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# Hölderian weak invariance principle under the Maxwell and Woodroofe condition

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**Abstract.** We investigate the weak invariance principle in Hölder spaces under some reinforcement of the Maxwell and Woodroofe condition. Optimality of the obtained condition is established.

## 1 Introduction and main results

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T\colon \Omega \to \Omega$ be a measure-preserving bijective and bi-measurable map. Let $\mathcal{M}$ be a sub-$\sigma$-algebra of $\mathcal{F}$ such that $T\mathcal{M} \subset \mathcal{M}$. If $f\colon \Omega \to \mathbf{R}$ a measurable function, we denote $S_n(T, f) := \sum_{j=0}^{n-1} f \circ T^j$ and

$$W(n, f, T, t) := S_{[nt]}(T, f) + \big(nt - [nt]\big) f \circ T^{[nt]}. \tag{1.1}$$

We shall write $S_n(f)$ and $W(n, f, t)$ for simplicity, except when $T$ is replaced by $T^2$.

An important problem in probability theory is the understanding of the asymptotic behavior of the process $(n^{-1/2} W(n, f, t), t \in [0, 1])_{n \geq 1}$. Conditions on the quantities $\mathbf{E}[S_n(f) | T\mathcal{M}]$ and $S_n(f) - \mathbf{E}[S_n(f) | T^{-n}\mathcal{M}]$ have been investigated. The first result in this direction was obtained by Maxwell and Woodroofe (2000): if $f$ is $\mathcal{M}$-measurable and

$$\sum_{n=1}^{+\infty} \frac{\|\mathbf{E}[S_n(f) | \mathcal{M}]\|_2}{n^{3/2}} < +\infty, \tag{1.2}$$

then $(n^{-1/2} S_n(f))_{n \geq 1}$ converges in distribution to $\eta^2 N$, where $N$ is normally distributed and independent of $\eta$. Then Volný (2006) proposed a method to treat the nonadapted case. Peligrad and Utev (2005) proved the weak invariance principle under condition (1.2). The nonadapted case was addressed in Volný (2007). Peligrad and Utev also showed that condition (1.2) is optimal among conditions on the growth of the sequence $(\|\mathbf{E}[S_n(f) | \mathcal{M}]\|_2)_{n \geq 1}$: if

$$\sum_{n=1}^{+\infty} a_n \frac{\|\mathbf{E}[S_n(f) | \mathcal{M}]\|_2}{n^{3/2}} < \infty \tag{1.3}$$

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for some sequence $(a_n)_{n\geq 1}$ converging to 0, the sequence $(n^{-1/2}S_n(f))_{n\geq 1}$ is not necessarily stochastically bounded (Theorem 1.2. of Peligrad and Utev (2005)). Volný constructed Volný (2010) an example satisfying (1.3) and such that the sequence $(\|S_n(f)\|_2^{-1}S_n(f))_{n\geq 1}$ admits two subsequences which converge weakly to two different distributions.

Let us denote by $\mathcal{H}_\alpha$ the space of Hölder continuous functions, that is, the functions $x\colon [0,1]\to\mathbf{R}$ such that $\|x\|_{\mathcal{H}_\alpha} := \sup_{0\leq s<t\leq 1}|x(t)-x(s)|/(t-s)^\alpha + |x(0)|$ is finite. Since the paths of Brownian motion belong almost surely to $\mathcal{H}_\alpha$ for each $\alpha\in(0,1/2)$ as well as $W(n,f,\cdot)$, we can investigate the weak convergence of the sequence $(n^{-1/2}W(n,f,\cdot))_{n\geq 1}$ in the the space $\mathcal{H}_\alpha$, for $0<\alpha<1/2$. The case of i.i.d. sequences and stationary martingale difference sequences have been addressed respectively, by Račkauskas and Suquet (Theorem 1 of Račkauskas and Suquet (2003)) and Giraudo (Theorem 2.2 of Giraudo (2016b)). In this note, we focus on conditions on the sequences $(\mathbf{E}[S_n(f)|\mathcal{M}])_{n\geq 1}$ and $(S_n(f)-\mathbf{E}[S_n(f)|T^{-n}\mathcal{M}])_{n\geq 1}$.

**Theorem 1.1.** *Let $p>2$ and $f\in\mathbf{L}^p$. If*

$$\begin{aligned}\sum_{k=1}^{+\infty}\frac{\|\mathbf{E}[S_k(f)|\mathcal{M}]\|_p}{k^{3/2}}&<+\infty,\\ \sum_{k=1}^{+\infty}\frac{\|S_k(f)-\mathbf{E}[S_k(f)|T^{-k}\mathcal{M}]\|_p}{k^{3/2}}&<+\infty,\end{aligned}\tag{1.4}$$

*then the sequence $(n^{-1/2}W(n,f))_{n\geq 1}$ converges weakly to the process $\sqrt{\eta}W$ in $\mathcal{H}_{1/2-1/p}$, where $W$ is the Brownian motion and the random variable $\eta$ is independent of $W$ and is given by $\eta=\lim_{n\to+\infty}\mathbf{E}[S_n(f)^2|\mathcal{I}]/n$ (where $\mathcal{I}$ is the $\sigma$-algebra of invariant sets and the limit is in the $\mathbf{L}^1$ sense).*

Of course, if $f$ is $\mathcal{M}$-measurable, all the terms of the second series vanish and we only have to check the convergence of the first series.

**Remark 1.2.** If the sequence $(f\circ T^j)_{j\geq 0}$ is a martingale difference sequence with respect to the filtration $(T^{-i}\mathcal{M})$, then condition (1.4) is satisfied if and only if the function $f$ belongs to $\mathbf{L}^p$, hence we recover the result of Giraudo (2016b). However, if the sequence $(f\circ T^j)_{j\geq 0}$ is independent, (1.4) is stronger than the sufficient condition $t^p\mu\{|f|>t\}\to 0$. This can be explained by the fact that the key maximal inequality (2.9) does not include the quadratic variance term which appears in the martingale inequality. In Remark 1 (after the proof of Theorem 1) in Peligrad, Utev and Wu (2007), a version of (2.9) with this term is obtained. In our context it seems that it does not follow from an adaptation of the proof.

