_1) = K_\varepsilon^Q(z - z_1) \,,$$
$$\big((\widehat{\mathcal{W}}^{(1)} \overset{\bullet}{\underset{\circ}{\text{}}})(z)\big)(z_1) = K^Q(z - z_1) \,.$$

The first bound in (6.20) then follows from the fact that

$$\big\langle (\widehat{\mathcal{W}}^{(1)} \overset{\bullet}{\underset{\circ}{\text{}}})(z), (\widehat{\mathcal{W}}^{(1)} \overset{\bullet}{\underset{\circ}{\text{}}})(\bar{z}) \big\rangle = \int K^Q(z - z_1) K^Q(\bar{z} - z_1) \, \mathrm{d}z_1 = \mathcal{Q}_2^0(z - \bar{z})$$

(via the change of variables $z_1 \mapsto z - z_1$). The right-hand side being bounded as shown in Lemma 6.2, the bound (6.20) is satisfied for $\lambda = 0$ provided $\bar{\kappa} \leqslant -2|\overset{\bullet}{\underset{\circ}{\text{}}}|_{\mathfrak{s}} = 1 + 2\kappa$. In order to prove the second bound in (6.20), we use the bound (6.47) in Lemma 6.3, which yields

$$\big\langle (\delta \widehat{\mathcal{W}}^{(\varepsilon;1)} \overset{\bullet}{\underset{\circ}{\text{}}})(t, x), (\delta \widehat{\mathcal{W}}^{(\varepsilon;1)} \overset{\bullet}{\underset{\circ}{\text{}}})(\bar{t}, \bar{x}) \big\rangle \lesssim \varepsilon^{2\theta}$$

for any $\theta \in (0, \frac{1}{2})$. Next we consider the term $\overset{\bullet}{\vee}$. Then a decomposition analogous to the one given in (6.30) shows that we have

$$\big((\widehat{\mathcal{W}}^{(\varepsilon;2)} \overset{\bullet}{\vee})(z)\big)(z_1, z_2) = K_\varepsilon(z - z_1) K_\varepsilon^Q(z - z_2) \,,$$
$$(\widehat{\mathcal{W}}^{(\varepsilon;0)} \overset{\bullet}{\vee})(z) = \int K_\varepsilon(-z_1) K_\varepsilon^Q(-z_1) \, \mathrm{d}z_1 - C_1'(\varepsilon) \,,$$

where we have used the change of variables $z_1 \mapsto z_1 - z$. Note that the second line is independent of z, as a consequence of translation invariance, so that we can drop the argument z from the notation. The integral in the second line is equal to $\mathcal{Q}_1^\varepsilon(0)$, which is bounded uniformly in ε by Lemma 6.2. We may thus simply choose $C_1'(\varepsilon) = 0$, which amounts to setting

$$(\widehat{\Pi}_0^{(\varepsilon)} \overset{\bullet}{\vee})(z) = \iint K_\varepsilon(z - z_1) K_\varepsilon^Q(z - z_2) \xi(z_1) \diamond \xi(z_2) \, \mathrm{d}z_1 \, \mathrm{d}z_2 + \widehat{\mathcal{W}}^{(\varepsilon;0)} \overset{\bullet}{\vee}$$
$$= \chi_\varepsilon(z) \chi_\varepsilon^Q(z) \,.$$

Indeed, the first bound in (6.20) for $k = 0$ is satisfied with $\lambda = 0$, provided $\bar{\kappa} \leqslant -2|\overset{\bullet}{\vee}|_{\mathfrak{s}} = 2 + 4\kappa$. Note that being allowed to set $C_1'(\varepsilon) = 0$ here is precisely the point where we use the fact that certain terms do not have to be renormalised due to the bounds in Lemma 6.2. Of course, another possibility would be to set $C_1'(\varepsilon) = \widehat{\mathcal{W}}^{(\varepsilon;0)} \overset{\bullet}{\vee}$, which would result in $\widehat{\Pi}_0 \overset{\bullet}{\vee}$ belonging to the second *homogeneous* Wiener chaos. Since $\mathcal{Q}_1^\varepsilon(0)$ is bounded, however, the choice of $C_1'(\varepsilon)$ does not really matter, since it only results in a shift of the random variable defining $\widehat{Z}$. Whatever the choice, the bounds (6.20) can be checked by similar arguments as in the previous case.

The symbols $\overset{\bullet}{\vee}$, $\overset{}{\vee}$, $\overset{}{\vee}$, $\overset{}{\vee}$ as well as $\overset{}{\vee} X_i$ and $\overset{}{\vee} X_i$ can be treated in pretty much the same way. It thus remains to consider the symbols in the families of $\overset{}{\vee}$, $\overset{}{\vee}$ and $\overset{}{\vee}$. As shown in the proof of [19, Thm. 10.22], the symbol $\overset{}{\vee}$ already fulfils the bounds (6.20) as a consequence of the choice of $C_1(\varepsilon)$. Since the kernel K^Q is less singular than K, the same conclusion holds for all symbols of the same family.

The symbol $\mathord{\text{⚓}}$ requires the introduction of a further renormalisation constant $C_2(\mathord{\text{⚓}})$ which has order $|\log \varepsilon|$. We thus have to check what happens to the other symbols in the same family. Consider for instance the case of $\mathord{\text{⚓}}$. A computation similar to the one made in (6.36) and (6.37) yields the expression

$$(\widehat{\Pi}_0^\varepsilon \mathord{\text{⚓}})(z) = \big[(K * (\chi_\varepsilon \diamond \chi_\varepsilon^Q))(z) - (K * (\chi_\varepsilon \diamond \chi_\varepsilon^Q))(0)\big]\chi_\varepsilon^{\diamond 2}(z) - C_2(\mathord{\text{⚓}}) \, .$$

