Electronic Journal of Probability

Electron. J. Probab. **21** (2016), no. 52, 1–17.
ISSN: 1083-6489 DOI: 10.1214/16-EJP4

Near-extreme eigenvalues in the beta-ensembles

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Abstract

For beta-ensembles with convex polynomial potentials, we prove a large deviation principle for the empirical spectral distribution seen from the rightmost particle. This modified spectral distribution was introduced by Perret and Schehr (J. Stat. Phys. 2014) to study the crowding near the maximal eigenvalue, in the case of the GUE. We prove also convergence of fluctuations.

Keywords: beta-ensembles; spectral distribution; top eigenvalue; large deviations.
AMS MSC 2010: 15B52; 60G57; 60F10.
Submitted to EJP on February 2, 2016, final version accepted on July 23, 2016.

1 Introduction

In random matrix models, the most popular statistics is the empirical spectral distribution (ESD). For a $N \times N$ matrix M_N with real eigenvalues $(\lambda_1, \cdots, \lambda_N)$, it is:

$$\mu^{(N)} := \frac{1}{N} \sum_{k=1}^{N} \delta_{\lambda_k} . \tag{1.1}$$

The first step in asymptotic study is to prove the convergence of $\mu^{(N)}$ and also of the so called integrated density of states $\mathbb{E}\mu^{(N)}$. The limiting distribution σ is most often compactly supported. A second step is to prove the convergence of the largest eigenvalue $\lambda_{(N)} = \max(\lambda_1, \cdots, \lambda_N)$ to the end of the support of σ. At a more precise level, it is sometimes possible to establish large deviations. In the so-called β-models, the density of eigenvalues is

$$\mathbb{P}_{V,\beta}^{N}(d\lambda_1, \cdots, d\lambda_N) = (Z_{V,\beta}^{N})^{-1} |\Delta(\lambda)|^{\beta} \exp\left(-\frac{N\beta}{2} \sum_{1}^{N} V(\lambda_k)\right) \prod_{1}^{N} d\lambda_k \tag{1.2}$$

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where $\Delta(\lambda)$ is the Vandermonde determinant. Under convenient assumptions on the potential V, the ESD satisfy the large deviation Principle (LDP) with speed $\beta N^2/2$ and good rate function

$$I_V(\mu) = -\Sigma(\mu) + \int V d\mu - c_V \quad \text{if} \quad \int |V| d\mu < \infty\,, \tag{1.3}$$

where Σ is the logarithmic entropy

$$\Sigma(\mu) = \int\int \ln(|x-y|) d\mu(x) d\mu(y)\,, \tag{1.4}$$

and

$$c_V = \inf_\nu -\Sigma(\nu) + \int V d\nu\,. \tag{1.5}$$

Moreover I_V achieves its minimum 0 at a unique probability measure μ_V which is compactly supported, and which is consequently the limit of $\mu^{(N)}$.

The most famous example is the Gaussian Unitary Ensemble which corresponds to $V(x) = x^2/2$ and $\beta = 2$. The limiting distribution μ_V is then the semicircle distribution :

$$\mu_{\mathrm{SC}}(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\ 1_{[-2,2]}(x)\, dx\,, \tag{1.6}$$

and the result of large deviations is due to [2], with $c_V = 3/4$.

Moreover under appropriate conditions again, the support of μ_V is an interval $[a_V, b_V]$ and the maximal eigenvalue $\lambda_{(N)}$ converges to b_V.

To analyze the "crowding" phenomenon near the largest eigenvalue, Perret and Schehr proposed in [10] and [11] to study the empirical measure :

$$\mu_N := \frac{1}{N-1}\sum_{k=1}^{N-1} \delta_{\lambda_{(N)}-\lambda_{(k)}} \in M_1(\mathbb{R}^+)\,, \tag{1.7}$$

where $\lambda_{(1)} < \lambda_{(2)} < \cdots < \lambda_{(N)}$ are the eigenvalues of M_N ranked increasingly. They considered the Gaussian case with the Dyson values $\beta = 1, 2, 4$ and made a complete study of $\mathbb{E}\mu_N$, in the limit $N \to \infty$ both in the bulk and at the edge.

In the present paper, we consider more general potentials V, actually convex polynomials of even degree. We first prove that μ_N converges in probability to the pushforward ν_V of μ_V by the mapping $x \mapsto b_V - x$. Then we prove that the family of distributions of $(\mu_N)_N$ satisfies the LDP with speed N^2 and a “new" rate function which we call I_V^{DOS}, referring to the name “Density of States near the maximum" given by Perret and Schehr to $\mathbb{E}\mu_N$. There are two striking facts. The first one is that the LDP is obtained for a Wasserstein topology (and not for the usual weak topology). This ensures in particular that the rate function is lower semicontinuous. The second one is that the LDP is weak i.e. we do not have a large deviation upperbound for closed sets but only for compact sets. This implies that we could not deduce the convergence to the limit from the LDP as usual. In the Gaussian case, we have $V(x) = x^2/2$ and

$$I_V^{\mathrm{DOS}}(\nu) = -\Sigma(\nu) + \frac{1}{2}\operatorname{Var}\nu - \frac{3}{4}\,, \tag{1.8}$$

where for $\nu \in M_1(\mathbb{R}^+)$ such that $\int x d\nu(x) < \infty$, we define

$$\operatorname{Var}\nu = \int x^2 d\nu(x) - \left(\int x d\nu(x)\right)^2 \in [0,\infty]\,. \tag{1.9}$$

Section 2 is devoted to LDPs: Proposition 2.7 and Corollary 2.8 study the pair $(\lambda^{(N)}, \mu_N)$, which prepares the main result, the LDP for $(\mu_N)_N$ in Theorem 2.9. The proofs are in Section 3. To complete the description of the asymptotic behavior of μ_N, we prove also the convergence of fluctuations in Section 4. Finally, in the Appendix (Section 5), we gather some properties of the Wasserstein distance on probability measures.

2 Assumptions and main result

To begin with, let us recall the definition of the Wasserstein distance.

Definition 2.1. *Let $p \in [1, \infty[$, and $M_1^p(\mathbb{R}) = \{\nu \in M_1(\mathbb{R}), \int |x|^p d\nu(x) < \infty\}$. For two probabilities μ and ν in $M_1^p(\mathbb{R})$, the Wasserstein distance of order p is defined by*

$$d_{W_p}(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}} |x-y|^p d\pi(x,y) \right)^{1/p} \tag{2.1}$$

where $\Pi(\mu, \nu)$ is the set of probabilities on $\mathbb{R}^2$ with first marginal μ and second marginal ν.

Besides, we denote by d the usual distance for the weak topology, given by Lipschitz bounded functions. It is known that

$$d \leq d_{W_1} \leq d_{W_q} \text{ for } q \geq 1 \,. \tag{2.2}$$

We assume that

Assumption 2.2. *V is a convex polynomial of even degree $p \geq 2$.*

This assumption guarantees that μ_V is unique, with support $[a_V, b_V]$. Moreover we have:

Theorem 2.3. *The sequence of distributions of $(\mu^{(N)})_N$ satisfies a large deviation principle in $(M_1(\mathbb{R}), d)$, with speed $\beta N^2/2$ with good rate function I_V given by* (1.3).