**Remark 1.3.** In Giraudo (2016b), the conclusion of Theorem 1.1 was obtained for an $\mathcal{M}$-measurable $f$ under the condition

$$\sum_{i=1}^{\infty}\left\|\mathbf{E}[f|T^i\mathcal{M}]-\mathbf{E}[f|T^{i+1}\mathcal{M}]\right\|_p<\infty, \tag{1.5}$$

which holds as soon as

$$\sum_{k=1}^{+\infty}\frac{\|\mathbf{E}[f\circ T^k|\mathcal{M}]\|_p}{k^{1/p}}<+\infty, \tag{1.6}$$

while (1.4) holds as soon as

$$\sum_{k=1}^{+\infty}\frac{\|\mathbf{E}[f\circ T^k|\mathcal{M}]\|_p}{\sqrt{k}}<+\infty. \tag{1.7}$$

Therefore, (1.7) gives a better sufficient condition than (1.6) if we seek for conditions relying only on $(\|\mathbf{E}[f\circ T^k|\mathcal{M}]\|_p)_{k\geq 1}$.

However, (1.5) gives the existence of a martingale approximation in the following sense: there exists a martingale difference $m\in\mathbf{L}^p(\mathcal{M})$ such that

$$\left\|\|W(n,f)-W(n,m)\|_{\mathcal{H}_{1/2-1/p}}\right\|_{p,\infty}=o(\sqrt{n}). \tag{1.8}$$

Indeed, define for an integrable function $h$ and a non-negative integer $i$, $P_i(h):=\mathbf{E}[h|T^i\mathcal{M}]-\mathbf{E}[h|T^{i+1}\mathcal{M}]$. If $f$ satisfies (1.5), then we set $m:=\sum_{i\geq 0}P_0(U^if)$. Then for any $K\geq 1$, the equality $f-m=\sum_{i=0}^{K}(P_i(f)-P_0(U^if))+\sum_{i=K+1}^{+\infty}(P_i(f)-P_0(U^if))$ holds. Since $\sum_{i=0}^{K}(P_i(f)-P_0(U^if))$ may be written as $(I-U)g_K$, where $g_K$ is such that $t^p\mu\{|g_K|>t\}\to 0$ as $t$ goes to infinity, we get, by inequalities (2.4) and (2.5) of Giraudo (2016b) that

$$\begin{aligned}&\limsup_{n\to+\infty}\frac{1}{\sqrt{n}}\left\|\|W(n,f)-W(n,m)\|_{\mathcal{H}_{1/2-1/p}}\right\|_{p,\infty}\\&\quad\leq\sum_{i\geq K+1}\limsup_{n\to+\infty}\frac{1}{\sqrt{n}}\Big(\left\|\|W\big(n,P_i(f)\big)\|_{\mathcal{H}_{1/2-1/p}}\right\|_{p,\infty}\\&\qquad+\left\|\|W\big(n,P_0(U^i(f))\big)\|_{\mathcal{H}_{1/2-1/p}}\right\|_{p,\infty}\Big).\end{aligned}$$

We conclude by Proposition 2.3 of Giraudo (2016b).

The following condition (in the spirit of Maxwell and Woodroofe's one) is sufficient for a martingale approximation in the sense of (1.8):

$$\sum_{k=1}^{+\infty}\frac{\|\mathbf{E}[S_k(f)|\mathcal{M}]\|_p}{k^{1+1/p}}<+\infty. \tag{1.9}$$

Indeed, Theorem 2.3 of Cuny and Merlevède (2014) gives a martingale differences sequence $(m\circ T^i)_{i\geq 0}$ such that $\lim_{n\to+\infty}n^{-1/p}\|S_n(f-m)\|_p=0$. Using Serfling arguments (see Serfling (1970)), we get that (1.9) implies

$$\lim_{n\to+\infty} n^{-1/p} \Big\| \max_{1\le i\le n} \big|S_i(f-m)\big| \Big\|_p = 0. \tag{1.10}$$

Note that for a function $h$, by Lemma A.2 of Markevičiūtė, Suquet and Račkauskas (2012),

$$n^{-1/2} \big\| \|W(n,h)\|_{\mathcal{H}_{1/2-1/p}} \big\|_{p,\infty} \le 2n^{-1/p} \Big\| \max_{1\le j\le n} \big|S_j(f)\big| \Big\|_{p,\infty},$$

hence by (1.10), the martingale approximation (1.8) holds.

Furthermore, using the construction given in Durieu and Volný (2008), Durieu (2009), in any ergodic dynamical system of positive entropy one can construct a function satisfying condition (1.4) but not (1.5) and vice versa.

**Remark 1.4.** For the $\rho$-mixing coefficient defined by

$$\rho(n) = \sup\big\{\mathrm{Cov}(X,Y)/\big(\|X\|_2\|Y\|_2\big), X \in \mathbf{L}^2(\sigma\big(f\circ T^i, i\le 0\big),$$
$$Y \in \mathbf{L}^2\big(\sigma\big(f\circ T^i, i\ge n\big)\big)\big\},$$

Lemma 1 of Peligrad, Utev and Wu (2007) shows that for an adapted process, condition (1.4) is satisfied if the series $\sum_{n=1}^{\infty} \rho^{2/p}(2^n)$ converges. However, the conclusion of Theorem 1.1 holds if $t^p\mu\{|f|>t\}\to 0$ and $\sum_{n=1}^{\infty}\rho(2^n)$ converges (see Theorem 2.3, Giraudo (2016a)), which is less restrictive.

It turns out that even in the adapted case, condition (1.4) is sharp among conditions on $\|\mathbf{E}[S_k(f)|\mathcal{M}]\|_p$ in the following sense.

**Theorem 1.5.** *For each sequence $(a_n)_{n\ge1}$ converging to 0 and each real number $p>2$, there exists a strictly stationary sequence $(f\circ T^j)_{j\ge0}$ and a sub-$\sigma$-algebra $\mathcal{M}$ such that $T\mathcal{M}\subset\mathcal{M}$,*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{3/2}} \|\mathbf{E}[S_n(f)|\mathcal{M}]\|_p < \infty, \tag{1.11}$$

*but the sequence $(n^{-1/2}W(n,f,t))_{n\ge1}$ is not tight in $\mathcal{H}_{1/2-1/p}$.*

**Remark 1.6.** Using the inequalities in Peligrad, Utev and Wu (2007) in order to bound $\|\mathbf{E}[S_n(f)|T\mathcal{M}]\|_2$, we can see that the constructed $f$ in the proof of Theorem 1.5 satisfies the classical Maxwell and Woodroofe condition (1.2) (the fact that $p$ is strictly greater than 2 is crucial), hence the weak invariance principle in the space of continuous functions takes place.

However, it remains an open question whether condition (1.11) implies the central limit theorem or the weak invariance principle (in the space of continuous functions).

