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POWER VARIATION FOR A CLASS OF STATIONARY INCREMENTS LÉVY DRIVEN MOVING AVERAGES

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In this paper, we present some new limit theorems for power variation of kth order increments of stationary increments Lévy driven moving averages. In the infill asymptotic setting, where the sampling frequency converges to zero while the time span remains fixed, the asymptotic theory gives novel results, which (partially) have no counterpart in the theory of discrete moving averages. More specifically, we show that the first-order limit theory and the mode of convergence strongly depend on the interplay between the given order of the increments $k \geq 1$, the considered power $p > 0$, the Blumenthal–Getoor index $\beta \in [0, 2)$ of the driving pure jump Lévy process L and the behaviour of the kernel function g at 0 determined by the power α. First-order asymptotic theory essentially comprises three cases: stable convergence towards a certain infinitely divisible distribution, an ergodic type limit theorem and convergence in probability towards an integrated random process. We also prove a second-order limit theorem connected to the ergodic type result. When the driving Lévy process L is a symmetric β-stable process, we obtain two different limits: a central limit theorem and convergence in distribution towards a $(k - \alpha)\beta$-stable totally right skewed random variable.

CONTENTS

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1. Introduction and main results. In the recent years, there has been an increasing interest in limit theory for power variations of stochastic processes. Power variation functionals and related statistics play a major role in analyzing the fine properties of the underlying model, in stochastic integration concepts and statistical inference. In the last decade, asymptotic theory for power variations of various classes of stochastic processes has been intensively investigated in the literature. We refer, for example, to [5, 24, 25, 31] for limit theory for power variations of Itô semimartingales, to [3, 4, 17, 21, 30] for the asymptotic results in the framework of fractional Brownian motion and related processes, and to [15, 16, 38] for investigations of power variation of the Rosenblatt process.

In this paper, we study the power variation of a class of *stationary increments Lévy driven moving averages*. More specifically, we consider an infinitely divisible process with stationary increments $(X_t)_{t\geq 0}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, given as

$$X_t = \int_{-\infty}^{t} \{g(t-s) - g_0(-s)\}\, dL_s. \tag{1.1}$$

Here, $L = (L_t)_{t\in\mathbb{R}}$ is a symmetric Lévy process on $\mathbb{R}$ with $L_0 = 0$ and without Gaussian component. Furthermore, $g, g_0 : \mathbb{R} \to \mathbb{R}$ are deterministic measurable functions vanishing on $(-\infty, 0)$. In the further discussion, we will need the notion of *Blumenthal–Getoor index* of L, which is defined via

$$\beta := \inf\left\{r \geq 0 : \int_{-1}^{1} |x|^r \nu(dx) < \infty\right\} \in [0, 2],$$

where ν denotes the Lévy measure of L. When $g_0 = 0$, the process X is a moving average, and in this case X is a stationary process. If $g(s) = g_0(s) = s_+^{\alpha}$, X is a so-called fractional Lévy process. In particular, when L is a β-stable Lévy process with $\beta \in (0, 2)$, X is called a linear fractional stable motion and it is self-similar with index $H = \alpha + 1/\beta$; see, for example, [34] (since in this case the stability index and the Blumenthal–Getoor index coincide, they are both denoted by β).

Probabilistic analysis of stationary increments Lévy driven moving averages such as semimartingale property, fine scale structure and integration concepts, have been investigated in several papers. We refer to the work of [6, 8–10, 27] among many others. However, only few results on the power variations of such processes are presently available. Exceptions to this are [8], Theorem 5.1, and [19], Theorem 2; see Remark 2.1 for a closer discussion of a result from [8], Theorem 5.1. These two results are concerned with certain power variations of a fractional Lévy process and have some overlap with our Theorem 1.1(ii) for the linear fractional stable motion, but we apply different proofs. The aim of this paper is to derive a rather complete picture of the first-order asymptotic theory for power variation of the process X, and, in some cases, the associated second-order limit theory.

To describe our main results, we need to introduce some notation and a set of assumptions. In this work, we consider the kth order increments $\Delta_{i,k}^n X$ of X, $k \in \mathbb{N}$, that are defined by

$$\Delta_{i,k}^n X := \sum_{j=0}^{k} (-1)^j \binom{k}{j} X_{(i-j)/n}, \qquad i \geq k.$$

For instance, we have that $\Delta_{i,1}^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$ and $\Delta_{i,2}^n X = X_{\frac{i}{n}} - 2X_{\frac{i-1}{n}} + X_{\frac{i-2}{n}}$. Our main functional is the power variation computed on the basis of kth order filters:

$$V(p;k)_n := \sum_{i=k}^{n} \left|\Delta_{i,k}^n X\right|^p, \qquad p > 0. \tag{1.2}$$

Now, we introduce the following set of assumptions on g and ν:

Assumption (A): The function $g : \mathbb{R} \to \mathbb{R}$ satisfies $g \in C^k((0,\infty))$ and

$$g(t) \sim c_0 t^\alpha \qquad \text{as } t \downarrow 0 \text{ for some } \alpha > 0 \text{ and } c_0 \neq 0, \tag{1.3}$$

where $g(t) \sim f(t)$ as $t \downarrow 0$ means that $\lim_{t\downarrow 0} g(t)/f(t) = 1$. For some $\theta \in (0,2]$, $\limsup_{t\to\infty} \nu(x : |x| \geq t)t^\theta < \infty$ and $g - g_0$ is a bounded function in $L^\theta(\mathbb{R}_+)$. Finally, there exists a $\delta > 0$ such that $|g^{(k)}(t)| \leq K t^{\alpha-k}$ for all $t \in (0,\delta)$, $|g'|$ and $|g^{(k)}|$ are in $L^\theta((\delta,\infty))$ and are decreasing on (δ,∞).

Assumption (A-log): In addition to (A), suppose that $\int_\delta^\infty |g^{(k)}(s)|^\theta \log(1/|g^{(k)}(s)|)\, ds < \infty$.

Assumption (A) ensures in particular that the process X is well defined; cf. Section 3. When L is a β-stable Lévy process, we always choose $\theta = \beta$ in assumption (A). Before we introduce the main results, we need some more notation. Let $h_k : \mathbb{R} \to \mathbb{R}$ be given by

$$h_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x-j)_+^\alpha, \qquad x \in \mathbb{R}, \tag{1.4}$$

where $y_+ = \max\{y, 0\}$ for all $y \in \mathbb{R}$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $(L_t)_{t\geq 0}$, $(T_m)_{m\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that exhausts the jumps of $(L_t)_{t\geq 0}$. That is, $\{T_m(\omega) : m \geq 1\} \cap \mathbb{R}_+ = \{t \geq 0 : \Delta L_t(\omega) \neq 0\}$ and $T_m(\omega) \neq T_n(\omega)$ for all $m \neq n$ with $T_m(\omega) < \infty$, where $\Delta L_t := L_t - L_{t-}$ and

$L_{t-} := \lim_{s\uparrow t, s<t} L_s$ for all $t \geq 0$. Let $(U_m)_{m\geq 1}$ be independent and uniform $[0, 1]$-distributed random variables, defined on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space, which are independent of the σ-algebra $\mathcal{F}$.

The following two theorems summarized our first- and second-order limit theorems for the power variation $V(p; k)_n$. We would like to emphasize part (i) of Theorem 1.1 and part (i) of Theorem 1.2, which both have a very different structure than the corresponding results in the context of, for example, semimartingales or Gaussian processes. We refer to [1, 33] and to Section 3 for the definition of $\mathcal{F}$-stable convergence in law which will be denoted $\xrightarrow{\mathcal{L}-s}$.

THEOREM 1.1 (First-order asymptotics). *Suppose* (A) *is satisfied and assume that the Blumenthal–Getoor index satisfies* $\beta < 2$. *We obtain the following three cases*:

(i) *Suppose that* (A-log) *holds if* $\theta = 1$. *If* $\alpha < k - 1/p$ *and* $p > \beta$, *we obtain the* $\mathcal{F}$-*stable convergence*

$$n^{\alpha p} V(p; k)_n \xrightarrow{\mathcal{L}-s} |c_0|^p \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^p V_m, \tag{1.5}$$
$$V_m := \sum_{l=0}^{\infty} \bigl|h_k(l + U_m)\bigr|^p.$$

(ii) *Suppose that* L *is a symmetric* β-*stable Lévy process with scale parameter* $\sigma > 0$, *that is*, $\mathbb{E}[\exp(iuL_1)] = \exp(-\sigma^\beta |u|^\beta)$ *for all* $u \in \mathbb{R}$. *If* $\alpha < k - 1/\beta$ *and* $p < \beta$, *then it holds that*

$$n^{-1+p(\alpha+1/\beta)} V(p; k)_n \xrightarrow{\mathbb{P}} m_p,$$

where $m_p = |c_0|^p \sigma^p (\int_{\mathbb{R}} |h_k(x)|^\beta \, dx)^{p/\beta} \mathbb{E}[|Z|^p]$ *and* Z *is a symmetric* β-*stable random variable with scale parameter* 1.

(iii) *Suppose that* $p \geq 1$. *If* $p = \theta$ *suppose in addition that* (A-log) *holds*. *For all* $\alpha > k - 1/(\beta \vee p)$, *we deduce that*

$$n^{-1+pk} V(p; k)_n \xrightarrow{\mathbb{P}} \int_0^1 |F_u|^p \, du, \tag{1.6}$$

where $(F_u)_{u\in\mathbb{R}}$ *is a measurable process satisfying*

$$F_u = \int_{-\infty}^u g^{(k)}(u - s) \, dL_s, \qquad u \in \mathbb{R}, \quad \text{and} \quad \int_0^1 |F_u|^p \, du < \infty \qquad \text{a.s.}$$

We remark that, except the critical cases where $p = \beta$, $\alpha = k - 1/p$ and $\alpha = k - 1/\beta$, Theorem 1.1 covers all possible choices of $\alpha > 0$, $\beta \in [0, 2)$ and $p \geq 1$. We also note that the limiting random variable in (1.5) is infinitely divisible; see Section 2 for more details. In addition, we note that there is no convergence in probability in (1.5) due to the fact that the random variables $V_m, m \geq 1$, are independent of L and the properties of stable convergence. Moreover, under assumption $\alpha \in (0, k - 1/2)$ the case $p = 2$, which corresponds to quadratic variation, always falls under the scope of Theorem 1.1(i). To be used in the next theorem, we recall that a totally right skewed ρ-stable random variable S with $\rho > 1$, mean zero and scale parameter $\sigma > 0$ has characteristic function given by

$$\mathbb{E}\big[e^{i\theta S}\big] = \exp\big(-\sigma^\rho |\theta|^\rho \big(1 - i \operatorname{sign}(\theta) \tan(\pi\rho/2)\big)\big), \qquad \theta \in \mathbb{R}.$$

For part (ii) of Theorem 1.1, we also show the second-order asymptotic results under the additional condition $p < \beta/2$. We remark that for $k = 1$ we are automatically in the regime of Theorem 1.2(i).

