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Weighted maximal inequality for differentially subordinate martingales*

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Abstract

We establish a weighted maximal L^1-inequality for differentially subordinate martingales taking values in $\mathbb{R}^\nu$, $\nu \geq 1$, under the assumption that the weight satisfies Muckenhoupt's condition A_1. An optimal dependence of the constant on the A_1 characteristics is identified.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, equipped with a continuous-time filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all the events of probability 0. Suppose that X, Y are adapted local martingales taking values in $\mathbb{R}^\nu$ (for some $\nu \geq 1$), whose trajectories are right-continuous and have limits from the left. We will use the notation $X^* = \sup_{s\geq 0} |X_s|$ and $X_t^* = \sup_{0\leq s\leq t} |X_s|$ for the maximal and the truncated maximal function of X. Let $[X]$, $[Y]$ denote the quadratic variation processes (square brackets) of X and Y; see Dellacherie and Meyer [5] for details when $\nu = 1$, and extend the definition to the higher-dimensional setting by $[X] = \sum_{n=1}^{\nu} [X^n]$, where X^n denotes the n-th coordinate of X. Following Bañuelos and Wang [3] and Wang [17], the process Y is differentially subordinate to X, if, with probability 1, the difference $[X] - [Y] = ([X]_t - [Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t. This notion arises naturally in the context of stochastic integration. Suppose that X is a local martingale, H is a predictable process and let $Y = H \cdot X$ be the stochastic integral of H with respect to X:

$$Y_t = H_0 X_0 + \int_0^t H_s \mathrm{d}X_s, \qquad t \geq 0.$$

If H takes values in $[-1, 1]$, then Y is differentially subordinate to X; this follows at once from the identity

$$[X]_t - [Y]_t = \int_0^t (1 - H_s^2) \mathrm{d}[X]_s, \qquad t \geq 0.$$

Differential subordination of Y to X implies many interesting inequalities between the processes (e.g., moment, weak-type, exponential, etc.), which can be applied in many

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areas of mathematics. See e.g. the monograph [13] by the author and the papers [1], [2], [3], [6], [16], [17] for an overview of the results in this direction. In this paper, our particular emphasis will be put on maximal inequalities. In [4], Burkholder introduced a general method of proving such estimates in the context of stochastic integrals and exploited it to establish the following result.

Theorem 1.1. *Let X be a real-valued martingale and $Y = H\cdot X$, where H is a predictable process with values in $\{-1,1\}$. Then we have*

$$||Y||_1 \leq \eta ||X^*||_1,$$

where $\eta = 2.536\ldots$ is the unique solution of the equation

$$\eta - 3 = -\exp\Big(\frac{1-\eta}{2}\Big).$$

The constant is the best possible.

If the martingales X and Y are assumed to have continuous trajectories, the constant changes. Here is the result of [12], under the less restrictive assumption of differential subordination and in the wider context of vector-valued processes.

Theorem 1.2. *If X, Y are continuous-path local martingales taking values in $\mathbb{R}^\nu$ such that Y is differentially subordinate to X, then*

$$||Y||_1 \leq \sqrt{2}||X^*||_1. \tag{1.1}$$

The constant $\sqrt{2}$ is optimal.

See also [10], [11] and [15] for related results and extensions. We will be interested in the following *weighted* version of (1.1):

$$||Y||_{L^1(W)} \leq C||X^*||_{L^1(W)}. \tag{1.2}$$

Here W is a weight, i.e., a uniformly integrable, positive, mean-one and continuous-path martingale $W = (W_t)_{t\geq 0}$, and we have used the standard notation

$$||Y||_{L^1(W)} = \sup_{t\geq 0} \mathbb{E}|Y_t|W_\infty \quad \text{and} \quad ||X^*||_{L^1(W)} = \mathbb{E}X^*W_\infty$$

for the weighted L^1 norms of Y and X^*. It is not difficult to see that the above bound cannot hold with some finite C for all processes W. A natural assumption on the weight (in the context of the above L^1 estimate) is that it belongs to the class A_1. This class was originally introduced by Muckenhoupt [9] in the analytic setting, and its probabilistic counterpart is due to Izumisawa and Kazamaki. Following [7] and [8], W satisfies the A_1 condition if there is a finite constant c such that $\mathbb{P}(W_t^* \leq cW_t \text{ for all } t\geq 0) = 1$. The least c with this property is denoted by $[W]_{A_1}$ and called the A_1 characteristics of W.