## 2 Proofs

We may observe that condition (1.4) implies by Theorem 1 of Peligrad, Utev and Wu (2007) that the sequence $(S_n(f)/\sqrt{n})_{n\geq 1}$ is bounded in $\mathbf{L}^p$; nevertheless the counter-example given in Theorem 2.6 of Giraudo (2016a) shows that we cannot deduce the weak invariance principle from this.

We shall rather work with a tightness criterion. The analogue of the continuity modulus in $C[0,1]$ is $\omega_\alpha$, defined by

$$\omega_\alpha(x,\delta) = \sup_{0<|t-s|<\delta} \frac{|x(t)-x(s)|}{|t-s|^\alpha}, \qquad x\colon [0,1]\to \mathbf{R}, \delta\in (0,].$$

Define $\mathcal{H}^o_\alpha[0,1] := \{x \in \mathcal{H}_\alpha[0,1], \lim_{\delta\to 0}\omega_\alpha(x,\delta) = 0\}$. We shall essentially work with the space $\mathcal{H}^o_\alpha[0,1]$ which, endowed with $\|\cdot\|_\alpha\colon x\mapsto \omega_\alpha(x,1)+|x(0)|$, is a separable Banach space (while $\mathcal{H}_\alpha[0,1]$ is not). Since the canonical embedding $\iota\colon \mathcal{H}^o_\alpha[0,1]\to\mathcal{H}_\alpha[0,1]$ is continuous, each convergence in distribution in $\mathcal{H}^o_\alpha[0,1]$ also takes place in $\mathcal{H}_\alpha[0,1]$.

Let us state the tightness criterion we shall use (Theorem 13 of Suquet (1999)).

**Proposition 2.1.** *Let $\alpha\in(0,1)$. A sequence of processes $(\xi_n)_{n\geq 1}$ with paths in $\mathcal{H}^o_\alpha[0,1]$ and such that $\xi_n(0)=0$ for each $n$ is tight in $\mathcal{H}^o_\alpha[0,1]$ if and only if*

$$\forall\varepsilon>0, \qquad \lim_{\delta\to 0}\sup_{n\to+\infty}\mu\{\omega_\alpha(\xi_n,\delta)>\varepsilon\} = 0. \tag{2.1}$$

In order to prove the weak convergence in $\mathcal{H}^o_\alpha[0,1]$, it suffices to prove the convergence of the finite dimensional distributions and establish tightness in this space.

### 2.1 A maximal inequality

For $p>2$, we define

$$\|h\|_{p,\infty} := \sup_{\substack{A\in\mathcal{F}\\ \mu(A)>0}} \frac{1}{\mu(A)^{1-1/p}}\mathbf{E}[|h|\mathbf{1}_A]. \tag{2.2}$$

This norm is linked to the tail function of $h$ by the following inequalities (see Exercise 1.1.12, p. 13 in Grafakos (2014)):

$$\left(\sup_{t>0} t^p\mu\{|h|>t\}\right)^{1/p} \leq \|h\|_{p,\infty} \leq \frac{p}{p-1}\left(\sup_{t>0} t^p\mu\{|h|>t\}\right)^{1/p}. \tag{2.3}$$

As a consequence, if $N$ is an integer and $h_1,\dots,h_n$ are functions, then

$$\left\|\max_{1\leq j\leq N}|h_j|\right\|_{p,\infty} \leq \frac{p}{p-1}N^{1/p}\max_{1\leq j\leq N}\left\||h_j|\right\|_{p,\infty}. \tag{2.4}$$

For a positive $n \geq 1$, a function $f\colon \Omega \to \mathbf{R}$ and a measure-preserving map $T$, we define

$$M(n, f, T) := \max_{0 \leq i < j \leq n} \frac{|S_j(T, f) - S_i(T, f)|}{(j-i)^{1/2-1/p}}. \tag{2.5}$$

By Lemma A.2 of Markevičiūtė, Suquet and Račkauskas (2012), the Hölderian norm of a polygonal line is reached at two vertices, hence

$$M(n, f, T) = n^{1/2-1/p} \| W(n, f, T, \cdot) \|_{\mathcal{H}_{1/2-1/p}}. \tag{2.6}$$

Applying Proposition 2.3 of Giraudo (2016b), we can find for each $p > 2$ a constant $C_p$ depending only on $p$ such that if $(m \circ T^i)_{i \geq 1}$ is a martingale difference sequence, then for each $n$,

$$\frac{1}{\sqrt{n}} \left\| \| W(n, m, T, \cdot) \|_{\mathcal{H}_{1/2-1/p}} \right\|_{p,\infty} \leq C_p \|m\|_p. \tag{2.7}$$

In the sequel, fix such a constant $C_p$ that we shall choose greater than $6 \cdot 2^{1/p} p/(p-1)$. We denote by $U$ the Koopman operator associated with $T$, that is, for each $f\colon \Omega \to \mathbf{R}$ and each $\omega \in \Omega$, $(Uf)(\omega) = f(T\omega)$.

**Definition 2.2.** Let $H$ be a closed subspace of $\mathbf{L}^p$. Let $P$ be a linear operator from $H$ to itself. We say that $(H, P)$ satisfies condition (C) if:

1. the inclusion $U^{-1}H \subset H$ holds (respectively, the inclusion $UH \subset H$ holds);
2. $P$ is power bounded on $H$, that is, for each $h \in H$,

$$K(P) := \sup_{n \geq 1} \sup_{h \in H \setminus \{0\}} \frac{\|P^n h\|_p}{\|h\|_p} < +\infty\,; \tag{2.8}$$

3. if $h \in H$ is such that $Ph = 0$, then the sequence $(h \circ T^i)_{i \geq 0}$ is a martingale difference sequence with respect to the filtration $(T^{-i}\mathcal{M})_{i \geq 0}$ (respectively, $(T^{-i-1}\mathcal{M})_{i \geq 0}$);
4. $PU^{-1}f = f$ for each $f \in H$ (respectively, $PUf = f$ for each $f \in H$).

Let us give two examples of subspace $H$ and operator $P$ satisfying condition (C).

1. Let $H$ be the subspace of $\mathbf{L}^p$ which consists of $\mathcal{M}$-measurable functions and $Ph := \mathbf{E}[Uh|\mathcal{M}]$. Then $(H, P)$ satisfies condition (C).
2. Let $H$ be the subspace of $\mathbf{L}^p$ which consists of functions $h$ such that $\mathbf{E}[h|\mathcal{M}] = 0$ and $Ph := U^{-1}h - \mathbf{E}[U^{-1}h|\mathcal{M}]$. Then $(H, P)$ satisfies condition (C).

The goal of this subsection is to establish the following maximal inequality.

