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Deterministic time intervals on which a class of persistent processes are away from their origins

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Abstract

There are three results each concerning large but remote deterministic time intervals at which excursions of a process away from the origin must occur. The first result gives a sufficient condition for a persistent random walk with a finite fourth moment. In this instance the aforementioned time intervals include an additional requirement that the walk is far away from the origin. The second result gives a necessary and a sufficient condition for similar excursions in the case of Brownian motion. The third result gives a necessary and a sufficient condition for time intervals to be free of the zeros of a class of persistent natural scale linear diffusions on the line and is equivalent to the determination of recurrent sets at infinity of the inverse local time.

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1 The results

A random walk $\mathbf{S}_n = \mathbf{S}_0 + \xi_1 + \cdots + \xi_n$, $n = 0, 1, \ldots$, on the line with steps a sequence $\{\xi_n\}$ of i.i.d. random variables with mean 0 returns infinitely often to arbitrarily small neighborhoods of its origin, but given non-random pairwise disjoint time intervals J_k moving out to infinity, steadily increasing in length, the possibility arises of finding the walk far away from its origin during these particular time intervals.

Theorem 1.1. *Let $J_k = [a_k,\, a_k + b_k]$ and define events*

$$A_k = A(a_k, b_k, c_k) = \left\{ |\mathbf{S}_n| < c_k \text{ for some } n \in J_k \right\}. \tag{1.1}$$

Assume $E\,\xi_1 = 0$ and $E\,\xi_1^4 < \infty$. If for the positive sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$:

$$a_k \uparrow \infty, \qquad a_k + b_k < a_{k+1}, \qquad c_k = \mathrm{O}\left(b_k^{1/2}\right) = \mathrm{O}\left(a_k^{1/4}\right), \tag{1.2}$$

as $k \to \infty$, and the series

$$\sum_k \left(\frac{\log\log a_k}{a_k}\right)^{1/4} \tag{1.3}$$

converges, then $P\{A_k \text{ i.o.}\} = 0$, ($\text{i.o.} =$ "infinitely often" with its usual meaning).

The method selected for proving this theorem gives a very short, concise demonstration but it does not lend itself to proving a converse. However, for Brownian motion a converse presents itself at the cost of a little tediousness in the proof.

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Theorem 1.2. *Let* $\{\mathbf{B}(t),\ t \geq 0\}$ *be standard Brownian motion on the line with probabilities* $P_x[\,\cdot\,] = P[\,\cdot\,\big|\,\mathbf{B}(0) = x]$ *. Define*

$$A_k = \big\{\, |\mathbf{B}(t)| \leq c_k \ \text{ for some } \ t \in [a_k, a_k + b_k]\,\big\}\,.$$

If $a_{k+1} - a_k - b_k > \theta a_{k+1}$*,* $a_{k+1} - a_k > b_{k+1} \geq b_k$*,* $c_k \leq \sqrt{b_k}$*,* $c_k \leq c_{k+1}$*, for some fixed* $0 < \theta < 1$ *and all* k*, then* $P\{A_k \text{ i.o. }\}$ *equals 0 or 1 according as the series* $\sum_k \sqrt{b_k/a_k}$ *converges or diverges.*

When $c_k = 0\ \forall\, k$, Theorem 1.2 takes a simpler form. More generally:

Theorem 1.3. *Let* $\{\mathbf{X}(t),\ t \geq 0\}$ *be the persistent non-singular diffusion on* $(-\infty, \infty)$ *(or on any interval containing 0) with natural scale and speed measure* m*. Assume* m *is absolutely continuous with R.-N. derivative* $m' = \mathrm{d}m/\,\mathrm{d}x$ *such that for some* $q > -1$ *and positive constants* c*,* C

$$c|x|^q \leq m'(x) \leq C|x|^q \tag{1.4}$$

for all x *in the state interval. If* $[a_k, a_k + b_k],\ k = 1, 2\dots$ *are disjoint intervals marching out to infinity, and*

$$B_k = \{\mathbf{X}(t) = 0 \ \text{ for some } \ t \in [a_k, a_k + b_k]\}\,,$$

and if the series

$$\sum_k (b_k/a_k)^{(q+1)/(q+2)} \tag{1.5}$$

converges, then $P[\,B_k \text{ i.o. } k \uparrow \infty\,] = 0$*. If also* $k \mapsto (a_k + b_k)/a_{k+1}$ *is bounded away from 1 and if* (1.5) *diverges, then* $P[\,B_k \text{ i.o. } k \uparrow \infty\,] = 1$*.*

Theorem 1.1 gives an answer to the implied question at the beginning and allows the conclusion that a finite random variable κ exists such that w.p.1 *for all* $k \geq \kappa$, $|S_n| \geq c_k$ *for every* $n \in J_k$. This result relates to a result of K. L. Chung and P. Erdös, [3]. They show that under mild growth conditions on a deterministic sequence of times, $\{2n_i\}$, a simple random walk (unit step with probability 1/2 in either direction) returns to its starting position at infinitely many of these times, w.p.1, if and only if $\sum n_i^{-1/2}$ diverges.

Theorem 3 is equivalent to the assertion that the set $B = \bigcup_k [a_k, a_k + b_k]$ is *recurrent/transient at infinity* (terminology of [7]), for the inverse local time at 0 of the diffusion. Inverse local time at 0 for a diffusion is a special type of subordinator and recurrence of sets at infinity is a complicated problem for general subordinators. Though Wiener type tests in terms of capacities are known, except in special cases, it seems difficult to get concrete expressions in terms of the sequences $\{a_k\}$, $\{b_k\}$ and the fundamental object associated with subordinators, namely the Lévy measure. Some preliminary work is included in §4, Theorem 4.1, which is used in the proof of Theorem 1.3 in the last section.

The problem of recurrence at infinity of sets for Lévy processes is closely related, but not does not seem equivalent, to the gap problems discussed in [5] or [11].

2 Proof of Theorem 1.1

Without loss of generality we may suppose that the a_k and b_k are actulally positive integers. Next, note that the event " A_k occurs infinitely often $k \uparrow \infty$" belongs to the σ-field of permutable events based on the i.i.d. sequence $\{\xi_n\}$ and therefore has probability 0 or 1.

