PARISI FORMULA FOR THE GROUND STATE ENERGY IN THE MIXED *p*-SPIN MODEL

BY ANTONIO AUFFINGER[1] AND WEI-KUO CHEN[2]

Northwestern University and University of Minnesota

We show that the thermodynamic limit of the ground state energy in the mixed *p*-spin model can be identified as a variational problem. This gives a natural generalization of the Parisi formula at zero temperature.

1. Introduction and main result. The mixed *p*-spin model is defined on the hypercube $\Sigma_N := \{-1, +1\}^N$ for $N \geq 1$ and its Hamiltonian is given by

$$H_N(\sigma) = X_N(\sigma) + h \sum_{i=1}^{N} \sigma_i,$$

where $X_N = (X_N(\sigma) : \sigma \in \Sigma_N)$ is a centered Gaussian process indexed by Σ_N,

$$X_N(\sigma) = \sum_{p \geq 2} \frac{c_p}{N^{(p-1)/2}} \sum_{1 \leq i_1, \ldots, i_p \leq N} g_{i_1, \ldots, i_p} \sigma_{i_1} \cdots \sigma_{i_p}$$

for i.i.d. standard Gaussian random variables $g_{i_1, \ldots, i_p}$ for $1 \leq i_1, \ldots, i_p \leq N$ and $p \geq 2$. Here, h denotes the strength of the external field and the sequence $(c_p)_{p \geq 2}$ is assumed to decay fast enough, for instance, $\sum_{p \geq 2} 2^p c_p^2 < \infty$, to guarantee the infinite sum X_N converges a.s. With this assumption, one readily computes that

$$\mathbb{E} X_N(\sigma^1) X_N(\sigma^2) = N\xi(R_{1,2}),$$

where

$$\xi(s) = \sum_{p \geq 2} c_p^2 s^p$$

and $R_{1,2} = N^{-1} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2$ is the overlap between σ^1 and σ^2. To avoid triviality, we shall assume that $c_p \neq 0$ for at least one $p \geq 2$. The classical Sherrington–Kirkpatrick (SK) model corresponds to $\xi(s) = s^2/2$.

A quantity of great interest in the mixed p-spin model is the large N limit (known as the thermodynamic limit) of the ground state energy

$$L_N := \max_{\sigma \in \Sigma_N} \frac{H_N(\sigma)}{N}.$$

In the past decades, there have been several numerical studies and analytic predictions of this limit in the physics literature, especially in the case of the SK model without external field ($h = 0$) [6, 11, 13, 14, 21]. This quantity also obtained great relevance in problems coming from computer science (see for instance [7, 12] and the references therein).

In order to get an explicit expression for

$$GSE := \lim_{N \to \infty} L_N,$$

the usual approach is to consider the free energy

$$F_N(\beta) = \frac{1}{\beta N} \log \sum_{\sigma \in \Sigma_N} \exp \beta H_N(\sigma),$$

where $\beta > 0$ is called the inverse temperature. It is well known that the thermodynamic limit of the free energy can be computed through the famous Parisi formula. More precisely, denote by $\mathcal{M}$ the collection of all probability distribution functions on $[0, 1]$. Define the Parisi functional by

$$\mathcal{P}_\beta(\alpha) = \frac{\log 2}{\beta} + \Psi_{\alpha,\beta}(0, h) - \frac{1}{2} \int_0^1 \beta\alpha(s)s\xi''(s)\,ds$$

for $\alpha \in \mathcal{M}$, where $\Psi_{\alpha,\beta}(t, x)$ is the weak solution to the following nonlinear parabolic PDE:

$$(1.1) \qquad \partial_t \Psi_{\alpha,\beta}(t, x) = -\frac{\xi''(t)}{2}\big(\partial_{xx}\Psi_{\alpha,\beta}(t, x) + \beta\alpha(t)\big(\partial_x \Psi_{\alpha,\beta}(t, x)\big)^2\big)$$

for $(t, x) \in [0, 1) \times \mathbb{R}$ with boundary condition

$$\Psi_{\alpha,\beta}(1, x) = \frac{\log \cosh \beta x}{\beta}.$$

For results on the regularity of $\Psi_{\alpha,\beta}$, we refer the readers to [1, 9]. The Parisi formula says that

$$(1.2) \qquad\qquad F(\beta) = \lim_{N \to \infty} F_N(\beta) = \inf_{\alpha \in \mathcal{M}} \mathcal{P}_\beta(\alpha) \qquad \text{a.s.}$$

Predicted by Parisi in [22], this formula was established by Talagrand [23] in the case of the mixed even p-spin model, that is, $c_p = 0$ for all odd $p \geq 3$, following the beautiful discovery of Guerra's replica symmetry breaking bound [8]. Its validity to general mixtures was obtained by Panchenko [17]. For fixed $\beta > 0$,

we denote the minimizer of (1.2) by $\alpha_{P,\beta}$. Uniqueness of this minimizer was established by Auffinger–Chen [2]. Throughout this paper, we call $\alpha_{P,\beta}$ the Parisi measure at temperature β.