Writing out the integrals and applying (6.38), we can decompose this expression into a sum of terms in the Wiener chaoses of order $0, 2$ and 4. In particular, the term in the 0th Wiener chaos is given by

$$(\widehat{\mathcal{W}}^{(\varepsilon;0)}\mathord{\text{⚓}})(z) = 2 \iiint K(z - \bar{z})K_\varepsilon(\bar{z} - z_1)K_\varepsilon^Q(\bar{z} - z_2)K_\varepsilon(z - z_1)K_\varepsilon(z - z_2)\,\mathrm{d}\bar{z}\,\mathrm{d}z_1\,\mathrm{d}z_2$$
$$+ 2 \iiint K(-\bar{z})K_\varepsilon(\bar{z} - z_1)K_\varepsilon^Q(\bar{z} - z_2)K_\varepsilon(z - z_1)K_\varepsilon(z - z_2)\,\mathrm{d}\bar{z}\,\mathrm{d}z_1\,\mathrm{d}z_2$$
$$- C_2(\mathord{\text{⚓}}) \, .$$

The first integral on the right-hand side does not depend on z, as can be seen by the change of variables $(\bar{z}, z_1, z_2) \mapsto (\bar{z} + z, z_1 + z, z_2 + z)$. As a consequence, it is equal to $2I_{01}(\varepsilon)$, and is thus uniformly bounded in ε as shown in Lemma 6.2. The second integral is also bounded, so that we may choose $C_2(\mathord{\text{⚓}}) = 0$ (or any other finite value). The terms in the other two Wiener chaoses are already bounded in the case of $\mathord{\text{⚓}}$, so they remain bounded in the present case. It is now quite obvious that the other symbols of the family do not need to be renormalised either, as the relevant integrals are obtained from the above ones by substituting the appropriate kernels K by K^Q.

Finally, we have to deal with symbols in the family of $\mathord{\text{⚓}}$. However, it is shown in the proof of [19, Thm. 10.22] that the constant $C_2(\mathord{\text{⚓}})$ suffices to renormalise $\mathord{\text{⚓}}$ as well, so the situation is completely analogous for the other symbols of the family. $\qquad\square$

6.3 Computation of the renormalised equations

In order to complete the proof of the main results, it remains to derive the classical equation with mollified noise that corresponds to the renormalised models. In other words, for a given $M = M_\varepsilon \in \mathfrak{R}$, we have to characterise the solution map $\bar{\mathcal{S}}_M^L$ satisfying

$$\mathcal{R}^M \mathcal{S}_M^L(u_0, v_0, MZ^\varepsilon) = \bar{\mathcal{S}}_M^L(u_0, v_0, \xi^\varepsilon) \, , \tag{6.49}$$

where $\mathcal{R}^M$ is the reconstruction operator of the model Π^M ($= \widehat{\Pi}^{(\varepsilon)}$) and $\mathcal{S}_M^L$ is the map giving the fixed point of the renormalised equation in terms of initial data and forcing term. Relation (6.49) can also be represented as the following commutative diagram:

$$
\begin{array}{ccc}
(u_0, v_0, MZ^\varepsilon) & \xmapsto{\;\;\mathcal{S}_M^L\;\;} & (U_M^*, V_M^*) \\[2pt]
{\scriptstyle M\Psi}\Big\uparrow & & \Big\downarrow{\scriptstyle \mathcal{R}^M} \\[2pt]
(u_0, v_0, \xi^\varepsilon) & \xmapsto[\;\;\bar{\mathcal{S}}_M^L\;\;]{} & (\hat{u}^\varepsilon, \hat{v}^\varepsilon)
\end{array}
$$

In order to determine the classical equation, we will need the following algebraic result.

Lemma 6.5. *For all $\tau \in \mathcal{F}_0$, one has*

$$(\Pi_z^M \tau)(z) = (\Pi_z M\tau)(z) \, . \tag{6.50}$$

Proof. The claim can be checked directly by comparing the expressions given in (6.12) and (6.13). More generally, the first relation in (6.4) shows that (6.50) holds for all τ such that $\Delta^M(\tau) = M\tau \otimes \mathbf{1}$, which happens to be the case for all symbols excepts those of the families $\math{\vcenter{\hbox{⸎}}}$, $\math{\vcenter{\hbox{⸎}}}$ and $\math{\vcenter{\hbox{⸎}}}$. In the exceptional cases, the relation has to be checked by an explicit computation; in general it holds only when the model is evaluated at the base point, i.e. we may have $(\Pi_z^M \tau)(\bar{z}) \neq (\Pi_z M\tau)(\bar{z})$ if $z \neq \bar{z}$. For instance, in the case $\tau = \math{\vcenter{\hbox{⸎}}}$, the action of the renormalisation map is given by

$$M\,\overset{\bullet\bullet}{\math{\vcenter{\hbox{⸎}}}}\,\bullet = \overset{\bullet\bullet}{\math{\vcenter{\hbox{⸎}}}}\,\bullet - 2C_1'\,\overset{\bullet}{\math{\vcenter{\hbox{⸎}}}}\,\bullet - C_1''\,\math{\vcenter{\hbox{⸎}}}\,\bullet \;.$$

One can then check that in the case $\tau = \math{\vcenter{\hbox{⸎}}}$, (6.5) is satisfied by setting $\Delta^M\,\math{\vcenter{\hbox{⸎}}} = M\,\math{\vcenter{\hbox{⸎}}} \otimes \mathbf{1}$ and $\hat{M}\mathcal{J}(\math{\vcenter{\hbox{⸎}}}) = \mathcal{J}(M\,\math{\vcenter{\hbox{⸎}}})$. Using this and the definition of the map Δ, a somewhat lengthy computation shows that

$$\Delta^M\,\overset{\bullet\bullet}{\math{\vcenter{\hbox{⸎}}}}\,\bullet = M\,\overset{\bullet\bullet}{\math{\vcenter{\hbox{⸎}}}}\,\bullet \otimes \mathbf{1} + 2C_1'\,\math{\vcenter{\hbox{⸎}}}X_i \otimes \mathcal{J}_i(\math{\vcenter{\hbox{⸎}}}) + C_1''\,\math{\vcenter{\hbox{⸎}}}X_i \otimes \mathcal{J}_i(\math{\vcenter{\hbox{↑}}})$$