According to the Skorokhod Embeding Theorem, [2], on some probability space $(\Omega, \mathcal{A}, P)$ a standard Brownian motion $(\mathbf{B}(t),\, t \geq 0)$ and an increasing sequence of stopping times $\{T_n\}$ with mutually independent identically distributed increments $\{T_n - T_{n-1}\}$ exist such that, $T_0 = 0$ and

$$\{\mathbf{S}_n;\ n = 0, 1, \ldots\} \stackrel{\mathrm{d}}{=} \{\mathbf{B}(T_n);\ n = 0, 1, \ldots\}\,. \tag{2.1}$$

Moreover, $E\,T_n = n\,E\,\xi_1^2 = n$ and because the ξ_i have a finite moment of order 4, the increments of T have a finite moment of order 2. Clearly,

$$\begin{aligned} A_k &= \{|\mathbf{S}_n| \le c_k \ \text{ for some } \ n \in [a_k, a_k + b_k] \\ &\implies \{\,|\mathbf{B}(t)| \le c_k \ \text{ for some } \ t \in [T(a_k), T(a_k + b_k)]\}\,. \end{aligned}$$

However, the Hartman-Wintner Law of the Iterated Logarithm entails existence of a finite random variable ρ and a finite positive number q such that

$$|T_n - n| \le q\sqrt{n \log\log n} \quad \forall\, n \ge \rho\,.$$

It follows that if infinitely many of the events A_k occur, then infinitely many of the events D_k must occur where

$$\begin{aligned} D_k =&\big\{|\mathbf{B}(t)| \le c_k \ \text{ for some } t \text{ in the interval} \\ &[a_k - q\sqrt{a_k \log\log a_k},\ a_k + b_k + q\sqrt{(a_k + b_k)\log\log(a_k + b_k)}]\,\big\}\,. \end{aligned}$$

However,

$$P\big\{\,|\mathbf{B}(t)| \le c \ \text{ for some } \ t \in [\alpha, \alpha + \beta]\,\big\} \le \frac{c + (2/\sqrt{\pi})\beta^{1/2}}{\alpha^{1/2}} \tag{2.2}$$

(See below for the proof.) Set

$$\alpha = a_k - q\sqrt{a_k \log\log a_k} \ge a_k/2, \ \text{ eventually },$$

$$\beta = b_k + q\sqrt{(a_k + b_k)\log\log(a_k + b_k)} + q\sqrt{a_k \log\log a_k}$$

Then by (1.2)

$$\begin{aligned} c_k + \sqrt{\beta} &= \mathrm{O}\left(\sqrt{b_k}\right) + \mathrm{O}\left(\sqrt{a_k + \sqrt{(a_k + b_k)\log\log(a_k + b_k)}}\right) \\ &= \mathrm{O}\left(a_k^{1/4}\right) + \mathrm{O}\left(\left(a_k \log\log a_k\,\right)^{1/4}\right) = \mathrm{O}\left(\left(a_k \log\log a_k\,\right)^{1/4}\right) \end{aligned}$$

and

$$P\{D_k\} \le \frac{c + (2/\sqrt{\pi})\beta^{1/2}}{\alpha^{1/2}} = \mathrm{O}\left(\left[\frac{\log\log a_k}{a_k}\right]^{1/4}\right)\,.$$

The Borel-Cantelli Lemma takes care of the rest. □

Proof of (2.2)

Put $g(x) = (2\pi)^{-1/2}\mathrm{e}^{-x^2/2}$, $g_\alpha(x) = \alpha^{-1/2}g(x/\alpha^{1/2})$, and

$$G^*(x) = 2\int_x^\infty g(z)\,\mathrm{d}z, \quad x \ge 0$$

Recall that if $\tau_z = \min(t : \mathbf{B}_t = z)$ then

$$P[\tau_z \le t] = P[\max_{s\le t}\mathbf{B}(s) \ge z] = 2P[\sqrt{t}\,\mathbf{B}(1) > x] = G^*(x/\sqrt{t})\,. \tag{2.3}$$

For future use here and in the next section, we note the inequalities:

$$\frac{g(x)}{x+1} < G^*(x) < 2\sqrt{2}g(x)\,. \quad \forall\, x > 0 \tag{2.4}$$

(These follow quickly from [9], page 17, Problem 1, or do it yourself.) Let $\alpha > 1$, $\beta > 1$; then by symmetry and the Markov property one gets

$$P\{\,|\mathbf{B}(t)| < c \text{ for some } t \in [\alpha, \alpha+\beta]\,\} = \int_{-\infty}^{\infty} P_x\{\,|\mathbf{B}(t)| < c \text{ for some } t \le \beta\,\}\, g_\alpha(x)\,\mathrm{d}x$$

$$= \int_{|x|\le c} g_\alpha(x)\,\mathrm{d}x + 2\int_c^\infty P_x\{\tau_c \le \beta\}\, g_\alpha(x)\,\mathrm{d}x < c\,\alpha^{-1/2} + 2\int_0^\infty G^*(x/\beta^{1/2})\, g_\alpha(x+c)\,\mathrm{d}x$$

$$< \alpha^{-1/2}\left(c + 4\sqrt{2}\int_0^\infty g(x/\beta^{1/2})\, g(x/\alpha^{1/2})\,\mathrm{d}x \right)$$

$$= \alpha^{-1/2}\left[c + \,(2/\sqrt{\pi})\,\left(\alpha^{-1}+\beta^{-1}\right)^{-1/2}\right] \;<\; \alpha^{-1/2}\left(c + (2/\sqrt{\pi})\,\beta^{1/2}\right) .$$

3 Proof of Theorem 1.2

The main tools are strong Markov, (2.3), some simple inequalities, a standard extension of the Borel-Cantelli Lemma, [10]. I omit some details.

