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On diffusion limited deposition

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Abstract

We propose a simple model of columnar growth through *diffusion limited aggregation* (DLA). Consider a graph $G_N \times \mathbb{N}$, where the basis has N vertices $G_N := \{1, \dots, N\}$, and two vertices (x,h) and (x',h') are adjacent if $|h-h'| \le 1$. Consider there a simple random walk *coming from infinity* which *deposits* on a growing cluster as follows: the cluster is a collection of columns, and the height of the column first hit by the walk immediately grows by one unit. Thus, columns do not grow laterally.

We prove that there is a critical time scale $N/\log(N)$ for the maximal height of the piles, i.e., there exist constants $\alpha < \beta$ such that the maximal pile height at time $\alpha N/\log(N)$ is of order $\log(N)$, while at time $\beta N/\log(N)$ is larger than N^χ for some positive χ. This suggests that a *monopolistic regime* starts at such a time and only the highest pile goes on growing. If we rather consider a walk whose height-component goes down deterministically, the resulting *ballistic deposition* has maximal height of order $\log(N)$ at time N.

These two deposition models, diffusive and ballistic, are also compared with uniform random allocation and Polya's urn.

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1 Introduction

Motivation. A celebrated model of deposition via diffusion was proposed in the early 80's by Witten and Sanders [31]. The aggregate, denoted $A(K)$, made of K sites of $\mathbb{Z}^d$ is built inductively as follows. Choose $A(1) = \{0\}$ and assume $A(K)$. Let $\partial A(K)$

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denote its outer boundary. Informally, launch a simple random walk, $n \mapsto S(n)$, far away from the origin, and stop it when it reaches $\partial A(K)$, say on random site Y. We set $A(K+1) = A(K) \cup \{Y\}$. In other words, if $\tau_{\partial A(K)}$ is the time at which the walk hits $\partial A(K)$, then for $y \in \partial A(K)$,

$$P\big(A(K+1) = A(K) \cup \{y\} \big| A(K)\big) = \lim_{\|x\| \to \infty} P_x\big(S(\tau_{\partial A(K)}) = y \big| \tau_{\partial A(K)} < \infty\big).$$

Simulations show that the cluster looks like a ramified tree with long branches. Heuristically, the origin of reinforcement is clear. Think of the walk in terms of its radial component, which performs an almost symmetric one-dimensional walk, and its transverse component. Either the random walk sticks soon after reaching the outer radius of the cluster, and it has to settle on a tip, or it takes time before settling and its radial component diffuses, and has more chances to visit the extremal shells, hence increasing the probability of attaching a tip rather than an inside site. This explains reinforcement, but does not explain why this reinforcement is enough to produce a ramified tree structure. It is clear also, at the heuristic level, that we face two problems: controlling the number of tips in the growing cluster, and controlling in a quantitative way the reinforcement of these tips.

One natural way to measure the dimension of the cluster is to find the scaling of the radius of $A(K)$, and look for $\bar{d}$ such that

$$\text{Radius}(A(K)) \sim K^{1/\bar{d}}. \tag{1.1}$$

If $A(K)$ were a ball, then $\bar{d} = d$, and the conjecture is that $\bar{d} < d$. Now, physicists have a much sharper conjecture

$$\bar{d}_c = d - \frac{d-1}{d+1}. \tag{1.2}$$

In dimension 2, $\bar{d}_c = 5/3$, and simulations give $\bar{d} = 1.7$.

Kesten in [14, 15, 16] considered the problem, and showed that the arms of the cluster are not too long. More precisely, his result reads

$$\bar{d} \geq \begin{cases} 3/2 \text{ for } d = 2\,, & (\bar{d}_c = 2 - 1/3) \\ 2 \text{ for } d = 3\,, & (\bar{d}_c = 5/2) \\ d/2 \text{ for } d \geq 3\,, & (\bar{d}_c \leq d - 3/5). \end{cases} \tag{1.3}$$

By reversing time, (see [21] and assume $d \geq 3$) one writes the probability of adding $Y = y$ to the cluster as

$$P\big(A(K+1) = A(K) \cup \{y\} \big| A(K)\big) = \frac{P_y\big(\tau_{\partial A(K)} = \infty\big)}{\sum_{z \in \partial A} P_z\big(\tau_{\partial A(K)} = \infty\big)}. \tag{1.4}$$

The difficulty is to estimate the escape probability when the set A is not a sphere, or some simple geometric shape. Let us mention an interesting result about holes in the DLA cluster, where a hole is a finite maximal connected subset of the complement of $A(K)$. Erbez-Wagner [12] showed that in dimension two, almost surely the number of holes tends to infinity with K.

Barlow, Pemantle and Perkins in [7] studied DLA on a regular d-ary tree where the conductance between edges joining generation n and $n+1$ is α^{-n} for $\alpha < 1$. These authors showed that the infinite cluster has a unique infinite line of descent. Even though there is an explicit formula for the harmonic measure, the proof that $r(A(K))$ scales like K with normal fluctuations is non-trivial.

Benjamini and Yadin in [9] proposed another toy model for DLA. They considered a cylinder $G_N \times \mathbb{N}$, where the graph G_N has constant degree, N vertices, and is fast mixing: the mixing-time should be less than $\log^{2-\epsilon}(|G_N|)$ for some positive ϵ (the class of d-regular random graphs works). They showed that if we send $H \times |G_N|$ simple walks from infinity, then the height of the aggregate is larger than $H\log(\log(|G_N|))$ for any H and N large enough.

There is a two-dimensional model, the Hastings-Levitov model, which takes advantage of the conformal invariance of two-dimensional brownian motion, and Riemann's mapping Theorem to map the complement of the cluster into the complement of the unit disk, and then attach on the unit circle a *stick* at a random uniform angle. Recently, Norris and Turner [26] studied very precisely the limiting cluster obtained by iteration of randomly rotated conformal mappings.

In a series of three recent papers, Amir, Angel, Benjamini and Kozma [1, 2, 3] studied DLA on $\mathbb{Z}$ with long-range random walks. The cluster is no longer connected, and they discover many phase transitions in the growth rate of the cluster according to the tail decay of the increment of the walk.

Our model is a further simplification of Benjamini and Yadin's model [9] in two ways: (i) no lateral hair are produced, and (ii) the basis graph has no geometry. In our toy model of DLA, the radial component does a one-dimensional random walk, and the transverse component samples uniformly the section of our graph. Still we believe that our model is interesting, and one can answer some of the following questions in a quantitative way.

- What is the origin of reinforcement?
- What is the critical height to overcome ?
- What are the different regimes in the cluster's growth?

Models. We shall consider two deposition models, *diffusive deposition* and *ballistic deposition*.

We start with defining diffusive deposition. Our graph is a half–cylinder $G_N \times \mathbb{N}$, where the basis has N vertices $G_N := \{1, \dots, N\}$, and two vertices (x, h) and (x', h') are adjacent if $|h - h'| = 1$. The set $G_N \times \{0\}$ is called *the ground*.

Let $n \mapsto A(n)$ be the evolution of random subsets of $G_N \times \mathbb{N}$ that we call *the cluster*. The cluster is built inductively with $A(0) = G_N \times \{0\}$. For an integer k, the cluster $A(k)$ is made of columns, that is,

$$A(k) = \bigcup_{i=1}^{N} \{i\} \times \{0, \dots, \sigma_i(k)\} \quad \text{with} \quad \sum_{i=1}^{N} \sigma_i(k) = k. \tag{1.5}$$

We shall write for simplicity $A(k) = (\sigma_1(k), \dots, \sigma_N(k))$.

Assume that $A(k)$ is built. We consider a simple random walk $n \mapsto S_n = (X_n, Z_n)$ on our graph. In other words,

1. $\{X_n\}$ an i.i.d. sequence uniformly distributed on G_N;
2. $\{Z_{n+1} - Z_n\}$ i.i.d. uniformly on $\{-1, 1\}$;
3. the initial condition Z_0 is above the maximal height of the cluster $A(k)$. For definiteness we take $Z_0 = \max_i \sigma_i(k) + 1$.

The following rule of *aggregation*, or *deposition*, makes the cluster grow. The walk S_n, roams until it hits the cluster $A(k)$. Let (X^*, Z^*) be the hitting site on $A(k)$, and necessarily $0 \le Z^* \le \sigma_{X^*}(k)$. We build $A(k+1)$ by increasing the height of column X^* by one unit. That is

$$\sigma_i(k+1) = \sigma_i(k) \quad \text{for any } i \neq X^* \quad \text{and} \quad \sigma_{X^*}(k+1) = \sigma_{X^*}(k) + 1 \ .$$

We shall also say that the walk *attaches* to the column, or pile, at X^*. The walk with the aggregation rule is called an *explorer*. We shall denote by P the probability associated with this process.

In diffusive deposition there are two relevant phenomena: one is diffusion, the other is deposition which happens instantly and this explains the name *diffusion limited deposition*.

Ballistic deposition is defined similarly, with the same notation, but with a totally asymmetric walk $\{Z_{n+1} - Z_n = -1\}$. One could consider a continuum of biased models with a drift parameter.

Definitions and notation. We use σ, η to denote configurations, i.e., $\sigma, \eta \in \mathbb{N}^N$, $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$. We also let $|\sigma| := \sum_{i=1}^N \sigma_i$. The symbol $\bar\sigma$ will denote the configuration obtained by ordering the components of σ so that $\bar\sigma_1 \geq \bar\sigma_2 \geq \cdots \geq \bar\sigma_N$. We call $\mathcal{O}_N$ the set of ordered configurations $\eta \in \mathbb{N}^N$, namely, such that $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_N$.

Given a configuration σ, we denote by $\zeta(\sigma)$ the *height occupation* of σ, i.e.,

$$\zeta_j(\sigma) = \sum_{i=1}^N 1\!\!1_{\{\sigma_i \geq j\}} \quad . \tag{1.6}$$

Note that $\sum_{j\geq 1} \zeta_j(\sigma) = |\sigma|$, and that $\zeta(\sigma) = \zeta(\bar\sigma)$. Given two configurations σ and η such that $|\sigma| = |\eta|$, we say that σ is more *monopolistic* than η, writing $\sigma \succ \eta$, when

$$\forall k = 1, \dots, N \qquad \sum_{i=1}^k \bar\sigma_i \geq \sum_{i=1}^k \bar\eta_i. \tag{1.7}$$

Equivalently, one realizes $\bar\eta$ from $\bar\sigma$ by moving particles from the highest columns to the lowest ones.

Urn models are paradigms of reinforcement phenomena (see for instance the survey [27]), and our deposition models actually can be stochastically compared with urns with N colors. We briefly recall Polya's urn with N colors: starting with one ball of each color, at each unit time one draws a ball and put it back in the urn with an additional ball of the same color. Calling η_i, with $i = 1, \dots, N$, the number of added balls of color i after $|\eta|$ draws, the probability of drawing a ball of color i is

$$q_i^P(\eta) = \frac{\eta_i + 1}{\sum_{j=1}^N \eta_j + N} \quad . \tag{1.8}$$

We consider also a generalized urn by replacing the r.h.s. in (1.8) by $f(\eta_i)/\sum_{j=1}^N f(\eta_j)$ where $f : \mathbb{N} \to \mathbb{R}^+$ is a function such that $f(0) = 1$. When $f(x) = x^2 + 1$, we call the model the *quadratic urn*. When $f \equiv 1$, we call the model the *uniform random allocation* and denote by $q_i^U(\eta) = 1/N$ the corresponding probability of drawing a ball of color i at any time. Finally, we say that a process $t \mapsto \sigma(t)$ is more *monopolistic* than process $t \mapsto \eta(t)$ if there is a coupling of the processes such that for any $t > 0$ we have $\sigma(t) \succ \eta(t)$, if this is the case initially.

Main results. In this Section we collect our main results. The first Theorem gives an estimate of the number of explorers necessary, in diffusive deposition, to form a cluster with at least one column proportional to a power of N.

Theorem 1.1. *Consider diffusive deposition. There are constants $\alpha < \beta$, such that almost surely, when N is large enough*

$$\max_{i \in G_N} \sigma_i\Big(\frac{\alpha N}{\log(N)}\Big) \leq 3 \log(N), \tag{1.9}$$

and there exists a positive constant χ such that

$$\max_{i\in G_N} \sigma_i\Big(\frac{\beta N}{\log(N)}\Big) \geq N^{\chi}. \tag{1.10}$$

In ballistic deposition, we prove that the growth of the height of the cluster is much slower. Indeed, it is unlikely that N explorers produce a column of height $\log(N)$.

Theorem 1.2. *Consider ballistic deposition. There exists a positive constant A such that almost surely, when N is large enough*

$$\max_{i\in G_N} \sigma_i(N) \leq A\log(N). \tag{1.11}$$

Remark 1.3. For the *radial* component, we could have chosen $\{Z_{n+1}-Z_n\}$ i.i.d. with some finite range law without affecting our results.