satisfies the second relation in (6.5) when $\tau = \overset{\bullet\bullet}{\math{\vcenter{\hbox{⸎}}}}\,\bullet$. By (6.4), this implies

$$\Pi_z^M\,\overset{\bullet\bullet}{\math{\vcenter{\hbox{⸎}}}}\,\bullet = (\Pi_z \otimes f_z)\Delta^M\,\overset{\bullet\bullet}{\math{\vcenter{\hbox{⸎}}}}\,\bullet = \Pi_z M\,\overset{\bullet\bullet}{\math{\vcenter{\hbox{⸎}}}}\,\bullet + 2C_1'\Pi_z(\math{\vcenter{\hbox{⸎}}}X_i)\langle f_z, \mathcal{J}_i\,\math{\vcenter{\hbox{⸎}}}\rangle + C_1''\Pi_z(\math{\vcenter{\hbox{⸎}}}X_i)\langle f_z, \mathcal{J}_i\,\math{\vcenter{\hbox{↑}}}\rangle \;,$$

where we used the fact that $\langle f_z, \mathbf{1}\rangle = 1$. Since $(\Pi_z\,\math{\vcenter{\hbox{⸎}}}X_i)(\bar{z}) = (\Pi_z\,\math{\vcenter{\hbox{⸎}}})(\bar{z})(\bar{x}_i - x_i)$ by (3.22), we see that the last two terms on the right-hand side indeed vanish when $\bar{z} = z$. $\qquad\square$

As a consequence of (6.50), when f is a function, as is the case for $\varepsilon > 0$, we have

$$(\mathcal{R}^M f)(z) = (\Pi_z^M f(z))(z) = (\Pi_z M f(z))(z) = (\mathcal{R}M f)(z) \;. \tag{6.51}$$

This directly implies the following result, which is the equivalent of Proposition 5.2 for the renormalised equation.

Proposition 6.6 (Renormalised equation). *Assume that $\varepsilon > 0$ and that (5.1) admits a fixed point (U, V) where $U, V \in \mathcal{D}^{\gamma,\eta}$ for some $\gamma, \eta \in \mathbb{R}$. Then $(\hat{u}, \hat{v}) = (\mathcal{R}^M U, \mathcal{R}^M V)$ satisfies the classical SPDE*

$$\begin{aligned}
\partial_t \hat{u} &= \Delta \hat{u} + \widehat{F}(\hat{u}, \hat{v}) + \xi^\varepsilon \;, \\
\partial_t \hat{v} &= A_1 \hat{u} + A_2 \hat{v} \;,
\end{aligned} \tag{6.52}$$

provided $\widehat{F}$ is such that

$$\widehat{F}(MU, MV) = MF(U, V) + \varrho(U, V) \;, \tag{6.53}$$

with ϱ having only components of strictly positive homogeneity.

Proof. Applying M to the expansions of U and V given in (5.6), we see that MU differs from U and MV differs from V only by components of homogeneity $\frac{3}{2} - \kappa$ at least. Applying $\mathcal{R}$ to MU and MV and using (6.51), we obtain that as in the proof of Proposition 5.2

$$\begin{aligned}
\hat{u}(z) &= \chi_\varepsilon(z) + \varphi(z) \;, \\
\hat{v}(z) &= \chi_\varepsilon^Q(z) + \psi(z) \;.
\end{aligned}$$

On the other hand, applying M to the right-hand side of (5.3) and using the fact that M commutes with $\mathcal{I}$ and $\mathcal{E}$ and leaves polynomials invariant, we get

$$\begin{aligned}
MU &= \mathcal{I}\boldsymbol{R}^+\big[\Xi + MF(U, V)\big] + \widehat{R}(MU, MV) + Gu_0 \;, \\
MV &= \mathcal{E}\boldsymbol{R}^+ MU_- + \mathcal{Q}\boldsymbol{R}^+ MU_+ + \widehat{Q}v_0 \;.
\end{aligned}$$

Applying the reconstruction operator, we thus obtain

$$\hat{u}(t,x) = G * \boldsymbol{R}^{+}\big[\xi^{\varepsilon}(t,x) + (\mathcal{R}MF(U,V))(t,x)\big] + Gu_0(t,x) \ ,$$

$$\hat{v}(t,x) = \int_0^t Q(t-s)\hat{v}(s,x)\,\mathrm{d}s + \widehat{Q}v_0(t,x) \ .$$

The claim then follows from the fact that (6.53) implies

$$(\mathcal{R}MF(U,V))(z) = (\mathcal{R}\widehat{F}(MU,MV))(z) = \widehat{F}(\hat{u}(z),\hat{v}(z)) \ ,$$

where the last equality follows from the same argument as the one used in the proof of Proposition 5.2. □

Finally, we compute the expression of $\widehat{F}$. To simplify the computations, we assume that $C_1 = C_1(\vee)$ and $C_2 = C_2(\vee\!\!\!\!\!\cdot)$ are the only non-zero renormalisation constants, which is a possible choice as seen in the proof of Proposition 6.4. This is also the place where the assumption $\gamma_2 = 0$ turns out to be necessary in order to obtain a simple renormalised equation.

