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TESTING FOR TIME-VARYING JUMP ACTIVITY FOR PURE JUMP SEMIMARTINGALES[1]

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In this paper, we propose a test for deciding whether the jump activity index of a discretely observed Itô semimartingale of pure-jump type (i.e., one without a diffusion) varies over a fixed interval of time. The asymptotic setting is based on observations within a fixed time interval with mesh of the observation grid shrinking to zero. The test is derived for semimartingales whose "spot" jump compensator around zero is like that of a stable process, but importantly the stability index can vary over the time interval. The test is based on forming a sequence of local estimators of the jump activity over blocks of shrinking time span and contrasting their variability around a global activity estimator based on the whole data set. The local and global jump activity estimates are constructed from the real part of the empirical characteristic function of the increments of the process scaled by local power variations. We derive the asymptotic distribution of the test statistic under the null hypothesis of constant jump activity and show that the test has asymptotic power of one against fixed alternatives of processes with time-varying jump activity.

1. Introduction. In this paper, we derive a test for deciding whether the jump activity index of a pure-jump Itô semimartingale X remains constant over a fixed time interval. The jump activity of the process X is measured by the instantaneous Blumenthal–Getoor (BG) index defined as

$$\inf\left\{r > 0 : \int_{\mathbb{R}} \big(|x|^r \wedge 1\big)\nu_t^X(dx) < \infty\right\}, \tag{1.1}$$

where the jump compensator of X is given by $dt \otimes \nu_t^X(dx)$ for some predictable random measure $\nu_t^X(dx)$ on $\mathbb{R}$. The BG index takes values in the interval $[0, 2]$, and for general semimartingales it can be random and depend on time. However, in the important case when X is a Lévy process, the instantaneous BG index is a nonrandom constant. Moreover, the BG index of X continues to be a nonrandom constant in much more general settings. Indeed this is the case when the jump component of X is given by a stochastic integral with respect to a Lévy process or when it is a time-changed Lévy process with the time change being an absolutely continuous process.

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All prior work on estimating the BG index from high-frequency data [Aït-Sahalia and Jacod (2009), Bull (2016), Jing, Kong and Liu (2011), Jing et al. (2012), Kong, Liu and Jing (2015), Todorov (2015), Todorov and Tauchen (2011), Woerner (2003, 2007)] has been done in the setting where the instantaneous BG index is a nonrandom constant with the additional restriction that the driving Lévy process is locally stable, that is, it is a Lévy process whose Lévy measure around zero behaves like that of a stable process (the local stable assumption can be viewed as the counterpart of the assumption of regular variation of the tails used typically in extreme value theory). The goal of this paper is to design a test for deciding whether the instantaneous BG index is constant or not on a given interval from discrete observations of the process with mesh of the observation grid shrinking to zero.

In particular, we consider the class of pure-jump Itô semimartingales:

$$dX_t = \alpha_t\,dt + \sigma_{t-}\,dS_t + dY_t, \tag{1.2}$$

where S is a pure-jump process with characteristic triplet [Jacod and Shiryaev (2003), Definition II.2.6] $(0, 0, dt \otimes \nu_t(dx))$ with respect to some truncation function (see definition in next section) and $\nu_t(dx)$ is given by

$$\nu_t(dx) = \frac{A}{|x|^{\beta_t+1}}\,dx, \tag{1.3}$$

for some $A > 0$ and β being a stochastic process with càglàd paths, and with β_t taking values in the interval $(1, 2)$ for every $t \geq 0$. Y in (1.2) is a "residual" jump process, in the sense that its activity is below that of S on the observed time interval, and it can have dependence with the rest of the components of X. Finally, α and σ are processes with càdlàg paths. In the setting of (1.2)–(1.3), the instantaneous BG index of X at time t is given by β_t. Hence, our testing problem reduces to deciding whether the process β is constant or not on the observed time interval.

Designing a test for the process β being constant is complicated for at least two reasons. First, the jump activity determines the asymptotic order of magnitude of the increments at high-frequencies but it is unknown to the statistician. Hence, the test statistic should contain some form of self-scaling to ensure non-degenerate limit behavior (at least under the null hypothesis). Second, we want to perform the test while allowing for time-varying process σ of unknown type. The jump compensator (intensity) of X then has two sources of variation: one is given by the presence of the time-varying process σ and the other one is due to the time-variation of the BG index of X. Essentially, our test statistic should separate nonparametrically the two sources of variation in the jump compensator of X around zero.

The test we develop in this paper makes use of the self-normalized statistics proposed in Todorov (2015) for efficient estimation of the BG index when the latter is constant. These statistics are formed from the first differences of the high-frequency increments of X scaled by local power variations. The local power variations are formed from blocks of increments, of asymptotically shrinking time span, preceding the ones that are scaled. The scaling serves as self-normalization since it removes the effect of σ on the limit of sample averages formed from (known) transforms of the scaled increments.

Using the scaled increments, we form two types of estimators for the jump activity. The first is global, that is, it makes use of all the high-frequency data. The second jump activity estimator is local, that is, it is based on a block of scaled high-frequency increments with asymptotically shrinking time span. The blocks for the local power variations, used to scale the high-frequency increments, and the ones for constructing the local jump activity estimates from the scaled increments are allowed to be of different size and can overlap. Finally, both the global and the local jump activity estimates are based on the real part of the characteristic function of the scaled increments.

When the BG index of X remains constant on the fixed time interval, then both the local and global estimators are valid, with the former being obviously much noisier. When the BG index of X varies over the interval, then the local estimators recover the time-varying BG index while the global estimator converges to a random variable, taking values in $[0, 2]$, whose value depends on the trajectory of β. Given this different behavior of the local and global jump activity estimators, our test is based on the integrated squared difference between them. When the BG index of X varies, the latter integrated squared difference converges to a measure of dispersion of the BG index on the time interval. When the BG index remains constant on the interval, then the squared differences need to be centered by an estimate of their nonrandom asymptotic variance. In this case of constant BG index, the partial sums of the centered squared differences between the local and global estimates behave asymptotically like a discrete martingale. Therefore, when scaled up appropriately, the sum of the centered squared differences of the local and global BG index estimates converges to a normal variable. Thus, a feasible test for time-varying BG index can be based on this sum.

Overall, our test builds on: (1) self-normalization to separate the time-variation in the jump compensator due to σ and β and (2) time-aggregation, that is, we use statistics formed at different time scales, to form an estimate of variability of functionals of the process β on the observed interval.

The rest of the paper is organized as follows. In Section 2, we introduce our setting and state the necessary assumptions. In Section 3, we construct our statistics for local and global estimation of the jump activity. The asymptotic properties of these statistics are presented in Section 4. This section also contains a test based on the derived limit results for deciding whether the BG index of the observed process varies. Section 5 evaluates the performance of the test on simulated data. Section 6 contains the proofs.

2. Setting and assumptions. We start with stating the assumptions that we need for our results. The process X in (1.2) is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Below we denote with κ a symmetric truncation function, that is, $\kappa : \mathbb{R} \to \mathbb{R}$ is some symmetric C^3 function with compact support and $\kappa(x) = 0$ for x in a neighborhood of zero.

We begin with our assumption concerning the jump activity of the processes S and Y in (1.2).

ASSUMPTION A. (a) S in (1.2) is an Itô semimartingale with characteristic triplet $(0, 0, dt \otimes \nu_t(dx))$ with ν_t given by (1.3) for some positive constant A and some process β with càglàd paths and $\beta_t \in (1+\epsilon, 2)$ for $\forall t \geq 0$ and some $\epsilon > 0$. We further assume $\mathbb{E}|\beta_t - \beta_s|^2 \leq K|t-s|$ for $s, t \geq 0$ and a constant $K > 0$.

(b) $Y_t = \int_0^t \int_{\mathbb{R}} x\mu^Y(ds, dx)$ where μ^Y is an integer-valued measure on $\mathbb{R}_+ \times \mathbb{R}$ with jump compensator $dt \otimes \nu_t^Y(dx)$. ν_t^Y satisfies $\int_{\mathbb{R}}(|x|^{\beta_t'} \wedge 1)\nu_t^Y(dx) < \infty$ for every $t \geq 0$ and some nonnegative process β' with càglàd paths satisfying $\sup_{t\geq 0} \beta_t' < 1$ for $\forall t \geq 0$.

Part (a) of Assumption A allows for stochastic time-varying BG index. The condition on the time-variation in β imposed in part (a) of the assumption is satisfied for a wide range of processes. For example, it holds for the class of Itô semimartigales. It holds also for processes driven by fractional Brownian motion. The restriction $\beta_t > 1$ is nontrivial and is discussed later in Section 4.

We note that Assumption A allows Y and S to have dependence. Therefore, as shown in Todorov and Tauchen (2012), we can accommodate in our setup time-changed Lévy models with absolute continuous time-change process, which are widely used in applied work.

Finally, part (b) of the assumption restricts Y to be of finite variation, and for the convergence in probability results below we can further relax this restriction.

We next state our assumption for the dynamics of α and σ.

ASSUMPTION B. The processes α and σ are Itô semimartingales of the form

$$
\begin{aligned}
\alpha_t &= \alpha_0 + \int_0^t b_s^\alpha\, ds + \int_0^t \eta_s^\alpha\, dW_s + \int_0^t \widetilde{\eta}_s^\alpha\, d\widetilde{W}_s \\
&\quad + \int_0^t \int_E \kappa\bigl(\delta^\alpha(s,x)\bigr)\underline{\widetilde{\mu}}(ds,dx) + \int_E \kappa'\bigl(\delta^\alpha(s,x)\bigr)\underline{\mu}(ds,dx), \\
\sigma_t &= \sigma_0 + \int_0^t b_s^\sigma\, ds + \int_0^t \eta_s^\sigma\, dW_s + \int_0^t \widetilde{\eta}_s^\sigma\, d\widetilde{W}_s \\
&\quad + \int_0^t \int_E \kappa\bigl(\delta^\sigma(s,x)\bigr)\underline{\widetilde{\mu}}(ds,dx) + \int_E \kappa'\bigl(\delta^\sigma(s,x)\bigr)\underline{\mu}(ds,dx),
\end{aligned}
\tag{2.1}
$$

where $\kappa'(x) = x - \kappa(x)$, and:

(a) $|\sigma_t|^{-1}$ and $|\sigma_{t-}|^{-1}$ are strictly positive;

(b) W and $\widetilde{W}$ are two independent Brownian motions; $\underline{\mu}$ is Poisson measure on $\mathbb{R}_+ \times E$, having arbitrary dependence with the jump measure of S, with compensator $dt \otimes \lambda(dx)$ for some σ-finite measure λ on E;

(c) $\delta^\alpha(t,x)$ and $\delta^\sigma(t,x)$ are predictable, left-continuous with right limits in t with $|\delta^\alpha(t,x)| + |\delta^\sigma(t,x)| \le \gamma_k(x)$ for all $t \le T_k$, where $\gamma_k(x)$ is a deterministic function on $\mathbb{R}$ with $\int_{\mathbb{R}}(|\gamma_k(x)|^r \wedge 1)\lambda(dx) < \infty$ for some $0 \le r < 2$ and T_k is a sequence of stopping times increasing to $+\infty$;

(d) b^α, b^σ, η^α, η^σ, $\widetilde{\eta}^\alpha$ and $\widetilde{\eta}^\sigma$ are processes with càdlàg paths and further there exists a sequence of stopping times T_k increasing to infinity and a sequence of positive numbers Γ_k such that for $s, t < T_k$ we have $\mathbb{E}|\eta_t^\sigma - \eta_s^\sigma|^2 + \mathbb{E}|\widetilde{\eta}_t^\sigma - \widetilde{\eta}_s^\sigma|^2 \le \Gamma_k|t-s|^\varsigma$ for some $\varsigma > 0$.

Assumption B is very general and it is satisfied in the case when the pair (α, σ) follows a Lévy-driven SDE, which is the typical way of modeling dynamics in applications. We note that Assumption B significantly generalizes the analogous assumption in Todorov (2015) by allowing α and σ to contain diffusions and further leaving their jump activities essentially unrestricted.

3. The statistics.

We continue next with the construction of our statistics. The estimation in the paper is based on observations of X at the equidistant grid times $0, \frac{1}{n}, \dots, 1$ with $n \to \infty$, and we denote $\Delta_n = \frac{1}{n}$.