This result is Th. 2.6.1 in [1]. As we will prove in the following section, it can be improved:

Corollary 2.4. *The LDP still holds in M_1^q, endowed with the distance d_{W_q} for any $q < p$.*

For the largest eigenvalue, we have:

Proposition 2.5. *Under Assumption 2.2,*

1. *$\lambda_{(N)}$ converges in probability to b_V,*
2. *the sequence of distributions of $(\lambda_{(N)})_N$ satisfies a large deviation principle with speed $\beta N/2$, with a good rate function J_V^+ satisfying $J_V^+(x) = +\infty$ for $x < b_V$, i.e.*
$$\lim_N \frac{2}{\beta N} \ln \mathbb{P}_{V,\beta}(\lambda_{(N)} > x) = -J_V^+(x) \quad , \quad x > b_V \tag{2.3}$$
with $J_V^+(b_V) = 0$,
3. *the sequence of distributions of $(\lambda_{(N)})_N$ satisfies a large deviation principle with speed $\beta N^2/2$, with a rate function J_V^-, on the left of b_V i.e.*
$$\lim_N \frac{2}{\beta N^2} \ln \mathbb{P}_{V,\beta}(\lambda_{(N)} \leq x) = -J_V^-(x) := - \inf_{\mu : \mathrm{Supp}\, \mu \subset (-\infty, x]} I_V(\mu) \quad , \quad x < b_V \,. \tag{2.4}$$

Points 1 and 2 are in [1] Prop. 2.6.6, but are more readable in [3] Prop. 2.1. For Point 3, see [9] Rem. 2.3, [7] Sect. 4.2, [5] and [14]).

We are now interested in the behavior of μ_N. First, we have the following convergence result:

Proposition 2.6. *We denote by $\tau_c\mu$ the probability defined by*

$$\int f(x) \tau_c \mu(dx) = \int f(c-x) \mu(dx) \,.$$

Then, as $N \to \infty$, μ_N converges weakly in probability to the probability measure :

$$\nu_V := \tau_{b_V} \mu_V \,. \tag{2.5}$$

Our main result rules the large deviations of the pair $(\lambda_{(N)}, \mu_N)$. We equip $M_1^p(\mathbb{R}^+)$ with d_{W_p} and denote by $B(\mu;\delta)$ the ball around μ of radius δ.

We define

$$\mathcal{I}_V(c,\nu) = I_V(\tau_c \nu)\,, \tag{2.6}$$

which, since Σ is invariant by the transformation τ_c, is also

$$\mathcal{I}_V(c,\nu) = -\Sigma(\nu) + \int V d\tau_c \nu - c_V\,. \tag{2.7}$$

Proposition 2.7. *We have*

1. *For any $c \in \mathbb{R}$ and $\mu \in M_1^p(\mathbb{R}^+)$,*

$$\lim_{\delta \searrow 0, \delta' \searrow 0} \liminf_{N\to\infty} \frac{2}{\beta N^2} \ln(\mathbb{P}^N_{V,\beta}(\lambda_{(N)} \in [c-\delta', c+\delta'], \mu_N \in B_{W_p}(\mu,\delta))) \geq -\mathcal{I}_V(c,\mu)\,. \tag{2.8}$$

2. *For any closed set $F \subset \mathbb{R}$ and $\mu \in M_1^p(\mathbb{R}^+)$,*

$$\lim_{\delta \searrow 0} \limsup_{N\to\infty} \frac{2}{\beta N^2} \ln(\mathbb{P}^N_{V,\beta}(\lambda_{(N)} \in F, \mu_N \in B_{W_p}(\mu,\delta))) \leq -\inf_{c\in F} \mathcal{I}_V(c,\mu)\,. \tag{2.9}$$

Corollary 2.8. *The sequence of distributions of $(\lambda_{(N)}, \mu_N)_N$ satisfies a weak LDP on $\mathbb{R} \times M_1^p(\mathbb{R}^+)$ equipped with the product topology, at speed $\beta N^2/2$ with rate function $\mathcal{I}_V$.*

From these results, we may deduce on the one hand a weak LDP for the random measure μ_N, and on the other hand a conditional LDP for μ_N, knowing $\lambda_{(N)}$.

Theorem 2.9. *The sequence of distributions of $(\mu_N)_N$ satisfies a weak LDP in $(M_1^p(\mathbb{R}^+), d_{W_p})$ at speed $\beta N^2/2$ with rate function*

$$I_V^{\mathrm{DOS}}(\nu) := \inf_{c\in(-\infty,\infty)} \mathcal{I}_V(c,\nu) = -\Sigma(\nu) + G_V(\nu) - c_V\,, \tag{2.10}$$

with

$$G_V(\nu) := \inf_{c\in(-\infty,\infty)} \int V d\tau_c \nu\,. \tag{2.11}$$

The properties of I_V^{DOS} and G_V are ruled by the following lemma:

Lemma 2.10. *Let $p-1 \leq q \leq p$.*

1. *The infimum in* (2.11) *is reached at a unique point which we call $\kappa_V(\nu)$.*
2. *$\nu \mapsto \kappa_V(\nu)$ is continuous for d_{W_q}.*
3. *$\nu \mapsto G_V(\nu) = \int V(\kappa_V(\nu) - x) d\nu(x)$ is lower semicontinuous for d_{W_q}.*
4. *I_V^{DOS} is well defined on $M_1^q(\mathbb{R})$ with values in $[0,+\infty]$ and lower semicontinuous for the W_q topology.*
5. *For $b \geq b_V$,*
$$I_V^{\mathrm{DOS}}(\tau_b \mu_V) = 0\,.$$

It follows from property 5 in Lemma 2.10 that I_V^{DOS} is not a good rate function since the level sets are not compact. Then, (μ_N) is not exponentially tight in scale N^2 (see [6, Lemma 1.2.18]) and we do not know if the large deviations upper bound in Theorem 2.9 is true for closed sets. Nevertheless, we can prove exponential tightness in a weaker topology, conditionally that $\lambda_{(N)}$ remains bounded, which leads to:

Proposition 2.11. *Let $p-1 \leq q < p$.*
For any closed set $\mathbf{F}$ of $M_1^q(\mathbb{R}^+)$ and any $C > b_V$, we have

$$\limsup_{N\to\infty} \frac{2}{\beta N^2} \ln(\mathbb{P}^N_{V,\beta}(\mu_N \in \mathbf{F} \,|\lambda_{(N)} \in [-C,C])) \leq -\inf_{\mu\in\mathbf{F}} I_V^{\mathrm{DOS}}(\mu)\,. \tag{2.12}$$

The conditional large deviations are ruled by the following theorem.