Put $I = [a, a+b]$, $I' = [a', a'+b']$, $a+b < a'$, and for $c > 0$, $c' > 0$, denote by A, A' the events: $|\mathbf{B}_s| < c$, $(< c')$ for some $s \in I$, $(s \in I')$, respectively. Assume that the parameters satisfy the conditions of the theorem:

$$a' - a - b > \theta a', \quad a' - a > b' > b, \quad c < \sqrt{b}, \quad c' < \sqrt{b'}, \quad c \le c' \tag{3.1}$$

for some θ, $0 < \theta < 1$. The key to Theorem 1.2 is the following:

$$P\{A\,A'\} \le M P\{A\}\, P\{A'\} \tag{3.2}$$

for a constant M, which may depend on θ, but not on a, a', b, b', c, c'.

Proof of (3.2)

For a path to be in A and A' we have $\mathbf{B}(a) = x$, (for some x) then a (first after time a) hit in $(-c, c)$ at some time $\sigma = s \in I$. It must happen also that $\mathbf{B}(a') = y$ for some y followed by a first hit after time a' in $(-c', c')$ at some time σ' in I'. But the strong Markov property and time homogeneity implies that $\mathbf{B}$ starts from scratch at time s and the subsequent development is identical in law to the process time shifted by s. Given $\mathbf{B}(a) = x$, the event $\sigma = s \in [a, a+b]$ is the same as $\sigma = r = s - a \in [0, b]$ for a Brownian $\widetilde{\mathbf{B}}$ starting at place x at time 0. Similarly, given the position y of the path at time a' (= time $a' - s = a' - a - r$ for $\widetilde{\mathbf{B}}$), the probability of a hit of $(-c', c')$ during $[a', a'+b']$ is the same as an *unconditional* hit during $[0, b']$ for a brand new Brownian starting at place y. Hitting an interval, $[-c, c]$ say, *from outside* the interval is, by path continuity, the same as hitting one or the other boundary points $\pm c$. Hence for $x > c$:

$$P_x[\sigma \le r] = P_x[\tau_c \le r] = P[\tau_{c-x} \le r] = P[\tau_{x-c} \le r] = G^*((x-c)r^{-1/2}), \tag{3.3}$$

and for $x \le c$, $P_x[\sigma \in \mathrm{d}r] = \delta_0(\mathrm{d}r)$ = unit mass at 0. (Recall $P = P_0$.) Similarly

$$P_y[\sigma' \le b'] = \begin{cases} G^*((y-c')b'^{\,-1/2}), & y > c' \\ 1, & |y| \le c' \\ G^*((-c'-y)b'^{\,-1/2}), & y < -c'. \end{cases} \tag{3.4}$$

Note that $P[\tau_z \in \mathrm{d}r] = -\,\mathrm{d}G^*(z/\sqrt{r})/\,\mathrm{d}r = r^{-3/2}\, z\, g(z/r^{1/2}) = r^{-1}\, z\, g_r(z)$.

Now we can begin the estimations. The letter K, with or without subscripts or superscripts, denotes a positive *numerical* constant independent of $a, a', b, b', c, c', r, \theta$ *but these* K*s are not necessarily the same at each appearance.*

To reduce some of the notation, we will frequently omit the differentials $\mathrm{d}x$, etc. The indicated integral limits suffice to specify the integration variables. The necessity to keep down the length of this paper (12 pages) required brevity in proofs so the reader may need to take pencil in hand to check details he or she does not quite believe.

From symmetry:

$$\begin{aligned} P\{AA'\} = 2 \iiint_{[x>c,\ r\le b,\ y\in\mathcal{R}]} & P[\mathbf{B}_a \in \mathrm{d}x] P_x[\sigma \in \mathrm{d}r] P_c[\mathbf{B}_{a'-a-r} \in \mathrm{d}y] P_y[\sigma' \le b'] \\ + \int_{[\,|x|\le c,\ y\in\mathcal{R}]} & P[\mathbf{B}_a \in \mathrm{d}x] P_x[\mathbf{B}_{a'-a} \in \mathrm{d}y] P_y[\sigma' \le b'] \end{aligned} \tag{3.5}$$

We will write $\int_1$ for the first term on the RHS of (3.5) and $\int_2$ for the second term.

The triple integral $\int_1$ in (3.5)

The y part does not involve x and $x > c$. Also $P[\mathbf{B}_a \in \mathrm{d}x] P_x[\sigma \in \mathrm{d}r] = g_a(x)\,\mathrm{d}x\, P[\tau_{x-c} \in \mathrm{d}r]$. We make the change of variables $x \mapsto x + c$ with a *new* $x \ge 0$. (One ought never to introduce a new symbol with a trivial change of variables.) In light of (3.4), the y integral splits into into three integrals corresponding to $y \ge c'$, $y \le -c'$, and $|y| \le c'$. The first two of these combine into one with $y \ge c'$ but with an additional integrand and in that integral we replace y by $y + c'$ with a new $y \ge 0$:

$$\begin{aligned} \int_1 &= 2 \iiint_{[x>c,\ r\le b,\ y\ge c']} \\ &\overset{x\to x+c}{=} 2 \iiint_{[x\ge 0,\ r\le b,\ y\ge c'\,]} g_a(x+c)\, P[\tau_x \in \mathrm{d}r]\, G^*\left(\frac{y - c't}{\sqrt{b'}}\right) \big[\, g_{a'-a-r}(y-c) + g_{a'-a-r}(y+c)\,\big] \\ &\quad + 2 \iiint_{[x\ge 0,\ r\le b,\ |y|\le c'\,]} g_a(x+c)\, P[\tau_x \in \mathrm{d}r]\, g_{a'-a-r}(y-c) \\ &= 2 \iiint_{[x\ge 0,\ r\le b,\ y\ge 0\,]} g_a(x+c)\, P[\tau_x \in \mathrm{d}r]\, G^*\big(y/\sqrt{b'}\big) \big[\, g_{a'-a-r}(y+c'-c) + g_{a'-a-r}(y+c'-c)\,\big] \\ &\quad + 2 \iiint_{[x\ge 0,\ r\le b,\ |y|\le c'\,]} g_a(x+c)\, P[\tau_x \in \mathrm{d}r]\, g_{a'-a-r}(y-c) \\ &\le 4 \iiint_{[x\ge\, 0,\ r\le b,\ y\ge\, 0\,]} g_a(x)\, P[\tau_x \in \mathrm{d}r]\, G^*\big(y/\sqrt{b'}\big)\, g_{a'-a-r}(y) \\ &\quad + 2 \iiint_{[x\ge 0,\ r\le b,\ |y|\le c']} g_a(x)\, P[\tau_x \in \mathrm{d}r]\, g_{a'-a-r}(y-c) \\ &\equiv J_1 + J_2 \end{aligned} \tag{3.6}$$

The inequality results from $g_{a'-a-r}(y + c' - c') + g_{a'-a-r}(y + c' - c) \le 2g_{a'-a-r}(y)$ for $c' - c \ge 0$, $y \ge 0$, and $g_a(x + c) \le g(x)$, $x \ge 0$.