THEOREM 1.2 (Second-order assymptotics). *Suppose that assumption* (A) *is satisfied and L is a symmetric β-stable Lévy process with scale parameter $\sigma > 0$. Let $f : [0, \infty) \mapsto \mathbb{R}$ be given by $f(t) = g(t)t^{-\alpha}$ for $t > 0$. Suppose that $\lim_{t \downarrow 0} f^{(j)}(t)$ exists in $\mathbb{R}$ for all $j = 1, \dots, k$ and that $|g^{(k)}(t)| \leq K t^{\alpha - k}$ for all $t > 0$. For all $p < \beta/2$, we have the following two cases*:

(i) *Suppose that $\alpha \in (k - 2/\beta, k - 1/\beta)$. Then it holds that*

$$n^{1 - \frac{1}{(k-\alpha)\beta}} \big(n^{-1+p(\alpha+1/\beta)} V(p; k)_n - m_p\big) \xrightarrow{d} S,$$

where S is a totally right skewed $(k - \alpha)\beta$-stable random variable with mean zero and scale parameter $\tilde{\sigma} \in (0, \infty)$, which is defined in Remark 2.3.

(ii) *For $\alpha \in (0, k - 2/\beta)$, we deduce that*

$$\sqrt{n}\big(n^{-1+p(\alpha+1/\beta)} V(p; k)_n - m_p\big) \xrightarrow{d} \mathcal{N}(0, \eta^2),$$

where η^2 is the finite positive constant defined in Remark 2.3.

This paper is structured as follows. The methodology of the proofs, related results and some potential statistical applications are discussed in Section 2. Section 3 introduces some preliminaries. We state the proof of Theorem 1.1 in Section 4, while the proof of Theorem 1.2 is demonstrated in Section 5.

2. Methodology, related literature and statistical applications. In this section, we highlight the basic ideas behind the proofs of Theorems 1.1 and 1.2, discuss some related results and present some potential statistical applications. In case of Theorem 1.1, we assume for the ease of exposition that $k = 1$ and set $\Delta_i^n X := \Delta_{i,1}^n X$ and $V(p)_n := V(p; 1)_n$.

We start with the intuition behind Theorem 1.1(iii). First, we will show that the process X is differentiable almost everywhere and $X' = F \in L^p([0, 1])$ almost surely, where the F has been introduced at (1.6); see Lemma 4.3 for a detailed exposition. By using this, Theorem 1.1(iii) is deduced by a Riemann integrability type argument.

The result of Theorem 1.1(ii) is shown via a tangent process technique. Let us consider the process

$$\tilde{X}_t := c_0 \int_{-\infty}^{t} \{(t-s)_+^{\alpha} - (-s)_+^{\alpha}\}\, dL_s, \tag{2.1}$$

which is a linear fractional β-stable motion under assumptions of Theorem 1.1(ii). We recall that $(\tilde{X}_t)_{t\geq 0}$ has stationary increments, symmetric β-stable marginals and it is self-similar with index $H = \alpha + 1/\beta \in (1/2, 1)$, that is, $(\tilde{X}_{at})_{t\geq 0} \stackrel{d}{=} a^H (\tilde{X}_t)_{t\geq 0}$ for any $a \in \mathbb{R}_+$. We will prove that $\Delta_i^n X$ are close to $\Delta_i^n \tilde{X}$ in probability as $n \to \infty$. From this, we prove the statement of Theorem 1.1(ii) by using the self-similarity of $\tilde{X}$ and the mixing property of the increments $(\tilde{X}_t - \tilde{X}_{t-1})_{t\geq 1}$.

REMARK 2.1. Theorem 5.1 of [8] studies the first-order asymptotic of the power variation of some fractional fields $(X_t)_{t\in\mathbb{R}^d}$. In the case $d = 1$, they consider fractional Lévy processes $(X_t)_{t\in\mathbb{R}}$ of the form

$$X_t = \int_{\mathbb{R}} \{|t-s|^{H-1/2} - |s|^{H-1/2}\}\, dL_s, \tag{2.2}$$

where L is a truncated β-stable Lévy process. This setting is close to fit into the framework of the present paper (1.1) with $\alpha = H - 1/2$ except for the fact that the stochastic integral (2.2) is over the whole real line. However, the proof of Theorem 1.1(i) still holds for X in (2.2) with the obvious modifications of h_k and V_m in (1.4) and (1.5), respectively. For $p < \beta$, Theorem 5.1 of [8] claims that $2^{n\alpha p} V(p; 2)_{2^n} \to C$ almost surely, where C is a positive constant. However, this obviously contradicts Theorem 1.1(i). It seems that the last three lines of the proof of [8], Theorem 5.1, are erroneous, since the derived estimates are not uniform in the parameters which are required for the stated conclusion to hold; see [8], page 372.

We describe the intuition behind the statement of Theorem 1.1(i) in the following simple setting: We consider the driving motion $L_t = \mathbb{1}_{[T,\infty)}(t)\Delta L_T$, where T has a density on the interval $(0, 1)$ (note that L is not a Lévy process). We also assume for simplicity that $g(x) = g_0(x) = c_0 x_+^{\alpha}$. Let i_n be the random index satisfying $T \in [(i_n - 1)/n, i_n/n)$. Then $\Delta_j^n X = 0$ for $j < i_n$ and

$$\Delta_{i_n+l}^n X = c_0 \Delta L_T \left(\left(\frac{i_n + l}{n} - T \right)_+^{\alpha} - \left(\frac{i_n + l - 1}{n} - T \right)_+^{\alpha} \right), \qquad l \geq 0.$$

Now, we use the following result, which is essentially due to Tukey [39] (see also [18] and Lemma 4.1 below): Let Z be a random variable with an absolutely continuous distribution and let $\{x\} := x - \lfloor x \rfloor \in [0, 1)$ denote the fractional part of $x \in \mathbb{R}$. Then $\{nZ\} \xrightarrow{\mathcal{L}-s} U \sim \mathcal{U}([0, 1])$ and U is independent of Z. Since

$i_n - nT = 1 - \{nT\}$ and $1 - U \sim \mathcal{U}([0,1])$, we conclude the stable convergence in law

$$n^{\alpha p} V(p)_n \xrightarrow{\mathcal{L}-s} |c_0 \Delta L_T|^p \sum_{l=0}^{\infty} \big|(l+U)_+^{\alpha} - (l-1+U)_+^{\alpha}\big|^p.$$

A formal proof of Theorem 1.1(i) for a general Lévy process L requires a decomposition of the jump measure associated with L into big and small jumps, and a certain time separation between the big jumps.

REMARK 2.2. We remark that the distribution of the limiting variable in (1.5) does not depend on the chosen sequence $(T_m)_{m\geq 1}$ of stopping times, which exhausts the jump times of L. Furthermore, the limiting random variable in (1.5) is infinitely divisible with Lévy measure $(\nu \otimes \eta) \circ ((y, v) \mapsto |c_0 y|^p v)^{-1}$, where η denotes the law of V_1. In fact, if W denotes the limiting random variable in (1.5), then W has characteristic function given by

$$\mathbb{E}\big[\exp(i\theta W)\big] = \exp\bigg(\int_{\mathbb{R}_0 \times \mathbb{R}} \big(e^{i\theta |c_0 y|^p v} - 1\big)\nu(dy)\eta(dv)\bigg).$$

To show this, let Λ be the Poisson random measure on $[0,1] \times \mathbb{R}_0$ given by $\Lambda = \sum_{m=1}^{\infty} \delta_{(T_m, \Delta L_{T_m})}$ which has mean measure $\lambda \otimes \nu$. Here, $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and λ denotes the Lebesgue measure on $[0,1]$. Set $\Theta = \sum_{m=1}^{\infty} \delta_{(T_m, \Delta L_{T_m}, V_m)}$. Then Θ is a Poisson random measure with mean measure $\lambda \otimes \nu \otimes \eta$, due to [35], Theorem 36. Thus, the above claim follows from the stochastic integral representation $W = \int_{[0,1]\times\mathbb{R}_0\times\mathbb{R}} (|c_0 y|^p v)\Theta(ds, dy, dv)$.

The results of Theorem 1.2 are related to the weak limit theory for statistics of *discrete* moving averages. In a discrete framework, a variety of different limit processes may appear. They include Brownian motion, mth order Hermite processes, stable Lévy processes with various stability indexes and fractional Brownian motion. We refer to the papers [2, 22, 23, 28, 36, 37] for an overview. First, we present some of the main steps in the proof of Theorem 1.2(i). By means of several projection techniques, which are described in Section 5.1 [cf. (5.7)], we show that the rescaled version of $V(p;k)_n$ is asymptotically equivalent to a sum of i.i.d. random variables. Then the statement of Theorem 1.2(i) is shown using a standard result [34], Theorem 1.8.1, by identifying the tail behaviour of the summands. This proof strategy is similar to the one investigated in [37] in the discrete time setting. However, strong modifications are required due to unboundedness of the function $H : x \mapsto |x|^p - m_p$, infinite second moments of L, triangular nature of summands in (1.2) and different set of conditions. To prove Theorem 1.2(ii), we show that the increments $(\Delta_{i,k}^n X)_{i\geq k}$ are well approximated by an m-dependent process $(\Delta_{i,k}^n X(m))_{i\geq k}$, which is obtained by a truncation of the integration region. This part is also carried out by using projection techniques. We conclude the proof of Theorem 1.2(ii) by showing a central limit theorem for power variation of $(\Delta_{i,k}^n X(m))_{i\geq k}$ and then let m converge to infinity.

REMARK 2.3 (The constants in Theorem 1.2). In order to introduce the constant $\tilde{\sigma}$ appearing in Theorem 1.2(i), we set

$$\kappa = k_\alpha^{1/(k-\alpha)}(k-\alpha)^{-1}\int_0^\infty \Phi(y)y^{-1-1/(k-\alpha)}\,dy,$$

where

$$\Phi(y) := \mathbb{E}\big[\big|\Delta_{k,k}^1\tilde{X}+y\big|^p - \big|\Delta_{k,k}^1\tilde{X}\big|^p\big], \qquad y\in\mathbb{R},$$

$k_\alpha = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)$, and $(\tilde{X}_t)$ is a linear fractional stable motion defined in (2.1) with $c_0 = 1$ and L being a standard symmetric β-stable Lévy process. In addition, set $\tau_\rho = (\rho-1)/\Gamma(2-\rho)|\cos(\pi\rho/2)|$ for $\rho\in(0,2)\setminus\{1\}$ and $\tau_1 = 2/\pi$, where Γ denotes the gamma function. Then the scale parameter $\tilde{\sigma}$ is defined via

$$\tilde{\sigma} = |c_0|^p\sigma^p\left(\frac{\tau_\beta}{\tau_{(1-\alpha)\beta}}\right)^{\frac{1}{(1-\alpha)\beta}}\kappa.$$

The function $\Phi(y)$ can be computed explicitly; see (5.12). This representation shows that $\Phi(y) > 0$ for all $y > 0$, and hence the limiting variable S in Theorem 1.2(i) is not degenerate, because $\tilde{\sigma} > 0$. The constant η^2 in Theorem (1.2)(ii) is given by $\eta^2 = \lim_{m\to\infty}\eta_m^2$, where for all $m\geq 1$, η_m^2 is defined in (5.54).