Theorem 2.12. *Let $p-1 \leq q < p$. Let $\mathbf{F}$ be closed and $\mathbf{G}$ be open in $M_1^q(\mathbb{R}^+)$.*

1. *If $c > b_V$, we have*

$$\limsup_N \frac{2}{\beta N^2} \ln \mathbb{P}^N_{V,\beta}\left(\mu_N \in \mathbf{F} \mid \lambda_{(N)} \in [c, c+\delta]\right) \leq -\inf_{\mu\in\mathbf{F}} \mathcal{I}_V^\delta(c,\mu) \tag{2.13}$$

$$\liminf_N \frac{2}{\beta N^2} \ln \mathbb{P}^N_{V,\beta}\left(\mu_N \in \mathbf{G} \mid \lambda_{(N)} \in [c, c+\delta]\right) \geq -\inf_{\mu\in\mathbf{G}} \mathcal{I}_V^\delta(c,\mu)\,, \tag{2.14}$$

where $\mathcal{I}_V^\delta(c,\mu) := \inf_{a\in[c,c+\delta]} \mathcal{I}_V(a,\mu)$ satisfies

$$\lim_{\delta\to 0} \mathcal{I}_V^\delta(c,\mu) = \mathcal{I}_V(c,\mu)\,. \tag{2.15}$$

2. *If $c < b_V$, set*

$$\begin{aligned} \mathcal{J}_V(c,\mu) \quad &:= \quad \mathcal{I}_V(c,\mu) - \inf_{\nu\in M_1^p(\mathbb{R}^+)} \mathcal{I}_V(c,\nu) \\ &= \quad \mathcal{I}_V(c,\mu) - J_V^-(c), \end{aligned} \tag{2.16}$$

(see Remark 2.13).

Then

$$\limsup_N \frac{2}{\beta N^2} \ln \mathbb{P}^N_{V,\beta}\left(\mu_N \in \mathbf{F} \mid \lambda_{(N)} \in [c-\delta, c]\right) \leq -\inf_{\mu\in\mathbf{F}} \mathcal{J}_V^\delta(c,\mu) \tag{2.17}$$

$$\liminf_N \frac{2}{\beta N^2} \ln \mathbb{P}^N_{V,\beta}\left(\mu_N \in \mathbf{G} \mid \lambda_{(N)} \in [c-\delta, c]\right) \geq -\inf_{\mu\in\mathbf{G}} \mathcal{J}_V^\delta(c,\mu)\,, \tag{2.18}$$

where $\mathcal{J}_V^\delta(c,\mu) := \inf_{a\in[c-\delta,c]} \mathcal{I}_V(a,\mu) - J_V^-(c)$ satisfies

$$\lim_{\delta\to 0} \mathcal{J}_V^\delta(c,\mu) = \mathcal{J}_V(c,\mu)\,. \tag{2.19}$$

Remark 2.13. Let us compute the projection on $\mathbb{R}$ of the rate function (2.7) i.e.

$$J_V(c) := \inf_{\mu\in M_1^p(\mathbb{R}^+)} \mathcal{I}_V(c,\mu)\,.$$

We have, by invariance

$$J_V(c) = \inf_{\nu:\exists\mu\in M_1^p(\mathbb{R}^+):\mu=\tau_c\nu} -\Sigma(\nu) + \int V d\nu - c_V\,.$$

Recall that the support of μ_V is assumed to be $[a_V, b_V]$. Then, either $c \geq b_V$ and we can take $\nu = \mu_V$, $J_V(c) = 0$, or $c < b_V$ and we get

$$J_V(c) = \inf_{\nu:\mathrm{Supp}\,\nu\subset(-\infty,c)} I_V(\nu) := J_V^-(c)\,.$$

We recover Point 3 of Prop. 2.5.

As noticed in Prop. 2.1 and Rem. 2.3 in [9] there is a unique μ such that $J_V(c) = I_V(\mu)$, let us call it μ^c. In the Gaussian case, its explicit expression is in [5] (up to some notational changes).

Remark 2.14. Let us notice that for fixed c, $\mathcal{I}_V(c,\cdot)$ and $\mathcal{J}_V(c,\cdot)$ may be seen as conditional rate functions. From (2.6), we conclude that

$$\mathcal{I}_V(c,\mu) = 0 \ \text{ iff } \ \mu = \tau_c \mu_V\,,$$

whereas from (2.16) and the above remark, we conclude that

$$\mathcal{J}_V(c,\mu) = 0 \ \text{ iff } \ \mu = \tau_c \mu^c\,.$$

Remark 2.15. Let us now give some additional comments relative to the Gaussian case: $V(x) = x^2/2$. In this case, $\kappa_V(\mu)$ is the mean of μ i.e. $m(\mu) = \int x d\mu(x)$ and $G_V(\mu)$ is its variance. Therefore,

$$I_V^{\mathrm{DOS}}(\mu) = -\Sigma(\mu) + \frac{1}{2}\operatorname{Var}(\mu) - \frac{3}{4}\,,$$

and

$$d\nu_V(x) = \frac{\sqrt{4x-x^2}}{2\pi} 1_{[0,4]}(x)\,dx\,.$$

As we have seen above, the rate function I_V^{DOS} is zero for probabilities of the form $\mu = \tau_b \mu_{\mathrm{SC}}$, $b \geq 2$ and thus is not a convex rate function. Notice that this particular functional is semicontinuous not only for d_{W_1} but also for the weak topology. It is a consequence of the semicontinuity of Var. To prove this fact, use the representation

$$\operatorname{Var}(\mu) = \frac{1}{2}\,\mathbb{E}(X-Y)^2$$

where X and Y are two real random variables independent and μ distributed, and then apply Fatou's Lemma.

A nice consequence is that the weak LDP satisfied by μ_N holds also in the weak topology (see p. 127 Remark (b) in [6]).

3 Proofs

In this section, we begin with the proofs of the easiest results and we end with the proof of the main result.

3.1 Proof of Corollary 2.4

The set

$$K_M = \{\mu \in M_1(\mathbb{R}), \int |V(x)| d\mu(x) \leq M\}$$

is compact for the weak topology and is used to prove the exponential tightness for $\mu^{(N)}$ in [1] p. 78. Actually K_M is also a compact set for the q-Wasserstein distance for $q < p$ (see the Appendix). It is then enough to apply Theorem 4.2.4 of [6].

3.2 Proof of Proposition 2.6

Let f a bounded Lipschitz function with Lipschitz constant and uniform bound less than 1. Then,

$$\begin{aligned}
&|\frac{1}{N-1}\sum_{1}^{N-1} f(\lambda_{(N)} - \lambda_{(k)}) - \frac{1}{N}\sum_{1}^{N} f(b_V - \lambda_{(k)})| \\
&= \quad |\frac{1}{N-1}\sum_{1}^{N-1} (f((\lambda_{(N)} - \lambda_{(k)}) - f(b_V - \lambda_{(k)})) +
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{N(N-1)}\sum_{1}^{N-1}(f(b_V-\lambda_{(k)})-f(b_V-\lambda_{(N)}))| \\
\leq \quad & |\lambda_{(N)}-b_V|+\frac{2}{N}\,.
\end{aligned}$$

Therefore $d(\mu_N,\tau_{b_V}\mu^{(N)})\leq|\lambda_{(N)}-b_V|+\frac{2}{N}$.

