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# Coarsening with a frozen vertex

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**Abstract**

In the standard nearest-neighbor coarsening model with state space $\{-1,+1\}^{\mathbb{Z}^2}$ and initial state chosen from symmetric product measure, it is known (see [2]) that almost surely, every vertex flips infinitely often. In this paper, we study the modified model in which a single vertex is frozen to $+1$ for all time, and show that every other site still flips infinitely often. The proof combines stochastic domination (attractivity) and influence propagation arguments.

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## 1 Introduction

As in our earlier paper [1], we study and compare the long time behavior of two continuous time Markov coarsening models with state space $\Omega = \{-1,+1\}^{\mathbb{Z}^d}$. One, $\sigma(t)$, is the standard model in which at time zero $\{\sigma_x(0) : x \in \mathbb{Z}^d\}$ is an i.i.d. set with $\theta \equiv P(\sigma_x(0) = +1) = 1/2$ and then vertices update to agree with a strict majority of their $2d$ nearest neighbors or, in case of a tie, choose their value by tossing a fair coin. The modified model, $\sigma'(t)$, is the same except that $\sigma'$ at the origin $(0, 0....0)$ is frozen to $+1$ for all $t \geq 0$.

For $d = 2$, it is an old result [2] that in the standard $\sigma(t)$ model, almost surely, every vertex changes sign infinitely many times as $t \to \infty$. The main result of this paper (see Theorem 2.7) is that the same is true for the frozen model $\sigma'(t)$ on $\mathbb{Z}^2$. It is believed (see, for example, Sec. 6.2 of [3]), but not proved, that the $d = 2$ behavior of $\sigma$ remains valid at least for some values of $d > 2$. If this were so, then the arguments of this paper would show the same for the corresponding $\sigma'$ model.

In the previous paper [1] we considered models with infinitely many frozen vertices and in this paper a model with a single frozen vertex. It would be of interest to study models with finitely many, but more than one, frozen vertices; in this regard, see the remark following the proof of Theorem 2.8 below.

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## 2 Results

In this section we fix $d = 2$. We also use the standard convention that the updates are made when independent rate one Poisson process clocks at each vertex ring.

Let $A_T$ denote the event that the "right" neighbor of the origin (at $x = (1, 0)$) is $-1$ for some $t \geq T$. Let $A^1_T \subset A_T$ denote the event that the right neighbor of the origin is the *first* neighbor to be $-1$ at some time $t \geq T$ (more precisely, that no other neighbor is $-1$ at an earlier time in $[T, \infty]$). Let $B_{L,s}$ for $s \in \{-1, +1\}^{\Lambda_L}$ (where $\Lambda_L = \{-L, -L+1, ...., L\}^2$) denote the event that $\sigma'(0)|_{\Lambda_L} = s$ and write $B_{L,+}$ when $s \equiv +1$. We denote the probability measure for the frozen origin $\sigma'(\cdot)$ model by $P'$ and that for the regular coarsening model $\sigma(\cdot)$ by $P$.

**Lemma 2.1.** For all $L$,

$$P(A^1_0|B_{L,+}) \geq 1/4\,.$$

*Proof.* The result is an easy consequence of symmetry among the four neighbors of the origin and the fact that $P(A_0) = 1$ (indeed, for all $T$, $P(A_T) = 1$ — see [2]). □

Let $\Sigma^L_T$ denote the sigma-field generated by the initial spin values and clock rings and coin tosses up to time $T$ inside the box $\Lambda_L$.

**Proposition 2.2.** For any $T$, $L$,

$$P'(A_T|\Sigma^L_T) \geq 1/4 \text{ a.s.}\,.$$

*Proof.* Let $\tilde{\sigma}^L_T(.)$ denote the model with the spin values at all sites in $\Lambda_L$ frozen to $+1$ from time $0$ up to time $T$ and with the spin value at the origin remaining frozen at $+1$ thereafter. Denote the corresponding probability measure by $\tilde{P}^L_T$. Under the standard coupling, $\tilde{\sigma}(\cdot)$ stochastically dominates $\sigma'(\cdot)$, so we have

$$P'(A_T|\Sigma^L_T) \geq \tilde{P}^L_T(A_T) \geq \tilde{P}^L_T(A^1_T).$$

To continue the proof, we will use the following result about the "propagation speed" of influence between different spatial regions:

**Lemma 2.3.** Let $D^L_T$ denote the event that $\sigma_x(t) = +1\, \forall x \in \Lambda_L, \forall t \in [0, T]$. Then

$$\forall L, T, \varepsilon, \exists L' \text{ such that } P(D^L_T|B_{L',+}) \geq 1 - \varepsilon\,.$$

*Proof.* Let $L' >> L$ and note that given $B_{L',+}$, $(D^L_T)^c$ can occur only if there is a nearest neighbor (self-avoiding) path between the boundaries of the two sets, $\mathbb{Z}^2 \setminus \Lambda_{L'}$ and $\Lambda_L$, along which there are clock rings occurring in succession between times $0$ and $T$. Any such path is at least of length $L' - L$ (i.e., contains at least $L' - L$ vertices besides the starting one).

Consider a particular path $\gamma$ of length $m \geq L' - L$. For each $m$ there are no more than $3^m$ such paths from each boundary point and the time it takes for successive clock rings along $\gamma$ is at least $S_m = \sum_{i=1}^m \tau_i$ where the $\tau_i$ are i.i.d. exponential random variables with parameter 1. By the exponential Markov inequality, for any $\alpha > 0$,

$$P(\sum_{i=1}^m \tau_i < T) = P(-\sum_{i=1}^m \tau_i > -T) \leq \frac{E(e^{-\alpha\sum_{i=1}^m \tau_i})}{E(e^{-\alpha T})} = e^{\alpha T} E\{e^{-\alpha\tau_i}\}^m = \frac{e^{\alpha T}}{(1+\alpha)^m}.$$

Therefore, since there are at most $CL'$ possible starting points (for some constant $C$),

$$P((D^L_T)^c|B_{L',+}) \leq C\,L' \sum_{m=L'-L}^{\infty} 3^m \frac{e^{\alpha T}}{(1+\alpha)^m} = C(\alpha, T, L)L'(\frac{3}{1+\alpha})^{L'}\,,$$

where $C(\alpha, T, L)$ is a constant depending on $\alpha$, $T$ and $L$. Taking $\alpha > 2$ and the limit as $L' \to \infty$ completes the proof of the lemma. □

*Proof. (Continuation of proof of Proposition 2.2.)*

Pick $\epsilon > 0$ and fix $T$ and $L$. By Lemma 2.3, $\exists\, L'$ such that

$$P(D_T^L | B_{L',+}) \geq 1 - \epsilon\,.$$

Therefore, given $B_{L',+}$, with probability at least $1-\epsilon$, $\sigma_t(\cdot)$ positively dominates $\tilde{\sigma}_L^T(\cdot)$ for $0 \leq t < S$, where $S = \inf\{t > 0 \,|\, \sigma_t(0,0) = -1\}$, and so

$$\tilde{P}_L^T(A_T^1) \geq P(A_T^1 | B_{L',+}) - \epsilon \geq 1/4 - \epsilon\,.$$

Taking the limit as $\epsilon \to 0$ completes the proof of Proposition 2.2. □

Now let $\Sigma_T$ denote the sigma field generated by the initial assignment of spins on $\mathbb{Z}^2$ and the clock rings and coin tosses on $\mathbb{Z}^2$ up to time $T$.

**Proposition 2.4.** For all $T$,

$$P'(A_T | \Sigma_T) \geq 1/4 \text{ a.s.}$$

*Proof.* For $L \geq 1$ let $X_L = P'(A_T | \Sigma_T^L)$. $\{\Sigma_T^L, L \geq 1\}$ is an increasing filtration of sigma fields, and $E(X_{L+1} | \Sigma_T^L) = X_L$. By the martingale convergence theorem, $\lim_{L\to\infty}(X_L) = X_\infty = P'(A_T|\Sigma_T)$ and since $X_L \geq 1/4$ for all $L$, we have $P'(A_T|\Sigma_T) \geq 1/4$. □

Let $A_{T,T'}$ denote the event that the right neighbor of the origin is $-1$ for some time $t \in [T, T']$. The following is immediate from Proposition 2.4.