Letting N and then β tend to infinity, the simple bound

$$L_N \leq F_N(\beta) \leq L_N + \frac{\log 2}{\beta}$$

yields that

$$GSE = \lim_{\beta \to \infty} F(\beta) \qquad \text{a.s.},$$

from which the Parisi formula deduces

$$(1.3) \qquad GSE = \lim_{\beta \to \infty} \inf_{\alpha \in \mathcal{M}} \mathcal{P}_\beta(\alpha) \qquad \text{a.s.}$$

In this paper, we show that (1.3) can be expressed as a variational problem. This gives a natural generalization of Parisi's formulation to the ground state energy. To prepare for the statements of our main results, we introduce the space $\mathcal{U}$ that collects all nonnegative and nondecreasing functions γ on $[0, 1)$ that are right continuous and satisfy $\int_0^1 \gamma(t)\,dt < \infty$. We endow this space with the L^1-distance d. Let $\mathcal{U}_d$ be the set of all step-like $\gamma \in \mathcal{U}$, that is, γ is a piecewise constant function with finite jumps. For each $\gamma \in \mathcal{U}_d$, consider the following fully nonlinear parabolic PDE:

$$(1.4) \qquad \partial_t \Psi_\gamma(t, x) = -\frac{\xi''(t)}{2} \big(\partial_x^2 \Psi_\gamma(t, x) + \gamma(t)\big(\partial_x \Psi_\gamma(t, x)\big)^2 \big),$$

for $(t, x) \in [0, 1) \times \mathbb{R}$ with boundary condition

$$\Psi_\gamma(1, x) = |x|.$$

Using the Cole–Hopf transformation, Ψ_γ can be solved explicitly in the classical sense. As we will show below that $\gamma \in (\mathcal{U}_d, d) \mapsto \Psi_\gamma(t, x)$ defines a Lipschitz functional with uniform Lipschitz constant for all $(t, x) \in [0, 1] \times \mathbb{R}$, one may extend Ψ_γ uniquely and continuously to arbitrary $\gamma \in \mathcal{U}$. We shall call Ψ_γ the Parisi PDE solution at zero temperature throughout this paper. With this construction, we can now define a continuous functional $\mathcal{P}$ on $\mathcal{U}$ by

$$\mathcal{P}(\gamma) = \Psi_\gamma(0, h) - \frac{1}{2} \int_0^1 t\xi''(t)\gamma(t)\,dt.$$

Our main result is stated as follows.

THEOREM 1 (Parisi formula). *We have that*

$$(1.5) \qquad GSE = \inf_{\gamma \in \mathcal{U}} \mathcal{P}(\gamma) \qquad \text{a.s.}$$

Moreover, there exists a $\gamma_0 \in \mathcal{U}$ that reaches the minimum of $\mathcal{P}$ over $\mathcal{U}$ and it is the vague limit of an infinite sequence $(\beta_l \alpha_{P,\beta_l})_{l \geq 1}$ for some $(\beta_l)_{l \geq 1}$ with $\lim_{l \to \infty} \beta_l = \infty$.

We mention that the Parisi formula for the ground state energy in the spherical version of the mixed p-spin model has been established recently in Chen–Sen [5] and Jagannath–Tobasco [10]. The approaches in both works rely on the Crisanti–Sommers representation for the thermodynamic limit of the free energy, where the functional has an explicit and simple expression in terms of ξ and α. This representation is not available in our setting, which leads to a substantially more demanding problem that requires a different approach. The proof of Theorem 1 is based on the establishment of upper and lower inequalities between the two sides of (1.5). The upper bound is not difficult and has already been obtained by Guerra [8], Theorem 6, via choosing suitable candidates in $\mathcal{M}$ since one is taking infimum of the functional. The lower bound, in contrast, carries all the challenges as one would unavoidably need to handle the sequence $\mathcal{P}_\beta(\alpha_{P,\beta})$, which involves the nonlinear PDE

$$(1.6) \quad \partial_t \Psi_{\alpha_{P,\beta},\beta}(t,x) = -\frac{\xi''(t)}{2}\big(\partial_{xx}\Psi_{\alpha_{P,\beta},\beta}(t,x) \\ + \beta\alpha_{P,\beta}(t)\big(\partial_x\Psi_{\alpha_{P,\beta},\beta}(t,x)\big)^2\big).$$

More precisely, it is known that $q_{P,\beta} := \inf\{t \in [0,1] : \alpha_{P,\beta}(t) = 1\} < 1$ and one can solve the PDE in (1.6) for $t \in [q_{P,\beta}, 1]$ to get

$$\Psi_{\alpha_{P,\beta},\beta}(t,x) = \frac{1}{\beta}\log\cosh\beta x + \frac{\beta}{2}\big(\xi'(1) - \xi'(t)\big)$$

(see [24], Chapter 14). The major obstacle here is that we do not know the quantitative behavior of $\beta\alpha_{P,\beta}(t)$ for t being close to $q_{P,\beta}$ from below, when β tends to infinity. This makes it very hard to track the effect of this singularity by a direct analysis of the PDE solution. To overcome this issue, we construct a representation of the Parisi PDE in terms of the stochastic optimal control problem introduced in [2, 3]. Under this framework, we are able to deal with the large β limit of the Parisi functional and we remove this singularity as a marvel cancellation happens between the nonlinear PDE and the linear term in the functional $\mathcal{P}_\beta$. The details are in Section 3, where we present the argument of obtaining the lower bound. Although we only consider the mixed p-spin model in this paper, we believe that the present approach could also be useful in deriving similar results as Theorem 1 from the existing Parisi formulas for the free energies in other mean-field spin glass models [15, 18–20].

REMARK 1. We comment that one may as well formulate the Parisi functional $\mathcal{P}$ by constructing the PDE solution Ψ_γ directly from the equation (1.4) rather than using the above approximation procedure. However, as γ could tend to infinity when t approaches 1 from below and the boundary condition $|x|$ is not differentiable at 0, the construction of the PDE solution and its regularity properties require extra effort. For this reason and clarity, we use the Lipschitz property of the PDE [see (2.7)] to construct the functional $\mathcal{P}$.