From the convergence of $\mu^{(N)}$ to μ_V, and $\lambda_{(N)}$ to b_V, we deduce that μ_N converges to $\tau_{b_V}\mu_V$. □

3.3 Proof of Lemma 2.10

1) Notice that the uniqueness of κ_V comes from the convexity of V.
2) Let $f(c)=\int V(c-x)d\nu(x)$. Then $\kappa_V(\nu)$ is the solution of

$$f'(c)=\int V'(c-x)d\nu(x)=0\,.$$

Since the polynomial V' is of degree $p-1$ and, for d_{W_q}, the functions $\nu\mapsto\int x^k d\nu(x)$ are continuous for $k\leq p-1$, this implies the continuity of $\nu\mapsto\kappa_V(\nu)$.
3) Denote by $m_k(\nu)$ the kth moment of ν. We can write $\int V(\kappa_V(\nu)-x)d\nu(x)$ as

$$a_p m_p(\nu)+F(m_1(\nu),\ldots,m_{p-1}(\nu),\kappa_V(\nu))$$

where F is a polynomial function and $a_p>0$. The function $m_p(\nu)$ is lower semicontinuous as the supremum of the continuous functions $\int(|x|^p\wedge M)d\nu(x)$. The functions $\kappa_V(\nu)$ and $m_k(\nu)$, $k\leq p-1$ are continuous in ν for d_{W_q}. Therefore, G_V is lower semicontinuous.
4) We refer to [1] for the same properties of I_V, using e.g. for the positivity that $I_V^{\mathrm{DOS}}(\mu)=I_V(\tau_{\kappa_V(\mu)}(\mu))$.

From [2], $-\Sigma(\mu)$ is lower semicontinous for the topology of the weak convergence, and therefore is lower semicontinuous for the stronger topology W_q.

At last, G_V is lower semicontinuous from the 3).
5) First notice that $\tau_b\mu_V$ has a support in $\mathbb{R}^+$ iff $b\geq b_V$. From (2.10) and (2.6) we have then

$$I_V^{\mathrm{DOS}}(\tau_b\mu_V)=\inf_c I_V(\tau_c\tau_b\mu_V)\,,$$

and this infimum is 0, reached at $c=b$ since τ_b is an involution.

We could have also argued that, since the sequence μ_N converges to $\tau_{b_V}\mu_V$ and I_V^{DOS} is the rate function in the LDP for μ_N, this insures that $I_V^{\mathrm{DOS}}(\tau_{b_V}\mu_V)=0$.

3.4 Proof of Theorem 2.9

It is enough to take $c=\kappa_V(\mu)$ in the lower bound (2.8) in Proposition 2.7 to obtain:

$$\lim_{\delta\searrow 0}\liminf_{N\to\infty}\frac{2}{\beta N^2}\ln(\mathbb{P}^N_{V,\beta}(\mu_N\in B_{W_p}(\mu,\delta)))\geq -I_V^{\mathrm{DOS}}(\mu)\,,$$

which implies the lower bound for open sets.

For the upperbound, we take $F=\mathbb{R}$ in (2.9). □

3.5 Proof of Proposition 2.7

Since the potential V is assumed to be a convex polynomial, it is lower bounded by $V_{\min}$. Changing V into $V-V_{\min}$ induces a change of c_V into $c_V-V_{\min}$, so in the above proofs we may and shall assume $V\geq 0$.

If $\mathbb{P}^N_{V,\beta}$ is the distribution defined in (1.2), we denote by $\bar{Q}_N$ the non normalized measure $\bar{Q}^N_{V,\beta} = Z^N_{V,\beta}\mathbb{P}^N_{V,\beta}$. From [1, p.81], we know that:

$$\lim_{N\to\infty} \frac{2}{\beta N^2} \ln(Z^N_{V,\beta}) = -c_V \, .$$

Therefore, it is enough to prove the weak LDP for the measure $\bar{Q}_N$.

The proof will consist in two parts: the lower bound and the upper bound.

3.5.1 Proof of the lower bound

We need an approximation lemma whose second statement is an easy consequence of Lemma 3.3 in [2] (see also [1] p. 79). Indeed, the statement is given there for the distance of the weak convergence. Since the measure ν (and therefore its approximation) has compact support, the same is true for the Wasserstein distance.

Lemma 3.1. *i) Let $\mu \in M_1^p(\mathbb{R}^+)$, for any $\delta > 0$, there exists a compactly supported probability ν such that $d_{W_p}(\mu,\nu) \le \delta$.*

ii) Let ν be probability on a compact set in $\mathbb{R}^+$, with no atoms. Let $(x^{i,N})$ the sequence of real numbers defined by

$$x_{1,N} = \inf\{x \ge 0 |\, \nu([0,x]) \ge \frac{1}{N}\}\, ,$$

$$x^{i+1,N} = \inf\{x \ge x^{i,N} |\, \nu(]x^{i+1,N}, x]) \ge \frac{1}{N},\ 1 \le i \le N-2\}\, .$$

Then,

$$x^{1,N} < x^{N-2,N} < \ldots < x^{N-1,N}\, ,$$

and for any $\delta > 0$ and N large enough,

$$d_{W_p}\left(\nu, \frac{1}{N-1}\sum_{i=1}^{N-1} \delta_{x^{i,N}}\right) \le \delta.$$

Proof of *i)*
For $M > 0$ and $\mu \in M_1^p(\mathbb{R}^+)$, we denote by $\widetilde{\mu}^M$ the compactly supported probability defined by $d\widetilde{\mu}^M = (\mu([-M,M]))^{-1} 1_{\{|x|\le M\}} d\mu$.

It is easy to see that $\widetilde{\mu}^M$ converges weakly to μ as M tends to ∞. Moreover, by dominated convergence theorem,

$$\frac{1}{\mu([-M,M])} \int |x|^p 1_{\{|x|\le M\}} d\mu(x) \to_{M\to\infty} \int |x|^p d\mu(x).$$

This implies the convergence in W_p distance (see Proposition 5.3, ii)). □

To prove the lower bound (2.8), we will repeat almost verbatim the proof of [1] pp. 79-81, but follow step by step the rôle played by $\lambda_{(N)}$. We assume that $\mathcal{I}(c,\mu) < \infty$ so that μ has no atoms. We can also assume that μ is compactly supported, by considering $\widetilde{\mu}^M$ defined in Lemma 3.1. One can check that $\mathcal{I}(c,\widetilde{\mu}^M) \to \mathcal{I}(c,\mu)$.

Recall that

$$\mu_N = \frac{1}{N-1}\sum_{k=1}^{N-1} \delta_{\lambda_{(N)} - \lambda_{(k)}}$$

where the $\lambda_{(k)}$ are the increasing sequence of eigenvalues.

