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Weak and strong disorder for the stochastic heat equation and continuous directed polymers in $d \geq 3^*$

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Abstract

We consider the smoothed multiplicative noise stochastic heat equation

$$\mathrm{d}u_{\varepsilon,t} = \frac{1}{2}\Delta u_{\varepsilon,t}\mathrm{d}t + \beta\varepsilon^{\frac{d-2}{2}}\, u_{\varepsilon,t}\,\mathrm{d}B_{\varepsilon,t}, \quad u_{\varepsilon,0} = 1,$$

in dimension $d \geq 3$, where $B_{\varepsilon,t}$ is a spatially smoothed (at scale ε) space-time white noise, and $\beta > 0$ is a parameter. We show the existence of a $\bar{\beta} \in (0,\infty)$ so that the solution exhibits weak disorder when $\beta < \bar{\beta}$ and strong disorder when $\beta > \bar{\beta}$. The proof techniques use elements of the theory of the Gaussian multiplicative chaos.

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1 Motivation and introduction

We consider the stochastic heat equation (SHE) with multiplicative noise, written formally as

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x)\,\eta(t,x). \tag{1.1}$$

Here η is the "space-time white noise", which formally is the centered Gaussian process with covariance function $\mathbb{E}(\eta(s,x)\eta(t,y)) = \delta_0(t-s)\delta_0(x-y)$ for $s,t > 0$ and $x, y \in \mathbb{R}^d$. We emphasize that (1.1) is a formal expression, and in attempting to give it a precise meaning one is immediately faced with the problem of multiplication of distributions.

Besides the intrinsic interest in the SHE, we recall that the Cole-Hopf transformation $h := -\log u$ formally transforms the SHE to the non-linear Kardar-Parisi-Zhang (KPZ) equation, which can be written as

$$\partial_t h(t,x) = \frac{1}{2}\Delta h(t,x) - \frac{1}{2}(\partial_x h(t,x))^2 + \eta, \tag{1.2}$$

and appears in dimension $d = 1$ as the limit of front propagation in certain exclusion processes ([BG97], [ACQ11]). While a-priori the equation (1.2) is not well posed due to

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the presence of products of distributions, much recent progress has been achieved in giving an intrinsic precise interpretation to it in dimension $d = 1$ ([H13])

As discussed in [AKQ14] and [CSZ15], the equations (1.1) and (1.2) share close analogies to the well-studied *discrete directed polymer*, which can be defined as the transformed path measure

$$\mu_n(\mathrm{d}\omega) = \frac{1}{Z_n} \exp\left\{\beta \sum_{i=1}^{n} \eta(i, \omega_i)\right\} \mathrm{d}P_0. \tag{1.3}$$

Here the white noise (the *disorder*) is replaced by i.i.d. random variables $\eta = \{\eta(n,x)\colon n \in \mathbb{N}, x \in \mathbb{Z}^d\}$, P_0 denotes the law of a simple random walk starting at the origin corresponding to a d-dimensional path $\omega_n = (\omega_i)_{i \leq n}$, while $\beta > 0$ stands for the strength of the disorder. It is well-known that, when $d \geq 3$ the normalized partition function $Z_n / \mathbb{E} Z_n$ converges almost surely to a random variable Z_∞, which, when β is small enough, is positive almost surely (i.e., *weak disorder* persists [IS88, B89]), while for β large enough, $Z_\infty = 0$ (i.e., *strong disorder holds* [CSY04]). Related results for a continuous directed polymer in a field of random traps appear in [CY13].

We return to the study of the stochastic heat equation in the continuum $\mathbb{R}^d$, written as a stochastic differential equation

$$\mathrm{d}u_t = \frac{1}{2} \Delta u_t \mathrm{d}t + \beta\, u_t\, \mathrm{d}B_t, \tag{1.4}$$

where B_t is a cylindrical Wiener process in $L^2(\mathbb{R}^d)$. Since the solution to (1.4) is not well defined, a standard approach to treat this equation is to introduce a regularization of the process B_t, followed by a suitable rescaling of the coupling coefficients and subsequently passing to a limit as the regularization is turned off. In one space dimension $d = 1$, this task was carried out by Bertini-Cancrini ([BC95]) by expressing the regularized process by a Feynman-Kac formula; after a simple renormalization (*the Wick exponential*), a meaningful expression was obtained when the mollification was removed. The renormalized Feynman-Kac formula defines a process with continuous (in space and time) trajectories and it solves the equation (1.4) (when the stochastic differential is interpreted in the Ito sense). Extending this procedure to $d = 2$ (where small scale singularities coming from the noise are stronger), Bertini-Cancrini ([BC98]) introduced a rescaling of the coupling constant

$$\beta = \beta(\varepsilon) = \left(\frac{2\pi}{\log \varepsilon^{-1}} + \frac{C}{(\log \varepsilon^{-1})^2}\right)^{1/2} \qquad C \in \mathbb{R}$$

which vanishes as $\varepsilon \to 0$. It turned out that the covariance $\mathbb{E}[Z_\varepsilon(t,x) Z_\varepsilon(t,y)]$ of the regularized field Z_ε converges to a non-trivial limit as the mollification is removed, but the limiting law of Z_ε was not identified in [BC98]. The latter identification was recently carried out by Caravenna, Sun and Zygouras ([CSZ15]) (see also Feng [F15]), who proved that, in $d = 2$, if β_ε is chosen to be $\beta \sqrt{2\pi\, [\log(1/\varepsilon)]^{-1}}$, then for $\beta < 1$, Z_ε converges in law to a random variable with an explicit distribution, while for $\beta \geq 1$, Z_ε converges in law to 0.

The results of this article concern related statements for $d \geq 3$ pertaining to the smoothened and rescaled equation

$$\begin{aligned} \mathrm{d}u_{\varepsilon,t} &= \frac{1}{2} \Delta u_{\varepsilon,t} + \beta\, \varepsilon^{\frac{d-2}{2}}\; u_{\varepsilon,t}\, \mathrm{d}B_{\varepsilon,t} \\ u_{\varepsilon,0} &= 1 \end{aligned}$$

Write $u_\varepsilon(x) := u_{\varepsilon,1}(x)$. Our main result shows that for every $x \in \mathbb{R}^d$, for any β small enough $u_\varepsilon(x)$ converges in distribution to a non-degenerate random variable

$Z_\infty = Z_\infty(\beta)$, i.e., *weak disorder* prevails, while for β large enough, $u_\varepsilon(x)$ converges in probability to 0, i.e., *strong disorder* takes place. We also show that for β small enough and any suitable test function f, $u_\varepsilon(f) = \int f(x)u_\varepsilon(x)\mathrm{d}x$ converges in probability to $\int f(x)\,\mathrm{d}x$. We remark that our results, unlike [CSZ15], do not charaterize the limiting non-degenerate random variable $Z_\infty(\beta)$, nor do they identify the exact critical threshold for the value of β (which happens to be 1 in $d = 2$), where the departure from weak disorder to strong disorder takes place.