Applying (2.4) we have in the integral J_1 of (3.6)

$$J_1 < 4 \iiint_{[x\ge 0,\ r\le b]} g_a(x)\, P[\tau_x \in \mathrm{d}r] \bigg(2\sqrt{2} \int_{y\ge 0} g(y/\sqrt{b'}) g_{a'-a-r}(y) \bigg)$$

$$= K' \iint\limits_{[x\geq 0,\ r\leq b]} \frac{g_a(x)\, P[\tau_x \in \mathrm{d}r]}{(a'-a-r)^{1/2}} \int\limits_{y\geq 0} \mathrm{e}^{-[1/b'+1/(a'-a-r)]y^2/2}\,\mathrm{d}y$$

$$< \frac{K''\sqrt{b'}\, J_3}{(a'+b'-a-b)^{1/2}}, \quad J_3 = \iint\limits_{[x\geq 0,\ r\leq b]} g_a(x)\, P[\tau_x \in \mathrm{d}r]$$

For J_2 in (3.6) note that for $0 \leq r \leq b \leq a'-a$ and any y,

$$g_{a'-a-r}(y-c) \leq \frac{1}{\sqrt{2\pi(a'-a-r)}} \leq \frac{1}{\sqrt{2\pi(a'-a-b)}}$$

Hence

$$J_2 = 2 \iiint\limits_{[x\geq 0,\ r\leq b,\ |y|\leq c']} g_a(x)\, P[\tau_x \in \mathrm{d}r]\, g_{a'-a-r}(y-c)$$

$$< \frac{Kc'}{(a'-a-b)^{1/2}} \iint\limits_{[x\geq 0,\ r\leq b]} g_a(x)\, P[\tau_x \in \mathrm{d}r] = \frac{Kc'\, J_3}{(a'-a-b)^{1/2}}$$

for some constant K. But $P[\tau_x \in \mathrm{d}r] = r^{-3/2}\, z\, g(z/r^{1/2})$, so

$$J_3 = \iint\limits_{[x\geq 0,\ r\leq b]} r^{-3/2} x\, g_a(x) g(xr^{-1/2}) = \frac{1}{2\pi\sqrt{a}} \iint\limits_{[\,x\geq 0,\ r\leq b\,]} r^{-3/2}\, x\, \mathrm{e}^{-[1/a+1/r]x^2/2}$$

$$= Ka^{-1/2} \int_0^b r^{-3/2}[1/a+1/r)]^{-1}\,\mathrm{d}r < K'\sqrt{b/a}\,.$$

It follows from these inequalities and (3.1) that

$$\int_1 = J_1 + J_2 \,<\, K\left(\frac{\sqrt{b'}}{(a'+b'-a-b)^{1/2}} + \frac{c'}{(a'-a-b)^{1/2}}\right)\sqrt{b/a} < K_1\theta^{-1/2}(bb'/aa')^{1/2}$$

for a number K_1 independent of θ, a', b', etc.

Next

$$\int_2 = \int\limits_{|x|\leq c} P[\mathbf{B}_a \in \mathrm{d}x] \int\limits_{y\in\mathcal{R}} P_x[\mathbf{B}_{a'-a} \in \mathrm{d}y] P_y[\sigma' \leq b']$$
$$= 2\Big[\iint\limits_{[\,0\leq x\leq c,\ y\geq 0\,]} + \iint\limits_{[\,-c\leq x\leq 0,\ y>0\,]}\Big] g_a(x) g_{a'-a}(y-x) P_y[\sigma' \leq b']\,\mathrm{d}x\,\mathrm{d}y \tag{3.7}$$

First integral on RHS of (3.7)

Using $g_a(x) \leq (2\pi a)^{-1/2}$, $g_{a'-a} \leq [2\pi(a'-a)]^{-1/2}$, and $g_{a'-a}(y-x) \leq g_{a'-a}(y-c')$ (because $0 \leq x \leq c \leq c' \Rightarrow y-x \geq y-c' \geq 0$ and monotonicity of $g_{a'-a}$ on positive axis), one finds

$$\iint_{[\,0\leq x\leq c,\ y\geq 0\,]} = \iint_{[\,0\leq x\leq c,\ y\leq c'\,]} + \iint_{[\,0\leq x\leq c,\ y\geq c'\,]}$$

$$< \frac{Kc\,c'}{\sqrt{a(a'-a)}} + \frac{K'c}{\sqrt{a}} \int\limits_{[\,y\geq c'\,]} g_{a'-a}(y-c') G^*\left(\frac{y-c'}{\sqrt{b'}}\right)$$

$$\overset{y\to y+c'}{=} \frac{Kc\,c'}{\sqrt{a(a'-a)}} + \frac{K'c}{\sqrt{a}} \int_0^\infty g_{a'-a}(y) G^*\left(\frac{y}{\sqrt{b'}}\right)\mathrm{d}y$$

Hence

$$\iint_{[\,0\le x\le c,\ y\ge 0\,]} < \frac{K c\, c'}{\sqrt{a(a'-a)}} + \frac{K'\, c}{\sqrt{a(a'-a)}} \int_0^\infty \mathrm{e}^{-[1/b'+1/(a'-a)]y^2/2}\,\mathrm{d}y$$

$$= \frac{K c\, c'}{\sqrt{a(a'-a)}} + \frac{K''\, c}{\{a(a'-a)[1/b'+1/(a'-a)]\}^{1/2}} < K'''\,\theta^{-1/2}(bb'/aa')^{1/2}$$

