A Local Limit Theorem for sums of independent random vectors

Dmitry Dolgopyat

Abstract

We prove a local limit theorem for sums of independent random vectors satisfying appropriate tightness assumptions. In particular, the local limit theorem holds in dimension 1 if the summands are uniformly bounded.

Keywords: local limit theorem; characteristic function; lattice distribution; concentration inequality.

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1 Introduction

1.1 The main result

A classical Local Limit Theorem says that the distribution of the sum of i.i.d. random variables considered at a small scale is approximately invariant with respect to translations by a large subgroup of $\mathbb{R}^d$. Several authors addressed a generalization of this result for non-identically distributed terms (see e.g. [1, 2, 4, 5, 6, 7, 8, 9, 11] and references therein). Here we show that a reasonable theory can be obtained if we impose appropriate tightness assumptions on individual summands.

Consider a sum $S_N = \sum_{j=1}^{N} X_j$ where $X_j$ are independent, $\mathbb{R}^d$ valued random variables such that

\[ E(X_j) = 0, \]
\[ E(|X_j|^3) \leq m_3 \]

and there exists a constant $\varepsilon_0 > 0$ such that for each $s \in \mathbb{R}^d$

\[ E((X_j, s)^2) \geq \varepsilon_0 |s|^2. \]
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Note that in the presence of (1.2) condition (1.3) is equivalent to existence of \( \varepsilon_1, \varepsilon_2 > 0 \) such that for each proper affine subspace \( II \subset \mathbb{R}^d \) we have

\[
P(d(X_j, II) \leq \varepsilon_1) \leq 1 - \varepsilon_2. \tag{1.4}
\]

Let \( V_N \) denote the covariance matrix

\[
V_{N,t_1,t_2} = \sum_{j=1}^{N} \mathbb{E}(X_j(t_1)X_j(t_2))
\]

(here and below we denote by \( X(j) \) the \( t \)-th coordinate of vector \( X \)).

We call a closed subgroup \( H \subset \mathbb{R}^d \) sufficient if there is a deterministic sequence \( a_N \) such that \( S_N - a_N \mod H \) converges almost surely. The minimal subgroup, denoted by \( H_\text{s} \), is defined as the intersection of all sufficient subgroups.

**Proposition 1.1.** (a) If \( H \) is sufficient then \( \mathbb{R}^d/H \) is compact.

(b) The minimal subgroup is sufficient.

If \( H \) is a proper subgroup of \( \mathbb{R}^d \) we call the sequence \( \{X_N\} \) arithmetic, otherwise it is called nonarithmetic\(^2\).

Due to Proposition 1.1 there exists a bounded sequence \( a_N \) such that \( S_N - a_N \mod H \) converges almost surely. Fix such a sequence and denote the limiting random variable by \( S \).

We refer the reader to Subsection 1.3 for examples of computation of the minimal subgroup for \( d = 1 \).

Given a random variable \( Y \) let \( C_Y \) be the convolution operator

\[
C_Y(g)(x) = \mathbb{E}(g(x + Y)).
\]

We denote by \( C(\mathbb{R}^d) \) (respectively \( C^r(\mathbb{R}^d) \)) the space of continuous (respectively \( r \) times differentiable) functions on \( \mathbb{R}^d \). The subscript 0 indicates that we consider only functions of compact support in the corresponding space.

**Theorem 1.2.** For each \( g \in C_0(\mathbb{R}^d) \) for each sequence \( z_N = O(\sqrt{N}) \) such that \( z_N - a_N \in H \) we have

\[
\lim_{N \to \infty} \left[ \mathbb{E}(g(S_N - z_N)) - \frac{u_N(z_N)}{u_N(z_N)} \right] = \int_H C_S(g)(h)d\lambda_H(h)
\]

where \( \lambda_H \) is the Haar measure on \( H \) and \( u_N(z) \) is the density of the normal random variable with zero mean and covariance \( V_N \).

In particular, in the non-arithmetic case for each sequence \( z_N = O(\sqrt{N}) \) we have

\[
\lim_{N \to \infty} \left[ \mathbb{E}(g(S_N - z_N)) - \frac{u_N(z_N)}{u_N(z_N)} \right] = \int_{\mathbb{R}^d} g(x)dx.
\]

The Haar measure in the above theorem is defined as follows. \( H \) is isomorphic to the product of \( \mathbb{Z}^{d_1} \times \mathbb{R}^{d-d_1} \). \( \lambda_H \) is the product of the counting measure on the first factor and the Lebesgue measure on the second factor normalized as follows. Choose a set \( D \) so that each \( x \in \mathbb{R}^d \) can be uniquely written as \( x = h + \theta \) where \( h \in H, \theta \in D \). \( \lambda_D \) is normalized so that

\[
\int_{\mathbb{R}^d} g(x)dx = \int_H \int_D g(h + \theta)d\lambda_H(h)d\lambda_D(\theta) \tag{1.5}
\]

where \( \lambda_D \) is the Lebesgue measure on \( D \) normalized to have total volume 1.

\(^2\)Sometimes in the literature the term arithmetic is reserved to the case where \( H \) is a discrete subgroup of \( \mathbb{R}^d \) while the case where it has both discrete and continuous parts is called mixed but in our presentation we will not distinguish between those two cases.
1.2 One dimensional case

If $d = 1$ there are several simplifications. Namely $V_N$ is a scalar and $H$ is either $\mathbb{R}$ or $h\mathbb{Z}$ for some $h \in \mathbb{R}$. So Theorem 1.2 can be restated as follows.

**Corollary 1.3.** Either

(i) for each $g \in C_0(\mathbb{R})$ for each sequence $z_N$ such that $\lim_{N \to \infty} \frac{z_N}{\sqrt{V_N}} = z$

$$
\lim_{N \to \infty} \left[ \sqrt{V_N} E(g(S_N - z_N)) \right] = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) dx
$$

(1.6)

or (ii) there exists $h > 0$ and a bounded sequence $a_N$ such that $S_N - a_N \pmod{h}$ converges almost surely to a random variable $S$ and for each $g \in C_0(\mathbb{R})$ for each sequence $z_N$ such that $z_N = a_N + k_N h$ with $k_N \in \mathbb{Z}$ and $\lim_{N \to \infty} \frac{z_N}{\sqrt{V_N}} = z$

$$
\lim_{N \to \infty} \left[ \sqrt{V_N} E(g(S_N - z_N)) \right] = \frac{h e^{-z^2/2}}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} C_S(g(jh)).
$$

In Section 8 we deduce the following consequence of this result.

**Corollary 1.4.** Let $X_j$ be independent random variables of zero mean which are uniformly bounded (that is, there is $K$ such that $|X_j| \leq K$ with probability one). Then either $S_N$ converges almost surely to some random variable $S$ in which case

$$
\sqrt{V_N} E(g(S_N)) \to \sqrt{V(S)} E(g(S))
$$

(1.7)

or $S_N$ satisfies the conclusions of Corollary 1.3.