2 Main results

2.1 Preliminaries We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a cylindrical Wiener process $B = (B_t)_{t\geq 0}$ on $L^2(\mathbb{R}^d)$. The latter is defined as the centered Gaussian process with covariance

$$\mathbb{E}\bigg(B_s(f)B_t(g)\bigg) = \big(s \wedge t\big) \int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x \qquad f, g \in \mathcal{S}(\mathbb{R}^d).$$

Here $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing functions in $\mathbb{R}^d$. To define B pointwise in $\mathbb{R}^d$, we need the regularization

$$B_{\varepsilon,t}(x) = B_t\big(\phi_\varepsilon(x - \cdot)\big),$$

with respect to some mollifier

$$\phi_\varepsilon = \varepsilon^{-d}\phi(x/\varepsilon).$$

Here ϕ is some smooth, non-negative, compactly supported and even function such that $\int_{\mathbb{R}^d} \phi(x)\mathrm{d}x = 1$. Then $\int_{\mathbb{R}^d} \phi_\varepsilon(x)\mathrm{d}x = 1$, and $\phi_\varepsilon \Rightarrow \delta_0$ weakly as probability measures. Furthermore, for any $\varepsilon > 0$, $B_\varepsilon = (B_{\varepsilon,t})_{t\geq 0}$ is also a centered Gaussian process with covariance

$$\mathbb{E}\big(B_{s,\varepsilon}(x)B_{t,\varepsilon}(y)\big) = \big(s \wedge t\big)V_\varepsilon(x - y)$$

where we introduced

$$V = \phi \star \phi, V_{\varepsilon,\delta} = \phi_\varepsilon \star \phi_\delta, V_\varepsilon = V_{\varepsilon,\varepsilon}. \tag{2.1}$$

Note that $V_\varepsilon(x) = \varepsilon^{-d}V(x/\varepsilon)$.

For any $\beta > 0$ and $\varepsilon > 0$, we consider the stochastic differential equation

$$\begin{aligned} \mathrm{d}u_{\varepsilon,t} &= \frac{1}{2}\Delta u_{\varepsilon,t}\mathrm{d}t + \beta\varepsilon^{\frac{d-2}{2}}\; u_{\varepsilon,t}\,\mathrm{d}B_{\varepsilon,t} \\ u_{\varepsilon,0} &= 1, \end{aligned} \tag{2.2}$$

where the stochastic differential is interpreted in the classical Ito sense (since our smoothing of B was done in space only, the well-defined solution $u_{\varepsilon,t}$ is adapted to the natural filtration $\mathcal{G}_t = \sigma(\{B_{\varepsilon,s}(x), x \in \mathbb{R}^d, s \leq t\})$. See e.g. [N97] for details. Our goal is to study the behavior of $u_{\varepsilon,1}(x)$ as the mollification parameter ε is turned off. For this, we will use a convenient Feynman-Kac representation of $u_{\varepsilon,t}(x)$, which we introduce in Section 2.3 after stating our main results.

2.2 Main results: weak and strong disorder Henceforth we fix $d \geq 3$ and set $u_\varepsilon(x) := u_{\varepsilon,1}(x)$ and, for any $f \in \mathcal{S}(\mathbb{R}^d)$, we write $u_\varepsilon(f) = \int_{\mathbb{R}^d} u_\varepsilon(x)f(x)\mathrm{d}x$. Here is the statement of our first main result.

Theorem 2.1 (Convergence to the heat equation in the weak disorder phase)**.** *There exists $\beta_\star \in (0,\infty)$ such that for all $\beta < \beta_\star$ and any $f \in \mathcal{S}(\mathbb{R}^d)$, $u_\varepsilon(f)$ converges in probability to $\int_{\mathbb{R}^d} f(x)dx$ as $\varepsilon \to 0$. Furthermore, for any $\beta < \beta_\star$ and any $x \in \mathbb{R}^d$, $u_\varepsilon(x)$ converges in distribution to a nondegenerate random variable Z_∞ which is positive almost surely.*

Remark 2.2. The first statement in Theorem 2.1 implies that u_ε converges in the sense of distributions to the solution of the heat equation. Although for simplicity we content ourselves with the initial condition $Z_\varepsilon(0,x) = 1$ in (2.2), the same statement continues to hold for reasonably nice initial condition $u_\varepsilon(0,x) = u_0(x)$.

Remark 2.3. While we do not discuss it in detail, the Feynman-Kac representation of $u_\varepsilon(x)$ that we introduce in the next subsection shows that $u_\varepsilon(x)$ and $u_\varepsilon(y)$ become asymptotically independent as $\varepsilon \to 0$; this explains the fact that smoothing with f makes $u_\varepsilon(f)$ deterministic.

The proof of Theorem 2.1 is based on an L^2 computation and is presented in Section 3.

Theorem 2.4 (The strong disorder phase)**.** *There is $\beta^* > 0$ such that for all $\beta > \beta^*$ and any fixed $x \in \mathbb{R}^d$, $u_\varepsilon(x) \to 0$ in probability.*

The proof of Theorem 2.4 is presented in Section 4. This proof avoids the use of the well-known *fractional moment method* which pervades the proofs of strong disorder assertions in realm of the aforementioned literature on the discrete directed polymer models, and instead uses the theory of *Gaussian multiplicative chaos* (GMC).

As a by-product of our arguments, we have the following corollary.

Corollary 2.5. There is a $\bar{\beta} \in (0,\infty)$ such that, as $\varepsilon \to 0$, $u_\varepsilon(0)$ converges to 0 in probability for all $\beta > \bar{\beta}$ while $u_\varepsilon(0)$ converges in distribution to a non-degenerate, strictly positive random variable $Z_\infty = Z_\infty(\beta)$ when $\beta < \bar{\beta}$.

The constant $\bar{\beta}$ is given as the threshold for the uniform integrability of a certain family of martingales $Z_{\varepsilon,\beta}$; we refer to the proof of Corollary 2.5 for details, which can also be found at the end of Section 4. We leave unresolved the question of what happens at $\beta = \bar{\beta}$.