**Proposition 2.3.** *Let $T\colon \Omega \to \Omega$ be a bijective and bi-measurable measure-preserving map. Let $H$ be a closed subspace of $\mathbf{L}^p$. Let $r$ be a positive integer.*

*For each, operator $P$ from $H$ to itself such that $(H, P)$ satisfies condition (C), each $f \in H$ and each integer $n$ satisfying $2^{r-1} \leq n < 2^r$,*

$$\begin{aligned} &\|M(n, f, T)\|_{p,\infty} \\ &\quad \leq C_p n^{1/p}\left((1+K(P))\|f\|_p + K_p \sum_{j=0}^{r-1} 2^{-j/2}\left\|\sum_{i=0}^{2^j-1} P^i f\right\|_p\right), \end{aligned} \tag{2.9}$$

*where $K_p = 2^{1/p-1/2} + 2^{1/2}(1+K(P))$.*

If $H$ is a closed subspace of $\mathbf{L}^p$ and $P\colon H \to H$ an operator such that $(H, P)$ satisfies condition (C), we define for $f \in H$ the quantity

$$\|f\|_{\mathrm{MW}(p,P)} := \sum_{j=0}^{+\infty} 2^{-j/2}\left\|\sum_{i=0}^{2^j-1} P^i f\right\|_p \tag{2.10}$$

and the vector space

$$\mathrm{MW}(p, P) := \{f \in H \mid \|f\|_{\mathrm{MW}(p,P)} < +\infty\}. \tag{2.11}$$

Note that $\mathrm{MW}(p, P)$ endowed with $\|\cdot\|_{\mathrm{MW}(p,P)}$ is a Banach space.

Combining Proposition 2.3 and (2.6), we derive the following bound for the Hölderian norm of the partial sum process.

**Corollary 2.4.** *Let $H$ be a closed subspace of $\mathbf{L}^p$ and let $P$ be an operator from $H$ to itself such that $(H, P)$ satisfies the condition (C). Then there exists a constant $C = C(p, P)$ such that for each $n$, and each $h \in H$,*

$$\left\|\left\|\frac{1}{\sqrt{n}} W(n, h)\right\|_{\mathcal{H}_{1/2-1/p}}\right\|_{p,\infty} \leq C\|h\|_{\mathrm{MW}(p,P)}. \tag{2.12}$$

The proof of Proposition 2.3 is in the same spirit as the proof of Theorem 1 of Peligrad, Utev and Wu (2007), which is done by dyadic induction. To do so, we start from the following lemma.

**Lemma 2.5.** *For each positive integer $n$, each function $h\colon \Omega \to \mathbf{R}$ and each measure-preserving map $T\colon \Omega \to \Omega$, the following inequality holds:*

$$M(n, h, T) \leq 6 \max_{0\leq k\leq n}\left|h \circ T^k\right| + \frac{1}{2^{1/2-1/p}} M\left(\left[\frac{n}{2}\right], h + h\circ T, T^2\right). \tag{2.13}$$

**Proof.** First, notice that if $1 \leq j \leq n$, then $j = 2[\frac{j}{2}]$ or $j = 2[\frac{j}{2}] + 1$, hence

$$\left|S_j(h) - S_{2[\frac{j}{2}]}(h)\right| \leq \max_{0\leq k\leq n}\left|h \circ T^k\right|. \tag{2.14}$$

Similarly, we have

$$\left|S_i(h) - S_{2[\frac{i+2}{2}]}(h)\right| \le 2 \max_{0\le k\le n} \left|h \circ T^k\right|. \tag{2.15}$$

It thus follows that

$$M(n,h,T) \le 4 \max_{0\le k\le n} \left|h \circ T^k\right| + \max_{0\le i<j\le n} \frac{|S_{2[\frac{j}{2}]}(h) - S_{2[\frac{i+2}{2}]}(h)|}{(j-i)^{1/2-1/p}}. \tag{2.16}$$

Notice that if $j \ge i+4$, then

$$1 \le \left[\frac{j}{2}\right] - \left[\frac{i+2}{2}\right] \le \frac{j-i}{2}, \tag{2.17}$$

and we derive the bound

$$\begin{aligned}
&\max_{0\le i<j\le n} \frac{|S_{2[\frac{j}{2}]}(h) - S_{2[\frac{i+2}{2}]}(h)|}{(j-i)^{1/2-1/p}} \\
&\quad\le \frac{1}{2^{1/2-1/p}} \max_{0\le u<v\le [\frac{n}{2}]} \frac{|S_v(T^2, h+h\circ T) - S_u(T^2, h+h\circ T)|}{(v-u)^{1/2-1/p}} \\
&\qquad + \max_{\substack{0\le i<j\le n \\ j\le i+4}} \left|S_{2[\frac{j}{2}]}(h) - S_{2[\frac{i+2}{2}]}(h)\right|.
\end{aligned}$$

Since for $j \le i+4$, the number of terms of the form $h \circ T^q$ involved in $S_{2[\frac{j}{2}]}(h) - S_{2[\frac{i+2}{2}]}(h)$ is at most 2, we conclude that

$$\begin{aligned}
\max_{0\le i<j\le n} \frac{|S_{2[\frac{j}{2}]}(h) - S_{2[\frac{i+2}{2}]}(h)|}{(j-i)^{1/2-1/p}} &\le \frac{1}{2^{1/2-1/p}} M\left(\left[\frac{n}{2}\right], h+h\circ T, T^2\right) \\
&\quad + 2 \max_{0\le k\le n} \left|h \circ T^k\right|.
\end{aligned}$$

Combining this inequality with (2.16), we obtain (2.13), which concludes the proof of Lemma 2.5. □

Now, we establish inequality (2.9) by induction on $r$.

**Proof of Proposition 2.3.** We first assume that $PU^{-1} = \mathrm{Id}$ and $U^{-1}H \subset H$. We check the case $r=1$. Then necessarily $n=1$ and the expression $M(n,f,t)$ reduces to $f$. Since $C_p$ and $K_p$ are greater than 1, the result is a simple consequence of the triangle inequality applied to $f - U^{-1}Pf$ and $U^{-1}Pf$.