1.3 Examples

Here we provide several examples of computing the minimal subgroup, the normalizing sequence $a_N$ and the shape of local distribution $S$.\(^3\)

They provide a good illustration of versatility of Corollary 1.4, even though the computations in each individual example presented below could be done by hand. Namely, all cases where $H \neq \mathbb{R}$ follow immediately from Kolmogorov’s Three Series Theorem. The cases where $H = \mathbb{R}$ seem a little more tricky and could be most easily analyzed with the help of Lemma 3.2.

**Example 1.5.** $X_1$ has a continuous distribution and $X_n$ for $n \geq 2$ are i.i.d and $P(X_n \in a + h\mathbb{Z}) = 1$ where $h$ is the maximal number with this property. Then

$$
H = h\mathbb{Z}, \quad a_N = Na \pmod{h}, \quad S = X_1.
$$

**Example 1.6.** $X_n$ are integer valued and $|X_n| \leq M$ with probability 1. According to Corollary 1.4 there are two cases

(I) $\sum\limits_N (X_N - E(X_N))$ converges\(^4\). Let $b_N$ be the closest integer to $E(X_N)$. Then either $X_N = b_N$ or $|X_N - E(X_N)| \geq 1/2$. Therefore the case (b1) is characterized by the condition

$$
\sum\limits_N \left( 1 - \max\limits_k P(X_N = k) \right) < \infty.
$$

\(^3\)The reader should keep in mind that the choices of $a_N$ and $S$ are not unique. Namely, we can replace $(a_N, S)$ by $(a_N + \tilde{a}_N, c, S - c)$ where $c$ is an arbitrary constant and $\tilde{a}_N$ is a sequence converging to 0. In Examples 1.5-1.8 we give one possible choice.

\(^4\)Note that we do not assume here that $X_N$ have zero mean since $E(X_N)$ need not be an integer, so we can not reduced the general case to the zero mean case by subtracting the mean.
The minimal subgroup is $h \mathbb{Z}$ for some $h \leq 2M$. Note that the same argument as in (b1) shows that $h \mathbb{Z}$ is sufficient iff
\[ \sum_N \left( 1 - \max_k P(X_N \equiv k \mod h) \right) \] converges.

We now distinguish to further subcases:

(Iia) The series (1.8) converges only for $h = 1$. In this case $S = 0$ and we obtain the classical arithmetic local limit theorem
\[ \sqrt{V_N} P(S_N = k_N) \to \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \text{ if } \frac{k_N}{\sqrt{V_N}} \to z. \]

(Iib) The maximal $h$ for which the series (1.8) converges is larger than 1. In this case $H = h \mathbb{Z}$ with $h$ as above,
\[ a_N = \sum_{n=1}^N k_n \mod h, \text{ where } k_n = \arg \max P(X_n \equiv k \mod h) \]
and $S = \sum_{n=1}^\infty (X_n - k_n)$ (note that due to Borel-Cantelli Lemma this sum has only finitely many non-zero terms with probability 1).

The LLT in Example 1.6 is proven in [10] (except that our results are slightly more precise in case (Iib). The fact that (1.2) and (1.3) are sufficient for the LLT is noted in [12] which obtains the LLT under slightly weaker conditions than (1.2) and (1.3) (under the assumption that $X_N$ are integer valued!).

Example 1.7. $X_n = \xi_n + \varepsilon_n \eta_n$ where $\{\xi_n\}$ and $\{\eta_n\}$ are i.i.d random variables, $\xi$s and $\eta$s are independent, $\xi_n$ take values $\pm 1$ with probability $\frac{1}{2}$ and $\eta_n$ have continuous distribution with finite third moment. Then either

(I) $\sum_n \varepsilon_n^2$ converges and
\[ H = 2\mathbb{Z}, \quad a_N = N \mod 2, \quad S = \sum_{n=1}^\infty \varepsilon_n \eta_n \]
or (II) $\sum_n \varepsilon_n^2$ diverges in which case $H = \mathbb{R}$ and we are in the non-arithmetic situation.

Example 1.8.
\[ P(X_n = -1) = \frac{1}{2} + p_n, \quad P(X_n = 1 + \varepsilon_n) = \frac{1}{2} - p_n, \text{ where } \varepsilon_n = \frac{4p_n}{1 - 2p_n} \]

(so that $E(X_n) = 0$). We assume that $p_n \to 0$. Then either

(I) $\sum_n \varepsilon_n^2$ converges (which is equivalent to the convergence of $\sum_n p_n^2$). Then
\[ H = 2\mathbb{Z}, \quad a_N = \left( N + \frac{1}{2} \sum_{n=1}^N \varepsilon_n \right) \mod 2, \quad S = \sum_{n=1}^\infty \varepsilon_n \left( 1_{X_n=1+\varepsilon_n} - \frac{1}{2} \right) \]
or (II) $\sum_n \varepsilon_n^2$ diverges in which case $H = \mathbb{R}$ and we are in the non-arithmetic situation.

1.4 Plan of the paper

In Section 2 we prove Proposition 1.1. In Section 3 we show that the non-arithmetic case is characterized by the condition that the characteristic function of $S_N$ tends to 0 everywhere except for the origin. In Section 4 we show that if the characteristic function
is large at some point then it decays rapidly nearby. This estimate is used in Section 5 to prove the Local Limit Theorem for test functions whose Fourier transform is compactly supported. In Section 6 we use an approximation argument to prove the Local Limit Theorem for continuous functions of compact support. The proof relies on an auxiliary estimate saying that a probability to visit a cube of a unit size is $O(dV_N^{-1/2})$. That estimate is established in Section 7. Finally, in Section 8 we prove Corollary 1.4.

Throughout the paper $\hat{g}$ denotes the Fourier transform of a function $g$. $\mathcal{U}_c(A)$ denotes $\varepsilon$-neighborhood of a set $A \subset \mathbb{R}^d$. $B_R$ is a ball of radius $R$ centered at the origin.

## 2 Minimal subgroup

We need the following deterministic fact.

**Lemma 2.1.** Let $H, \tilde{H}$ be closed subgroups of $\mathbb{R}^d$ such that $\mathbb{R}^d/H$ is a compact subgroup, where $H = \tilde{H} \cap \tilde{H}$. Let $s_N$ be a sequence such that both $s_N \mod \tilde{H}$ and $s_N \mod \tilde{H}$ converge. Then $s_N \mod H$ converges.

