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Spectral Density for Random Matrices with Independent Skew-Diagonals

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Abstract

We consider the empirical eigenvalue distribution of random real symmetric matrices with stochastically independent skew-diagonals and study its limit if the matrix size tends to infinity. We allow correlations between entries on the same skew-diagonal and we distinguish between two types of such correlations, a rather weak and a rather strong one. For weak correlations the limiting distribution is Wigner's semi-circle distribution; for strong correlations it is the free convolution of the semi-circle distribution and the limiting distribution for random Hankel matrices.

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1 Introduction

Wigner's semi-circle law is possibly the most famous principle in random matrix theory. It states that, for various random matrix ensembles, the empirical eigenvalue distribution tends to a universal limit if the matrix size tends to infinity. Its proof started with the pioneering works of Wigner himself [20, 21] and experienced many generalizations, e.g. by Arnold [2]. For Wigner ensembles it was recently accomplished under rather mild regularity assumptions in a series of papers by Tao and Vu [18, 19] and Erdős et al. (see e.g. the survey [8]).

However, one may ask to which extent the independence of the matrix entries, which is assumed in Wigner ensembles, is necessary for the limiting spectral density to be the semi-circle. Hence, matrices with a dependence structure of some kind have attracted attention over the last years and were e.g. studied in [5, 11, 12, 13, 14, 16]. One possible approach to matrices with correlated real-valued entries is to allow that entries on the same (skew-)diagonal are correlated, while entries on different (skew-)diagonals are independent. The most distinct type of such matrices are random Toeplitz matrices T_n and random Hankel matrices H_n. These matrices appear e.g. as autocovariance matrices in time series analysis and as information matrices in a polynomial regression model [3]. Their study was proposed by Bai [3] and later addressed by Bryc, Dembo and Jiang [7]. It was shown that if the entries have variance one, the empirical eigenvalue distributions of T_n and H_n converge weakly with probability one to non-random limits, both differing from the semi-circle distribution. Starting from the results for Toeplitz

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matrices, matrices with independent diagonals have been studied by Friesen and Löwe [9, 10].

In this paper, we consider real symmetric random matrices with independent skew-diagonals instead of independent diagonals. We assume that each matrix entry is centered with the same variance and that the k-th moments are uniformly bounded for all matrix sizes. Our main results state that the empirical eigenvalue distribution converges weakly, with probability one, to a non-random distribution. Here, we distinguish two ensembles, one allowing for weak correlations, the other one allowing for strong correlations.

By weak correlations, we mean that the covariance of two entries on the same skew-diagonal depends on their distance only and decays sufficiently fast. In this case, we show that the limiting spectral density is given by the semi-circle. When we consider a type of rather strong correlations between entries on the same skew-diagonal, we assume that these correlations depend on the matrix size only and converge as the matrix size tends to infinity. Here, the limiting spectral distribution is given by a combination of the semi-circle distribution and the limiting distribution for Hankel matrices.

In our proof, we use the method of moments and we can follow the basic ideas of [7, 9, 10], while several (technical) difficulties arise for the current ensembles. The main difference is, of course, the (in)dependence structure of a matrix $(a_n(i,j))_{1\leq i,j\leq n}$:

$$a_n(i,j), a_n(i',j') \text{ stoch. independent, if } \begin{cases} |i-j| \neq |i'-j'| & \text{for ind. diagonals} \\ i+j \neq i'+j' & \text{for ind. skew-diag.} \end{cases}. \quad (1.1)$$

Although these defining relations appear quite similar, the implications are more involved leading to two major difficulties.

Firstly, some of the calculation in [9, 10] particularly depend on the appearance of the differences $i-j$ rather than the sums $i+j$ in (1.1), which necessitates e.g. Lemma 4.4. In this Lemma it is shown that a certain quantity only vanishes for $n \to \infty$ rather than being zero for all n as in [10]. Secondly, the usual symmetry condition $a_n(i,j) = a_n(j,i)$ affects the matrix ensembles in different ways: For independent diagonals, a $n \times n$ matrix is built from n independent families of random variables (one for each diagonal in the upper triangular matrix). By the symmetry condition, two respective diagonals are equal. In the case of independent skew-diagonals, there are $2n-1$ independent families (one for the upper half of each skew-diagonal) and the symmetry affects entries on the *same* skew-diagonal. This necessitates further symmetry considerations (see Lemma 4.3 onward).

For the convenience of the reader, we follow the line of arguments presented in [9, 10]. We adapt the proofs to the current ensembles and insert new ideas when necessary. More detailed comments on the differences between the methods in [9, 10] and the methods in this paper are given in Remark 5.3.

Matrices from the ensembles considered in this paper and in [9, 10] can actually be generated in several ways. Ensembles with weak correlations along the (skew-)diagonals can e.g. be built from independent families of stationary Gaussian Markov processes with mean zero and variance one. One can also fill the independent (skew-)diagonals with random variables from the Curie-Weiss model with inverse temperature $\beta > 0$; these exhibit the required strong correlations. Details on these examples can be found in [9, 10].

This paper is organized as follows: In Sec. 2 we introduce our model of matrices and state our main results (Th. 2.2 and Th. 2.3). In Sec. 3, we introduce the notion of partitions to model the dependence structure of the matrix entries and derive an intermediate result for the expected k-th moment of the empirical eigenvalue distribution; this calculation is completed in Sec. 4. In Sec. 5 we show the required weak convergence with probability one. In the case of strong correlations we further show some results for the limiting distribution.

2 Statement of results

For $n \in \mathbb{N}$ let $a_n(p,q)$ with $1 \le p \le q \le n$ denote real random variables, which are centered with variance one and the k-th moments are uniformly bounded. Moreover, we assume that the skew-diagonals are independent. Technically, these assumptions read:

(A1) $\mathbb{E}(a_n(p,q)) = 0, \quad \mathbb{E}((a_n(p,q)^2) = 1, \quad 1 \le p \le q \le n$

(A2) $m_k := \sup_{n\in\mathbb{N}} \max_{1\le p\le q\le n} \mathbb{E}\left(|a_n(p,q)|^k\right) < \infty$

(A3) The families $\{a_n(p,q) : p+q = r\}$ are independent for $r = 2, 3, \dots, 2n$.