Now, assume that Proposition 2.3 holds for some $r$ and let us show that it takes place for $r+1$. We thus consider an integer $n$ such that $2^r \le n < 2^{r+1}$, a function $f \in H$, a measure-preserving map $T\colon \Omega \to \Omega$ bijective and bi-measurable, and a sub-$\sigma$-algebra $\mathcal{M}$ satisfying $T\mathcal{M} \subset \mathcal{M}$, a closed subspace $H$ of $\mathbf{L}^2$ such that $U^{-1}H \subset H$ and an operator $P\colon H \to H$ such that $(H,P)$ satisfies condition (C) with $PU^{-1} = \mathrm{Id}$ and we have to show that (2.9) holds with $r+1$ instead of $r$. First, using inequality $M(n, f) \leq M(n, f - U^{-1}Pf) + M(n, U^{-1}Pf)$ and Lemma 2.5 with $h := U^{-1}Pf$, we derive

$$\begin{aligned} M(n, f, T) \leq{}& M\big(n, f - U^{-1}Pf, T\big) + 6 \max_{0\leq k\leq n} \big|U^{-1}Pf \circ T^k\big| \\ &+ \frac{1}{2^{1/2-1/p}} M\left(\left[\frac{n}{2}\right], (I+U)U^{-1}Pf, T^2\right), \end{aligned} \tag{2.18}$$

hence taking the norm $\|\cdot\|_{p,\infty}$, we obtain by (2.4) that

$$\begin{aligned} \|M(n, f, T)\|_{p,\infty} \leq{}& \big\|M\big(n, f - U^{-1}Pf, T\big)\big\|_{p,\infty} \\ &+ 6(n+1)^{1/p}\frac{p}{p-1}\big\|U^{-1}Pf\big\|_p \\ &+ \frac{1}{2^{1/2-1/p}} \left\|M\left(\left[\frac{n}{2}\right], (I+U)U^{-1}Pf, T^2\right)\right\|_{p,\infty}. \end{aligned} \tag{2.19}$$

By inequality (2.7) and accounting the fact that $6\cdot(n+1)^{1/p}p/(p-1) \leq C_p n^{1/p}$, we obtain

$$\begin{aligned} \|M(n, f, T)\|_{p,\infty} \leq{}& C_p n^{1/p}\big\|f - U^{-1}Pf\big\|_p + C_p n^{1/p}\|Pf\|_p \\ &+ \frac{1}{2^{1/2-1/p}} \left\|M\left(\left[\frac{n}{2}\right], (I+U)U^{-1}Pf, T^2\right)\right\|_{p,\infty}. \end{aligned} \tag{2.20}$$

Since $2^{r-1} \leq [n/2] < 2^r$, we may apply the induction hypothesis to the integer $[n/2]$, the function $h := (I+U^{-1})Pf$, $T^2$ instead of $T$ and $P^2$ instead of $P$. This gives

$$\begin{aligned} &\left[\frac{n}{2}\right]^{-1/p} \left\|M\left(\left[\frac{n}{2}\right], h, T^2\right)\right\|_{p,\infty} \\ &\quad\leq C_p\big(1 + K\big(P^2\big)\big)\|h\|_p \end{aligned} \tag{2.21}$$

$$\qquad + C_p\widetilde{K_p}\sum_{j=0}^{r-1} 2^{-j/2}\left\|\sum_{i=0}^{2^j-1} P^{2i}\big(I+U^{-1}\big)Pf\right\|_p, \tag{2.22}$$

where $\widetilde{K_p} = 2^{1/p-1/2} + 2^{1/2}(1 + K(P^2))$. Notice that $\|h\|_p \leq 2\|Pf\|_p$, and by item 4 of Definition 2.2, it follows that

$$\sum_{i=0}^{2^j-1} P^{2i}\big(I+U^{-1}\big)Pf = \sum_{i=0}^{2^j-1}\big(P^{2i+1}f + P^{2i}f\big) = \sum_{i=0}^{2^{j+1}-1} P^i f. \tag{2.23}$$

Accounting the inequality $K(P^2) \le K(P)$ and $\widetilde{K_p} \le K_p$, we have

$$\left[\frac{n}{2}\right]^{-1/p} \left\| M\left(\left[\frac{n}{2}\right], h, T^2\right) \right\|_{p,\infty}$$
$$\le 2(1+K(P))C_p\|Pf\|_p + C_pK_p \sum_{j=0}^{r-1} 2^{-j/2} \left\| \sum_{i=0}^{2^{j+1}-1} P^i f \right\|_p$$
$$= 2(1+K(P))C_p\|Pf\|_p + 2^{1/2}C_pK_p \sum_{j=1}^{r} 2^{-j/2} \left\| \sum_{i=0}^{2^{j}-1} P^i f \right\|_p$$

and we infer

$$\left\| M\left(\left[\frac{n}{2}\right], h, T^2\right) \right\|_{p,\infty} \le \left(\frac{n}{2}\right)^{1/p} (2(1+K(P)) - K_p\sqrt{2}))C_p\|Pf\|_p$$
$$+ n^{1/p}2^{1/2-1/p}C_pK_p \sum_{j=0}^{r} 2^{-j/2} \left\| \sum_{i=0}^{2^{j}-1} P^i f \right\|_p . \tag{2.24}$$

Plugging this into (2.20), we derive

$$\|M(n, f, T)\|_{p,\infty}$$
$$\le C_pn^{1/p}(1+K(P))\|f\|_p + n^{1/p}C_pK_p \sum_{j=0}^{r} 2^{-j/2} \left\| \sum_{i=0}^{2^{j}-1} P^i f \right\|_p \tag{2.25}$$
$$+ C_pn^{1/p}(1 + 2^{1-1/p}(1+K(P)) - 2^{1/2-1/p}K_p)\|Pf\|_p.$$

The definition of $K_p$ implies that $2^{1/p-1/2} - \sqrt{2}(1+K(P)) - K_p = 0$, hence (2.9) is established. This concludes the proof of Proposition 2.3 in the case $PU^{-1} = \mathrm{Id}$.

When $PU = \mathrm{Id}$ and $UH \subset H$ we do the same proof, but replacing each occurrence of $U^{-1}$ by $U$. This ends the proof of Proposition 2.3. □

## 2.2 Proof of Theorem 1.1

Since the convergence of the finite dimensional distributions is contained in the main result of Volný (2007), the only difficulty in proving Theorem 1.1 is to establish tightness. To this aim, we shall proceed as in the proof of Theorem 5.3 in Cuny (2014).