REMARK 2. Determining uniqueness of the minimizer of (1.5) needs regularity properties of the solution Ψ_γ as those used in [2]. The proof of uniqueness in [2] carries through once these properties are established. Reportedly, all these have been successfully carried out recently in Chen–Handschy–Lerman [4].

OPEN PROBLEMS. Understanding physical interpretations to the variational representation (1.5) is highly relevant. Moreover, in view of the approaches to the Parisi formula (1.2) by Talagrand [23] and Panchenko [17], it is also of great interest to see whether their methods can be applied here to obtain Theorem 1.

2. Variational representation for the Parisi PDE.

In this section, we will derive a variational representation for the PDE (1.1) in the form of stochastic optimal control. This formulation appeared initially in [3] and was used to establish the strict convexity of the Parisi functional $\mathcal{P}_\beta$ in [2]. See a simplified argument of [2] in [9]. Different than the derivations in [2, 3, 9], here we present an approach that relies only on Itô's formula. Consider the following Parisi PDE:

$$(2.1) \qquad \partial_t \Psi(t, x) = -\frac{\xi''(t)}{2} \left(\partial_x^2 \Psi(t, x) + \eta(t) \left(\partial_x \Psi(t, x) \right)^2 \right)$$

for $(t, x) \in [0, 1) \times \mathbb{R}$ with boundary condition $\Psi(1, x) = f(x)$, where f and η are specified by one of the following two cases:

(A1) $f(x) = \beta^{-1} \log \cosh \beta x$ and $\eta = \beta \alpha$ for some $\beta > 0$ and $\alpha \in \mathcal{M}$,
(A2) $f(x) = |x|$ and $\eta \in \mathcal{U}_d$.

It is known [1, 9] that given (A1), the solution Ψ has the properties that $\partial_x^j \Psi \in C([0, 1] \times \mathbb{R})$ for all $j \geq 0$ and $|\partial_x \Psi| \leq 1$. Likewise, since $\gamma \in \mathcal{U}_d$ is a step function, Ψ can be solved via the Cole–Hopf transformation, from which it can be checked that $\partial_x^j \Psi \in C([0, 1) \times \mathbb{R})$ for all $j \geq 0$ and $|\partial_x \Psi| \leq 1$.

Let $W = (W_t)_{0 \leq t \leq 1}$ be a standard Brownian motion. For $0 \leq s < t \leq 1$, denote by $D[s, t]$ the collection of all progressive measurable processes u on $[0, 1]$ with respect to the filtration generated by W and satisfying $\sup_{0 \leq t \leq 1} |u(t)| \leq 1$. For any $x \in \mathbb{R}$ and $u \in D[s, t]$, define

$$F^{s,t}(u, x) = \mathbb{E}\left[C^{s,t}(u, x) - L^{s,t}(u) \right],$$

where

$$C^{s,t}(u, x) = \Psi\left(t, x + \int_s^t \eta(r)\xi''(r)u(r)\, dr + \int_s^t \xi''(r)^{1/2}\, dW_r \right),$$

$$L^{s,t}(u) = \frac{1}{2} \int_s^t \eta(r)\xi''(r)\mathbb{E}u(r)^2\, dr.$$

Note that these functionals are well-defined as $\int_0^1 \eta(r)\, dr < \infty$ and $|u(r)| \leq 1$ for all $r \in [0, 1]$.

THEOREM 2 (Variational formula).   *Let f and η satisfy (A1) or (A2). We have that*

$$\Psi(s, x) = \max\{F^{s,t}(u, x)|u \in D[s, t]\}.$$

PROOF.   For simplicity, we shall only consider the case that η is continuous on $[0, 1]$. Under this assumption, the PDE (2.1) is valid in the classical sense and this allows us to use Itô's formula. The general case can be treated by an approximation argument identical to [2], Theorem 3. Let $u \in D[s, t]$. For notational convenience, we denote

$$Y(r) = x + \int_s^r \eta(w)\xi''(w)u(w)\,dw + \int_s^r \xi''(w)^{1/2}\,dW_w.$$

Define

$$Z(r) = \Psi(r, Y(r)) + \frac{1}{2}\int_s^r \eta(w)\xi''(w)(\partial_x\Psi(w, Y(w)) - u(w))^2\,dw$$

$$- \int_s^r \xi''(w)^{1/2}\partial_x\Psi(w, Y(w))\,dW_w - \frac{1}{2}\int_s^r \eta(w)\xi''(w)u(w)^2\,dw.$$

Using Itô's formula, we obtain that

$$\Psi(w, Y(w)) = \partial_w\Psi(w, Y(w))\,dw + \partial_x\Psi(w, Y(w))\,dY(w)$$

$$+ \frac{1}{2}\partial_{xx}\Psi(w, Y(w))\,d\langle Y, Y\rangle_w.$$

Here, from the PDE (2.1), the right-hand side becomes

$$-\frac{\xi''}{2}\big(\partial_w^2\Psi(w, Y) + \eta(w)(\partial_x\Psi(w, Y))^2\big)$$

$$+ \eta\xi''u\partial_x\Psi(w, Y)\,dw + \xi''^{1/2}\partial_x\Psi(w, Y)\,dW_w + \frac{\xi''}{2}\partial_{xx}\Psi(w, Y)\,dw$$

$$= -\frac{1}{2}\eta\xi''\big((\partial_x\Psi(w, Y))^2 - 2u\partial_x\Psi(w, Y)\big)\,dw + \xi''^{1/2}\partial_x\Psi(w, Y)\,dW_w$$

$$= -\frac{1}{2}\eta\xi''(\partial_x\Psi(w, Y) - u)^2\,dw + \xi''^{1/2}\partial_x\Psi(w, Y)\,dW_w + \frac{1}{2}\eta\xi''u^2\,dw.$$