Theorem 1.1 occurs in a regime where less than N explorers are thrown in the graph. We call this the *early regime* which is to be thought of as the configurations where an additional explorer has good chances to settle on the ground $G_N \times \{0\}$. To motivate other results, let us explain the different steps leading to a column of height N^{χ}. A first step is to reach a *subcritical* height $\log(N)/\log(\log(N))$. We obtain that a large number of columns reach this height by comparison with random allocation. We obtain interesting comparison with other urns, with the observation that ballistic deposition looks like Polya's urn with N colors, whereas diffusive deposition looks like a quadratic urn (see below (1.8) for the definition). Then, one of these subcritical columns reaches the *critical height* $\log(N)$. Since our estimate requires the configuration to stay in the early regime, one has to bound the number of *critical columns*. We show that the number of critical columns is less than $N^{1-2\chi}$ for some positive χ, and this implies that the evolution remains in the early regime as long as the highest column has not crossed N^{χ}. We now can state our comparison result.

Theorem 1.4. *Both deposition models (diffusive and ballistic) are more monopolistic than Polya's urn, which itself is more monopolistic than random allocation.*

The following corollary is a side result interesting on its own right which seems new, to the best of our knowledge.

Proposition 1.5. *Polya's urn with N colors is monotone with respect to the order $\succ$.*

Related models. There are many models of cluster growth similar in definition to DLA. They differ according to the law of Y, the site we add on the boundary of the cluster A. This can also be expressed according to the site, say X, from where the random walks are launched and lead to different phenomenology.

- If $X = 0$, we rather define a dual model of *erosion*. The cluster represents the eroded materia, and $A(0) = \emptyset$. Each new walk starts at 0, and settles on the first visited site outside the cluster (a site which we interpreted as being eroded). This is internal DLA, and was introduced by Meakin and Deutch in[25]. The cluster is spherical as was first seen Lawler, Bramson and Griffeath in [22]. The fluctuations were studied in [4, 5, 6] and independently in [18, 19, 20].
- If X is uniformly drawn in the cluster, then Benjamini, Duminil-Copin, Kozma, Lucas in [8] showed that the cluster is spherical.
- If Y is uniform on the boundary of the cluster, then this is the celebrated Eden model [11], which was proposed in the '60, and studied first by Richardson [28].

- If particles do not erode immediately the materia, but do it with an exponential clock, and if they can be activated again when another walk stands on their site, this is Activated Random Walks. This model has been introduced by Spitzer in the 70, and much discussed in the physics literature as an example of *self-organized criticality*. This has been studied mathematically by Rolla and Sidoravicius [29] (and references therein), and recently by Sidoravicius and Teixera [30] among others. Recent efforts have focused on the case of an initial condition drawn from a product Poisson measure. As one tunes the density there is phase transition between settlement of explorers (in any finite box), and their perpetual activity.

Pictures and simulations. In order to illustrate our main results, we show some numerics. In particular, we emphasize the freezing phenomenon which leads to the monopolistic regime: after a given time the highest pile grows linearly catching all particles. We stress that simulations do not capture quantitative aspects of the problem (scaling or exponents), but serve merely as qualitative illustrations.

For both the diffusive and the ballistic model we have simulated the systems for $N = 50, 100, 200, \ldots, 1000$. For the diffusive model we have considered also the cases $N = 2000, 4000, 5000, 6000, 8000, 10000$. In all the cases averages have been computed over 10^4 independent realizations of the process. We have checked in all the cases that the sample is large enough to get stable averages.

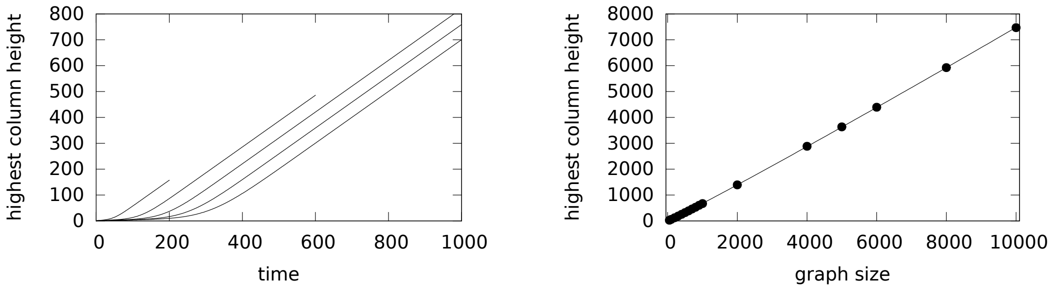

Figure 1: Numerical simulations for the diffusive model. Left panel: the highest column height is plotted as function of time (number of explorers). The five plotted curves, from the left to the right, refer to $N = 100, 300, 500, 700, 900$, respectively. Right panel: solid disks refer to the simulated highest column height at time N, namely, after N explorers have been sent, for different values of the size of the graph N. The solid line is an eye–guide obtained by plotting the fitting function $0.498 \times N^{1.044}$.

Simulations show clearly that the ballistic model reaches the monopolistic regime much later than the diffusive one. Indeed in both models, compare the left panels in Figures 1 and 2, the height of the highest pile attains a linear behavior after an initial transient. This late time regime is the one in which all the particles are caught by the highest pile. Data show that the time length of the transient is much smaller in the diffusive model.

We have also tested numerically our main results in Theorems 1.1 and 1.2. Indeed, we have computed, by averaging over different realizations of the process, the typical height of the highest pile at time N. Rigorous results suggest that this quantity should scale as a power law in the diffusive case and logarithmically in the ballistic one. The related numerical results are shown in the right panels in Figures 1 and 2. The qualitative agreement between simulations and theoretical results is striking. We stress again that

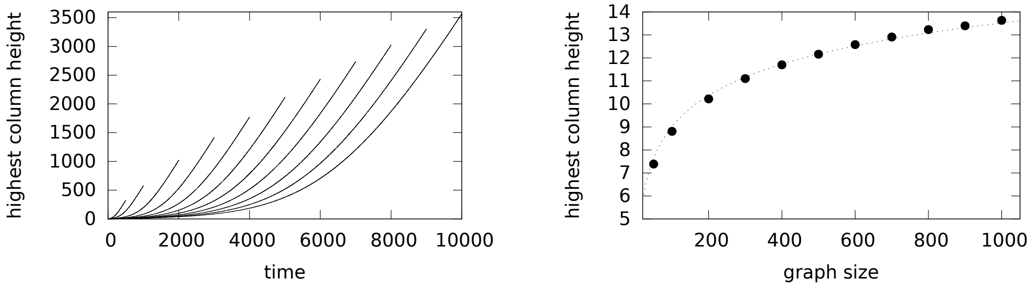

Figure 2: Numerical simulations for the ballistic model for $N = 50, 100, 200, \dots, 1000$ with averages computed over 10^4 independent realizations of the process. Left panel: the highest column height is plotted as function of time (number of explorers). The plotted curves, from the left to the right, refer to $N = 50, 100, 200, \dots, 1000$, respectively. Right panel: solid disks refer to the simulated highest column height at time N, namely, after N explorers have been sent, for different values of the size of the graph N. The solid line is an eye–guide obtained by plotting the function $1.957 \log(N)$.

the numerical results cannot be interpreted as a quantitative description of the model behavior since, for instance, too small values of the graph size N have been considered.

Finally, in the diffusive case we have also tested numerically our results about the critical character of the time scale $N/\log(N)$. In Figure 3 we compare the highest column height measured at times $N/\log(N)$ and $2N/\log(N)$. In the first case the numerical (solid circles) data can be perfectly fitted by a logarithmic function. In the latter case, on the other hand, the poor logarithmic fitting is opposed to a perfect power law one of the numerical data (solid squares). This result is in perfect agreement with the one proved in Theorem 1.1 and, in particular, it suggests that $1 < \alpha < \beta < 2$. We stress that our numerics cannot in any case be considered quantitative, indeed, we have no clue to state that, by considering larger sizes of the graph, our numerical results would be confirmed.

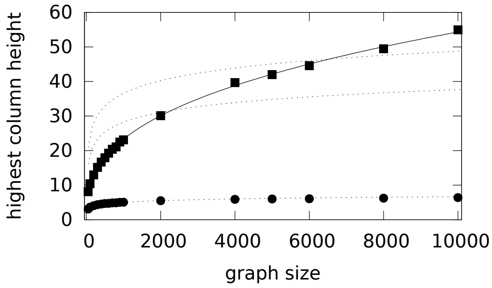

Figure 3: Numerical simulations for the diffusive model. Solid disks and squares refer, respectively, to the simulated highest column height at times $N/\log(N)$ and $2N/\log(N)$ for different values of the size of the graph N. The solid line is an eye–guide obtained by plotting the fitting function $1.847 \times N^{0.367}$. The two dotted lines are the graph of the two functions $0.724 \log(N)$, $4.085 \log(N)$, and $5.292 \log(N)$.

Plan. The rest of the paper is organized as follows. In Section 2 we present our main tool, which is the probability an explorer hits the ground, as well as a heuristic explanation for the logarithmic scale of the *critical* height. In Section 3, we present comparison with urns models. The proof of Proposition 1.5 is given in Section 3.2. The proof of the Theorem 1.4 is given in Section 3.3. In Section 4, we establish our main tool, and related estimates. We study the very early regime in Section 5. Then, we study the growth of cluster in Section 6, and the reason why a large number of columns cannot overcome height $\log(N)$. Finally, we gather all the needed estimates to prove Theorem 1.1 in Section 7.

2 Key tools and sketch

The key to Theorem 1.1 is an estimate of the probability of attaching to a given column. Before, we need a lower bound on the probability of hitting first the ground.

Lemma 2.1. *Consider diffusive deposition with a configuration σ such that $|\sigma| < N/2$ explorers,*

$$P_g(\sigma) := P\big(\text{ Explorer hits the ground } \big| \sigma\big) \geq \exp\Big(-\frac{1}{N}\big(\sum_{j=1}^{N}\sigma_j(\sigma_j+1)\big)\big(1+O(\frac{|\sigma|}{N})\big)\Big). \tag{2.1}$$

The time spent on the slab $G_N \times \{0,\dots,\sigma_i\}$ before touching the ground is typically σ_i^2 for a SRW, but only if the walk has good chances to cross the whole slab. Our key attachment estimate follows.

Lemma 2.2. *Consider diffusive deposition. Let σ be a configuration such that $|\sigma| < N/2$. Then there exists a positive constant κ_D such that*

$$P\big(\text{ explorer attaches pile } i \;\big|\; \sigma\big) \geq \kappa_D \frac{\sigma_i^2+1}{N} \times \exp\Big(-\frac{3}{N}\big(\sum_{j=1}^{N}\sigma_j(\sigma_j+1)\big)\big(1+O(\frac{|\sigma|}{N})\big)\Big). \tag{2.2}$$

In a sense (2.2) and (2.1) are saying opposite things: the former inequality tells how easy it is to get trapped, whereas the latter tells how easy it is to reach the ground.

Imagine a regime where $\sum_i \sigma_i^2 \gg N$. In view of (2.1) the probability of hitting the ground would be small, and very likely the walk would not go below H, where H is such that

$$\sum_i \big(\sigma_i - H\big)_+^2 \sim N. \tag{2.3}$$

In other words, H of (2.3) would play the role of an effective ground. We then replace (2.2) by the following estimate.