We will consider two types of matrices with different conditions on the covariances of entries from the same skew-diagonal. These conditions are:

(C1) There exists a function $c_n : \mathbb{N} \to \mathbb{R}$ such that

(i) $|\text{Cov}(a_n(p,q), a_n(r,s))| = c_n(|p-r|) = c_n(|q-s|)$ for $p \le q, r \le s$ with $p+q = r+s$

(ii) $\sum_{\tau=0}^{n-1} c_n(\tau) = o(n), \quad n \in \mathbb{N}$

(C2) There exists $(c_n)_{n\in\mathbb{N}}$ such that for all $p, p', q, q' \in \{1, \dots, n\}$ with $p+q = p'+q'$ and $(p,q) \notin \{(p',q'), (q',p')\}$ we have:

$$\text{Cov}(a_n(p,q), a_n(p',q')) = c_n.$$

Moreover, the limit $c := \lim_{n\to\infty} c_n < \infty$ exists.

Remark 2.1. If (C2) is satisfied, we have $0 \le c \le 1$ (see Remark 2.1 in [9]). Indeed, $0 \le c$ is a consequence of $0 \le \mathbb{V}\left(\sum_{p=1}^n a_n(p,p)\right) = n + n(n-1)c_n$ and $c_n \le 1$ is a consequence of Hölder's inequality.

We study the eigenvalues of the symmetric random $n \times n$ matrix X_n obtained from $a_n(p,q)_{1\le p\le q\le n}$ by rescaling $X_n(p,q) = \frac{1}{\sqrt{n}} a_n(p,q)$, $1 \le p \le q \le n$. For the ordered eigenvalues of X_n, denoted by $\lambda_1^{(n)} \le \ldots \le \lambda_n^{(n)}$, we introduce the empirical eigenvalue distribution

$$\mu_n(X_n) := \frac{1}{n} \sum_{k=1}^n \delta_{\lambda_k^{(n)}}.$$

Our main theorems then state the weak convergence of μ_n under (C1) resp. under (C2). If (C1) is satisfied, the limiting distribution is Wigner's semi-circle distribution.

Theorem 2.2. *Suppose that (A1)–(A3) and (C1) are satisfied. Then, with probability one, μ_n converges weakly to the standard semi-circle distribution μ with density*

$$\frac{d\mu}{dx} \frac{1}{2\pi} \sqrt{4-x^2}\, \chi_{[-2,2]}(x). \tag{2.1}$$

Theorem 2.3. *Suppose that (A1)–(A3) and (C2) are satisfied. Then, with probability one, μ_n converges weakly to a non-random probability measure ν_c.*

If (C2) is satisfied, we can show further results for v_c.

Theorem 2.4. *In the situation of Theorem 2.3, the limiting measure v_c, $0 \le c \le 1$, is the free convolution of the measures $v_{0,1-c}$ and $v_{1,c}$, we write $v_c = v_{0,1-c} \boxplus v_{1,c}$. Here, $v_{0,1-c}$ denotes the rescaled semi-circle with variance $1-c$ and $v_{1,c}$ the rescaled measure* *for Hankel matrices γ_H with variance c as derived in [7]. Moreover, v_c is a symmetric measure. If $c > 0$, v_c has an unbounded support, and if $0 \leq c < 1$, its density is smooth.*

Remark 2.5. Here, neither the notion of free probability nor of the free convolution is introduced and the reader is referred to [15] for details on this topic.

3 Preliminaries, notation and combinatorics

Following the suggestions of [7, 9, 10], we will prove Theorems 2.2 and 2.3 by the method of moments. The k-th moment of the expected empirical distribution μ_n, i.e. $E\left[\int x^k d\mu_n(X_n)\right]$, is given by

$$\frac{1}{n}\mathbb{E}\left[\operatorname{tr}(X_n^k)\right] = \frac{1}{n^{\frac{k}{2}+1}} \sum_{p_1,\ldots,p_k=1}^{n} \mathbb{E}\left[a_n(p_1,p_2)a_n(p_2,p_3)\ldots a_n(p_k,p_1)\right]. \tag{3.1}$$

To simplify the notation we set

$$\tau_n(k) \coloneqq \{(P_1,\ldots,P_k) : P_j = (p_j,q_j) \in \{1,\ldots,n\}^2, q_j = p_{j+1}\}.$$

Hence, in (3.1) for $P_i = (p_i, p_{i+1})$ we can write $a_n(P_i)$ instead of $a_n(p_i, p_{i+1})$ and we sum over all $(P_1,\ldots,P_k) \in \tau_n(k)$. Throughout this paper, we identify $k+1$ with 1.

In order to display the dependence structure of the matrix entries, we use the notion of partitions as suggested by [7]. We want to express that $a_n(P_i)$ and $a_n(P_j)$ denote entries on the same skew-diagonal, by i and j being in the same partition block. More precisely, for a partition π of $\{1,\ldots,k\}$ we call $(P_1,\ldots,P_k) \in \tau_n(k)$ π*-consistent* if

$$p_i + q_i = p_j + q_j \quad \Leftrightarrow \quad i \sim_\pi j. \tag{3.2}$$

We write $i \sim j$ instead of $i \sim_\pi j$ if the partition π can be recovered from the context. With

$$\mathcal{P}(k) \coloneqq \{\pi : \pi \text{ is a partition of } \{1,\ldots,k\}\},$$
$$S_n(\pi) \coloneqq \{(P_1,\ldots,P_k) \in \tau_n(k) : (P_1,\ldots,P_k) \text{ is } \pi\text{-consistent}\}, \quad \pi \in \mathcal{P}(k),$$

$$\mathbb{E}\left[\int x^k d\mu_n(X_n)\right] = \frac{1}{n^{\frac{k}{2}+1}} \sum_{\pi\in\mathcal{P}(k)} \sum_{(P_1,\ldots,P_k)\in S_n(\pi)} \mathbb{E}\left[a_n(P_1)a_n(P_2)\ldots a_n(P_k)\right]. \tag{3.3}$$

We will argue that only the pair partitions give a non-vanishing contribution in (3.3).