**Proof.** Let $p : \mathbb{R}^d \to \mathbb{R}^d/H$, $\tilde{p} : \mathbb{R}^d/H \to \mathbb{R}^d/\tilde{H}$, $\tilde{\tilde{p}} : \mathbb{R}^d/H \to \mathbb{R}^d/\tilde{\tilde{H}}$ be natural projections,

$$\bar{s} = \lim_{N \to \infty} s_N \mod \tilde{H}, \quad \tilde{s} = \lim_{N \to \infty} s_N \mod \tilde{H}, \quad \tilde{\tilde{s}} = \tilde{p}^{-1} \bar{\tilde{s}}, \quad \tilde{s} = \tilde{\tilde{p}}^{-1} \tilde{\bar{s}}.$$  

Note that Card$(\tilde{S} \cap \tilde{\tilde{S}}) \leq 1$. On the other hand for each $\varepsilon > 0$

$$p(s_N) \in \mathcal{U}_c(\tilde{S}) \cap \mathcal{U}_c(\tilde{\tilde{S}})$$

provided that $N$ is large enough. It follows that $\tilde{S}$ and $\tilde{\tilde{S}}$ do indeed intersect and $\lim_{N \to \infty} p(s_N) = \tilde{S} \cap \tilde{\tilde{S}}$. \hfill $\square$

**Proof of Proposition 1.1.** (a) If $\mathbb{R}^d/H$ was not compact then we may assume after an appropriate change of variables that all vectors in $H$ have zero last coordinate. That is, $S_{N,(d)} - a_{N,(d)}$ converges almost surely. By (1.2) and (1.3) we can choose $R$ so large that denoting $X_N = X_{N,(d)}1_{|X_{N,(d)}| \leq R}$ we have $V(X_N) \geq \varepsilon_0/2$. Thus $\sum N V(X_N)$ diverges and so $S_{N,(d)} - a_{N,(d)}$ diverges due to Kolmogorov’s Three Series Theorem.

To prove (b) let $\tilde{H}, \tilde{\tilde{H}}$ be sufficient subgroups such that $S_N - \tilde{a}_N \mod \tilde{H}$ and $S_N - \tilde{\tilde{a}}_N \mod \tilde{\tilde{H}}$ converge. Let

$$\tilde{b}_N = \tilde{a}_N - \tilde{a}_{N-1}, \quad \bar{\tilde{b}}_N = \bar{\tilde{a}}_N - \bar{\tilde{a}}_{N-1}, \quad H = \tilde{H} \cap \tilde{\tilde{H}}.$$  

We claim that $\mathbb{R}^d/H$ is compact. Indeed take $R$ so large that

$$\mathbb{P}(|X_N| \geq R) \leq \varepsilon_2/2$$

where $\varepsilon_2$ is the constant from (1.4). By our assumptions for each $\delta_1, \delta_2$

$$\mathbb{P}(X_N \in \tilde{b}_N + \mathcal{U}_{\delta_1}(\tilde{H})) \geq 1 - \delta_2, \quad \mathbb{P}(X_N \in \bar{\tilde{b}}_N + \mathcal{U}_{\delta_1}(\tilde{H})) \geq 1 - \delta_2$$

provided that $N$ is large enough. Hence if $2\delta_2 + \varepsilon_2/2 < 1$ then

$$\mathbb{P} \left( X_N \in \left( \tilde{b}_N + \mathcal{U}_{\delta_1}(\tilde{H}) \right) \cap \left( \bar{\tilde{b}}_N + \mathcal{U}_{\delta_1}(\tilde{H}) \right) \cap B_R \right) > 0.$$
Therefore the set \( (\tilde{b}_N + \mathcal{U}_{\delta}(\tilde{H})) \cap (\tilde{b}_N + \mathcal{U}_{\delta}(\tilde{H})) \cap B_R \) is non empty, it contains a point \( \tilde{b}_N \). Then
\[
P(X_N \in \tilde{b}_N + (\mathcal{U}_{2\delta}(\tilde{H}) \cap \mathcal{U}_{2\delta}(\tilde{H}))) \geq 1 - 2\delta_2. \tag{2.1}
\]
Take \( \delta_1 \) so small that
\[
(\mathcal{U}_{2\delta_1}(\tilde{H}) \cap \mathcal{U}_{2\delta_1}(\tilde{H})) \cap B_{2R} \subset \mathcal{U}_{\delta_1}(\tilde{H}). \tag{2.2}
\]
Now note that if \( \mathbb{R}^d / H \) was not compact there would be a proper subspace \( L \supset H \) and so (2.1) and (2.2) would contradict (1.4) with \( \Pi = \tilde{b}_N + L \).

Our next claim is that \( H \) is sufficient. Indeed pick \( \omega \) so that both \( S_N(\omega) - \tilde{a}_N \mod \tilde{H} \) and \( S_N(\omega) - \tilde{a}_N \mod \tilde{H} \) converge. Then for almost every \( \omega \) both \( S_N(\omega) - S_N(\omega) \mod \tilde{H} \) and \( S_N(\omega) - S_N(\omega) \mod \tilde{H} \) converge. Now Lemma 2.1 tells us that \( S_N - a_N \mod H \) converges almost surely where \( a_N = S_N(\omega) \). Hence \( H \) is sufficient.

Observe that \( H_0 = \mathbb{R}^d \) is sufficient. If it is not minimal there is a proper sufficient subgroup \( H_1 \subset H_0 \). If \( H_1 \) is minimal we are done. Otherwise there is \( H'_1 \subset H_1 \) which is sufficient and by the foregoing discussion \( H_2 = (H_1 \cap H'_1) \) is sufficient. Continuing we obtain a chain of proper subgroups
\[
H_0 \supset H_1 \supset H_2 \supset \ldots
\]
such that \( H_k \) is sufficient for each \( k \). Note that either \( \dim(H_k) < \dim(H_{k-1}) \) or \( \text{Vol}(H_{k-1}/H_k) \) is an integer greater than 1. On the other hand the proof of part (a) shows that if \( R \) is large enough then \( H_k \) has a basis in \( B_R \) for each \( k \). Thus the chain can not be continued indefinitely ending at some finite \( r \). Then \( H_r \) is minimal and it is sufficient by construction. \( \square \)

3 Distinguishing between the arithmetic and non-arithmetic cases

We start with an auxiliary estimate.