**Proposition 2.6.** *Let $T$ be a measure preserving map, $H$ a closed subspace of $\mathbf{L}^p$ ($p > 2$) and let $P$ be an operator from $H$ to itself such that $(H, P)$ satisfies condition (C). Assume that h is an element of $H$ such that $\|h\|_{\mathrm{MW}(p,P)} < +\infty$.*

*Then the sequence $(n^{-1/2}W(n, h))_{n\ge1}$ is tight in $\mathcal{H}_{1/2-1/p}$.*

**Proof.** Let us define $V_n := \sum_{i=0}^{n-1} P^i$. Using $\|V_n V_k\|_p \leq K(P)\min\{k\|V_n\|_p, n\|V_k\|_p\}$, we derive that for each $f \in \mathrm{MW}(p, P)$,

$$\frac{\|V_{2^n} f\|_{\mathrm{MW}(p,P)}}{2^n} \leq K(P)\left(\frac{\|V_{2^n} f\|_p}{2^{n/2}} + \sum_{k\geq n+1} \frac{\|V^{2^k} f\|_p}{2^{k/2}}\right) \tag{2.26}$$

which goes to 0 as $n$ goes to infinity. If $m \geq 1$ is an integer and if $n$ is such that $2^n \leq m < 2^{n+1}$, then

$$\frac{\|V_m f\|_{\mathrm{MW}(p,P)}}{m} \leq \frac{K(P)}{m}\sum_{k=0}^{n}\|V_{2^k} f\|_{\mathrm{MW}(p,P)} \leq \frac{K(P)}{m}\sum_{k=0}^{n} 2^k \varepsilon_k, \tag{2.27}$$

where $(\varepsilon_k)_{k\geq 1}$ is a sequence converging to 0. This entails that the operator $P$ is mean-ergodic on $\mathrm{MW}(p, P)$. Furthermore, since $P$ has no non trivial fixed points on the Banach space $(\mathrm{MW}(p, P), \|\cdot\|_{\mathrm{MW}(p,P)})$, we derive by Theorem 1.3, p. 73 of Krengel (1985) that the subspace $(I - P)\mathrm{MW}(p, P)$ is dense in $\mathrm{MW}(p, P)$ for the topology induced by the norm $\|\cdot\|_{\mathrm{MW}(p,P)}$.

Let $h \in H$ be such that $\|h\|_{\mathrm{MW}(p,P)} < +\infty$ and $x > 0$. We can find $f \in (I - P)\mathrm{MW}(p, P)$ such that $\|h - f\|_{\mathrm{MW}(p,P)} < x$. Consequently, using Corollary 2.4, we derive that for each positive $\varepsilon$ and $\delta$,

$$\begin{aligned} &\mu\left\{\omega_{1/2-1/p}\left(\frac{1}{\sqrt{n}}W(n,h),\delta\right) > 2\varepsilon\right\} \\ &\quad\leq \varepsilon^{-p}x + \mu\left\{\omega_{1/2-1/p}\left(\frac{1}{\sqrt{n}}W(n,f),\delta\right) > \varepsilon\right\}. \end{aligned} \tag{2.28}$$

Now, since the function $f$ belongs to $(I - P)\mathrm{MW}(p, P)$, we can find $f' \in \mathrm{MW}(p, P)$ such that $f = f' - Pf'$. If $PU^{-1} = \mathrm{Id}$, then we write $f = f' - U^{-1}Pf' + (U^{-1} - I)f'$ and if $PU = \mathrm{Id}$, then $f = f' - UPf' + (U - I)f'$. In other words, $f$ admits a martingale-coboundary decomposition in $\mathbf{L}^p$ (since $f'$ belongs to $\mathbf{L}^p$). Consequently, by Corollary 2.5 of Giraudo (2016b), the sequence $(n^{-1/2}W(n, f))_{n\geq 1}$ is tight in $\mathcal{H}_{1/2-1/p}$. By Proposition 2.1 and (2.28), we derive that for each positive $\varepsilon$ and $x$,

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\mu\left\{\omega_{1/2-1/p}\left(\frac{1}{\sqrt{n}}W(n,h),\delta\right) > 2\varepsilon\right\} \leq \varepsilon^{-p}x. \tag{2.29}$$

Since $x$ is arbitrary we conclude the proof of (2.6) by using again Proposition 2.1. □

**Proof of Theorem 1.1.** Writing $f = \mathbf{E}[f|\mathcal{M}] + f - \mathbf{E}[f|\mathcal{M}]$, the proof reduces (as mentioned in the beginning of the section) to establish tightness in $\mathcal{H}^o_{1/2-1/p}[0,1]$ of the sequences $(W_n)_{n\geq 1} := (n^{-1/2}W(n, \mathbf{E}[f|\mathcal{M}]))_{n\geq 1}$ and $(W'_n)_{n\geq 1} := (n^{-1/2}W(n, f - \mathbf{E}[f|\mathcal{M}]))_{n\geq 1}$.

- Tightness of $(W_n)_{n\geq 1}$. We define

$$P(f) := \mathbf{E}[Uf|\mathcal{M}] \quad \text{and} \tag{2.30}$$

$$H := \{f \in \mathbf{L}^p,\ f \text{ is } \mathcal{M}\text{-measurable}\}. \tag{2.31}$$

Then $(H, P)$ satisfies condition (C). Since

$$\sum_{i=0}^{n-1} P^i(\mathbf{E}[f|\mathcal{M}]) = \mathbf{E}[S_n(f)|\mathcal{M}], \tag{2.32}$$

the convergence of the first series in (1.4) is equivalent to $f \in \mathrm{MW}(p, P)$ (by Lemma 2.7 of Peligrad and Utev (2005)). By Proposition 2.6, we derive that the sequence $(W_n)_{n\geq 1}$ is tight in $\mathcal{H}^o_{1/2-1/p}[0, 1]$.

- Tightness of $(W'_n)_{n\geq 1}$. We define

$$P(f) := U^{-1}f - \mathbf{E}[U^{-1}f|\mathcal{M}] \quad \text{and} \tag{2.33}$$

$$H := \{f \in \mathbf{L}^p, \mathbf{E}[f|\mathcal{M}] = 0\}. \tag{2.34}$$

Since for each $f \in H$ and each $k \geq 1$, $\|P^k f\|_p \leq 2\|f\|_p$, $(H, P)$ satisfies condition (C) (see the proof of Proposition 2 in Volný (2007) for the other conditions). Since $P(\mathbf{E}[f|\mathcal{M}]) = 0$, we have

$$\sum_{i=1}^{n} P^i(f - \mathbf{E}[f|\mathcal{M}]) = \sum_{i=1}^{n} P^i f \tag{2.35}$$

$$= U^{-n}\big(S_n(f) - \mathbf{E}[S_n(f)|T^{-n}\mathcal{M}]\big), \tag{2.36}$$

hence the convergence of the second series in (1.4) implies that $f$ belongs to $\mathrm{MW}(p, P)$ (by Lemma 37 of Merlevède and Peligrad (2013)). By Proposition 2.6, we derive that the sequence $(W'_n)_{n\geq 1}$ is tight in $\mathcal{H}^o_{1/2-1/p}[0, 1]$.