In other words, $dZ(r) = 0$ and this implies that $Z(t) = Z(s) = \Psi(s, x)$. Taking expectation of this equation gives

(2.2)
$$\Psi(s, x) = \mathbb{E}\Psi(t, Y(t)) - \frac{1}{2}\int_s^t \eta(r)\xi''(r)\mathbb{E}u(r)^2\,dr$$

$$+ \frac{1}{2}\int_s^t \eta(r)\xi''(r)\mathbb{E}(\partial_x\Psi(r, Y(r)) - u(r))^2\,dr,$$

so

$$\Psi(s, x) \geq \sup_{u \in D[s,t]} \left(\mathbb{E}\Psi\left(t, x + \int_s^t \eta(r)\xi''(r)u(r)\,dr + \int_s^t \xi''(r)^{1/2}\,dW_r\right)\right.$$

(2.3)
$$\left. - \frac{1}{2} \int_s^t \eta(r)\xi''(r)\mathbb{E}u(r)^2\,dr \right).$$

To obtain the optimality, in the case of either $(A1)$ or $(A2)$ with $t < 1$, we consider

(2.4) $$u^*(r) = \partial_x \Psi(r, X(r)),$$

where $(X(r))_{s \leq r \leq t}$ is the strong solution to

$$dX(r) = \eta(r)\xi''(r)\partial_x\Psi(r, X(r))\,dr + \xi''(r)^{1/2}\,dW(r),$$

$$X(s) = x.$$

Simply notice that if $u = u^*$, then $Y(r) = X(r)$ for $s \leq r \leq t$ such that $\partial_x\Psi(r, Y(r)) = u^*$ for all $s \leq r \leq t$. From this and (2.2), the equality of (2.3) follows. For the case $(A2)$ with $t = 1$, we take u^* to be the same as (2.4) for $s \leq r < 1$ and set $u^*(1) = 0$. Letting $u = u^*$, one sees that $Y(r) = X(r)$ for $s \leq r < 1$ such that $\partial_x\Psi(r, Y(r)) = u^*$ for all $s \leq r < 1$. The optimality remains true. Note that the reason why one could not take the same u^* directly from (2.4) on the whole $[s, 1]$ is because $\Psi(1, x) = |x|$ is not differentiable at 0. $\square$

REMARK 3. In [2, 3, 9], the fact that the process u^* attains the maximum value of $F^{s,t}(\cdot, x)$ was established by a direct verification using Itô's formula. From the above proof, equation (2.2) quantifies the distance between the PDE and the functional $F^{s,t}$ for any u. Furthermore, it also explains how to choose the right candidate to reach the optimality.

As an immediate consequence of Theorem 2, we obtain the variational representations for the PDEs $\Psi_{\alpha,\beta}$ for $\alpha \in \mathcal{M}$ and Ψ_γ for $\gamma \in \mathcal{U}_d$.

COROLLARY 1. Let $s \in [0, 1]$ and $x \in \mathbb{R}$. For any $\alpha \in \mathcal{M}$, we have that

$$\Psi_{\alpha,\beta}(s, x) = \sup_{u \in D[s,1]} \left(\frac{1}{\beta}\mathbb{E}\log\cosh\beta\left(x + \int_s^1 u(r)\xi''(r)\beta\alpha(r)\,dr \right.\right.$$

(2.5)
$$\left.\left. + \int_s^1 \xi''(r)^{1/2}\,dW_r \right) - \frac{1}{2}\int_s^1 \mathbb{E}u(r)^2\xi''(r)\beta\alpha(r)\,dr \right).$$

COROLLARY 2. Let $s \in [0, 1]$ and $x \in \mathbb{R}$. For any $\gamma, \gamma' \in \mathcal{U}_d$, we have that

$$\Psi_\gamma(s, x) = \sup_{u \in D[s,1]} \left(\mathbb{E}\left|x + \int_s^1 u(s)\xi''(r)\gamma(r)\,dr + \int_s^1 \xi''(r)^{1/2}\,dW_r\right| \right.$$

(2.6)
$$\left. - \frac{1}{2}\int_s^1 \mathbb{E}u(r)^2\xi''(r)\gamma(r)\,dr \right)$$

and

(2.7)
$$\left|\Psi_\gamma(s, x) - \Psi_{\gamma'}(s, x)\right| \leq 2\xi''(1) d(\gamma, \gamma').$$

Here, (2.7) can be checked directly from (2.6) by noting that $|x|$ is Lipschitz 1 and using the triangle inequality. With the Lipschitz property (2.7), one can now extend the solution Ψ_γ continuously and uniquely to all $\gamma \in \mathcal{U}$ by an approximation procedure using $\mathcal{U}_d$. It is then clear that both (2.6) and (2.7) are valid for any $\gamma \in \mathcal{U}$.