Corollary 2.3. *In the diffusive case, and for any positive H,*

$$P\big(\text{ explorer attaches pile } i|\sigma\big) \geq \kappa_D \frac{(\sigma_i - H)_+^2}{N} \times \exp\big(-\frac{3}{N}\sum_{x=1}^{N}\big(\sigma_x - H\big)_+^2\big(1+O(\frac{|\sigma|}{N})\big)\big). \tag{2.4}$$

Sketch. We wish to sketch heuristically the reason why $\log(N)$ is the critical height at which a monopole forms. We fix a given column, say column 1, and we estimate the number of explorers needed to produce a given height. Lemma 2.2 allows us to bound this number by a sum of independent geometric variables, for which we know everything. Indeed, introduce τ_1 the number of explorers needed so that the height of our distinguished site reaches height 1, that is $\tau_1 := \inf\{n > 0 : \ \sigma_1(n) = 1\}$. By induction, for any integer h knowing τ_h we define $\tau_{h+1} := \inf\{n > 0 : \ \sigma_1(\tau_h + n) - \sigma_1(\tau_h) = 1\}$. Assume now that we are in a regime where hitting the ground is likely. The estimate

(2.2) means that for any integers h, n

$$P\big(\tau_{h+1} > n |\ \tau_1, \dots, \tau_h\big) \le \big(1 - \kappa_D \frac{h^2}{N}\big)^n. \qquad (2.5)$$

Let us now introduce independent geometric variables $\{\tilde\tau_h,\ h \ge 1\}$ with $E[\tilde\tau_{h+1}] = N/(\kappa_D h^2)$. Then, we will show that for any height H

$$P\big(\sum_{i=1}^{H} \tau_i \le X\big) \ge P\big(\sum_{i=1}^{H} \tilde\tau_i \le X\big). \qquad (2.6)$$

Recall that $\{\sum_{i=1}^{H} \tau_i \le X\}$ means that X explorers produce a column of height H at site 1. Now, we want to find X such that a given height H is likely to be reached. This would be the case if the probability that any distinguished site reaches height H is above $1/N$. Thus, we look for X such that

$$P\big(\sum_{i=1}^{H} \tilde\tau_i \le X\big) \sim \frac{1}{N}. \qquad (2.7)$$

Let us write X as $N/f(N)$, and try to guess the size of $f(N)$ which produces a monopole. Note also that for any $H > f(N)$,

$$E\big[\sum_{i=f(N)}^{H} \tilde\tau_i\big] = \sum_{i=f(N)}^{H} \frac{N}{\kappa_D i^2} \le \frac{N}{\kappa_D f(N)}.$$

Thus, $\{\sum_{i=1}^{H} \tilde\tau_i \le N/f(N)\}$ imposes a constraint only on the first $f(N)$ variables in (2.7). We then have to estimate $f(N)$ such that

$$P\big(\sum_{i=1}^{f(N)} \frac{\tilde\tau_i}{N} \le \frac{1}{f(N)}\big) \sim \frac{1}{N}.$$

We use now that $\{\tilde\tau_i\}$ are independent geometric variables

$$\begin{aligned} P\big(\tilde\tau_1 + \dots + \tilde\tau_{f(N)} \le \frac{N}{f(N)}\big) &\ge \prod_{i=1}^{f(N)} P\big(\tilde\tau_i \le \frac{N}{f^2(N)}\big) \ge \prod_{i=1}^{f(N)} \Big(1 - \big(1 - \frac{\kappa_D i^2}{N}\big)^{N/f^2(N)}\Big) \\ &\ge \prod_{i=1}^{f(N)} \Big(1 - \exp(-\kappa_D \frac{i^2}{f^2(N)})\Big) \sim \prod_{i=1}^{f(N)} \Big(\kappa_D \frac{i^2}{f^2(N)}\Big) \\ &= \Big(\kappa_D \frac{1}{f^2(N)}\Big)^{f(N)} (f(N)!)^2. \end{aligned} \qquad (2.8)$$

Now, using Stirling's formula, we obtain

$$P\big(\tilde\tau_1 + \dots + \tilde\tau_H \le X\big) \ge \Big(\kappa_D \frac{1}{e^2}\Big)^{f(N)} = \exp\Big(-\log\big(\frac{e^2}{\kappa_D}\big) \times f(N)\Big). \qquad (2.9)$$

Thus, if $f(N)$ is of order $\log(N)$, it is likely that one monopole forms.

3 Comparison with urns

In this section, we establish a coupling between our deposition processes and simpler ones which preserves a natural order on ordered configurations, to be defined below.

We consider growth evolution on $\mathbb{N}^N$ such that at each unit time we add a unit height to a configuration, say η, at a given site, say i, with a probability $p_i(\eta)$ which depends only on the value η_i, and on the unordered set $\{\eta_j, j \neq i\}$. In this case, it is useful to reorder the indices through a permutation of the indices to obtain configurations whose heights are in decreasing order. We call $p = \{(p_1(\eta), \dots, p_N(\eta)),\ \eta \in \mathbb{N}^N\}$ the law of the growth process.

3.1 Comparing evolutions

It will be important to compare configurations with the same number of explorers. Our main results are the following.

Proposition 3.1. *Consider two processes $t \mapsto \eta(t)$ and $t \mapsto \sigma(t)$ on $\mathbb{N}^N$ evolving, respectively, according to the laws $p = \big(p_1(\cdot), \dots, p_N(\cdot)\big)$ and $q = \big(q_1(\cdot), \dots, q_N(\cdot)\big)$. Assume that for any $\eta, \sigma \in \mathcal{O}_N$ such that $\eta \prec \sigma$ we have that*

$$\forall k = 1, \dots, N \qquad \sum_{i=1}^{k} p_i(\eta) \leq \sum_{i=1}^{k} q_i(\sigma). \tag{3.1}$$

Then, the process $\sigma(t)$ is more monopolistic than $\eta(t)$,i.e. there is a coupling between the two processes such that $\sigma(t) \succ \eta(t)$ for any t.

Lemma 3.2. *Let $\{p_1, \dots, p_N\}$ and $\{q_1, \dots, q_N\}$ two sets of positive numbers both summing up to 1. Assume that*

$$\frac{q_1}{p_1} \geq \frac{q_2}{p_2} \geq \dots \geq \frac{q_N}{p_N}. \tag{3.2}$$

Then, for any $k = 1, \dots, N$, we have that

$$\sum_{i=1}^{k} q_i \geq \sum_{i=1}^{k} p_i. \tag{3.3}$$

Proof. The proof is by induction on N. Assume the Lemma is true with $N-1$ sets of positive numbers, and define the renormalized $N-1$ numbers

$$\tilde{p}_i = p_i \times \frac{1}{1-p_N}, \quad \text{and} \quad \tilde{q}_i = q_i \times \frac{1}{1-q_N}. \tag{3.4}$$

The induction hypothesis states that for $k = 1$ to $N-1$

$$\sum_{i=1}^{k} \tilde{q}_i \geq \sum_{i=1}^{k} \tilde{p}_i.$$

In other words,

$$\sum_{i=1}^{k} q_i \geq \frac{(1-q_N)}{(1-p_N)} \sum_{i=1}^{k} p_i. \tag{3.5}$$

The question is whether $1 - q_N \geq 1 - p_N$ or $q_N/p_N \leq 1$ which follows from (3.2) since

$$1 = \sum_{i=1}^{N} p_i \frac{q_i}{p_i} \geq \sum_{i=1}^{N} p_i \frac{q_N}{p_N} = \frac{q_N}{p_N}. \qquad \square$$

As a corollary of Proposition 3.1 and Lemma 3.2 we have the following result.

Corollary 3.3. *With the notation of Proposition 3.1, assume that for any $\eta, \sigma \in \mathcal{O}_N$ such that $\eta \prec \sigma$ we have that*

$$\frac{p_i(\eta)}{p_{i+1}(\eta)} \leq \frac{q_i(\sigma)}{q_{i+1}(\sigma)} \quad \text{for } i = 1, \dots, N-1.$$

Then, there is an order preserving coupling between $\eta(t)$ and $\sigma(t)$.

In order to prove Proposition 3.1 we need some notation and some simple observations. We define the action $\mathcal{A}_j : \mathbb{N}^N \to \mathbb{N}^N$ of adding one explorer to site j: $(\mathcal{A}_j\eta)_i = \eta_i + \delta_{i,j}$. Note that $\mathcal{A}_j$ does not leave $\mathcal{O}_N$ invariant.

Assume that $\eta \in \mathcal{O}_N$ and define

$$I(\eta) = \{i \in \{2, \dots, N\} : \eta_{i-1} > \eta_i\} \cup \{1\}. \tag{3.6}$$

Also, for $i \in \{1, \dots, N\}$, let $d(\eta, i) = \max I(\eta) \cap \{1, \dots, i\}$. In other words, $d(\eta, i)$ is the last position of a height decrease up to position i. Note that for $\eta \in \mathcal{O}_N$ we have

$$\overline{\mathcal{A}_i\eta} = \mathcal{A}_{d(\eta,i)}\eta \in \mathcal{O}_N. \tag{3.7}$$

For $\eta \in \mathcal{O}_N$, note that if $i \le j$, then $d(\eta, i) \le d(\eta, j)$. Also, if $i \le j$, then $\mathcal{A}_j\eta \prec \mathcal{A}_i\eta$.

The main observation about ordering is the following.

Lemma 3.4. *Assume that $\eta, \sigma \in \mathcal{O}_N$ with $\eta \prec \sigma$. If $i \le j$, then $\overline{\mathcal{A}_j\eta} \prec \overline{\mathcal{A}_i\sigma}$.*

This lemma is based on the following simple observation.

Lemma 3.5. *Assume that $\eta, \sigma \in \mathcal{O}_N$ with $\eta \prec \sigma$. If some integers $i < j$ satisfying $d(\eta, j) \le i$, are such that*

$$\sum_{k=1}^{i} \eta_k = \sum_{k=1}^{i} \sigma_k, \tag{3.8}$$

then $i \ge d(\sigma, j)$.

Proof of Lemma 3.5. Assume for a moment that $i > 1$. Since $\eta \prec \sigma$, we have

$$\sum_{k=1}^{i-1} \eta_k \le \sum_{k=1}^{i-1} \sigma_k, \quad \text{and} \quad \sum_{k=1}^{i+1} \eta_k \le \sum_{k=1}^{i+1} \sigma_k. \tag{3.9}$$

Rewrite now (3.8) as

$$\sum_{k=1}^{i-1} \eta_k + \ \eta_i = \sum_{k=1}^{i-1} \sigma_k + \ \sigma_i, \quad \text{and} \quad \sum_{k=1}^{i+1} \eta_k - \ \eta_{i+1} = \sum_{k=1}^{i+1} \sigma_k - \ \sigma_{i+1}.$$

Using (3.9), we have both that $\sigma_i \le \eta_i$, and $\eta_{i+1} \le \sigma_{i+1}$. Since $\sigma \in \mathcal{O}_N$, we have $\sigma_{i+1} \le \sigma_i$. Now, if $i = 1$, we have $\sigma_i = \eta_i$ so that $\sigma_i \le \eta_i$ is again true. So that we reach

$$\eta_{i+1} \le \sigma_{i+1} \le \sigma_i \le \eta_i. \tag{3.10}$$

Now $d(\eta, j) \le i < j$ means that $\eta_i = \eta_{i+1} = \dots = \eta_j$, and with (3.10) this implies that $\sigma_i = \sigma_{i+1}$, and by induction, we reach that

$$\eta_i = \sigma_i = \sigma_{i+1} = \dots = \sigma_j. \tag{3.11}$$

These last equalities mean that $i \ge d(\sigma, j)$. □

Proof of Lemma 3.4. To simplify notation assume $\eta, \sigma \in \mathcal{O}_N$. We have already observed that for $i \le j$, $\mathcal{A}_j\eta \prec \mathcal{A}_i\eta$. Thus, we only need to prove that $\mathcal{A}_j\eta \prec \mathcal{A}_j\sigma$. If $d(\eta, j) = j$, then $d(\eta, j) \ge d(\sigma, j)$, and the result is obvious. Assume henceforth that $d(\eta, j) < j$. If for all $k = d(\eta, j), \dots, j-1$, we have that

$$\sum_{i=1}^{k} \eta_i < \sum_{i=1}^{k} \sigma_i,$$

then the result is also obvious. In the opposite case, let k in $[d(\eta,j),j[$, be the first index for which we have

$$\sum_{i=1}^{k}\eta_i=\sum_{i=1}^{k}\sigma_i,$$

then, Lemma 3.5 implies that $k\geq d(\sigma,j)$, and the lemma follows. □

Proof of Proposition 3.1. By way of induction, assume that up to time t, we have $\eta(t)\prec\sigma(t)$. Draw a uniform random variable U in $[0,1[$, and define two random variables J,J^* as follows:

- if $U\in[p_1(\eta(t))+\dots+p_{i-1}(\eta(t)),p_1(\eta(t))+\dots+p_i(\eta(t))[$ then $J=i$ (we set $p_0=0$);
- if $U\in[q_1(\sigma(t))+\dots+q_{j-1}(\sigma(t)),q_1(\sigma(t))+\dots+q_j(\sigma(t))[$ then $J^*=j$.

Then, (3.1) implies that $J\geq J^*$. We set

$$\eta(t+1)=\overline{\mathcal{A}_J\eta(t)}\qquad\text{and}\quad\sigma(t+1)=\overline{\mathcal{A}_{J^*}\sigma(t)}. \tag{3.12}$$

Then, Lemma 3.4 yields that $\eta(t+1)\prec\sigma(t+1)$. □

3.2 Comparing Polya's urn with random allocation

By *random allocation*, we mean repeated draws of one out of N colors, labelled from 1 to N, uniformly at random. In other words, at each draw, the probability to pick up color i is $1/N$. The law for Polya's urn and random allocation are denoted respectively q^P and q^U with

$$\forall\sigma\in\mathbb{N}^N,\qquad q_i^P(\sigma)=\frac{\sigma_i+1}{N+\sum_{i\leq N}\sigma_i}\qquad\text{and}\qquad q_i^U(\sigma)=\frac{1}{N}\quad.$$

Lemma 3.6. *Polya's urn with N colors is more monopolistic than random allocation of N colors.*

Proof. Note that for any $\sigma,\eta\in\mathcal{O}_N$ such that $\eta\prec\sigma$

$$\frac{q_i^P(\sigma)}{q_{i+1}^P(\sigma)}\geq\frac{q_i^U(\eta)}{q_{i+1}^U(\eta)}\iff\frac{\sigma_i+1}{\sigma_{i+1}+1}\geq 1, \tag{3.13}$$

which clearly holds since $\sigma\in\mathcal{O}_N$. Thus, Corollary 3.3 implies the lemma. □

Proof of Proposition 1.5. Note that if $\eta\succ\sigma$ and $|\eta|=|\sigma|$, we have

$$\forall k=1,\dots,N,\qquad\sum_{i=1}^{k}q_i^P(\eta)=\frac{k+\sum_{i=1}^k\eta_i}{N+\sum_{i=1}^N\eta_i}\geq\frac{k+\sum_{i=1}^k\sigma_i}{N+\sum_{i=1}^N\sigma_i}=\sum_{i=1}^{k}q_i^P(\sigma).$$

This establishes Proposition 1.5 saying that Polya's evolution with N colors preserves the order. □

3.3 Comparing deposition models with Polya's urn

Recall the definition of ballistic and diffusive deposition given in Section 1. Denote their law, respectively, by p^B and p^D. We show that both ballistic and diffusive deposition are more monopolistic than Polya's urn, which is one of the statements of Theorem 1.4.