At the present stage, nonparametric estimation of the model (1.1) seems to be out of reach in terms of estimating the Lévy measure ν and kernel function g. For instance, when g has compact support we may only recover finitely many jumps of L, which exceed any given positive threshold. Thus, it becomes impossible to estimate the Lévy measure ν from $(X_{i/n})_{i=1}^n$, $n\in\mathbb{N}$ (we refer to [7] for the estimation of ν for a class of short-range dependence processes and low frequency observations $(X_t)_{t=1,\dots,T}$ with $T\to\infty$). Despite this fact, we will in the following briefly mention two simple consequences of the results of Section 1. Motivated by the linear fractional β-stable motion, we investigate estimation procedures for the parameters α and β in the framework of the underlying process $(X_t)_{t\geq 0}$ with $\alpha > 0$ and $H = \alpha + 1/\beta < 1$. We note that in this setting it must hold that $\beta\in(1,2)$ and $\alpha < 1 - 1/(p\vee\beta)$.

We start with a direct inference procedure that is based on a log scale estimator. Let $k = 1$ and define $S_{\alpha,\beta}(n,p) := -\log V(p)_n/\log n$ for any $p > 0$. Then Theorem 1.1(i) and (ii) implies the convergence $S_{\alpha,\beta}(n,p)\xrightarrow{\mathbb{P}} S_{\alpha,\beta}(p)$ for $p\neq\beta$, where $S_{\alpha,\beta}(p) = \alpha p$ for $p > \beta$ and $S_{\alpha,\beta}(p) = pH - 1$ for $p\leq\beta$. Next, we define the set $J := \{(\alpha,\beta)\in\mathbb{R}^2 : \ \beta\in[1,2], \ \alpha\in[0, 1-1/\beta]\}$ and let (α_0,β_0) denote the true parameter of the model (1.1), where it is assumed that (α_0,β_0) is an inner point of J. Now, a random vector $(\hat{\alpha}_n,\hat{\beta}_n)$ defined via

$$(\hat{\alpha}_n,\hat{\beta}_n)\in\underset{(\alpha,\beta)\in J}{\operatorname{argmin}}\big\|S_{\alpha_0,\beta_0}(n) - S_{\alpha,\beta}\big\|_{L^2([\underline{p},\overline{p}])}$$

for some $\underline{p} \in (0,1)$, $\overline{p} > 2$ and with $S_{\alpha_0,\beta_0}(n) := S_{\alpha_0,\beta_0}(n,\cdot)$, is a consistent estimator of the parameter (α_0, β_0). However, obtaining second-order limit theorems for $(\hat{\alpha}_n, \hat{\beta}_n)$ seem to be a very nontrivial issue. The above mentioned approach is somewhat similar to the estimation method proposed in [20].

We conjecture that the above log scale estimator has a slow rate of convergence, for example, logarithmic rate. On the other hand, the parameter $H = \alpha + 1/\beta \in (1/2, 1)$ might be estimated with a faster rate of convergence by applying the following ratio statistic approach. Recalling that $\beta \in (1,2)$, we deduce under conditions of Theorem 1.1(ii) that $R(n,p) := V(p)_n / V(p)_{2n} \xrightarrow{\mathbb{P}} 2^{1-pH}$ for any $p \in (0,1]$. Thus, we can immediately conclude the consistency result

$$\hat{H}_n := \frac{1}{p}\left(1 - \frac{\log R(n,p)}{\log 2}\right) \xrightarrow{\mathbb{P}} H.$$

This type of idea is rather standard in the framework of a fractional Brownian motion with Hurst parameter H. Using Theorem 1.2(i), we deduce that $\hat{H}_n - H$ is of order $O_{\mathbb{P}}(n^{1/(1-\alpha)\beta-1})$ when $p \in (0, 1/2]$. Furthermore, Theorem 1.2(ii) shows that the order $O_{\mathbb{P}}(n^{1/(1-\alpha)\beta-1})$ can be improved to $O_{\mathbb{P}}(n^{-1/2})$ when the first-order increments are replaced by kth order increments, $k \geq 2$, in the definition of the statistic $R(n,p)$. However, obtaining confidence regions for H is a much more delicate issue, which will not be considered in this paper. In particular, the parameters of the limiting distribution need to be estimated.

3. Preliminaries. Throughout the following, sections all positive constants will be denoted by K, although they may change from line to line. Moreover, we will assume, without loss of generality, that $c_0 = \delta = \sigma = 1$. Recall that $g(t) = g_0(t) = 0$ for all $t < 0$ by assumption.

For a sequences of random variables $(Y_n)_{n\in\mathbb{N}}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we write $Y_n \xrightarrow{\mathcal{L}-s} Y$ if Y_n converges $\mathcal{F}$-stably in law to Y. That is, Y is a random variable defined on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$ such that for all $\mathcal{F}$-measurable random variables U we have the joint convergence in law $(Y_n, U) \xrightarrow{d} (Y, U)$. For $A \in \mathcal{F}$, we will say that $Y_n \xrightarrow{\mathcal{L}-s} Y$ on A, if $Y_n \xrightarrow{\mathcal{L}-s} Y$ under $\mathbb{P}_{|A}$, where $\mathbb{P}_{|A}$ denotes the conditionally probability measure $B \mapsto \mathbb{P}(B \cap A)/\mathbb{P}(A)$, when $\mathbb{P}(A) > 0$. We refer to the work [1, 33] for a detailed exposition of stable convergence. In addition, $\xrightarrow{\mathbb{P}}$ will denote convergence in probability. We will write $V(Y, p; k)_n = \sum_{i=k}^n |\Delta_{i,k}^n Y|^p$ when we want to stress that the power variation is built from a process Y. On the other hand, when k and p are fixed we will sometimes write $V(Y)_n = V(Y, p; k)_n$ to simplify the notation.

First of all, it follows from [32], Theorem 7, that the process X introduced in (1.1) is well defined if and only if, for all $t \geq 0$,

$$\int_{-t}^{\infty} \int_{\mathbb{R}} (|f_t(s)x|^2 \wedge 1)\nu(dx)\,ds < \infty, \tag{3.1}$$

where $f_t(s) = g(t+s) - g_0(s)$. By assumption (A) f_t is a bounded function in $L^\theta(\mathbb{R}_+)$. For all $\varepsilon > 0$, assumption (A) implies that

$$\int_{\mathbb{R}} (|yx|^2 \wedge 1)\nu(dx) \le K\big(\mathbb{1}_{\{|y|\le 1\}}|y|^\theta + \mathbb{1}_{\{|y|>1\}}|y|^{\beta+\varepsilon}\big),$$

which shows (3.1) since f_t is a bounded function in $L^\theta(\mathbb{R}_+)$. Now, for all $n, i \in \mathbb{N}$, we set for $x \in \mathbb{R}$

$$g_{i,n}(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} g\big((i-j)/n - x\big), \tag{3.2}$$

$$h_{i,n}(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \big((i-j)/n - x\big)_+^\alpha, \qquad g_n(x) = n^\alpha g(x/n). \tag{3.3}$$

In addition, for each function $\phi : \mathbb{R} \to \mathbb{R}$ define $D^k\phi : \mathbb{R} \to \mathbb{R}$ by

$$D^k\phi(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \phi(x-j). \tag{3.4}$$

In this notation, the function h_k, defined in (1.4), is given by $h_k = D^k\phi$ with $\phi : x \mapsto x_+^\alpha$.

LEMMA 3.1. *Assume that g satisfies condition* (A). *Then there exists a finite constant $K > 0$ such that, for all $n \ge 1$ and $i = k, \dots, n$,*

$$\big|g_{i,n}(x)\big| \le K(i/n - x)^\alpha, \qquad x \in \big[(i-k)/n, i/n\big], \tag{3.5}$$

$$\big|g_{i,n}(x)\big| \le Kn^{-k}\big((i-k)/n - x\big)^{\alpha-k}, \qquad x \in \big(i/n - 1, (i-k)/n\big), \tag{3.6}$$

$$\begin{aligned}\big|g_{i,n}(x)\big| \le Kn^{-k}\big(&\mathbb{1}_{[(i-k)/n-1, i/n-1]}(x) \\ &+ g^{(k)}\big((i-k)/n - x\big) \\ &\times \mathbb{1}_{(-\infty,(i-k)/n-1)}(x)\big), \qquad x \in (-\infty, i/n - 1].\end{aligned} \tag{3.7}$$

The same estimates trivially hold for the function $h_{i,n}$.

PROOF. Inequality (3.5) follows directly from condition (1.3) of (A). The second inequality (3.6) is a straightforward consequence of Taylor's expansion of order k and the condition $|g^{(k)}(t)| \le Kt^{\alpha-k}$ for $t \in (0,1)$. The third inequality (3.7) follows again through Taylor's expansion and the fact that the function $g^{(k)}$ is decreasing on $(1, \infty)$. $\square$

4. Proof of Theorem 1.1. In this section, we will prove the assertions of Theorem 1.1.

4.1. *Proof of Theorem* 1.1(i). The proof of Theorem 1.1(i) is divided into the following three steps. In Step (i), we show Theorem 1.1(i) for the compound Poisson case, which stands for the treatment of big jumps of L. Step (ii) consists of an approximating lemma, which proves that the small jumps of L are asymptotically negligible. Step (iii) combines the previous results to obtain the general theorem.

Before proceeding with the proof, we will need the following preliminary lemma. Let $\{x\} := x - \lfloor x \rfloor \in [0, 1)$ denote the fractional part of $x \in \mathbb{R}$. The lemma below follows along the lines of [18, 39].

LEMMA 4.1. *For $d \geq 1$, let $V = (V_1, \ldots, V_d)$ be an absolutely continuous random vector in $\mathbb{R}^d$ with a density $v : \mathbb{R}^d \to \mathbb{R}_+$. Suppose that there exists an open convex set $A \subseteq \mathbb{R}^d$ such that v is continuous differentiable on A and vanish outside A. Then, as $n \to \infty$,*

$$\big(\{nV_1\}, \ldots, \{nV_d\}\big) \xrightarrow{\mathcal{L}-s} U = (U_1, \ldots, U_d),$$

where $U_1, \ldots, U_d$ are independent $\mathcal{U}([0, 1])$-distributed random variables which are independent of $\mathcal{F}$.