This ends the proof of Theorem 1.1. □

### 2.3 Proof of Theorem 1.5

We take a similar construction as in the proof of Proposition 1 of Peligrad, Utev and Wu (2007). We consider a nonnegative sequence $(a_n)_{n\geq 1}$, and a sequence $(u_k)_{k\geq 1}$ of real numbers such that

$$\begin{aligned} &u_1 = 1, \qquad u_2 = 2, \qquad u_k^{p/2+1} + 1 < u_{k+1} \qquad \text{for } k \geq 3 \quad \text{and} \\ &a_t \leq k^{-2} \qquad \text{for } t \geq u_k. \end{aligned} \tag{2.37}$$

Notice that since $p > 2$, the conditions (2.37) are more restrictive than that of the proof of Proposition 1 of Peligrad, Utev and Wu (2007). If $i = u_j$ for some $j \geq 1$, then we define $p_i := cj/u_j^{1+p/2}$ and $p_i = 0$ otherwise. Let $(Y_k)_{k\geq 0}$ be a discrete time Markov chain with the state space $\mathbf{Z}^+$ and transition matrix given by $p_{k,k-1} =$

1 for $k \geq 1$ and $p_{0,j-1} := p_j$, $j \geq 1$. We shall also consider a random variable $\tau$ which takes its values among nonnegative integers, and whose distribution is given by $\boldsymbol{\mu}(\tau = j) = p_j$. Then the stationary distribution exists and is given by

$$\pi_j = \pi_0 \sum_{i=j+1}^{\infty} p_i, \qquad j \geq 1, \text{ where } \pi_0 = 1/\mathbf{E}[\tau]. \tag{2.38}$$

We start from the stationary distribution $(\pi_j)_{j\geq 0}$ and we take $g(x) := \mathbf{1}_{x=0} - \pi_0$, where $\pi_0 = \mu\{Y_0 = 0\}$. We then define $f \circ T^j = X_j := g(Y_j)$.

It is already checked in Peligrad, Utev and Wu (2007) that the sequence $(X_j)_{j\geq 0}$ satisfies (1.11), where $\mathcal{M} = \sigma(X_k, k \leq j)$ and $S_n = \sum_{j=1}^{n} X_j$. To conclude the proof, it remains to check that the sequence $(n^{-1/2} W(n, f, T))_{n\geq 1}$ is not tight in $\mathcal{H}^o_{1/2-1/p}$, which will be done by disproving (2.1) for a particular choice of $\varepsilon$. To this aim, we define

$$T_0 = 0, \qquad T_k = \min\{t > T_{k-1} \mid Y_t = 0\}, \qquad \tau_k = T_k - T_{k-1}, \qquad k \geq 1. \tag{2.39}$$

Then $(\tau_k)_{k\geq 1}$ is an independent sequence and each $\tau_k$ is distributed as $\tau$ and

$$S_{T_k} = \sum_{j=1}^{k} (1 - \pi_0 \tau_j) = k - \pi_0 T_k. \tag{2.40}$$

Let us fix some integer $K$ greater than $\mathbf{E}[\tau]$. Let $\delta \in (0, 1)$ be fixed and $n$ an integer such that $1/n < \delta$. Then the inequality

$$\begin{aligned} &\frac{1}{(nK)^{1/p}} \max_{\substack{0\leq i<j\leq nK \\ j-i\leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} \\ &\quad\geq \frac{1}{(nK)^{1/p}} \mathbf{1}\{T_n \leq Kn\} \\ &\qquad \times \max_{1\leq k\leq n} \frac{|S_{T_k} - S_{T_{k-1}}|}{(T_k - T_{k-1})^{1/2-1/p}} \mathbf{1}\{|T_k - T_{k-1}| \leq n\delta\} \end{aligned} \tag{2.41}$$

takes place. By (2.39) and (2.40), this can be rewritten as

$$\begin{aligned} &\frac{1}{(nK)^{1/p}} \max_{\substack{0\leq i<j\leq nK \\ j-i\leq n\delta}} \frac{|S_j - S_i|}{(j-i)^{1/2-1/p}} \\ &\quad\geq \frac{1}{(nK)^{1/p}} \mathbf{1}\{T_n \leq Kn\} \max_{1\leq k\leq n} \frac{|1 - \pi_0 \tau_k|}{\tau_k^{1/2-1/p}} \mathbf{1}\{\tau_k \leq n\delta\}. \end{aligned} \tag{2.42}$$

Defining for a fixed $C$ the event

$$A_n(C) := \left\{ \frac{|1 - \pi_0 \tau|}{\tau^{1/2-1/p}} \geq C (Kn)^{1/p} \right\} \cap \{\tau \leq n\delta\}, \tag{2.43}$$

we obtain by independence of $(\tau_k)_{k\geq 1}$

$$\mu\left\{\frac{1}{(nK)^{1/p}}\max_{\substack{0\leq i<j\leq nK\\ j-i\leq n\delta}}\frac{|S_j-S_i|}{(j-i)^{1/2-1/p}}\geq C\right\}$$
$$\geq 1-\big(1-\mu\big(A_n(C)\big)\big)^n-\mu\{T_n>Kn\}. \tag{2.44}$$

By the law of large numbers, we obtain, accounting $K>\mathbf{E}[\tau]$, that

$$\limsup_{n\to\infty}\mu\left\{\frac{1}{(nK)^{1/p}}\max_{\substack{0\leq i<j\leq nK\\ j-i\leq n\delta}}\frac{|S_j-S_i|}{(j-i)^{1/2-1/p}}\geq C\right\}$$
$$\geq \limsup_{n\to\infty}1-\big(1-\mu\big(A_n(C)\big)\big)^n. \tag{2.45}$$

We choose $C:=\pi_0/(2K^{1/p})$. Considering the integers $n$ of the form $[u_j^{(p+2)/2}]$, we obtain in view of (2.45):

$$\limsup_{n\to\infty}\mu\left\{\frac{1}{(nK)^{1/p}}\max_{\substack{0\leq i<j\leq nK\\ j-i\leq n\delta}}\frac{|S_j-S_i|}{(j-i)^{1/2-1/p}}\geq \frac{\pi_0}{2K^{1/p}}\right\}$$
$$\geq \limsup_{j\to\infty}1-\left(1-\mu\left(A_{[u_j^{(p+2)/2}]}\left(\frac{\pi_0}{2K^{1/p}}\right)\right)\right)^{[u_j^{(p+2)/2}]}. \tag{2.46}$$