Proposition 6.7 (Expression for the renormalised F). *Let F be the cubic polynomial given in (5.5) with either $d = 2$, or $d = 3$ and $\gamma_2 = 0$, and let M be the renormalisation transformation defined in (6.11), where the only nonzero constants are $C_1 = C_1(\vee)$ and $C_2 = C_2(\vee\!\!\!\!\!\cdot)$. Then*

$$\widehat{F}(u,v) = F(u,v) - c_0(\varepsilon) - c_1(\varepsilon)u - c_2(\varepsilon)v \ , \tag{6.54}$$

where the coefficients are given by

$$\begin{aligned}
c_0(\varepsilon) &= \beta_1(C_1 + 3\gamma_1 C_2) \ , \\
c_1(\varepsilon) &= 3\gamma_1(C_1 + 3\gamma_1 C_2) \ , \\
c_2(\varepsilon) &= \gamma_2 C_1 \ .
\end{aligned} \tag{6.55}$$

Proof. The fact that (U,V) differs from (MU,MV) only by terms of order $\frac{3}{2} - \kappa$ implies

$$\widehat{F}(U,V) = MF(U,V) + \varrho'(U,V) \ ,$$

where all components of $\varrho'(U,V)$ have strictly positive homogeneity. Thus we only have to compute the difference $MF(U,V) - F(U,V)$, which reduces to determining the terms of $F(U,V)$ which are modified by the renormalisation map, or, more precisely, which terms of homogeneity less or equal 0 appear when applying M.

The computation is straightforward, though somewhat lengthy. In practice, it suffices to expand the quadratic terms of $F(U,V)$ up to order $\frac{1}{2} - \kappa$, and cubic terms up to order $0 - 2\kappa$. To explain where the condition $\gamma_2 = 0$ arises if $d = 3$, we allow for general $\gamma_2 \in \mathbb{R}$ for the time being.

Consider first the quadratic part $F_2(U,V) = \beta_1 U^2 + \beta_2 UV + \beta_3 V^2$ of $F(U,V)$. It turns out that when the renormalisation map M is applied, only the term proportional to $\vee$ of $F_2(U,V)$ yields a contribution of not strictly positive homogeneity. Namely, one has

$$MF_2(U,V) - F_2(U,V) = -\beta_1 C_1 \mathbf{1} + \varrho_2(U,V) \ ,$$

where $\varrho_2(U,V)$ has strictly positive homogeneity. It remains to determine the contribution of the cubic part $F_3(U,V) = \gamma_1 U^3 + \gamma_2 U^2 V + \gamma_3 UV^2 + \gamma_4 V^3$. Substituting U and V by their expansion (5.6) and sorting terms by homogeneity yields an expression of the form

$$F_3(U,V) = F_{3;-\frac{3}{2}}(U,V) + F_{3;-1}(U,V) + F_{3;-\frac{1}{2}}(U,V) + F_{3;0}(U,V) + \varrho_3(U,V) \ ,$$

where each $F_{3;\beta}$ contains only terms of homogeneity $\beta - \mathcal{O}(\kappa)$ and the remainder ϱ_3 has strictly positive homogeneity. We treat all of these terms separately. For instance, we have $F_{3;-\frac{3}{2}}(U,V) = \gamma_1\,\vartheta + \gamma_2\,\vartheta + \gamma_3\,\vartheta + \gamma_4\,\vartheta$, where the first two terms have to be renormalised, yielding

$$MF_{3;-\frac{3}{2}}(U,V) - F_{3;-\frac{3}{2}}(U,V) = -3\gamma_1 C_1\,\uparrow - \gamma_2 C_1\,\varphi + \varrho_{3;-\frac{3}{2}}(U,V)\,.$$

The term $F_{3;-1}(U,V)$ yields a contribution

$$MF_{3;-1}(U,V) - F_{3;-1}(U,V) = -3\gamma_1 C_1\varphi\mathbf{1} - \gamma_2 C_1\psi\mathbf{1} + \varrho_{3;-1}(U,V)$$

stemming from the terms $\vee$. Consider next the term $F_{3;0}(U,V)$. It contains 2 terms proportional to $\vee$, whose renormalisation yields the contribution

$$MF_{3;0}(U,V) - F_{3;0}(U,V) = -\big[3\gamma_1 b_1 + \gamma_2\hat{b}_1\big]C_2\mathbf{1} + \varrho_{3;0}(U,V)\,.$$

Finally, in the term $F_{3;-\frac{1}{2}}(U,V)$, it is the terms in $\vee$ and $\vee$ that need to be renormalised. One obtains

$$\begin{aligned}
MF_{3;-\frac{1}{2}}(U,V) - F_{3;-\frac{1}{2}}(U,V) = & -\big[9\gamma_1 a_1 + 3\gamma_2\hat{a}_1\big]C_2\,\uparrow \\
& -\big[9\gamma_1 a_2 + 3\gamma_2\hat{a}_2\big]C_2\,\varphi + \varrho_{3;-\frac{1}{2}}(U,V)\,.
\end{aligned}$$

In the case $\gamma_2 = 0$, using $a_i = \gamma_i$ and replacing the b_i by their expressions (5.7), and adding the last four expressions, we see that we can always factor out the expression $\varphi\mathbf{1} + \uparrow$, which coincides with U up to terms of strictly positive homogeneity. The result follows.

In case $\gamma_2 \neq 0$, there are additional terms such as $\gamma_2 C_2\hat{b}_1\mathbf{1}$, which do not admit a simple expression in terms of U and V, so that the renormalised equation has no simple closed-form expression. However when $d = 2$, then we can choose $C_2 = 0$ because the symbol $\vee$ has strictly positive homogeneity and does not have to be renormalised. Then we can also factor out the quantity $\psi\mathbf{1} + \varphi$, yielding the expression for $c_2(\varepsilon)$ in (6.55). $\qquad\square$

7 Proof of the main results

Since Theorem 2.1 is a particular case of Theorem 2.2, we proceed directly to proving the latter result.