STEP (i): THE COMPOUND POISSON CASE. Let $L = (L_t)_{t \in \mathbb{R}}$ be a compound Poisson process and let $0 \leq T_1 < T_2 < \cdots$ denote the jump times of the Lévy process $(L_t)_{t \geq 0}$ chosen in increasing order. Consider a fixed $\varepsilon > 0$ and let $n \in \mathbb{N}$ satisfy $\varepsilon n > 4k$. We define

$$\Omega_\varepsilon := \big\{\omega \in \Omega : \text{ for all } j \geq 1 \text{ with } T_j(\omega) \in [0, 1] \text{ we have } \big|T_{j+1}(\omega) - T_j(\omega)\big| > \varepsilon \\ \text{and } \Delta L_s(\omega) = 0 \text{ for all } s \in [-\varepsilon, \varepsilon] \cup [1 - \varepsilon, 1]\big\}.$$

Notice that $\mathbb{P}(\Omega_\varepsilon) \uparrow 1$ as $\varepsilon \downarrow 0$. Now, we decompose $\Delta_{i,k}^n X = M_{i,n,\varepsilon} + R_{i,n,\varepsilon}$ with

$$M_{i,n,\varepsilon} = \int_{\frac{i}{n}-\varepsilon}^{\frac{i}{n}} g_{i,n}(s)\, dL_s, \qquad R_{i,n,\varepsilon} = \int_{-\infty}^{\frac{i}{n}-\varepsilon} g_{i,n}(s)\, dL_s,$$

and the function $g_{i,n}$ is introduced in (3.2). The term $M_{i,n,\varepsilon}$ represents the dominating quantity, while $R_{i,n,\varepsilon}$ turns out to be negligible.

The dominating term: We claim that on Ω_ε and as $n \to \infty$,

$$n^{\alpha p} \sum_{i=k}^{n} |M_{i,n,\varepsilon}|^p \xrightarrow{\mathcal{L}-s} Z \qquad \text{where } Z = \sum_{m: T_m \in (0,1]} |\Delta L_{T_m}|^p V_m, \tag{4.1}$$

where V_m, $m \geq 1$, are defined in (1.5). To show (4.1), let $i_m = i_m(\omega, n)$ denote the random index such that $T_m \in ((i_m - 1)/n, i_m/n]$. The following representation will be crucial: On Ω_ε, we have that

$$V_{n,\varepsilon} := n^{\alpha p} \sum_{i=k}^{n} |M_{i,n,\varepsilon}|^p = n^{\alpha p} \sum_{m: T_m \in (0,1]} |\Delta L_{T_m}|^p \left(\sum_{l=0}^{[\varepsilon n]+v_m} \big|g_{i_m+l,n}(T_m)\big|^p \right) \tag{4.2}$$

for some random indexes $v_m = v_m(\omega, n, \varepsilon) \in \{-2, -1, 0\}$ which are measurable with respect to T_m. Indeed, on Ω_ε and for each $i = k, \dots, n$, L has at most one jump in $(i/n - \varepsilon/2, i/n]$. For each $m \in \mathbb{N}$ with $T_m \in (0, 1]$, we have $T_m \in (i/n - \varepsilon, i/n]$ if and only if $i_m \le i < n(T_m + \varepsilon)$ (recall that $\varepsilon n > 4k$). Thus,

$$\sum_{i \in \{k,\dots,n\}: T_m \in (i/n - \varepsilon, i/n]} |M_{i,n,\varepsilon}|^p = |\Delta L_{T_m}|^p \left(\sum_{l=0}^{[\varepsilon n] + v_m} \left| g_{i_m + l, n}(T_m) \right|^p \right) \tag{4.3}$$

for some T_m-measurable random variable $v_m \in \{-2, -1, 0\}$. Thus, by summing (4.3) over all $m \in \mathbb{N}$ with $T_m \in (0, 1]$, (4.2) follows. In the following, we will show that

$$V_{n,\varepsilon} \xrightarrow{\mathcal{L}-s} Z \qquad \text{as } n \to \infty.$$

For $d \ge 1$, it is well known that the random vector $(T_1, \dots, T_d)$ is absolutely continuous with a C^1-density on the open convex set $A := \{(x_1, \dots, x_d) \in \mathbb{R}^d : 0 < x_1 < x_2 < \cdots < x_d\}$, which is vanishing outside A. Thus, by Lemma 4.1 we have

$$\big(\{nT_m\}\big)_{m \le d} \xrightarrow{\mathcal{L}-s} (U_m)_{m \le d} \qquad \text{as } n \to \infty, \tag{4.4}$$

where $(U_i)_{i \in \mathbb{N}}$ are i.i.d. $\mathcal{U}([0, 1])$-distributed random variables. By (1.3), we may write $g(x) = x_+^\alpha f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x) \to 1$ as $x \downarrow 0$. By definition of i_m, we have that $\{nT_m\} = nT_m - (i_m - 1)$ and, therefore, for all $l = 0, 1, 2, \dots$ and $j = 0, \dots, k$,

$$n^\alpha g\left(\frac{l + i_m - j}{n} - T_m \right) = \big(l - j + 1 - \{nT_m\}\big)_+^\alpha f\left(\frac{l - j}{n} + n^{-1}\big(1 - \{nT_m\}\big) \right).$$

By (4.4), $(U_m)_{m \le d} \overset{d}{=} (1 - U_m)_{m \le d}$ and $f(x) \to 1$ as $x \downarrow 0$ we obtain that

$$\left\{ n^\alpha g\left(\frac{l + i_m - j}{n} - T_m \right) \right\}_{l,m \le d} \xrightarrow{\mathcal{L}-s} \big\{ (l - j + U_m)_+^\alpha \big\}_{l,m \le d} \qquad \text{as } n \to \infty. \tag{4.5}$$

Equation (4.5) implies that

$$\big\{ n^\alpha g_{i_m + l, n}(T_m) \big\}_{l,m \le d} \xrightarrow{\mathcal{L}-s} \big\{ h_k(l + U_m) \big\}_{l,m \le d}, \tag{4.6}$$

with h_k being defined at (1.4). Due to the $\mathcal{F}$-stable convergence in (4.6), we obtain by the continuous mapping theorem that for each fixed $d \ge 1$ and as $n \to \infty$,

$$\begin{aligned} V_{n,\varepsilon,d} &:= n^{\alpha p} \sum_{m: m \le d, T_m \in [0,1]} |\Delta L_{T_m}|^p \left(\sum_{l=0}^{[\varepsilon d] + v_m} \left| g_{i_m + l, n}(T_m) \right|^p \right) \\ &\xrightarrow{\mathcal{L}-s} Z_d = \sum_{m: m \le d, T_m \in [0,1]} |\Delta L_{T_m}|^p \left(\sum_{l=0}^{[\varepsilon d] + v_m} \left| h_k(l + U_m) \right|^p \right). \end{aligned}$$

Moreover, for $\omega \in \Omega$ we have $Z_d(\omega) \uparrow Z(\omega)$ as $d \to \infty$. Recall that $|h_k(x)| \le K(x - k)^{\alpha - k}$ for $x > k + 1$, which implies that $Z < \infty$ a.s. since $p(\alpha - k) < -1$.

For all $l \in \mathbb{N}$ with $k \le l \le n$, we have $n^{\alpha p}|g_{i_m+l,n}(T_m)|^p \le K|l-k|^{(\alpha-k)p}$ due to (3.6) of Lemma 3.1. For all $d \ge 0$, set $C_d = \sum_{m>d: T_m \in [0,1]} |\Delta L_{T_m}|^p$ and note that $C_d \to 0$ a.s. as $d \to \infty$ since L is a compound Poisson process. Hence, we deduce

$$|V_{n,\varepsilon} - V_{n,\varepsilon,d}| \le K\left(C_d + C_0 \sum_{l=[\varepsilon d]-1}^{\infty} |l-k|^{p(\alpha-k)}\right) \to 0 \qquad \text{as } d \to \infty$$

since $p(\alpha - k) < -1$. Due to the fact that $n^{\alpha p} \sum_{i=k}^n |M_{i,n,\varepsilon}|^p = V_{n,\varepsilon}$ a.s. on Ω_ε and $V_{n,\varepsilon} \xrightarrow{\mathcal{L}-s} Z$, it follows that $n^{\alpha p} \sum_{i=k}^n |M_{i,n,\varepsilon}|^p \xrightarrow{\mathcal{L}-s} Z$ on Ω_ε, since $\Omega_\varepsilon \in \mathcal{F}$. This proves (4.1).

The rest term: In the following, we will show that

$$n^{\alpha p} \sum_{i=k}^{n} |R_{i,n,\varepsilon}|^p \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty. \tag{4.7}$$

The fact that the random variables in (4.7) are usually not integrable makes the proof of (4.7) considerably more complicated. Similar to (3.7) of Lemma 3.1, we have that

$$n^k |g_{i,n}(s)| \mathbb{1}_{\{s \le i/n - \varepsilon\}} \le K\big(\mathbb{1}_{\{s \in [-1,1]\}} + \mathbb{1}_{\{s<-1\}} |g^{(k)}(-s)|\big) =: \psi(s),$$

where $K = K_\varepsilon$. We will use the function ψ several times in the proof of (4.7), which will be divided into the two special cases $\theta \in (0,1]$ and $\theta \in (1,2]$. Suppose first that $\theta \in (0,1]$. To show (4.7), it suffices to prove that

$$\sup_{n \in \mathbb{N}, i \in \{k,\dots,n\}} n^k |R_{i,n,\varepsilon}| < \infty \qquad \text{a.s.} \tag{4.8}$$

since $\alpha < k - 1/p$. To show (4.8), we will first prove that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \big(|\psi(s)x| \wedge 1\big) \nu(dx)\, ds < \infty. \tag{4.9}$$

Choose $\tilde{K}$ such that $\psi(x) \le \tilde{K}$ for all $x \in \mathbb{R}$. For $u \in [-\tilde{K}, \tilde{K}]$, we have that

$$\begin{aligned} \int_{\mathbb{R}} \big(|ux| \wedge 1\big) \nu(dx) &\le K \int_1^\infty \big(|xu| \wedge 1\big) x^{-1-\theta}\, dx \\ &\le \begin{cases} K|u|^\theta, & \theta \in (0,1), \\ K|u|^\theta \log(1/u), & \theta = 1, \end{cases} \end{aligned} \tag{4.10}$$

where we have used that $\theta \le 1$. By (4.10) applied to $u = \psi(s)$ and assumption (A), it follows that (4.9) is satisfied. Since L is a symmetric compound Poisson process, we can find a Poisson random measure μ with mean measure $\lambda \otimes \nu$ such that for all $-\infty < u < t < \infty$, $L_t - L_u = \int_{(u,t] \times \mathbb{R}} x \mu(ds, dx)$. Due to [26], Theorem 10.15, (4.9) ensures the existence of the stochastic integral

$\int_{\mathbb{R}\times\mathbb{R}} |\psi(s)x|\mu(ds,dx)$. Moreover, $\int_{\mathbb{R}\times\mathbb{R}} |\psi(s)x|\mu(ds,dx)$ can be regarded as an ω by ω integral with respect to the measure μ_ω. Now, we have that

$$\begin{aligned} |n^k R_{i,n,\varepsilon}| &\le \int_{(-\infty,i/n-\varepsilon]\times\mathbb{R}} |n^k g_{i,n}(s)x|\mu(ds,dx) \\ &\le \int_{\mathbb{R}\times\mathbb{R}} |\psi(s)x|\mu(ds,dx) < \infty, \end{aligned} \tag{4.11}$$

which shows (4.8), since the right-hand side of (4.11) does not depend on i and n.