Assume for a moment the following lemma.

Lemma 3.7. *For any $k=1,\dots,N$ and $\eta\in\mathcal{O}_N$*

$$\sum_{i=1}^{k}p_i^B(\eta)\geq\sum_{i=1}^{k}q_i^P(\eta)\quad\textit{and}\quad\sum_{i=1}^{k}p_i^D(\eta)\geq\sum_{i=1}^{k}q_i^P(\eta)\quad. \tag{3.14}$$

By Proposition 1.5, Lemma 3.7, and Proposition 3.1 we have that both ballistic and diffusive deposition are more monopolistic than Polya's urn.

To state a preliminary simple observation, we need more notation. For $\eta \in \mathcal{O}_N$, let $p_{i,k}^{B,D}(\eta)$ be the probability that the explorer hits site i at height k, for $k \in \mathbb{N}$.

Lemma 3.8. *For any $i, \in \{1, \dots, N\}$ and $\eta(i) \geq k > k' \geq 0$ we have*

$$p_{i,k'}^B(\eta) < p_{i,k}^B(\eta), \quad \text{and if } \eta(j) \geq k \quad p_{i,k}^B(\eta) = p_{j,k}^B(\eta) \tag{3.15}$$

Similarly,

$$p_{i,k'}^D(\eta) < p_{i,k}^D(\eta), \quad \text{and if } \eta(j) \geq k \quad p_{i,k}^D(\eta) = p_{j,k}^D(\eta) \tag{3.16}$$

Proof of Lemma 3.7. We consider first the ballistic case. In view of Lemma 3.2, we need to show that

$$\frac{p_i^B(\eta)}{p_{i+1}^B(\eta)} \geq \frac{\eta_i + 1}{\eta_{i+1} + 1} = \frac{q_i^P(\eta)}{q_{i+1}^P(\eta)} \quad . \tag{3.17}$$

In order to prove (3.17), we need to show that

$$\frac{\sum_{k=1}^{\eta_i} p_{i,k}^B(\eta)}{\eta_i + 1} \geq \frac{\sum_{k=1}^{\eta_{i+1}} p_{i+1,k}^B(\eta)}{\eta_{i+1} + 1} \tag{3.18}$$

By Lemma 3.8, this inequality has the structure

$$\frac{a_1 + \cdots + a_n}{n} \geq \frac{a_{m+1} + \cdots + a_n}{n - m}$$

for $n > m \geq 1$ and $a_1 \geq a_2 \geq \cdots \geq a_n$. The validity of such an inequality is immediate once we let $\mu = (a_{m+1} + \cdots + a_n)/(n-m)$, note $a_1 \geq \cdots \geq a_m \geq \mu$, and write

$$\frac{a_1 + \cdots + a_n}{n} = \frac{a_1 + \cdots + a_m + \mu(n-m)}{n} \geq \frac{\mu m + \mu(n-m)}{n} = \mu$$

Finally, by using (3.17) and Lemma 3.2 the first of equations (3.14) follows immediately. The diffusive case can be treated in the same way. This completes the proof of the lemma. $\square$

Proof of Lemma 3.8. First we prove the lemma for the ballistic case. Note first that

$$p_{i,k}^B(\eta) = \frac{1}{N} P(\text{explorer reaches height } k+1 \mid \eta).$$

Now, since the function $k \mapsto P(\text{explorer reaches height } k \mid \eta)$ is increasing, the lemma follows at once.

Now, we consider diffusive deposition, and assume $k > k'$. First, assume $k - k'$ an even number. To prove the Lemma, we associate uniquely to each path $s' = \{(x'_j, z'_j)\}_{j=1,\dots,n}$, hitting η at site i at height k' in a time n, a path $s = \{(x_j, z_j)\}_{j=1,\dots,n}$, of the same length, and therefore the same probability, hitting η in i at height k. Indeed, given the path s', we construct s in the following way. Call n_1 the time of last passage of s' through the intermediate height $H = (k + k')/2$. s is equal to s' up to time n_1 while after n_1 it uses opposite height increments than the original s', i.e. $z_{j+1} - z_j = -(z'_{j+1} - z'_j)$ for all $n_1 \leq j < n$ keeping the same horizontal increments. We therefore obtain a path ending in i and height k. Note that such a path avoids η until it hits site i at height k, because η is a union of columns. If $k - k'$ is an odd number, we do a similar construction but we have to associate with a collection of paths, of length n, one single path s of length $n - 1$. The collection is obtained considering together all the paths coinciding on $[0, n]$ except at time $n_1 + 1$ on the coordinate x_{n_1+1}, where n_1 is now the last hitting time of the level $\frac{k+k'+1}{2}$ of the vertical process. We now call s' such a collection of paths. We construct s as follows. It coincides with the paths s' up to time n_1, and is afterwards the reflection of $s' :]n_1+1, n]$ with respect to height $\frac{k+k'+1}{2}$. In other words, $x_j = x'_{j+1},\ z_j - z_{j-1} = -(z'_{j+1} - z'_j)$ for any $j = n_1+1, ..., n-1$ so that $x_{n-1} = i$. Clearly the probability of s is larger than or equal to the sum of the probabilities over the collection of paths s', since s is one step shorter, and the sum is over x_{n_1+1} on $N - \zeta_{n_1+1}(\eta)$ sites. $\square$

4 Estimating unit growth

In this section we discuss how heights grow. We consider the random walk $S_n = (X_n, Z_n)$, and for an integer k, we call H_k the first time the walk reaches height k. In other words,

$$H_k = \inf\{n \geq 0 :\ Z_n = k\}. \tag{4.1}$$

Since the X-component is uniform on the base, giving the configuration σ, the ordered one $\bar{\sigma}$, or the height occupation ζ (defined in (1.6)) is equivalent, and we use $P_g(\sigma)$ or $P_g(\zeta)$ indifferently to denote the probability an explorer hits the ground. Lemma 2.1 is obtained as a simple application of Jensen's inequality whereas Lemma 2.2 requires Kesten-Kozlov-Spitzer representation of the local times [17].

Proof of Lemma 2.1. For an integer $k \leq \bar{\sigma}_1$ let $l(k)$ be the number of visits of height k by the random walk before H_0. We have the representation

$$P_g(\zeta) = E\big[\prod_{k=1}^{\bar{\sigma}_1}\big(1-\frac{\zeta_k}{N}\big)^{l(k)}\big] = E\big[\exp\Big(\sum_{k=1}^{\bar{\sigma}_1} l(k)\log\big(1-\frac{\zeta_k}{N}\big)\Big)\big]. \tag{4.2}$$

Our hypothesis $|\sigma| < N/2$ implies that for $k \geq 1$, $\zeta_k \leq \zeta_1 \leq N/2$, and since $\log(1-x) \geq -x - x^2$ for $0 \leq x \leq 1/2$, we have using Jensen's inequality

$$P_g(\zeta) \geq \exp\Big(-\sum_{k=1}^{\bar{\sigma}_1} E[l(k)]\big(\frac{\zeta_k}{N} + \frac{\zeta_k^2}{N^2}\big)\Big). \tag{4.3}$$

Now note that $E[l(k)] = 2k$. Indeed, the height of the random walk being a simple random walk on $\mathbb{N}$, we have for $k \leq \bar{\sigma}_1$, by conditioning on the first step

$$\begin{aligned} E[l(k)|Z_0=\bar{\sigma}_1] =& E[l(k)|Z_0=k] = 1 + \frac{1}{2}\big(E[l(k)|Z_0=k+1] + E[l(k)|Z_0=k-1]\big) \\ =& 1 + \frac{1}{2}\big(E[l(k)|Z_0=k] + P(H_k < H_0|Z_0=k-1)E[l(k)|Z_0=k]\big) \\ =& 1 + \frac{1}{2}\big(E[l(k)|Z_0=k] + (1-\frac{1}{k})E[l(k)|Z_0=k]\big). \end{aligned} \tag{4.4}$$

The equality $E[l(k)] = 2k$ for $k \leq \bar{\sigma}_1$ follows at once. Note now that

$$\sum_{k\geq 1} 2k\zeta_k = \sum_{i=1}^{N} \sigma_i(\sigma_i+1).$$

Finally, (2.1) follows as we note that

$$\sum_{k\geq 1} 2k\frac{\zeta_k^2}{N^2} \leq \frac{\zeta_1}{N}\sum_{k\geq 1} 2k\frac{\zeta_k}{N} \leq 2\frac{|\sigma|}{N}\sum_{i=1}^{N}\frac{\sigma_i(\sigma_i+1)}{N}. \qquad \square$$

On Kesten-Kozlov-Spitzer representation. Let $u(h)$ be the number of *up-crossings* of height h before touching the base. In other words, we define $u(0)=0$ and for $h>0$

$$u(h)=\sum_{i=1}^{H_0} \mathbb{1}_{(Z_{i-1},Z_i)=(h,h+1)}. \tag{4.5}$$

Similarly, *down-crossings* of height h correspond to jumps from h to $h-1$ before time H_0. One way to realize the random walk $n\mapsto Z_n$ is to assign the sequence of up and down-crossings on each height. Thus, we consider $\{\{\xi_i^k,\ i\in\mathbb{N}\},\ k\in\mathbb{N}\}$ a collection of i.i.d. geometric variables, with law $P(\xi=n)=1/2^{n+1}$ for $n\in\mathbb{N}$. Now, the sequence of up and down-crossings at height k is as follows: ξ_0^k up-crossings, then one down-crossing, then ξ_1^k up-crossings, the one down-crossing, then ξ_2^k up-crossings... and so on and so forth. The key observation is that each ξ_i^k, for $i\geq 1$, is preceded by an up-crossing of the height $k-1$. In other words,

$$u(k)=\xi_0^k+\sum_{j=1}^{u(k-1)}\xi_j^k,\quad \text{and}\quad u(1)=\xi_0^1. \tag{4.6}$$

We set $\mathcal{G}(h)=\sigma(\xi_i^k,\ k\leq h,\ i\in\mathbb{N})$ the σ-field representing the choices of moves on the first h heights. Kesten-Kozlov-Spizter representation expresses the local times of Z in terms of the u. Thus, if $l(k)$ represents the number of visits of height k before H_0, for a walk with starting level above $\bar\sigma_1$, then

$$\forall k\geq 1,\quad l(k)=u(k)+u(k-1)+1. \tag{4.7}$$

Then, with notation $x_i=1-\zeta_i/N$

$$\begin{aligned} P_g(\zeta)=&E\big[\prod_{k=1}^{\bar\sigma_1}x_k^{l(k)}\big]=E\big[\prod_{k=1}^{\bar\sigma_1-1}x_k^{l(k)}x_{\bar\sigma_1}^{u(\bar\sigma_1)+u(\bar\sigma_1-1)+1}\big]\\ =&E\big[\prod_{k=1}^{\bar\sigma_1-1}x_k^{l(k)}x_{\bar\sigma_1}^{u(\bar\sigma_1-1)+1}E\big[x_{\bar\sigma_1}^{\xi_0^{\bar\sigma_1}+\dots+\xi_{u(\bar\sigma_1-1)}^{\bar\sigma_1}}\big|\mathcal{G}(\bar\sigma_1-1)\big]\big]. \end{aligned} \tag{4.8}$$

Since $E[z^\xi]=1/(2-z)$, we have

$$E\big[x_{\bar\sigma_1}^{\xi_0^{\bar\sigma_1}+\dots+\xi_{u(\bar\sigma_1-1)}^{\bar\sigma_1}}\big|\mathcal{G}(\bar\sigma_1-1)\big]=\frac{1}{(2-x_{\bar\sigma_1})^{1+u(\bar\sigma_1-1)}}. \tag{4.9}$$

We set $a(k)=0$ for $k\geq\bar\sigma_1$, whereas for any $k<\bar\sigma_1$

$$e^{-a(k-1)}=\frac{x_k}{2-x_ke^{-a(k)}}, \tag{4.10}$$

so that

$$P_g(\zeta)=e^{-a(\bar\sigma_1-1)}E\big[\prod_{k=1}^{\bar\sigma_1-1}x_k^{l(k)}e^{-a(\bar\sigma_1-1)u(\bar\sigma_1-1)}\big], \tag{4.11}$$

and by induction, we obtain

$$P_g(\zeta)=\exp\big(-\sum_{k\geq 0}a(k)\big). \tag{4.12}$$

Note that (4.10) reads for $1\leq k\leq\bar\sigma_1$

$$e^{-a(k)}+e^{a(k-1)}=\frac{2}{x_k},\quad \text{and}\quad e^{a(k-1)}-e^{a(k)}=e^{-a(k+1)}-e^{-a(k)}+\frac{2}{x_k}-\frac{2}{x_{k+1}}, \tag{4.13}$$

and $a(k-1)\geq a(k)\geq 0$ follows by induction from (4.10). Inequality (2.1) implies that

$$\sum_{k\geq 0} a(k) \leq \frac{2}{N}\sum_{j\geq 1} j\zeta_j + \frac{2}{N^2}\sum_{j\geq 1} j\zeta_j^2. \tag{4.14}$$