Then, if $C = [c - \delta', c + \delta']$ and $B := B_{W_p}(\mu, \delta)$,

$$\begin{aligned}
&\bar{Q}_N(\lambda_{(N)} \in C, \mu_N \in B) \\
&= \quad N! \int_{\Delta_N \cap \{\frac{1}{N-1}\sum_{k=1}^{N-1} \delta_{\lambda_N - \lambda_k} \in B, \lambda_N \in C\}} \widehat{L}_N(\lambda_1, \dots \lambda_N) d\lambda_1 \dots d\lambda_N
\end{aligned} \tag{3.1}$$

where

$$\Delta_N = \{\lambda_1 < \lambda_2 < \dots < \lambda_N\}, \tag{3.2}$$

and

$$\begin{aligned}
\widehat{L}_N(\lambda) &= \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp(-\frac{\beta N}{2} \sum_{i=1}^N V(\lambda_i)) \qquad (3.3)\\
&= \prod_{i<j<N-1} |\lambda'_i - \lambda'_j|^\beta \prod_{i<N} |\lambda'_i|^\beta \exp(-\frac{\beta N}{2} (\sum_{i=1}^{N-1} V(\lambda_N - \lambda'_i) + V(\lambda_N))) \\
&:= L_N((\lambda'_i)_{i<N}, \lambda_N)
\end{aligned}$$

where $\{\lambda'_i = \lambda_N - \lambda_i, i < N\}$ is a decreasing family of positive numbers. Since the density $L_N((\lambda'_i)_{i<N}, \lambda_N)$ is symmetric in (λ'_i), we can write:

$$\begin{aligned}
&\bar{Q}_N(\lambda_{(N)} \in C, \mu_N \in B) \\
&= \quad N \int_{\mathbb{R}_+^{N-1} \times C \cap \{\frac{1}{N-1}\sum_{k=1}^{N-1} \delta_{\lambda'_k} \in B\}} L_N(\lambda'_1, \dots, \lambda'_{N-1}, \lambda_N)) d\lambda'_1 \dots d\lambda'_{N-1} d\lambda_N .
\end{aligned}$$

From Lemma 3.1, for $N \geq N_\delta$,

$$\left\{(\lambda'_k)_k : \ |\lambda'_k - x^{k,N}| \leq \frac{\delta}{2}, \forall k < N\right\} \subset \left\{(\lambda'_k)_k : \ \frac{1}{N-1}\sum_{k=1}^{N-1} \delta_{\lambda'_k} \in B\right\},$$

and we can write L_N as

$$\begin{aligned}
L_N \quad &= \quad \prod_{i<j<N} |(\lambda'_i - x^{i,N}) - (\lambda'_j - x^{j,N}) + x^{i,N} - x^{j,N}|^\beta \prod_{i<N} |\lambda'_i|^\beta \\
&\times \quad \exp(-\frac{\beta N}{2}(\sum_{i=1}^{N-1} V(\lambda_N - x^{i,N} - (\lambda'_i - x^{i,N})) + V(\lambda_N - c + c))) .
\end{aligned}$$

Set $y_i = \lambda'_i - x^{i,N}$, $i < N$ and $y_N = \lambda_N - c$. Then,

$$\begin{aligned}
&\bar{Q}(\mu_N \in B) \\
&\geq \quad N \int_{\Delta_N(\delta)} \prod_{i<j<N} |y_i - y_j + x^{i,N} - x^{j,N}|^\beta \prod_{i \leq N-1} |y_i + x^{i,N}|^\beta \\
&\qquad \exp(-\frac{\beta N}{2}(\sum_{i=1}^{N-1}(V((y_N + c) - (y_i + x^{i,N})) + V(y_N + c))) \prod_{i \leq N} dy_i
\end{aligned}$$

where

$$\begin{aligned}
\Delta'_N \quad &= \quad \{y_1 < y_2 < \dots < y_{N-1}\} \\
\Delta_N(\delta) \quad &= \quad \{(y_1, \dots, y_N) : (y_i)_{i<N} \in [0, \delta/2]^{N-1}, y_N \in [-\delta, \delta]\} \cap \Delta'_N .
\end{aligned}$$

Since on Δ'_N, the (y_i) and the $(x^{i,N})$ form both increasing sequences, we have the lower bound:

$$|y_i - y_j + x^{i,N} - x^{j,N}| \geq \sup\{|x^{i,N} - x^{j,N}|, |y_i - y_j|\}$$

and we use the same minoration as in [1] for the term

$$
\begin{aligned}
A &:= \prod_{i<j<N} |y_i - y_j + x^{i,N} - x^{j,N}|^{\beta} \\
&\geq \prod_{i+1<j<N} |x^{i,N} - x^{j,N}|^{\beta} \prod_{i<N-1} |x^{i,N} - x^{i+1,N}|^{\beta/2} \prod_{i<N-1} |y_i - y_{i+1}|^{\beta/2}.
\end{aligned}
$$

For the second term, we use, since the y_i and $x^{i,N}$ are positive,

$$
\prod_{i<N-1} |y_i + x^{i,N}|^{\beta} \geq \prod_{i<N-1} |y_i|^{\beta}.
$$

We get:

$$
\bar{Q}(\lambda_{(N)} \in C, \mu_N \in B) \geq P_{N,1} P_{N,2},
$$

where

$$
\begin{aligned}
P_{N,1} &= N \exp -\frac{\beta N}{2} \left(\sum_{i=1}^{N-1} V(c - x^{i,N}) + V(c) \right) \\
&\quad \times \prod_{i+1<j<N} |x^{i,N} - x^{j,N}|^{\beta} \prod |x^{i,N} - x^{i+1,N}|^{\beta/2}
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
P_{N,2} &= \int_{\Delta_N(\delta)} \prod_{i<N-1} |y_i - y_{i+1}|^{\beta/2} |y_i|^{\beta} \\
&\quad \times \exp -\frac{\beta N}{2} \sum_{i=1}^{N-1} V(c - x^{i,N} - y_i + y_N) - V(c - x^{i,N}) \\
&\quad \times \exp -\frac{\beta N}{2} \left[V(y_N + c) - V(c) \right] \prod_{i \leq N} dy_i.
\end{aligned}
$$

Since we have assumed that μ is compactly supported, the sets $\{x^{i,N}, 1 \leq i \leq N-1\}$ are uniformly bounded and by continuity of V,

$$
\lim_{\delta \to 0} \sup_N \sup_{1 \leq i \leq N} \sup_{|x| \leq \delta} |V(c - x^{i,N} + x) - V(c - x^{i,N}))| = 0,
$$

and

$$
\lim_{\delta \to 0} \sup_{|x| \leq \delta} |V(c + x) - V(c)| = 0.
$$

Moreover, writing $u_1 = y_1, u_{i+1} = y_{i+1} - y_i$, with $\delta'' = \min(\delta/2, \delta')$

$$
\begin{aligned}
&\int_{\{(y_i)_{i<N} \in [0, \frac{\delta}{2}]^{N-1}\} \cap \Delta'_N \cap \{y_N \in [-\delta, \delta]\}} \prod_{i<N-1} |y_i - y_{i+1}|^{\beta/2} |y_i|^{\beta} \prod_{i \leq N} dy_i \\
&\geq \int_{\{(u_i)_{i<N} \in [0, \frac{\delta''}{N}]^N} u_1^{\beta} (u_2 \cdots u_{N-1})^{3\beta/2} u_N^{\beta/2} \prod_{i \leq N} du_i \\
&\geq C_1^{-N} \left(\frac{\delta''}{N} \right)^{C_1 N}
\end{aligned}
$$

for some constant C_1, which yields

$$
\lim_{\delta, \delta' \to 0} \liminf_N \frac{2}{\beta N^2} \log P_{N,2} \geq 0.
$$