Suppose that $\theta \in (1,2]$. Similar as before, it suffices to show that

$$\sup_{n\in\mathbb{N}, i\in\{k,\dots,n\}} \frac{n^k|R_{i,n,\varepsilon}|}{(\log n)^{1/q}} < \infty \qquad \text{a.s.,} \tag{4.12}$$

where $q > 1$ denotes the conjugated number to $\theta > 1$ determined by $1/\theta + 1/q = 1$. In the following, we will show (4.12) using the majorizing measure techniques developed in [29]. In fact, our arguments are closely related to their Section 4.2. Set $T = \{(i,n) : n \ge k, i = k,\dots,n\}$. For $(i,n) \in T$, we have

$$\frac{n^k|R_{i,n,\varepsilon}|}{(\log n)^{1/q}} = \left|\int_{\mathbb{R}} \zeta_{i,n}(s)\,dL_s\right|, \qquad \zeta_{i,n}(s) := \frac{n^k}{(\log n)^{1/q}} g_{i,n}(s)\mathbb{1}_{\{s\le i/n-\varepsilon\}}.$$

For $t = (i,n) \in T$, we will sometimes write $\zeta_t(s)$ for $\zeta_{i,n}(s)$. Let $\tau : T\times T \to \mathbb{R}_+$ denote the metric given by

$$\begin{aligned} &\tau\big((i,n),(j,m)\big) \\ &\quad = \begin{cases} \log(n-k+1)^{-1/q} + \log(m-k+1)^{-1/q}, & (i,n) \ne (j,l), \\ 0, & (i,n) = (j,l). \end{cases} \end{aligned}$$

Moreover, let m be the probability measure on T given by $m(\{(i,n)\}) = Kn^{-3}$ for a suitable constant $K > 0$. Set $B_\tau(t,r) = \{s \in T : \tau(s,t) \le r\}$ for $t \in T$, $r > 0$, $D = \sup\{\tau(s,t) : s,t \in T\}$ and

$$I_q(m,\tau;D) = \sup_{t\in T} \int_0^D \left(\log \frac{1}{m(B_\tau(t,r))}\right)^{1/q} dr.$$

In the following, we will show that m is a so-called majorizing measure, which means that $I_q(m,\tau,D) < \infty$. For $r < (\log(n-k+1))^{-1/q}$, we have $B_\tau((i,n),r) = \{(i,n)\}$. Therefore, $m(B_\tau((i,n),r)) = Kn^{-3}$ and

$$\begin{aligned} &\int_0^{(\log(n-k+1))^{-1/q}} \left(\log \frac{1}{m(B_\tau((i,n),r))}\right)^{1/q} dr \\ &\quad = \int_0^{(\log(n-k+1))^{-1/q}} (3\log n + \log K)^{1/q}\,dr. \end{aligned} \tag{4.13}$$

For all $r \geq (\log(n-k+1))^{-1/q}$, $(k,k) \in B_\tau((i,n),r)$, and hence $m(B_\tau((i,n),r)) \geq m(\{(k,k)\}) = K(k+1)^{-3}$. Therefore,

$$\begin{aligned}\int_{(\log(n-k+1))^{-1/q}}^{D} \left(\log \frac{1}{m(B_\tau((i,n),r))}\right)^{1/q} dr \\ \leq \int_{(\log(n-k+1))^{-1/q}}^{D} \big(3\log(k+1) + \log K\big)^{1/q} dr.\end{aligned} \tag{4.14}$$

By (4.13) and (4.14), it follows that $I_q(m,\tau,D) < \infty$. For $(i,n) \neq (j,l)$, we have that

$$\begin{aligned}\frac{|\zeta_{i,n}(s) - \zeta_{j,l}(s)|}{\tau((i,n),(j,l))} &\leq n^k |g_{i,n}(s)| \mathbb{1}_{\{s \leq i/n - \varepsilon\}} + l^k |g_{j,l}(s)| \mathbb{1}_{\{s \leq j/l - \varepsilon\}} \\ &\leq K\psi(s).\end{aligned} \tag{4.15}$$

For fixed $t_0 \in T$, we let $\|\zeta\|_\tau(s) = D^{-1}|\zeta_{t_0}(s)| + \sup_{t_1,t_2 \in T:\tau(t_1,t_2)\neq 0} |\zeta_{t_1}(s) - \zeta_{t_2}(s)|/\tau(t_1,t_2)$ be a Lipschitz type norm on T. By (4.15), it follows that $\|\zeta\|_\tau(s) \leq K\psi(s)$, and hence

$$\int_{\mathbb{R}} \|\zeta\|_\tau^\theta(s)\, ds \leq K\left(2 + \int_1^\infty |g^{(k)}(s)|^\theta ds\right) < \infty. \tag{4.16}$$

By [29], Theorem 3.1, Equation (3.11), together with $I_q(m,\tau,D) < \infty$ and (4.16) we deduce (4.12), which completes the proof of (4.7).

End of the proof: Recall the decomposition $\Delta_{i,n}^n X = M_{i,n,\varepsilon} + R_{i,n,\varepsilon}$. Equation (4.1), (4.7) and an application of Minkowski inequality yield that

$$n^{\alpha p} V(p;k)_n \xrightarrow{\mathcal{L}-s} Z \qquad \text{on } \Omega_\varepsilon \text{ as } n \to \infty. \tag{4.17}$$

Since $\mathbb{P}(\Omega_\varepsilon) \uparrow 1$ as $\varepsilon \downarrow 0$, (4.17) implies that $n^{\alpha p} V(p;k)_n \xrightarrow{\mathcal{L}-s} Z$. We have now completed the proof for a particular choice of stopping times $(T_m)_{m\geq 1}$. However, the result remains valid for any choice of $\mathbb{F}$-stopping times, since the distribution of Z is invariant with respect to reordering of stopping times. □

Step (ii): *An approximation*. To prove Theorem 1.1(i) in the general case, we need the following approximation result. Consider a general symmetric Lévy process $L = (L_t)_{t\in\mathbb{R}}$ as in Theorem 1.1(i) and let N be the corresponding Poisson random measure $N(A) := \sharp\{t : (t, \Delta L_t) \in A\}$ for all measurable $A \subseteq \mathbb{R} \times (\mathbb{R} \setminus \{0\})$. By our assumptions (in particular, by symmetry), the process $X(j)$ given by

$$X_t(j) = \int_{(-\infty,t]\times[-\frac{1}{j},\frac{1}{j}]} \big\{\big(g(t-s) - g_0(-s)\big)x\big\} N(ds,dx) \tag{4.18}$$

is well defined. The following estimate on the processes $X(j)$ will be crucial.

LEMMA 4.2. *Suppose that $\alpha < k - 1/p$ and $\beta < p$. Then*

$$\lim_{j\to\infty} \limsup_{n\to\infty} \mathbb{P}\big(n^{\alpha p} V\big(X(j)\big)_n > \varepsilon\big) = 0 \qquad \textit{for all } \varepsilon > 0.$$

PROOF. By Markov's inequality and the stationary increments of $X(j)$, we have that

$$\begin{aligned}
&\mathbb{P}\big(n^{\alpha p} V\big(X(j)\big)_n > \varepsilon\big) \\
&\quad \le \varepsilon^{-1} n^{\alpha p} \sum_{i=k}^{n} \mathbb{E}\big[\big|\Delta_{i,k}^n X(j)\big|^p\big] \le \varepsilon^{-1} n^{\alpha p+1} \mathbb{E}\big[\big|\Delta_{k,k}^n X(j)\big|^p\big].
\end{aligned}$$

Hence, it is enough to show that

$$\lim_{j\to\infty} \limsup_{n\to\infty} \mathbb{E}\big[|Y_{n,j}|^p\big] = 0 \qquad \text{with } Y_{n,j} := n^{\alpha+1/p} \Delta_{k,k}^n X(j). \tag{4.19}$$

To show (4.19), it suffices to prove that

$$\begin{aligned}
\lim_{j\to\infty} \limsup_{n\to\infty} \xi_{n,j} = 0 \qquad \text{where } \xi_{n,j} = \int_{|x|\le 1/j} \chi_n(x)\nu(dx) \quad \text{and} \\
\chi_n(x) = \int_{-\infty}^{k/n} \big(\big|n^{\alpha+1/p} g_{k,n}(s)x\big|^p \mathbb{1}_{\{|n^{\alpha+1/p} g_{k,n}(s)x|\ge 1\}} \\
+ \big|n^{\alpha+1/p} g_{k,n}(s)x\big|^2 \mathbb{1}_{\{|n^{\alpha+1/p} g_{k,n}(s)x|\le 1\}}\big)\, ds,
\end{aligned}$$

which follows from the representation

$$Y_{n,j} = \int_{(-\infty,k/n]\times[-1/j,1/j]} \big(n^{\alpha+1/p} g_{k,n}(s)x\big) N(ds,dx)$$

and by [32], Theorem 3.3 and the remarks above it. Suppose for the moment that there exists a finite constant $K > 0$ such that

$$\chi_n(x) \le K\big(|x|^p + x^2\big) \qquad \text{for all } x \in [-1,1]. \tag{4.20}$$

Then

$$\limsup_{j\to\infty} \Big\{\limsup_{n\to\infty} \xi_{n,j}\Big\} \le K \limsup_{j\to\infty} \int_{|x|\le 1/j} \big(|x|^p + x^2\big)\nu(dx) = 0$$

since $p > \beta$. Hence, it suffices to show the estimate (4.20), which we will do in the following.

Let $\Phi_p : \mathbb{R} \to \mathbb{R}_+$ denote the function $\Phi_p(y) = |y|^2 \mathbb{1}_{\{|y|\le 1\}} + |y|^p \mathbb{1}_{\{|y|>1\}}$. We split χ_n into the following three terms which need different treatments:

$$\begin{aligned}
\chi_n(x) = \int_{-k/n}^{k/n} \Phi_p\big(n^{\alpha+1/p} g_{k,n}(s)x\big)\, ds + \int_{-1}^{-k/n} \Phi_p\big(n^{\alpha+1/p} g_{k,n}(s)x\big)\, ds \\
+ \int_{-\infty}^{-1} \Phi_p\big(n^{\alpha+1/p} g_{k,n}(s)x\big)\, ds =: I_{1,n}(x) + I_{2,n}(x) + I_{3,n}(x).
\end{aligned}$$

Estimation of $I_{1,n}$: By (3.5) of Lemma 3.1, we have that

$$|g_{k,n}(s)| \le K(k/n-s)^{\alpha}, \qquad s\in[-k/n,k/n]. \tag{4.21}$$

Since Φ_p is increasing on $\mathbb{R}_+$, (4.21) implies that

$$I_{1,n}(x) \le K\int_0^{2k/n} \Phi_p\big(xn^{\alpha+1/p}s^{\alpha}\big)\,ds. \tag{4.22}$$

By basic calculus, it follows that

$$\begin{aligned}
&\int_0^{2k/n} |xn^{\alpha+1/p}s^{\alpha}|^2 \mathbb{1}_{\{|xn^{\alpha+1/p}s^{\alpha}|\le 1\}}\,ds\\
&\quad\le K\big(\mathbb{1}_{\{|x|\le(2k)^{-\alpha}n^{-1/p}\}}x^2n^{2/p-1} + \mathbb{1}_{\{|x|>(2k)^{-\alpha}n^{-1/p}\}}|x|^{-1/\alpha}n^{-1-1/(\alpha p)}\big)\\
&\quad\le K\big(|x|^p+x^2\big).
\end{aligned} \tag{4.23}$$

Moreover,

$$\begin{aligned}
&\int_0^{2k/n} |xn^{\alpha+1/p}s^{\alpha}|^p \mathbb{1}_{\{|xn^{\alpha+1/p}s^{\alpha}|>1\}}\,ds\\
&\qquad\le \int_0^{2k/n}|xn^{\alpha+1/p}s^{\alpha}|^p\,ds \le K|x|^p.
\end{aligned} \tag{4.24}$$

By combining (4.22), (4.23) and (4.24), we obtain the estimate $I_{1,n}(x)\le K(|x|^p+x^2)$.