3. Proof of Parisi's formula. First, we construct a vaguely convergent subsequence of $(\beta\alpha_{P,\beta})_{\beta>0}$ as follows. Note that from Gaussian integration by parts, one has the following identity:

(3.1)
$$\beta\left(\xi(1) - \mathbb{E}\langle\xi(R_{1,2})\rangle_\beta\right) = \mathbb{E}\left\langle\frac{X_N(\sigma)}{N}\right\rangle_\beta,$$

where letting

$$G_{N,\beta}(\sigma) = \frac{\exp\beta H_N(\sigma)}{\sum_{\sigma\in\Sigma_N}\exp\beta H_N(\sigma)}$$

be the Gibbs measure, σ^1, σ^2 are two i.i.d. samplings from $G_{N,\beta}$ and $\langle\cdot\rangle_\beta$ is the Gibbs average with respect to this measure. To control (3.1), we observe that

$$\mathbb{E}\left\langle\frac{X_N(\sigma)}{N}\right\rangle_\beta \leq \mathbb{E}\max_{\sigma\in\Sigma_N}\frac{X_N(\sigma)}{N} \leq \sqrt{2\xi'(1)\log 2}.$$

Here, the second inequality is obtained by using the usual estimate for the size of the maximum of Gaussian process (see, e.g., [25]). It is well known (see [16]) that $\beta F(\beta)$ is differentiable in temperature, which yields

$$\lim_{N\to\infty}\mathbb{E}\langle\xi(R_{1,2})\rangle_\beta = \int_0^1 \xi(s)\alpha_{P,\beta}(ds).$$

Consequently, from (3.1) and integration by part,

$$\int_0^1 \beta\alpha_{P,\beta}(s)\xi'(s)\,ds = \beta\left(\xi(1) - \int_0^1 \xi(s)\alpha_{P,\beta}(ds)\right) \leq \sqrt{2\xi'(1)\log 2}.$$

From this inequality, since $\alpha_{P,\beta}$ is nondecreasing, it follows that

(3.2)
$$\beta\alpha_{P,\beta}(s) \leq \frac{\sqrt{2\xi'(1)\log 2}}{\xi(1) - \xi(s)}, \qquad \forall s \in [0, 1).$$

From the last two inequalities, we may use Helly's selection theorem combined with a diagonalization process to conclude that, without loss of generality,

(3.3)
$$\gamma_0 := \lim_{\beta\to\infty}\beta\alpha_{P,\beta}$$

exists vaguely on $[0, 1)$ and

$$(3.4) \qquad L_0 := \lim_{\beta \to \infty} \int_0^1 \beta \alpha_{P,\beta}(s) \xi''(s) \, ds$$

exists. The following lemma, though simple, will be of great use in our argument.

LEMMA 1. *Let* $(\alpha_\beta)_{\beta>0} \in \mathcal{M}$ *such that* $(\beta \alpha_\beta)$ *converges to* γ *vaguely on* $[0, 1)$ *for some* $\gamma \in \mathcal{U}$ *and*

$$\int_0^1 \beta \alpha_\beta(s) \xi''(s) \, ds \to L.$$

If ϕ *is any measurable function with* $\|\phi\|_\infty \leq 1$ *and* $\lim_{t \to 1-} \phi(t) = \phi(1)$ *a.s.,*
then

$$\lim_{\beta \to \infty} \int_0^1 \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds = \int_0^1 \xi''(s) \phi(s) \nu(ds),$$

where ν *is the measure induced by*

$$\nu(ds) = 1_{[0,1)}(s) \gamma(s) \, ds + \frac{1}{\xi''(1)} \left(L - \int_0^1 \gamma(s) \xi''(s) \, ds \right) \delta_1(ds)$$

for $\delta_1(ds)$ *the Dirac measure at* 1.

PROOF. Write

$$\int_0^1 \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds = \int_0^t \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds + \int_t^1 \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds.$$

Observe that

$$\left| \int_t^1 \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds - \phi(1) \left(\int_0^1 \beta \alpha_\beta(s) \xi''(s) \, ds - \int_0^t \beta \alpha_\beta(s) \xi''(s) \, ds \right) \right|$$

$$= \left| \int_t^1 \beta \alpha_\beta(s) \xi''(s) (\phi(s) - \phi(1)) \, ds \right|$$

$$\leq \int_t^1 \beta \alpha_\beta(s) \xi''(s) \, ds \max_{t \leq s \leq 1} |\phi(s) - \phi(1)|.$$

From this, it follows that by (3.2) and the dominated convergence theorem,

$$\limsup_{\beta \to \infty} \int_t^1 \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds$$

$$\leq \phi(1) \left(L - \int_0^t \xi''(s) \gamma(ds) \right) + L \max_{t \leq s \leq 1} |\phi(s) - \phi(1)|,$$

$$\liminf_{\beta \to \infty} \int_t^1 \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds$$

$$\geq \phi(1) \left(L - \int_0^t \xi''(s) \gamma(ds) \right) - L \max_{t \leq s \leq 1} |\phi(s) - \phi(1)|.$$

On the other hand, using (3.2) and the dominated convergence theorem again,

$$\limsup_{\beta \to \infty} \int_0^1 \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds$$

$$\leq \limsup_{\beta \to \infty} \int_0^t \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds + \limsup_{\beta \to \infty} \int_t^1 \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds$$

$$\leq \int_0^t \phi(s) \gamma(s) \xi''(s) \, ds + \phi(1) \left(L - \int_0^t \xi''(s) \phi(ds) \right)$$

$$+ L \max_{t \leq s \leq 1} |\phi(s) - \phi(1)|$$

and

$$\liminf_{\beta \to \infty} \int_0^1 \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds$$

$$\geq \liminf_{\beta \to \infty} \int_0^t \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds + \liminf_{\beta \to \infty} \int_t^1 \beta \alpha_\beta(s) \xi''(s) \phi(s) \, ds$$

$$\geq \int_0^t \phi(s) \gamma(s) \xi''(s) \, ds + \phi(1) \left(L - \int_0^t \xi''(s) \phi(ds) \right)$$

$$- L \max_{t \leq s \leq 1} |\phi(s) - \phi(1)|.$$

Since these two inequalities hold for all $t \in (0, 1)$, letting $t \to 1-$, the continuity of ϕ at 1 implies the announced result. $\square$

The proof of Theorem 1 is completed by the following two lemmas, upper and lower bounds. We deal with the upper bound first. It agrees with the one that appeared in [8], Theorem 6.