We will show that if W belongs to the class A_1, then (1.2) holds for all martingales X, Y satisfying the differential subordination. Actually, we will additionally study the following aspect of the weighted bound. Namely, there is a very interesting question of extracting the sharp dependence of the constant C on the characteristics $[W]_{A_1}$. More precisely: what is the least exponent κ for which there exists an absolute constant $\tilde{C}$ such that

$$||Y||_{L^1(W)} \leq \tilde{C}[W]_{A_1}^\kappa ||X||_{L^1(W)}$$

for all W and all X, Y satisfying the differential subordination?

The main result of this paper gives a full answer to this question.

Theorem 1.3. *Let X, Y be continuous-path martingales such that Y is differentially subordinate to X. Then for any A_1 weight W we have*

$$||Y||_{L^1(W)} \leq C[W]_{A_1}||X^*||_{L^1(W)}, \tag{1.3}$$

where $C = 5 + 2\ln(3/2) = 5.81093\ldots$. The dependence on the A_1 characteristics of the weight is optimal in the sense that for any $\kappa < 1$ and any $K > 0$, there is a weight W, a real-valued martingale X and a predictable sequence H with values in $\{-1, 1\}$ such that the stochastic integral $Y = H \cdot X$ satisfies

$$||Y||_{L^1(W)} > K[W]^{\kappa}_{A_1}||X^*||_{L^1(W)}.$$

We should emphasize here that the constant C we obtain above is not sharp, however, we believe that it is not far from the optimal one.

There is a well-known method of proving maximal inequalities for stochastic integrals and differentially subordinate martingales. This method, invented by Burkholder in [4] and modified by the author in [12, 13], allows to deduce a given estimate from the existence of a certain special function, enjoying appropriate majorization and concavity. However, we should stress here that all the works in which the method has been successfully implemented, concerned the unweighted setting. To the best of our knowledge, this paper contains the first example in which Burkholder's method has been successfully applied to yield a nontrivial weighted maximal bound for differentially subordinate martingales.

The inequality (1.3) is proved in the next section. The optimality of the exponent 1 is studied in Section 3. In the final part of the paper we sketch some ideas leading to the special function U on which the proof of (1.3) rests.

2 Proof of the maximal inequality

Let $c \geq 1$ be a fixed parameter and consider the set

$$\mathcal{D} = \{(x, y, z, w, v) \in \mathbb{R}^\nu \times \mathbb{R}^\nu \times (0, \infty)^3 : |x| \leq z,\ c^{-1} \leq w/v \leq 1\}.$$

As we have mentioned in the introduction, a crucial role in the proof of (1.3) is played by a special function. Let $U = U^{(c)} : \mathcal{D} \to \mathbb{R}$ be given by

$$U^{(c)}(x, y, z, w, v) = \frac{|y|^2 v \ln\left(\sqrt{2}c^{-1}w/v + 1 - c^{-2}\right) - \gamma|x|^2 v}{z} - \gamma z v, \tag{2.1}$$

where $\gamma = 2 + \ln(3/2)$. Let us study some crucial properties of the special function. Recall that C is the constant appearing in (1.3).

Lemma 2.1. (i) *For any x, $y \in \mathbb{R}^\nu$ satisfying $|y| \leq |x|$ and any w, $v > 0$ with $c^{-1} \leq w/v \leq 1$, we have*

$$U(x, y, |x|, w, v) \leq 0. \tag{2.2}$$

(ii) *For any $(x, y, z, w, v) \in \mathcal{D}$, we have*

$$U(x, y, z, w, v) \geq |y|w - Cczw. \tag{2.3}$$

(iii) *For any $x \in \mathbb{R}^\nu \setminus \{0\}$ and $y \in \mathbb{R}^\nu$ and any w, $v > 0$ satisfying $c^{-1} \leq w/v \leq 1$, we have*

$$U_z(x, y, |x|, w, v) \leq 0. \tag{2.4}$$

(iv) *For any x, $y \in \mathbb{R}^\nu$, $z > 0$ (satisfying $z \geq |x|$) and $w > 0$, we have*