3.1 Reduction to pair partitions

We observe that in (3.3) terms corresponding to partitions π with more than $\frac{k}{2}$ blocks, i.e. $\#\pi > \frac{k}{2}$, vanish. Indeed, in this case there is a partition block with a single element i, i.e. $a_n(P_i)$ is independent of all the other $a_n(P_j), j \neq i$ and by (A1) the respective term equals zero. We claim that for partitions with less than $k/2$ blocks we have

$$\frac{\#S_n(\pi)}{n^{\frac{k}{2}+1}} = o(1), \quad \text{if } \#\pi < \frac{k}{2}. \tag{3.4}$$

This can be seen from the following combinatorial arguments used to determine the number of $(P_1,\ldots,P_k)$ in $S_n(\pi)$:

- Once $P_i = (p_i, p_{i+1})$ is fixed, $P_{i+1} = (p_{i+1}, p_{i+2})$ is determined by the choice of p_{i+2}.
- We start with the choice of $P_1 = (p_1, p_2)$, for which there are at most n^2 possibilities.
- We proceed sequentially to determine $P_2, P_3, \ldots$ as follows: To determine P_i, if i is in the same block of π as some preceding index $j \in \{1,\ldots,i-1\}$, there is no choice left, as the indices need to satisfy $p_j + p_{j+1} = p_i + p_{i+1}$, where p_j, p_{j+1}, p_i are already known. Otherwise, there are at most n possible choices. Once P_1 is fixed, there are at most n possibilities for each 'new' partition block, i.e. $\#\pi - 1$ times.

Hence, we obtain $\#S_n(\pi) \leq n^2 \cdot n^{\#\pi-1} = n^{\#\pi+1}$, proving the claim in (3.4). By Hölder's inequality and (A2) (uniform boundedness of the moments) we have

$$|\mathbb{E}[a_n(P_1)\dots a_n(P_k)]| \leq \prod_{1\leq i\leq k} \left[\mathbb{E}|a_n(P_i)|^k\right]^{\frac{1}{k}} \leq m_k,$$

and

$$\frac{1}{n}\mathbb{E}\left[\operatorname{tr} X_n^k\right] = \frac{1}{n^{\frac{k}{2}+1}} \sum_{\substack{\pi\in\mathcal{P}(k)\\ \#\pi=\frac{k}{2}}} \sum_{(P_1,\dots,P_k)\in S_n(\pi)} \mathbb{E}\left[a_n(P_1)a_n(P_2)\dots a_n(P_k)\right] + o(1). \tag{3.5}$$

In particular, all odd moments vanish. Moreover, it suffices to consider pair partitions π in (3.5), i.e. partitions where each block has exactly two elements. Indeed, partitions with $\#\pi = \frac{k}{2}$ that are not pair partitions contain a block with a single element and do hence not contribute to (3.5).

3.2 Partitions and combinatorics

A recurring combinatorial consideration is the following: Suppose we want to determine the number of possible vectors $(P_1,\dots,P_k) \in S_n(\pi)$ for a given pair partition π. We write $P_i = (p_i, p_{i+1}), i = 1,\dots,k$ and state two counting principles:

(CP1) Assume that p_i is fixed for some $i \in \{1,\dots,k\}$. How many choices are there (at most) for $p_{i+1},\dots,p_{j+1}$, if $j > i$? There are at most n choices for p_{i+1}. Then, we proceed sequentially for $p_{i+2}, p_{i+3},\dots$: for each p_l we have at most n choices if l is not in the same block as any of the $i, i+1,\dots,l-1$. Otherwise there is no choice and p_l is already determined by the requirement of π-consistency. Hence, if r denotes the number of blocks that are occupied by $\{i,\dots,j\}$, we have at most n^r possibilities for $P_i,\dots,P_j$.

(CP2) Assume p_i is fixed, $j > i$ and $P_{i_1},\dots,P_{i_l}$ with $\{i_1,\dots,i_l\} \cap \{i,\dots,j\} = \emptyset$ is fixed. How many choices are there (at most) for $p_{i+1},\dots,p_{j+1}$? Again, we start with the choice of p_{i+1}: if $i+1$ is not equivalent to any of the $i_1,\dots,i_l$, there are at most n possibilities, otherwise p_{i+1} is already fixed by the π-consistency. For p_{i+2} there are at most n possibilities if $i+2$ is not equivalent to any of the indices $i+1, i_1,\dots,i_l$, otherwise there is no choice. Proceeding sequentially, we have n^{r-s} possibilities to choose $P_i,\dots,P_j$ if r denotes the number of partition blocks that are occupied by $\{i,\dots,j\}$ and s denotes the number of indices in $\{i,\dots,j\}$ that are equivalent to any of the $i_1,\dots,i_l$. In other words, $r-s$ is the number of partition blocks occupied by $\{i,\dots,j\}$, which have an empty intersection with $\{i_1,\dots,i_l\}$.

As already indicated pair partitions will be of particular interest for the proof (we denote the set of pair partitions on {1,. . . , k} by $\mathcal{PP}(k)$). We distinguish between *crossing* pair partitions and *non-crossing* pair partitions. A pair partition is said to be crossing if there exist indices $i < j < l < m$ with $i \sim l$ and $j \sim m$. We denote the set of crossing pair partitions by $\mathcal{CPP}(k)$ and the set of non-crossing pair partitions by $\mathcal{NCPP}(k)$. For a non-crossing pair partition $\pi \in \mathcal{NCPP}(k)$ and $(P_1,\dots,P_k) \in S_n(\pi)$ we have:

(NC1) There exist indices $i, j \in \{1,\dots,k\}$ with $i \sim j$ and $j = i+1$.

(NC2) If $i \sim j$ and $j = i+1$, we have $a_n(P_i) = a_n(P_j)$ and hence $\mathbb{E}[a_n(P_i)a_n(P_j)] = 1$. This is due to the fact that $i \sim i+1$ implies that for $P_i = (p_i, p_{i+1})$ and $P_{i+1} = (p_{i+1}, p_{i+2})$ we have $p_i = p_{i+2}$. By the symmetry of the considered matrix we have $a_n(P_i) = a_n(p_{i+2}, p_{i+1}) = a_n(P_{i+1})$.

(NC3) $i \sim (i+1)$ implies $(P_1,\dots,P_{i-1},P_{i+2},\dots,P_k) \in \tau_n(k-2)$ (this follows from (NC2)).

(NC4) $\#\mathcal{NCPP}(k) = C_{\frac{k}{2}}$, where $C_k := \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number (see e.g. Lemma 8.9 in [6]).