**Lemma 3.1.** Each random variable \( \mathcal{X} \) can be decomposed as \( \mathcal{X} = b + \mathcal{Y} + \mathcal{Z} \) where \( b \) is a constant, \( \mathcal{Z} \in 2\pi\mathbb{Z}, \vert \mathcal{Y} \vert \leq 2\pi, \text{E}(\mathcal{Y}) = 0, \) and
\[
\vert \text{E}(e^{i\mathcal{X}}) \vert \leq 1 - \frac{\text{E}(\mathcal{Y}^2)}{14}.
\]

**Proof.** Let \( \text{E}(e^{i\mathcal{X}}) = \rho e^{ib} \) where \( \rho, b \in \mathbb{R} \). Decompose \( \mathcal{X} - b = \mathcal{Y} + \mathcal{Z} \) where \( \mathcal{Z} \in 2\pi\mathbb{Z} \) and \( \vert \mathcal{Y} \vert \leq \pi \). Then
\[
\rho = \text{E}(e^{i(\mathcal{X} - b)}) = \mathbb{R}(\text{E}(e^{i(\mathcal{X} - b)})) = \text{E}(\text{cos}((\mathcal{X} - b))) = \text{E}(\text{cos}(\mathcal{Y})).
\]

Using that
\[
\cos(x) \leq 1 - \frac{x^2}{2} \text{ if } |x| \leq \pi \text{ we get } \rho < 1 - \frac{\text{E}(\mathcal{Y}^2)}{14} \leq 1 - \frac{\text{E}(\mathcal{Y}^2)}{14}. \text{ This proves the result with } \mathcal{Y} = \mathcal{Y} - \text{E}(\mathcal{Y}) \text{ and } b = b + \text{E}(\mathcal{Y}). \square
\]

We will refer to the decomposition of Lemma 3.1 as the **useful decomposition of \( \mathcal{X} \).**

The next result will help us to distinguish between the arithmetic and non-arithmetic cases.

\[
\cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24} = 1 - \frac{x^2}{2} \left(1 - \frac{x^2}{12}\right) \leq 1 - \frac{x^2}{2} \left(1 - \frac{\pi^2}{12}\right) \leq 1 - \frac{x^2}{2} \times \frac{1}{7}.
\]
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Lemma 3.2. Let \( X \) be independent random variables with zero mean. Let \( S_N = \sum_{n=1}^{N} X_n \). The following are equivalent
(a) There is a sequence \( a_N \) such that \( S_N - a_N \mod 2\pi \) converges;
(b) If \( X \) is \( b_N + Y_N + Z_N \) is a useful decomposition of \( X \) then \( \sum N(V(Y_N)) \) converges;
(c)\(^6\) \( \lim_{N_0 \to \infty} \lim_{N \to \infty} \left| E (e^{i(S_N - S_{N_0})}) \right| = 1. \)

Proof. If \( S_N - a_N \mod 2\pi \) converges then
\[
\lim_{N_0 \to \infty} \lim_{N \to \infty} \left| E (e^{i(S_N - S_{N_0})}) \right| = \lim_{N_0 \to \infty} \lim_{N \to \infty} \left| E (e^{i((S_N - a_N) - (S_{N_0} - a_{N_0}))}) \right| = 1.
\]
Therefore (a) implies (c).
\[\text{If } \lim_{N_0 \to \infty} \lim_{N \to \infty} \left| E (e^{i(S_N - S_{N_0})}) \right| = 1 \text{ then for large } N_0\]
\[\lim_{N \to \infty} \left| E (e^{i((S_N - a_N) - (S_{N_0} - a_{N_0}))}) \right| > 0.\]
Denote this limit by \( e^{-A} \). Combining Lemma 3.1 with the inequality \( 1 - x \leq e^{-x} \) we get
\[
\sum_{N} V(Y_N) \leq 14A. \tag{3.1}
\]
Therefore (c) implies (b).
Finally (b) implies (a) by Kolmogorov’s Three Series Theorem. \( \square \)

We now return to considering a sequence of independent random vectors \( X_n \) with \( S_N = \sum_{n=1}^{N} X_n \). Denote
\[\phi_n(s) = E(e^{i(s,X_n)}), \quad \Phi_N(s) = E(e^{i(s,S_N)}).\]

Corollary 3.3. (a) If \( \mathcal{H} = \mathbb{R}^d \) then \( \lim_{N \to \infty} \Phi_N(s) = 0 \) for \( s \neq 0 \).
(b) If \( \mathcal{H} = \mathbb{Z}^d + \mathbb{R}^{d-d_1} \) then \( \lim_{N \to \infty} \Phi_N(s) = 0 \) unless the last \( d - d_1 \) coordinates of \( s \) are 0 and the first \( d_1 \) coordinates belong to \( 2\pi\mathbb{Z}^{d_1} \).

Proof. By Lemma 3.2 if \( \lim_{N \to \infty} \left| \Phi_N(s) \right| > 0 \) then the group
\[\{ h : \langle h, s \rangle \in 2\pi\mathbb{Z} \}\]
is sufficient and so \( \langle h, s \rangle \in 2\pi\mathbb{Z} \) for \( h \in \mathcal{H} \). \( \square \)

4 A local estimate

One of standard proofs of the Central Limit Theorem relies on the following bound (see e.g. [3, Section XVI.6]).

Lemma 4.1. (a) \( \lim_{N \to \infty} \Phi_N \left( V_N^{-1/2} u \right) - e^{-u^2/2} = 0 \) uniformly on compact sets.
(b) There are positive constants \( c, \delta_0 \) such that if \( |s| \leq \delta_0 \) then
\[\left| \Phi_N(s) \right| \leq e^{-c(|V_N s|)}.\]
\(^6\)In other words \( E(e^{iX_N}) \) vanishes for at most finitely many \( N \) and if \( E(e^{iX_N}) \neq 0 \) for \( N > N_0 \) then
\[\lim_{N \to N_0} \left| E (e^{i(S_N - S_{N_0})}) \right| > 0.\]
\(^7\)Here and below \( \mathbb{Z}^{d_1} + \mathbb{R}^{d-d_1} \) denotes the set of vectors whose first \( d_1 \) coordinates are integers.
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In this section we extend this result to a neighborhood of an arbitrary point (rather than 0). So fix an arbitrary \( \bar{s} \in \mathbb{R}^d \).

**Lemma 4.2.** (a) Suppose that

\[
\langle X_N, \bar{s} \rangle = b_N + \mathcal{Y}_N + \mathcal{Z}_N
\]

where \( \mathcal{Z}_N \in 2\pi \mathbb{Z} \), \( \mathcal{Y}_N \) is bounded, \( \mathbb{E}(\mathcal{Y}_N) = 0 \) and

\[
\sum_{j=1}^{N} V(\mathcal{Y}_j) \leq \varepsilon.
\]

Let \( a_N = \sum_{j=1}^{N} b_j \). Then for each \( L > 0 \) there exists a constant \( C \) such that for \( |u| \leq L \) we have

\[
|\Phi_N (\bar{s} + V_N^{-1/2} u) e^{-ia_N} - e^{-u^2/2}| \leq C \left( \sqrt{\varepsilon} + \frac{1}{\sqrt{N}} \right).
\]

(b) There are positive constants \( M, c, \delta_0 \) such that if \( |\Phi_N (\bar{s})| = e^{-A_N} \) for some \( \bar{s} \in \mathbb{R}^d \) then for \( |\Delta| \leq \delta_0 \) we have

\[
|\Phi_N (\bar{s} + \Delta)| \leq e^{MA_N - c(V_N\Delta, \Delta)}.
\]