Estimation of $I_{2,n}$: By (3.6) of Lemma 3.1, it holds that

$$|g_{k,n}(s)| \le Kn^{-k}|s|^{\alpha-k}, \qquad s\in(-1,-k/n). \tag{4.25}$$

Again, due to the fact that Φ_p is increasing on $\mathbb{R}_+$, (4.25) implies that

$$I_{2,n}(x)\le K\int_{k/n}^1 \Phi_p\big(xn^{\alpha+1/p-k}s^{\alpha-k}\big)\,ds. \tag{4.26}$$

For $\alpha\neq k-1/2$, we have

$$\begin{aligned}
&\int_{k/n}^1 |xn^{\alpha+1/p-k}s^{\alpha-k}|^2\mathbb{1}_{\{|xn^{\alpha+1/p-k}s^{\alpha-k}|\le 1\}}\,ds\\
&\quad\le K\big(x^2n^{2(\alpha+1/p-k)} + \mathbb{1}_{\{|x|\le n^{-1/p}k^{-(\alpha-k)}\}}|x|^2n^{2/p-1}\\
&\qquad + \mathbb{1}_{\{|x|>n^{-1/p}k^{-(\alpha-k)}\}}|x|^{1/(k-\alpha)}n^{1/(p(k-\alpha))-1}\big)\\
&\quad\le K\big(x^2+|x|^p\big),
\end{aligned} \tag{4.27}$$

where we have used that $\alpha<k-1/p$. For $\alpha=k-1/2$, we have

$$\begin{aligned}
&\int_{\frac{k}{n}}^1 |xn^{\alpha+1/p-k}s^{\alpha-k}|^2\mathbb{1}_{\{|xn^{\alpha+1/p-k}s^{\alpha-k}|\le 1\}}\,ds\\
&\qquad\le x^2n^{2(\alpha+1/p-k)}\int_{\frac{k}{n}}^1 s^{-1}\,ds \le Kx^2,
\end{aligned} \tag{4.28}$$

where we again have used $\alpha < k - 1/p$ in the last inequality. Moreover,

$$\int_{k/n}^{1} \left|xn^{\alpha+1/p-k}s^{\alpha-k}\right|^p \mathbb{1}_{\{|xn^{\alpha+1/p-k}s^{\alpha-k}|>1\}}\, ds \le K|x|^p. \tag{4.29}$$

By (4.26), (4.27), (4.28) and (4.29), we obtain the estimate $I_{2,n}(x) \le K(|x|^p + x^2)$.

Estimation of $I_{3,n}$: For $s < -1$, we have that $|g_{k,n}(s)| \le Kn^{-k}|g^{(k)}(-k/n - s)|$, by (3.7) of Lemma 3.1, and hence

$$I_{3,n}(x) \le K \int_1^\infty \Phi_p\big(n^{\alpha+1/p-k}g^{(k)}(s)\big)\, ds. \tag{4.30}$$

We have that

$$\begin{aligned} &\int_1^\infty \left|xn^{\alpha+1/p-k}g^{(k)}(s)\right|^2 \mathbb{1}_{\{|xn^{\alpha+1/p-k}g^{(k)}(s)|\le 1\}}\, ds \\ &\quad \le x^2 n^{2(\alpha+1/p-k)} \int_1^\infty \left|g^{(k)}(s)\right|^2 ds. \end{aligned} \tag{4.31}$$

Since $|g^{(k)}|$ is decreasing on $(1,\infty)$ and $g^{(k)} \in L^\theta((1,\infty))$ for some $\theta \le 2$, the integral on the right-hand side of (4.31) is finite. For $x \in [-1, 1]$, we have

$$\begin{aligned} &\int_1^\infty \left|xn^{\alpha+1/p-k}g^{(k)}(s)\right|^p \mathbb{1}_{\{|xn^{\alpha+1/p-k}g^{(k)}(s)|>1\}}\, ds \\ &\quad \le |x|^p n^{p(\alpha+1/p-k)} \int_1^\infty \left|g^{(k)}(s)\right|^p \mathbb{1}_{\{|g^{(k)}(s)|>1\}}\, ds. \end{aligned} \tag{4.32}$$

From our assumptions, it follows that the integral in (4.32) is finite. By (4.30), (4.31) and (4.32), we have that $I_{3,n}(x) \le K(|x|^p + x^2)$ for all $x \in [-1, 1]$, which completes the proof of (4.20) and, therefore, also the proof of the lemma. □

Step (iii): *The general case*. In the following, we will prove Theorem 1.1(i) in the general case by combining the above Steps (i) and (ii).

PROOF OF THEOREM 1.1(i). Let $(T_m)_{m\ge1}$ be a sequence of $\mathbb{F}$-stopping times that exhausts the jumps of $(L_t)_{t\ge0}$. For each $j \in \mathbb{N}$, let $\hat{L}(j)$ be the Lévy process given by

$$\hat{L}_t(j) - \hat{L}_s(j) = \sum_{u\in(s,t]} \Delta L_u \mathbb{1}_{\{|\Delta L_u|>\frac{1}{j}\}}, \qquad s < t,$$

and set

$$\hat{X}_t(j) = \int_{-\infty}^t \big(g(t-s) - g_0(-s)\big)\, d\hat{L}_s(j).$$

Moreover, set $T_{m,j} = T_m$ when $|\Delta L_{T_m}| > 1/j$ and $T_{m,j} = \infty$ else. Note that $(T_{m,j})_{m\ge1}$ is a sequence of $\mathbb{F}$-stopping times that exhausts the jumps of $(\hat{L}_t(j))_{t\ge0}$.

Since $\hat{L}(j)$ is a compound Poisson process, Step (i) shows that

$$n^{\alpha p} V(\hat{X}(j))_n \xrightarrow{\mathcal{L}-s} Z_j := \sum_{m:T_{m,j}\in[0,1]} |\Delta \hat{L}_{T_{m,j}}(j)|^p V_m \qquad \text{as } n\to\infty, \tag{4.33}$$

where V_m, $m\geq 1$, are defined in (1.5). By definition of $T_{m,j}$ and monotone convergence we have, as $j\to\infty$,

$$Z_j = \sum_{m:T_m\in[0,1]} |\Delta L_{T_m}|^p V_m \mathbb{1}_{\{|\Delta L_{T_m}|>\frac{1}{j}\}} \xrightarrow{\text{a.s.}} \sum_{m:T_m\in[0,1]} |\Delta L_{T_m}|^p V_m =: Z. \tag{4.34}$$

Suppose first that $p\geq 1$ and decompose

$$\begin{aligned}(n^{\alpha p} V(X)_n)^{1/p} &= (n^{\alpha p} V(\hat{X}(j))_n)^{1/p} + ((n^{\alpha p} V(X)_n)^{1/p} - (n^{\alpha p} V(\hat{X}(j))_n)^{1/p}) \\ &=: Y_{n,j} + U_{n,j}.\end{aligned}$$

Equations (4.33) and (4.34) show

$$Y_{n,j} \xrightarrow[n\to\infty]{\mathcal{L}-s} Z_j^{1/p} \quad \text{and} \quad Z_j^{1/p} \xrightarrow[j\to\infty]{\mathbb{P}} Z^{1/p}. \tag{4.35}$$

Note that $X - \hat{X}(j) = X(j)$, where $X(j)$ is defined in (4.18). For all $\varepsilon>0$, we have by Minkowski's inequality

$$\begin{aligned}&\limsup_{j\to\infty}\limsup_{n\to\infty}\mathbb{P}(|U_{n,j}|>\varepsilon) \\ &\quad\leq \limsup_{j\to\infty}\limsup_{n\to\infty}\mathbb{P}(n^{\alpha p} V(X(j))_n > \varepsilon^p) = 0,\end{aligned} \tag{4.36}$$

where the last equality follows by Lemma 4.2. By a standard argument (see, e.g., [12], Theorem 3.2), (4.35) and (4.36) implies that $(n^{\alpha p} V(X)_n)^{1/p} \xrightarrow{\mathcal{L}-s} Z^{1/p}$ which completes the proof of Theorem 1.1(i) when $p\geq 1$. For $p<1$, Theorem 1.1(i) follows by (4.33), (4.34), the inequality $|V(X)_n - V(\hat{X}(j))_n| \leq V(X(j))_n$ and [12], Theorem 3.2. □

4.2. *Proof of Theorem* 1.1(ii). Suppose that $\alpha<k-1/\beta$, $p<\beta$ and L is a symmetric β-stable Lévy process. In the proof of Theorem 1.1(ii), we will use the following notation: For all $n\geq 1$, $r\geq 0$ set

$$\phi_r^n(s) = D^k g_n(r-s), \qquad \phi_r^\infty(s) = h_k(r-s), \tag{4.37}$$

where g_n and D^k are defined at (3.3) and (3.4), and the function h_k is defined in (1.4). For all $n\in\mathbb{N}\cup\{\infty\}$ and $t\geq 0$, set

$$Y_t^n = \int_{-\infty}^t \phi_t^n(s)\,dL_s. \tag{4.38}$$

By self-similarity of L of index $1/\beta$, we have for all $n\in\mathbb{N}$,

$$\{n^{\alpha+1/\beta}\Delta_{i,k}^n X : i=k,\dots,n\} \stackrel{d}{=} \{Y_i^n : i=k,\dots,n\}, \tag{4.39}$$

where $\overset{d}{=}$ means equality in distribution. For $\alpha < 1 - 1/\beta$, Y^∞ is the k-order increments of a linear fractional stable motion. For $\alpha \geq 1 - 1/\beta$, the linear fractional stable motion is not well defined, but Y^∞ is well defined since the function h_k is locally bounded and satisfies $|h_k(x)| \leq Kx^{\alpha-k}$ for all $x \geq k+1$, which implies that $h_k \in L^\beta(\mathbb{R})$. In the following, we will prove Theorem 1.1(ii) by approximating Y_t^n by Y_t^∞ and applying the ergodic properties of Y_t^∞.