Proof of Lemma 2.2. An explorer settling on the pile at site i, hits the i-th pile at a height between 1 and σ_i. Knowing that it settles at height k, it has chance $1/\zeta_k$ to settle on (i,k) since we are on the complete graph. We underestimate the probability of settling on σ_i, if we only consider trajectories hitting only one of the $\{\zeta_1, \zeta_2, \dots\}$ before H_0. Thus,

$$\begin{aligned} P\big(\text{explorer attaches pile } i \big|\ \sigma\big) &\geq \sum_{h=1}^{\sigma_i} \frac{1}{\zeta_h} E\big[\big(1 - x_h^{l(h)}\big)\prod_{j\neq h} x_j^{l(j)}\big] \\ &\geq \sum_{h=1}^{\sigma_i}\frac{1}{\zeta_h}\Big(E\big[\prod_{j\neq h} x_j^{l(j)} - P_g(\zeta)\Big). \end{aligned} \tag{4.15}$$

Fix $h > 0$, and write ζ^h for the height occupation such that

$$\forall k \neq h, \quad \zeta_k^h = \zeta_k, \quad \text{and} \quad \zeta_h^h = 0.$$

We rewrite (4.15) in terms of the function P_g as follows

$$P\big(\text{explorer attaches pile } i \big|\ \sigma\big) \geq P_g(\zeta)\sum_{h=1}^{\sigma_i}\frac{1}{\zeta_h}\big(\frac{P_g(\zeta^h)}{P_g(\zeta)} - 1\big). \tag{4.16}$$

For a given $h \leq \bar\sigma_1$, we now study the ratio $P_g(\zeta^h)/P_g(\zeta)$. As in (4.12) in the previous paragraph, we write $P_g(\zeta^h) = \exp(-\sum_{k\geq 0}\tilde a(k))$, with $\tilde a$ satisfying the relation (4.10) with ζ^h in place of ζ. In other words,

$$\forall k \geq h, \quad \tilde a(k) = a(k), \quad \text{and} \quad \exp(-\tilde a(h-1)) = \frac{1}{2 - e^{-a(h)}}. \tag{4.17}$$

For $k \leq h-1$,

$$e^{\tilde a(k-1)} + e^{-\tilde a(k)} = \frac{2}{x_k} = e^{a(k-1)} + e^{-a(k)}. \tag{4.18}$$

We set $\delta_k = a(k) - \tilde a(k)$, and from (4.18), we have that $\delta_k \geq 0$. In terms of δ_k, (4.18) reads for $k < h-1$

$$\exp(\delta_{k-1}) - 1 = e^{-\tilde a(k-1) - a(k)}\big(\exp(\delta_k) - 1\big), \tag{4.19}$$

whereas for $k = h-1$ we have

$$\exp(\delta_{h-1}) - 1 = \frac{2(1-x_h)}{x_h(2 - \exp(-a(h))} \leq 2. \tag{4.20}$$

Equality (4.20) implies (since $a(.) \leq 2$ by (4.18)) that for some constant κ_D

$$\delta_{h-1} \geq \kappa_D(1 - x_h) = \kappa_D\frac{\zeta_h}{N}. \tag{4.21}$$

We deduce that $\delta_{k-1} \leq \delta_k$ and since $x \mapsto (e^x - 1)/x$ is increasing, (4.19) implies

$$\frac{\delta_{k-1}}{\delta_k} \geq \exp\big(-\tilde a(k-1) - a(k)\big) \geq \exp\big(-a(k-1) - a(k)\big) \geq \exp\big(-2a(k-1)\big). \tag{4.22}$$

By induction on (4.22), and using (4.21), this implies that for each $h \leq \bar\sigma_1$

$$\begin{aligned}\frac{P_g(\zeta^h)}{P_g(\zeta)}-1 =&\exp\big(\sum_{k=0}^{h-1}\delta_k\big)-1\geq\sum_{k=1}^{h-1}\delta_k\\ \geq&\delta_{h-1}\sum_{k=0}^{h-1}\exp\big(-2\sum_{j=k}^{h-1}a(j)\big)\\ \geq&\kappa_D\zeta_h\frac{(h-1)}{N}\exp\big(-2\sum_{j=0}^{\bar\sigma_1-1}a(j)\big)\\ \geq&\kappa_D\zeta_h\frac{(h-1)}{N}\exp\Big(-4\sum_{j\geq1}\big(j\frac{\zeta_j}{N}+j\frac{\zeta^2(j)}{N^2}\big)\Big).\end{aligned} \tag{4.23}$$

Since $\zeta_j\leq N$, we have $\zeta_j^2/N\leq\zeta_j$, and (4.23) implies (2.2). □

Remark 4.1. In order to obtain Corollary 2.3, consider simply the height occupation $\{\zeta_{k+H},\ k\in\mathbb{N}\}$ and proceed along the exact same proof. This height occupation corresponds to $\{(\sigma_i-H)_+,\ i=1,\dots,N\}$.

Given $A>0$, we define the *early regime* as the following subset of configurations.

$$\mathcal{X}_{\mathrm{e}}(A)=\{\sigma:\ \sum_{i=1}^N\sigma_i^2\leq AN,\ \sum_{i=1}^N\sigma_i<\frac{N}{2}\}. \tag{4.24}$$

Define also $\kappa(A)$ to be $\kappa_D\exp(-2A)$.

Corollary 4.2. *For any $\sigma\in\mathcal{X}_{\mathrm{e}}(A)$, and $i\in\{1,\dots,N\}$*

$$P\big(\text{ explorer attaches pile } i\big|\sigma\big)\geq\kappa(A)\frac{\sigma_i^2}{N}. \tag{4.25}$$

4.1 Upper bound

Lemma 4.3. *Consider diffusive deposition. Let i be a fixed site and σ be a configuration such that $\sigma_i<\sqrt{N}$. Then*

$$P\big(\text{ explorer attaches to site } i\ \big|\ \sigma\big)\leq\kappa\frac{(\sigma_i\vee1)^2}{N}\quad. \tag{4.26}$$

with $\kappa=1+O(N^{-1/2})$.

Remark 4.4. Comparing lower and upper bound (Lemmas 2.2 and 4.3) on attachment probability we obtain a good control on this probability for configurations in the early regime, that is in $\mathcal{X}_{\mathrm{e}}(A)$ when A is small.

Proof of Lemma 4.3. If $\sigma_i=0$ then (4.26) is immediate with $\kappa=1$. If $\sigma_i\geq1$ the chances an explorer attaches to column i, in configuration σ, is smaller than if all columns distinct from i were set to zero. This is seen by coupling. First, for configuration σ, let σ^i denote the configuration where we annihilate all columns distinct from i. In other words

$$(\sigma^i)_k=0,\quad\text{when } k\neq i,\qquad\text{and}\quad(\sigma^i)_i=\sigma_i.$$

Therefore, in σ^i, the highest column is i with height $\sigma_i\geq1$. Now, the event hit column i in σ^i is the complement of the event hit the base first. Since $P_g(\sigma^i)$ is the probability the explorer hits first the base in configuration σ^i, by Lemma 2.1, we have that for some $\kappa>0$

$$\begin{aligned}P\big(\text{explorer attaches to i}\big|\sigma\big)\leq&P\big(\text{explorer attaches to i}\big|\sigma^i\big)\\ =&1-P_g(\sigma^i)\leq\frac{\sigma_i^2}{N}(1+O(N^{-1/2}),\end{aligned}$$

which completes the proof. □

Lemma 4.5. *Consider ballistic deposition. Consider a configuration* $\sigma \in \mathbb{N}^{G_N}$*, then*

$$P\big(\text{ explorer attaches to site } i \mid \sigma\big) \le \frac{\sigma_i + 1}{N} \quad . \tag{4.27}$$

Proof. To get attached to site i, the particle has to survive up to the time it reaches height σ_i, and then at each step–down has a chance $1/N$ to fall on column i provided it has avoided the other columns. Thus

$$\begin{aligned}
&P\big(\text{explorer attaches to site } i|\sigma\big)\\
&\quad = \Big[\prod_{h=\sigma_i+1}^{\max\sigma}\Big(1-\frac{\zeta_h}{N}\Big)\Big]\frac{1}{N}\Big(1+\Big(1-\frac{\zeta_{\sigma_i}}{N}\Big)+\Big(1-\frac{\zeta_{\sigma_i}}{N}\Big)\Big(1-\frac{\zeta_{\sigma_i-1}}{N}\Big)+\dots\Big)\\
&\quad \le \frac{1}{N}\big(\sigma_i+1\big).
\end{aligned}$$

which completes the proof. □

4.2 Stochastic domination

Both in ballistic and diffusive deposition, we have a simple upper bound on the probability of attaching to a given column (see Section 4.1), which depends only on its height. This, in turn, is used to bound the number of explorers necessary to increase the height by one unit in terms of a geometric random variable, for which everything can be computed explicitly. In other words, call τ_1 the number of explorers needed so that column 1 reaches height 1. Let τ_2 be the additional number of explorers needed to reach a height 2, and so on. Note that $\{\tau_1 > k\}$ means that out of k explorers none of them has reached site 1. These times are used to control the height of column 1 after k explorers have been sent,

$$\forall k \ge 1, \quad \forall H \ge 1, \qquad P_0(\sigma_1(k) > H) = P(\tau_1 + \dots + \tau_H < k).$$

We need to estimate the sum of the $\{\tau_i\}$ with the following general lemma.

Lemma 4.6. *Let* τ, T *be stopping times with respect to a filtration* $\{\mathcal{F}_n\}$*. Let* $\tau_1 := \tau$ *and if* $\theta(n)$ *is the time-shift by* n *units, define inductively*

$$\tau_n := \tau \circ \theta(\tau_1 + \dots + \tau_{n-1}).$$

Let $\{\tilde\tau_n,\ n \in \mathbb{N}\}$ *be independent random variables, which are also independent from* $\{\tau_n,\ n \in \mathbb{N}\}$*. Assume that for positive integers* $\xi \le \xi'$*, we have*

$$P\big(\tau_n > \xi\ , T > \xi' |\ \mathcal{F}_{\tau_1+\dots+\tau_{n-1}}\big) \ge P\big(\tilde\tau_n > \xi\big) P(T > \xi'). \tag{4.28}$$

Then, for any integer n *and* $\xi > 0$

$$P\Big(\sum_{i=1}^n \tau_i > \xi \mid T > \xi\Big) \ge P\Big(\sum_{i=1}^n \tilde\tau_i > \xi\Big). \tag{4.29}$$

Similarly, if instead of (4.28) we have

$$P\big(\tau_n > \xi\ , T > \xi' |\ \mathcal{F}_{\tau_1+\dots+\tau_{n-1}}\big) \le P\big(\tilde\tau_n > \xi\big) P\big(T > \xi'\big),$$

then

$$P\Big(\sum_{i=1}^n \tau_i > \xi |\ T > \xi\Big) \le P\Big(\sum_{i=1}^n \tilde\tau_i > \xi\Big).$$

Proof of Lemma 4.6. We prove (4.29) by induction. The step $n=1$ is obvious. Assume the inequality is true at step $n-1$. Using that the variables are positive,

$$\begin{aligned} P\Big(\sum_{i=1}^{n}\tau_i>\xi,\ T>\xi\Big) &= \sum_{K\le\xi} P\Big(\sum_{i=1}^{n-1}\tau_i=K,\ T>\xi,\ \tau_n>\xi-K\Big)+P\Big(\sum_{i=1}^{n-1}\tau_i>\xi,\ T>\xi\Big)\\ &= \sum_{K\le\xi} E\Big[P\Big(\tau_n>\xi-K,\ T>\xi-K\Big|\mathcal{F}_K\Big)\mathbb{1}_{\sum_{i=1}^{n-1}\tau_i=K,\ T>K}\Big]+P\Big(\sum_{i=1}^{n-1}\tau_i>\xi,\ T>\xi\Big)\\ &\ge \sum_{K\le\xi} E\Big[P\Big(\tilde\tau_n>\xi-K\Big)P\big(T>\xi-K\Big|\mathcal{F}_K\big)\mathbb{1}_{\sum_{i=1}^{n-1}\tau_i=K,\ T>K}\Big]+P\Big(\sum_{i=1}^{n-1}\tau_i>\xi,\ T>\xi\Big)\\ &= \sum_{K\le\xi} P\Big(\tilde\tau_n>\xi-K\Big)\times P\Big(\sum_{i=1}^{n-1}\tau_i=K, T>\xi\Big)+P\Big(\sum_{i=1}^{n-1}\tau_i>\xi,\ T>\xi\Big)\\ &= P\Big(\sum_{i=1}^{n-1}\tau_i+\tilde\tau_n>\xi,\ T>\xi\Big)\ . \end{aligned} \tag{4.30}$$

Now, we can exchange the role played by $\tilde\tau_n$ and by $\tau_1+\cdots+\tau_{n-1}$ in the previous argument, to use the induction hypothesis. Indeed,

$$\begin{aligned} P\big(\sum_{i=1}^{n-1}\tau_i+\tilde\tau_n>\xi,\ T>\xi\big) &= \sum_{K=1}^{\xi} P\big(\tilde\tau_n=K\big)\times P\big(\sum_{i=1}^{n-1}\tau_i>\xi-K,\ T>\xi\big)+P\big(\tilde\tau_n>\xi\big)\\ &\ge P\big(\sum_{i=1}^{n}\tilde\tau_i>\xi\big)P(T>\xi). \end{aligned} \tag{4.31}$$

The proof of the opposite inequalities follows the same steps. □

5 Very early regime

One important step in the cluster growth is to reach height $\log(N)$. We cover this intermediary step in the following proposition, even if it is included in Theorem 1.1.