3.1. *Global estimates of jump activity.*

We start with constructing an estimator of the jump activity based on all high frequency observations on the unit interval. This estimator was introduced in Todorov (2015) and is based on the real part of the empirical characteristic function of the increments scaled by local block-based volatility estimates, which we define as

$$(3.1)\qquad V_i^n(p) = \frac{1}{k_n} \sum_{j=i-k_n-1}^{i-2} |\Delta_j^n X - \Delta_{j-1}^n X|^p, \qquad i = k_n+3, \dots, n,\ p > 0,$$

for some $1 < k_n < n$ and where $\Delta_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$. The empirical characteristic function of the scaled and differenced increments is given by

$$(3.2)\qquad \widehat{\mathcal{L}}^n(p,u) = \frac{1}{n-k_n-2} \sum_{i=k_n+3}^{n} \cos\left(u \frac{\Delta_i^n X - \Delta_{i-1}^n X}{(V_i^n(p))^{1/p}}\right), \qquad u \in \mathbb{R}_+.$$

Using two different fixed numbers $u, v \in \mathbb{R}_+$, the global estimate of the jump activity is constructed as

$$(3.3)\qquad \widehat{\beta}^n(p,u,v) = \frac{\log(-\log(\widetilde{\mathcal{L}}^n(p,u))) - \log(-\log(\widetilde{\mathcal{L}}^n(p,v)))}{\log(u/v)},$$

where $\widetilde{\mathcal{L}}^n(p,u) = (\widehat{\mathcal{L}}^n(p,u) \wedge \frac{n-1}{n}) \vee \frac{1}{n}$.

As we will see in the next section, $\widehat{\beta}^n(p,u,v)$ will be a valid estimator of the jump activity only when the latter is constant on the unit time interval.

3.2. *Block-based estimates of jump activity.* We next introduce local, block-based, estimates of the jump activity. They will estimate the jump activity locally (in time), even when it varies. These estimators will be based on blocks of size m_n of scaled and differenced increments of X, where $k_n < m_n < n$. They are built from the block analogues of $\widehat{\mathcal{L}}^n(p,u)$ given by

$$\widehat{\mathcal{L}}_j^n(p,u) = \frac{1}{m_n}\sum_{i\in I_j^n}\cos\left(u\frac{\Delta_i^n X - \Delta_{i-1}^n X}{(V_i^n(p))^{1/p}}\right), \qquad u \in \mathbb{R}_+,\ j = 1,\dots,b_n, \tag{3.4}$$

where $I_j^n = k_n + 1 + \{(j-1)(m_n+1)+2,\dots,j(m_n+1)\}$ and the number of blocks over which the local jump activity estimation is performed is $b_n = \lfloor\frac{n-k_n-2}{m_n+1}\rfloor$. The local estimator of the jump activity is then simply the counterpart of $\widehat{\beta}^n(p,u,v)$ on the local block:

$$\widehat{\beta}_j^n(p,u,v) = \frac{\log(-\log(\widetilde{\mathcal{L}}_j^n(p,u))) - \log(-\log(\widetilde{\mathcal{L}}_j^n(p,v)))}{\log(u/v)}, \qquad j = 1,\dots,b_n, \tag{3.5}$$

where $u, v \in \mathbb{R}_+$ with $u \neq v$, and further we use the shorthand $\widetilde{\mathcal{L}}_j^n(p,u) = (\widehat{\mathcal{L}}_j^n(p,u) \wedge \frac{m_n-1}{m_n}) \vee \frac{1}{m_n}$.

We note that in the construction of our statistics we use two types of blocks which play separate roles. First, we use local blocks in order to purge the effect of the time-varying σ_t on our statistics. Second, we use local blocks to account for the presence of time-varying jump activity. Our test for time-varying jump activity will be based on the statistical significance of the difference between the local estimates $\widehat{\beta}_j^n(p,u,v)$ and the global one $\widehat{\beta}^n(p,u,v)$.

3.3. *Estimates for feasible inference.* We conclude this section with introducing estimates for the asymptotic variance of our test statistic from the high frequency data. We denote $\widehat{\zeta}_i^n(u) = \cos(u\frac{\Delta_i^n X - \Delta_{i-1}^n X}{(V_i^n(p))^{1/p}}) - \widehat{\mathcal{L}}^n(p,u)$. We then set $\widehat{\Xi}^n(p,u,v) = \widehat{\Xi}_0^n(p,u,v) + 2\widehat{\Xi}_1^n(p,u,v)$, where

$$\widehat{\Xi}_j^n(p,u,v) = \frac{1}{n-k_n-2-j}\sum_{i=k_n+3+j}^{n}\widehat{\zeta}_i^n(u)\widehat{\zeta}_{i-j}^n(v), \qquad j = 0, 1. \tag{3.6}$$

Finally, we denote

$$\overline{\Xi}^n(p,u,v) = \frac{\widehat{\Xi}^n(p,u,u)}{(\widehat{\mathcal{L}}^n(p,u)\log(\widehat{\mathcal{L}}^n(p,u)))^2} + \frac{\widehat{\Xi}^n(p,v,v)}{(\widehat{\mathcal{L}}^n(p,v)\log(\widehat{\mathcal{L}}^n(p,v)))^2} - 2\frac{\widehat{\Xi}^n(p,u,v)}{\widehat{\mathcal{L}}^n(p,u)\log(\widehat{\mathcal{L}}^n(p,u))\widehat{\mathcal{L}}^n(p,v)\log(\widehat{\mathcal{L}}^n(p,v))}. \tag{3.7}$$

4. Limit behavior of the statistics and testing for time-varying jump activity.

4.1. *The results.* We now derive the limit behavior of our statistics and use the limit results to construct a test for time-varying jump activity. Henceforth, all limits are understood to be for $n \to \infty$. To define the asymptotic limits of our statistics we need some notation. For some $\beta \in (1,2)$, we let $S_1^{(\beta)}$, $S_2^{(\beta)}$ and $S_3^{(\beta)}$ be random variables corresponding to the values of three independent Lévy processes at time 1, each of which with characteristic triplet $(0,0,dt \otimes \frac{A}{|x|^{1+\beta}}\,dx)$ with respect to some symmetric truncation function κ. Then we denote $\mu_{p,\beta} = (\mathbb{E}|S_1^{(\beta)} - S_2^{(\beta)}|^p)^{\beta/p}$, which does not depend on κ, and we further use the shorthand notation $\mathbb{E}(e^{iu(S_1^{(\beta)}-S_2^{(\beta)})}) = e^{-A_\beta u^\beta}$ for any $u > 0$ with A_β being a known function of A and β. Finally, using the expression for the pth moment of a symmetric β-stable random variable; see, for example, (25.6) in Sato (1999), we have

$$C_{p,\beta} = \frac{A_\beta}{\mu_{p,\beta}} = \left[\frac{2^p\Gamma(\frac{1+p}{2})\Gamma(1-\frac{p}{\beta})}{\sqrt{\pi}\Gamma(1-\frac{p}{2})}\right]^{-\beta/p}, \tag{4.1}$$

which depends only on p and β but not on the scale parameter of the stable random variables $S_1^{(\beta)}$ and $S_2^{(\beta)}$. With this notation, we set

$$\mathcal{L}(p,u) = \int_0^1 e^{-C_{p,\beta_s}u^{\beta_s}}\,ds, \qquad u \in \mathbb{R}_+, \tag{4.2}$$

which will be the probability limits of $\widehat{\mathcal{L}}^n(p,u)$. The probability limit of $\widehat{\beta}^n(p,u,v)$ is then given by

$$\overline{\beta}(p,u,v) = \frac{\log(-\log(\mathcal{L}(p,u))) - \log(-\log(\mathcal{L}(p,v)))}{\log(u/v)}. \tag{4.3}$$

We note that $\overline{\beta}(p,u,v) \equiv \beta$ when the process β remains constant on the interval $[0,1]$. Also, it is not difficult to show that $\overline{\beta}(p,u,v) \in (0,2)$ for arbitrary continuous process β on the interval $[0,1]$ taking values in $(0,2)$.

Finally, the probability limits of $\widehat{\Xi}_0^n(p,u,v)$ and $\widehat{\Xi}_1^n(p,u,v)$ are given by

$$\begin{aligned}
\Xi_0(p,u,v) &= \frac{1}{2}\mathcal{L}(p,u+v) + \frac{1}{2}\mathcal{L}(p,u-v) - \mathcal{L}(p,u)\mathcal{L}(p,v), \\
\Xi_1(p,u,v) &= \frac{1}{2}\int_0^1 e^{-\frac{C_{p,\beta_s}}{2}(u^{\beta_s}+v^{\beta_s}+|u+v|^{\beta_s})}\,ds \\
&\quad + \frac{1}{2}\int_0^1 e^{-\frac{C_{p,\beta_s}}{2}(u^{\beta_s}+v^{\beta_s}+|u-v|^{\beta_s})}\,ds - \mathcal{L}(p,u)\mathcal{L}(p,v).
\end{aligned} \tag{4.4}$$

We then define $\overline{\Xi}(p,u,v)$ from $\mathcal{L}(p,u)$, $\Xi_0(p,u,v)$ and $\Xi_1(p,u,v)$ exactly as we defined above $\overline{\Xi}^n(p,u,v)$ from $\widehat{\mathcal{L}}^n(p,u)$, $\widehat{\Xi}_0^n(p,u,v)$ and $\widehat{\Xi}_1^n(p,u,v)$. We note that $\overline{\Xi}(p,u,v)$ is finite-valued and strictly positive as soon as $u \neq v$.

The next theorem shows the limit behavior of $\widehat{\beta}^n(p,u,v)$ and $\overline{\Xi}^n(p,u,v)$ as well as that of an integrated measure of divergence between the local and global jump activity estimates.

THEOREM 1. *For the process X, assume Assumptions A and B hold. Let $k_n \asymp n^{\varpi}$ and $m_n \asymp n^{\varrho}$ as $n \to \infty$, for some $0 < \varpi < \varrho < 1$, with $p \in (0, 1/2)$. Then we have*

$$\begin{aligned}
\widehat{\beta}^n(p,u,v) &\xrightarrow{\mathbb{P}} \overline{\beta}(p,u,v), \\
\overline{\Xi}^n(p,u,v) &\xrightarrow{\mathbb{P}} \overline{\Xi}(p,u,v),
\end{aligned} \tag{4.5}$$

$$\frac{1}{b_n}\sum_{j=1}^{b_n}\big(\widehat{\beta}_j^n(p,u,v) - \widehat{\beta}^n(p,u,v)\big)^2 \xrightarrow{\mathbb{P}} \int_0^1 \big(\beta_s - \overline{\beta}(p,u,v)\big)^2\,ds. \tag{4.6}$$

To derive a test for presence of time-varying jump activity, we need a higher order asymptotic result for the sum on the left-hand side of (4.6) when β remains constant. To derive such a result, we make use of the fact that $\sqrt{m_n}(\widehat{\beta}_j^n(p,u,v) - \widehat{\beta}^n(p,u,v))$ is approximately normal with mean zero and variance $\overline{\Xi}(p,u,v)/(\log(u/v))^2$ when β is constant on $[0,1]$. Moreover, $m_n(\widehat{\beta}_j^n(p,u,v) - \widehat{\beta}^n(p,u,v))^2$ become asymptotically uncorrelated across blocks in spite of the dependence generated from the self-normalization. Therefore, the statistic

$$\mathcal{T}^n(p,u,v) = \frac{1}{\sqrt{2b_n}}\sum_{j=1}^{b_n}\left(\frac{m_n(\widehat{\beta}_j^n(p,u,v) - \widehat{\beta}^n(p,u,v))^2}{\overline{\Xi}^n(p,u,v)/(\log(u/v))^2} - 1\right), \tag{4.7}$$

converges in distribution to a standard normal random variable when β is constant on $[0,1]$. The formal result is given in the following theorem in which $\xrightarrow{\mathcal{L}-(s)}$ denotes stable convergence in law [see, e.g., Definition VIII.5.28 in Jacod and Shiryaev (2003)].