On the other hand, from the choice of the $x^{i,N}$, we have

$$\lim \frac{1}{N-1}\sum_{i=1}^{N-1} V(c-x^{i,N}) = \int V(c-x)d\mu(x)\,.$$

Finally the product in (3.4) can be managed exactly as in [1] p. 80. We conclude

$$\begin{aligned}&\lim_{\delta'\searrow 0}\lim_{\delta\searrow 0}\liminf_{N}\frac{2}{\beta N^2}\ln(\bar{Q}(\lambda_{(N)}\in C,\mu_N\in B)\\ \geq\;& -\int\int \ln(y-x)d\mu(x)d\mu(y) - \int V(c-x)d\mu(x)\end{aligned}$$

which is the expected lower bound. □

3.5.2 Proof of the upper bound

We start as in the proof of the lower bound with the representation (3.1). Formula (3.3) can be rewritten as

$$\begin{aligned}\frac{2}{\beta}\ln(\widehat{L}_N(\lambda)) &= 2(N-1)^2\int\int_{x<y}\ln(|x-y|)d\mu_N(x)d\mu_N(y)\\ &+ 2(N-1)\int\ln(|x|)d\mu_N(x)\\ &- (N-1)^2\int V(\lambda_N-x)d\mu_N(x)-(N-1)V(\lambda_N)\\ &- \sum_{i=1}^N V(\lambda_i)\end{aligned}\tag{3.5}$$

where $\mu_N=\frac{1}{N-1}\sum_{k=1}^{N-1}\delta_{\lambda_N-\lambda_k}$, since we are on the set Δ_N defined in (3.2).
Under $\bar{Q}_N$, the λ_i are a.s. distinct so $(\mu_N)^{\otimes 2}(\{(x,y);x=y\})=\frac{1}{N-1}$ a.s.. Therefore, for every $M\in\mathbb{R}$ we can write,

$$\begin{aligned}&-2\int\int_{x<y}\ln(|x-y|)d\mu_N(x)d\mu_N(y) =\\ &\qquad -2\int\int_{x<y}\ln(|x-y|)d\mu_N(x)d\mu_N(y)+M\left((\mu_N)^{\otimes 2}(\{(x,y);x=y\})-\frac{1}{N-1}\right)\\ \geq\;& \int\int_{\mathbb{R}^2}(-\ln(|x-y|)\wedge M)d\mu_N(x)d\mu_N(y)-\frac{M}{N-1}\\ :=\;& -\Sigma^M(\mu)-\frac{M}{N-1}\,.\end{aligned}\tag{3.6}$$

We have assumed $V\geq 0$ so the second term in the third line of (3.5) is non positive. For the first term of the same line, notice that

$$-(N-1)^2\int V(\lambda_N-x)d\mu_N(x)\quad\leq\quad -(N-1)^2\inf_{c\in F}\int V(c-x)d\mu_N(x)\,.$$

We bound the term in the second line of (3.5) by $(N-1)\int|x|d\mu_N(x)$. On the event $\{\mu_N\in B\}$, we have

$$\begin{aligned}\frac{2}{\beta}\log\widehat{L}_N(\lambda)\quad\leq\quad & -(N-1)^2\inf_{\nu\in B}\left(-\Sigma^M(\nu)+\inf_{c\in F}\int Vd\tau_c\nu\right)\\ &+(N-1)\sup_{\nu\in B}\int|x|d\nu(x)+(N-1)M\\ &-\sum_{i=1}^N V(\lambda_i)\,.\end{aligned}$$

Since

$$N! \int_{\Delta_N} \exp -\frac{\beta}{2} \sum_{i=1}^{N} V(\lambda_i) = \left[\int \exp -\frac{\beta}{2} V(\lambda) d\lambda \right]^N ,$$

and $\sup_{\nu \in B} \int |x| d\nu(x) < \infty$, we obtain:

$$\limsup_N \frac{2}{\beta N^2} \ln(\bar{Q}(\lambda_{(N)} \in F, \mu_N \in B) \leq -\inf_{\nu \in B} \left(-\Sigma^M(\nu) + \inf_{c \in F} \int V d\tau_c \nu \right) .$$

To let $\delta \to 0$, we need semicontinuity in ν. We know that Σ^M is lower semicontinuous. Assume that $F = [a, \infty)$. Since V is convex, the infimum $\inf_{c \geq a} \int V(c-x) d\nu(x)$ is reached at $c = \kappa_V(\nu)$ or at $c = a$, so that

$$\inf_{c \geq a} \int V(c-x) d\nu(x) = \int V(\max(a, \kappa_V(\nu)) - x) d\nu(x)$$

which is a lower semicontinuous function of ν.

We obtain:

$$\lim_{\delta \searrow 0} \limsup_N \frac{2}{\beta N^2} \ln(\bar{Q}(\lambda^{(N)} \in F, \mu_N \in B)) \leq \\ - \left(-\Sigma^M(\mu) + \inf_{c \in F} \int V(c-x) d\mu(x) \right) . \quad (3.7)$$

and since Σ^M grows to Σ as M goes to infinity, this yields the upper bound (2.9).

The same is true for $F =]-\infty, a]$.

Now, take F a non empty closed set. If $\kappa_V(\mu) \in F$, (3.7) is clearly true. If $\kappa_V(\mu) \notin F$, then $\kappa_V(\nu) \notin F$ for $\nu \in B$ for small δ. Denote by $a_- = \sup\{x < \kappa_V(\mu), x \in F\}$ and $a_+ = \inf\{x > \kappa_V(\mu), x \in F\}$. Then, $F \subset]-\infty, a_-] \cup [a_+, \infty[$ and

$$\lim_{\delta \searrow 0} \limsup_N \frac{1}{N^2} \ln(\bar{Q}(\mu_N \in B, \lambda_{(N)} \in F)) \leq \\ - \left(-\Sigma^M(\mu) + \int V(a_- - x) d\mu(x) \wedge \int V(a_+ - x) d\mu(x) \right) . \quad (3.8)$$

The last term in the above equation is $\inf_{c \in F} \int V(c-x) d\mu(x)$. □

3.6 Proof of Propostion 2.11 and Theorem 2.12

From Corollary 2.8, we have

$$\limsup \frac{2}{\beta N^2} \ln \mathbb{P}^N_{V,\beta}(\mu_N \in \mathbf{F}, \lambda_{(N)} \in [a,b]) \leq -\inf_{\mu \in \mathbf{F}, c \in [a,b]} \mathcal{I}_V(c, \mu) , \quad (3.9)$$

as soon as $\mathbf{F}$ is compact. To extend this property to closed sets, we follow the classical way and prove:

Lemma 3.2. *Let $q < p$. For any $-\infty < a < b < \infty$ and $M > 0$, there exists a compact set $\mathbf{K}_{a,b,M}$ of $M_1^q(\mathbb{R}^+)$ such that*

$$\limsup_N \frac{2}{\beta N^2} \ln(\mathbb{P}^N_{V,\beta}(\lambda_{(N)} \in [a,b], \mu_N \notin \mathbf{K}_{a,b,M})) \leq -M . \quad (3.10)$$

Proof. Let

$$\mathbf{K}_M := \{\mu \in M_1^p(\mathbb{R}^+) : \int |V| d\mu \leq M\} ,$$

which is compact of $M_1^q(\mathbb{R}^+)$ from Proposition 5.3.

With our assumptions on the potential V, there exists $c_1, c_2 > 0$ such that

$$|V(x)| \leq c_1 x^p + c_2\,,$$

Let $a < b$ and $C = \sup\{|a|, |b|\}$.