Proposition 5.1. *Consider diffusive deposition. There exist positive constants b and γ such that almost surely, for N large*

$$\max_{i\le N}\sigma_i\big(b\frac{N}{\log N}\big)\ge\gamma\log N.$$

This proposition is concerned with what we call the *very early regime*, and it is based on comparison with urn models.

Two scales play an important role in this section: the time scale $N/\log(N)$, and the space scale $\log(N)/\log\log(N)$. We therefore introduce notation

$$T_N=\frac{N}{\log(N)},\quad\text{and}\quad H_N=\frac{\log(N)}{\log\log(N)}. \tag{5.1}$$

The set of configurations σ with maximal height lower than $\gamma\log N$ and $|\sigma|\le\beta T_N$ is called the *very early regime* and is denoted by $\mathcal{X}_{\mathrm{ve}}(\gamma,\beta)$. In other words,

$$\mathcal{X}_{\mathrm{ve}}(\gamma,\beta)=\{\sigma:|\sigma|\le\beta T_N,\quad\max_x\sigma_x\le\gamma\log(N)\},$$

and note that

$$\forall\sigma\in\mathcal{X}_{\mathrm{ve}}(\gamma,\beta),\qquad\sum_{x=1}^{N}\sigma_x^2\le\big(\max_{x\le N}\sigma_x\big)|\sigma|\le\beta\gamma N.$$

If τ_A is the hitting time of set A, we show in this Section that there are constants $b < \beta$ and $\delta > 0$ such that

$$P\big(\tau_{\mathcal{X}_{\text{ve}}(\gamma,\beta)^c} > bT_N\big) \le \exp(-N^{\delta}). \tag{5.2}$$

Strategy of the proof. We divide time in two periods. In the first, of length T_N, a large number of columns, of order N^a with $0 < a < 1$, reach a height δH_N. This is the content of Lemma 5.3, whose main ingredient is a coupling between diffusive deposition and random allocation. In the second period, we use the estimate of Corollary 2.3 to control the growth of these columns together with sending Poisson waves of explorers to ensure the growth of each column independently.

Step 1: reaching height H_N. The random allocation evolution is denoted by $n \mapsto \eta(n)$. Our first lemma deals exclusively with random allocation.

Lemma 5.2. *For $\alpha \in [\frac{1}{2}, 1)$, and $\delta < (1-\alpha)/2$, we have almost surely, for N large enough*

$$|\{x : \eta_x(T_N) > \delta H_N\}| \ge N^{\alpha}. \tag{5.3}$$

Proof. Let X be a Poisson variable of parameter $T_N/2$. We have

$$P\big(|\{x : \eta_x(T_N) > \delta H_N\}| < N^{\alpha}\big) \le P\big(|\{x : \eta_x(X) > \delta H_N\}| < N^{\alpha}\big) + P\big(X > T_N\big). \tag{5.4}$$

Now, $\{\eta_x(X), x = 1, \dots, N\}$ are independent Poisson variables of parameter $1/2\log(N)$. A tedious but simple computation gives

$$P\big(\eta_1(X) \ge \delta H_N\big) = N^{-2\delta(1+o(1))}. \tag{5.5}$$

Now,

$$|\{x : \eta_x(X) > \delta H_N\}| = \sum_{x=1}^{N} 1\!\!1_{\{\eta_x(X) \ge \delta H_N\}}, \tag{5.6}$$

and by Bernstein's inequality, for $\alpha < 1 - 2\delta$, and

$$P\big(|\{x : \eta_x(X) > \delta H_N\}| < N^{\alpha}\big) \le \exp(-\frac{N^{1-2\delta} - N^{\alpha}}{2}). \tag{5.7}$$

Note also that from Chebychev's exponential inequality

$$P\big(X > T_N\big) \le \exp\Big(-\frac{3-e}{2}\,T_N\Big). \tag{5.8}$$

The statement follows. □

In Lemma 3.7, we establish that diffusive deposition, denoted $t \mapsto \sigma(t)$ is more monopolistic than random allocation. Thus, there is a coupling such that with probability 1, when $\sigma(0) = \eta(0)$, we have for any $t \ge 0$

$$\forall k \le N, \qquad \sum_{i=1}^{k} \bar\sigma_i(t) \ge \sum_{i=1}^{k} \bar\eta_i(t).$$

Assume now that $\sigma(T_N) \in \mathcal{X}_{\text{ve}}(\gamma,\beta)$, and that

$$L := |\{x : \eta_x(T_N) > \delta H_N\}| > L' := |\{x : \sigma_x(T_N) > \frac{\delta}{2} H_N\}|. \tag{5.9}$$

Then, by our coupling

$$\sum_{i=1}^{L} \bar\sigma_i(T_N) \ge \sum_{i=1}^{L} \bar\eta_i(T_N) \ge \delta H_N L, \quad \text{and} \quad \sum_{i=L'}^{L} \bar\sigma_i(T_N) \le \frac{\delta}{2} H_N L. \tag{5.10}$$

Then

$$\gamma \log(N)L' \geq \sum_{i=1}^{L} \bar{\sigma}_i(T_N) \geq \frac{\delta}{2} H_N L \Longrightarrow L' \geq \frac{\delta}{2\gamma \log\log(N)} L.$$

Thus, for any $\alpha > 1 - 2\delta$, we have that $L' > N^\alpha$. We therefore state the result as follows.

Lemma 5.3. *For $\alpha \in [\frac{1}{2}, 1)$, and $\delta < (1-\alpha)/2$, we have almost surely, for N large enough, and for the diffusive deposition $t \mapsto \sigma(t)$,*

$$|\{x : \sigma_x(T_N) > \frac{\delta}{2} H_N\}| > N^\alpha. \tag{5.11}$$

By applying Markov's property at time T_N

$$\begin{aligned} P\big(\tau_{\mathcal{X}_{\text{ve}}(\gamma,\beta)^c}(\sigma) > (b+1)T_N\big) &\leq P\big(|\{x :\ \eta_x(T_N) > \delta H_N\}| < N^\alpha\big) \\ &+ \sup\Big(P_\sigma\big(\sigma(bT_N) \in \mathcal{X}_{\text{ve}}(\gamma,\beta)\big) :\ \sigma \in \mathcal{X}_{\text{ve}}(\gamma,\beta),\ |\{x : \sigma_x > \frac{\delta}{2} H_N\}| > N^\alpha\Big). \end{aligned} \tag{5.12}$$

Step 2: poisson waves. We realize diffusive deposition for times in $[T_N, bT_N]$ by a sequence of Poisson waves, the k-th wave made of $X^{(k)}$ explorers, and $\{X^{(k)},\ k \geq 1\}$ an i.i.d sequence of Poisson random variables with parameter x_N going to infinity with N.

Our starting configuration denoted $\sigma^{(0)}$ satisfies

$$\sigma^{(0)} \in \mathcal{X}_{\text{ve}}(\gamma,\beta), \quad \Lambda_N := \{x : \sigma^{(0)}_x > \frac{\delta}{2} H_N\}, \quad \text{and} \quad |\Lambda_N| > N^\alpha.$$

Let $\sigma^{(k)}$ be the configuration of diffusive deposition starting from $\sigma^{(0)}$ after the k-th wave is sent, i.e.,

$$\sigma^{(k)} = \sigma\Big(\sum_{\ell=1}^{k} X^{(\ell)}\Big) \qquad \text{with} \quad |\sigma^{(k)}| = |\sigma^{(0)}| + \sum_{\ell=1}^{k} X^{(\ell)}$$

Define now, using κ_D of Corollary 2.3,

$$K_{\text{VE}} := \kappa_D \exp(-\gamma\beta).$$

We have for the diffusive deposition process for any $k = 1, 2, \ldots$, if $\sigma^{(k)} \in \mathcal{X}_{\text{ve}}(\gamma,\beta)$ then

$$p_i(\sigma(t)) \geq \frac{K_{\text{ve}}}{N} (\sigma_i^{(k-1)})^2 \qquad \forall t \in [|\sigma^{(k-1)}|, X^{(k)} + |\sigma^{(k-1)}|[,\ \forall i \in \Lambda_N. \tag{5.13}$$

This immediately follows from Corollary 2.3.

Consider an auxiliary growth process $\tilde{\sigma}(t)$ which evolves on the sites of $\Lambda_N \cup \{0\}$, defined iteratively as follows. Set $\sigma_0^{(0)} = 0$, and for $i \in \Lambda_N$, set $\tilde{\sigma}_i^{(0)} = \frac{\delta}{2} H_n$. Each explorer in the k–th wave is attached to site $i \in \Lambda_N$ with probability

$$p_i^A(k) = K_{\text{VE}} \frac{(\tilde{\sigma}_i^{(k-1)})^2}{N}, \tag{5.14}$$

whereas site 0 grows by one with probability $1 - \sum_{i \in \Lambda_N} p_i^A(k)$.

The following result is crucial.

Lemma 5.4. *There exists a coupling between $t \mapsto \sigma(t)$ and $t \mapsto \tilde{\sigma}(t)$ such that if $|\sigma^{(k)}| \leq \tau_{\mathcal{X}_{\text{ve}}(\gamma,\beta)^c}$ and t is within the k-th wave, i.e., $t \in [|\sigma^{(k-1)}|, X^{(k)} + |\sigma^{(k-1)}|[$*

$$\forall i \in \Lambda_N, \qquad \sigma_i(t) \geq \tilde{\sigma}_i(t). \tag{5.15}$$

Moreover, $\{\tilde{\sigma}_i,\ i \in \Lambda_N\}$ are independent, where we used the shorthand notation $\tilde{\sigma}_i = \{\tilde{\sigma}_i^{(k)}, k \leq \tau_{\mathcal{X}_{\text{ve}}(\gamma,\beta)^c}\}$.

Proof. The coupling part is simple and we omit it here. We denote by $\{Y_i^1,\ i \in \Lambda_N\}$ independent Poisson variables of parameter $x_N p_i^A(1)$. We denote by $\mathcal{G}_1$ the sigma-field generated by $X^{(1)}$ and by $\{Y_i^1,\ i \in \Lambda_N\}$. We now build $\mathcal{G}_k$ by induction. Assume that $\mathcal{G}_{k-1}$ has been built. Then conditioned on $\mathcal{G}_{k-1}$, we fix the height of all sites after the $k-1$-th wave. Draw a Poisson variable $X^{(k)}$ independent of $\mathcal{G}_{k-1}$, and denote by $\{Y_i^k,\ i \in \Lambda_N\}$ the independent Poisson variables of parameter $p_i^A(k)x_N$ which is itself $\mathcal{G}_{k-1}$ measurable. Note that Y_i^k depends only on $X^{(k)}$ and on the past through $Y_i^1 + \cdots + Y_i^{k-1}$, the height of site i after the $k-1$-th wave. In other words, for any real function f_i, there is a function ϕ_i such that

$$E\big[f_i\big(Y_i^k + \cdots + Y_i^k\big)|\mathcal{G}_{k-1}\big] = E\big[f_i\big(Y_i^k + \cdots + Y_i^k\big)|Y_i^1 + \cdots + Y_i^{k-1}\big] = \phi_i(Y_i^1 + \cdots + Y_i^{k-1}). \tag{5.16}$$

Note also, that if we integrate only over X_k, and for any real functions f_i, for $i \in \Lambda_N$ we have

$$\begin{aligned} E\big[\prod_{i\in\Lambda_N} f_i\big(Y_i^k + \cdots + Y_i^k\big)|\mathcal{G}_{k-1}\big] &= \prod_{i\in\Lambda_N} E\big[f_i\big(Y_i^k + \cdots + Y_i^k\big)|Y_i^1 + \cdots + Y_i^{k-1}\big] \\ &= \prod_{i\in\Lambda_N} \phi_i(Y_i^1 + \cdots + Y_i^{k-1}). \end{aligned} \tag{5.17}$$