Second integral on RHS of (3.7)

For $x \le 0$, we have $y - x \ge y \ge 0 \Longrightarrow g_{a'-a}(y-x) \le g_{a'-a}(y)$. Hence

$$\iint\limits_{[\,-c\le x\le 0,\ y>0\,]} g_a(x) g_{a'-a}(y-x) P_y[\sigma' \le b'] \ < \ \frac{K\,c}{\sqrt{a}} \int_0^\infty g_{a'-a}(y) P_y[\sigma' \le b']\,\mathrm{d}y$$

$$\le \frac{K\,c\,c'}{\sqrt{a(a'-a)}} + \frac{K\,c}{\sqrt{a}} \int\limits_{\{y\ge c'\}} g_{a'-a}(y) G^*\left(\frac{y-c'}{\sqrt{b'}}\right)$$

$$\le \frac{K\,c\,c'}{\sqrt{a(a'-a)}} + \frac{K''\,c}{\{a(a'-a)[1/b'+1/(a'-a)]\}^{1/2}} < K'''\,\theta^{-1/2}(bb'/aa')^{1/2}$$

and we now conclude

$$\int_2 \ \equiv\ 2 \iiint_{[x>c,\ r\le b,\ y\in\mathcal{R}]} P[\mathbf{B}_a \in\, \mathrm{d}x] P_x[\sigma \in\, \mathrm{d}r] P_c[\mathbf{B}_{a'-a-r} \in\, \mathrm{d}y] P_y[\sigma' \le b']$$

$$\le K_2\,\theta^{-1/2}(bb'/aa')^{1/2}$$

for some number K_2 and putting the estimates for $\int_1$, $\int_2$ together in (3.5), we get:

$$P\{AA'\} < K_3 \theta^{-1/2}(bb'/aa')^{1/2} \tag{3.8}$$

with $K_3 = K_1 + K_2$ independent of θ, a'..., etc.

Next, a lower bound for $P\{A\}$:

$$P\{A\} = \int\limits_{|x|\le c} + \int\limits_{|x|>c} g_a(x)\, P_x[\sigma \le b]\,\mathrm{d}x$$

$$= \int\limits_{|x|\le c} g_a(x)\,\mathrm{d}x + 2\int\limits_{x>c} g_a(x) G^*((x-c)/\sqrt{b})\,\mathrm{d}x$$

$$\ge 2\int\limits_{x>0} g_a(x+c) G^*(x/\sqrt{b})\,\mathrm{d}x \ge \frac{1}{\pi\sqrt{a}} \int\limits_{x>0} \frac{\mathrm{e}^{-(x+c)^2/2a - x^2/2b}\,\mathrm{d}x}{1+x/\sqrt{b}}$$

$$= \frac{(b/a)^{1/2}}{\pi} \int\limits_{x>0} \frac{\mathrm{e}^{-(b^{1/2}x+c)^2/2a - x^2/2}\,\mathrm{d}x}{1+x}$$

But $0 < b/a < 1,\ 0 < c < \sqrt{b} \Rightarrow (b^{1/2}x + c)^2/2a + \frac{1}{2}x^2 < x^2 + x + \frac{1}{2} \Longrightarrow$

$$P\{A\} \ge K_4\sqrt{b/a}, \qquad K_4 = (\mathrm{e}^{-1/2}/\pi)\int_0^\infty \frac{\mathrm{e}^{-x^2-x}\,\mathrm{d}x}{1+x} > \frac{1}{20} \tag{3.9}$$

and this also holds for $P\{A'\}$, same K_4, on replacing the b, a by b', a'. This proves (3.2) with a constant $M = \theta^{-1/2}K_3/K_4^2$. □

Incidentally, the work getting (3.8) and (3.9), or (2.2), also gives

$$P\{A\} \le K_5 \theta^{-1/2}(b/a)^{1/2} \tag{3.10}$$

for a numerical constant K_5.

The rest of the proof of Theorem 1.2 A well known extension of the Borel-Cantelli Lemma, [10], states that for events $\{A_k\}$ in a probability space, if

$$P\{A_k A_j\} \le M\, P\{A_k\}P\{A_j\} \tag{3.11}$$

for all k, j $k \neq j$ with a constant M independent of all k, j, then $P\{A_k \text{ i.o.}\}$ equals 0 or equals a positive number according as the series $\sum_{k=1}^{\infty} P\{A_k\}$ converges or diverges. For the events of Theorem 1.2 it follows from (3.2) that we have exactly the situation of (3.11). The event $\{A_k \text{ i.o. } k \uparrow \infty\}$ is a tail event for Brownian motion and therefore has probability 0 or 1, so if its probabilty is positive it must equal 1. The last assertion of the theorem follows from (3.9) and (3.10) which show that $P\{A_k\} \asymp \sqrt{b_k/a_k}$. □

4 Recurrence at infinity of sets for a class of subordinators

In this section $S = \{S_t = S(t)\}_{t \ge 0}$ denotes a *pure jump* (no continuous "drift") subordinator with infinite lifetime. References for this section are [1], [7], and [8] though our notation is different. Also in this section P_x denotes probabilites for S with $S(0) = x$, but $P_0 = P$. Under these assumptions the Laplace transform of S is given by

$$\begin{aligned} &E\mathrm{e}^{-\lambda S_t} = \mathrm{e}^{-tg(\lambda)}, \\ g(\lambda) = &\int_0^\infty (1 - \mathrm{e}^{-\lambda x})\nu(\,\mathrm{d}x) = \lambda \int_0^\infty \mathrm{e}^{-\lambda x}\,\widetilde{\nu}\,(x)\,\mathrm{d}x, \qquad \widetilde{\nu}\,(x) = \nu\{(x,\infty)\} \end{aligned} \tag{4.1}$$

where ν, the Lévy measure, satifies $\int \min(1,x)\nu(\,\mathrm{d}x) < \infty$.