THEOREM 2. *For the process X assume Assumptions A and B hold with the process β being constant on* $[0, 1]$. *Let $k_n \asymp n^{\varpi}$ and $m_n \asymp n^{\varrho}$ as $n \to \infty$, for some $\frac{1}{3} < \varpi < \frac{1}{2} < \varrho < 1$, and further assume the following holds true for $\beta = \beta_0$*:

$$(1-\varrho) \vee \frac{1-p}{2} < \varpi < \frac{1+\varrho}{4} \wedge \frac{4\varrho - 1}{3} \tag{4.8}$$

and

$$p < \frac{\beta}{4}, \qquad \sup_{t \in [0,1]} \beta_t' < \frac{\beta}{2}, \qquad \frac{p}{\sup_{t \in [0,1]} \beta_t'} - \frac{p}{\beta} > \frac{1-\varpi}{2}. \tag{4.9}$$

Then, *denoting with Z a standard normal variable*, *defined on an extension of the original probability space and independent of $\mathcal{F}$*, *we have*

$$\mathcal{T}^n(p, u, v) \stackrel{\mathcal{L}-(s)}{\longrightarrow} Z. \tag{4.10}$$

We note that both the local and global jump activity estimates contain asymptotic biases of order $O_p(1/k_n)$. This is due to the local scale estimation via $V_i^n(p)$. However, since our statistic $\mathcal{T}^n(p, u, v)$ depends only on their difference, $\widehat{\beta}_j^n(p, u, v) - \widehat{\beta}^n(p, u, v)$, we do not need to perform bias correction to eliminate these biases. This is very convenient for applications.

We further note that the limiting variance $\overline{\Xi}(p, u, v)$ does not contain a term due to the estimation error of the local power variation $V_i^n(p)$ that is used to scale the increments. This is because the first-order effect of $V_i^n(p)$ on the estimation of the jump activity (both locally and globally) is zero. Nevertheless, as seen from the conditions in (4.8) and (4.9), the power p restricts the range of growth of the sequences k_n and m_n.

The two sequences k_n and m_n control the asymptotic size of the errors in estimating locally the scale and the jump activity. To make the biases due to the variation of σ (which are hard to estimate feasibly) negligible, we need $\varpi < 1/2$ as well as the bounds from above for ϖ in (4.8). The latter disappear if the process σ is constant, for example, when X is a Lévy process. On the other hand, the bounds from below for ϖ in (4.8) ensure the variability in the local power variation $V_i^n(p)$ is sufficiently small. This is also the reason for requiring $p < \beta/4$.

One feasible choice for p, ϖ and ϱ is the following. We can first set p in the interval $(1/5, 1/4)$. For the given choice of p, we can then set ϖ in the region $((1-p)/2, 2/5)$. Finally, ϱ can take any values in the relatively wide interval $(1-\varpi, 1)$ which in particular contains the fixed interval $(5/8, 1)$.

Finally, the highest possible value for p such that $p < \beta/4$ guarantees the weakest form of the conditions (4.8) and (4.9) for the tuning parameters ϖ and ϱ as well as the weakest assumption for β_t' (but we always need $\sup_{t \in [0,1]} \beta_t' < \beta/2$). Since we do not know β, we need to set $p < 1/4$ (recall from Assumption A that $\beta > 1$). However, a simple adaptive-type approach can improve on this choice. In particular, starting with arbitrary small $p > 0$, we can estimate consistently β using the first block estimator $\widehat{\beta}_1^n(p,u,v)$. We can then set p arbitrary close to, but below, $\widehat{\beta}_1^n(p,u,v)/4$ and perform the test on the rest of the data. Given the consistency of $\widehat{\beta}_1^n(p,u,v)$ and by conditioning, the result of Theorem 2 will continue to hold with this adaptive choice of p.

Combining Theorems 1 and 2, we have the validity of a test for time-varying jump activity, based on $\mathcal{T}^n(p,u,v)$. To state it, we introduce the two sets:

$$\Omega^c = \{\omega : \beta_t(\omega) = \beta_0(\omega), \forall t \in [0,1]\}, \qquad \Omega^v = \Omega \setminus \Omega^c. \tag{4.11}$$

COROLLARY 1. *For the process X assume Assumptions A and B hold. Let $k_n \asymp n^{\varpi}$ and $m_n \asymp n^{\varrho}$ as $n \to \infty$, for some $\frac{1}{3} < \varpi < \frac{1}{2} < \varrho < 1$ such that (4.8) holds true, $p < 1/4$, and β' satisfies the second condition in (4.9) if the process β is constant on $[0,1]$.*

Denote with z_α the α-quantile of standard normal distribution. We have

$$\begin{aligned} &\mathbb{P}(\mathcal{T}^n(p,u,v) > z_{1-\alpha}|\Omega^c) \longrightarrow \alpha, \qquad \text{if } \mathbb{P}(\Omega^c) > 0,\\ &\mathbb{P}(\mathcal{T}^n(p,u,v) > z_{1-\alpha}|\Omega^v) \longrightarrow 1. \end{aligned} \tag{4.12}$$

4.2. *Discussion.* The developed test applies to processes of pure-jump type, that is, processes that do not include a diffusion component. We further restrict attention to pure-jump specifications with BG index above 1 (so finite variation jump specifications for X are excluded). We briefly discuss the role of these two assumptions.

As shown in Todorov (2015) when X contains a diffusive component, then the jump activity estimators used here converge to the value of 2. Therefore, when X contains a diffusion and a jump process with time-varying jump activity, our test will fail to reject the null of constant jump activity. On the other hand, if on the interval $[0,1]$, the diffusive component of X is nonzero only on some part of it (with length less than 1), then our test will reject the null of constant jump activity. For this reason, the test should be performed for processes for which it is known that there is no diffusion component and a test for this is easy to construct; see, for example, Todorov and Tauchen (2011) and references therein.

Developing a test for time-varying jump activity in presence of a diffusion in X will involve different methods than the ones developed here. In such a case, one first have to account for the role of the diffusion which dominates the increments at high frequencies to derive jump activity estimates. Then an approach similar to the one proposed here of comparing global and local jump activity estimators can be adopted for testing the time-varying jump activity hypothesis in presence of a diffusion.

The second nontrivial assumption for the process X, that we impose in our setup, is the requirement $\beta_t > 1$. In general, the asymptotic behavior of estimators for jump activity, when the latter is below 1, worsens; see, for example, Aït-Sahalia and Jacod (2009), Bull (2016) and Todorov (2013). In our particular case, the main source of the problem is the deterioration in the rate of convergence of the local power variation that we use to scale the increments with (and in addition the leading terms driving its asymptotic behavior around its limit become biases that are hard to quantify). Of course, one can test the null that a (constant) jump activity is above 1, using, for example, the estimators in Todorov (2013) and Todorov (2015), and then proceed with the test proposed here. For many financial data sets, for example, volatility derivatives, this assumption for the jump activity seems to be satisfied; see, for example, Todorov and Tauchen (2011).

5. Monte Carlo. We now test the performance of our test on simulated data from models with the following dynamics:

$$dX_t = \sigma_{t-}\,dL_t, \qquad d\sigma_t = -0.03\sigma_t\,dt + dZ_t, \tag{5.1}$$

where L and Z are two independent of each other processes. Z is a Lévy process with Lévy density given by $\nu^Z(x) = 0.0293\frac{e^{-3x}}{x^{1.5}}1_{\{x>0\}}$, and hence σ is a Lévy-driven Ornstein–Uhlenbeck process with a tempered stable driving Lévy subordinator. The parameters governing the dynamics of σ imply $\mathbb{E}(\sigma_t) = 1$ and half-life of shock in σ of around one month (when unit of time is a day) and are adopted from the Monte Carlo setup in Todorov (2015).

In all considered scenarios, the process L is a pure-jump process with no drift and jump compensator $dt \otimes \nu_t^L(dx)$, for $\nu_t^L(dx) = \frac{\lambda^{2-\beta_t}}{2\Gamma(2-\beta_t)}\frac{e^{-\lambda|x|}}{|x|^{\beta_t+1}}\,dx$ where $\lambda > 0$ is some constant and β_t is some deterministic process taking values in $(0, 2)$. In this specification, β_t coincides with the instantaneous BG index of L (and of X). The parametrization of the jump compensator of L ensures $\int_{\mathbb{R}} x^2\nu_t^L(dx) = 1$ for every t, even when the process β varies over time, so that the "instantaneous variance" of L does not change over the time interval. We note that when the process β is constant, L is a tempered stable process used extensively in empirical work.

In all considered cases, we set $\lambda = 0.25$. We consider two scenarios under the null: (N1) $\beta_t = 1.2$ and (N2) $\beta_t = 1.5$, for $t \in [0, T]$, where $[0, T]$ is the fixed time interval over which X is observed. We consider two scenarios under the alternative hypothesis of time-varying BG index: (A1) $\beta_t = 1.2 + 0.6 \times \frac{t}{T}$, and (A2) $\beta_t = 1.3 + 0.4 \times \frac{t}{T}$.

To show that the model given by (5.1) satisfies our Assumptions A and B, first note that using the decomposition in Section 1 of the supplementary appendix of Todorov and Tauchen (2012) and after appropriately extending the probability space, we can decompose $L_t = \widetilde{L}_t + L_t^{(1)} - L_t^{(2)}$ where $\widetilde{L}$, $L^{(1)}$, and $L^{(2)}$ are processes with zero first two characteristics and Lévy densities of $\frac{\lambda^{2-\beta_t}}{2\Gamma(2-\beta_t)}\frac{1}{|x|^{\beta_t+1}}$, $\frac{\lambda^{2-\beta_t}}{2\Gamma(2-\beta_t)}\frac{1-e^{-\lambda|x|}}{|x|^{\beta_t+1}}$ and $\frac{\lambda^{2-\beta_t}}{\Gamma(2-\beta_t)}\frac{1-e^{-\lambda|x|}}{|x|^{\beta_t+1}}$, respectively. We further denote with $\widehat{L}_t$ the process whose jump at time t is equal to that of the process $\widetilde{L}_t$

divided by $(\frac{\lambda^{2-\beta_t}}{2\Gamma(2-\beta_t)})^{1/\beta_t}$. This process has Lévy density of $\frac{1}{|x|^{\beta_t+1}}$. We thus finally can represent X via $dX_t = \widehat{\sigma}_{t-} d\widehat{L}_t + dY_t$, where $\widehat{\sigma}_t = (\frac{\lambda^{2-\beta_t}}{2\Gamma(2-\beta_t)})^{1/\beta_t}\sigma_t$ and $Y_t = \int_0^t \sigma_{s-}(dL_s^{(1)} - dL_s^{(2)})$. Thus, Assumptions A and B hold with $\beta_t' \equiv \beta_t - 1$.

In the Monte Carlo, we set $T = 22$ which corresponds approximately to a time span of one month, given our time convention. We consider $\Delta_n = 1/100$ and $\Delta_n = 1/400$, which correspond to sampling approximately every five minutes and one minute, respectively, in a typical financial setting. We further set $p = 0.24$, and $k_n = 50$ for $\Delta_n = 1/100$ and $k_n = 85$ for $\Delta_n = 1/400$. For the jump activity estimator we use $u = 0.2$ and $v = 0.8$. Finally, we experiment with $b_n = 7$ and $b_n = 9$ for $\Delta_n = 1/100$ and with $b_n = 11$ and $b_n = 14$ for $\Delta_n = 1/400$.