**Proof.** We start with (b). Let \( \langle X_N, \bar{s} \rangle = b_N + \mathcal{Y}_N + \mathcal{Z}_N \) be a useful decomposition of \( \langle X_N, \bar{s} \rangle \). Then

\[
\phi_j (\bar{s} + \Delta) = e^{ib_j} \mathbb{E}(e^{i\mathcal{Y}_j + \mathcal{X}_j})
\]

where

\[
\mathcal{X}_j = \langle \Delta, \mathcal{X}_j \rangle.
\]

Next,

\[
e^{i\mathcal{Y}_j + \mathcal{X}_j} = 1 + i(\mathcal{Y}_j + \mathcal{X}_j) - \frac{1}{2} [\mathcal{Y}_j^2 + \mathcal{X}_j^2 + 2(\mathcal{X}_j \mathcal{Y}_j)] + \mathcal{O}(|\mathcal{X}_j + \mathcal{Y}_j|^3).
\]

Note that

\[
|\mathcal{X}_j + \mathcal{Y}_j|^3 \leq 8 \max (|\mathcal{X}_j|^3, |\mathcal{Y}_j|^3) = \mathcal{O} (|\Delta|^3 |\mathcal{X}_j|^3 + |\mathcal{Y}_j|^3).
\]

Thus (1.2) gives

\[
\mathbb{E} \left( e^{i\mathcal{Y}_j + \mathcal{X}_j} \right) = 1 - \frac{1}{2} \left[ \mathbb{E}(\mathcal{X}_j^2) + 2\mathbb{E}(\mathcal{X}_j \mathcal{Y}_j) \right] + \mathcal{O} (|\Delta|^3 + \mathbb{E}(\mathcal{Y}_j^2)).
\]

Denoting \( p_j = -\frac{1}{2} \left[ \mathbb{E}(\mathcal{X}_j^2) + 2\mathbb{E}(\mathcal{X}_j \mathcal{Y}_j) \right] \) and writing the remainder term as \( \mathcal{P}_j + i\mathcal{Q}_j \) where \( (\mathcal{P}_j, \mathcal{Q}_j) = \mathcal{O} (|\Delta|^3 + \mathbb{E}(\mathcal{Y}_j^2)) \) are real we get

\[
\mathbb{E} \left( e^{i\mathcal{Y}_j + \mathcal{X}_j} \right) = \sqrt{1 + 2p_j + 2\mathcal{P}_j + 2p_j^2 + \mathcal{P}_j^2 + \mathcal{Q}_j^2} = 1 + p_j + \mathcal{O}(p_j^2 + \mathcal{P}_j + \mathcal{Q}_j^2)
\]

where the last step uses that \( p_j^2 = \mathcal{O} (|\Delta|^3 + \mathbb{E}(\mathcal{Y}_j^2)). \)

Next, the inequality

\[
\ln(1 + x) \leq x
\]

gives

\[
\ln \mathbb{E} \left( e^{i\mathcal{Y}_j + \mathcal{X}_j} \right) \leq -\frac{1}{2} \left[ \mathbb{E}(\mathcal{X}_j^2) + 2\mathbb{E}(\mathcal{X}_j \mathcal{Y}_j) \right] + \mathcal{O} (|\Delta|^3 + \mathbb{E}(\mathcal{Y}_j^2)).
\]
Therefore
\[
\ln |\Phi_N(\bar{\mathbf{s}} + \Delta)| \leq - \sum_{j=1}^{N} \left[ \frac{1}{2} \left[ E(X_j^2) + 2E(X_j Y_j) \right] + O \left( \Delta^3 + E(Y_j^2) \right) \right].
\]

Denoting \( V_N = \sum_{j=1}^{N} E(X_j^2) \), \( W_N = \sum_{j=1}^{N} E(Y_j^2) \) and using Cauchy-Schwartz inequality and the fact that \(|\Delta|^2 N = O(V_N)\), due to (1.3), we get
\[
\ln |\Phi_N(\bar{\mathbf{s}} + \Delta)| \leq - \frac{V_N}{2} + O \left( |\Delta|V_N + W_N + \sqrt{W_N V_N} \right).
\]

Since for each \( R \)
\[
\sqrt{W_N V_N} \leq \frac{1}{2} \left[ \frac{V_N}{R} + R W_N \right]
\]
we see that for small \( \Delta \) we have
\[
\ln |\Phi_N(\bar{\mathbf{s}} + \Delta)| \leq - \frac{V_N}{4} + O \left( W_N \right). \tag{4.7}
\]

Next, Lemma 3.1 tells us that
\[
W_N \leq 14A_N \tag{4.8}
\]
so (4.3) follows from (4.7).

To prove part (a) we use (4.5) where \( \mathcal{Y}_N \) is from (4.1) and \( \mathcal{X}_N \) is given by (4.4). The fact that \( \mathcal{Y}_N \) was a part of a useful decomposition was used in part (b) only to get (4.8). Here we have a stronger bound (4.2) by the assumptions of part (a). In particular, (4.2) implies that \( E(Y_j^2) \leq \varepsilon \) so all terms in (4.5) are small. Accordingly we can use the Taylor expansion of \( \ln(1 + x) \) to conclude that
\[
\ln \phi_j(\bar{\mathbf{s}} + \Delta) - ib_j = - \frac{E(X_j^2)}{2} + O \left( E(X_j Y_j) + |\Delta|^3 + E(Y_j^2) \right).
\]

Hence
\[
\ln \Phi_N(\bar{\mathbf{s}} + \Delta) - ia_N + \frac{V_N}{2} = O \left( \sum_{j=1}^{N} E(X_j Y_j) \right) + O \left( N \Delta^3 \right) + O \left( \sum_{j=1}^{N} E(Y_j^2) \right).
\]

Using (4.2) to estimate the third term, Cauchy-Schwartz to estimate the first term and the fact that \(|\Delta|^2 N = O(V_N)\) to estimate the second term we get
\[
\ln \Phi_N(\bar{\mathbf{s}} + \Delta) - ia_N = - \frac{V_N}{2} + O \left( |\Delta|V_N + \varepsilon + \sqrt{\varepsilon} V_N \right)
\]
as stated. \( \square \)

**Corollary 4.3.** Suppose that
\[
(X_N, \bar{\mathbf{s}}) = b_N + \mathcal{Y}_N + \mathcal{Z}_N
\]
where \( \mathcal{Z}_N \in 2\pi \mathbb{Z} \), \( \mathcal{Y}_N \) is bounded, \( E(\mathcal{Y}_N) = 0 \) and \( \sum_N \mathcal{Y}_N \) converges to \( \tilde{S} \) almost surely. Then
\[
\lim_{N \to \infty} \Phi_N \left( \bar{\mathbf{s}} + V_N^{-1/2} u \right) e^{-ia_N} = e^{-u^2/2} E \left( e^{i\tilde{S}} \right)
\]
uniformly on compact sets.
Proof. Given $\varepsilon > 0$ let $N$ be such that
\[
\sum_{N=N+1}^{\infty} V_j \leq \varepsilon \quad \text{and} \quad |E(e^{i\sum_{j=1}^N Y_j}) - E(e^{i\bar{s}})| \leq \varepsilon.
\]
Then $\Phi_N(s + V^{-1/2}_N u) e^{-iuN} = \Phi_N(s, u)$ for all $u$ such that
\[
\left[ \Phi_N(s + V^{-1/2}_N u) e^{-iu} \right] E(e^{i(\sum_{j=1}^\infty (N_j - S_N - (a_j - a_N))}) \quad = \quad \Phi''_{N,N}(s, u).
\]
By the corollary, this happens in particular in the non arithmetic case.