4 Calculating the expected moments of the spectral distribution

We return to the expected k-th moment of the spectral distribution given in (3.5). In the following lemma, we show that summing over all non-crossing pair partitions in (3.5) equals $C_{\frac{k}{2}}$ under both (C1) and (C2). The contribution of the crossing partitions is studied in subsection 4.1 for (C1) and in subsection 4.2 for (C2).

Lemma 4.1 (cf. Lemma 5.2 and 5.3 in [9]). *Under (C1) and (C2) we have for $k \in \mathbb{N}$ even*

$$\frac{1}{n}\mathbb{E}\left[\operatorname{tr} X_n^k\right] = C_{\frac{k}{2}} + \frac{1}{n^{\frac{k}{2}+1}} \sum_{\pi \in \mathcal{CPP}(k)} \sum_{(P_1,\ldots,P_k)\in S_n(\pi)} \mathbb{E}\left[a_n(P_1)a_n(P_2)\ldots a_n(P_k)\right] + o(1).$$

Proof. By successively applying (NC1)–(NC3), we have for any $\pi \in \mathcal{NCPP}(k)$ and $(P_1,\ldots,P_k) \in S_n(\pi)$ that $\mathbb{E}[a_n(P_1)\ldots a_n(P_k)] = 1$. We claim that

$$\lim_{n\to\infty} \frac{\#S_n(\pi)}{n^{\frac{k}{2}+1}} = 1, \quad \pi \in \mathcal{NCPP}(k). \tag{4.1}$$

Let $(P_1,\ldots,P_k)$ be in $S_n(\pi)$. By (NC1)–(NC3) we have $i \sim i+1$ for some $i \in \{1,\ldots,k\}$ and $P' \coloneqq (P_1,\ldots,P_{i-1},P_{i+2},\ldots,P_k) \in \tau_n(k-2)$. Moreover, we have $P' \in S_n(\pi')$, where $\pi' \coloneqq \pi \setminus \{\{i,i+1\}\} \in \mathcal{NCPP}(k-2)$ and all $l \geq i+2$ are relabeled to $l-2$. Thus, all possible $(P_1,\ldots,P_k) \in S_n(\pi)$ can be constructed from a choice of P' and a choice of p_{i+1}. For p_{i+1} there are $n - \frac{k-2}{2}$ possibilities, as we have to ensure that $p_i + p_{i+1}$ does not equal any of the $(k-2)/2$ values $p_j + p_{j+1}$ for $j \neq i$, $j \neq i+1$. This implies

$$\frac{\#S_n(\pi)}{n^{\frac{k}{2}+1}} = \frac{\#S_n(\pi')}{n^{\frac{k}{2}}} + o(1).$$

The claim in (4.1) then follows by induction and the fact that for $k = 2$ we have $\#S_n(\pi) = \{((p,q),(q,p)) : p,q \in \{1,\ldots,n\}\} = n^2$. Statement (NC4) completes the proof. ☐

4.1 The expected k-th moment of the spectral distribution under (C1)

In this subsection we assume that (C1) is satisfied and we show that for k even the expected k-th moment of μ_n is asymptotically given by $C_{\frac{k}{2}}$.

Lemma 4.2 (cf. Lemma 3.3 and 3.4 in [10]). *Under (C1), we have for $k \in \mathbb{N}$ even*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\operatorname{tr} X_n^k\right] = C_{\frac{k}{2}}.$$

Proof. By Lemma 4.1 it suffices to show that for each $\pi \in \mathcal{CPP}(k)$

$$\lim_{n\to\infty} \frac{1}{n^{\frac{k}{2}+1}} \sum_{(P_1,\ldots,P_k)\in S_n(\pi)} \mathbb{E}\left[a_n(P_1)a_n(P_2)\ldots a_n(P_k)\right] = 0.$$

Let $\pi \in \mathcal{CPP}(k)$. We will define partitions $\pi^{(1)},\ldots,\pi^{(r)}$ by successively deleting blocks of π such that we arrive at some partition $\pi^{(r)} \in \mathcal{CPP}(k-2r)$, for which adjacent elements $j, j+1$ are in different blocks. Suppose that $l \sim_\pi l+1$ for some $l \in \{1,\ldots,k\}$, otherwise we set $r = 0, \pi^{(r)} = \pi$. Then we obtain $\pi^{(1)}$ from π by deleting the block $\{l, l+1\}$

$$\pi^{(1)} \coloneqq \pi \setminus \{\{l,l+1\}\}$$

and relabeling all $j \geq l+2$ to $j-2$. Hence $\pi^{(1)} \in \mathcal{CPP}(k-2)$. Correspondingly, we delete P_l, P_{l+1} from $(P_1,\ldots,P_k)$ to obtain (see (NC3))

$$(P_1,\ldots,P_k)^{(1)} \coloneqq (P_1,\ldots,P_{l-1},P_{l+2},\ldots,P_k) \in S_n(\pi^{(1)}).$$

We repeat this procedure to obtain $\pi^{(2)}$ and $(P_1, \ldots, P_k)^{(2)}, \pi^{(3)}$ and $(P_1, \ldots, P_k)^{(3)}$, ... until we arrive at a partition $\pi^{(r)} \in \mathcal{CCP}(k-2r)$ where none of the blocks contains adjacent elements. Then $(P_1, \ldots, P_k)^{(r)} \in S_n(\pi^{(r)})$. Since π is a crossing partition, at least two blocks of π remain after the elimination process and we have $r \leq \frac{k}{2} - 2$. Using the same arguments as in the proof of Lemma 4.1 leads to the following estimate for given $(Q_1, \ldots, Q_{k-2r}) \in \tau_n(k-2r)$:

$$\#\{(P_1, \ldots, P_k) \in S_n(\pi) : (P_1, \ldots, P_k)^{(r)} = (Q_1, \ldots, Q_{k-2r})\} \leq n^r.$$

By (NC2) we have for $(P_1, \ldots, P_k)^{(r)} = (Q_1, \ldots, Q_{k-2r})$

$$\mathbb{E}[a_n(P_1) \ldots a_n(P_k)] = \mathbb{E}[a_n(Q_1) \ldots a_n(Q_{k-2r})]$$

and hence (with $P = (P_1, \ldots, P_k)$, $Q = (Q_1, \ldots, Q_{k-2r})$)

$$\sum_{P \in S_n(\pi)} |\mathbb{E}\left[a_n(P_1) \ldots a_n(P_k)\right]| \leq n^r \sum_{Q \in S_n(\pi^{(r)})} |\mathbb{E}\left[a_n(Q_1) \ldots a_n(Q_{k-2r})\right]|. \tag{4.2}$$