On Figure 1, we plot the local (block-based) and global jump activity estimates on single realizations from scenarios N2 and A2 for frequency $\Delta_n = 1/400$. The constant jump activity case reveals the statistical uncertainty of measuring locally the jump activity. All local estimates in this scenario are, however, centered around the true constant value. By contrast, in the case of time-varying jump activity, the local estimates are centered around the time-varying level of the jump activity process β. As a result, the difference between them and the global estimate $\widehat{\beta}^n(p, u, v)$

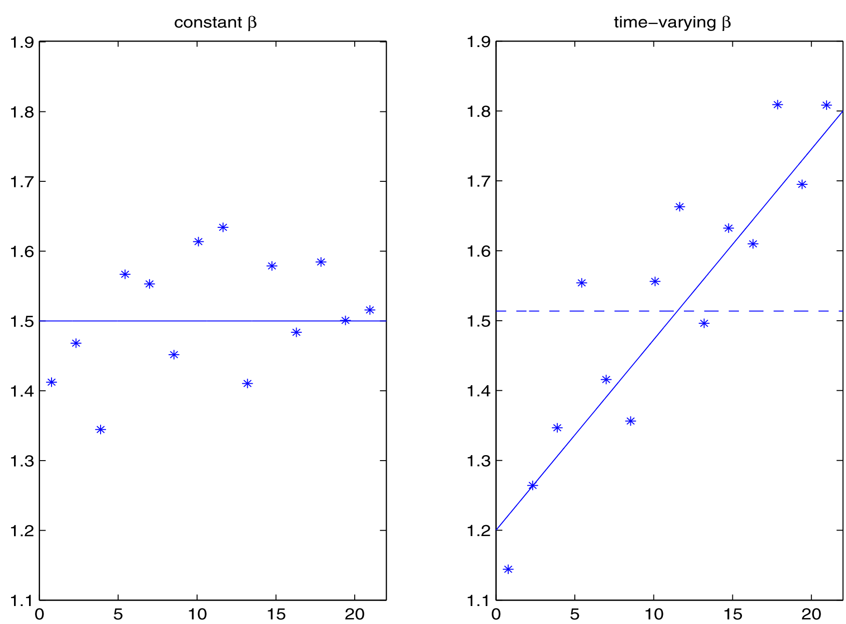

FIG. 1. *Local Jump Activity Estimates. The solid line corresponds to the true value of the jump activity index over the interval of length $T = 22$; the stars correspond to block-based estimates of jump activity for the case $\Delta_n = 1/400$ and $b_n = 14$ (and $p = 0.24$, $k_n = 85$, $u = 0.2$ and $v = 0.8$); the dashed line corresponds to the global $\widehat{\beta}^n(p, u, v)$. The constant β corresponds to a simulation from scenario N2 and the time-varying β to a simulation from scenario A2.*

TABLE 1
Monte Carlo results

	$\Delta_n = 1/100$				$\Delta_n = 1/400$			
	$b_n = 7$		$b_n = 9$		$b_n = 11$		$b_n = 14$	
	Size of test				Size of test			
Scenario	**5%**	**1%**	**5%**	**1%**	**5%**	**1%**	**5%**	**1%**
N1 ($\beta = 1.2$)	3.88	1.62	4.54	1.62	4.10	1.32	4.70	1.60
N2 ($\beta = 1.5$)	3.30	1.08	3.36	0.94	4.54	1.16	4.02	1.08
A1	58.80	42.50	52.40	35.20	99.86	99.44	86.66	85.96
A2	25.10	14.20	22.98	10.94	84.34	72.26	79.62	65.10

contains a bias which when aggregated across the blocks in $\mathcal{T}^n(p,u,v)$ makes it explode asymptotically.

The results from the Monte Carlo are reported in Table 1. We notice satisfactory performance of the test under all scenarios of constant jump activity. The test is slightly under-sized for the coarser frequency $\Delta_n = 1/100$ at the 5% nominal level. However, as expected from theory, the deviations of the empirical rejection rates from the nominal levels of the test (under the null scenarios) shrink when the sampling frequency increases to $\Delta_n = 1/400$. We note that there is no significant difference in the performance of the test for the different block sizes. The test has also satisfactory performance under the two alternative scenarios. Not surprisingly, the power of the test is higher for scenario A1 which has more dispersion of the jump activity over the interval than scenario A2. In both A1 and A2 and for the two different sampling frequencies, the power of the test is slightly higher for the case with lower b_n, although the differences are not very big. Smaller b_n allows to reduce the sampling error of measuring locally the jump activity via the block-based estimates, and thus better identify the differences in the jump activity across the blocks. On the other hand, smaller b_n means the local jump activity effectively averages the time-varying activity over longer time intervals, thus ignoring the time variation within them. For our alternative scenarios, the gains are larger than the costs (in terms of power) for the smaller of the block sizes considered.

6. Proofs. In the proofs, we use the shorthand notation $\mathbb{E}_i^n(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_{i\Delta_n})$ and $\mathbb{P}_i^n(\cdot) = \mathbb{P}(\cdot|\mathcal{F}_{i\Delta_n})$. We also set

$$i_j^n = k_n + 1 + (j-1)(m_n+1), \tag{6.1}$$

so that $I_j^n = \{i_j^n + 2, \dots, i_j^n + m_n + 1\}$.

Further, in the proofs we will denote with K a (finite) positive constant that does not depend on n and might change from line to line. Finally, to simplify the notation, henceforth we will use $\widehat{\beta}_j^n$, $\widehat{\beta}^n$ and $\overline{\beta}$ instead of $\widehat{\beta}_j^n(p,u,v)$, $\widehat{\beta}^n(p,u,v)$ and $\overline{\beta}(p,u,v)$, respectively.

6.1. *Localization.* By standard localization techniques, it suffices to prove the results in the paper under the following strengthened version of Assumption B.

ASSUMPTION SB. We have Assumption B and:

(a) $|\sigma_t|$ and $|\sigma_t|^{-1}$ are uniformly bounded;
(b) $|\delta^\alpha(t,x)| + |\delta^\sigma(t,x)| \leq \gamma(x)$ for all t, where $\gamma(x)$ is a deterministic bounded function on $\mathbb{R}$ with $\int_{\mathbb{R}} |\gamma(x)|^r \lambda(dx) < \infty$ for some $0 \leq r < 2$;
(c) $b^\alpha, b^\sigma, \eta^\alpha, \eta^\sigma, \widetilde{\eta}^\alpha$ and $\widetilde{\eta}^\sigma$ are bounded;
(d) the process $\int_{\mathbb{R}}(|x|^{\beta_t'} \wedge 1)\nu_t^Y(dx)$ is bounded and the jumps of Y are bounded;
(e) $\mathbb{E}|\eta_t^\sigma - \eta_s^\sigma|^2 + \mathbb{E}|\widetilde{\eta}_t^\sigma - \widetilde{\eta}_s^\sigma|^2 < \Gamma|t-s|^\iota$ for some positive constant Γ and every $s, t \geq 0$;

Henceforth, the proofs will be conducted under Assumption SB.

6.2. *Proof of Theorem 1.* We start with the decomposition

$$(\widehat{\beta}_j^n - \widehat{\beta}^n)^2 = (\widetilde{\beta}_j^n - \widehat{\beta}^n)^2 + (\widehat{\beta}_j^n - \widetilde{\beta}_j^n)^2 + 2(\widetilde{\beta}_j^n - \widehat{\beta}^n)(\widehat{\beta}_j^n - \widetilde{\beta}_j^n),$$

where $\widetilde{\beta}_j^n = \beta_{i_j^n \Delta_n}$. Provided we have $\widehat{\beta}^n \xrightarrow{\mathbb{P}} \overline{\beta}$, then since the process β has càglàd paths, by Riemann integrability, we have

$$\frac{1}{b_n}\sum_{j=1}^{b_n}(\widetilde{\beta}_j^n - \widehat{\beta}^n)^2 \xrightarrow{\mathbb{P}} \int_0^1 (\beta_s - \overline{\beta})^2\,ds.$$

Since $|\widehat{\beta}_j^n| \leq K\log(m_n)$ [recall the definition of $\widetilde{L}_j^n(p,u)$] and using a Taylor series expansion for the difference $\widehat{\beta}_j^n - \widetilde{\beta}_j^n$, it is easy to see that to prove the theorem it suffices to establish the following for some arbitrarily small $\iota > 0$ and $u, v \in \mathbb{R}_+$:

$$\text{(6.2)}\qquad \begin{cases} \mathbb{E}\bigl|\widehat{\mathcal{L}}_j^n(p,u) - e^{-C_{p,\widetilde{\beta}_j^n} u^{\widetilde{\beta}_j^n}}\bigr| \leq K/\log(m_n)^{2+\iota}, \\ \widehat{\mathcal{L}}^n(p,u) \xrightarrow{\mathbb{P}} \mathcal{L}(p,u) \quad \text{and} \quad \overline{\Xi}^n(p,u,v) \xrightarrow{\mathbb{P}} \Xi(p,u,v). \end{cases}$$

To establish the above result, we will need some preliminary bounds which we turn to next.

Upon suitably extending the probability space and using Grigelionis decomposition [Theorem 2.1.2 of Jacod and Protter (2012)], we can represent $S_t = S_t^{(1)} + S_t^{(2)}$ where

$$\text{(6.3)}\quad S_t^{(1)} = \int_0^t \int_{\mathbb{R}} \kappa\bigl(\delta(s,x)\bigr)\widetilde{\mu}(ds,dx), \qquad S_t^{(2)} = \int_0^t \int_{\mathbb{R}} \kappa'\bigl(\delta(s,x)\bigr)\mu(ds,dx),$$

with $\delta(t,x) = (\beta_t|x|)^{-1/\beta_t}\operatorname{sign}(x)$ and μ being a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with Lévy measure $A\,dt \otimes dx$, and further $\widetilde{\mu}(dt,dx) = \mu(dt,dx) -$

$A\,dt\,dx$. We further denote for $t \in [(i-2)\Delta_n, i\Delta_n]$

$$S_t^{(1,n)} = S_{(i-2)\Delta_n}^{(1)} + \int_{(i-2)\Delta_n}^t \int_{\mathbb{R}} \kappa\big(\delta((i-2)\Delta_n, x)\big)\widetilde{\mu}(ds, dx), \tag{6.4}$$

and we define similarly $S_t^{(2,n)}$ from $S_t^{(2)}$. Then using Lemmas 2.1.5 and 2.1.7 of Jacod and Protter (2012) and the smoothness assumption for the process β, we have

$$\mathbb{E}_{i-2}^n \big|S_t^{(1)} - S_t^{(1,n)}\big|^2 \le K\Delta_n^2, \qquad \mathbb{E}_{i-2}^n \big|S_t^{(2)}\big| + \mathbb{E}_{i-2}^n \big|S_t^{(2,n)}\big| \le K\Delta_n. \tag{6.5}$$

Another application of Lemmas 2.1.5 and 2.1.7 of Jacod and Protter (2012) plus the fact that $\sup_{t\in[0,1]} \beta_t' < 1$ as well as Assumption SB for σ, yields

$$\mathbb{E}_{i-2}^n \left| \int_{(i-1)\Delta_n}^{i\Delta_n} (\sigma_{u-} - \sigma_{(i-2)\Delta_n})\, dS_u^{(1)} \right|^2 \le K\Delta_n^2,$$

$$\mathbb{E}_{i-2}^n \left| \int_{(i-1)\Delta_n}^{i\Delta_n} (\sigma_{u-} - \sigma_{(i-2)\Delta_n})\, dS_u^{(2)} \right| + \mathbb{E}_{i-1}^n |\Delta_i^n Y| \le K\Delta_n,$$

$$\mathbb{E}_{i-2}^n \left| \int_{(i-1)\Delta_n}^{i\Delta_n} (\alpha_u - \alpha_{u-\Delta_n})\, du \right| \le K\Delta_n^{3/2}.$$

Next, applying a second-order Taylor expansion, using the fact that β_t takes values in the interval $(1, 2)$ for $\forall t \ge 0$, as well as the smoothness assumption for β in Assumption A(a), we have for any $q \in (0, 1]$ and for $j \le i$

$$\mathbb{E}_j^n \big|\Delta_n^{-q/\beta_j^n + q/\beta_i^n} - 1\big| \le K\big(|j-i|\log(n) + |j-i|^2 \Delta_n^{-q/2}\log^2(n)\big), \tag{6.6}$$

where we set $\beta_i^n = \beta_{i\Delta_n}$ (this shorthand is used for the rest of the proof of the theorem) and the constant K does not depend on i and j.

Given the above bounds and the smoothness assumption for β in Assumption A(a), we have for any $q \in (0, 1]$

$$\begin{aligned} &\big|\mathbb{E}_{i-2}^n \cos\big(\Delta_n^{-1/\beta_{i-2}^n}(\Delta_i^n X - \Delta_{i-1}^n X) f_{i-2}^n\big) \\ &\quad - \exp\big(-A_{\beta_{i-2}^n} |\sigma_{(i-2)\Delta_n} f_{i-2}^n|^{\beta_{i-2}^n}\big)\big| \le K\Delta_n^{1-\frac{1}{\beta_{i-2}^n}}, \end{aligned} \tag{6.7}$$

$$\big|\Delta_n^{-q/\beta_{i-2}^n} \mathbb{E}_{i-2}^n |\Delta_i^n X - \Delta_{i-1}^n X|^q - |\sigma_{(i-2)\Delta_n}|^q \mu_{q,\beta_{i-2}^n}^{q/\beta_{i-2}^n}\big| \le K\Delta_n^{q-\frac{q}{\beta_{i-2}^n}}, \tag{6.8}$$

$$\begin{aligned} &\mathbb{E}_{i-k_n-3}^n \big|\Delta_n^{-q/\beta_{i-k_n-3}^n} \mathbb{E}_{i-2}^n |\Delta_i^n X - \Delta_{i-1}^n X|^q - |\sigma_{(i-2)\Delta_n}|^q \mu_{q,\beta_{i-k_n-3}^n}^{q/\beta_{i-k_n-3}^n}\big| \\ &\quad \le K\big(\sqrt{k_n\Delta_n}\log(n) \vee (k_n\Delta_n)\Delta_n^{-q/2}\log^2(n) \vee \Delta_n^{q-q/\beta_{i-k_n-3}^n}\big), \end{aligned} \tag{6.9}$$

for f_i^n denoting some bounded random variable adapted to $\mathcal{F}_{i\Delta_n}$.