Note that $\Phi''_{N,N}(s, u)$ depends only on $\sum_{j=N+1}^\infty X_j$ so
\[
\lim_{N \to \infty} \Phi''_{N,N}(s, u) = E(e^{i\sum_{j=1}^\infty Y_j}) = E(e^{i\bar{s}}) + O(\varepsilon).
\]
On the other hand Lemma 4.2(a) (applied to $\sum_{j=N+1}^\infty X_j$) gives
\[
|\Phi''_{N,N}(s, u) - e^{-u^2/2}| = O\left(\sqrt{\varepsilon} + (N - N)^{-1/2}\right).
\]
Since $\varepsilon$ can be chosen arbitrary small the result follows. \qed

5 Observables with compact Fourier transform

Here we prove that formulas of Theorem 1.2 are valid if $\hat{g}$ is continuous and has compact support. So we suppose that $\text{supp}(\hat{g}) \in [-K, K]^d$ for some $K$.

5.1 Non-arithmetic case

Assume first, that $\lim_{N \to 0} \Phi_N(s) = 0$ for all $s \neq 0$. By Corollary 3.3 this happens, in particular, in the non arithmetic case. Note that since $|\Phi_N|$ is monotone in $N$ the convergence is uniform on $[-K, K]^d \setminus (-\delta_0, \delta_0)^d$ for each $\delta_0 > 0$. We select $\delta_0$ so that the conditions of Lemma 4.1(b) and 4.2(b) are satisfied. Divide $[-K, K]^d$ into boxes $\{I_j\}$ of side $\delta_1$ where $\delta_1 \leq \delta_0/2d$ so that $I_0$ is the box centered at 0. Then
\[
E(g(S_N - z_N)) = \frac{1}{(2\pi)^d} \int_{[-K,K]^d} \hat{g}(-s) e^{-i(s,z_N)} \Phi_N(s) ds = \frac{1}{(2\pi)^d} \sum_{j \in I_j} \hat{g}(-s) e^{-i(s,z_N)} \Phi_N(s) ds.
\]

We claim that the main contribution comes from
\[
\int_{I_0} \hat{g}(-s) e^{-i(s,z_N)} \Phi_N(s) ds = \bar{J}_{L,N} + \bar{J}_{L,N}
\]
where $J_L$ denotes the integral over the set
\[
Q_L := \{s : V^{1/2}_N s \in [-L, L]^d\}
\]
and $\bar{J}_{L,N}$ denotes the integral over $I_0 - Q_L$. Making the change of variables $V^{1/2}_N s = u$ we get by Lemma 4.1(a)
\[
\det(V^{1/2}_N) J_{L,N} = \int_{[-L,L]^d} \hat{g}(-V^{-1/2}_N u) e^{-i(V^{-1/2}_N u,z_N)} \Phi_N(V^{-1/2}_N u) du = \hat{g}(0) \left[ \int_{[-L,L]^d} e^{-u^2/2 - i(u,z_N)} du \right] \left(1 + o_{N \to \infty}(1)\right) = \hat{g}(0) e^{-\bar{z}_N^2/2} \left(\frac{2\pi}{d/2} + o_{L \to \infty}(1) + o_{N \to \infty}(1)\right).
\]
where
\[ z_N = V_{1/2}^{-1} z_N. \]  
(5.1)

On the other hand, by Lemma 4.1(b)
\[ \det(V_{1/2}^1) \int_{[0,1]^d} e^{-cu^2} du = o_{L \to \infty}(1). \]

Since this holds for all \( L \) we can let \( L \to \infty \) to conclude that
\[ \lim_{N \to \infty} e^{z_N^2} \det(V_{1/2}^1) \int_{[0,1]^d} \hat{g}(-s) \Phi_N(s) ds = (2\pi)^{d/2} \hat{g}(0) = (2\pi)^{d/2} \int_{\mathbb{R}^d} g(x) dx. \]

It remains to show that the contributions of \( I_j \) with \( j \neq 0 \) are smaller.

**Lemma 5.1.** If \( \mathcal{J} \) be a cube of size \( \delta_1 \) such that \( \Phi_N(s) \) converges to 0 on \( \mathcal{J} \). Then
\[ \lim_{N \to \infty} \det(V_{1/2}^1) \int_{\mathcal{J}} |\Phi_N(s)| ds = 0. \]

**Proof.** Let
\[ e^{-\mathfrak{A}_N} = \max_{\mathcal{J}} |\Phi_N(s)| \text{ and } s_N = \arg \max_{\mathcal{J}} |\Phi_N(s)|. \]

Split \( \int_{\mathcal{J}} |\Phi_N(s)| ds = \tilde{J}_N + \bar{J}_N \) where \( \tilde{J}_N \) denotes the integral over the set
\[ \Omega_N := \{ c(V_N \Delta, \Delta) < 2\mathfrak{A}_N \} \text{ where } \Delta = s - s_N, \]
and \( \bar{J}_N \) denotes the integral over \( \mathcal{J} - \Omega_N \). Since \( \Omega_N \) is contained in a ball or radius \( O\left(\frac{\sqrt{\mathfrak{A}_N}}{N}\right) \) we have
\[ \det(V_{1/2}^1) \tilde{J}_N = O\left(\frac{\mathfrak{A}_N^{d/2} e^{-\mathfrak{A}_N}}{N} \right) \to 0 \]
since \( \mathfrak{A}_N \to \infty \) as \( N \to \infty \). On the other hand, by Lemma 4.2(b)
\[ |\bar{J}_N| \leq \text{Const} \int_{c(V_N \Delta, \Delta) \geq 2\mathfrak{A}_N} e^{-c(V_N \Delta, \Delta)} d\Delta \leq \frac{\text{Const}}{N^{d/2}} \int_{|u| > e\sqrt{\mathfrak{A}_N}} e^{-cu^2} du = O\left(\frac{\mathfrak{A}_N^{d/2}}{N^{d/2}} e^{-c\mathfrak{A}_N}\right). \]

Combining the estimates for \( \bar{J}_N \) and \( \bar{J}_N \) we obtain the lemma. \( \square \)

Lemma 5.1 shows that the main contribution to \( E(g(S_N)) \) comes from \( I_0 \) so that
\[ e^{z_N^2} \det(V_{1/2}^1) E(g(S_N - z_N)) \to \left(\frac{\sqrt{2\pi}}{2\pi}\right)^d \hat{g}(0) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} g(x) dx \]
as claimed.