An unbounded Borel set $B \subset (0,\infty)$ is recurrent at infinity for S if there is never a *last time* at which S is in B, i.e., $P[\sup(t : S_t \in B) = \infty] = 1$ and B is transient if this probability equals 0. See [7]. (Being a tail event for the subordinator, the probability equals either 0 or 1.) Because the motivation is the application to the exclusion/non-exclusion of a deterministic family of large intervals from the zero set of a Markov process, Theorem 1.3, we confine attention to sets of the form $B = \bigcup_k\, [a_k, a_k + b_k]$ for given positive sequences $\{a_k\}$, $\{b_k\}$ satisying:

$$a_k + b_k < a_{k+1} \;\; \forall\, k \;\text{ and }\;\; a_k,\; b_k \to \infty, \;\text{ as }\; k \to \infty\,. \tag{4.2}$$

For $a > S_0$ let $\sigma_a = \inf(t : S_t > a)$ and $Z_a = Z(a) = S(\sigma_a) - a$, the overshoot amount at level a. Clearly B is recurrent at infinity if and only if $P[\,Z(a_k) \le b_k \text{ i.o. } k \uparrow \infty\,] = 1$. (Note that $P[Z(a_k) = 0 \text{ for some } k] = 0$. See [4], Theorem 3, p. 14.)

In addition to ν we will also need the renewal function:

$$U(x) = \int_0^\infty P[S_t \le x]\,\mathrm{d}t\,.$$

From Corollary 1, p.13, [4] (with $\delta = 0$) or Proposition 1.4 of [1], we get the bounds

$$U(x) \asymp \frac{x}{I(x)} \quad \forall\, x > 0, \qquad I(x) = \int_0^x \widetilde{\nu}\,(y)\,\mathrm{d}y \tag{4.3}$$

meaning that the LHS is bounded above and below by a positive constant multiple of the RHS.

The event $Z_a \le b$ occurs if and only if the *range* of the subordinator has non-empty intersection with $[a, a+b]$. Hence by [1], Lemma 5.5, page 45, we have:

$$\frac{U(a+b) - U(a)}{U(b)} \le P[Z_a \le b] \le \frac{U(a+2b) - U(a)}{U(b)}\,. \tag{4.4}$$

We now assume that U has a positive monotone (decreasing) density, i.e.

$$U(x) = \int_0^x u(y)\,\mathrm{d}y, \ \forall\, x > 0, \quad \text{and} \quad 0 < u(x_2) \le u(x_1) \quad \text{whenever} \quad x_1 \le x_2\,. \tag{4.5}$$

This is a very strong assumption but it does hold in the case that S is the inverse local time at a non-singular point of a linear diffusion. (More about this later.) With this assumption in place we have $U(a+b) - U(a) \ge u(a+b)b, \quad U(a+2b) - U(a) \le u(a)2b$ which, with (4.3), gives

$$c_1 u(a+b) I(b) \le P[Z_a \le b] \le c_2 u(a) I(b)\,. \tag{4.6}$$

This and the B.-C. Lemma, show that convergence of $\sum_k u(a_k) I(b_k)$ implies transience of B.

For the converse it seems most straighforward to make use of Lamperti's extension of the B.-C. Lemma again and for this it suffices to establish an inequality

$$P[Z(a_k) \le b_k,\ Z(a_j) \le b_j] \le C\, P[Z(a_k) \le b_k]\, P[Z(a_j) \le b_j] \quad \forall\, j > k \tag{4.7}$$

for some constant C independent of j and k.

This task requires an explicit formula for the distribution of Z_a, not simply the inequalities (4.4). Here is the formula from [8], Lemma 2.3:

$$F(z;a) \equiv P[Z_a \le z] = \int_a^{a+z} \widetilde{\nu}\,(a+z-x)\,\mathrm{d}U(x) \tag{4.8}$$

In addition to (4.2) we now also impose

$$\limsup_k (a_k + b_k)/a_{k+1} < 1, \qquad 0 < b_k = \mathrm{O}(a_k)\,. \tag{4.9}$$

Fix k_0, θ so that $0 < \theta < 1$ and $(a_k + b_k)/a_{k+1} \le 1 - \theta, \quad \forall\, k \ge k_0$. Let $a, b; a', b'$ stand for $a_k, b_k; a_j, b_j$, with $j > k \ge k_0$, respectively. Note that $a + b < a'$, and $\theta a' \le a' - a - b$. The stationary independent increments property implies that the conditional law of the process $\{S(t+t_0),\ t \ge 0\}$ *given* $\{S(s),\ s \le t_0,\ \&\ S(t_0) = x\}$, coincides with the unconditional law of $\{x + S(t),\ t \ge 0\}$ *given* $S(0) = 0$, $t_0 < \infty$ a stopping time. Since $\sigma_a \le \sigma_{a'} < \infty$, it follows that

$$\begin{aligned} P\big[\, Z_a \le b,\ Z_{a'} \le b' \,\big] &= E\Big\{ P_{x+a}\big[\, Z_{a'} \le b' \mid \mathcal{F}_{\sigma_a} \big] \Big\lfloor_{\, x = Z_a};\ Z_a \le b \Big\} \\ &= E\Big\{ P\big[Z_{a'-a-x} \le b' \big] \Big\lfloor_{\, x = Z_a};\ Z_a \le b \Big\} \\ &= \int_0^b F(b'; a' - a - x)\,\mathrm{d}_x F(x;a) \end{aligned}$$

where $\mathcal{F}_{\sigma_a}$ is the usual "stopped" σ-field generated by $\{S(t),\ t \le \sigma_a\}$. Assuming (4.5), it follows from (4.8) that $F(z;a) = \int_0^z \widetilde{\nu}\,(x) u(a+z-x)\,\mathrm{d}x$ is non-increasing in a, and, because $a' - a - x \ge \theta a'$ for $0 \le x \le b$, we conclude

$$P\big[\, Z_a \le b,\ Z_{a'} \le b' \,\big] \le F(b'; \theta a') F(b; a)\,. \tag{4.10}$$