Using the above bounds, we can now show (6.2). First, using Burkholder–Davis–Gundy inequality for discrete martingales and the above bounds, we have

$$\mathbb{E}\left|\sum_{i\in I_j^n}\left[\cos\left(u\frac{\Delta_i^n X-\Delta_{i-1}^n X}{(V_i^n(p))^{1/p}}\right)-\mathbb{E}_{i-2}^n\left(\cos\left(u\frac{\Delta_i^n X-\Delta_{i-1}^n X}{(V_i^n(p))^{1/p}}\right)\right)\right]\right|$$
$$\leq K\sqrt{m_n},$$

$$\Delta_n^{-\frac{2p}{\beta_{i-k_n-3}^n}}\mathbb{E}_{i-k_n-3}^n\left|V_i^n(p)-\frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2}\mathbb{E}_{i-2}^n\left|\Delta_i^n X-\Delta_{i-1}^n X\right|^p\right|^2$$
$$\leq K\frac{[(k_n\Delta_n)\Delta_n^{-p}\log^2(n)]\vee 1}{k_n},$$

and from here for some sufficiently small $\epsilon>0$, and taking into account the restriction on k_n and p, the fact that the process $|\sigma|$ is bounded from below and (6.6) and (6.9), we have

$$\mathbb{P}_{i-k_n-3}^n\left(\Delta_n^{-p/\beta_{i-2}^n}V_i^n(p)<\epsilon\right)\leq K\Delta_n^{\iota},$$

for some sufficiently small $\iota>0$.

Combining the above bounds and using the smoothness assumption for σ, we have for some sufficiently small $\iota>0$

$$\mathbb{E}\left|\widehat{\mathcal{L}}_j^n(p,u)-e^{-C_{p,\widetilde{\beta}_j^n}u^{\widetilde{\beta}_j^n}}\right|\leq K\Delta_n^{\iota},$$

which establishes the first part of (6.2). To establish the remaining results in (6.2), we use the following bound for some sufficiently small $\iota>0$:

$$\mathbb{E}\Bigg|\mathbb{E}_{i-3}^n\left(\cos\left(u\frac{\Delta_i^n X-\Delta_{i-1}^n X}{(V_i^n(p))^{1/p}}\right)\cos\left(v\frac{\Delta_{i-1}^n X-\Delta_{i-2}^n X}{(V_{i-1}^n(p))^{1/p}}\right)\right)$$
$$-\frac{e^{-\frac{C_{p,\beta_{i-3}^n}}{2}(u^{\beta_{i-3}^n}+v^{\beta_{i-3}^n}+|u+v|^{\beta_{i-3}^n})}+e^{-\frac{C_{p,\beta_{i-3}^n}}{2}(u^{\beta_{i-3}^n}+v^{\beta_{i-3}^n}+|u-v|^{\beta_{i-3}^n})}}{2}\Bigg|$$
$$\leq K\Delta_n^{\iota},$$

with the result following from the inequalities above. From here, the last two results in (6.2) follow easily.

6.3. *Proof of Theorem* 2: *Decompositions and notation*. First, we denote the bias term

$$\mathcal{B}^n=\frac{1}{2k_n}\left(\frac{\beta}{p}\right)^2 C_{p,\beta}(v^\beta-u^\beta)\Sigma_{p,\beta},$$

where we set

$$\Sigma_{p,\beta} = \mathbb{E}\left(\left|\frac{S_1^{(\beta)} - S_2^{(\beta)}}{\mu_{p,\beta}^{1/\beta}}\right|^p - 1\right)^2 + 2\mathbb{E}\left\{\left(\left|\frac{S_1^{(\beta)} - S_2^{(\beta)}}{\mu_{p,\beta}^{1/\beta}}\right|^p - 1\right)\left(\left|\frac{S_2^{(\beta)} - S_3^{(\beta)}}{\mu_{p,\beta}^{1/\beta}}\right|^p - 1\right)\right\},$$

and we further use the shorthand $\widetilde{\mathcal{B}}^n = \mathcal{B}^n / \log(u/v)$.

We also introduce the following three different approximations of the local power variation $V_i^n(p)$ which will be used in various stages of the proof:

$$\begin{pmatrix} \widetilde{V}_i^n(p) \\ \widehat{V}_i^n(p) \\ |\overline{\sigma}|_i^p \end{pmatrix} = \frac{1}{k_n} \sum_{j=i-k_n-1}^{i-2} \begin{pmatrix} \mathbb{E}_{j-2}^n |\Delta_j^n X - \Delta_{j-1}^n X|^p \\ |\sigma_{(j-2)\Delta_n}|^p |\Delta_j^n S - \Delta_{j-1}^n S|^p / \mu_{p,\beta}^{p/\beta} \\ |\sigma_{(j-2)\Delta_n}|^p \end{pmatrix}.$$

Using the decomposition

$$(\widehat{\beta}_j^n - \widehat{\beta}^n)^2 = (\widehat{\beta}_j^n - \beta - \widetilde{\mathcal{B}}^n)^2 + (\beta - \widehat{\beta}^n + \widetilde{\mathcal{B}}^n)^2 + 2(\widehat{\beta}_j^n - \beta - \widetilde{\mathcal{B}}^n)(\beta - \widehat{\beta}^n + \widetilde{\mathcal{B}}^n),$$

it is clear that (4.10) will follow if we can show for $\forall \iota > 0$

$$\begin{cases} \sum_{j=1}^{b_n} (\widehat{\beta}_j^n - \beta - \widetilde{\mathcal{B}}^n) = o_p\left(\frac{n^{1/2+\iota}}{m_n}\right), \\ \widehat{\beta}^n - \beta - \widetilde{\mathcal{B}}^n = o_p\left(\frac{1}{(nm_n)^{1/4}}\right), \\ \widehat{\mathcal{T}}^n(p,u,v) \xrightarrow{\mathcal{L}-(s)} Z, \\ \overline{\Xi}^n(p,u,v) \xrightarrow{\mathbb{P}} \overline{\Xi}(p,u,v), \end{cases} \tag{6.10}$$

where we denote

$$\widehat{\mathcal{T}}^n(p,u,v) = \frac{1}{\sqrt{2b_n}} \sum_{j=1}^{b_n} \left(\frac{m_n(\widehat{\beta}_j^n - \beta - \widetilde{\mathcal{B}}^n)^2}{\overline{\Xi}(p,u,v)/(\log(u/v))^2} - 1\right).$$

Now we decompose the difference $\widehat{\beta}_j^n - \beta - \widetilde{\mathcal{B}}^n$. For some sufficiently small $\epsilon > 0$, and n sufficiently high so that $\epsilon > 1/m_n$, using a Taylor series expansion we have

$$\widehat{\beta}_j^n - \beta - \widetilde{\mathcal{B}}^n = \widehat{\beta}_j^{(n,1)} + \widehat{\beta}_j^{(n,2)} + \widehat{\beta}_j^{(n,3)} + \widehat{\beta}_j^{(n,4)},$$

$$\widehat{\beta}_j^{(n,1)} = \frac{1}{\log(u/v)}\left(\frac{\widehat{\mathcal{L}}_j^n(p,u) - \mathcal{L}(p,u)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))} - \frac{\widehat{\mathcal{L}}_j^n(p,v) - \mathcal{L}(p,v)}{\mathcal{L}(p,v)\log(\mathcal{L}(p,v))} - \mathcal{B}^n\right),$$

$$\widehat{\beta}_j^{(n,2)} = -\frac{1+\log(\overline{\mathcal{L}}_j^n(p,u))}{2\log(u/v)} \frac{(\widehat{\mathcal{L}}_j^n(p,u)-\mathcal{L}(p,u))^2}{(\overline{\mathcal{L}}_j^n(p,u)\log(\overline{\mathcal{L}}_j^n(p,u)))^2} 1_{\{\mathcal{C}_j^n\}}$$

$$+ \frac{1+\log(\overline{\mathcal{L}}_j^n(p,v))}{2\log(u/v)} \frac{(\widehat{\mathcal{L}}_j^n(p,v)-\mathcal{L}(p,v))^2}{(\overline{\mathcal{L}}_j^n(p,v)\log(\overline{\mathcal{L}}_j^n(p,v)))^2} 1_{\{\mathcal{C}_j^n\}},$$

$$\widehat{\beta}_j^{(n,3)} = -\widehat{\beta}_j^{(n,1)} 1_{\{(\mathcal{C}_j^n)^c\}},$$

$$\widehat{\beta}_j^{(n,4)} = (\widehat{\beta}_j^n - \beta - \widetilde{\mathcal{B}}^n) 1_{\{(\mathcal{C}_j^n)^c\}},$$

where $\overline{\mathcal{L}}_j^n(p,u)$ is between $\widehat{\mathcal{L}}_j^n(p,u)$ and $\mathcal{L}(p,u)$ and $\overline{\mathcal{L}}_j^n(p,v)$ is between $\widehat{\mathcal{L}}_j^n(p,v)$ and $\mathcal{L}(p,v)$ and we denote the set

$$\mathcal{C}_j^n = \{\widehat{\mathcal{L}}_j^n(p,u) \in [\epsilon, 1-\epsilon] \cap \widehat{\mathcal{L}}_j^n(p,v) \in [\epsilon, 1-\epsilon]\}.$$

We further decompose

$$\widehat{\mathcal{L}}_j^n(p,u) - \mathcal{L}(p,u) = \widehat{\mathcal{L}}_j^{(n,1)}(p,u) + \widehat{\mathcal{L}}_j^{(n,2)}(p,u) + \widehat{\mathcal{L}}_j^{(n,3)}(p,u),$$

where $\widehat{\mathcal{L}}_j^{(n,k)}(p,u) = \frac{1}{m_n}\sum_{i\in I_j^n} z_i^k(u)$ for $k = 1, 2, 3$ with

$$z_i^1(u) = \left[\cos\left(u\frac{\Delta_i^n X - \Delta_{i-1}^n X}{(V_i^n(p))^{1/p}}\right) - \exp\left(-\frac{A_\beta u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{\Delta_n^{-1}(V_i^n(p))^{\beta/p}}\right)\right] 1_{\{\Delta_n^{-p/\beta} V_i^n(p) > \epsilon\}},$$

$$z_i^2(u) = \left[\exp\left(-\frac{A_\beta u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{\Delta_n^{-1}(V_i^n(p))^{\beta/p}}\right) - \exp(-C_{p,\beta} u^\beta)\right] 1_{\{\Delta_n^{-p/\beta} V_i^n(p) > \epsilon\}},$$

$$z_i^3(u) = \left[\cos\left(u\frac{\Delta_i^n X - \Delta_{i-1}^n X}{(V_i^n(p))^{1/p}}\right) - \exp(-C_{p,\beta} u^\beta)\right] 1_{\{\Delta_n^{-p/\beta} V_i^n(p) \le \epsilon\}},$$

with ϵ being the constant used in the definition of the set $\mathcal{C}_j^n$ (a different $\epsilon > 0$ will work also). With the same ϵ, we further decompose

$$z_i^2(u)/(\mathcal{L}(p,u)\log(\mathcal{L}(p,u))) = \widetilde{z}_i^{(2,a)}(u) + \widetilde{z}_i^{(2,b)}(u) + \widetilde{z}_i^{(2,c)}(u),$$

$$\widetilde{z}_i^{(2,a)}(u) = \left(e^{-\frac{A_\beta u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{\Delta_n^{-1}(V_i^n(p))^{\beta/p}}} - e^{-\frac{C_{p,\beta} u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{(|\overline{\sigma}|_i^p)^{\beta/p}}}\right) / (\mathcal{L}(p,u)\log(\mathcal{L}(p,u))),$$

$$\widetilde{z}_i^{(2,b)}(u) = \left(e^{-\frac{C_{p,\beta} u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{(|\overline{\sigma}|_i^p)^{\beta/p}}} - e^{-C_{p,\beta} u^\beta}\right) / (\mathcal{L}(p,u)\log(\mathcal{L}(p,u))),$$

$$\widetilde{z}_i^{(2,c)}(u) = -(\widetilde{z}_i^{(2,a)}(u) + \widetilde{z}_i^{(2,b)}(u)) 1_{\{\Delta_n^{-p/\beta} V_i^n(p) \le \epsilon\}}.$$