### 5.2 Arithmetic case

Next, we consider the arithmetic case. Let \( \mathcal{H} \) be the minimal subgroup. After a linear change of variables we can assume that \( \mathcal{H} = \mathbb{Z}^{d_1} + \mathbb{R}^{d-d_1} \). Let \( X_N = b_N + Y_N + Z_N \) be the decomposition of \( X_N \) such that \( X_N,l = b_N,l + Y_N,l + Z_N,l \) is a useful decomposition.

\(^8\text{Here } \mathbb{Z}^{d_1} \text{ is the set of vectors whose first } d_1 \text{ coordinates are integers and the last } d - d_1 \text{ coordinates are zero and } \mathbb{R}^{d-d_1} \text{ is the set of vectors whose first } d_1 \text{ coordinates are zero.} \)
for \( l \leq d_1 \) and \( b_{N,(l)} = Y_{N,(l)} = 0 \) for \( l > d_1 \). Let \( \tilde{S}_N = (S_N - a_N) \mod \mathcal{H} \). Due to Lemma 3.2 we may (and will) assume that \( a_N \) is chosen so that

\[
\tilde{S}_N = \sum_{j=1}^{N} Y_j \mod \mathcal{H}.
\]

Lemma 5.1 shows that the main contribution to

\[
\det \left( V_N^{1/2} \right) \mathbb{E}(g(S_N - z_N))
\]

comes from small cubes \( I(s_m) \) centered at points \( s_m \) where

\[
\lim_{N \to \infty} |\Phi_N(s_m)| > 0.
\]

By Corollary 3.3 these points have form \( s_m = 2\pi m \) with \( m \in \mathbb{Z}^{d_1} \). The contribution of \( m = 0 \) is \( \frac{e^{-\frac{1}{2}T^2/2}}{(2\pi)^d/2} \mathbb{E}(0) \) as before.

For \( m \neq 0 \) note that \( e^{i(s_m \cdot z_N - a_N)} = 1 \). Let \( \Delta = s - s_m \). Then

\[
\frac{1}{(2\pi)^d} \int_{I(s_m)} \hat{g}(-s)e^{-i(s \cdot z_N)} \mathbb{E}(e^{i(s \cdot S_N)}) ds = \frac{1}{(2\pi)^d} \int_{I(s_m)} \hat{g}(-s)e^{-i(\Delta \cdot (z_N - a_N))} \mathbb{E}(e^{i(s \cdot (S_N - a_N))}) ds.
\]

Denoting

\[
Q_{m,L,N} = \{ s : V_N^{1/2} \Delta \in [-L, L]^d \}
\]

we decompose the last integral as \( \tilde{J}_{m,L,N} + \tilde{J}_{m,L,N} \) where \( \tilde{J}_{m,L,N} \) is the integral over \( Q_{m,L,N} \) and \( \tilde{J}_{m,L,N} \) is the integral over \( I(s_m) - Q_{m,L,N} \). By Corollary 4.3

\[
\frac{\det \left( V_N^{1/2} \right) J_{j,L,N}}{(2\pi)^d} = \frac{\hat{g}(-s_m)\mathbb{E}(e^{i(s_m \cdot S)}) + o_{N \to \infty}(1)}{(2\pi)^d} \int_{[-L, L]^d} e^{-u^2/2 - i(s \cdot u)} du = e^{-\frac{1}{2}T^2} \frac{\hat{g}(-s_m)\mathbb{E}(e^{i(s_m \cdot S)})}{(2\pi)^{d/2}} + o_{N \to \infty,L \to \infty}(1)
\]

where \( \tilde{z}_N \) is defined by (5.1). On the other hand by Lemma 4.2(b)

\[
\det \left( V_N^{1/2} \right) |\tilde{J}_{m,L,N}| \leq \text{Const} \int_{\mathbb{R}^d} e^{-cu^2} du = o_{L \to \infty}(1).
\]

Since this holds for all \( L \) we can let \( L \to \infty \) to conclude that

\[
\lim_{N \to \infty} e^{\frac{1}{2}T^2} \frac{\det \left( V_N^{1/2} \right)}{(2\pi)^d} \int_{U(s_j)} \hat{g}(-s) e^{-is\tilde{z}_N} \mathbb{E}(e^{is\tilde{S}_N}) ds
\]

\[
= \frac{\hat{g}(-s_m)\mathbb{E}(e^{i(s_m \cdot S)})}{(2\pi)^{d/2}} = \frac{\hat{C}_g(-s_m)}{(2\pi)^{d/2}}.
\]

Note that the argument above relies on Corollary 4.3, so it only works under the assumption that \( |\Phi_N(s_m)| \neq 0 \). However if \( \Phi_N(s_m) \to 0 \) then the limit in (5.2) is zero due to Lemma 5.1. Hence

\[
\lim_{N \to \infty} e^{\frac{1}{2}T^2} \det \left( V_N^{1/2} \right) \mathbb{E}(g(S_N - z_N)) = \sum_{m \in \mathbb{Z}^{d_1}} \frac{\hat{C}_g(2\pi m)}{(2\pi)^{d/2}}.
\]
Define the following function on $\mathbb{R}^{d_1}$

$$\mathcal{G}(x') = \int_{\mathbb{R}^{d-d_1}} (Cs_g)(x',x'')dx''.$$  \hspace{1cm} (5.3)

Then

$$\sum_{m \in \mathbb{Z}^{d_1}} \hat{Cs}_g(2\pi m) = \sum_{m \in \mathbb{Z}^{d_1}} \hat{G}(2\pi m) = \sum_{m \in \mathbb{Z}^{d_1}} G(m) = \int_{\mathcal{H}} C_s(g(h))d\lambda_H.$$  

Here the first equality holds since we have identified $m \in \mathbb{Z}^{d_1}$ with $(m,0) \in \mathbb{R}^d$, the second equality follows by the Poisson Summation Formula and the third equality follows by (5.3) and (1.5). This proves Theorem 1.2 for the functions with compactly supported Fourier transform.

### 6 Proof of the Local Limit Theorem

Here we finish the proof of Theorem 1.2.

We need the following \textit{a priori} estimate proven in Section 7.

**Lemma 6.1.** There is a constant $D$ such that for any cube $Q$ of unit size

$$\mathbb{P}(S_N \in Q) \leq \frac{D}{N^{d/2}}.$$  

To fix the notation we consider a non-arithmetic case, the argument in the arithmetic case is similar.