Proof of Theorem 2.2. Consider first the situation on a fixed, sufficiently small time interval $[0,T]$. Fix $\varepsilon > 0$ and let

$$(U^\varepsilon, V^\varepsilon) = \mathcal{S}_T(u_0, v_0, M_\varepsilon Z^\varepsilon)\,,$$
$$(U, V) = \mathcal{S}_T(u_0, v_0, \widehat{Z})\,,$$

where $\mathcal{S}_T$ is the solution map given in Proposition 5.6 (we have suppressed the dependence on W, which is the same for both fixed points). For any $\delta > 0$, the bound (5.16), Markov's inequality and Proposition 6.4 imply

$$\mathbb{P}\Big\{\|U^\varepsilon\,;U\|_{\gamma,\eta;T} > C_0\delta\Big\} \leqslant \frac{1}{\delta}\mathbb{E}\|M_\varepsilon Z^\varepsilon\,;\widehat{Z}\|_{2\gamma+\alpha;O} \lesssim \frac{\varepsilon^\theta}{\delta}\,.$$

Furthermore, since $V = \mathcal{K}_\gamma^Q \boldsymbol{R}^+ U + \widehat{Q}v_0$, the multilevel Schauder estimate in Proposition 4.11 yields a bound of the form

$$\|V^\varepsilon\,;V\|_{\gamma,\eta;\mathfrak{K}} \leqslant \overline{C}_1\|U^\varepsilon\,;U\|_{\gamma,\eta;\overline{\mathfrak{K}}} + \overline{C}_2\|M_\varepsilon Z^\varepsilon\,;\widehat{Z}\|_{\gamma;\overline{\mathfrak{K}}}\,,$$

which implies a similar bound for the probability of $\|V^\varepsilon \,; V\|_{\gamma,\eta;T}$ becoming large. It follows that $(U^\varepsilon, V^\varepsilon)$ converges to (U, V) in probability as $\varepsilon \to 0$.

Next, let $(u, v) = (\mathcal{R}U, \mathcal{R}V)$ and $(u^\varepsilon, v^\varepsilon) = (\mathcal{R}U^\varepsilon, \mathcal{R}V^\varepsilon)$ (where we do not indicate the model-dependence of the reconstruction operators $\mathcal{R}$). The fact that $(u^\varepsilon, v^\varepsilon)$ converges to (u, v) in probability as $\varepsilon \to 0$ is a consequence of the bound [19, (3.4)] in the reconstruction theorem, combined with the definitions (3.20) of $\|\Pi\|_{\gamma;\mathfrak{K}}$ and (4.28) of $\|\cdot\|_{\gamma,\eta;\mathfrak{K}}$.

The convergence result can be extended from a fixed time interval $[0, T]$ to the random time interval up to the first exit from a ball of radius L in exactly the same way as in [19, Section 7.3]. In particular, the required continuous dependence on the model is proved in [19, Cor. 7.12].

Proposition 6.6 shows that $(u^\varepsilon, v^\varepsilon)$ satisfies the system (2.11), where the explicit expression of $\widehat{F}$ is a consequence of Proposition 6.7. The expressions (2.5) and (2.12) for the renormalisation constants in dimension $d = 3$ follow from (6.55), where the only nonzero terms are $C_1(\varepsilon)$, defined in (6.33), and $C_2(\varepsilon)$, defined in (6.41).

In the case $d = 2$, Table 1 shows that only the terms $\mathbf{1}$, $\vee$ and $\mathbf{V}$ need to be renormalised, so that the expression (2.7) for the renormalisation constant follows by taking $C_2(\varepsilon) = 0$ in (6.55). $\qquad\square$

Proof of Theorem 2.3. The only difference with the previous case is that we have to take into account the presence of the n different operators $\mathcal{E}_i$, $i = 1, \dots, n$, introduced in Remark 4.2. This means that the fixed-point equation for V in (5.1) is replaced by n different equations of the form

$$V_i = \mathcal{K}_\gamma^{Q_i} \boldsymbol{R}^+ U + \widehat{Q}_i v_0 \,, \qquad i = 1, \dots, n \,.$$

Since n is finite, it is straightforward to check that the proofs of Proposition 5.2 and Proposition 5.6 carry over to this situation.

It remains to check the expressions (2.18) for the renormalisation constants. To compute the diverging part of $\widehat{F}(U, V) - F(U, V)$, it is in fact sufficient to determine the coefficients of $\vee$, $\mathbf{V}$ and $\mathbf{V}$ in $F(U, V)$, since these are the only ones that need to be renormalised. These can only result from the terms $\beta_1 U^2 + \gamma_1 U^3$ in $F(U, V)$, since all other terms will contain at least one of the $\mathcal{E}_i$. It is now sufficient to check that U has the same expansion as in (5.6), except that the first relation in (5.7) becomes

$$b_1 = \beta_1 + 3\varphi\gamma_1 + \sum_{i=1}^n \psi_i \gamma_{2,i} \,,$$

where ψ_i is the coefficient of $\mathbf{1}$ in V_i. The result then follows as in the proof of Proposition 6.7, taking into account the modified value of b_1. $\qquad\square$

A Proof of Lemma 4.8

The proof is by induction on the size of the regularity structure, which is constructed recursively, starting with Ξ, $\mathbf{1}$ and the X_i, and applying the operators $\mathcal{I}$ and $\mathcal{E}$ as well as multiplication to already existing symbols. We show that the properties (4.21) hold on the polynomial regularity structure and for $\tau = \Xi$, and that they are stable under the algebraic operations extending this structure.

1. The claim certainly holds on the polynomial part of the regularity structure, since

$$(\Pi_{z+h}^\varepsilon X^k)(\bar{z} + h) = (\bar{z} + h - z - h)^k = (\Pi_z^\varepsilon X^k)(\bar{z}) \,,$$
$$\Gamma_{z+h,\bar{z}+h}^\varepsilon X^k = (X - \bar{z} - h + z + h)^k = \Gamma_{z\bar{z}}^\varepsilon X^k \,,$$

and

$$\Gamma^\varepsilon_{z+h,\bar{z}+h} X^k = (\mathrm{Id} \otimes \gamma^\varepsilon_{z+h,\bar{z}+h}) \Delta X^k = X^k + \mathbf{1} \langle \gamma^\varepsilon_{z+h,\bar{z}+h}, X^k \rangle \,,$$

$$\Gamma^\varepsilon_{z,\bar{z}} X^k = (\mathrm{Id} \otimes \gamma^\varepsilon_{z,\bar{z}}) \Delta X^k = X^k + \mathbf{1} \langle \gamma^\varepsilon_{z,\bar{z}}, X^k \rangle \,.$$

2. Using the fact that $\Delta(\Xi) = \Xi \otimes \mathbf{1}$, it is also immediate to check that (4.21) holds for $\tau = \Xi$ (this reflects translation invariance of the noise).