With this notation, we split further $\widehat{\beta}_j^{(n,1)}$ into three parts, denoted with $\widehat{\beta}_j^{(n,1)}(a)$, $\widehat{\beta}_j^{(n,1)}(b)$ and $\widehat{\beta}_j^{(n,1)}(c)$, and given by

$$\widehat{\beta}_j^{(n,1)}(a) = \frac{1}{\log(u/v)}\left(\frac{\widehat{\mathcal{L}}_j^{(n,1)}(p,u)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))} - \frac{\widehat{\mathcal{L}}_j^{(n,1)}(p,v)}{\mathcal{L}(p,v)\log(\mathcal{L}(p,v))}\right),$$

$$\widehat{\beta}_j^{(n,1)}(b) = \frac{1}{\log(u/v)}\left(\frac{\widehat{\mathcal{L}}_j^{(n,2)}(p,u)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))} - \frac{\widehat{\mathcal{L}}_j^{(n,2)}(p,v)}{\mathcal{L}(p,v)\log(\mathcal{L}(p,v))} - \mathcal{B}^n\right),$$

$$\widehat{\beta}_j^{(n,1)}(c) = \frac{1}{\log(u/v)}\left(\frac{\widehat{\mathcal{L}}_j^{(n,3)}(p,u)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))} - \frac{\widehat{\mathcal{L}}_j^{(n,3)}(p,v)}{\mathcal{L}(p,v)\log(\mathcal{L}(p,v))}\right).$$

Finally, denoting $\beta' = \sup_{t\in[0,1]} \beta'_t$ and for arbitrary small $\iota > 0$, we set

$$\alpha_n = \Delta_n^{\frac{\beta}{2}\frac{p+1}{\beta+1}\wedge(\frac{p}{\beta'}\wedge 1-\frac{p}{\beta})-\iota}.$$

6.4. *Auxiliary results for the proof of Theorem 2*. We state the auxiliary results in a sequence of lemmas whose proofs are given in Section 6.7 and the supplementary material [Todorov (2017)].

LEMMA 1. *Under the conditions of Theorem 2, we have for some sufficiently small constants $\varepsilon > 0$ and $\iota > 0$ and $u \in \mathbb{R}_+$*

$$\Delta_n^{-p/\beta}\mathbb{E}_{i-2}^n\big||\Delta_i^n X - \Delta_{i-1}^n X|^p - |\sigma_{(i-2)\Delta_n}|^p|\Delta_i^n S - \Delta_{i-1}^n S|^p\big| \le K\alpha_n, \tag{6.11}$$

$$\Delta_n^{-p/\beta}\big|\mathbb{E}_{i-2}^n\big((\sigma_{i\Delta_n} - \sigma_{(i-2)\Delta_n})|\Delta_i^n X - \Delta_{i-1}^n X|^p\big)\big| \le K\Delta_n^{1/2+\iota}, \tag{6.12}$$

$$\begin{aligned}&\Delta_n^{-xp/\beta}\mathbb{E}_{i-k_n-3}^n|V_i^n(p) - \widetilde{V}_i^n(p)|^x + \mathbb{E}_{i-k_n-3}^n|\Delta_n^{-p/\beta}\widehat{V}_i^n(p) - |\overline{\sigma}|_i^p|^x\\ &\quad\le Kk_n^{-x/2}, \qquad x\in[2,\beta/p),\end{aligned} \tag{6.13}$$

$$\mathbb{P}_{i-k_n-3}^n\big(\Delta_n^{-p/\beta}V_i^n(p) \le \varepsilon\big) \le Kk_n^{-\beta/(2p)+\iota}, \tag{6.14}$$

$$\mathbb{P}\big(\widehat{\mathcal{L}}_j^n(p,u) < \varepsilon \cup \widehat{\mathcal{L}}_j^n(p,u) > 1-\varepsilon\big) \le K(nm_n)^{-1/2-\iota}. \tag{6.15}$$

To state the next lemma, we need some more notation. We denote

$$\widetilde{\Xi}_{k,i}(p,u,v) = \Xi_k\left(p, \frac{u|\sigma_{(i-2)\Delta_n}|}{\Delta_n^{-1/\beta}(\widehat{V}_i^n(p))^{1/p}}, \frac{v|\sigma_{(i-2)\Delta_n}|}{\Delta_n^{-1/\beta}(\widehat{V}_i^n(p))^{1/p}}\right),$$

for $k = 0, 1$, and we set $\widetilde{\Xi}_i(p,u,v) = \widetilde{\Xi}_{0,i}(p,u,v) + 2\widetilde{\Xi}_{1,i}(p,u,v)$. Finally,

$$\begin{aligned}\overline{\Xi}_i = {}& \frac{\widetilde{\Xi}_i(p,u,u)}{(\mathcal{L}(p,u)\log(\mathcal{L}(p,u)))^2} + \frac{\widetilde{\Xi}_i(p,v,v)}{(\mathcal{L}(p,v)\log(\mathcal{L}(p,v)))^2}\\ &- 2\frac{\widetilde{\Xi}_i(p,u,v)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))\mathcal{L}(p,v)\log(\mathcal{L}(p,v))}.\end{aligned}$$

LEMMA 2. *Under the conditions of Theorem 2, if we further denote*

$$x_i = \frac{z_i^1(u)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))} - \frac{z_i^1(v)}{\mathcal{L}(p,v)\log(\mathcal{L}(p,v))},$$

we have for $j = 1, \dots, b_n$, $u, v \in \mathbb{R}_+$, *and some sufficiently small* $\iota > 0$

$$\left|\mathbb{E}_{i-2}^n\big(z_i^1(u)\big)\right| \le K\Delta_n^{1/2+\iota}, \tag{6.16}$$

$$\begin{aligned} &\mathbb{E}\left|\mathbb{E}_{i_j^n}^n\left(\frac{1}{\sqrt{m_n}}\sum_{i\in I_j^n} x_i\right)^2 - \frac{1}{m_n}\sum_{i\in I_j^n}\overline{\Xi}_i\right| \\ &\qquad \le K\big[\alpha_n \vee k_n^{-1} \vee \sqrt{m_n}\Delta_n^{1/2+\iota}\big], \end{aligned} \tag{6.17}$$

$$\sum_{j=1}^{b_n}\sum_{i\in I_j^n}\big(\overline{\Xi}_i - \overline{\Xi}(p,u,v)\big) = O_p\left(\sqrt{k_n n} \vee \frac{n}{k_n}\right), \tag{6.18}$$

$$\begin{aligned} &\mathbb{E}\left|\mathbb{E}_{i_j^n}^n\left(\frac{1}{\sqrt{m_n}}\sum_{i\in I_j^n} x_i\right)^4 - 3\overline{\Xi}^2(p,u,v)\right| \\ &\qquad \le K\big(\alpha_n \vee k_n^{-1/2} \vee (k_n\Delta_n)^{1/2} \vee m_n^{1/2}\Delta_n^{1/2+\iota} \vee m_n^{-1/2}\big), \end{aligned} \tag{6.19}$$

$$\mathbb{E}\left|\frac{1}{\sqrt{m_n}}\sum_{i\in I_j^n}\big(z_i^1(u) - \mathbb{E}_{i-2}^n\big(z_i^1(u)\big)\big)\right|^q \le K, \qquad q \ge 0. \tag{6.20}$$

For the next lemma, we introduce the following additional notation:

$$\mathcal{B}^n(u) = \frac{1}{k_n}\left[\frac{1}{2}\frac{\beta}{p}\left(\frac{\beta}{p}+1\right) - \frac{1}{2}\left(\frac{\beta}{p}\right)^2 C_{p,\beta}u^\beta\right]\Sigma_{p,\beta},$$

and note $\mathcal{B}^n \equiv \mathcal{B}^n(u) - \mathcal{B}^n(v)$.

LEMMA 3. *Under the conditions of Theorem 2, we have for some sufficiently small* $\iota > 0$ *and* $u \in \mathbb{R}_+$

$$\begin{aligned} &\left|\mathbb{E}_{i-k_n-3}^n\big(\widetilde{z}_i^{(2,a)}(u) - \widetilde{z}_i^{(2,a)}(v) - \mathcal{B}^n\big)\right| \\ &\quad \le K\left(\alpha_n^2 \vee \frac{\alpha_n}{\sqrt{k_n}} \vee \Delta_n^{1/2+\iota} \vee k_n^{-3/2}\right), \end{aligned} \tag{6.21}$$

$$\left|\mathbb{E}_{i-k_n-3}^n\big(\widetilde{z}_i^{(2,a)}(u) - \mathcal{B}^n(u)\big)\right| \le K\big(\alpha_n \vee \sqrt{\Delta_n} \vee k_n^{-3/2}\big), \tag{6.22}$$

$$\begin{aligned} &\mathbb{E}\big(\widetilde{z}_i^{(2,a)}(u) - \widetilde{z}_i^{(2,a)}(v) - \mathcal{B}^n\big)^2 \\ &\qquad \le K\left[\alpha_n^4 \vee k_n^{-2} \vee \alpha_n^2 k_n\Delta_n \vee \frac{\sqrt{\Delta_n}}{k_n} \vee \frac{(k_n\Delta_n)^{3/4}}{k_n}\right], \end{aligned} \tag{6.23}$$

$$\mathbb{E}\big(\widetilde{z}_i^{(2,a)}(u) - \mathcal{B}^n(u)\big)^4 \le K\big[\alpha_n^4 \vee k_n^{-2}\big], \tag{6.24}$$

$$\begin{cases} \mathbb{E}^n_{i-k_n-3}\big|\widetilde{z}_i^{(2,b)}(u) - \widetilde{z}_i^{(2,b)}(v)\big|^q \le K(k_n\Delta_n), & q \ge 1, \\ \big|\mathbb{E}^n_{i-k_n-3}\big(\widetilde{z}_i^{(2,b)}(u)\big)\big| + \mathbb{E}^n_{i-k_n-3}\big|\widetilde{z}_i^{(2,b)}(u)\big|^q \le K(k_n\Delta_n), & q \ge 2, \end{cases} \tag{6.25}$$

$$\mathbb{E}^n_{i-k_n-3}\big|\widetilde{z}_i^{(2,c)}(u)\big|^q \le K k_n^{-\beta/(2p)+\iota}, \qquad q > 0. \tag{6.26}$$

LEMMA 4. *Under the conditions of Theorem 2, we have*

$$\widehat{\beta}^n - \beta - \widetilde{\mathcal{B}}_n = O_p\bigg(\Delta_n^{1/2-\iota} \vee \frac{\alpha_n}{\sqrt{k_n}}\bigg), \qquad \forall \iota > 0. \tag{6.27}$$

LEMMA 5. *Under the conditions of Theorem 2, for any bounded martingale M and every $t \in [0,1]$, we have*

$$\frac{1}{\sqrt{b_n}} \sum_{j=1}^{\lfloor t b_n \rfloor} \mathbb{E}^n_{i_j^n}\big[\overline{\chi}_j^n (M_{k_n+1+j(m_n+1)} - M_{k_n+1+(j-1)(m_n+1)})\big] \xrightarrow{\mathbb{P}} 0, \tag{6.28}$$

where $\overline{\chi}_j^n = (m_n(\widehat{\beta}_j^{(n,1)}(a))^2 - \frac{1}{m_n}\sum_{i \in I_j^n} \overline{\Xi}_i/(\log(u/v))^2)$.