**Corollary 2.5.**

$$\lim_{T'\to\infty} P'(A_{T,T'} | \Sigma_T) \geq 1/4 \text{ a.s.}\,.$$

**Lemma 2.6.** For any $T \geq 0$ and $\gamma > 0$, $\exists$ a deterministic $T'$ such that

$$P\{\omega : P'(A_{T,T'} | \Sigma_T) \geq 1/8\} \geq 1 - \gamma\,.$$

*Proof.* This is a straightforward consequence of the preceding corollary. □

**Theorem 2.7.** For any $T$,

$$P'(A_T) = 1, \text{and hence } P'(\cap_{T>0} A_T) = 1\,.$$

It follows that with probability one, $\sigma'_{(1,0)}(t)$ changes sign infinitely many times as $t \to \infty$.

*Proof.* Given $T$ and $\epsilon > 0$ construct a sequence of deterministic times $\{T_i; i \geq 0\}$ so that

1. $T_0 = T$, and
2. $P'\{\omega : P'(A_{T_{i-1},T_i} | \Sigma_{T_{i-1}}) \geq 1/8\} \geq 1 - \frac{\epsilon}{2^i}$.

Condition now on the event (of probability at least $1 - \sum_{i=1}^\infty \frac{\epsilon}{2^i} = 1 - \epsilon$) that $P'(A_{T_{i-1},T_i} | \Sigma_{T_{i-1}}) \geq 1/8$ for all $i$. On this conditioned probability space, letting $\tilde{W}_i = 1$ (and otherwise 0) if $A_{T_{i-1},T_i}$ occurs, we note that the $\tilde{W}_i$'s stochastically dominate i.i.d. $\{0,1\}$-valued $W_i$'s with $\text{Prob}(W_i = 1) = 1/8$. Thus

$$P'(A_{T_{i-1},T_i} \text{occurs for only finitely many } i) \leq \epsilon\,.$$

Letting $\epsilon \to 0$ completes the proof of the first part of the theorem. The second part then follows because by stochastic domination (attractivity) and the results of [2], $\sigma'_{(0,0)}(t_i)$ equals $+1$ for an infinite sequence of $t_i \to \infty$. □

The next theorem follows from a modified version of the proof of Theorem 2.7.

**Theorem 2.8.** Every site in $\mathbb{Z}^2 \setminus \{(0,0)\}$ flips infinitely many times in $\sigma'(\cdot)$ with probability one.

*Proof.* For any site $z$ other than the origin, and for $L$ much larger than say the Euclidean norm of $z$, we consider the unfrozen $\sigma$ model in which at time zero all the vertex values are set to $+1$ in the box of side length $2L$, centered at $z/2$ (so that the origin and z are located symmetrically with respect to this box). Then with probability $1/2$ the vertex at $z$ flips to $-1$ before the one at the origin flips and until just after that time, there is no difference between the frozen (at the origin) $\sigma'$ model and the unfrozen $\sigma$ model. Hence there is probability at least $1/2$ in $\sigma'$ that $z$ will flip to minus. By applying the methods used in the proof of Theorem 2.7 (but with $1/4$ now replaced by $1/2$), we conclude that $z$ will flip infinitely many times with probability one. □

We note that the line of reasoning in the proof of the last theorem could have also been used to give a modified proof of Theorem 2.7 with $1/4$ replaced by $1/2$. A more interesting remark is the following.

**Remark 2.1.** For the process $\sigma''$ with some finite set $\mathcal{S}$ of vertices frozen to $+1$, it is possible to show by an extension of the arguments used in this paper that there is a finite deterministic $\mathcal{S}' \supseteq \mathcal{S}$ such that all sites in $\mathbb{Z}^2 \setminus \mathcal{S}'$ flip infinitely many times in $\sigma''(\cdot)$ with probability one. In some cases, $\mathcal{S}'$ must be strictly larger than $\mathcal{S}$ — e.g., when $\mathcal{S} = \{(-L,-L),(-L+L),(+L,-L),(+L,+L)\}$, $\mathcal{S}'$ includes all of $\Lambda_L$. One may also consider processes where some vertices are frozen to $-1$ and some to $+1$. We expect to to pursue these issues in a future paper.

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