Since $\tau\geq 1$ almost surely, the following inclusions take place for $n>(2/\pi_0)^p$:

$$\begin{aligned}
A_n\big(\pi_0/\big(2K^{1/p}\big)\big) &\supset \big\{\pi_0\tau^{1/2+1/p}-\tau^{-1/2+1/p}\geq \pi_0/\big(2K^{1/p}\big)(Kn)^{1/p}\big\}\cap\{\tau\leq n\delta\}\\
&\supset\left\{\tau^{1/2+1/p}\geq\frac{1+\pi_0 n^{1/p}/2}{\pi_0}\right\}\cap\{\tau\leq n\delta\}\\
&\supset\big\{\tau^{1/2+1/p}\geq n^{1/p}\big\}\cap\{\tau\leq n\delta\}\\
&=\big\{n^{2/(p+2)}\leq\tau\leq n\delta\big\}.
\end{aligned}$$

Consequently, for $j$ large enough,

$$\mu\left(A_{[u_j^{(p+2)/2}]}\left(\frac{\pi_0}{2K^{1/p}}\right)\right)\geq\mu\big\{[u_j^{(p+2)/2}]^{2/(p+2)}\leq\tau\leq[u_j^{(p+2)/2}]\delta\big\}. \tag{2.47}$$

Since $\tau$ takes only integer values among $u_l$'s and $[u_j^{(p+2)/2}]\delta<u_{j+1}$ (by (2.37) and the fact that $\delta<1$), we obtain in view of (2.46), that

$$\limsup_{n\to\infty}\mu\left\{\frac{1}{(nK)^{1/p}}\max_{\substack{0\leq i<j\leq nK\\ j-i\leq n\delta}}\frac{|S_j-S_i|}{(j-i)^{1/2-1/p}}\geq \frac{\pi_0}{2K^{1/p}}\right\}$$
$$\geq\limsup_{j\to\infty}1-\big(1-\mu\{\tau=u_j\}\big)^{[u_j^{(p+2)/2}]} \tag{2.48}$$
$$=1-\liminf_{j\to\infty}\big(1-cju_j^{-1-p/2}\big)^{[u_j^{(p+2)/2}]}.$$

Noticing that for a fixed $J$,

$$\liminf_{j\to\infty}\left(1-cju_j^{-1-p/2}\right)^{[u_j^{(p+2)/2}]} \leq \limsup_{j\to\infty}\left(1-cJu_j^{-1-p/2}\right)^{[u_j^{(p+2)/2}]} = e^{-cJ}, \tag{2.49}$$

we deduce that the last term of (2.48) is equal to 1. Since

$$\frac{1}{(nK)^{1/p}} \max_{\substack{0\leq i<j\leq nK\\ j-i\leq n\delta}} \frac{|S_j-S_i|}{(j-i)^{1/2-1/p}} \leq \omega_{1/2-1/p}\left(\frac{1}{\sqrt{nK}}W(nK,f),\delta\right), \tag{2.50}$$

we derive that (2.1) does not hold with $\varepsilon=\pi_0/(2K^{1/p})$. This finishes the proof of Theorem 1.5.

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## References

Cuny, C. (2014). Limit theorems under the Maxwell–Woodroofe condition in Banach spaces. *Ann. Probab.* To appear.

Cuny, C. and Merlevède, F. (2014). On martingale approximations and the quenched weak invariance principle. *Ann. Probab.* **42**, 760–793. MR3178473

Durieu, O. (2009). Independence of four projective criteria for the weak invariance principle, *ALEA* Lat. *Am. J. Probab. Math. Stat.* **5**, 21–26. MR2475604

Durieu, O. and Volný, D. (2008). Comparison between criteria leading to the weak invariance principle. *Ann. Inst. Henri Poincaré Probab. Stat.* **44**, 324–340. MR2446326

Giraudo, D. (2016a). Holderian weak invariance principle under a Hannan type condition. *Stochastic Process. Appl.* **126**, 290–311. MR3426520

Giraudo, D. (2016b). Holderian weak invariance principle for stationary mixing sequences. *J. Theoret. Probab.* To appear. Available at https://hal.archives-ouvertes.fr/hal-01075583.

Grafakos, L. (2014). *Classical Fourier Analysis*, 3rd ed. *Graduate Texts in Mathematics* **249**. New York: Springer. MR3243734

Krengel, U. (1985). *Ergodic Theorems*. *de Gruyter Studies in Mathematics* **6**. Berlin: Walter de Gruyter & Co. MR0797411

Markevičiūtė, J., Suquet, C. and Račkauskas, A. (2012). Functional central limit theorems for sums of nearly nonstationary processes. *Lith. Math. J.* **52**, 282–296. MR3020943

Maxwell, M. and Woodroofe, M. (2000). Central limit theorems for additive functionals of Markov chains. *Ann. Probab.* **28**, 713–724. MR1782272

Merlevède, F. and Peligrad, M. (2013). Rosenthal-type inequalities for the maximum of partial sums of stationary processes and examples. *Ann. Probab.* **41**, 914–960. MR3077530

Peligrad, M. and Utev, S. (2005). A new maximal inequality and invariance principle for stationary sequences. *Ann. Probab.* **33**, 798–815. MR2123210

Peligrad, M., Utev, S. and Wu, W. B. (2007). A maximal $\mathbf{L}_p$-inequality for stationary sequences and its applications. *Proc. Amer. Math. Soc.* **135**, 541–550. MR2255301

Račkauskas, A. and Suquet, C. (2003). Necessary and sufficient condition for the Lamperti invariance principle. *Teor. Ĭmovīr. Mat. Stat.* **68**, 115–124. MR2000642

Serfling, R. J. (1970). Moment inequalities for the maximum cumulative sum. *Ann. Math. Stat.* **41**, 1227–1234. MR0268938

Suquet, Ch. (1999). Tightness in Schauder decomposable Banach spaces. In *Proceedings of the St. Petersburg Mathematical Society, Vol. V*. *Amer. Math. Soc. Transl. Ser. 2* **193**, 201–224. Providence, RI: Amer. Math. Soc. MR1736910

Volný, D. (2006). Martingale approximation of non adapted stochastic processes with nonlinear growth of variance, dependence in probability and statistics. In *Lecture Notes in Statist., Vol. 187*, 141–156. New York: Springer. MR2283254

Volný, D. (2007). A nonadapted version of the invariance principle of Peligrad and Utev. *C. R. Math. Acad. Sci. Paris* **345**, 167–169. MR2344817

Volný, D. (2010). Martingale approximation and optimality of some conditions for the central limit theorem. *J. Theoret. Probab.* **23**, 888–903. MR2679961

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