Remark 2.6. Clearly $\bar{\beta}$ depends on the dimension $d \geq 3$ and on the mollifier ϕ since scaling the latter amounts to modifying β. As mentioned in Section 1, it remains an open problem to determine the exact value of $\overline{\beta} \in (0,\infty)$ and to identify the exact distribution of the positive random variable Z_∞ appearing in Corollary 2.5.

2.3 A Feynman-Kac representation For any $x \in \mathbb{R}^d$, let P_x denote the Wiener measure corresponding to a d-dimensional Brownian motion $(W_t)_{t\geq 0}$ starting at x and independent of the cylindrical Wiener process B. E_x will denote the corresponding expectation. For fixed W, set

$$M_{\varepsilon,t}(W) = \int_0^t \int_{\mathbb{R}^d} \phi_\varepsilon(W_s - x)\ \dot{B}(t-s,\mathrm{d}x)\mathrm{d}s \tag{2.3}$$

as a Wiener integral. For two fixed W and W', the covariance is given by

$$\mathbb{E}\left(M_{\varepsilon,t}(W)\cdot M_{\delta,t}(W')\right) = \int_0^t V_{\varepsilon,\delta}(W_s - W'_s)\mathrm{d}s \tag{2.4}$$

(recall (2.1). Here and later, we write $\mathbb{E}$ for integration over B only, keeping W fixed). We also note that, for any fixed W,

$$\mathbb{E}\left(M_{\varepsilon,t}^2(W)\right) = tV_\varepsilon(0) = t(\phi_\varepsilon \star \phi_\varepsilon)(0),$$

which diverges like ε^{-d} as $\varepsilon \to 0$.

We now turn to (2.2) and write its renormalized Feynman-Kac solution, see [BC95], as

$$\begin{aligned} u_{\varepsilon,t}(x) &= E_x\left[\exp\left\{\beta\varepsilon^{(d-2)/2}M_{\varepsilon,t}(W)\ -\frac{\beta^2\varepsilon^{d-2}}{2}\mathbb{E}(M_{\varepsilon,t}(W)^2)\right\}\right] \\ &= E_x\left[\exp\left\{\beta\varepsilon^{(d-2)/2}\int_0^t\int_{\mathbb{R}^d}\phi_\varepsilon(W_s - x)\ \dot{B}(t-s,\mathrm{d}x)\mathrm{d}s - \frac{\beta^2\varepsilon^{d-2}}{2}\ tV_\varepsilon(0)\right\}\right]. \end{aligned} \tag{2.5}$$

Note that $\mathbb{E}[u_{\varepsilon,t}(x)] = 1$.

For our purposes, it is convenient to introduce another representation of $u_{\varepsilon,t}$. Note that by rescaling of time and space, $\varepsilon^{-1}W_s$ has the same distribution as $W_{s\varepsilon^{-2}}$, while $\dot{B}(s, \mathrm{d}x)\mathrm{d}s$ has the same distribution as

$$\varepsilon^{d/2+1}\dot{B}\big(s\varepsilon^{-2}, \mathrm{d}\varepsilon^{-1}x\big)\mathrm{d}(\varepsilon^{-2}s).$$

Then, by (2.3), for a fixed W,

$$M_{\varepsilon,t}(W) \overset{(\mathrm{d})}{=} \frac{1}{\varepsilon^{(d-2)/2}} \int_0^{t\varepsilon^{-2}} \int_{\mathbb{R}^d} \phi\big(y - \varepsilon^{-1}W_{s\varepsilon^2}\big)\dot{B}(t/\varepsilon^2 - s, \mathrm{d}y)\mathrm{d}s$$

Hence (2.5) implies that

$$u_{\varepsilon,t}(x) \overset{(\mathrm{d})}{=} E_{\frac{x}{\varepsilon}}\Big[\exp\Big\{\beta \int_0^{t\varepsilon^{-2}} \int_{\mathbb{R}^d} \phi\big(y - W_s\big)\dot{B}(t/\varepsilon^2 - s, \mathrm{d}y)\mathrm{d}s - \frac{\beta^2}{2\varepsilon^2}tV(0)\Big\}\Big]. \tag{2.6}$$

Recall that $u_\varepsilon(x) = u_{\varepsilon,1}(x)$. Using the invariance of the distribution of $\dot{B}$ under time reversal, we obtain that for any fixed ε, the spatially-indexed process $\{u_\varepsilon(x)\}$ possesses the same distribution as the process $\{Z_\varepsilon(x/\varepsilon)\}$, where

$$Z_\varepsilon(x) = E_x\Big[\exp\Big\{\beta \int_0^{\varepsilon^{-2}} \int_{\mathbb{R}^d} \phi\big(y - W_s\big)\dot{B}(s, \mathrm{d}y)\mathrm{d}s - \frac{\beta^2}{2\varepsilon^2}V(0)\Big\}\Big] \tag{2.7}$$

3 Proof of Theorem 2.1: the second moment method

We start with an elementary computation.

Lemma 3.1. *Fix $d \geq 3$. If $\beta > 0$ is chosen small enough, for any $x \in \mathbb{R}^d$, the family $\{u_\varepsilon(x)\}_{\varepsilon>0}$ remains bounded in $L^2(\mathbb{P})$.*

Proof. Let W and W' be two independent standard Brownian motions with $P_0 \otimes P_0$ denoting their joint law. Then, writing $\eta_\varepsilon = \varepsilon^{(d-2)/2}$ and $M_\varepsilon(W) = M_{\varepsilon,1}(W)$,

$$\begin{aligned}
&\mathbb{E}\big[u_\varepsilon(0)^2\big] \\
&= \mathbb{E}\Big[\Big\{E_0 \exp\Big(\beta\eta_\varepsilon M_\varepsilon(W) \; - \frac{\beta^2\eta_\varepsilon^2}{2}\; V_\varepsilon(0)\Big)\Big\}^2\Big] \\
&= \big(E_0 \otimes E_0\big)\Big[\mathbb{E}\Big\{\exp\Big(\beta\eta_\varepsilon M_\varepsilon(W) \; - \frac{\beta^2\eta_\varepsilon^2}{2}\; V_\varepsilon(0)\Big)\exp\Big(\beta\eta_\varepsilon M_\varepsilon(W') \; - \frac{\beta^2\eta_\varepsilon^2}{2}\; V_\varepsilon(0)\Big)\Big\}\Big] \\
&= \big(E_0 \otimes E_0\big)\Big[\exp\Big\{\beta^2\eta_\varepsilon^2 \int_0^1 V_\varepsilon(W_s - W'_s)\mathrm{d}s\Big\}\Big] = E_0\Big[\exp\Big\{\beta^2\eta_\varepsilon^2 \int_0^1 V_\varepsilon(\sqrt{2}W_s)\mathrm{d}s\Big\}\Big]
\end{aligned}$$