We choose $i \sim_{\pi^{(r)}} i+j$ such that $j \geq 2$ is minimal. By Hölder's inequality we have

$$|\mathbb{E}[a_n(Q_s)a_n(Q_t)]| \leq (\mathbb{E}[a_n(Q_s)^2])^{1/2}(\mathbb{E}[a_n(Q_t)^2])^{1/2} = 1$$

and hence, as k is even,

$$|\mathbb{E}\left[a_n(Q_1)a_n(Q_2) \ldots a_n(Q_{k-2r})\right]| \leq |\mathbb{E}\left[a_n(Q_i)a_n(Q_{i+j})\right]| = |\mathrm{Cov}(a_n(Q_i), a_n(Q_{i+j}))|.$$

Inserting this estimate into (4.2) we obtain (set $Q := (Q_1, \ldots, Q_{k-2r})$)

$$\sum_{Q \in S_n(\pi^{(r)})} |\mathbb{E}\left[a_n(Q_1)a_n(Q_2) \ldots a_n(Q_{k-2r})\right]| \leq \sum_{Q \in S_n(\pi^{(r)})} |\mathbb{E}\left[a_n(Q_i)a_n(Q_{i+j})\right]|.$$

As before, we denote $Q_l = (q_l, q_{l+1})$ for $l = 1, \ldots, k-2r$, where $k-2r$ is identified with 1. If we could use $|\mathbb{E}\left[a_n(Q_i)a_n(Q_{i+j})\right]| = c_n(|q_i - q_{i+j}|)$, we could calculate the number of points $(Q_1, \ldots, Q_{k-2r}) \in S_n(\pi^{(r)})$ for given q_i, q_{i+j} and finally use (C1) to obtain $\sum_{q_i, q_{i+j}=1}^{n} c_n(|q_i - q_{i+j}|) = o(n^2)$. However, this is only valid for $q_i \leq q_{i+1}$ and $q_{i+j} \leq q_{i+j+1}$ or for $q_i \geq q_{i+1}$ and $q_{i+j} \geq q_{i+j+1}$. Hence, we have to take the ordering of the q_i, q_{i+1} and q_{i+j}, q_{i+j+1} into account. We distinguish two types of pairs (Q_i, Q_{i+j}): We call (Q_i, Q_{i+j}) positive if $\mathrm{sgn}(q_i - q_{i+1}) = \mathrm{sgn}(q_{i+j} - q_{i+j+1})$ and negative otherwise. Then we have

$$|\mathrm{Cov}(Q_i, Q_{i+j})| = \begin{cases} c_n(|q_i - q_{i+j}|), & \text{if } (Q_i, Q_{i+j}) \text{ positive} \\ c_n(|q_i - q_{i+j+1}|), & \text{if } (Q_i, Q_{i+j}) \text{ negative} \end{cases}.$$

We claim the following estimate: For given q_i, q_{i+j} there are less than $n^{\frac{k}{2}-r-1}$ tuples $(Q_1, \ldots, Q_{k-2r}) \in S_n(\pi^{(r)})$ with (Q_i, Q_{i+j}) positive. Similarly, for given q_i, q_{i+j+1} there are less than $n^{\frac{k}{2}-r-1}$ tuples $(Q_1, \ldots, Q_{k-2r}) \in S_n(\pi^{(r)})$ with (Q_i, Q_{i+j}) negative. We start with the case (Q_i, Q_{i+j}) positive and q_i, q_{i+j} fixed. We have n possible choices for q_{i+1}. By $i \sim i+j$, this determines the value of q_{i+j+1} (recall q_{i+j} is fixed) and hence Q_i, Q_{i+j} are fixed. Since j is chosen to be minimal, the $j-1$ elements in $\{i+1, \ldots, i+j-1\}$ lie in $j-1$ different partition blocks. Hence, we have n possibilities for each of the $j-2$ points $q_{i+2}, \ldots, q_{i+j-1}$. So far, there were $n \cdot n^{j-2}$ possibilities and we fixed $Q_i, \ldots, Q_{j+i}$. We want to apply counting principle (CP2) to determine the number of possible choices for the remaining pairs $Q_{i+j+1}, \ldots, Q_{k-2r}, Q_1, \ldots, Q_{i-1}$. Hence, we have to determine the number of partition blocks occupied by $i+j+1, \ldots, k-2r, 1, \ldots, i-1$, that have an empty intersection with the set $\{i,\dots,i+j\}$. From the total of $\frac{k}{2}-r$ partition blocks of $\pi^{(r)}$ one block is occupied by i and $i+j$ and the $j-1$ blocks occupied by $\{i+1,\dots,i+j-1\}$ each contain one element in $\{i+j+1,\dots,k-2r,1,\dots,i-1\}$ as well. Thus, (CP2) gives $n^{\frac{k}{2}-r-1-(j-1)}=n^{\frac{k}{2}-r-j}$ possibilities to fix $Q_{i+j+1},\dots,Q_{k-2r},Q_1,\dots,Q_{i-1}$. Hence, for fixed q_i,q_{i+j} we have a total of $n\cdot n^{j-2}n^{(\frac{k}{2}-r-j)}=n^{\frac{k}{2}-r-1}$ possibilities to choose $Q_1,\dots,Q_{k-2r}$. By (ii) in (C1)

$$n^r\sum_{\substack{Q\in S_n(\pi^{(r)})\\ Q_i,Q_{i+j}\text{ positive}}}|\mathbb{E}\left[a_n(Q_i)a_n(Q_{i+j})\right]|\le n^{\frac{k}{2}-1}\sum_{q_i,q_{i+j}=1}^{n}c_n(|q_i-q_{i+j}|)\le n^{\frac{k}{2}}\sum_{\tau=0}^{n-1}c_n(\tau)=o(n^{\frac{k}{2}+1}).$$

In the case (Q_i,Q_{i+j}) negative, we can proceed similarly. □

4.2 The expected k-th moment of the spectral distribution under (C2)

Before we proceed to calculate the expected k-th moment of μ_n under (C2), we state a combinatorial lemma. Throughout this section we write $P_i=(p_i,p_{i+1}), i=1,\dots,k$. We want to pay special attention to pairs P_i,P_j with $i\sim j$ and

$$P_i=P_j\quad\text{or}\quad P_i=\overline{P}_j:=(p_{j+1},p_j). \tag{4.3}$$

The lemma states that if a block $\{i,j\}$ of a partition π with (4.3) is crossed by some other block, the number of points $(P_1,\dots,P_k)\in S_n(\pi)$ is of order $o(n^{\frac{k}{2}+1})$.