$$U_v(x, y, z, w, w) \leq 0. \tag{2.5}$$

(v) *For any* $(x,y,z,w,v)\in\mathcal{D}$*, the* $(\nu+1)\times(\nu+1)$ *matrix*

$$\mathcal{M}(x,y,z,w,v)=\begin{bmatrix}(U_{xx}+U_{yy})(x,y,z,w,v) & U_{yw}(x,y,z,w,v)\\ U_{wy}(x,y,z,w,v) & U_{ww}(x,y,z,w,v)\end{bmatrix}$$

is nonpositive-definite. (Here $U_{xx}+U_{yy}$ *denotes the matrix* $[U_{x_ix_j}+U_{y_iy_j}]_{1\leq i,j\leq\nu}$*, and* U_{yw}*,* U_{wy} *stand for column and vector with entries* U_{y_1w}*,* U_{y_2w}*, ...,* $U_{y_\nu w}$*, respectively).*

Proof. The proof of (2.2) is very simple: since $\ln(1+s)\leq s$ for any positive s, we may write

$$\ln\left(\sqrt{2}c^{-1}w/v+1-c^{-2}\right)\leq\sqrt{2}c^{-1}w/v\leq\sqrt{2}c^{-1}\leq\gamma$$

and hence

$$U(x,y,z,w,v)\leq\frac{(|y|^2-|x|^2)\gamma v}{z}\leq 0.$$

To show (2.3), note that $\ln(1+s)\geq s/(1+s)$ for all $s>0$, which implies

$$\ln\left(\sqrt{2}c^{-1}w/v+1-c^{-2}\right)\geq\frac{\sqrt{2}c^{-1}w/v-c^{-2}}{1+\sqrt{2}c^{-1}w/v-c^{-2}}.$$

But $c^{-2}\leq c^{-1}w/v$ and

$$1+\sqrt{2}c^{-1}w/v-c^{-2}\leq 1+\sqrt{2}c^{-1}-c^{-2}\leq 3/2, \tag{2.6}$$

so the preceding estimate gives

$$\ln\left(\sqrt{2}c^{-1}w/v+1-c^{-2}\right)\geq\frac{(\sqrt{2}-1)c^{-1}w/v}{3/2}\geq\frac{c^{-1}}{4}\frac{w}{v}.$$

Therefore, the majorization (2.3) will be established if we manage to show that

$$\frac{c^{-1}w}{4}|y|^2-|y|zw+Ccz^2w\geq\gamma(|x|^2+z^2)v.$$

But observe that

$$\begin{aligned}\frac{c^{-1}w}{4}|y|^2-|y|zw+Ccz^2w&=\frac{c^{-1}w}{4}\left(|y|-2cz\right)^2+(Cc-c)z^2w\\&\geq(C-1)cz^2w=2\gamma cz^2w\geq 2\gamma z^2v\geq\gamma(|x|^2+z^2)v.\end{aligned}$$

The inequality (2.4) is evident once one computes the partial derivative with respect to z:

$$U_z(x,y,|x|,w,v)=-\frac{|y|^2v\ln(\sqrt{2}c^{-1}w/v+1-c^{-2})}{z^2}.$$

To show (2.5), we derive that

$$zU_v(x,y,z,w,w)=|y|^2\left[\ln(\sqrt{2}c^{-1}+1-c^{-2})-\frac{\sqrt{2}c^{-1}}{\sqrt{2}c^{-1}+1-c^{-2}}\right]-\gamma|x|^2-\gamma z^2$$

and hence we will be done if we show that the expression in the square brackets is nonpositive. This is elementary: substitute $a=c^{-1}\in[0,1]$ and consider the function

$$\xi(a)=\ln(\sqrt{2}a+1-a^2)-\frac{\sqrt{2}a}{\sqrt{2}a+1-a^2}.$$

It suffices to note that $\xi(0)=0$ and

$$\xi'(a)=\frac{2a^2(-2\sqrt{2}+a)}{(\sqrt{2}a+1-a^2)^2}\leq 0$$

provided $a \in (0,1)$. This yields (2.5). Finally, we turn our attention to the property (v). We easily check that $\mathcal{M}(x,y,z,w,v)$ equals