3. We show that if the relations (4.21) hold for τ, then they also hold with τ and $\bar{\tau}$ replaced by $\mathcal{I}\tau$ and $\mathcal{J}_k \tau$. Indeed,

$$\langle f^\varepsilon_{z+h}, \mathcal{J}_k \tau \rangle = - \int D^k K(z + h - \bar{z})(\Pi^\varepsilon_{z+h}\tau)(\bar{z}) \, \mathrm{d}\bar{z}$$

$$= - \int D^k K(z - \tilde{z})(\Pi^\varepsilon_{z+h}\tau)(\tilde{z} + h) \, \mathrm{d}\tilde{z}$$

$$= - \int D^k K(z - \tilde{z})(\Pi^\varepsilon_z \tau)(\tilde{z}) \, \mathrm{d}\tilde{z}$$

$$= \langle f^\varepsilon_z, \mathcal{J}_k \tau \rangle \,,$$

and

$$(\Pi^\varepsilon_{z+h}\mathcal{I}\tau)(\bar{z} + h) = \int K(\bar{z} + h - z')(\Pi^\varepsilon_{z+h}\tau)(z') \, \mathrm{d}z' + \sum_\ell \frac{(\bar{z} - z)^\ell}{\ell!} \langle f^\varepsilon_{z+h}, \mathcal{J}_\ell \tau \rangle$$

$$= \int K(\bar{z} - \tilde{z})(\Pi^\varepsilon_z \tau)(\tilde{z}) \, \mathrm{d}\tilde{z} + \sum_\ell \frac{(\bar{z} - z)^\ell}{\ell!} \langle f^\varepsilon_z, \mathcal{J}_\ell \tau \rangle$$

$$= (\Pi^\varepsilon_z \mathcal{I}\tau)(\bar{z}) \,.$$

Furthermore, it is shown in the proof of [19, Thm. 8.24] that

$$\Gamma^\varepsilon_{z+h,\bar{z}+h}(\mathcal{I}\tau) = (\mathrm{Id} \otimes \gamma^\varepsilon_{z+h,\bar{z}+h})(\mathcal{I} \otimes \mathrm{Id})\Delta\tau + \sum_\ell \frac{(X - z_\gamma)^\ell}{\ell!} \langle \gamma^\varepsilon_{z+h,\bar{z}+h}, \mathcal{J}_\ell \tau \rangle \,,$$

where $z_\gamma \in \mathbb{R}^{d+1}$ has components $(z_\gamma)_i = -\langle \gamma^\varepsilon_{z+h,\bar{z}+h}, X_i \rangle$. The first term on the right-hand side is equal to $\mathcal{I}\Gamma^\varepsilon_{z+h,\bar{z}+h}(\tau)$. The induction hypothesis implies that all terms are equal to their value for $h = 0$, proving the required identity for $\Gamma^\varepsilon_{z\bar{z}}(\mathcal{I}\tau)$. The identity for $\gamma^\varepsilon_{z\bar{z}}(\mathcal{J}_k \tau)$ follows in a similar manner.

4. In a similar way, using the definition (4.6) of $\Pi^\varepsilon_z(\mathcal{E}\tau)$, it is straightforward to check that if τ satisfies (4.21), then $\mathcal{E}\tau$ satisfies (4.21) as well.

5. If τ, σ satisfy (4.21), then

$$(\Pi^\varepsilon_{z+h}\tau\sigma)(\bar{z} + h) = (\Pi^\varepsilon_{z+h}\tau)(\bar{z} + h)(\Pi^\varepsilon_{z+h}\sigma)(\bar{z} + h) = (\Pi^\varepsilon_z \tau)(\bar{z})(\Pi^\varepsilon_z \sigma)(\bar{z}) = (\Pi^\varepsilon_z \tau\sigma)(\bar{z}) \,.$$

Furthermore,

$$\Gamma^\varepsilon_{z+h,\bar{z}+h}(\tau\sigma) = (\mathrm{Id} \otimes \gamma^\varepsilon_{z+h,\bar{z}+h})\Delta(\tau\sigma) = (\mathrm{Id} \otimes \gamma^\varepsilon_{z\bar{z}})\Delta(\tau\sigma) = \Gamma^\varepsilon_{z\bar{z}}(\tau\sigma) \,.$$

Here we have used the fact that the induction hypothesis applies because the second component of $\Delta(\tau)$ always depends only on already computed quantities, cf. Table 1. The translation invariance of $\gamma^\varepsilon_{z\bar{z}}(\tau\sigma)$ follows. $\qquad\square$

B Proof of Lemma 6.2

It follows from the definition of K that it satisfies $|K(z)| \lesssim \|z\|_{\mathfrak{s}}^{-3}$. Computing the derivatives of K, one sees that it is singular of order -3 in the sense of [19, Def. 10.12]. Thus the first bound in (6.42) follows from [19, Lemma 10.17].