This means that what happens on different sites of Λ_N is independent. □

Now, each Poisson wave we send has about x_N explorers, and we expect to send about $(b-1)T_N/x_N$ waves. Recall that for N large, we have a.s. that $\sigma^{(0)} \in \mathcal{X}_{\text{ve}}(\gamma, \beta)$ and $|\Lambda_N| > N^\alpha$. Therefore, if t_N denotes the integer part of $bT_N/(2ex_N)$, and for simplicity $H = \frac{\delta}{2} H_N$

$$\begin{aligned} P_{\sigma^{(0)}}\big(\tau_{\mathcal{X}_{\text{ve}}(\gamma,\beta)^c} > bT_N\big) &\le P\Big(\sum_{k=1}^{t_N} |X^{(k)}| > bT_N\Big) + P_{\sigma^{(0)}}\Big(\max_{i\in\Lambda_N}\big(H + \sum_{k=1}^{t_N} Y_i^k\big) \ < \ \gamma N\Big) \\ &\le e^{-bT_N/2} + \prod_{i\in\Lambda_N}\Big(1 - P_{\sigma^{(0)}}\Big(\big(H + \sum_{k=1}^{t_N} Y_i^k\big) \ \ge \ \gamma N\Big)\Big). \end{aligned} \tag{5.18}$$

Step 3: dealing with one site. We show that for a function $\epsilon(\gamma)$ going to 0 with γ, for all $i \in \Lambda_N$

$$P\Big(\sum_{k=1}^{t_N} Y_i^k \ge \gamma \log(N) - H\Big) \ge \exp\Big(-\epsilon(\gamma)\log N\Big). \tag{5.19}$$

We define the successive wave numbers at which the column at 1 grows. Let τ be the *number of waves* needed so as to increase by at least one the height of site 1. Then, let $\tau_1 = \tau$ and $\tau_n = \tau \circ \theta(\tau_1 + \cdots + \tau_{n-1})$. Note that for any integer n

$$P(\tau_1 > n) = P\big(\tilde{Y}_i^{(1)} = 0, \ldots, \tilde{Y}_i^{(n)} = 0\big) = \exp\big(-n\frac{K_{\text{VE}} H^2 x_N}{N}\big), \tag{5.20}$$

where we used (5.14). Note that at the number of waves $t = \tau_1 + \cdots + \tau_{k-1}$, the configuration $\tilde{\sigma}_i^{(t)}$ is larger or equal than $H + k - 1$. We have, using Lemma 5.4

$$P\big(\tau_k > n\big|\mathcal{G}_{k-1}\big) \le \exp\big(-n\frac{K_{\text{VE}}(H+k-1)^2 x_N}{N}\big). \tag{5.21}$$

Then, we are in the setting of Lemma 4.6, and have a comparison with independent geometric random variables $\{\tilde{\tau}_k,\ k \ge 1\}$ with

$$E[\tilde{\tau}_k] = \frac{N}{K_{\text{VE}}(H+k-1)^2 x_N}.$$

Then, (with the abuse of taking $\gamma\log(N)$ to be integer)

$$
\begin{aligned}
P\Big(\sum_{k=1}^{t_N} Y_i^k \geq \gamma\log(N)-H\Big) &\geq P\big(\tau_1+\cdots+\tau_{\gamma\log N}\leq t_N\big) \geq P\big(\tilde\tau_1+\cdots+\tilde\tau_{\gamma\log N}\leq t_N\big)\\
&\geq \prod_{k=1}^{\gamma\log N} P\big(\tilde\tau_k\leq \frac{t_N}{\gamma\log N}\big) \geq \prod_{k=1}^{\gamma\log N}\Big(1-\exp\big(-t_N\frac{K_{\mathrm{VE}}(H+k)^2x_N}{\gamma N\log(N)}\big)\Big)\\
&\geq \prod_{k=1}^{\gamma\log N}\Big(1-\exp\big(-\frac{\gamma b K_{\mathrm{VE}}}{2e}\frac{k^2}{\gamma^2\log(N)}\big)\Big). \qquad (5.22)
\end{aligned}
$$

Now use the estimate $1-e^{-x}\geq e^{-a}x$ for $0\leq x\leq a$ with $a=\gamma bK_{\mathrm{VE}}/2e$. Now,

$$
\begin{aligned}
\prod_{k=1}^{\gamma\log N}\Big(1-\exp\big(-\frac{\gamma bK_{\mathrm{VE}}}{2e}\frac{k^2}{\gamma^2\log(N)}\big)\Big) \geq& (ae^{-a})^{\gamma log(N)}\frac{(\gamma\log(N))!^2}{(\gamma^2\log^2(N))^{\gamma\log(N)}} \qquad (5.23)\\
\geq& \exp(-\epsilon(\gamma)\log(N)),
\end{aligned}
$$

where we took $\epsilon(\gamma)=\gamma\log(ae^{-a})-2\gamma$.

The proof of the Proposition 5.1 is completed if $\alpha>\epsilon(\gamma)$.

6 Growing columns

In this section we present a simple way to bound the height of the maximal pile, based on stochastic domination, for both ballistic and diffusive deposition. Indeed, for both models we construct a sequence of *inter-arrival* times of explorers on a given column, say column number 1, stochastically dominated by independent geometric variables.

We now state three propositions that are crucial in the proofs of Theorems 1.2 and 1.1. The propositions bound the probabilities of building a high pile, and are proven at the end of this section.

Proposition 6.1. *Consider ballistic deposition. There is a constant $\kappa>0$ such that for all $H>2$, and any site $i\in G_N$,*

$$P\big(\sigma_i(N)>H\big)\leq\exp(-\kappa H). \qquad (6.1)$$

Proposition 6.1 implies that, for a given column, N particles are not enough to reach a maximal height of order $\log(N)/\kappa$.

Proposition 6.2. *Consider diffusive deposition. For $H<N^{\frac12-\epsilon}$, and any site $i\in G_N$, and X a positive integer*

$$P\big(\sigma_i(X)>H\big)\leq\exp\Big(\kappa\frac{X}{N}H^2-\frac{\pi}{4}H\Big), \qquad (6.2)$$

with $\kappa=1+O(N^{-1/2})$.

Proposition 6.2 implies that for α small, $\alpha N/\log(N)$ particles are not enough to reach a maximal height of order $\log(N)$.

Finally, we consider diffusive deposition, with a configuration in $\mathcal{X}_{\mathrm{e}}(A)$, and with one distinguished site, say i, above height $\gamma\log(N)$. We show that as long as we do not leave the *early regime*, see equation (4.24) we have a *fast growth*. Let $\tau_{\mathcal{X}_{\mathrm{e}}{}^c}$ the time at which you exit the early regime.

Proposition 6.3. *Let χ,γ,C be any positive constants with $\chi<1/2$. Assume that there is a distinguished site, say i^*, with $\sigma_{i^*}\geq\gamma\log(N)$. Then, we have*

$$P_\sigma\Big(\sigma_{i^*}\big(C\frac{N}{\log(N)}\big)<N^\chi\big|\tau_{\mathcal{X}_{\mathrm{e}}{}^c}>C\frac{N}{\log(N)}\Big)\leq\exp\Big(-\gamma\big(\kappa(A)\frac{\gamma C}{2}-\frac{\pi}{2}\big)\log(N)\Big). \qquad (6.3)$$

6.1 Growing a column in ballistic deposition

Proof of Proposition 6.1. By lemma 4.5

$$P\big(\tau_1 > k\big) \geq \big(1 - \frac{1}{N}\big)^k \qquad \text{so that} \qquad E[\tau_1] > N. \tag{6.4}$$

and in general

$$P\big(\tau_i > k\big) \geq \big(1 - \frac{i}{N}\big)^k. \qquad \text{so that} \qquad E[\tau_i] > \frac{N}{i}. \tag{6.5}$$

This implies that (6.1) is a large deviation event since

$$\sum_{i=1}^{H} E[\tau_i] \geq N \log(H).$$

More precisely by Lemma 4.6 we have

$$P\big(\tau_1 + ... + \tau_H < N\big) \leq P\big(\tilde{\tau}_1 + ... + \tilde{\tau}_H < N\big)$$

with $\{\tilde{\tau}_i,\ i = 1, \ldots, H\}$ independent geometric variables of mean $\frac{N}{i}$. By the exponential Chebyshev's inequality we get, for every $\lambda > 0$

$$P\big(\tilde{\tau}_1 + ... + \tilde{\tau}_H < N\big) \leq e^{\lambda N} \prod_{i=1}^{H} E[e^{-\lambda \tilde{\tau}_i}]$$

Note that for a geometric variable X of mean $1/p$

$$E[\exp(-\lambda X)] = \frac{p}{e^{\lambda} - (1-p)}.$$

When λ is positive, $\exp(\lambda) - 1 \geq \lambda$, and we have

$$E[\exp(-\lambda X)] \leq 1 - \frac{\lambda}{\lambda + p}.$$

Now,

$$\prod_{i=1}^{H} E[\exp(-\lambda \tilde{\tau}_i)] \leq \prod_{i=1}^{H} \big(1 - \frac{\lambda}{\lambda + i/N}\big) \leq \exp(-\lambda \sum_{i=1}^{H} \frac{1}{\lambda + i/N}).$$

We choose $\lambda = i_0/N$ so that by the asymptotic of the harmonic series

$$\lambda \sum_{i=1}^{H} \frac{1}{\lambda + i/N} = i_0 \sum_{i=i_0+1}^{i_0+H} \frac{1}{i} \geq i_0 \log\big(\frac{i_0 + H + 1}{i_0 + 1}\big).$$

So, we obtain

$$P\big(\tilde{\tau}_1 + \cdots + \tilde{\tau}_H \leq N\big) \leq \exp\Big(-i_0\big[\log\big(\frac{i_0 + 1 + H}{i_0 + 1}\big) - 1\big]\Big).$$

Now, if we choose i_0 to be the integer part of αH, for some constant α, then

$$P\big(\tilde{\tau}_1 + \cdots + \tilde{\tau}_H \leq N\big) \leq \exp\Big(-\alpha H (1 - \frac{1}{\alpha H})\big(\log(\frac{1+\alpha}{\alpha}) - 1\big)\Big).$$

If α is sufficiently small, say $\alpha = \frac{1}{2}$, then $\log((1+\alpha)/\alpha) - 1 > 0$, and Lemma 6.1 is established. □

6.2 Growing a column in diffusive deposition

Proof of Proposition 6.2. We follow the arguments of the previous proof. By using Lemma 4.3 for any $i \geq 1$, and integer n

$$P\big(\tau_i > n\big) \geq \big(1 - \kappa \frac{i^2}{N}\big)^n, \qquad \text{with} \quad \kappa = 1 + O(N^{-1/2}).$$

By Lemma 4.6, we have for any H, X

$$P\big(\tau_1 + ... + \tau_H < X\big) \leq P\big(\tilde{\tau}_1 + ... + \tilde{\tau}_H < X\big),$$

with $\{\tilde{\tau}_i,\ i \geq 1\}$ independent geometric variables with $E[\tilde{\tau}_i] = N/(\kappa i^2)$.

Then, for every $\lambda > 0$, by Chebyshev's inequality

$$P\big(\tau_1 + \cdots + \tau_H < X\big) \leq \exp\big(\lambda X - \lambda \sum_{i=1}^{H} \frac{1}{\lambda + \kappa i^2/N}\big). \tag{6.6}$$

We set $\lambda = \kappa H^2/N$ and we note that

$$H^2 \sum_{i=1}^{H} \frac{1}{H^2 + i^2} = \sum_{i=1}^{H} \frac{1}{1 + (\frac{i}{H})^2} \geq H \int_0^1 \frac{dx}{1+x^2} = H\frac{\pi}{4}. \tag{6.7}$$

We conclude obtaining (6.2). □

Proof of Proposition 6.3. Again, let $\{\tau_1, \tau_2, \ldots, \}$ be the random number of explorers linked with growing a column at i^* from σ with an initial state with $\sigma_{i^*} = \gamma \log(N)$. By Corollary 4.2 we have for any integer $m < X$

$$P_\sigma\big(\tau_k > m \big| \tau_1, \ldots, \tau_{k-1}, \tau_{\mathcal{X}_e^c} > X\big) \leq \Big(1 - \kappa(A)\frac{(\gamma \log N + k - 1)^2}{N}\Big)^m = P(\tilde{\tau}_k > m)$$

Therefore, by Lemma 4.6

$$P_\sigma\big(\sum_{k=1}^{N^\chi} \tau_k > X \big| \tau_{\mathcal{X}_e^c} > X\big) \leq P\big(\sum_{k=1}^{N^\chi} \tilde{\tau}_k > X\big).$$

By Chebyshev inequality,

$$\mathbb{E}e^{\lambda \tilde{\tau}_k} = \frac{p_k}{e^{-\lambda} - (1 - p_k)} = 1 + \frac{a}{p_k - a}, \quad \text{with} \quad p_k := \kappa(A)\frac{(\gamma \log N + k - 1)^2}{N},$$

assuming $a := 1 - e^{-\lambda} < p_k$. Thus,

$$\mathbb{E}e^{\lambda \tilde{\tau}_k} < \exp\{a/(p_k - a)\} \ .$$

Hence,

$$P\Big(\sum_{k=1}^{N^\chi} \tilde{\tau}_k > X\Big) \leq \exp\Big(-\lambda X + \sum_{k=1}^{N^\chi} \frac{a}{p_k - a}\Big) \ .$$

We choose $a = \kappa(A)(\gamma^2 \log^2 N)/N$ (and $a \geq \lambda/2$) and $X = CN/\log N$, and we have

$$\sum_{k=1}^{N^\chi} \frac{a}{p_k - a} \leq \gamma \log N \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}\gamma \log N,$$

so that

$$\begin{aligned} P\Big(\sum_{k=1}^{N^\chi} \tilde{\tau}_k > X\Big) &\leq \exp\Big(-\kappa(A)\frac{\gamma^2 \log^2 N}{2N} C \frac{N}{\log N} + \frac{\pi}{2}\gamma \log N\Big) \\ &= \exp\Big(-\gamma \log N[\kappa(A)\frac{\gamma C}{2} - \frac{\pi}{2}]\Big) \end{aligned}$$

□

6.3 Growing a tower in diffusive deposition

In this section, we bound the probability of forming a *high tower* of explorers in diffusive deposition.