where the third identity follows by (2.4). Hence, by (2.1), Brownian scaling and change of variables, we infer that

$$\mathbb{E}\big[u_\varepsilon^2(0)\big] = E_0\Big[\exp\Big\{\beta^2 \int_0^{1/\varepsilon^2} V(\sqrt{2}W_s)\mathrm{d}s\Big\}\Big] \leq E_0\Big[\exp\Big\{\beta^2 \int_0^{\infty} V(\sqrt{2}W_s)\mathrm{d}s\Big\}\Big].$$

Since V is a bounded function of compact support, it is easy to check (using that $d \geq 3$, see e.g. (3.5) below) that for β small enough,

$$\sup_{x\in\mathbb{R}^d} E_x\Big\{\beta^2 \int_0^\infty V(W_s)\mathrm{d}s\Big\} \leq \eta < 1. \tag{3.1}$$

Hence, by Portenko's lemma ([P76]),

$$\sup_{x\in\mathbb{R}^d} E_x\Big[\exp\Big\{\beta^2 \int_0^\infty V(W_s)\mathrm{d}s\Big\}\Big] \leq \frac{1}{1-\eta} < \infty. \tag{3.2}$$

This proves the lemma. ☐

Remark 3.2. Let us remark that $u_\varepsilon(0)$ is not a Cauchy sequence in $L^2(\mathbb{P})$. Indeed, a simple computation using (2.4) shows that

$$\begin{aligned}
&\mathbb{E}\big[(u_\varepsilon - u_\delta)^2\big] \\
&= E_0 \otimes E_0\bigg[\exp\bigg\{\beta^2\eta_\varepsilon^2 \int_0^t V_\varepsilon\big(W_s - W_s'\big)\mathrm{d}s\bigg\} - \exp\bigg\{\beta^2\eta_\varepsilon\eta_\delta \int_0^t V_{\varepsilon,\delta}\big(W_s - W_s'\big)\mathrm{d}s\bigg\}\bigg] \\
&+ E_0 \otimes E_0\bigg[\exp\bigg\{\beta^2\eta_\delta^2 \int_0^t V_\delta\big(W_s - W_s'\big)\mathrm{d}s\bigg\} - \exp\bigg\{\beta^2\eta_\varepsilon\eta_\delta \int_0^t V_{\varepsilon,\delta}\big(W_s - W_s'\big)\mathrm{d}s\bigg\}\bigg]
\end{aligned}$$

The difference of the two terms in the first line (and likewise, the second line) does not go to zero. For instance, if ϕ_ε is a centered Gaussian mollifier with variance ε^2, then in the first line, again by Brownian scaling, the second term (with the expectation) becomes (recall (2.1))

$$(E_0 \otimes E_0)\bigg[\exp\bigg\{\beta^2 \frac{\eta_\varepsilon\eta_\delta}{\eta^2_{\sqrt{\varepsilon^2+\delta^2}}} \int_0^{t/(\varepsilon^2+\delta^2)} V\big(W_s - W_s'\big)\mathrm{d}s\bigg\}\bigg]$$

while the first term becomes

$$(E_0 \otimes E_0)\bigg[\exp\bigg\{\beta^2 \int_0^{t/\varepsilon^2} V\big(W_s - W_s'\big)\mathrm{d}s\bigg\}\bigg].$$

From these expressions one can see that $\mathbb{E}\big[(u_\varepsilon - u_\delta)^2\big]$ does not vanish, e.g., in the iterated limit $\lim_{\varepsilon\to 0}\lim_{\delta\to 0}$.

Remark 3.3. The discussion in Remark 3.2 shows that $u_\varepsilon(0)$ does not converge in $L^2(\mathbb{P})$ as $\varepsilon \to 0$. In fact, it does not converge in probability either. Indeed, $u_\varepsilon(0)$ has, for fixed ε, the same distribution as $Z_\varepsilon(0)$ of (2.7). Noting that $\varepsilon \mapsto Z_\varepsilon(0)$ is a martingale with $\sup_\varepsilon \mathbb{E}(Z_\varepsilon(0)^2) < \infty$, it follows that $Z_\varepsilon(0)$ converges in $L^2(\mathbb{P})$ and in particular $Z_\varepsilon(0)^2$ is uniformly integrable. Therefore, the sequence $u_\varepsilon(0)^2$ is uniformly integrable, and in particular cannot converge in probability since it does not converge in $L^2(\mathbb{P})$.

We turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Let us denote by $\widehat{u}_\varepsilon(x) = u_\varepsilon(x) - \mathbb{E}(u_\varepsilon(x)) = u_\varepsilon(x) - 1$ and $\widehat{u}_\varepsilon(f) = \int_{\mathbb{R}^d} f(x)\widehat{u}_\varepsilon(x)\,\mathrm{d}x$. Then $\mathbb{E}\big(\widehat{u}_\varepsilon(f)\big) = 0$. Note that, for the proof of the first part of Theorem 2.1, it suffices to show that

$$\mathbb{E}\big[\widehat{u}_\varepsilon(f)^2\big] \to 0 \tag{3.3}$$

as $\varepsilon \to 0$. Let us prove this fact. Exactly similar computations as in the proof of Lemma 3.1 imply that

$$\begin{aligned}
\mathbb{E}\big[\widehat{u}_\varepsilon(f)^2\big] &= \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)f(y)\mathbb{E}\big[u_\varepsilon(x)u_\varepsilon(y)\big]\mathrm{d}x\mathrm{d}y - \bigg(\int_{\mathbb{R}^d} f(x)\mathrm{d}x\bigg)^2 \\
&= \int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)f(y)E_{\frac{x-y}{\varepsilon}}\bigg[\mathrm{e}^{\frac{1}{2}\beta^2\int_0^{2/\varepsilon^2} V(W_s)\mathrm{d}s}\bigg]\mathrm{d}x\mathrm{d}y - \bigg(\int_{\mathbb{R}^d} f(x)\mathrm{d}x\bigg)^2
\end{aligned} \tag{3.4}$$