The second bound in (6.42) is a crucial estimate, which allows us to avoid renormalisation of certain symbols involving the operator $\mathcal{E}$. Therefore, we provide additional details in the proof of this bound. Recalling the definitions of K^Q and Q we have

$$K^Q(t,x) = \int_{\mathbb{R}} Q(s,x)K_\varepsilon(t-s,x)\,\mathrm{d}s = \int_0^{2T \wedge (t+\varepsilon^2)} Q(s,x)K_\varepsilon(t-s,x)\,\mathrm{d}s$$

since $K_\varepsilon(t,x)$ is supported on $\{t > -\varepsilon^2\}$. By a change of variable $s \mapsto t-s$ in the last integral and using the boundedness of Q on $[0,T]$ for a final time $T > 0$, it follows that

$$K^Q(t,x) \asymp \int_{-\varepsilon^2}^t K_\varepsilon(s,x)\,\mathrm{d}s,$$

where we recall that the notation $f \asymp g$ indicates that we have both $f \lesssim g$ and $g \lesssim f$. Therefore, combining the last characterisation of $K^Q(t,x)$ with the first bound in (6.42) yields

$$\begin{aligned}
|K_\varepsilon^Q(t,x)| &\lesssim \int_{-\varepsilon^2}^t \frac{1}{(|s|^{1/2} + |x| + \varepsilon)^3}\,\mathrm{d}s \\
&\lesssim \int_{-\varepsilon^2}^{(|x|+\varepsilon)^2} \frac{1}{(|x|+\varepsilon)^3}\,\mathrm{d}s + \int_{(|x|+\varepsilon)^2}^t \frac{1}{|s|^{3/2}}\,\mathrm{d}s \\
&\lesssim \frac{1}{|x|+\varepsilon}\,.
\end{aligned}$$

and this concludes the proof of the second bound in (6.42). The first bound in (6.43) follows from

$$\int K_\varepsilon(z)^2\,\mathrm{d}z \lesssim \int \frac{1}{(\|z_1\|_{\mathfrak{s}} + \varepsilon)^6}\,\mathrm{d}z_1 \asymp \int_0^1 \int_0^1 \frac{r^2\,\mathrm{d}r}{(t^{1/2}+r+\varepsilon)^6}\,\mathrm{d}t \asymp \frac{1}{\varepsilon}\,.$$

The second bound in (6.43) is again important, since it leads to terms which involve $\mathcal{E}$ but which do not have to be renormalised, so we provide more details in the proof of this bound. Note that

$$\begin{aligned}
\int K_\varepsilon(t,x)K_\varepsilon^Q(t,x)\,\mathrm{d}x &\lesssim \int \frac{1}{(|t|^{1/2}+|x|+\varepsilon)^3}\frac{1}{(|x|+\varepsilon)}\,\mathrm{d}x\,, \\
&\lesssim \int \frac{1}{(|t|^{1/2}+\|x\|+\varepsilon)^3}\frac{1}{(\|x\|+\varepsilon)}\,\mathrm{d}x \\
&\lesssim \iint \frac{1}{(|t|^{1/2}+r+\varepsilon)^3}\frac{1}{(r+\varepsilon)}\sin(\theta_1)r^2\,\mathrm{d}\theta\,\mathrm{d}r\,,
\end{aligned}$$

using the equivalence of norms and spherical coordinates $(r,\theta) = (r,\theta_1,\theta_2)$ in $\mathbb{R}^3$. Therefore, we find the estimate

$$\int K_\varepsilon(t,x)K_\varepsilon^Q(t,x)\,\mathrm{d}x \lesssim \int_0^1 \frac{r^2\,\mathrm{d}r}{(r+|t|^{1/2}+\varepsilon)^3(r+\varepsilon)} \asymp \frac{1}{|t|^{1/2}+\varepsilon}$$

Integrating over time then yields

$$\iint K_\varepsilon(t,x)K_\varepsilon^Q(t,x)\,\mathrm{d}x\,\mathrm{d}t \lesssim \int_{-\varepsilon^2}^1 \frac{1}{|t|^{1/2}+\varepsilon}\,\mathrm{d}t \lesssim 1\,.$$

which proves the second bound in (6.43). The third bound in (6.44) follows from the fact that

$$\int K_\varepsilon^Q(t,x)^2\,\mathrm{d}x \lesssim \int_0^1 \frac{r^2\,\mathrm{d}r}{(r+\varepsilon)^2} \asymp 1\,.$$

To obtain the bound on $\mathcal{Q}_0$, we distinguish the regimes $\|z\|_{\mathfrak{s}} \leqslant \varepsilon$ and $\|z\|_{\mathfrak{s}} > \varepsilon$. If $\|z\|_{\mathfrak{s}} \leqslant \varepsilon$, the same computation as the one made for the integral of $K_\varepsilon(z)^2$ shows that $|\mathcal{Q}_0(z)| \lesssim 1/\varepsilon$, while for $\|z\|_{\mathfrak{s}} > \varepsilon$, we can use [19, Lemma 10.14] to obtain

$$|\mathcal{Q}_0(z)| \lesssim \int \frac{1}{\|z_1\|_{\mathfrak{s}}^3} \frac{1}{\|z - z_1\|_{\mathfrak{s}}^3}\, \mathrm{d}z_1 \lesssim \frac{1}{\|z\|_{\mathfrak{s}}} \ .$$

Using spherical coordinates, one can show that for all $a, b \geqslant 0$ and all $n \geqslant 1$, one has

$$I_n := \int \frac{1}{(|x_1| + a)^n} \frac{1}{|x - x_1| + b}\, \mathrm{d}x_1 \asymp \int_0^1 \frac{r^2\, \mathrm{d}r}{(r + a)^n (r + |x| + b)} \ .$$

From this we easily get that whenever $a \geqslant b \geqslant 0$,

$$I_1 \lesssim 1 + |x| + a^2 \ , \qquad I_3 \lesssim \frac{1 + |\log a|}{|x| + a} \ ,$$

and thus

$$|\mathcal{Q}_1^\varepsilon(t, x)| \lesssim \int_0^1 \frac{1 + |\log(t^{1/2} + \varepsilon)|}{|x| + t^{1/2} + \varepsilon}\, \mathrm{d}t \lesssim 1 \ ,$$
$$|\mathcal{Q}_2^\varepsilon(t, x)| \lesssim 1 + |x| + \varepsilon^2 \lesssim 1 \ .$$