For $N \geq 2$, using the convexity of x^p

$$\{\mu_N(V) \geq M\} \cap \{\lambda_{(N)} \in [a,b]\} \subset \{\mu^{(N)}(V) \geq M'\}$$

where

$$M = c_1(C^p 2^{p-1} + 2^{p-1}M') + c_2\,.$$

It remains to use the exponential tightness for the ESD $\mu^{(N)}$, see [1], p. 77) where it is shown that:

$$\limsup_N \frac{2}{\beta N^2} \ln(\mathbb{P}^N_{V,\beta}(\mu^{(N)} \notin \mathbf{K}_{\bar{M}})) \leq -M$$

where $\bar{M}$ is an affine function of M. From Lemma 3.2, (3.9) is satisfied for $\mathbf{F}$ a closed set of $M_1^q(\mathbb{R}^+)$. □

3.6.1 Proof of Proposition 2.11

By Proposition 2.5, we know that for $\Delta = [-C, C]$ large enough, then $\mathbb{P}^N_{V,\beta}(\lambda_{(N)} \in \Delta) \to 1$, so that

$$\lim_N \frac{2}{\beta N^2} \ln \mathbb{P}^N_{V,\beta}(\lambda_{(N)} \in \Delta) = 0\,,$$

and then, for a closed set $\mathbf{F}$,

$$\begin{aligned}\limsup_N \frac{2}{\beta N^2} \ln \mathbb{P}^N_{V,\beta}(\mu_N \in \mathbf{F} \mid \lambda_{(N)} \in \Delta) = \\ \limsup_N \frac{2}{\beta N^2} \ln \mathbb{P}^N_{V,\beta}(\mu_N \in \mathbf{F}, \lambda_{(N)} \in \Delta) \\ \leq -\inf_{\mu \in \mathbf{F}, c \in \Delta} \mathcal{I}_V(c,\mu)\,,\end{aligned} \tag{3.11}$$

from (3.9) for closed sets. Now, we use the easy bound

$$\inf_{\mu \in \mathbf{F}, c \in \Delta} \mathcal{I}_V(c,\mu) \geq \inf_{\mu \in \mathbf{F}, c \in (-\infty,\infty)} \mathcal{I}_V(c,\mu) = \inf_{\mu \in \mathbf{F}} I_V^{\mathrm{DOS}}(\mu)\,.$$

3.6.2 Proof of Theorem 2.12

We use Proposition 2.5 to estimate the probabilities of the conditioning events. On the one hand (3.9) for closed sets and (2.3) lead to (2.13) and on the other hand (3.9), (2.4) and Remark 2.13 lead to (2.17), using that J_V^- is decreasing on $[-\infty, b_V]$.

For the lower bounds, we use the lower bound coming from the LDP for both variables (Theorem 2.7 (1) or Corollary 2.8) and (2.3) and (2.4), respectively.

It remains to prove (2.15) and (2.19), but it is straightforward since

$$\begin{aligned}\lim_{\delta \to 0} \inf_{a \in [c, c+\delta]} \int V(a-x)d\mu(x) = \lim_{\delta \to 0} \inf_{a \in [c-\delta, c]} \int V(a-x)d\mu(x) \\ = \int V(c-x)d\mu(x)\,.\end{aligned}$$

4 Fluctuations

We want to study the fluctuations of μ_N around its limit ν_V given in (2.5). There are two contributions: the fluctuations of the largest eigenvalue and the fluctuations of the ESD. This yields a dichotomy according to the behavior of the test function. For the sake of simplicity, we choose a simple assumption on the test function f which is far from optimal. For V and β we introduce a new assumption:

Assumption 4.1. *V satisfies Assumption 2.2 and $\beta = 1, 2, 4$, or $V(x) = x^2/2$ and $\beta > 0$.*

Proposition 4.2. *Let f be a bounded $\mathcal{C}^2$ function with two bounded derivatives.*

1. *If V and β satisfy Assumption 4.1 and if $\nu_V(f') \neq 0$,*

$$N^{2/3}\left(\mu_N(f) - \nu_V(f)\right) \Rightarrow \nu_V(f')\mathrm{TW}_\beta\ ,$$

where TW_β denotes the Tracy-Widom distribution of index β (see [12] for a definition), and where $\Rightarrow$ denotes the convergence in distribution.

2. *If V satisfies Assumption 2.2 and $\beta > 0$ and if $\nu_V(f') = 0$,*

$$N\left(\mu_N(f) - \nu_V(f)\right) \Rightarrow \mathcal{N}(-f(0) + m_V(f), \sigma_V^2(f))$$

where

$$\sigma_V^2(f) = \frac{1}{4\beta}\sum_{k=1}^{\infty} k a_k^2\,, \tag{4.1}$$

with

$$a_k = \frac{2}{\pi}\int_0^\pi f\left(\frac{b_V - a_V}{2}(1-\cos\theta)\right)\cos k\theta\, d\theta\,,$$

and

$$m_V(f) = \left(\frac{2}{\beta} - 1\right)\int f(b_V - t)d\gamma_V(t) \tag{4.2}$$

where γ_V is a signed measure on $[a_V, b_V]$ given by formula (3.54) in [8].

Let us notice, from Remark 3.5 in [8], that in the Gaussian case, $V(x) = x^2/2$, then

$$d\gamma_V(t) = \frac{1}{4}\delta_{-2} + \frac{1}{4}\delta_2 - \frac{1}{2\pi}\frac{dt}{\sqrt{4-t^2}}\,.$$

Proof. Let $H > b_V - a_V$. Set K_H the random set defined by

$$K_H = \{(\lambda_{(1)}, \cdots, \lambda_{(N)}) : b_V - H \leq \lambda_{(1)}\ ;\ \lambda_{(N)} \leq b_V + H\}\,,$$

and

$$M = \max\left\{\frac{1}{2}\sup_{x\in[-2H,2H]}|f''(x)|, \sup_{x\in[-H,H]}|f(x)|, \sup_{x\in[-H,H]}|xf'(x)|\right\}\,.$$

Setting

$$S_N(f) := (N-1)\mu_N(f) = \sum_{1}^{N-1} f(\lambda_{(N)} - \lambda_{(k)})\,,$$

we make a Taylor expansion of f :

$$f(\lambda_{(N)} - \lambda_{(k)}) \quad = \quad f(b_V - \lambda_{(k)}) + \varepsilon_N f'(b_V - \lambda_{(k)}) + r_{k,N}(f)\,,$$

with

$$\varepsilon_N := \lambda_{(N)} - b_V$$

and

$$|r_{k,N}(f)|1_{K_H} \leq M\varepsilon_N^2 \,.$$

Adding,

$$\begin{aligned} S_N(f) &= \sum_1^{N-1} f(b_V - \lambda_{(k)}) + \varepsilon_N \sum_1^{N-1} f'(b_V - \lambda_{(k)}) + \sum_1^{N-1} r_{k,N} \\ &= \sum_{i=1}^{N} f(b_V - \lambda_i) + \varepsilon_N \sum_{i=1}^{N} f'(b_V - \lambda_i) + R_N(f) \end{aligned} \tag{4.3}$$

where

$$R_N(f) := \sum_1^{N-1} r_{k,N}(f) \ - f(-\varepsilon_N) - \varepsilon_N f'(-\varepsilon_N)\,,$$

satisfies

$$|R_N(f)|1_{K_H} \leq M(N\varepsilon_N^2 + |\varepsilon_N| + 1)\,. \tag{4.4}$$

Setting

$$\begin{aligned} \Delta_N(f) &:= \sum_{i=1}^{N} f(b_V - \lambda_i) - N\nu_V(f) \\ &= \sum_{i=1}^{N} f(b_V - \lambda_i) - N\int f(b_V - x)d\mu_V(x)\,, \end{aligned} \tag{4.5}$$

(4.3) gives

$$S_N(f) - N\nu_V(f) = N\varepsilon_N\nu_V(f') + \Delta_N(f) + \varepsilon_N\Delta_N(f') + R_N(f)\,. \tag{4.6}$$

The two sources of fluctuations are the convergences of ε_N (rescaled) and $\Delta_N(f)$.