If $z = (x-y)/\varepsilon$, then,

$$E_z\bigg[\int_0^\infty V(W_s)\,\mathrm{d}s\bigg] = C_d \int \mathrm{d}y\,\frac{V(y)}{|y-z|^{d-2}} \to 0 \quad \text{as } |z| \to \infty. \tag{3.5}$$

By applying Portenko's lemma again ([P76]), we see that for β small enough

$$\sup_x E_x\bigg[\mathrm{e}^{\frac{\beta^2}{2}\int_0^\infty V(W_s)\mathrm{d}s}\bigg] = \sup_x E_0\bigg[\mathrm{e}^{\frac{\beta^2}{2}\int_0^\infty V(W_s+x)\mathrm{d}s}\bigg] < \infty, \tag{3.6}$$

which implies the uniform integrability of $e^{\frac{\beta^2}{2}\int_0^\infty V(W_s+x)ds}$. Together with (3.5), we get for any even smaller β that

$$E_z\left[\mathrm{e}^{\frac{\beta^2}{2}\int_0^\infty V(W_s)\mathrm{d}s}\right] \to 1 \tag{3.7}$$

as $|z| \to \infty$. Combining (3.4), (3.6) and (3.7), we use the bounded convergence theorem to conclude (3.3). This proves the first part of Theorem 2.1.

For the second part, note that (2.6) implies that for fixed ε, $u_{\varepsilon,1}(0)$ is equal in distribution to Z_ε. Since the process $\{Z_\varepsilon\}_\varepsilon$ is a positive martingale (with respect to a filtration indexed by $1/\varepsilon^2$), it converges almost surely to a limit Z_∞. By Lemma 3.1, Z_ε is (uniformly in ε) $L^2(\mathbb{P})$ bounded for β small enough, and therefore Z_∞ does not vanish identically. By the 0-1 law as in the proof of Theorem 2.4 (see (4.1)), we conclude that $P(Z_\infty = 0) = 0$. Thus $u_\varepsilon(0)$ converges in distribution to Z_∞. Further, since $u_\varepsilon(x) \overset{d}{=} u_\varepsilon(0)$ by translation invariance, the same applies to $u_\varepsilon(x)$. Finally, since Z_∞ is the $L^2(\mathbb{P})$ limit of the sequence Z_ε, and the variance of the latter remains bounded away from 0 as $\varepsilon \to 0$, we conclude that Z_∞ is non-degenerate. □

4 Proof of Theorem 2.4 and Corollary 2.5: Gaussian multiplicative chaos

The starting point is the representation (2.7) for $Z_\varepsilon = Z_\varepsilon(0)$. For $d \geq 3$, which we assume throughout, we will show that there is a $\beta^* > 0$ such that for all $\beta > \beta^*$, $Z_\varepsilon \to 0$ in probability.

In order to prove this result, we represent Z_ε as a Gaussian Multiplicative Chaos (GMC), see [K85, S14] for background. Let $\mathcal{E} = C_0([0,\infty);\mathbb{R}^d)$ and recall that P_0 denotes the standard Wiener measure on $\mathcal{E}$ corresponding to the d-dimensional Brownian motion $W = (W_t)_{t\geq 0}$. Set

$$\Lambda_\varepsilon = \exp\left\{\beta\int_0^{\varepsilon^{-2}}\int_{\mathbb{R}^d}\phi\big(y - W_s\big)\dot{B}(s,\mathrm{d}y)\mathrm{d}s - \frac{\beta^2}{2\varepsilon^2}V(0)\right\}$$

and recall that $Z_\varepsilon \overset{(d)}{=} E_0\Lambda_\varepsilon$. Introduce the random measure M_ε with $\mathrm{d}M_\varepsilon = \Lambda_\varepsilon\,\mathrm{d}P_0$ on $\mathcal{E}$ and note that $Z_\varepsilon = \int_{\mathcal{E}} M_\varepsilon(\mathrm{d}W)$.

Introduce the event $\mathcal{V} := \{Z_\varepsilon \not\to_{\varepsilon\to 0} 0\}$. Since $\mathcal{V}$ is a tail event for the process $t \to B(t,\cdot)$, one has

$$\mathbb{P}(\mathcal{V}) \in \{0,1\}. \tag{4.1}$$

(To see the claim concerning the tail event, define the space-time shift $\theta^{y,s}$ formally by $\theta^{y,s}B(dx,dt) = B(y+dx, s+dt) - B(y,s)$ and note that $Z_\varepsilon(B) \to_{\varepsilon\to 0} 0$ iff $Z_\varepsilon(\theta^{y,s}B) \to_{\varepsilon\to 0} 0$ for Lebesgue almost every y and every $s > 0$.)

Note that $\varepsilon^{-1} \mapsto Z_\varepsilon$ is a strictly positive martingale of mean 1. Introduce on $\Omega\times\mathcal{E}$ the measure

$$\mathrm{d}\mathbb{Q}_\varepsilon := \Lambda_\varepsilon\,\mathrm{d}(\mathbb{P}\otimes P_0).$$

Let the measure $\overline{\mathbb{Q}}_\varepsilon$ be its marginal on Ω, i.e. $\mathrm{d}\overline{\mathbb{Q}}_\varepsilon = Z_\varepsilon\mathrm{d}\mathbb{P}$.

Lemma 4.1. *If the sequence $(Z_\varepsilon)_\varepsilon$ is uniformly integrable under $\mathbb{P}$, then under $\overline{\mathbb{Q}}_\varepsilon$, $(Z_\varepsilon)_\varepsilon$ is uniformly bounded in probability. In other words,*

$$\lim_{m\to\infty}\sup_\varepsilon \overline{\mathbb{Q}}_\varepsilon(Z_\varepsilon > m) = 0.$$

Proof. Assume that Z_ε is uniformly integrable. Then, by the la Vallée-Poussin theorem, there exists a convex increasing function $h:\mathbb{R}_+\to\mathbb{R}_+$, such that $h(x)/x \to \infty, x\to\infty$

and $\sup_\varepsilon \mathbb{E}h(Z_\varepsilon) = C < \infty$. Then,

$$C \geq \mathbb{E}h(Z_\varepsilon) = \int \frac{h(Z_\varepsilon)}{Z_\varepsilon} d\overline{\mathbb{Q}}_\varepsilon .$$

The conclusion follows. □

Remark 4.2. The implication in Lemma 4.1 is an "if and only if" statement; we only stated the direction that we need.