It remains to bound the terms $I_{ij}(\varepsilon)$. It is in fact sufficient to check that

$$I_{00}(\varepsilon) \lesssim \int_0^1 \int_0^1 \frac{r^2\, \mathrm{d}r}{(t^{1/2} + r)^3 (t^{1/2} + r + \varepsilon)^2}\, \mathrm{d}t \lesssim |\log \varepsilon| \ ,$$
$$I_{01}(\varepsilon) \lesssim \int_0^1 \int_0^1 \frac{r^2\, \mathrm{d}r}{(t^{1/2} + r)^3 (t^{1/2} + r + \varepsilon)}\, \mathrm{d}t \lesssim 1 \ ,$$

since the other terms can be bounded above by $I_{01}(\varepsilon)$. Elementary computations show that these bounds indeed hold true. $\qquad\square$

C Proof of Lemma 6.3

Let us recall Definition 10.12 in [19]. Given a scaling $\mathfrak{s}$, a smooth function $K : \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{R}$ is said to be (singular) of order ζ if for every sufficiently small multiindex k, one has

$$|D^k K(z)| \lesssim \|z\|_{\mathfrak{s}}^{\zeta - |k|_{\mathfrak{s}}} \qquad \forall z \text{ such that } \|z\|_{\mathfrak{s}} \leqslant 1 \ .$$

More precisely, we require that there exists $m \geqslant 0$ such that

$$\|K\|_{\zeta; m} := \sup_{k:\, |k|_{\mathfrak{s}} \leqslant m} \sup_{z \in \mathbb{R}^{d+1}} \|z\|_{\mathfrak{s}}^{|k|_{\mathfrak{s}} - \zeta} |D^k K(z)| < \infty \ .$$

Note that in [19, Def 10.12], the second supremum indeed runs over the whole space, but in practice we will only apply it to compactly supported functions. Lemma 10.17 in [19] states that if K is of order $\zeta \in (-|\mathfrak{s}|, 0)$, then $K_\varepsilon = K * \varrho_\varepsilon$ has bounded derivatives of all orders, satisfying

$$|D^k K_\varepsilon(z)| \lesssim (\|z\|_{\mathfrak{s}} + \varepsilon)^{\zeta - |k|_{\mathfrak{s}}} \|K\|_{\zeta; |k|_{\mathfrak{s}}} \ .$$

Furthermore, for all $\bar{\zeta} \in [\zeta - 1, \zeta)$ and $m \geqslant 0$, one has

$$\|K - K_\varepsilon\|_{\bar{\zeta}; m} \lesssim \varepsilon^{\zeta - \bar{\zeta}} \|K\|_{\zeta; m + \max\{\mathfrak{s}_i\}} \ . \tag{C.1}$$

Consider now the quantity

$$K^Q(t,x) - K_\varepsilon^Q(t,x) = \int_0^t \big[K(s,x) - K_\varepsilon(s,x) \big] Q(t-s)\,\mathrm{d}s$$

(in fact, the lower bound can be replaced by $0 \vee (t - 2T)$). We have

$$\big| K^Q(t,x) - K_\varepsilon^Q(t,x) \big| \lesssim \int_0^t \big| K(s,x) - K_\varepsilon(s,x) \big|\,\mathrm{d}s \, .$$

We already know that K is singular of order -3, and thus $\|K\|_{-3;m}$ is bounded for any m. In particular, we have by (C.1) with $m = 0$ the bound

$$\|K - K_\varepsilon\|_{\bar\zeta;0} \lesssim \varepsilon^{-3-\bar\zeta}\|K\|_{-3;2}$$

for any $\bar\zeta \in [-4, -3)$, which implies

$$\big| K(s,x) - K_\varepsilon(s,x) \big| \leqslant \frac{\|K - K_\varepsilon\|_{\bar\zeta;0}}{\|z\|_{\mathfrak{s}}^{-\bar\zeta}} \lesssim \frac{\varepsilon^{-3-\bar\zeta}}{\|z\|_{\mathfrak{s}}^{-\bar\zeta}}\|K\|_{-3;2} \, .$$

Taking $\bar\zeta = -3 - \theta$ for some $\theta \in (0, 1]$, we obtain

$$\big| K(s,x) - K_\varepsilon(s,x) \big| \lesssim \frac{\varepsilon^\theta}{\|z\|_{\mathfrak{s}}^{3+\theta}}\|K\|_{-3;2} \, ,$$

and thus, since $\|K\|_{-3;2} \lesssim 1$,

$$\big| K^Q(t,x) - K_\varepsilon^Q(t,x) \big| \lesssim \varepsilon^\theta \int_0^t \frac{\mathrm{d}s}{s^{(3+\theta)/2} + |x|^{3+\theta}} \lesssim \frac{\varepsilon^\theta}{|x|^{1+\theta}} \, ,$$

proving (6.46). To prove (6.47), we use spherical coordinates as in the proof of Lemma 6.2 to get, for any constant c of order 1,

$$\int_{|x_1| \leqslant c} \frac{\mathrm{d}x_1}{|x_1|^{1+\theta}|x - x_1|^{1+\theta}} \asymp \int_0^1 \frac{r^2}{r^{1+\theta}} \int_0^\pi \frac{\sin\phi\,\mathrm{d}\phi}{(r^2 + |x|^2 - 2r|x|\cos\phi)^{(1+\theta)/2}}\,\mathrm{d}r$$

$$\asymp \int_0^1 \frac{(r + |x|)^{1-\theta} - \big| r - |x| \big|^{1-\theta}}{2|x|r^\theta}\,\mathrm{d}r$$

$$\lesssim \int_0^{2|x|} \frac{|x|^{1-\theta}}{|x|r^\theta}\,\mathrm{d}r$$

$$+ \int_{2|x|}^1 \frac{r^{1-\theta}}{|x|r^\theta}\left[\left(1 + \frac{|x|}{r}\right)^{1-\theta} - \left(1 - \frac{|x|}{r}\right)^{1-\theta}\right]\mathrm{d}r$$

$$\asymp |x|^{1-2\theta} + \int_{2|x|}^1 \frac{\mathrm{d}r}{r^{2\theta}}$$

$$\lesssim 1$$

for $|x| \lesssim 1$, provided $2\theta < 1$. The result follows since $t \mapsto K^Q(t,x)$ has compact support. □

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