On the one hand we know (Prop. 2.5) that

$$\varepsilon_N \to 0 \ \text{ in probability}\,, \tag{4.7}$$

and the fluctuations are ruled by

$$N^{2/3}\varepsilon_N \Rightarrow \mathrm{TW}_\beta \ . \tag{4.8}$$

(see [4] in the cases $\beta = 1, 2, 4$, and [12] for the Gaussian case and $\beta > 0$).

On the other hand, under our assumptions on V and f,

$$\Delta_N(f) \Rightarrow \mathcal{N}(m_V(f); \sigma_V^2(f))\,, \tag{4.9}$$

where $m_V(f)$ and $\sigma^2(f)$ are given by (4.2) and (4.1), respectively (see ([8] Theorem 2.4)).
1. If $\nu_V(f') \neq 0$, we set, for the sake of simplicity

$$N^{-1/3}[S_N(f) - N\nu(f)] = N^{2/3}\varepsilon_N\nu_V(f') + R'_N(f)\,.$$

We have then, if $\Phi(f) := \mathbb{E}[\exp[\mathrm{i}\nu(f')\mathrm{TW}_\beta],$

$$\begin{aligned} &\mathbb{E}\mathrm{e}^{\mathrm{i}N^{-1/3}[S_N(f)-N\nu(f)]} - \Phi(f) = \\ &\quad \mathbb{E}\mathrm{e}^{\mathrm{i}N^{2/3}\varepsilon_N\nu(f')} - \Phi(f) \\ &\quad + \mathbb{E}\left(\mathrm{e}^{\mathrm{i}N^{2/3}\varepsilon_N\nu(f')}\left[\mathrm{e}^{\mathrm{i}R'_N(f)} - 1\right]1_{K_H}\right) \\ &\quad + \mathbb{E}\left(\mathrm{e}^{\mathrm{i}N^{2/3}\varepsilon_N\nu(f')}\left[\mathrm{e}^{\mathrm{i}R'_N(f)} - 1\right]1_{K_H^c}\right) \ . \end{aligned} \tag{4.10}$$

- The first term converges to zero, thanks to (4.8).
- The second term is bounded by $\mathbb{E}\left(|R'_N(f) \wedge 2| 1_{K_H}\right)$ which tends to zero since on K_H

$$\begin{aligned} |R'_N(f)| &\leq N^{-1/3}|\Delta_N(f)| + N^{-1/3}\varepsilon_N \Delta_N(f') \\ &\quad + MN^{2/3}(\varepsilon_N)^2 + MN^{-1/3}(|\varepsilon_N| + 1) \end{aligned}$$

  and each of these terms tends to zero in probability, thanks to (4.9) and (4.7).
- The third one is bounded by $2\mathbb{P}((K_H)^c)$ which tends to zero, since the extreme eigenvalues tend to the endpoints of the support.

This allows to conclude that

$$N^{2/3}\left(\mu_N(f) - \tau_{b_V}\mu_V(f)\right) \Rightarrow \nu_V(f')\mathrm{TW}_\beta \ .$$

2. If $\nu(f') = 0$, then,

$$S_N(f) - N\nu(f) = \Delta_N(f) - f(0) + \varepsilon_N \Delta_N(f') + \tilde{R}_N(f)\,, \tag{4.11}$$

with

$$\tilde{R}_N(f) := \sum_{1}^{N-1} r_{k,N}(f) \ - \left(f(-\varepsilon_N) - f(0)\right) - \varepsilon_N f'(-\varepsilon_N)\,.$$

From (4.9),

$$\Delta_N(f) - f(0) \Rightarrow \mathcal{N}(-f(0) + m_V(f); \sigma_V^2(f))\,.$$

Moreover $\varepsilon_N \Delta_N(f')$ and $\tilde{R}_N(f) 1_{K_H}$ tend to 0 in probability. The rest of the proof goes as before. □

5 Appendix

We give some properties of the Wasserstein distance d_{W_p}.

Definition 5.1. *Let $p \in [1, \infty[$, and $M_1^p(\mathbb{R}) = \{\nu \in M_1(\mathbb{R}), \int |x|^p d\nu(x) < \infty\}$. For two probabilities μ and ν in $M_1^p(\mathbb{R})$, the Wasserstein distance of order p is defined by*

$$d_{W_p}(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}} |x - y|^p d\pi(x, y)\right)^{1/p} \tag{5.1}$$

where $\Pi(\mu, \nu)$ is the set of probabilities on $\mathbb{R}^2$ with first marginal μ and second marginal ν.

Remark 5.2. For the Wasserstein distance of order 1, we have the duality formula

$$d_{W_1}(\mu, \nu) = \sup_{\|f\|_{\mathrm{Lip}} \leq 1} \left(\int f d\mu - \int f d\nu\right) .$$

We now give a characterization of the convergence of probabilities in the topology induced by d_{W_p} on $M_1^p(\mathbb{R})$. We refer to [13, Def. 6.8 and Theorem 6.9].
In the following, we denote by $\mu_n \to \mu$ the weak convergence of probabilities, i.e. against bounded continuous functions.

Proposition 5.3. *Let $(\mu_n)_{n \geq 0}$ a sequence of probabilities in $M_1^p(\mathbb{R})$ and $\mu \in M_1^p(\mathbb{R})$. The following assertions are equivalent:*

i) $d_{W_p}(\mu_n, \mu) \to 0$,

ii) $\mu_n \to \mu$ *and* $\displaystyle\int |x|^p d\mu_n(x) \to \int |x|^p d\mu(x)$, *iii)* $\mu_n \to \mu$ *and* $\limsup_{n\to\infty} \int |x|^p d\mu_n(x) \le \int |x|^p d\mu(x)$*,*

iv) $\mu_n \to \mu$ *and* $\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\ge R} |x|^p d\mu_n(x) = 0$*,*

v) For all continuous functions f *with* $|f(x)| \le C(1+|x|^p)$*, one has*

$$\int f(x)d\mu_n(x) \to \int f(x)d\mu(x)\,.$$

The condition in *iv)* is the condition of tightness, or relative compactness, in $(M_1^p(\mathbb{R}), d_{W_p})$. In particular, it follows that, for any $M \in \mathbb{R}$, the set

$$K_M := \{\mu, \int |x|^p d\mu(x) \le M\}$$

is a compact set in $(M_1^q(\mathbb{R}), d_{W_q})$ for any $q < p$.

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Acknowledgments. We thank Gregory Schehr for the presentation of its model at the working group MEGA and Michel Ledoux for pointing out the semicontinuity in Remark 2.15.

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