Another preparatory step that we need is the following proposition, whose statement and proof closely follow [CY06, Prop. 3.1].

Proposition 4.3. The sequence $\{Z_\varepsilon\}$ is uniformly integrable under $\mathbb{P}$ if and only if $\mathbb{P}(\mathcal{V}) = 1$.

Proof. If $\{Z_\varepsilon\}$ is uniformly integrable under $\mathbb{P}$ then its limit is necessarily non degenerate, i.e. $\mathbb{P}(\mathcal{V}) > 0$. Then, $\mathbb{P}(\mathcal{V}) = 1$ by (4.1).

To prove the reverse implication, recall the random variables $Z_\varepsilon(x)$ (with $x \in \mathbb{R}^d$), see (2.7). With $t = 1/\varepsilon^2$, we write $\bar{Z}_t(x) = Z_\varepsilon(x)$. It is enough to prove the uniform integrability for the sequence $\bar{Z}_n(0)$. Following [CY06], Let $\bar{Z}_\infty(B)$ denote the limit of $\bar{Z}_n(0)$ (which exists a.s.) and, for $z \in \mathbb{R}^d$, let $X_{n,z} = \bar{Z}_\infty(\theta_{n,z}B)/\mathbb{E}\bar{Z}_\infty$, where $\theta_{n,z}$ denote the temporal (by n) and spatial (by z) shift of B. Set, for $x, z \in \mathbb{R}^d$,

$$e_{n,x,z}(B) = E_x \left(\exp\left\{ \beta \int_0^1 \int_{\mathbb{R}^d} \phi(y - W_s) \dot{B}(s+n-1, \mathrm{d}y) \mathrm{d}s - \frac{\beta^2 V(0)}{2} \right\} \big| W_1 = z \right).$$

We have that $\mathbb{E}X_{n,z} = 1$ and $X_{n,x} \geq E_x(e_{n+1,x,W_1} \cdot X_{n+1,W_1})$ by Fatou. Denote by $\mathcal{G}_t$ the natural filtration induced by $t \to B(t, \cdot)$. By construction, $X_{n,\cdot}$ is independent of $\mathcal{G}_n$, and $\mathbb{E}(X_{n,z}|\mathcal{G}_n) = \mathbb{E}X_{n,z} = 1$. Now, iterating, we get by the Markov property

$$X_{0,0} \geq E_0(e_{1,0,W_1} e_{2,W_1,W_2} \cdots e_{n,W_{n-1},W_n} X_{n,W_n}) .$$

Thus,

$$\mathbb{E}(X_{0,0}|\mathcal{G}_n) \geq E_0(e_{1,0,W_1} e_{2,W_1,W_2} \cdots e_{n,W_{n-1},W_n}) = \bar{Z}_n .$$

It follows that the sequence $\bar{Z}_n$ is uniformly integrable under $\mathbb{P}$. □

Remark 4.4. An alternative proof of Proposition 4.3 can be obtained by using [K87, Thm. 2] and an appropriate 0-1 law with respect to the Brownian path W.

The following proposition is the heart of the proof of Theorem 2.4.

Proposition 4.5. There exists β^* such that for $\beta > \beta^*$ and any $m > 0$,

$$\overline{\mathbb{Q}}_\varepsilon(Z_\varepsilon > m) \to_{\varepsilon \to 0} 1.$$

We first complete the proof of Theorem 2.4, and then provide the proof of Proposition 4.5.

Proof of Theorem 2.4 (assuming Proposition 4.5): Assume that Z_ε does not converge to 0 almost surely. Then, by Proposition 4.3, it is uniformly integrable and, by Lemma 4.1, it is uniformly bounded in probability under $\overline{\mathbb{Q}}_\varepsilon$. In particular, there exists $K > 0$ such that $\overline{\mathbb{Q}}_\varepsilon(Z_\varepsilon > K) < 1/2$. This contradicts Proposition 4.5. □

Before providing the proof of Proposition 4.5, we need to introduce some notation and prove some preparatory lemmas. Introduce the stopping times $\tau_\delta(W, W') = \inf\{t > 0 : |W_t - W'_t| \geq \delta\}$. We need an estimate on the tail of $\tau := \tau_\delta$ conditionally on W, presented in the next lemma; in its statement and in its proof, $P_0^{\otimes 2}$ denotes the measure $P_0 \otimes P_0$ on (W, W')

Lemma 4.6. *There exists a random variable $\chi = \chi(W)$ and a constant $\kappa > 0$, such that*

$$P_0^{\otimes 2}\big(\tau \geq t | W\big) \geq \chi(W) e^{-\kappa t}.$$

Proof. Define

$$\kappa_1 = \liminf_{t \to \infty} \frac{1}{t} \log P_0^{\otimes 2}\big(\tau \geq t | W\big). \tag{4.2}$$

Note that since κ_1 is measurable with respect to the tail σ-field of W, it is deterministic, possibly equal to $-\infty$. We will show that $\kappa_1 > -\infty$. Taking then $\kappa = -2\kappa_1$ then proves the lemma.

With $|\cdot|$ denoting the Euclidean norm in $\mathbb{R}^d$, let

$$W_t^{1,2} = \left\{ \varphi : \varphi(0) = 0, \int_0^t |\dot\varphi(s)|^2 \mathrm{d}s < \infty \right\},$$

where $\dot\varphi$ denotes the time-derivative of φ. We also use the notation $\|\varphi\|_{\infty,t} = \sup_{s\in[0,t]} |\varphi(s)|$. Fix a (possibly random, but independent of W') function $\varphi \in W_t^{1,2}$. Then, by an application of the Cameron-Martin theorem in classical Wiener space,

$$\begin{aligned}
& P_0(\|W' - \varphi\|_{\infty,t} \leq \delta/2) \\
= \;& \int \mathrm{e}^{\int_0^t \dot\varphi(s)\mathrm{d}W'(s) - \frac{1}{2}\int_0^t |\dot\varphi(s)|^2 \mathrm{d}s} \mathbb{1}_{\{\|W'\|_{\infty,t} \leq \delta/2\}} \mathrm{d}P_0(W') \\
= \;& \mathrm{e}^{-\frac{1}{2}\int_0^t |\dot\varphi(s)|^2 \mathrm{d}s} \int \mathrm{e}^{\int_0^t \dot\varphi(s)\mathrm{d}W'(s)} \mathbb{1}_{\{\|W'\|_{\infty,t} \leq \delta/2\}} \mathrm{d}P_0(W') \\
= \;& \mathrm{e}^{-\frac{1}{2}\int_0^t |\dot\varphi(s)|^2 \mathrm{d}s}\; P_0\big(\|W'\|_{\infty,t} \leq \delta/2\big)\; E_0\Big[\mathrm{e}^{\int_0^t \dot\varphi(s)\,\mathrm{d}W'(s)} \big| \big\{\|W'\|_{\infty,t} \leq \delta/2\big\}\Big] \\
\geq \;& e^{-\frac{1}{2}\int_0^t |\dot\varphi(s)|^2 \mathrm{d}s} P_0\big(\|W'\|_{\infty,t} \leq \delta/2\big),
\end{aligned} \tag{4.3}$$

where the last inequality used Jensen's inequality and invariance of the set $\|W'\|_{\infty,t} \leq \delta/2$ with respect to the map $W' \mapsto -W'$.