To show that $Y_k^n \to Y_k^\infty$ in L^p as $n \to \infty$, we will use the fact that for any deterministic function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\varphi \in L^\beta(\mathbb{R})$, $\int_\mathbb{R} \varphi(s)\, dL_s$ is a symmetric β-stable random variable with scale parameter $\|\varphi\|_{L^\beta(\mathbb{R})}$, that is, for all $u \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(iu \int_\mathbb{R} \varphi(s)\, dL_s\right)\right] = \exp\left(-|u|^\beta \int_\mathbb{R} |\varphi(s)|^\beta\, ds\right). \tag{4.40}$$

We recall that for $\phi : s \mapsto s_+^\alpha$ we have $D^k\phi = h_k \in L^\beta(\mathbb{R})$ and $c_0 = 1$ by assumption. For all $s \in \mathbb{R}$, we let $\psi_n(s) = g_n(s) - s_+^\alpha$. By the scaling properties of β-stable random variables, we have for all $p < \beta$ that

$$\mathbb{E}\big[|Y_k^n - Y_k^\infty|^p\big] = K\left(\int_0^\infty |D^k\psi_n(s)|^\beta\, ds\right)^{p/\beta}. \tag{4.41}$$

To show that the right-hand side of (4.41) converges to zero we note that

$$\begin{aligned}\int_{n+k}^\infty |D^k g_n(s)|^\beta\, ds &\leq Kn^{\beta(\alpha-k)} \int_{n+k}^\infty \big|g^{(k)}\big((s-k)/n\big)\big|^\beta\, ds \\ &= Kn^{\beta(\alpha-k)+1} \int_1^\infty \big|g^{(k)}(s)\big|^\beta\, ds \to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

This implies that

$$\begin{aligned}&\int_{n+k}^\infty |D^k\psi_n(s)|^\beta\, ds \\ &\quad\leq K\left(\int_{n+k}^\infty |D^k g_n(s)|^\beta\, ds + \int_{n+k}^\infty |D^k\phi(s)|^\beta\, ds\right) \xrightarrow[n\to\infty]{} 0.\end{aligned} \tag{4.42}$$

By (3.6) of Lemma 3.1, it holds that $|D^k g_n(s)| \leq K(s-k)^{\alpha-k}$ for $s \in (k+1, n)$. Therefore, for $s \in (0, n]$ we have

$$|D^k\psi_n(s)| \leq K\big(\mathbb{1}_{\{s\leq k+1\}} + \mathbb{1}_{\{s>k+1\}}(s-k)^{\alpha-k}\big), \tag{4.43}$$

where the function on the right-hand side of (4.43) is in $L^\beta(\mathbb{R}_+)$. For fixed $s \geq 0$, $\psi_n(s) \to 0$ as $n \to \infty$ by assumption (1.3), and hence $D^k\psi_n(s) \to 0$ as $n \to \infty$. By (4.43) and the dominated convergence theorem, this shows that $\int_0^n |D^k\psi_n(s)|^\beta\, ds \to 0$. Hence, by (4.41) and (4.42) we have

$$\mathbb{E}\big[|Y_k^n - Y_k^\infty|^p\big] \to 0 \qquad \text{as } n \to \infty, \tag{4.44}$$

which implies that

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=k}^{n}\left|Y_i^n - Y_i^\infty\right|^p\right] = \frac{1}{n}\sum_{i=k}^{n}\mathbb{E}\left[\left|Y_i^n - Y_i^\infty\right|^p\right] \le \mathbb{E}\left[\left|Y_k^n - Y_k^\infty\right|^p\right] \to 0 \tag{4.45}$$

as $n \to \infty$. Moreover, $(Y_t^\infty)_{t\in\mathbb{R}}$ is mixing since it is a symmetric stable moving average; see, for example, [14]. This implies, in particular, that the discrete time stationary sequence $\{Y_j\}_{j\in\mathbb{Z}}$ is mixing, and hence ergodic. According to Birkhoff's ergodic theorem (cf. [26], Theorem 10.6),

$$\frac{1}{n}\sum_{i=k}^{n}\left|Y_i^\infty\right|^p \xrightarrow{\text{a.s.}} \mathbb{E}\left[\left|Y_k^\infty\right|^p\right] \in (0,\infty) \qquad \text{as } n \to \infty. \tag{4.46}$$

We note that the expectation $\mathbb{E}[|Y_k^\infty|^p]$ at (4.46) coincides with the definition of m_p in Theorem 1.1(ii); cf. [34], Property 1.2.17 and 3.2.2. By (4.45), Minkowski's inequality and (4.46), we deduce $n^{-1}\sum_{i=k}^{n}|Y_i^n|^p \xrightarrow{\mathbb{P}} m_p$ as $n \to \infty$. By (4.39), it follows that

$$n^{-1+p(\alpha+1/\beta)}V(X)_n = \frac{1}{n}\sum_{i=k}^{n}\left|n^{\alpha+1/\beta}\Delta_{i,k}^n X\right|^p \stackrel{d}{=} \frac{1}{n}\sum_{i=k}^{n}\left|Y_i^n\right|^p \xrightarrow{\mathbb{P}} m_p$$

as $n \to \infty$. This completes the proof of Theorem 1.1(ii).

4.3. *Proof of Theorem* 1.1(iii). We will derive Theorem 1.1(iii) from the two lemmas below. For $k \in \mathbb{N}$ and $p \in [1,\infty)$, let $W^{k,p}$ denote the Wiener space of functions $\zeta : [0,1] \to \mathbb{R}$ which are k-times absolutely continuous with $\zeta^{(k)} \in L^p([0,1])$ where $\zeta^{(k)}(t) = \partial^k\zeta(t)/\partial t^k$ for Lebesgue a.e. $t \in [0,1]$. We recall that a function $\zeta : [0,1] \to \mathbb{R}$ is absolutely continuous if there exists an integrable function κ such that for all $t \in [0,1]$ we have

$$\zeta(t) = \zeta(0) + \int_0^t \kappa(s)\,ds, \tag{4.47}$$

and in this case, ζ is differentiable Lebesgue a.e. with $\zeta' = \kappa$ a.e. A function ζ is said to be two times absolutely continuous if ζ is absolutely continuous and κ in (4.47) can be chosen absolutely continuous. Similarly, we define k-times absolutely continuity. First we will show that, under the conditions of Theorem 1.1(iii), $X \in W^{k,p}$ almost surely.

LEMMA 4.3. *Suppose that $p \neq \theta$, $p \ge 1$ and* (A) *holds. If $\alpha > k - 1/(p \vee \beta)$, then*

$$X \in W^{k,p} \quad \textit{a.s.} \quad \textit{and} \quad \frac{\partial^k}{\partial t^k}X_t = \int_{-\infty}^{t} g^{(k)}(t-s)\,dL_s \qquad \lambda \otimes \mathbb{P}\text{-}\textit{a.s.} \tag{4.48}$$

Equation (4.48) *remains valid for $p = \theta$ if, in addition,* (A-log) *holds.*

PROOF. We will not need the assumption (1.3) on g in the proof. For notation simplicity, we only consider the case $k = 1$, since the general case follows by similar arguments. To prove (4.48), it is sufficient to show that the three conditions (5.3), (5.4) and (5.6) from [13], Theorem 5.1, are satisfied (this result uses the condition $p \geq 1$). In fact, the representation (4.48) of $(\partial/\partial t)X_t$ follows by the equation below (5.10) in [13]. In our setting, the function $\dot{\sigma}$ defined in [13], Equation (5.5), is constant, and hence (5.3), (5.4) and (5.6) in [13] simplify to

$$\int_{\mathbb{R}} \nu\Big(\Big(\frac{1}{\|g'\|_{L^p([s,1+s])}}, \infty\Big)\Big)\, ds < \infty, \tag{4.49}$$

$$\int_0^\infty \int_{\mathbb{R}} \big(|xg'(s)|^2 \wedge 1\big)\nu(dx)\, ds < \infty, \tag{4.50}$$

$$\int_0^1 \int_{\mathbb{R}} |g'(t+s)|^p \Big(\int_{r/|g'(t+s)|}^{1/\|g'\|_{L^p([s,1+s])}} x^p \nu(dx)\Big)\, ds\, dt < \infty \tag{4.51}$$

for all $r > 0$. When the lower bound in the inner integral in (4.51) exceeds the upper bound the integral is set to zero. Since $\alpha > 1 - 1/\beta$, we may choose $\varepsilon > 0$ such that $(\alpha - 1)(\beta + \varepsilon) > -1$. To show (4.49), we use the estimates $\|g'\|_{L^p([s,1+s])} \leq K(\mathbb{1}_{\{s\in[-1,1]\}} + \mathbb{1}_{\{s>1\}}|g'(s)|)$ for $s \in \mathbb{R}$ and

$$\nu\big((u,\infty)\big) \leq \begin{cases} Ku^{-\theta}, & u \geq 1, \\ Ku^{-\beta-\varepsilon}, & u \in (0,1], \end{cases}$$

which both follow from assumption (A). Hence, we deduce that

$$\begin{aligned} &\int_{\mathbb{R}} \nu\Big(\Big(\frac{1}{\|g'\|_{L^p([s,1+s])}}, \infty\Big)\Big)\, ds \\ &\quad \leq \int_{-1}^1 \nu\Big(\Big(\frac{1}{K}, \infty\Big)\Big)\, ds + \int_1^\infty \nu\Big(\Big(\frac{1}{K|g'(s)|}, \infty\Big)\Big)\, ds \\ &\quad \leq 2\nu\Big(\Big(\frac{1}{K}, \infty\Big)\Big) \\ &\qquad + K\int_1^\infty \big(|g'(s)|^\theta \mathbb{1}_{\{K|g'(s)|\leq 1\}} + |g'(s)|^{\beta+\varepsilon}\mathbb{1}_{\{K|g'(s)|>1\}}\big)\, ds \end{aligned}$$

which is finite and thereby shows (4.49) [recall that $|g'|$ is decreasing on $(1,\infty)$]. To show (4.50), we will use the following two estimates:

$$\begin{aligned} &\int_0^1 \big(|s^{\alpha-1}x|^2 \wedge 1\big)\, ds \\ &\quad \leq \begin{cases} K\big(\mathbb{1}_{\{|x|\leq 1\}}|x|^{1/(1-\alpha)} + \mathbb{1}_{\{|x|>1\}}\big), & \alpha < 1/2, \\ K\big(\mathbb{1}_{\{|x|\leq 1\}}x^2\log(1/x) + \mathbb{1}_{\{|x|>1\}}\big), & \alpha = 1/2, \\ K\big(\mathbb{1}_{\{|x|\leq 1\}}x^2 + \mathbb{1}_{\{|x|>1\}}\big), & \alpha > 1/2, \end{cases} \end{aligned} \tag{4.52}$$

and

$$\int_{\{|x|>1\}} \big(|xg'(s)|^2 \wedge 1\big)\nu(dx) \le K \int_1^\infty \big(|xg'(s)|^2 \wedge 1\big)x^{-1-\theta}\,dx \le K|g'(s)|^\theta. \tag{4.53}$$