$$\begin{bmatrix} \frac{2v(\ln(\sqrt{2}c^{-1}w/v+1-c^{-2})-\gamma)}{z}\cdot \mathrm{Id} & \frac{2\sqrt{2}yc^{-1}}{z(\sqrt{2}c^{-1}w/v+1-c^{-2})} \\ \frac{2\sqrt{2}yc^{-1}}{z(\sqrt{2}c^{-1}w/v+1-c^{-2})} & -\frac{2c^{-2}|y|^2}{zv(\sqrt{2}c^{-1}w/v+1-c^{-2})^2} \end{bmatrix},$$

where Id denotes the identity matrix of dimension $\nu\times\nu$. Since $\ln(\sqrt{2}c^{-1}+1-c^{-2}) \leq \sqrt{2}c^{-1} \leq \gamma$, it is enough to prove that $(-1)^{\nu+1}\det\mathcal{M}(x,y,z,w,v) \geq 0$, by virtue of Sylvester's criterion. Most of the entries of the matrix are zero, and it is not difficult to compute the determinant. If we substitute $A = (2v(\ln(\sqrt{2}c^{-1}w/v+1-c^{-2})-\gamma))/z < 0$, and use cofactor expansion along the last row, we see that

$$\begin{aligned} &(-1)^{\nu+1}\det\mathcal{M}(x,y,z,w,v) \\ &= (-1)^{\nu+1}\left[-A^\nu\cdot\frac{2|y|^2c^{-2}}{zv(\sqrt{2}c^{-1}w/v+1-c^{-2})^2} - A^{\nu-1}\cdot\frac{8|y|^2c^{-2}}{z^2(\sqrt{2}c^{-1}w/v+1-c^{-2})^2}\right] \\ &= \frac{4|y|^2c^{-2}(-A)^{\nu-1}}{z^2(\sqrt{2}c^{-1}w/v+1-c^{-2})^2}\left[\gamma-\ln(\sqrt{2}c^{-1}w/v+1-c^{-2})-2\right]. \end{aligned}$$

Thus, we must prove that the expression in the square brackets is nonnegative. But this is immediate when one recalls the definition of γ and notes that by (2.6), $\ln(\sqrt{2}c^{-1}w/v+1-c^{-2}) \leq \ln(3/2)$.

The proof is complete. □

Proof of (1.3). Fix an arbitrary $\varepsilon > 0$ and pick martingales X, Y, W as in the statement. Let $c = [W]_{A_1}$. We will apply Itô's formula to the composition of $U^{(c)}$ and the process $P_t = (X_t, Y_t, X_t^*\vee\varepsilon, W_t, W_t^*)$, $t\geq 0$. For any $t \geq 0$ we have $X_t^*\vee\varepsilon > 0$ and $W_t^* \leq cW_t$, by the very definition of $[W]_{A_1}$. Hence the process P takes values in the domain of U and the composition $U(P)$ makes sense. Furthermore, U has the necessary regularity: actually, the formula (2.1) can be used for all $(x,y,z,w,v)\in\mathbb{R}^\nu\times\mathbb{R}^\nu\times(0,\infty)^3$ and defines a C^∞ function there, so the use of Itô's formula is permitted. As the result, we obtain

$$U(P_t) = I_0 + I_1 + I_2 + I_3/2,$$

where

$$\begin{aligned} I_0 &= U(P_0), \\ I_1 &= \int_0^t U_x(P_s)\cdot \mathrm{d}X_s + \int_0^t U_y(P_s)\cdot\mathrm{d}Y_s + \int_0^t U_w(P_s)\mathrm{d}W_s, \\ I_2 &= \int_0^t U_z(P_s)\mathrm{d}(X_s^*\vee\varepsilon) + \int_0^t U_v(P_s)\mathrm{d}W_s^*, \\ I_3 &= \int_0^t U_{xx}(P_s)\mathrm{d}[X]_s + \int_0^t U_{yy}(P_s)\mathrm{d}[Y]_s + \int_0^t U_{ww}(P_s)\mathrm{d}[W]_s \\ &\quad + 2\int_0^t U_{yw}(P_s)\mathrm{d}[Y,W]_s. \end{aligned}$$