To make more progress, it becomes necessary to somehow replace $F(b'; \theta a')$ by the smaller $F(b'; a')$ in the last inequality. From (4.6) we find that

$$\frac{F(b'; \theta a')}{F(b'; a')} \le \frac{u(\theta a')}{u(a' + b')} \tag{4.11}$$

We now make one final assumption:

$$\limsup_{x\to\infty} \frac{u(x/2)}{u(x)} < \infty \tag{4.12}$$

and this, along with $b_k = \mathrm{O}(a_k)$ and monotonicity of u, implies that the RHS of (4.11) is bounded independent of a', b', i.e., $F(b';\theta a') \le CF(b';a')$ with C independent of a', b' (but not independent of θ). (See [6], page 289, the definition and the exercises on that page.) Therefore (4.7) is indeed correct, the RHS being the same as $C\,F(b_k;a_k)F(b_j;a_j)$, and our work to this point almost completes the proof of the:

Theorem 4.1. *If $B = \bigcup_k [a_k, a_k + b_k]$ with sequences $\{a_k,\, b_k\}$ satisfying* (4.2) *and* (4.9)*, and if the subordinator S satisfies* (4.1) *and the renewal function satisfies* (4.5) *and* (4.12)*, then B is recurrent at infinity if and only if*

$$\sum_k u(a_k)I(b_k) = \infty \tag{4.13}$$

If $a \to \infty$, $b = \mathrm{O}(a)$ then from (4.6) and (4.12) we get that $P[Z(a_k) \le b_k] \asymp u(a_k)I(b_k)$ for $k \to \infty$, so the series $\sum_k P[Z(a_k) \le b_k]$ and $\sum_k u(a_k)I(b_k)$ converge or diverge together. Lamperti's theorem and (4.7) show that divergence implies $P[Z(a_k) \le b_k \ \text{i.o.}\ k \uparrow \infty\,] > 0$. But $[Z(a_k) \le b_k \ \text{i.o.}\,]$ is a tail event for the subordinator and therefore has probability 0 or 1. Consequently, divergence implies the probability equals 1, and now the proof is complete. □

5 Proof of Theorem 1.3

The proof makes use of the observation mentioned before and proved in [9], § 6, page 225, that the set of zeros of the diffusion $\mathbf{X}$ corresponds to the closure of the *range* of a subordinator, to wit, the inverse local time, L^{-1}, of $\mathbf{X}$ at 0. The assertion of the theorem is equivalent to the statement that $B = \bigcup_k [a_k, a_k + b_k]$ is either a transient or recurrent set (at infinity) for L^{-1} depending on the behavior of the series (1.5).

For the Browninan motion $\mathbf{B}$, denote by $\ell(t,x)$ or $\ell_x(t)$ the standard Brownian local time, jointly continuous in (t,x), normalized as in [9]. We may suppose, without loss of generality, that the diffusion $\mathbf{X}$ has the representation:

$$\begin{aligned}
&\mathbf{X}(t) = \mathbf{B} \circ \mu^{-1}(t), \quad \mu^{-1}(t) = \min(s : \mu(s) = t) \\
&\mu(s) = \int \ell(s,x)\,\mathrm{d}m(x)
\end{aligned}$$

the integral being over the entire state interval. (What is being set-up here is quite general – the assumption (1.4) will not come into play for awhile.) Let L denote local time for $\mathbf{X}$ at 0. Then $L(t) = \ell_0 \circ \mu^{-1}(t)$ and

$$\begin{aligned}
L^{-1}(s) &= \min(t : L(t) > s) = \mu(\ell_0^{-1}(t)) \\
&= \int \ell_x(\ell_0^{-1}(s))\,\mathrm{d}m(x); \quad \ell_0^{-1}(s) = \min(t : \ell_0(t) > s)
\end{aligned} \tag{5.1}$$

The Lévy-Khintchine formula for L^{-1} has the form (4.1) (no drift or killing terms as occur in [1], Theorem 1.2) because m has no atom at 0 and the diffusion is persistent. The measure ν also has infinite mass. (For a proof see [9], Chapter 6, and in particular §6.2 and Problem 2 on page 218. See also Chapter 14 in [12].) As before $\widetilde{\nu}(x) = \nu\{(x,\infty)\}$ which is finite for $x > 0$

Put

$$\Delta(s,x) = \ell_x(\ell_0^{-1}(s)) - \ell_x(\ell_0^{-1}(s-)) = \lim_{r\uparrow s} \ell_x(\ell_0^{-1}(s)) - \ell_x(\ell_0^{-1}(r))$$

From (5.1) we get:

$$L^{-1}(s) - L^{-1}(s-) = \int \Delta(s,x)\,\mathrm{d}m(x)$$

the integral being over the state interval of $\mathbf{X}$. Let $\mathbf{X}^*$ be another diffusion on natural scale with the same state interval and with the associated objects decked out with $*$. We may suppose $\mathbf{X}^*$ is also represented as a time change of the *same* Brownian motion as $\mathbf{X}$. If for some constant $K > 0$, $\mathrm{d}m(x) \le K\,\mathrm{d}m^*(x)$ for all x, then

$$L^{-1}(s) - L^{-1}(s-) = \int \Delta(s,x)\,\mathrm{d}m(x) \le K \int \Delta(s,x)\,\mathrm{d}m^*(x) = K\big(L^{*-1}(s) - L^{*-1}(s-)\big)$$

for all s. Thus, for any $x > 0$, the number of jumps in $L^{-1}(t)$ of magnitude greater than x during some interval $0 \le t \le s$, does not exceed the number of jumps of magnitude greater than x/K of L^{*-1} during the same interval. The expected number of such jumps per unit time equals $\widetilde{\nu}(x)$ for $\mathbf{X}$ and $\widetilde{\nu}^*(x/K)$ for $\mathbf{X}^*$ [1] and this argument proves:

Lemma 5.1. *If* $\mathrm{d}m/\mathrm{d}m^* \le K$ *(a.e.* m^**) for a constant* $K > 0$*, then* $\widetilde{\nu}(x) \le \widetilde{\nu}^*(x/K)$ $\forall\, x > 0$*. A similar result holds with* $\le$ *replaced by* $\ge$ *throughout.*