Introduce the random field

$$Y_{s,t}(W) = \inf \left\{ \int_s^t |\dot\varphi(u)|^2 \;\mathrm{d}u : \varphi(s) = W_s, \varphi(t) = W_t, \sup_{u\in[s,t]} |W(u) - \varphi(u)| \leq \delta/2 \right\}.$$

Since Y is subadditive in the sense that $Y_{s,t} \leq Y_{s,u} + Y_{u,t}$ for $u \in (s,t)$, Kingman's subadditive ergodic theorem implies that

$$t^{-1} Y_{0,t} \to_{t\to\infty} \kappa_2, \quad a.s. \tag{4.4}$$

for a deterministic κ_2. We claim that κ_2 is finite. To see that, note that $\kappa_2 \leq E Y_{0,1}$. Set $X := \sqrt{Y_{0,1}} : \mathcal{E} \to \mathbb{R}^d$ and note that $X < \infty$, P_0-a.s, and that

$$|X(W+\varphi) - X(W)| \leq \left(\int_0^1 |\dot\varphi(t)|^2 dt \right)^{1/2}$$

for P_0-almost all W and all φ in the Cameron-Martin space $W_1^{1,2}$. Thus, denoting by $\mathrm{med}(X)$ the median of X we have by isoperimetry for the Gaussian measure, see e.g. [Bo98, Theorem 4.5.6] that $X - \mathrm{med}(X)$ possesses Gaussian tails, and therefore $EX^2 = EY_{0,1} < \infty$.

We can now conclude. Let $\varphi^{(t)} = \varphi^{(t)}(W)$ be such that $\varphi^{(t)}(0) = 0$, $\varphi^{(t)}(t) = W(t)$ and $Y_{0,t} = \int_0^t |\dot\varphi(s)|^2 ds$. (Such $\varphi^{(t)}$ exists by lower-semicontinuity of the L^2 norm, although this is not essential to our argument and we could just assume that the last integral is smaller than $2Y_{0,t}$.) We have, by (4.3),

$$P_0^{\otimes 2}\big(\tau \geq t|W\big) = P_0^{\otimes 2}\big(\|W' - W\|_{\infty,t} \leq \delta|W\big) \geq P_0^{\otimes 2}\big(\|W' - \varphi^{(t)}\|_{\infty,t} \leq \delta/2|W\big) \\ \geq e^{-\frac{1}{2}Y_{0,t}} P_0(\|W'\|_{\infty,t} \leq \delta/2\big).$$

Thus, by (4.2) and (4.4),

$$\kappa_1 = \liminf_{t\to\infty} \frac{1}{t} \log P^{\otimes 2}(\tau \geq t|W) \geq -\frac{\kappa_2}{2} + \lim_{t\to\infty} \frac{1}{t} \log P_0(\|W'\|_{\infty,t} \leq \delta/2).$$

The last probability on the right hand side is $P_0(\sigma > t)$, where σ denotes the first exit time of the standard Brownian motion W' from the ball of radius $\delta/2$ around the origin. It is well-known (for example, by the spectral theorem for $-\frac{1}{2}\Delta$) that $\lim_{t\to\infty} \frac{1}{t} \log P_0\{\sigma > t\} = -\lambda_1$, where $\lambda_1 > 0$ is the principal eigenvalue of $-\frac{1}{2}\Delta$ with Dirichlet boundary conditions on the same ball. It follows that $\kappa_1 > -\infty$ and Lemma 4.6 is proved. □

Henceforth, we set $t = \varepsilon^{-2}$. Next, on $\mathcal{E} \times \mathcal{E}$, introduce the kernels

$$K_\varepsilon(W, W') = \int_0^{1/\varepsilon^2} \int_{\mathbb{R}^d} \phi(x - W_s)\phi(x - W'_s) dx\, ds.$$

Note that, by Cauchy-Schwarz, $K_\varepsilon(W, W') \leq V(0)t$.

Lemma 4.7. *There exists $\delta > 0$ such that on the event $\{\tau_\delta(W, W') \geq t\}$, one has $K_\varepsilon(W, W') \geq 2V(0)t/3$.*

Proof. Recall that $V(0) = \int_{\mathbb{R}^d} \phi^2(y)dy$. On the other hand, for θ small enough,

$$\inf_{f:\, \forall s,\ |f(s)|\leq\theta} \int_0^t \int_{\mathbb{R}^d} \phi(y)\phi\big(y + f(s)\big) dy ds \geq t\big(V(0) - O(\theta)\big).$$

This completes the proof. □

Finally we turn to the proof of Proposition 4.5.

Proof of Proposition 4.5: Since we will use two independent copies W, W' of Brownian motions, we write throughout $\Lambda_\varepsilon = \Lambda_\varepsilon(W)$, $\Lambda_\varepsilon(W')$ to emphasize which Brownian motion participates in the definition of Λ_ε.

The starting point of the proof is the remark that by the Cameron-Martin change of measure [Bo98], the law of $\dot{B}(x, s)$ under $\mathbb{Q}_\varepsilon$ is the same as the law of $\dot{B}(x, s) + \beta\phi(x - W_s)$ under $\mathbb{P} \otimes P_0$ when restricted to the σ-algebra generated by $\dot{B}\restriction_{\mathbb{R}^d\times[0,t]}$.