Here $\int_0^t U_{xx}(P_s)\mathrm{d}[X]_s$ is the shortened notation for $\sum_{i,j=1}^\nu \int_0^t U_{x_ix_j}(P_s)\mathrm{d}[X^i,X^j]$, and similarly for $\int_0^t U_{yy}(P_s)\mathrm{d}[Y]_s$ and $\int_0^t U_{yw}(P_s)\mathrm{d}[Y,W]_s$. Note that the remaining second-order terms are equal to 0, either due to vanishing of the corresponding partial derivatives, or to the fact that the processes $X^*\vee\varepsilon$, W^* are nondecreasing (and hence of finite variation). Let us analyze the terms I_0 through I_3 separately. First, observe that

$$I_0 = U(X_0, Y_0, |X_0|\vee\varepsilon, W_0, W_0) \leq 0,$$

due to (2.2). The term I_1 is a local martingale, by the properties of stochastic integrals. To handle I_2, note that by the continuity of paths, the times at which the process $X^*\vee\varepsilon$ increases are contained in the set $\{s : |X_s| = X_s^*\}$; however, for such s we have $U_z(P_s) \leq 0$, by virtue of (2.4), so the first integral in I_2 is nonpositive. An analogous reasoning exploiting (2.5) shows that the second integral also has this property and hence $I_2 \leq 0$. To deal with I_3, observe that $U_{xx} = [U_{x_ix_j}]_{1\leq i,j\leq\nu}$ is a negative multiple of the identity matrix and hence, by the differential subordination of Y to X,

$$\int_0^t U_{xx}(P_s)\mathrm{d}[X]_s \leq \int_0^t U_{xx}(P_s)\mathrm{d}[Y]_s.$$

Furthermore, by Lemma 2.1 (v), we have

$$\int_0^t [U_{xx}(P_s) + U_{yy}(P_s)]\mathrm{d}[Y]_s + \int_0^t U_{ww}(P_s)\mathrm{d}[W]_s + 2\int_0^t U_{yw}(P_s)\mathrm{d}[Y,W]_s \leq 0,$$

which implies $I_3 \leq 0$. Putting all the above facts together, if $(\tau_n)_{n\geq 1}$ denotes the localizing sequence for the local martingale I_1, then

$$\mathbb{E}U(P_{\tau_n\wedge t}) \leq 0, \qquad n = 1, 2, \ldots.$$

By (2.3), this yields $\mathbb{E}|Y_{\tau_n\wedge t}|W_{\tau_n\wedge t} \leq Cc\mathbb{E}X^*_{\tau_n\wedge t}W_{\tau_n\wedge t}$ for all n. But W is uniformly integrable, so

$$\mathbb{E}|Y_{\tau_n\wedge t}|W_\infty = \mathbb{E}|Y_{\tau_n\wedge t}|W_{\tau_n\wedge t} \leq Cc\mathbb{E}X^*_{\tau_n\wedge t}W_{\tau_n\wedge t} = Cc\mathbb{E}X^*_{\tau_n\wedge t}W_\infty \leq Cc\mathbb{E}X^*W_\infty.$$

Letting $n\to\infty$ and applying Fatou's lemma, we get $\mathbb{E}|Y_t|W_\infty \leq C[W]_{A_1}\mathbb{E}X^*W_\infty$. Since t was arbitrary, the claim follows. □

3 On the optimality of the exponent

Let $c > 1$ be a fixed parameter, take a large positive integer N and set $\delta = c/N$. Let B be a standard, one-dimensional Brownian motion starting from 1. Consider the family $(\tau_n)_{n=0}^N$ of stopping times given by $\tau_0 \equiv 0$ and

$$\tau_n = \inf\{t : B_t \leq c^{-1}(1+\delta)^{n-1} \text{ or } B_t = (1+\delta)^n\}, \qquad n = 1, 2, \ldots, N.$$

Let $W = (B_{\tau_N\wedge t})_{t\geq 0}$ and let X, Y be martingales starting from 0, satisfying

$$\mathrm{d}X_t = (-1)^{n-1}\mathrm{d}Y_t = \frac{(-1)^{n-1}}{(1+\delta)^{n-1}}\mathrm{d}W_t$$

for $t \in [\tau_{n-1}, \tau_n)$, $n = 1, 2, \ldots, N$. Finally, put $X_t = X_{\tau_N -}$ and $Y_t = Y_{\tau_N -}$ for $t \geq \tau_N$.