LEMMA 2 (Upper bound).   *We have that*

$$GSE \leq \inf_{\gamma \in \mathcal{U}} \left(\Psi_\gamma(0, h) - \frac{1}{2} \int_0^1 \gamma(s) s \xi''(s) \, ds \right).$$

PROOF.   Assume that $\gamma \in \mathcal{U}$ with $\gamma(1-) < \infty$. For $\beta > \gamma(1-)$, define

$$\alpha_\beta(s) = \frac{\gamma(s)}{\beta} 1_{[0,1)}(s) + \delta_1(s).$$

Since γ is nonnegative and nondecreasing with right continuity, we see that $\alpha_\beta \in \mathcal{M}$. From the PDE (1.1), $\Psi_{\alpha_\beta, \beta}$ is given by

$$\partial_t \Psi_{\alpha_\beta, \beta}(t, x) = -\frac{\xi''(t)}{2} \big(\partial_x^2 \Psi_{\alpha_\beta, \beta}(t, x) + \gamma(t) \big(\partial_x \Psi_{\alpha_\beta, \beta}(t, x) \big)^2 \big)$$

for $(t, x) \in [0, 1) \times \mathbb{R}$ with boundary condition $\Psi_{\alpha_\beta, \beta}(1, x) = \beta^{-1} \log \cosh \beta x$. In other words, $\Psi_{\alpha_\beta, \beta}$ follows the same PDE. As now the boundary condition satisfies $\lim_{\beta \to \infty} \beta^{-1} \log \cosh \beta x = |x|$, it can be seen that $\lim_{\beta \to \infty} \Psi_{\alpha_\beta, \beta} = \Psi_\gamma$ pointwise, where Ψ_γ is the solution to (1.4). We emphasize that this convergence can be carried out by utilizing the representations in Corollaries 1 and 2 and applying the dominated convergence theorem since the process $u \in D[t, 1]$ is uniformly bounded by 1. On the other hand, note that

$$\int_0^1 \beta \alpha_\beta(s) s \xi''(s) \, ds = \int_0^1 s \xi''(s) \gamma(s) \, ds.$$

Thus, we can conclude that

$$\lim_{\beta \to \infty} \mathcal{P}_\beta(\alpha_\beta) = \lim_{\beta \to \infty} \Psi_{\alpha_\beta, \beta}(0, h) - \lim_{\beta \to \infty} \int_0^1 \beta \alpha_\beta(s) s \xi''(s) \, ds$$

$$= \Psi_\gamma(0, h) - \frac{1}{2} \int_0^1 s \xi''(s) \gamma(s) \, ds$$

and so

(3.5)
$$\Psi_\gamma(0, h) - \frac{1}{2} \int_0^1 s \xi''(s) \gamma(s) \, ds = \lim_{\beta \to \infty} \mathcal{P}_\beta(\alpha_\beta)$$

$$\geq \lim_{\beta \to \infty} \inf_{\alpha \in \mathcal{M}} \mathcal{P}_\beta(\alpha) = GSE.$$

To establish the same inequality for any $\gamma \in \mathcal{U}$ without the assumption $\gamma(1-) < \infty$, we may apply the truncation $\gamma_n = \min(\gamma, n)$ of γ to (3.5),

$$\Psi_{\gamma_n}(0, h) - \frac{1}{2} \int_0^1 s \xi''(s) \gamma_n(s) \, ds \geq GSE.$$

Since $(\gamma_n)_{n \geq 1}$ converges to γ under the distance d, using (2.7) for γ_n and γ leads to (3.5) for any $\gamma \in \mathcal{U}$. This completes our proof. $\square$

Next, we establish the lower bound, which is the most critical part in our approach.

LEMMA 3 (Lower bound). *We have that*

$$GSE \geq \inf_{\gamma \in \mathcal{U}} \left(\Psi_\gamma(0, h) - \frac{1}{2} \int_0^1 \xi''(s) s \gamma(s) \, ds \right).$$

PROOF. Recall γ_0 and L_0 from (3.3) and (3.4). Define

$$\nu_0(ds) = \gamma_0(s) \mathbf{1}_{[0,1)}(s) \, ds + \xi''(1)^{-1} \left(L_0 - \int_0^1 \gamma_0(s) \xi''(s) \, ds \right) \delta_1(ds).$$

For each $n \geq 1$, consider the function g_n defined by

$$g_n(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 2nx + 1, & \text{if } -n^{-1} \leq x < 0, \\ -1, & \text{if } x < -n^{-1}. \end{cases}$$