6.5. *Proof of Theorem 2 continued.* Using the results of Lemma 3, and imposing the restriction $k_n \asymp n^{\varpi}$ with $\varpi \in (1/3, 1/2)$, we have for some sufficiently small $\iota > 0$

$$\begin{aligned} &\mathbb{E}\bigg(\frac{1}{m_n}\sum_{i \in I_j^n}\big(\widetilde{z}_i^{(2,a)}(u) - \widetilde{z}_i^{(2,a)}(v) - \mathcal{B}^n\big)\bigg)^2 \\ &\quad \le K\bigg\{\frac{k_n}{m_n}\bigg[\frac{1}{k_n^2} \vee \frac{(k_n\Delta_n)^{3/4}}{k_n}\bigg] \vee \bigg[\alpha_n^4 \vee \frac{\alpha_n^2}{k_n} \vee \Delta_n^{1+\iota} \vee \frac{1}{k_n^3}\bigg]\bigg\}, \end{aligned} \tag{6.29}$$

$$\mathbb{E}\bigg(\frac{1}{m_n}\sum_{i \in I_j^n}\widetilde{z}_i^{(2,a)}(u) - \mathcal{B}^n(u)\bigg)^4 \le K\big(\alpha_n^4 \vee m_n^{-2}\big), \tag{6.30}$$

$$\mathbb{E}\bigg|\frac{1}{m_n}\sum_{i \in I_j^n}\big(\widetilde{z}_i^{(2,b)}(u) - \widetilde{z}_i^{(2,b)}(v)\big)\bigg|^q \le K(k_n\Delta_n), \qquad q \ge 1, \tag{6.31}$$

$$\mathbb{E}\bigg|\frac{1}{m_n}\sum_{i \in I_j^n}\widetilde{z}_i^{(2,b)}(u)\bigg|^q \le K\bigg(\frac{k_n^{\frac{q}{2}}}{m_n^{\frac{q}{2}}}\frac{k_n}{n} \vee \frac{k_n^q}{n^q}\bigg), \qquad q \ge 2. \tag{6.32}$$

$$\mathbb{E}\bigg|\frac{1}{m_n}\sum_{i \in I_j^n}\widetilde{z}_i^{(2,c)}(u)\bigg|^q \le K\bigg(\frac{k_n^{\frac{q}{2}}}{m_n^{\frac{q}{2}}}k_n^{-\beta/(2p)+\iota} \vee k_n^{-q\beta/(2p)+\iota}\bigg), \qquad q \ge 2. \tag{6.33}$$

Using Lemma 1, we have

$$\mathbb{E}^n_{i-k_n-3}\big|z^3_i(u)\big|^q \le K k_n^{-\beta/(2p)+\iota}, \qquad q>0, \tag{6.34}$$

and, therefore,

$$\mathbb{E}\bigg|\frac{1}{m_n}\sum_{i\in I^n_j} z^3_i(u)\bigg|^q \le K\bigg(\frac{k_n^{\frac{q}{2}}}{m_n^{\frac{q}{2}}}k_n^{-\beta/(2p)+\iota} \vee k_n^{-q\beta/(2p)+\iota}\bigg), \qquad q\ge 2. \tag{6.35}$$

We are now ready to prove the theorem. We make the decomposition

$$\big(\widehat{\beta}^n_j-\beta-\widetilde{\mathcal{B}}^n\big)^2=\big(\widehat{\beta}^{(n,1)}_j(a)\big)^2+2\widehat{\beta}^{(n,1)}_j(a)R^n_j+\big(R^n_j\big)^2,$$

where we denote

$$R^n_j=\widehat{\beta}^{(n,1)}_j(b)+\widehat{\beta}^{(n,1)}_j(c)+\widehat{\beta}^{(n,2)}_j+\widehat{\beta}^{(n,3)}_j+\widehat{\beta}^{(n,4)}_j.$$

In what follows, we set $\eta_n=n^\iota$ for some arbitrary small $\iota>0$. Combining the bounds in (6.29), (6.30), (6.32)–(6.35) and using Lemmas 1 and 2, we have $\mathbb{E}|R^n_j|^2\le K(nm_n)^{-1/2-\iota}$, provided $p<\frac{\beta}{4}$, $\varpi\in(1/3,1/2)$ and $\frac{p}{\beta'}\wedge 1-\frac{p}{\beta}>\frac{1}{4}$, and the following conditions hold:

$$\eta_n\frac{n}{k_n^2 m_n}\to 0, \qquad \eta_n\frac{\alpha_n^2\sqrt{nm_n}}{k_n}\to 0, \qquad \eta_n\frac{k_n^4}{nm_n}\to 0.$$

Similarly, using the bounds in (6.29)–(6.35) and Lemmas 1 and 2 as well as Hölder inequality, we have $\mathbb{E}|\widehat{\beta}^{(n,1)}_j(a)R^n_j|\le K(nm_n)^{-1/2-\iota}$, provided $p<\frac{\beta}{4}$, $\varpi\in(1/3,1/2)$ and $\frac{p}{\beta'}\wedge 1-\frac{p}{\beta}>\frac{1}{4}$, and the following conditions hold:

$$\eta_n\frac{n}{k_n m_n}\to 0, \qquad \eta_n\frac{k_n^3 n}{m_n^4}\to 0, \qquad \eta_n\frac{\alpha_n\sqrt{n}}{\sqrt{k_n}}\to 0.$$

Therefore, taking into account the requirements for p, k_n and m_n in the theorem, we have altogether

$$\frac{m_n}{\sqrt{b_n}}\sum_{j=1}^{b_n}\big(2\widehat{\beta}^{(n,1)}_j(a)R^n_j+\big(R^n_j\big)^2\big)=o_p(1).$$

Using the bound in (6.18) and the requirements for p, k_n, m_n, we have

$$\frac{1}{\sqrt{b_n}m_n}\sum_{j=1}^{b_n}\sum_{i\in I^n_j}\big(\overline{\Xi}_i-\overline{\Xi}(p,u,v)\big)=o_p(1).$$

Next, we can use the bounds in Lemmas 2 and 5, the requirements for p, k_n and m_n in the theorem (which in particular imply $\alpha_n\sqrt{n/m_n}\to 0$), together with $\mathbb{E}|\overline{\Xi}_i -$

$\Xi(p,u,v)| \le K(\sqrt{k_n\Delta_n} \vee k_n^{-1/2})$ and the boundedness of $\overline{\Xi}_i$, and apply a stable CLT [Theorem IX.7.28 of Jacod and Shiryaev (2003)] to get

$$\frac{1}{\sqrt{2b_n}}\sum_{j=1}^{b_n}\left(\frac{m_n(\widehat{\beta}_j^{(n,1)}(a))^2 - \frac{1}{m_n}\sum_{i\in I_j^n}\overline{\Xi}_i/(\log(u/v))^2}{\overline{\Xi}(p,u,v)/(\log(u/v))^2}\right) \xrightarrow{\mathcal{L}-(s)} Z.$$

Combining the last three results, we have $\widehat{\mathcal{T}}^n(p,u,v) \xrightarrow{\mathcal{L}-(s)} Z$.

Using the bounds in (6.15), (6.16), (6.21)–(6.26) and (6.34), and since $p<\beta/3$, $\varpi \in (1/3, 1/2)$ and $\eta_n\alpha_n\sqrt{n}/\sqrt{k_n} \to 0$, we also have

$$\sum_{j=1}^{b_n}(\widehat{\beta}_j^n - \beta - \widetilde{\mathcal{B}}^n) = o_p(n^{1/2+\iota}m_n^{-1}), \qquad \forall \iota > 0.$$

Further from Lemma 4, provided $\alpha_n(nm_n)^{1/4}/\sqrt{k_n} \to 0$, we have $\widehat{\beta}^n - \beta - \widetilde{\mathcal{B}}^n = o_p((nm_n)^{-1/4})$. Finally, from (6.2) we also have $\overline{\Xi}^n(p,u,v) \xrightarrow{\mathbb{P}} \overline{\Xi}(p,u,v)$. Combining the above results we have altogether (6.10), and hence the result to be proved.

6.6. *Proof of Corollary* 1. First, given Theorem 2, we show that we have $\widehat{\mathcal{T}}^n(p,u,v) \xrightarrow{\mathcal{L}-(s)} Z$ in restriction to the set Ω^c. Indeed, we can construct a process X' satisfying $X'_s = X_s$ for all $0\le s\le 1$ on Ω^c and having the same constant β on Ω^v as well. Then we can apply Theorem 2 for X' and from the properties of the stable convergence, the result also holds in restriction to Ω^c (on which set X' coincides with X). Hence, the claim follows. From here, the result of the corollary is easily shown by using also Theorem 1 and by applying the Portmanteau lemma.

6.7. *Proof of the auxiliary results in Section* 6.4. In this section, we provide the proofs of the auxiliary results stated in Section 6.4.

6.7.1. *Proof of Lemma* 1. We start with (6.11). First, similar to the proof of Theorem 1, we split $S_t = S_t^{(1)} + S_t^{(2)}$ where $S_t^{(1)} = \int_0^t\int_{\mathbb{R}}\kappa(x)\widetilde{\mu}(ds,dx)$ and μ is Poisson random measure with Lévy measure $dt \otimes \frac{A}{|x|^{\beta+1}}dx$. Then using Lemmas 2.1.5 and 2.1.7 of Jacod and Protter (2012) and Assumption SB, we have for arbitrary small $\iota > 0$ and $l = 0, 1$,

$$\begin{aligned}\mathbb{E}_{i-2}^n\left|\int_{(i-2+l)\Delta_n}^{(i-1+l)\Delta_n}(\sigma_{u-} - \sigma_{(i-2)\Delta_n})\,dS_u^{(1)}\right|^q \\ \le K\Delta_n^{q/2+(q/\beta)\wedge 1-\iota}, \qquad q\in(0,2],\end{aligned} \tag{6.36}$$

$$\mathbb{E}_{i-2}^n\left|\int_{(i-2+l)\Delta_n}^{(i-1+l)i\Delta_n}(\sigma_{u-} - \sigma_{(i-2)\Delta_n})\,dS_u^{(2)}\right|^q \le K\Delta_n^{q/2+1}, \qquad q\in(0,1], \tag{6.37}$$

$$\mathbb{E}^n_{i-2}\left|\int_{(i-1)\Delta_n}^{i\Delta_n}(\alpha_u-\alpha_{u-\Delta_n})\,du\right|^q \le K\Delta_n^{3q/2}, \qquad q\in(0,2], \tag{6.38}$$

$$\mathbb{E}^n_{i-1}\left|\Delta_i^n Y\right|^q \le K\Delta_n^{(q/\beta')\wedge 1-\iota}, \qquad q>0 \text{ and } \beta'=\sup_{t\in[0,1]}\beta'_t. \tag{6.39}$$

Next, we introduce $\chi_1=\sigma_{(i-2)\Delta_n}(\Delta_i^n S-\Delta_{i-1}^n S)$ and

$$\begin{aligned}\chi_2 &= \int_{(i-1)\Delta_n}^{i\Delta_n}(\sigma_{u-}-\sigma_{(i-2)\Delta_n})\,dS_u^{(1)}-\int_{(i-2)\Delta_n}^{(i-1)\Delta_n}(\sigma_{u-}-\sigma_{(i-2)\Delta_n})\,dS_u^{(1)}\\ &\quad+\int_{(i-1)\Delta_n}^{i\Delta_n}(\alpha_u-\alpha_{u-\Delta_n})\,du,\\ \chi_3 &= \int_{(i-1)\Delta_n}^{i\Delta_n}(\sigma_{u-}-\sigma_{(i-2)\Delta_n})\,dS_u^{(2)}-\int_{(i-2)\Delta_n}^{(i-1)\Delta_n}(\sigma_{u-}-\sigma_{(i-2)\Delta_n})\,dS_u^{(2)}\\ &\quad+\Delta_i^n Y-\Delta_{i-1}^n Y.\end{aligned}$$

We finally use the shorthand $\widetilde{\chi}_i=\Delta_n^{-1/\beta}\chi_i$ for $i=1,\dots,3$. With this notation, using the results in (6.37) and (6.39), we first have

$$\mathbb{E}^n_{i-2}\big||\widetilde{\chi}_1+\widetilde{\chi}_2+\widetilde{\chi}_3|^p-|\widetilde{\chi}_1+\widetilde{\chi}_2|^p\big|\le K\Delta_n^{\frac{p}{\beta'}\wedge 1-\frac{p}{\beta}-\iota}. \tag{6.40}$$