Let us gain some intuition about the processes introduced above. Let us first look at W. Clearly, this process is a weight and it behavior is as follows. It starts from 1, and, on the time interval $[\tau_0, \tau_1)$, it evolves until it reaches c^{-1} or $1+\delta$. If the first possibility occurs, the process W stops; otherwise, it continues its evolution on $[\tau_1, \tau_2)$ until it reaches $c^{-1}(1+\delta)$ or $(1+\delta)^2$. In the first case the process terminates, while in the second it continues its movement on $[\tau_2, \tau_3)$ until it reaches $c^{-1}(1+\delta)^2$ or $(1+\delta)^3$, and so on, until N steps of this type are conducted. The above description immediately implies that $[W]_{A_1} \le c(1+\delta)$. Indeed, we have $W_0 = W_0^*$ and, for any $n = 1, 2, \ldots, N$, if $t \in (\tau_{n-1}, \tau_n]$, then $W_t \ge c^{-1}(1+\delta)^{n-1}$ and $W_t^* \le (1+\delta)^n$.

Now, let us look at the pair (X, Y). It starts from $(0,0)$ and, for $t \in [\tau_0, \tau_1)$, we have $dX_t = dY_t = dW_t$ or, equivalently, $X_t = Y_t = W_t - 1$. So, the pair (X, Y) moves along the line of slope 1 until it reaches one of the points $(c^{-1}-1, c^{-1}-1)$, (δ, δ). If it visits the first of these points (which means that $W_{\tau_1} = c^{-1}$), then the pair stops, since so does W. However, if $(X_{\tau_1}, Y_{\tau_1}) = (\delta, \delta)$, then the time interval $[\tau_1, \tau_2)$ is nonempty: for t belonging to this interval we have $dX_t = -dY_t = -dW_t/(1+\delta)$, or

$$X_t = \delta - \frac{W_t - W_{\tau_1}}{1+\delta}, \quad Y_t = \delta + \frac{W_t - W_{\tau_1}}{1+\delta}.$$

Therefore, on $[\tau_1, \tau_2)$, (X, Y) evolves along the line of slope -1 until it visits $(\delta + 1 - c^{-1}, \delta - 1 + c^{-1})$ or $(0, 2\delta)$. If the first possibility occurs, the pair terminates; but if $(X_{\tau_2}, Y_{\tau_2}) = (0, 2\delta)$, the movement is continued. In general, if n is an odd integer, then on $[\tau_{n-1}, \tau_n)$ the process (X, Y) moves along a line segment of slope 1 joining $(c^{-1}-1, c^{-1}-1+(n-1)\delta)$ and $(\delta, n\delta)$; it is killed when hitting the first point, and continues otherwise. If n is even, then for $t \in [\tau_{n-1}, \tau_n)$ the pair (X, Y) evolves along a line segment of slope -1, with endpoints $(1 - c^{-1} + \delta, (n-1)\delta - 1 + c^{-1})$, $(0, n\delta)$ (the first of which is absorbing, while the second is not). Directly from this description, we see that $X^* \le 1 - c^{-1} + \delta$, and hence $X^* \le 1$ provided N is sufficiently large. This implies $||X^*||_{L^1(W)} \le \mathbb{E}W_\infty = 1$.

Now, take a look at the event $A = \{W_{\tau_N} = (1+\delta)^N\}$. It follows from the above analysis that on this set we have $W_{\tau_n} = (1+\delta)^n$ for all $n = 1, 2, \ldots, N$. Consequently,

$$\begin{aligned}\mathbb{P}(A) &= \prod_{n=1}^{N} \mathbb{P}(W_{\tau_n} = (1+\delta)^n | W_{\tau_{n-1}} = (1+\delta)^{n-1}) \\ &= \prod_{n=1}^{N} \frac{(1+\delta)^{n-1} - (1+\delta)^{n-1}c^{-1}}{(1+\delta)^n - c^{-1}(1+\delta)^{n-1}} = \left(\frac{1-c^{-1}}{1-c^{-1}+\delta}\right)^N.\end{aligned}$$