Note that $(g_n)_{n \geq 1}$ is a sequence of continuous functions with $\|g_n\|_\infty \leq 1$ such that

$$\lim_{n \to \infty} g_n(x) = \text{sign}(x),$$

where

$$\text{sign}(x) := \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Let $u \in D[0, 1]$. For any $\varepsilon \in (0, 1)$ and $n \geq 1$, define

$$\phi_{\varepsilon,n}(s) = u(s)1_{[0,\varepsilon)}(s)$$
$$+ g_n\left(h + \int_0^s u(r)\xi''(r)\gamma_0(r)\,dr + \int_0^s \xi''(r)^{1/2}\,dW_r\right)1_{[\varepsilon,1]}(s).$$

Then $\phi_{\varepsilon,n} \in D[0, 1]$ and $\lim_{s \to 1-} \phi_{\varepsilon,n}(s) = \phi_{\varepsilon,n}(1)$ since g_n is continuous on $\mathbb{R}$. In addition, the following limits hold:

$$\phi_n(s) := \lim_{\varepsilon \to 1-} \phi_{\varepsilon,n}(s) = u(s)1_{[0,1)}(s) + g_n(S)1_{\{1\}}(s)$$

and

$$\lim_{n \to \infty} \phi_n(s) = u(s)1_{[0,1)}(s) + \text{sign}(S)1_{\{1\}}(s),$$

where

$$S := h + \int_0^1 u(s)\xi''(s)\gamma_0(s)\,ds + \int_0^1 \xi''(s)^{1/2}\,dW_s.$$

Now, using (1.3) and the formula (2.5) applied to $\Psi_{\alpha_{P,\beta},\beta}(0, h)$, the Fatou lemma and Lemma 1 together imply

$$GSE = \lim_{\beta \to \infty} \mathcal{P}_\beta(\alpha_{P,\beta})$$

$$= \lim_{\beta \to \infty}\left(\frac{\log 2}{\beta} + \Psi_{\alpha_{P,\beta},\beta}(0, h) - \frac{1}{2}\int_0^1 \beta\alpha_{P,\beta}(s)\xi''(s)s\,ds\right)$$

$$\geq \lim_{\beta \to \infty}\left(\frac{1}{\beta}\mathbb{E}\log\cosh\beta\left(h + \int_0^1 \beta\alpha_{P,\beta}(s)\xi''(s)\phi_{\varepsilon,n}(s)\,ds\right.\right.$$

$$\left.\left.+ \int_0^1 \xi''(s)^{1/2}\,dW_s\right) - \frac{1}{2}\int_0^1 \beta\alpha_{P,\beta}(s)\xi''(s)\big(\mathbb{E}\phi_{\varepsilon,n}(s)^2 + s\big)\,ds\right)$$

$$\geq \mathbb{E}\left|h + \int_0^1 \phi_{\varepsilon,n}(s)\xi''(s)v_0(ds) + \int_0^1 \xi''(s)^{1/2}\,dW_s\right|$$
$$- \frac{1}{2}\int_0^1 (\mathbb{E}\phi_{\varepsilon,n}(s)^2 + s)\xi''(s)v_0(ds).$$

By the dominated convergence theorem, letting $\varepsilon \to 1-$ and then $n \to \infty$ implies

$$GSE \geq \mathbb{E}\left|S + \mathrm{sign}(S)\xi''(1)v_0(1)\right|$$
$$- \frac{1}{2}\int_0^1 (\mathbb{E}u(s)^2 + s)\xi''(s)\gamma_0(s)\,ds - \frac{1}{2}(\mathbb{E}(\mathrm{sign}(S))^2 + 1)\xi''(1)v_0(1).$$

Since

(3.6)
$$\left|S + \mathrm{sign}(S)\xi''(1)v_0(1)\right|$$
$$= (S + \xi''(1)v_0(1))1_{\{S>0\}} - (S - \xi''(1)v_0(1))1_{\{S<0\}}$$
$$+ \xi''(1)v_0(1)1_{\{S=0\}}$$
$$= |S| + \xi''(1)v_0(1)$$

and

(3.7)
$$(\mathbb{E}(\mathrm{sign}(S))^2 + 1)\xi''(1)v_0(1) = 2\xi''(1)v_0(1),$$

it follows that the terms $\xi''(1)v_0(1)$ cancel each other and we obtain

$$GSE \geq \mathbb{E}|S| - \frac{1}{2}\int_0^1 (\mathbb{E}u(s)^2 + s)\xi''(s)\gamma_0(s)\,ds$$
$$= \mathbb{E}\left|h + \int_0^1 u(s)\xi''(s)\gamma_0(s)\,ds + \int_0^1 \xi''(s)^{1/2}\,dW_s\right|$$
$$- \frac{1}{2}\int_0^1 (\mathbb{E}u(s)^2 + s)\xi''(s)\gamma_0(s)\,ds.$$

Since this holds for all $u \in D[0,1]$, we get that by noting (2.6) holds for any $\gamma \in \mathcal{U}$,

$$GSE \geq \Psi_{\gamma_0}(0,h) - \frac{1}{2}\int_0^1 s\xi''(s)\gamma_0(s)\,ds$$
$$\geq \inf_{\gamma \in \mathcal{U}}\left(\Psi_\gamma(0,h) - \frac{1}{2}\int_0^1 s\xi''(s)\gamma(s)\,ds\right).$$

This completes our proof. $\square$

REMARK 4. From the construction of $\phi_{\varepsilon,n}$, one sees that the crucial observations (3.6) and (3.7) allow us to cancel out the common terms $\xi''(1)v_0(1)$ arising from the jump of v_0 at 1. This explains how the effect of the singularity of the Parisi PDE near 1 in the limiting procedure is eliminated by the linear term of the Parisi functional as mentioned in the Introduction.