We note that it is sufficient to prove Theorem 1.2 for $g \in C^{d+1}_0(\mathbb{R}^d)$. Indeed if $g \in C_0(\mathbb{R}^d)$ and $\text{supp}(g) \subseteq [-K,K]^d$ then for each $\varepsilon > 0$ we can find $\tilde{g} \in C^{d+1}_0(\mathbb{R}^d)$ with $\text{supp}(\tilde{g}) \subseteq(-(K+1),(K+1)]^d$ and $\|g - \tilde{g}\|_{L^\infty} \leq \varepsilon$. Then

$$\det \left( V_N^{1/2} \right) E(g(S_N - z_N))$$  \hspace{1cm} (6.1) 

$$= \det \left( V_N^{1/2} \right) E(\tilde{g}(S_N - z_N)) + \det \left( V_N^{1/2} \right) O(\varepsilon) \mathbb{P} \left( S_N \in z_N + [-(K+1),(K+1)]^d \right).$$

The second term is $O(\varepsilon)$ by Lemma 6.1. So if Theorem 1.2 is valid for $C^{d+1}_0$ functions then

$$\det \left( V_N^{1/2} \right) E(g(S_N - z_N)) = e^{-z_N^2/2} \int_{[-(K+1),K+1]^d} \tilde{g}(x)dx + o_{N \to \infty}(1) + O(\varepsilon).$$

Since

$$\left| \int_{[-(K+1),K+1]^d} \tilde{g}(x)dx - \int_{[-(K+1),K+1]^d} g(x)dx \right| \leq \varepsilon(2(K+1))^d$$

the theorem holds for all continuous functions.

So let $g \in C^{d+1}_0(\mathbb{R}^d)$. Then for each $\varepsilon$ there is $\tilde{g}$ such that $\tilde{g}$ has compact support and $|g(x) - \tilde{g}(x)| \leq \frac{\varepsilon}{1 + |x|^{d+1}}$. Denoting by $Q_m$ the unit cube centered at $m$ we get

$$\det \left( V_N^{1/2} \right) |E(g(S_N - z_N)) - E(\tilde{g}(S_N - z_N))|$$  

$$\leq \sum_{m \in \mathbb{Z}^d} \frac{\varepsilon}{1 + |m|^{d+1}} \mathbb{P}(S_N - z_N \in Q_m) = O \left( \sum_{m \in \mathbb{Z}^d} \frac{\varepsilon}{1 + |m|^{d+1}} \right) = O(\varepsilon)$$

where the penultimate step uses Lemma 6.1. Also

$$\int_{\mathbb{R}^d} |g(x) - \tilde{g}(x)|dx \leq \varepsilon \int_{\mathbb{R}^d} dx \frac{dx}{1 + |x|^{d+1}} = O(\varepsilon).$$
LLT for sums of independent random vectors

Since
\[
\frac{E(g(S_N - z_N))}{u(z_N)} \to \int_{\mathbb{R}^d} \hat{g}(x)dx
\]
due to the results of Section 5, Theorem 1.2 holds on \(C_{d+1}^0(\mathbb{R}^d)\) and, hence, on \(C_0(\mathbb{R}^d)\).

7 Concentration inequality

The proof of Lemma 6.1 in arbitrary dimension is the same as the proof for \(d = 1\) given in [9, Section III.1] but we reproduce the proof here for completeness.

Proof of Lemma 6.1. It is enough to prove the claim for cubes of any fixed size \(\rho\) since the unit cube can be covered by a finite number of cubes of size \(\rho\). Let
\[
g(x) = \prod_{i=1}^{d} \left(1 - \frac{\cos(\hat{\delta} x_{(i)})}{\hat{\delta}^2 x_{(i)}^2}\right)
\]
where \(\hat{\delta} = \delta_0/d\) and \(\delta_0\) is the constant of Lemma 4.1(b). Then
\[
\hat{g}(s) = (\pi \hat{\delta})^d \prod_{i=1}^{d} \left(1 - \frac{|s_{(i)}|}{\hat{\delta}}\right) 1_{|s_{(i)}| \leq \hat{\delta}}.
\]
Hence for each \(a\)
\[
E(g(S_N - a)) = \int_{\mathbb{R}^d} \hat{g}(s)e^{i(s,a)}\Phi_N(s)ds \leq \int_{\max_{j} |s_{(j)}| < \delta_0} \hat{g}(s)|\Phi_N(s)|ds
\]
since \(\hat{g}\) is real and supported inside the cube of size \(2\delta_0\). Thus (1.3) and Lemma 4.1(b) imply that there is a constant \(D\) such that
\[
E(g(S_N - a)) \leq \frac{D}{N^{d/2}}
\]
On the other hand \(g(0) = \frac{1}{\rho^d}\) so there is a constant \(\rho\) such that \(g(x) > \frac{1}{4}\) on the cube of size \(\rho\) centered at \(0\). Hence if \(Q\) is a cube of size \(\rho\) centered at \(a\) then
\[
E(g(S_N - a)) \geq \frac{P(S_N \in Q)}{4^d}.
\]
Combining the last two displays we obtain the result. \(\square\)

8 Bounded random variables

Proof of Corollary 1.4. If \(\sum_j V(X_j)\) converges then \(S_N\) converges almost surely by Kolmogorov’s Three Series Theorem and so (1.7) holds.

Therefore we assume that \(\sum_j V(X_j)\) diverges. Fix a large \(A\) and let \(k_n\) be a sequence such that denoting \(X_n = \sum_{j=k_{n-1}+1}^{k_n} X_j\) we have
\[
\frac{1}{A} \leq V(X_n) \leq A.
\]
Since
\[
E(X_n^2) = (E(X_n^2))^2 + \sum_{j=k_{n-1}+1}^{k_n} V(X_j^2) \leq A^2 + \sum_{j=k_{n-1}+1}^{k_n} E(X_j^2) \leq A^2 + \kappa^2 A
\]
\{X_n\} satisfies (1.1), (1.2) and (1.3). Accordingly \(\sum_{j=1}^{k_n} X_j\) satisfy the conclusions of Corollary 1.3. Note that this holds for any sequence \(k_N\) such that

\[
\frac{1}{A} \leq \sum_{j=k_{n-1}+1}^{k_n} E(X_j^2) \leq A
\]  

(8.1)

for some \(A\) and all \(n\). We claim that, in fact, the conclusions of Corollary 1.3 are satisfied for our original sum \(S_N\). Indeed, take an arbitrary sequence satisfying (8.1). Suppose, to fix our notation, that \(S_{k_n}\) satisfies a non-arithmetic Local Limit Theorem, the arithmetic case is similar. We claim that (1.6) holds. Otherwise there exist sequences \(\{N_l\}\) \(\{z_l\}\) such that \(z_l/\sqrt{V_{N_l}} \to z\) and a continuous function \(g\) of compact support such that

\[
\lim_{l \to \infty} \left[ \sqrt{V_{N_l}} E(g(S_{N_l} - z_l)) \right]
\]

fails to exist giving a contradiction with the assumption that (1.6) fails. Hence (1.6) holds as claimed. \(\square\)

References

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