If ν_q denotes the Lévy measure of the inverse local time at 0 of the diffusion on natural scale associated with the speed measure $|x|^q\,\mathrm{d}x$ with q as in (1.4), then

$$\widetilde{\nu}_q(x) = \kappa\, x^{-\beta}, \quad \beta = \frac{1}{q+2} \in (0,1]$$

for some constant κ depending only on q. See [9], §6. Applying Lemma 5.1 with $\mathrm{d}m^*$ equal to $Cx^q\,\mathrm{d}x$ for the uppper bound and equal to $cx^q\,\mathrm{d}x$ for the lower bound, gives:

Lemma 5.2. *There exist numbers* $C_1 > 0$*,* $C_2 > 0$ *depending only on* c*,* C*, and* q *of* (1.4)*, such that*

$$C_1 x^{-\beta} \le \widetilde{\nu}(x) \le C_2 x^{-\beta}\,. \tag{5.2}$$

Let U denote the renewal function of L^{-1}. From (5.2) and (4.3) we get

Corollary 5.3. *Under the assumptions of the last lemma, there exist numbers* C_3*,* C_4 *such that*

$$C_3 x^{\beta} \le U(x) \le C_4 x^{\beta} \quad \forall\, x \tag{5.3}$$

The inverse local time at a point of a diffusion (and more generally of a *gap* diffusion) is a special kind of subordinator and its Lévy measure and renewal measure have very nice properties. In particular the renewal function U has a density $u(x) = \mathrm{d}U(x)/\mathrm{d}x$ which is non-increasing on $(0,\infty)$. See [12], in particular, Theorem 10.3, (11.3 in 2nd ed.) See also [1] Corollary 9.7, page 78.

Lemma 5.4. *There exists a number* $C_5 > 0$ *such that*

$$C_5 x^{\beta-1} \le u(x) \le C_4 x^{\beta-1}, \quad x > 0 \quad (C_4 \text{ as above})\,. \tag{5.4}$$

Note that because $\beta - 1 < 0$, both $\widetilde{\nu}$ and u blow up at 0.

Proof. From (5.3) and monotonicity, $xu(x) \le \int_0^x u(y)\,\mathrm{d}y = U(x) \le C_4 x^{\beta}$ for any $x > 0$. On the other hand, for any $n > 1$, we have

$$(C_3 n^{\beta} - C_4)x^{\beta} \le U(nx) - U(x) = \int_x^{nx} u(y)\,\mathrm{d}y \le u(x)(n-1)x$$

and the LHS (5.4) holds with $C_5 = [C_3 n^{\beta} - C_4]/(n-1)$, for any fixed $n > (C_4/C_3)^{1/\beta}$. □

[1] The word *time* here refers to the parameter t of inverse local times not the t of $\mathbf{X}$ or $\mathbf{B}$.

Lemma 5.4 implies (4.12) of §4 and Lemma 5.2 implies $I(x) = \int_0^x \widetilde{\nu}(y)\,\mathrm{d}y \asymp x^{1-\beta}$. Hence

$$u(a_k)I(b_k) \asymp (b_k/a_k)^{1-\beta} = (b_k/a_k)^{\frac{q+1}{q+2}}$$

and Theorem 4.1 in in §4 now applies to yield the conclusion of Theorem 1.3.

Remark. If $\mathbf{X}$ is a mean 0 stable process of index $\alpha \in (1,2]$, then the conclusion of Theorem 1.3 remains valid for the zeros of $\mathbf{X}$ but with the exponent $(q+1)/(q+2)$ replaced by $1/\alpha$. The inverse local time at 0 of X is a stable subordinator of index $1-1/\alpha$ ([1], §8.1), so its Lévy measure $\widetilde{\nu}(x) = \text{Const.} \times x^{-1+1/\alpha}$ and the renewal density $u(x) = \text{Const.} \times x^{-1/\alpha}$. Our proof easily adapts to get the conclusion or one might also make a proof using the Wiener style test of [7], but the calculations of capacities of $\{\cup_k[a_k, a_k+b_k]\} \cap [2^n, 2^{n+1})$ required (see Lemma 2, p. 58 of [7]) will be left to the interested reader. Similar results involving regular variation of Lévy measures are perhaps possible but will also be left to the interested reader.

References

[1] Jean Bertoin, *Subordinators: Examples and applications*, Ecole d'été de Probabilitiés de St-Flour, 1997.

[2] Patrick Billingsley, *Probability and measure*, John Wiley & Sons, 1995. MR-1324786

[3] Kai Lai Chung and Paul Erdös, *On the application of the Borel-Cantelli Lemma*, Transaction of the American Mathematical Society **72** (1952), 179–186.

[4] Ronald A. Doney, *Fluctuation theory for Lévy processes*, Springer, 2007. MR-2320889

[5] K. Bruce Erickson, *Gaps in the range of nearly increasing processes with stationary independent increments*, Z. Wahrs. v. Gebiete **62** (1983), 449–463. MR-0690570

[6] W. Feller, *An introduction to probability theory and its applications*, 2nd ed., vol. II, John Wiley and Sons, New York, 1971.

[7] John Hawkes, *Polar sets, regular points, and recurrent sets for the symmetric and increasing stable processes*, Bull. London Math. Soc. **2** (1970), 53–59. MR-0261693

[8] John Hawkes, *On the potential theory of subordinators*, Z. Wahrs. v. Gebiete **33** (1975), 113–132. MR-0388553

[9] K. Itô and H. P. Mckean, Jr, *Diffusion processes and their sample paths*, 2nd ed., Springer, New York, 1974. MR-0345224

[10] John Lamperti, *Wiener's test and Markov chains*, J. Math. Analysis and Applications **6** (1963), 58–66. MR-0143258

[11] T. S. Mountford, P. O'hara, and S. Port, *Large gaps in the range of stable processes*, Journal of Theoretical Probability **10** (1997), 25–50. MR-1432614

[12] René L. Schilling, Renming Song, and Zoran Vondraček, *Bernstein functions: Theory and applications*, 1st ed., Studies in Mathematics, vol. 37, DeGruyter, Berlin/Boston, 2010. MR-2598208