Next, the above discussion concerning the behavior of (X, Y) implies that on the set A we have $Y_{\tau_n} = n\delta$ for all $n = 1, 2, \ldots, N$, which implies

$$||Y||_{L^1(W)} \ge \mathbb{E}Y_\infty W_\infty 1_A = N\delta(1+\delta)^N \cdot \left(\frac{1-c^{-1}}{1-c^{-1}+\delta}\right)^N.$$

Now recall that we have put $\delta = c/N$. Therefore, if N is sufficiently large, the latter expression can be made arbitrarily close to $c\exp(-(1-c^{-1})^{-1})$. Therefore, for any exponent $\kappa < 1$, we have

$$\frac{||Y||_{L^1(W)}}{||X^*||_{L^1(W)}[W]_{A_1}^\kappa} \ge c^{1-\kappa} \cdot \frac{\exp(-(1-c^{-1})^{-1})}{2},$$

for huge N. But the right-hand side explodes as $c \to \infty$; this proves that the exponent 1 in (1.3) is indeed the best possible.

4 On the search of a suitable function

The purpose of this section is to present some *informal* reasoning which has led us to the discovery of the special function $U = U^{(c)}$ satisfying the properties listed in Lemma 2.1. We would like to stress here that we have not tried to optimize the choice of various parameters that will arise below (which might lead to a slight improvement of the constant C in the statement of our main theorem). Instead, we rather focused on obtaining a relatively simple formula, for which the calculations will be not too complicated. It is convenient to split the search into several steps.

Step 1. It suffices to consider the case $\nu = 1$. The passage to higher dimensions will just require some minor and natural changes (one has to replace x^2, y^2 by $|x|^2$ and $|y|^2$, respectively). As a starting point, one looks at the appropriate function from the unweighted setting: as proved in [12], it is given by

$$\mathcal{U}(x, y, z) = \frac{y^2}{z} - \frac{x^2}{z} - z,$$

up to a multiplicative constant. It seems natural to expect that the function U we search for has a somewhat similar formula. A little thought and a look at (2.3) suggest that if we want to pass to the weighted setting, then each summand above should be multiplied a nonnegative expression depending on w and v, which is homogeneous of order 1: that is,

$$U(x, y, z, w, v) = \frac{y^2 v}{z}\varphi_1(w/v) - \frac{x^2 v}{z}\varphi_2(w/v) - zv\varphi_3(w/v),$$

for some $\varphi_i : [c^{-1}, 1] \to \mathbb{R}$ to be found, $i = 1, 2, 3$.

Step 2. Let us try to guess the functions φ_i. The first idea is to consider constant functions, but then the property (v) of Lemma 2.1 is not satisfied. The second thought is to assume that exactly *two* of φ_i's are constant, and it turns out that the choice $\varphi_2 \equiv \beta$, $\varphi_3 \equiv \gamma$ will do the job. Plugging this above and applying (2.4), we get

$$-\frac{y^2 v}{z^2}\varphi_1(w/v) + \beta v - \gamma v \leq 0.$$

The left-hand side is the largest when $y = 0$, and then the inequality is equivalent to $\beta \leq \gamma$. Let us assume that we actually have equality here, and let us write φ instead of φ_1. Then the formula for U becomes

$$U(x, y, z, w, v) = \frac{y^2 v}{z}\varphi(w/v) - \frac{\gamma x^2 v}{z} - \gamma zv.$$

Step 3. The next step is to find the formula for φ. An application of (2.5) enforces the condition

$$y^2(\varphi(1) - \varphi'(1)) \leq \gamma x^2 + \gamma z^2,$$

to be valid for all x, $y \in \mathbb{R}$ and positive $z \geq |x|$. Taking $y = 1$, $x = z$ and letting $z \to 0$ yields the inequality

$$\varphi(1) \leq \varphi'(1). \tag{4.1}$$

Now, let us look at the condition (v). The matrix $\mathcal{M}$ becomes

$$\mathcal{M}(x, y, z, w, v) = \frac{1}{z}\begin{bmatrix} -2\gamma v + 2v\varphi(w/v) & 2y\varphi'(w/v) \\ 2y\varphi'(w/v) & y^2 v^{-1}\varphi''(w/v) \end{bmatrix}.$$

To check whether this matrix is nonpositive-definite, we apply Sylvester's criterion. After some easy calculations, we see that (v) will hold if

$$\varphi''(s) \leq 0 \qquad \text{and} \qquad (-\gamma + \varphi(s))\varphi''(s) \geq 2(\varphi'(s))^2 \tag{4.2}$$

for all $s \in (c^{-1}, 1)$. Which function φ satisfies (4.1) and (4.2)? After some attempts, the author guessed that $\varphi(s) = \log(as + b)$ was a good choice, for some constants $a = a(c)$, $b = b(c)$ to be found.