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing concave function. Then, by the above remark,

$$\begin{aligned}
\int f(Z_\varepsilon) d\overline{\mathbb{Q}}_\varepsilon &= \int f(Z_\varepsilon) d\mathbb{Q}_\varepsilon = \int f\bigg(\int \Lambda_\varepsilon(W') dP_0(W')\bigg) d\mathbb{Q}_\varepsilon \\
&= \int f\bigg(\int \Lambda_\varepsilon(W') e^{\beta^2 K_\varepsilon(W,W')} dP_0(W')\bigg) d(\mathbb{P} \otimes P_0) \\
&\geq \int f\bigg(\int \Lambda_\varepsilon(W') e^{\beta^2 K_\varepsilon(W,W')} \mathbb{1}_{\{\tau(W,W')\geq t\}} dP_0(W')\bigg) d(\mathbb{P} \otimes P_0) \\
&\geq \int f\bigg(\int \Lambda_\varepsilon(W') e^{2\beta^2 V(0)t/3} \mathbb{1}_{\{\tau(W,W')\geq t\}} dP_0(W')\bigg) d(\mathbb{P} \otimes P_0) \\
&= \int f\bigg(e^{2\beta^2 V(0)t/3} \int \Lambda_\varepsilon(W') \mathbb{1}_{\{\tau(W,W')\geq t\}} dP_0(W')\bigg) d(\mathbb{P} \otimes P_0), \quad (4.5)
\end{aligned}$$

where in the first inequality we used that f is increasing, and in the last inequality we used the same together with Lemma 4.7 (recall $t = \varepsilon^{-2}$). On the other hand, f is concave and on the set $\{\tau \geq t\}$ the covariance kernel K_ε is bounded from above by the constant kernel $\widehat{K}_\varepsilon(W,W') := V(0)t$. Using Kahane's comparison inequality with kernels K_ε and $\widehat{K}_\varepsilon$ (see [K85] – it is stated there for *convex* functions, with the opposite sign; see also [S14, Theorem 28]), we get:

$$\int f(Z_\varepsilon)\mathrm{d}\overline{\mathbb{Q}}_\varepsilon \geq E_{G,W}\bigg[f\bigg(\mathrm{e}^{2\beta^2 V(0)t/3}\ \big(P_0 \otimes P_0\big)\big(\tau(W,W') \geq t|W\big)\ \mathrm{e}^{\beta(V(0)t)^{1/2}G - \beta^2 V(0)t/2}\bigg)\bigg], \tag{4.6}$$

where G is a standard centered Gaussian random variable which is independent of W, and the expectation $E_{G,W}$ is taken over both G and W. In particular,

$$\begin{aligned}\int f(Z_\varepsilon)\mathrm{d}\overline{\mathbb{Q}}_\varepsilon &\geq E_{G,W}\bigg[f\bigg(\mathrm{e}^{\beta^2 V(0)t/6}\ \big(P_0 \otimes P_0\big)\big(\tau(W,W') > t|W\big)\mathrm{e}^{\beta(V(0)t)^{1/2}G}\bigg)\bigg] \\ &\geq E_{G,W}\bigg[f\bigg(\chi(W)e^{-\kappa t}\mathrm{e}^{\beta^2 V(0)t/6}e^{\beta\sqrt{V(0)t}G}\bigg)\bigg].\end{aligned} \tag{4.7}$$

Note that the argument of f goes to infinity as $t \to \infty$ for almost every (G,W), if $\beta > \sqrt{6\kappa/V(0)}$. Using

$$f(x) = f_\alpha(x) = \begin{cases} \alpha^{-1}x, & x \leq \alpha \\ 1, & x \geq \alpha, \end{cases}$$

we conclude that

$$\lim_{\alpha\to\infty}\liminf_{\varepsilon\to 0}\int f_\alpha(Z_\varepsilon)\mathrm{d}\overline{\mathbb{Q}}_\varepsilon = 1.$$

This completes the proof. □

Proof of Corollary 2.5. Recall the random variable

$$Z_\varepsilon = Z_{\varepsilon,\beta}(B) = E_0\bigg[\exp\bigg\{\beta\int_0^{\varepsilon^{-2}}\int_{\mathbb{R}^d}\phi\big(y - W_s\big)\dot{B}(s,\mathrm{d}y)\mathrm{d}s - \frac{\beta^2}{2\varepsilon^2}V(0)\bigg\}\bigg].$$

Let

$$\overline{\beta} = \sup\bigg\{\beta > 0 : \big\{Z_{\varepsilon,\beta}\big\}_{\varepsilon>0} \text{ is uniformly integrable}\bigg\}.$$

In view of Theorem 2.1 and Theorem 2.4, we have $\overline{\beta} \in (0,\infty)$. Thus, the corollary will follow from the following fact.

$$\text{If } Z_{\varepsilon,\beta} \text{ is uniformly integrable for some } \beta > 0, \text{ then so is } Z_{\varepsilon,\beta'} \text{ for } \beta' < \beta. \tag{4.8}$$

To see (4.8), let B, B' be independent copies of B and let $\beta' = \rho\beta$ with $\rho < 1$. To emphasize the dependence of $Z_{\varepsilon,\beta}$ on B, we write $Z_{\varepsilon,\beta} = Z_{\varepsilon,\beta}(B)$. Note that

$$Z_{\varepsilon,\beta'}(B) = Z_{\varepsilon,\rho\beta}(B) = \mathbb{E}\left[Z_{\varepsilon,\beta}(\rho B + \sqrt{1-\rho^2}B')\,|B\right]$$

Since $\{Z_{\varepsilon,\beta}(B)\}_{\varepsilon>0}$ is uniformly integrable, there exists a positive increasing convex function f with $f(x)/x \to_{x\to\infty} \infty$ so that $\sup_\varepsilon \mathbb{E}f(Z_{\varepsilon,\beta}(B)) < \infty$. However, by Jensen's inequality and the last display,

$$\begin{aligned}\mathbb{E}[f(Z_{\varepsilon,\beta'}(B))] &= \mathbb{E}\bigg[f\left(\mathbb{E}\left(Z_{\varepsilon,\beta}(\rho B + \sqrt{1-\rho^2}B')\mid B\right)\right)\bigg] \\ &\leq \mathbb{E}\big[f(Z_{\varepsilon,\beta}(\rho B + \sqrt{1-\rho^2}B'))\big] = \mathbb{E}[f(Z_{\varepsilon,\beta}(B))]\,.\end{aligned}$$

It follows that $\sup_{\varepsilon>0} \mathbb{E}[f(Z_{\varepsilon,\beta'}(B))] < \infty$, which in turn implies the uniform integrability of $\{Z_{\varepsilon,\beta'}\}_{\varepsilon>0}$. This completes the proof. □

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