REMARK 5. It should be mentioned that while the expression (1.5) depends only on γ, the Parisi formula for the ground state energy in the spherical mixed p-spin model [5, 10] relies on γ and one extra variable, L, playing the role like L_0 in (3.4).

PROOF OF THEOREM 1. The equality (1.5) follows directly from Lemmas 2 and 3. In addition, from the proof of Lemma 3, the minimum of $\mathcal{P}$ over $\mathcal{U}$ is reached by γ_0 and γ_0 is the vague limit of some infinite sequence $(\beta_l)_{l \geq 1}$ with $\lim_{l \to \infty} \beta_l = \infty$. This ends our proof. $\square$

REFERENCES

[1] AUFFINGER, A. and CHEN, W.-K. (2015). On properties of Parisi measures. *Probab. Theory Related Fields* **161** 817–850. MR3334282

[2] AUFFINGER, A. and CHEN, W.-K. (2015). The Parisi formula has a unique minimizer. *Comm. Math. Phys.* **335** 1429–1444. MR3320318

[3] BOVIER, A. and KLIMOVSKY, A. (2009). The Aizenman–Sims–Starr and Guerra's schemes for the SK model with multidimensional spins. *Electron. J. Probab.* **14** 161–241. MR2471664

[4] CHEN, W.-K., HANDSCHY, M. and LERMAN, G. (2016). On the energy landscape of the mixed even p-spin model. Available at arXiv:1609.04368.

[5] CHEN, W.-K. and SEN, A. (2015). Parisi formula, disorder chaos and fluctuation for the ground state energy in the spherical mixed p-spin models. Available at arXiv:1512.08492.

[6] CRISANTI, A. and RIZZO, T. (2002). Analysis of the ∞-replica symmetry breaking solution of the Sherrington–Kirkpatrick model. *Phys. Rev. E* (3) **65** 046137. MR1918001

[7] DEMBO, A., MONTANARI, A. and SEN, S. (2017). Extremal cuts of sparse random graphs. *Ann. Probab.* **45** 1190–1217. MR3630296

[8] GUERRA, F. (2003). Broken replica symmetry bounds in the mean field spin glass model. *Comm. Math. Phys.* **233** 1–12. MR1957729

[9] JAGANNATH, A. and TOBASCO, I. (2016). A dynamic programming approach to the Parisi functional. *Proc. Amer. Math. Soc.* **144** 3135–3150. MR3487243

[10] JAGANNATH, A. and TOBASCO, I. (2016). Low temperature asymptotics of spherical mean field spin glasses. Available at arXiv:1602.00657.

[11] KIM, S.-Y., LEE, S. and LEE, J. (2007). Ground-state energy and energy landscape of the Sherrington–Kirkpatrick spin glass. *Phys. Rev. B* **76** 184412.

[12] MÉZARD, M. and MONTANARI, A. (2009). *Information, Physics, and Computation*. Oxford Univ. Press, Oxford. MR2518205

[13] OPPERMANN, R., SCHMIDT, M. J. and SHERRINGTON, D. (2007). Double criticality of the Sherrington–Kirkpatrick model at $T = 0$. *Phys. Rev. Lett.* **98** 127201.

[14] OPPERMANN, R. and SHERRINGTON, D. (2005). Scaling and renormalization group in replica-symmetry-breaking space: Evidence for a simple analytical solution of the Sherrington–Kirkpatrick model at zero temperature. *Phys. Rev. Lett.* **95** 197203.

[15] PANCHENKO, D. (2005). Free energy in the generalized Sherrington–Kirkpatrick mean field model. *Rev. Math. Phys.* **17** 793–857. MR2159369

[16] PANCHENKO, D. (2008). On differentiability of the Parisi formula. *Electron. Commun. Probab.* **13** 241–247.

[17] PANCHENKO, D. (2014). The Parisi formula for mixed p-spin models. *Ann. Probab.* **42** 946–958. MR3189062

[18] PANCHENKO, D. (2015). The free energy in a multi-species Sherrington–Kirkpatrick model. *Ann. Probab.* **43** 3494–3513. MR3433586

[19] PANCHENKO, D. (2015). Free energy in the Potts spin glass. Available at arXiv:1512.00370.

[20] PANCHENKO, D. (2015). Free energy in the mixed p-spin models with vector spins. Available at arXiv:1512.04441.

[21] PANKOV, S. (2006). Low-temperature solution of the Sherrington–Kirkpatrick model. *Phys. Rev. Lett.* **96** 197204.

[22] PARISI, G. (1979). Infinite number of order parameters for spin-glasses. *Phys. Rev. Lett.* **43** 1754–1756.

[23] TALAGRAND, M. (2006). The Parisi formula. *Ann. of Math.* (2) **163** 221–263.

[24] TALAGRAND, M. (2011). *Mean Field Models for Spin Glasses. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics* **55**. Springer, Berlin.

[25] TALAGRAND, M. and SPIN (2003). *Glasses: A Challenge for Mathematicians: Cavity and Mean Field Models. Ergebnisse der Mathematik und Ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics* **46**. Springer, Berlin.

DEPARTMENT OF MATHEMATICS
NORTHWESTERN UNIVERSITY
2033 SHERIDAN ROAD
EVANSTON, ILLINOIS 60208
USA
E-MAIL: auffing@math.northwestern.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MINNESOTA
127 VINCENT HALL
206 CHURCH ST. SE
MINNEAPOLIS, MINNESOTA 55455
USA
E-MAIL: wkchen@umn.edu