Step 4. Let us try to find the parameters a, b and γ, working with (4.1), (4.2) and the majorization condition (2.3). By the latter inequality, we have that $\varphi > 0$ on $[c^{-1}, 1]$ (take

$x = z \to 0$ and $y = 1$ in the majorization). Thus in particular we see that $\varphi(1) > 0$, which gives $a + b > 1$, and $\varphi(c^{-1}) > 0$, which is equivalent to

$$ac^{-1} + b \geq 1. \tag{4.3}$$

Next, (4.1) yields

$$\log(a + b) - \frac{a}{a + b} \leq 0, \tag{4.4}$$

which combined with $a + b > 1$ implies that a is positive. Let us further exploit the two inequalities above: they yield some crucial information on a and b. First, (4.4) implies that $b < 1$: indeed, for $b = 1$ the inequality does not hold, and the left-hand side is an increasing function of b. Thus, we have $b = 1 - \varepsilon$ for some $\varepsilon = \varepsilon(c) > 0$; then (4.3) gives $a \geq c\varepsilon$, and since the left-hand side of (4.4) is an increasing function of a, we get that

$$\log(1 + (c - 1)\varepsilon) \leq \frac{c\varepsilon}{1 + (c - 1)\varepsilon}.$$

Let us try to extract some information on the size of ε from this estimate. We must have $\varepsilon = o(c^{-1})$ as $c \to \infty$, since otherwise letting $c \to \infty$ above gives a contradiction. Now, transform the latter bound into the equivalent form

$$\frac{\log(1 + (c - 1)\varepsilon) - (c - 1)\varepsilon}{(c - 1)^2\varepsilon^2} \cdot (1 + (c - 1)\varepsilon) \leq \frac{1}{(c - 1)^2\varepsilon} - 1.$$

Letting $c \to \infty$, we see that the left-hand side converges to $-1/2$ and therefore we have $\limsup_{c\to\infty}(c - 1)^2\varepsilon \leq 2$. This suggests to take $\varepsilon = c^{-2}$ and, in the light of (4.3), $a = \alpha c^{-1}$ for some $\alpha = \alpha(c) \geq 1$. We assume that α is a constant function and come back to (4.4), obtaining that α must satisfy

$$\log(\alpha c^{-1} + 1 - c^{-2}) - \frac{\alpha c^{-1}}{\alpha c^{-1} + 1 - c^{-2}} \leq 0 \qquad \text{for } c \geq 1,$$

or, if we substitute $a = c^{-1} \in [0, 1]$, then

$$\xi(a) := \log(\alpha a + 1 - a^2) - \frac{\alpha a}{\alpha a + 1 - a^2} \leq 0$$

(a similar function, also denoted by ξ, has already appeared in Section 2). Since $\xi(0) = 0$, we see that the above inequality enforces $\xi'(a) \leq 0$ for a sufficiently close to 0. However, since

$$\xi'(a) = \frac{a(\alpha^2 - 2 - 4\alpha a + 2a^2)}{(\alpha a + 1 - a^2)^2},$$

the latter requirement will hold if $\alpha \leq \sqrt{2}$. This leads us to the choice $\alpha = \sqrt{2}$.

Finally, it remains to choose γ. The second inequality in (4.2) can be transformed into $\gamma - \log(as + b) \geq 2$, and this requirement is most restrictive when $s = 1$: $\gamma \geq 2 + \log(\sqrt{2}c^{-1} + 1 - c^{-2})$. One easily checks that the function $c \mapsto \sqrt{2}c^{-1} + 1 - c^{-2}$, $c \in [1, \infty)$, attains its maximal value $3/2$ for $c = \sqrt{2}$. This leads to our final choice $\gamma = 2 + \log(3/2)$, which produces the function U used in Section 2.

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