Lemma 4.3 (cf. Lemma 5.4 in [9]). *Let $k\in\mathbb{N}$, $\pi\in\mathcal{PP}(k)$ and $i<j$ with $i\sim j$. Set*

$$S_n(\pi,i,j):=\{(P_1,\dots,P_k)\in S_n(\pi):P_i=P_j\text{ or }P_i=\overline{P}_j\}.$$

If there exist i',j' with $i'\sim j', i<i'<j$ and either $j'<i$ or $j<j'$ (i.e. the block $\{i,j\}$ is crossed by the block $\{i',j'\}$), we have

$$\#S_n(\pi,i,j)=o(n^{\frac{k}{2}+1}).$$

Proof. To construct $(P_1,\dots,P_k)\in S_n(\pi,i,j)$, first choose p_i and p_{i+1}, each allowing for n possibilities. Then P_i is fixed and we choose one of the two possibilities $P_i=P_j$ or $P_i=\overline{P}_j$, fixing P_j. Let r denote the number of partition blocks occupied by $\{i+1,\dots,i'-1\}\cup\{j,\dots,i'+1\}$. By similar arguments as in (CP1) we have less than n^r choices to fix $P_{i+1},\dots,P_{i'-1},P_j,\dots,P_{i'+1}$. Hence, $P_{i'}$ is determined by consistency without any further choice. So far, we fixed P_l for l in $r+2$ different partition blocks. By (CP2) there are at most $n^{\frac{k}{2}-r-2}$ choices to fix all remaining points P_l. In total, there are $n^{\frac{k}{2}}$ possibilities to construct $(P_1,\dots,P_k)\in S_n(\pi,i,j)$. □

We continue by considering the r.h.s. of Lemma 4.1 and observe that for $\pi\in\mathcal{CPP}(k)$ and $(P_1,\dots,P_k)\in S_n(\pi)$ the term $\mathbb{E}\left[a_n(P_1)a_n(P_2)\dots a_n(P_k)\right]$ is a product of factors

$$\mathbb{E}\left[a_n(P_i)a_n(P_j)\right]=\begin{cases}1, & \text{if } P_i=P_j\text{ or }P_i=\overline{P}_j\\ c_n, & \text{else}\end{cases},\quad i\sim j. \tag{4.4}$$

Hence, we introduce the following notation for $\pi\in\mathcal{CPP}(k)$ and $(P_1,\dots,P_k)\in S_n(\pi)$:

$$m(P_1,\dots,P_k):=\#\{1\le i<j\le k:P_i=P_j\text{ or }P_i=\overline{P}_j\}\le\frac{k}{2},$$
$$A_n^{(l)}(\pi):=\{(P_1,\dots,P_k)\in S_n(\pi):m(P_1,\dots,P_k)=l\},\quad l\in\{1,\dots,\frac{k}{2}\}.$$

Then we can write for $\pi \in \mathcal{CPP}(k)$

$$\frac{1}{n^{\frac{k}{2}+1}} \sum_{(P_1,\dots,P_k)\in S_n(\pi)} \mathbb{E}\left[a_n(P_1)a_n(P_2)\dots a_n(P_k)\right] = \frac{1}{n^{\frac{k}{2}+1}} \sum_{l=0}^{k/2} c_n^{\frac{k}{2}-l} \# A_n^{(l)}(\pi).$$

Moreover, we set

$$\begin{aligned} B_n^{(l)}(\pi) \coloneqq \{&(P_1,\dots,P_k) \in S_n(\pi) : m(P_1,\dots,P_k) = l; \\ &P_i = P_j \text{ or } P_i = \overline{P}_j, i<j \;\Rightarrow\; j=i+1 \text{ or } \pi|_{\{i+1,\dots,j-1\}} \text{ is a pair partition}\}. \end{aligned}$$

By the crossing property of Lemma 4.3 we have

$$\frac{1}{n^{\frac{k}{2}+1}} \# \left(A_n^{(l)}(\pi) \setminus B_n^{(l)}(\pi) \right) \to 0, \quad n \to \infty. \tag{4.5}$$

In order to show that $n^{-(\frac{k}{2}+1)} \# B_n^{(l)}(\pi)$ vanishes for almost all values of l, we introduce the notion of *height* of a pair partition $\pi \in \mathcal{PP}(k)$:

$$h(\pi) \coloneqq \#\{1 \le i < j \le k, i \sim j : j = i+1 \text{ or } \pi|_{\{i+1,\dots,j-1\}} \text{ is a pair partition}\}.$$

As both $P_i = \overline{P}_j$ and $P_i = P_j$ imply $i \sim j$, we have $B_n^{(l)}(\pi) = \emptyset$ for $l > h(\pi)$. By (4.5) we have

$$\frac{1}{n^{\frac{k}{2}+1}} \sum_{l=0}^{\frac{k}{2}} c_n^{\frac{k}{2}-l} \# A_n^{(l)}(\pi) = \frac{1}{n^{\frac{k}{2}+1}} \sum_{l=0}^{h(\pi)} c_n^{\frac{k}{2}-l} \# B_n^{(l)}(\pi) + o(1). \tag{4.6}$$

We show that only $B_n^{(h(\pi))}(\pi)$ gives a non-vanishing contribution in (4.6).