Fix a region $\Lambda \subset \{1, \dots, N\}$ of size L. Let H be a fixed positive integer, and $\mathcal{C} = \Lambda \times \{0, \dots, H\}$. Given $\sigma \in \mathbb{N}^{G_N}$ we define

$$\sigma \wedge \mathcal{C} := \{i \in \Lambda : \sigma_i < H\} \subset G_N, \quad \text{and note that} \quad |\sigma \wedge \mathcal{C}| = \sum_{i \in \Lambda} \mathbb{1}_{\{\sigma_i < H\}}. \tag{6.8}$$

Proposition 6.4. *Consider diffusive deposition. For any positive H, X and ξ with $X < N/2$, we have that*

$$P(\forall x \in \Lambda,\ \sigma_x(X) \geq H) \leq \exp\Big(LH\big(\xi\frac{HX}{N} - \xi\log(\frac{1+\xi+1/H^2}{\xi+1/H^2})\big)\Big) \tag{6.9}$$

Moreover, for any positive a, γ, and a positive real χ satisfying $4\chi < a\gamma^2\exp(-2a\gamma)$, and the choice $H = \gamma\log(N)$, $X = aT_N$ and $L = N^{1-2\chi}$, we have

$$P\big(|\{i:\ \sigma_i(aT_N) > \gamma\log N\}| > N^{1-2\chi}\big) \leq \exp\big(-\chi N^{1-2\chi}\big). \tag{6.10}$$

Proof. Lemma 4.3 immediately yields

$$P\Big(\text{ explorer attaches to } \sigma \wedge \mathcal{C} \ \Big|\ \sigma\Big) \leq \kappa_A \sum_{x \in \sigma\wedge\mathcal{C}} \frac{\sigma_x^2 + 1}{N}$$

Note that

$$\sum_{x \in \sigma\wedge\mathcal{C}} \sigma_x^2 + 1 \leq L + H \sum_{x \in \sigma\wedge\mathcal{C}} \sigma_x.$$

We define $n_{\mathcal{C}}(\sigma) = \sum_{x \in \sigma\wedge\mathcal{C}} \sigma_x$, so that, for N large enough we have that

$$P\Big(\text{ explorer attaches to } \sigma \wedge \mathcal{C} \ \Big|\ \sigma\Big) \leq \frac{\kappa_A}{N}[L + Hn_{\mathcal{C}}(\sigma)]. \tag{6.11}$$

This allows us to define, as before, geometric random variables stochastically smaller than the number of explorers needed to settle one of them in $\mathcal{C}$. Let τ_1 be the number of explorers needed in order that one settles in $\mathcal{C}$, when we start with the empty configuration. By induction, when $k-1$ explorers are settled in $\mathcal{C}$, define τ_k to be the number of explorers needed to settle the k–th explorer in $\mathcal{C}$, and we do this up to time LH. Then for any configuration σ with $n_{\mathcal{C}}(\sigma) = k-1$, for any positive integer m

$$P(\tau_k > m \mid \sigma) \geq \big(1 - \frac{kH+L}{N}\big)^m = P(\tilde\tau_k > m). \tag{6.12}$$

We invoke again Lemma 4.6 to obtain

$$\begin{aligned}
P\big(\tau_1 + \dots + \tau_{HL} \leq X\big) &\leq P\big(\tilde\tau_1 + \dots + \tilde\tau_{HL} \leq X\big) \leq e^{\lambda X} \prod_{k=1}^{HL} E[\exp(-\lambda\tilde\tau_k)] \\
&\leq \exp\Big(\lambda X - \lambda \sum_{k=1}^{HL} \frac{1}{\lambda + (kH+L)/N}\Big) \qquad \text{choose} \quad \lambda = \frac{H^2L}{N}\xi \\
&\leq \exp\Big(\xi\frac{H^2LX}{N} - \xi HL \sum_{k=1}^{HL} \frac{1}{HL\xi + k + L\mathcal{H}}\Big) \\
&\leq \exp\Big(LH\big(\xi\frac{HX}{N} - \xi\log(\frac{1+\xi+1/H^2}{\xi+1/H^2})\big)\Big).
\end{aligned} \tag{6.13}$$

With the choice $H = \gamma \log(N)$, $L = N^{1-2\chi}$, and $X = aT_N$ we get

$$\begin{aligned}
P\big(|\{i:\ \sigma_i(aT_N) > \gamma \log N\}| > N^{1-2\chi}\big) &\le \binom{N}{L} \exp\Big(LH\big(\xi \frac{HX}{N} - \xi \log(\frac{1+\xi+1/H^2}{\xi+1/H^2})\big)\Big) \\
&\le \exp\Big(L\log(N/L) + LH\frac{HX}{N}\xi - LH\xi\log(\frac{1+\xi+1/H^2}{\xi+1/H^2})\big)\Big) \\
&\le \exp\Big(-L\log(N)\big(\gamma\xi\log(\frac{1+\xi}{\xi}) - 2\chi - a\gamma^2\xi\big)\Big). \qquad (6.14)
\end{aligned}$$

First choose $\xi = \exp(-2a\gamma)$, to get

$$\big(h\xi\log(\frac{1+\xi}{\xi}) - 2\chi - xh^2\xi\big) \ge \big(a\gamma^2\xi - 2\chi)\big).$$

Now choose $4\chi < (a\gamma^2)\exp(-2a\gamma)$, and the inequality (6.10) is obtained. $\square$

7 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Proof of (1.9): the statement follows immediately by Proposition 6.2 with $H = 3\log(N)$ and $X \le cN/\log(N)$, for c small enough. Indeed by Proposition 6.2 we have for the complementary event

$$\begin{aligned}
P(\exists i:\ \sigma_i(X) > 3\log N) \le& \sum_{i=1}^{N} P(\sigma_i(X) > 3\log N) \\
\le& N\exp\Big(3\log N\big(3\kappa c - \frac{\pi}{4}\big)\Big).
\end{aligned} \qquad (7.1)$$

Hence

$$P(\exists i:\ \sigma_i(X) > 3\log N) \le N^{1-3(\pi/4-3\kappa c)}.$$

This concludes the proof since the exponent of N is less than -1 when $9c\kappa < (3\pi/4-2)$. (recall that $\kappa = 1 + O(N^{-1/2})$).

Proof of (1.10). Recall that from Proposition 5.1 there is $b > 0$ (and (5.2) for the quantitative estimate), so that very likely $\tau_{\mathcal{X}_{\rm ve}(\gamma,\beta)^c} < bT_N$, where $T_N = N/\log(N)$. We therefore condition on the evolution up to $\tau_{\mathcal{X}_{\rm ve}(\gamma,\beta)^c}$.

$$\begin{aligned}
P\big(\max\sigma_x(aT_N+bT_N) < N^\chi\big) \le& P\big(\tau_{\mathcal{X}_{\rm ve}(\gamma,\beta)^c} < bT_N, \max\sigma_x(aT_N+bT_N) < N^\chi\big) \\
&+ P\big(\tau_{\mathcal{X}_{\rm ve}(\gamma,\beta)^c} \ge bT_N\big).
\end{aligned} \qquad (7.2)$$

Now, in the first term use Markov's property at time $\tau_{\mathcal{X}_{\rm ve}(\gamma,\beta)^c}$, calling for simplicity $\sigma(\mathcal{X}_{\rm e}) := \sigma(\tau_{\mathcal{X}_{\rm ve}(\gamma,\beta)^c})$

$$\begin{aligned}
&P\big(\tau_{\mathcal{X}_{\rm ve}(\gamma,\beta)^c} < bT_N, \max\sigma_x(aT_N+bT_N) < N^\chi\big) \\
&\qquad \le E\Big[\mathbb{1}_{\{\tau_{\mathcal{X}_{\rm ve}(\gamma,\beta)^c} < bT_N\}} P_{\sigma(\mathcal{X}_{\rm e})}\Big(\max\sigma_x(aT_N) < N^\chi\Big)\Big]
\end{aligned} \qquad (7.3)$$

Now, $\sigma(\mathcal{X}_{\rm e}) \in \mathcal{X}_{\rm e}(A)$, for $A > 2\gamma\beta$, and there is $i^* \in G_N$ such that $\sigma_{i^*}(\mathcal{X}_{\rm e}) \ge \gamma\log(N)$. Thus,

$$\begin{aligned}
P_{\sigma(\mathcal{X}_{\rm e})}\big(\max_x \sigma_x(aT_N) < N^\chi\big) \le& P_{\sigma(\mathcal{X}_{\rm e})}\big(\max_x \sigma_x(aT_N) < N^\chi \big|\ \tau_{\mathcal{X}_{\rm e}^c} > aT_N\big) P_{\sigma(\mathcal{X}_{\rm e})}\big(\tau_{\mathcal{X}_{\rm e}^c} > aT_N\big) \\
&+ P_{\sigma(\mathcal{X}_{\rm e})}\big(\tau_{\mathcal{X}_{\rm e}^c} < aT_N,\ \max_x \sigma_x(aT_N) < N^\chi\big). \qquad (7.4)
\end{aligned}$$

The first term on the right hand side of (7.4) is dealt with by Proposition 6.3.

The next lemma deals with the second term on the right hand side of (7.4).

Lemma 7.1. *Let* $\sigma \in \mathcal{X}_{\mathrm{e}}(A)$ *be a configuration such that* $\max \sigma_x = \gamma \log(N)$*, for some positive* γ*. For* $a > 0$ *such that* $a\gamma \leq A - 1$*, and* $4\chi < a\gamma^2 \exp(-2a\gamma)$*, we have*

$$P_\sigma\big(\tau_{\mathcal{X}_{\mathrm{e}}^c} < aT_N,\ \max_x \sigma_x(aT_N) < N^\chi\big) \leq \exp(-\chi N^{1-2\chi}). \tag{7.5}$$

Proof of Lemma 7.1. Note that when $a\gamma \leq A - 1$,

$$\{\tau_{\mathcal{X}_{\mathrm{e}}^c} < aT_N,\ \sigma_x(aT_N) < N^\chi\} \subset \{\big|\{x:\ \sigma_x(aT_N) > \gamma \log(N)\}\big| > N^{1-2\chi}\}. \tag{7.6}$$

Indeed, on the event $\{\tau_{\mathcal{X}_{\mathrm{e}}^c} < aT_N,\ \sigma_x(aT_N) < N^\chi\}$,

$$\begin{aligned} AN \leq \sum_i \sigma_i^2(\tau_{\mathcal{X}_{\mathrm{e}}^c}) = & \sum_{i:\, \sigma_i(\tau_{\mathcal{X}_{\mathrm{e}}^c}) \leq \gamma \log N} \sigma_i^2(\tau_{\mathcal{X}_{\mathrm{e}}^c}) + \sum_{i:\, \sigma_i(\tau_{\mathcal{X}_{\mathrm{e}}^c}) > \gamma \log N} \sigma_i^2(\tau_{\mathcal{X}_{\mathrm{e}}^c}) \\ \leq & \gamma(\log N) aT_N + N^{2\chi} |\{i:\ \sigma_i(\tau_{\mathcal{X}_{\mathrm{e}}^c}) > \gamma \log N\}| \\ \leq & a\gamma N + N^{2\chi} |\{i:\ \sigma_i(\tau_{\mathcal{X}_{\mathrm{e}}^c}) > \gamma \log N\}|. \end{aligned} \tag{7.7}$$

Proposition 6.4 deals with growing large towers, and by using inequality (6.10) the proof is complete. □

Proof of Theorem 1.2. The statement follows immediately by Proposition 6.1 with $H = \gamma \log(N)$ with $\gamma > 2/\kappa$. Indeed, by Proposition 6.1, there exists a constant κ such that

$$P(\exists i:\ \sigma_i(N) > \gamma \log N) \leq N e^{-\kappa\gamma \log N} = N^{1-\kappa\gamma},$$

and the proof concludes. □

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