So we are left with the difference $|\widetilde{\chi}_1+\widetilde{\chi}_2|^p-|\widetilde{\chi}_1|^p$. For it, we make use of the following algebraic inequality:

$$\begin{aligned}&\big||\widetilde{\chi}_1+\widetilde{\chi}_2|^p-|\widetilde{\chi}_1|^p\big|\\ &\quad\le K|\widetilde{\chi}_1|^{p-1}|\widetilde{\chi}_2|1_{\{|\widetilde{\chi}_1|>\epsilon,|\widetilde{\chi}_2|<0.5\epsilon\}}+|\widetilde{\chi}_2|^p(1_{\{|\widetilde{\chi}_1|\le\epsilon\}}+1_{\{|\widetilde{\chi}_2|>0.5\epsilon\}}),\end{aligned}$$

for any $\epsilon>0$ and $p\in(0,1]$, and where K that does not depend on ϵ. Since $p<1/\beta$ from (4.9), and using the bounds in (6.36) and (6.38) as well as Hölder inequality and the fact that $\mathbb{E}^n_{i-2}|\Delta_n^{-1/\beta}(\Delta_i^n S-\Delta_{i-1}^n S)|^q$ is a finite constant (that depends on q and β but not on n) for $q\in(-1,\beta)$ [see, e.g., (25.6) in Sato (1999)], we have

$$\begin{aligned}&\mathbb{E}^n_{i-2}\big(|\widetilde{\chi}_1|^{p-1}|\widetilde{\chi}_2|1_{\{|\widetilde{\chi}_1|>\epsilon,|\widetilde{\chi}_2|<0.5\epsilon\}}\big)\\ &\qquad\le\big(\mathbb{E}^n_{i-2}\big(|\widetilde{\chi}_1|^{\frac{(p-1)\beta}{\beta-1}}1_{\{|\widetilde{\chi}_1|>\epsilon\}}\big)\big)^{1-\frac{1}{\beta}}\big(\mathbb{E}^n_{i-2}|\widetilde{\chi}_2|^\beta\big)^{\frac{1}{\beta}}\\ &\qquad\le K\epsilon^{-(1/\beta-p)-\iota}\Delta_n^{1/2-\iota},\end{aligned} \tag{6.41}$$

$$\begin{aligned}&\mathbb{E}^n_{i-2}\big(|\widetilde{\chi}_2|^p1_{\{|\widetilde{\chi}_1|\le\epsilon\}}\big)\le K\big(\mathbb{P}^n_{i-2}\big(|\widetilde{\chi}_1|\le\epsilon\big)\big)^{1-\frac{p}{\beta}}\big(\mathbb{E}^n_{i-2}|\widetilde{\chi}_2|^\beta\big)^{\frac{p}{\beta}}\\ &\qquad\le K\epsilon^{1-p/\beta-\iota}\Delta_n^{p/2-\iota},\end{aligned} \tag{6.42}$$

$$\mathbb{E}^n_{i-2}\big(|\widetilde{\chi}_2|^p1_{\{|\widetilde{\chi}_2|>0.5\epsilon\}}\big)\le K\epsilon^{-(\beta-p)}\Delta_n^{\beta/2-\iota}, \tag{6.43}$$

where the constant K in (6.41)–(6.43) does not depend on ϵ. Upon setting $\epsilon = \Delta_n^{\frac{1}{2}\frac{\beta}{\beta+1}}$ in (6.41)–(6.43), and using (6.40), we get the result in (6.11).

We continue with (6.12) for which we introduce the following additional notation. For $s \in [(i-2)\Delta_n, i\Delta_n]$, we set

$$\widetilde{\sigma}_s = \sigma_{(i-2)\Delta_n} + \eta^{\sigma}_{(i-2)\Delta_n}(W_s - W_{(i-2)\Delta_n}) + \widetilde{\eta}^{\sigma}_{(i-2)\Delta_n}(\widetilde{W}_s - \widetilde{W}_{(i-2)\Delta_n}). \tag{6.44}$$

Using Assumption SB (the part about η^{σ} and $\widetilde{\eta}^{\sigma}$) and applying Burkholder–Davis–Gundy inequality and Corollary 2.1.9 of Jacod and Protter (2012), we have for $s \in [(i-2)\Delta_n, i\Delta_n]$

$$\left|\mathbb{E}^n_{i-2}(\widetilde{\sigma}_s - \sigma_{(i-2)\Delta_n})\right| + \mathbb{E}^n_{i-2}(\widetilde{\sigma}_s - \sigma_{(i-2)\Delta_n})^2 \leq K\Delta_n, \tag{6.45}$$

$$\mathbb{E}^n_{i-2}|\sigma_s - \widetilde{\sigma}_s|^q \leq K\Delta_n^{\frac{q+q\varsigma}{2}\wedge\frac{q}{r}\wedge 1}, \qquad q \in (1, 2]. \tag{6.46}$$

With these bounds, we can now show (6.12). Using the Itô semimartingale assumption for σ, Hölder inequality and the bounds in (6.36)–(6.39) and (6.46), as well as $p < \beta/2$ and the fact that W and $\widetilde{W}$ are independent of S, we have for some sufficiently small $\iota > 0$

$$\begin{aligned}
&\Delta_n^{-p/\beta}\mathbb{E}^n_{i-2}\big(|\sigma_{i\Delta_n} - \sigma_{(i-2)\Delta_n}|\big|\Delta^n_i X - \Delta^n_{i-1}X - \sigma_{(i-2)\Delta_n}(\Delta^n_i S - \Delta^n_{i-1}S)\big|^p\big) \\
&\qquad \leq K\Delta_n^{1/2+\iota}, \\
&\Delta_n^{-p/\beta}\mathbb{E}^n_{i-2}\big(|\sigma_{i\Delta_n} - \widetilde{\sigma}_{i\Delta_n}|\big|\Delta^n_i S - \Delta^n_{i-1}S\big|^p\big) \leq K\Delta_n^{1/2+\iota}, \\
&\mathbb{E}^n_{i-2}\big((\widetilde{\sigma}_{i\Delta_n} - \sigma_{(i-2)\Delta_n})\big|\Delta^n_i S - \Delta^n_{i-1}S\big|^p\big) = 0.
\end{aligned}$$

Using the above three bounds, we have altogether the result in (6.12).

Next, (6.13) follows directly by using successive application of Burkholder–Davis–Gundy inequality. We continue with (6.14). Using (6.11), we have

$$\left|\Delta_n^{-p/\beta}\widetilde{V}^n_i(p) - \mu^{p/\beta}_{p,\beta}|\overline{\sigma}|^p_i\right| \leq K\alpha_n. \tag{6.47}$$

Also, given the boundedness from below of the process $|\sigma|$ and provided ε is chosen sufficiently small, we have $\mu^{p/\beta}_{p,\beta}|\overline{\sigma}|^p_i > 3\varepsilon/2$. Combining this with the bounds in (6.13) and (6.47), we have the result in (6.14).

We are left with showing (6.15). We decompose $\widehat{\mathcal{L}}^n_j(p,u) = \sum_{p=1}^5 A_j^{(n,p)}$ where $A_j^{(n,p)} = \frac{1}{m_n}\sum_{i\in I^n_j} a_i^{(n,p)}$, for $p = 1,\dots,5$, with

$$\begin{aligned}
a_i^{(n,1)} &= \cos\left(u\frac{\Delta^n_i X - \Delta^n_{i-1}X}{(V^n_i(p))^{1/p}}\right) - \mathbb{E}^n_{i-2}\left(\cos\left(u\frac{\Delta^n_i X - \Delta^n_{i-1}X}{(V^n_i(p))^{1/p}}\right)\right), \\
a_i^{(n,2)} &= \mathbb{E}^n_{i-2}\left(\cos\left(u\frac{\Delta^n_i X - \Delta^n_{i-1}X}{(V^n_i(p))^{1/p}}\right)\right) - \exp\left(-\frac{A_\beta u^\beta|\sigma_{(i-2)\Delta_n}|^\beta}{\Delta_n^{-1}(V^n_i(p))^{\beta/p}}\right), \\
a_i^{(n,3)} &= \exp\left(-\frac{A_\beta u^\beta|\sigma_{(i-2)\Delta_n}|^\beta}{\Delta_n^{-1}(V^n_i(p))^{\beta/p}}\right) - \exp\left(-\frac{A_\beta u^\beta|\sigma_{(i-2)\Delta_n}|^\beta}{\Delta_n^{-1}(\widetilde{V}^n_i(p))^{\beta/p}}\right),
\end{aligned}$$

$$a_i^{(n,4)} = \exp\left(-\frac{A_\beta u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{\Delta_n^{-1}(\widetilde{V}_i^n(p))^{\beta/p}}\right) - \exp\left(-\frac{C_{p,\beta} u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{(|\overline{\sigma}|_i^p)^{\beta/p}}\right),$$

$$a_i^{(n,5)} = \exp\left(-\frac{C_{p,\beta} u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{(|\overline{\sigma}|_i^p)^{\beta/p}}\right).$$

If $\varepsilon > 0$ is chosen sufficiently small, given the boundedness of the process σ, we have $|A_j^{(n,5)}| > 3\varepsilon/2$ and $|A_j^{(n,5)}| < 1 - 3\varepsilon/2$. Hence to show (6.15), it suffices to show for some sufficiently small $\iota > 0$

$$\mathbb{P}\left(|A_j^{(n,p)}| > \frac{\varepsilon}{8}\right) \le K(nm_n)^{-1/2-\iota}, \qquad p = 1, \dots, 4. \tag{6.48}$$

(a) We start with $A_j^{(n,1)}$. Using successive application of Burkholder–Davis–Gundy inequality for discrete martingales as well as the boundedness of the cosine function, we have

$$\begin{aligned} &\mathbb{E}|A_j^{(n,1)}|^q \le K m_n^{-q/2} \\ &\quad \Longrightarrow \quad \mathbb{P}\left(|A_j^{(n,1)}| > \frac{\varepsilon}{8}\right) \le K m_n^{-q/2}, \qquad \forall q \ge 2. \end{aligned} \tag{6.49}$$

(b) We continue with $A_j^{(n,3)}$. Using a first-order Taylor expansion and the boundedness of the derivative of $f(x) = \exp(-\frac{A_\beta u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{x^{\beta/p}})$, we first have

$$|a_i^{(n,3)}|^q \le K\Delta_n^{-p/\beta}|V_i^n(p) - \widetilde{V}_i^n(p)|, \qquad q \ge 1.$$

We now can split $A_j^{(n,3)} = \frac{1}{m_n}\sum_{i\in I_j^n}\mathbb{E}_{i-k_n-3}^n(a_i^{(n,3)}) + \frac{1}{m_n}\sum_{i\in I_j^n}(a_i^{(n,3)} - \mathbb{E}_{i-k_n-3}^n(a_i^{(n,3)}))$. We can further split the last sum into $k_n + 1$ terms each of which can be viewed as the terminal value of a discrete martingale. From here, applying (6.13) derived earlier, we have

$$\mathbb{P}\left(|A_j^{(n,3)}| > \frac{\varepsilon}{8}\right) \le K\left(\left(\frac{k_n}{m_n}\right)^{q/2} k_n^{-1/2} \vee k_n^{-q/2}\right), \qquad q \ge 1. \tag{6.50}$$

(c) We show (6.48) for $A_j^{(n,4)}$. Using a first-order Taylor expansion we have

$$|a_i^{(n,4)}| \le K|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p) - \mu_{p,\beta}^{p/\beta}|\overline{\sigma}|_i^p|. \tag{6.51}$$

Then, by applying (6.11), we have for n sufficiently big $\mathbb{P}(|A_j^{(n,4)}| > \frac{\varepsilon}{8}) = 0$.

(d) We show (6.48) for $A_j^{(n,2)}$. For arbitrary $q \ge 1$, using the algebraic inequality $|\cos(x) - \cos(y)| \le K|x-y|^l$ for any $l \in [0,1]$ and $x, y \in \mathbb{R}$, the bounds in (6.36)–(6.39) as well as Assumption SB, we have for $\forall\ \iota > 0$

$$\mathbb{E}_{i-k_n-3}^n|a_i^{(n,2)} 1_{\{\Delta_n^{-p/\beta}|V_i^n(p)|>\varepsilon\}}|^q \le K\Delta_n^{1/2-\iota},$$

$$\mathbb{E}_{i-k_n-3}^n|a_i^{(n,2)} 1_{\{\Delta_n^{-p/\beta}|V_i^n(p)|\le\varepsilon\}}|^q \le K\mathbb{P}_{i-k_n-3}^n(\Delta_n^{-p/\beta}|V_i^n(p)| \le \varepsilon).$$

From here, using also (6.14) and a similar decomposition as that of $A_j^{(n,3)}$ above, we have

$$\mathbb{P}\left(|A_j^{(n,2)}| > \frac{\varepsilon}{8}\right) \leq K\left(\left(\frac{k_n}{m_n}\right)^{q/2} \gamma_n \vee \gamma_n^q\right), \qquad q \geq 1, \tag{6.52}$$

where we use the shorthand $\gamma_n = \Delta_n^{\frac{1}{2}-\iota} \vee k_n^{-\frac{\beta}{2p}+\iota}$.

Combining (6.49), (6.50), (6.51) with (6.11) and (6.52), we get (6.15).

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SUPPLEMENTARY MATERIAL

Supplement to "Testing for Time-Varying Jump Activity for Pure Jump Semimartingales" (DOI: 10.1214/16-AOS1485SUPP; .pdf). The proofs of the auxiliary results (Lemmas 2–5) given in Section 6.4 of the main text are relegated to the supplement [Todorov (2017)].

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