Lemma 4.4. *For $k \in \mathbb{N}$, $\pi \in \mathcal{PP}(k)$ and $l < h(\pi)$ we have:* $\lim_{n\to\infty} \frac{1}{n^{\frac{k}{2}+1}} \# B_n^{(l)}(\pi) = 0.$

Proof. Let $\pi \in \mathcal{PP}(k)$ and $l < h(\pi)$. We estimate the number of $(P_1,\dots,P_k) \in B_n^{(l)}(\pi)$. For $l < h(\pi)$, there are indices i, j that give a contribution to $h(\pi)$ but the corresponding pairs P_i, P_j do not contribute to $m(P_1,\dots,P_k)$, i.e. there exist $1 \le i < j \le k$ s. t. $i \sim j$ and

(i) $j = i+1$ or $\pi|_{\{i+1,\dots,j-1\}}$ is a pair partition, (ii) $P_i \ne P_j$ and $P_i \ne \overline{P}_j$.

In particular, $j = i+1$ cannot be satisfied, as this implies $P_i = \overline{P}_j$. Hence, we can assume $j > i+1$, $\pi|_{\{i+1,\dots,j-1\}}$ is a pair partition (with $(j-i-1)/2$ blocks) and, as a consequence of (ii), $p_{i+1} \ne p_j$. We observe that $\#(\pi|_{\{1,\dots,k\}\setminus\{i+1,\dots,j-1\}}) = \frac{k}{2} - \frac{j-i-1}{2}$. Then there are n possibilities to choose p_{i+1} and by (CP1) we have $n^{\frac{k}{2}-\frac{j-i-1}{2}}$ possibilities to successively choose $p_i, p_{i-1}, \dots, p_1, p_k, \dots, p_j$. Applying (CP2) to choose $p_{i+2},\dots,p_{j-1}$ would amount to $n^{\frac{j-i-1}{2}}$ possibilities, but we claim that there are actually only $Cn^{\frac{j-i-1}{2}-1}$ possibilities. Recalling that p_{i+1}, p_j are already known and distinct, we observe that we have

$$0 \ne p_{i+1} - p_j = \sum_{s=1}^{j-i-1} (-1)^s (p_{i+s} + p_{i+s+1}). \tag{4.7}$$

As $\pi|_{\{i+1,\dots,j-1\}}$ is a pair partition, neglecting their sign, each term $p_{i+s} + p_{i+s+1}$ appears exactly twice in the alternating sum in (4.7) and as the sum does not vanish, there are $1 \le \alpha, \beta \le j-i-1$ with $i+\alpha \sim i+\beta$ and $(-1)^\alpha = (-1)^\beta$. Then we have

$$p_{i+1} - p_j = 2(-1)^\alpha (p_{i+\alpha} + p_{i+\alpha+1}) + \sum_{\substack{s=1,\dots,j-i-1 \\ s\ne\alpha,\beta}} (-1)^s (p_{i+s} + p_{i+s+1}). \tag{4.8}$$

For each of the $\frac{j-i-1}{2}-1$ blocks $\{r,s\} \subset \{i+1,\dots,j-1\} \setminus \{i+\alpha, i+\beta\}$ we assign one of $2n$ possible values to $p_r + p_{r+1}$ (and hence to $p_s + p_{s+1}$), amounting to $(2n)^{\frac{j-i-1}{2}-1}$ possibilities. Then the sum in (4.8) is fixed and as we already know p_{i+1}, p_j, we can calculate $(p_{i+\alpha} + p_{i+\alpha+1})$ and hence $p_{i+\beta} + p_{i+\beta+1}$. Knowing p_{i+1}, p_j and $p_l + p_{l+1}$, $l = i+1, \dots, j-1$, the values of $p_{i+2}, \dots, p_{j-1}$ are uniquely determined. Hence, there was a total of $n^{\frac{k}{2}-\frac{j-i-1}{2}+1}(2n)^{\frac{j-i-1}{2}-1} = C_{i,j} n^{\frac{k}{2}}$ possibilities to choose $(P_1, \dots, P_k) \in B_n^{(l)}(\pi)$, where $C_{i,j}$ denotes some constant, depending on i and j only. Thus, we have

$$0 \le \lim_{n\to\infty} \frac{1}{n^{\frac{k}{2}+1}} \# B_n^{(l)}(\pi) \le \lim_{n\to\infty} C_{i,j} \frac{n^{\frac{k}{2}}}{n^{\frac{k}{2}+1}} = 0. \qquad \square$$

So far, we showed that

$$\frac{1}{n^{\frac{k}{2}+1}} \sum_{\pi \in \mathcal{CPP}(k)} \sum_{P \in S_n(\pi)} \mathbb{E}\left[a_n(P_1)\dots a_n(P_k)\right] = \frac{1}{n^{\frac{k}{2}+1}} \sum_{\pi \in \mathcal{CPP}(k)} c_n^{\frac{k}{2}-h(\pi)} \# B_n^{(h(\pi))}(\pi) + o(1).$$

Observe that we have by Lemma 4.4 and Lemma 4.3

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n^{\frac{k}{2}+1}} \# B_n^{(h(\pi))}(\pi) = \lim_{n\to\infty} \frac{1}{n^{\frac{k}{2}+1}} \# \left(B_n^{(h(\pi))}(\pi) \cup \left(\bigcup_{l<h(\pi)} B_n^{(l)}(\pi) \right) \right) \\
&= \lim_{n\to\infty} \frac{1}{n^{\frac{k}{2}+1}} \# \{ (P_1,\dots,P_k) \in S_n(\pi) : \\
&\qquad P_i = P_j \text{ or } P_i = \overline{P}_j, i<j \Rightarrow j = i+1 \text{ or } \pi|_{\{i+1,\dots,j-1\}} \text{ is a pair partition} \}. \\
&= \lim_{n\to\infty} \frac{1}{n^{\frac{k}{2}+1}} \# S_n(\pi) = p_H(\pi)
\end{aligned}$$

In the last line, $p_H(\pi)$ denotes the Hankel volume and the respective convergence is shown in Lemma 4.8 in [7]. Finally, using $c_n \to c$ (under (C2)), we have

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\operatorname{tr} X_n^k\right] = C_{\frac{k}{2}} + \sum_{\pi \in \mathcal{CPP}(k)} c^{\frac{k}{2}-h(\pi)} p_H(\pi) = \sum_{\pi \in \mathcal{PP}(k)} c^{\frac{k}{2}-h(\pi)} p_H(\pi) =: M_{k,c}. \qquad (4.9)$$

The second equality is due to the fact that all statements in subsection 4.2 remain valid for all pair partitions that are not necessarily crossing.

5 The proofs of Theorem 2.2, Theorem 2.3 and Theorem 2.4

Before we complete the proof of Th. 2.2 and Th. 2.3, we consider the limiting measures. Under (C1) the limiting measure is μ (see (2.1)), which is uniquely determined by its moments: the odd moments vanish and the $2k$-th moment is given by C_k (see e.g. [1, Section 2.1.1]). Under (C2), we show that the limiting measure v_c is given by

$$\int x^k dv_c = \begin{cases} 0 & k \text{ odd} \\ M_{k,c} & k \text{ even} \end{cases}. \qquad (5.1)$$

By checking the Carleman condition, we see that these moments uniquely determine v_c.

From (4.9) and (5.1) we can already deduce that v_0 is the semi-circle distribution and v_1 equals the measure γ_H from [7]. Moreover, v_c is symmetric for all $c \in [0,1]$ as all odd moments vanish and for $0 < c \le 1$ the support of v_c is unbounded. Indeed, as a bounded support of v_c would lead to $M_{2k} \le C^{2k}$, it suffices to verify $\limsup_{k\to\infty} (M_{2k})^{\frac{1}{k}} = \infty$. This relation is a consequence of $c^{\frac{k}{2}} \int x^k d\gamma_H(d) \le M_k$ and (see Proposition A.2 in [7]) $\limsup_{k\to\infty} (\int x^{2k} d\gamma_H(d))^{1/k} = \infty$. We can now prove Th. 2.2 and Th. 2.3, by showing that μ_n converges weakly, with probability one, to μ under (C1) resp. to v_c under (C2).

Proof of Th. 2.2 and Th. 2.3. We will use the concentration inequality obtained in Proposition 4.9 in [7], which can easily be extended to the case of matrices with independent skew-diagonals analogue to Lemma 3.5 in [10]. We have under both (C1) and (C2)

$$\mathbb{E}\left[\left(\operatorname{tr}(X_n^k) - \mathbb{E}(\operatorname{tr}(X_n^k))\right)^4\right] \le Cn^2, \quad \forall k \in \mathbb{N}. \tag{5.2}$$

As μ and v_c are uniquely determined by their moments, it suffices to show that $\frac{1}{n}\operatorname{tr}(X_n^k)$ converges almost surely to $\mathbb{E}[Y^k]$, where Y denotes a random variable distributed according to v_c (for (C2)) resp. according to μ (for (C1)). By Chebyshev's inequality and (5.2) we have for $\varepsilon > 0, k, n \in \mathbb{N}$

$$\mathbb{P}\left(\left|n^{-1}\operatorname{tr}\left(X_n^k\right) - \mathbb{E}\left(n^{-1}\operatorname{tr}\left(X_n^k\right)\right)\right| > \varepsilon\right) \le \frac{C}{\varepsilon^4 n^2}.$$

By the Borel-Cantelli Lemma we obtain $\frac{1}{n}\operatorname{tr}(X_n^k) - \mathbb{E}\left(\frac{1}{n}\operatorname{tr}(X_n^k)\right) \to 0, n \to \infty$ almost surely. Together with $\mathbb{E}\left(\frac{1}{n}\operatorname{tr}(X_n^k)\right) \to \mathbb{E}[Y^k]$ for $n \to \infty$, this completes the proof. □

It remains to prove following Lemma about the limiting measure v_c.

Lemma 5.1 (cf. Lemma 6.2 in [9]). *With the notation of Theorem 2.4 we have* $v_c = v_{0,1-c} \boxplus v_{1,c}$ *for* $0 \le c \le 1$. *Moreover,* v_c *has a smooth density if* $0 \le c < 1$.

Proof. Recall that $v_{0,1-c}$ denotes the rescaled semi-circle with variance $1-c$ and $v_{1,c}$ the rescaled Hankel distribution γ_H with variance c as derived in [7]. It suffices to show that the free cumulants of the free convolution of $v_{0,1-c}$ and $v_{1,c}$ coincide with the free cumulants of v_c. We apply the same arguments as in Lemma 6.2 in [9] (replacing p_T by p_H), that rely on [7, Lemma A.4] (see also p. 152 in [6]). Similarly, we conclude $(1-c)^k\kappa_{2k}(\mu) + c^k\kappa_{2k}(\gamma_H) = \kappa_{2k}(v_c)$, proving $v_c = v_{0,1-c} \boxplus v_{1,c}, 0 \le c \le 1$. Using general results about the free convolution with the semi-circle distribution provided in [4], we obtain that v_c has a smooth density for $0 \le c < 1$. □

Remark 5.2. The boundedness of the density of v_c is not derived here, because the boundedness of γ_H is not yet available in the literature. Note that for matrices with independent diagonals the boundedness of the corresponding density could be derived in [9], using [4] and the boundedness of γ_T [17].

As already noted, our line of arguments follows [9, 10] and in the following concluding remark we comment on the differences between the proofs.

Remark 5.3. The main difference between the current ensembles and those in [9, 10] is that instead of (3.2) the dependence structure in [9, 10] is given by $|p_i - q_i| = |p_j - q_j|$, iff $i \sim j$. Hence, the validity of all arguments from [9, 10] has to be verified for (3.2).

In the proof of Th. 2.2 (corresponding to the proofs in [10]), we do not/cannot introduce the sets $S_n^*(\pi) \subset S_n(\pi)$ and all needed relations have to be derived from (3.2) directly. The lack of S_n^* requires the distinction of positive/negative pairs in Lemma 4.2.

The proof of Th. 2.3 (corresponding to [9]) requires more modifications. Again, we do not introduce the sets $S_n^*(\pi)$, but in this case the implications are more severe. The most prominent one is that the analogue of [10, Lemma 5.5] is not valid and it has to be replaced by Lemma 4.4 (i.e. $\#B_n^{(l)}(\pi)$ is not necessarily zero for all n but vanishes in the limit $n \to \infty$), which required new ideas. Moreover, in Sec. 4.2 we have to additionally consider the pairs $\overline{P}_j$ in the definition of $S_n(\pi, i, j)$ in Lemma 4.3, in (4.4) and in all derived terms such as $m(P_1, \dots, P_k)$ and $B_n^{(l)}(\pi)$. In both cases, the extension to the almost sure convergence and the proof of Theorem 2.4 follow [9, 10].

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