For $\alpha < 1/2$, we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \big(|xg'(s)|^2 \wedge 1\big)\nu(dx)\,ds \\
&\quad\le K\bigg\{\int_{\mathbb{R}}\int_0^1 \big(|xs^{\alpha-1}|^2 \wedge 1\big)\,ds\,\nu(dx) \\
&\qquad + \int_1^\infty \int_{\{|x|\le 1\}} \big(|xg'(s)|^2 \wedge 1\big)\nu(dx)\,ds \\
&\qquad + \int_1^\infty \int_{\{|x|>1\}} \big(|xg'(s)|^2 \wedge 1\big)\nu(dx)\,ds\bigg\} \\
&\quad\le K\bigg\{\int_{\mathbb{R}} \big(\mathbb{1}_{\{|x|\le 1\}}|x|^{1/(1-\alpha)} + \mathbb{1}_{\{|x|>1\}}\big)\nu(dx) \\
&\qquad + \bigg(\int_1^\infty |g'(s)|^2\,ds\bigg)\bigg(\int_{\{|x|\le 1\}} x^2\nu(dx)\bigg) + \int_1^\infty |g'(s)|^\theta\,ds\bigg\} < \infty,
\end{aligned}$$

where the first inequality follows by assumption (A), the second inequality follows by (4.52) and (4.53), and the last inequality is due to the fact that $1/(1-\alpha) > \beta$ and $g' \in L^\theta((1,\infty)) \cap L^2((1,\infty))$. This shows (4.50). The two remaining cases $\alpha = 1/2$ and $\alpha > 1/2$ follow similarly.

Now, we will prove that (4.51) holds. Since $|g'|$ is decreasing on $(1,\infty)$, we have for all $t \in [0,1]$ that

$$\begin{aligned}
&\int_1^\infty |g'(t+s)|^p \bigg(\int_{r/|g'(t+s)|}^{1/\|g'\|_{L^p([s,1+s])}} x^p\nu(dx)\bigg)\,ds \\
&\quad\le \int_1^\infty |g'(s)|^p \bigg(\int_{r/|g'(1+s)|}^{1/|g'(s)|} x^p\nu(dx)\bigg)\,ds \\
&\quad\le \frac{K}{p-\theta}\int_1^\infty |g'(s)|^p \big(|g'(s)|^{\theta-p} \\
&\qquad - |g'(s+1)/r|^{\theta-p}\big)\mathbb{1}_{\{r/|g'(1+s)|\le 1/|g'(s)|\}}\,ds.
\end{aligned} \tag{4.54}$$

For $p > \theta$, (4.54) is less than or equal to

$$\frac{K}{p-\theta}\int_1^\infty |g'(s)|^\theta\,ds < \infty.$$

For $p < \theta$, (4.54) is less than or equal to

$$\frac{Kr^{p-\theta}}{\theta - p}\int_1^\infty |g'(s)|^p |g'(s+1)|^{\theta-p}\,ds \le \frac{Kr^{p-\theta}}{\theta-p}\int_1^\infty |g'(s)|^\theta\,ds < \infty,$$

where the first inequality is due to the fact that $|g'|$ is decreasing on $(1,\infty)$. Hence, we have shown that

$$\int_0^1\int_1^\infty |g'(t+s)|^p\Big(\int_{r/|g'(t+s)|}^{1/\|g'\|_{L^p([s,1+s])}} x^p\nu(dx)\Big)\,ds\,dt < \infty \tag{4.55}$$

for $p \neq \theta$. Suppose that $p > \beta$. For $t \in [0,1]$ and $s \in [-1,1]$, we have

$$\begin{aligned}\int_{r/|g'(t+s)|}^{1/\|g'\|_{L^p([s,1+s])}} x^p\nu(dx) &\le \int_1^{1/\|g'\|_{L^p([s,1+s])}} x^p\nu(dx) + \int_{r/|g'(t+s)|}^1 x^p\nu(dx)\\ &\le K\big(\|g'\|^{\theta-p}_{L^p([s,1+s])}+1\big)\end{aligned}$$

and hence

$$\int_0^1\int_{-1}^1 |g'(t+s)|^p\Big(\int_{r/|g'(t+s)|}^{1/\|g'\|_{L^p([s,1+s])}} x^p\nu(dx)\Big)\,ds\,dt \tag{4.56}$$

$$\le K\Big(\int_{-1}^1 \|g'\|^\theta_{L^p([s,s+1])}\,ds + \int_{-1}^1 \|g'\|^p_{L^p([s,1+s])}\,ds\Big) < \infty. \tag{4.57}$$

Suppose that $p \le \beta$. For $t \in [0,1]$ and $s \in [-1,1]$, we have

$$\int_{r/|g'(t+s)|}^{1/\|g'\|_{L^p([s,1+s])}} x^p\nu(dx) \le K\big(\|g'\|^{\theta-p}_{L^p([s,1+s])} + |g'(t+s)|^{\beta+\varepsilon-p}\big)$$

and hence

$$\int_0^1\int_{-1}^1 |g'(t+s)|^p\Big(\int_{r/|g'(t+s)|}^{1/\|g'\|_{L^p([s,1+s])}} x^p\nu(dx)\Big)\,ds\,dt \tag{4.58}$$

$$\le K\Big(\int_{-1}^1 \|g'\|^\theta_{L^p([s,s+1])}\,ds + \int_{-1}^1 \|g'\|^{\beta+\varepsilon}_{L^{\beta+\varepsilon}([s,1+s])}\,ds\Big) < \infty \tag{4.59}$$

since $(\alpha-1)(\beta+\varepsilon) > -1$. Thus, (4.51) follows by (4.55), (4.56)–(4.57) and (4.58)–(4.59).

For $p = \theta$, the above argument remains valid except for (4.55), where we need the additional assumption (A-log). This completes the proof. □

LEMMA 4.4. *For all $\zeta \in W^{k,p}$ we have, as $n \to \infty$,*

$$n^{-1+pk}V(\zeta,p;k)_n \to \int_0^1 |\zeta^{(k)}(s)|^p\,ds. \tag{4.60}$$

PROOF. First, we will assume that $\zeta \in C^{k+1}(\mathbb{R})$ and afterwards we will prove the lemma by an approximation argument. An application of Taylor's expansion gives $\Delta^n_{i,k}\zeta = n^{-k}\zeta^{(k)}((i-k)/n) + a_{i,n}$, where $a_{i,n} \in \mathbb{R}$ satisfies $|a_{i,n}| \le Kn^{-k-1}$. By Minkowski's inequality,

$$\left|\left(n^{kp-1}V(\zeta)_n\right)^{1/p} - \left(n^{kp-1}\sum_{j=k}^{n}\left|\zeta^{(k)}\left(\frac{i-k}{n}\right)\frac{1}{n^k}\right|^p\right)^{1/p}\right|$$
$$\le \left(n^{pk-1}\sum_{j=k}^{n}|a_{i,n}|^p\right)^{1/p} \to 0.$$

By continuity of $\zeta^{(k)}$, we have

$$n^{kp-1}\sum_{i=k}^{n}\left|\zeta^{(k)}\left(\frac{i-k}{n}\right)\frac{1}{n^k}\right|^p \to \int_0^1 |\zeta^{(k)}(s)|^p\,ds$$

as $n \to \infty$, which shows (4.60). The statement of the lemma for a general $\zeta \in W^{k,p}$ follows by approximating ζ through a sequence of $C^{k+1}(\mathbb{R})$-functions and Minkowski's inequality. This completes the proof. □

Lemmas 4.3 and 4.4 yield the statement of Theorem 1.1(iii).

5. Proof of Theorem 1.2. Throughout this section, we suppose that the assumptions stated in Theorem 1.2 hold. Without loss of generality, we will assume that the symmetric β-stable Lévy process L has scale parameter $\sigma = 1$ and (A) holds with $\delta = c_0 = 1$.

5.1. *Notation and outline of the proof.* In addition to the notation introduced in Section 4.2, we define the following truncated version of Y^n_r in (4.38) by

$$Y^{n,m}_r = \int_{r-m}^{r} \phi^n_r(s)\,dL_s, \qquad n \in \mathbb{N}\cup\{\infty\}, m, r \ge 0,$$

where the function ϕ^n_r has been introduced in (4.37). For $n, m \in \mathbb{N}$, we set

$$S_n = \sum_{r=k}^{n}\left(|Y^n_r|^p - \mathbb{E}\left[|Y^n_r|^p\right]\right) \quad \text{and} \quad S_{n,m} = \sum_{r=k}^{n}\left(|Y^{n,m}_r|^p - \mathbb{E}\left[|Y^{n,m}_r|^p\right]\right).$$

By (4.39), we have that

$$n^{p(\alpha+1/\beta)}V(p;k)_n \overset{d}{=} S_n + (n-k+1)\mathbb{E}\left[|Y^n_1|^p\right], \tag{5.1}$$

and hence when proving Theorem 1.2 we may instead analyse the right-hand side of (5.1). For all $n \in \mathbb{N}\cup\{\infty\}$, $j \ge 1$ and $m \ge 0$, we also set

$$\begin{aligned} \rho^n_j &= \|\phi^n_j\|_{L^\beta(\mathbb{R}\setminus[0,1])}, \qquad \rho^{n,m}_j = \|\phi^n_j\|_{L^\beta([j-m,j]\setminus[0,1])}, \\ U^n_{j,r} &= \int_r^{r+1}\phi^n_j(u)\,dL_u. \end{aligned} \tag{5.2}$$

For all $r \in \mathbb{R}$, we consider the following σ-algebras:

$$\mathcal{G}_r = \sigma(L_s - L_u : s, u \le r) \quad \text{and} \quad \mathcal{G}_r^1 = \sigma(L_s - L_u : r \le s, u \le r+1).$$

We note that $(\mathcal{G}_r^1)_{r \ge 0}$ is not a filtration. Let W denote a symmetric β-stable random variable with scale parameter $\rho \in (0, \infty)$ and $\Phi_\rho : \mathbb{R} \to \mathbb{R}$ be defined by

$$\Phi_\rho(x) = \mathbb{E}[|W + x|^p] - \mathbb{E}[|W|^p], \qquad x \in \mathbb{R}. \tag{5.3}$$

For all $n \ge 1, m, r \ge 0$ let

$$V_r^{n,m} = |Y_r^n|^p - |Y_r^{n,m}|^p - \mathbb{E}[|Y_r^n|^p - |Y_r^{n,m}|^p],$$

$$\zeta_{r,j}^{n,m} = \mathbb{E}[V_r^{n,m} | \mathcal{G}_{r-j+1}] - \mathbb{E}[V_r^{n,m} | \mathcal{G}_{r-j}] - \mathbb{E}[V_r^{n,m} | \mathcal{G}_{r-j}^1], \tag{5.4}$$

$$R_r^{n,m} = \sum_{j=1}^{\infty} \zeta_{r,j}^{n,m} \quad \text{and} \quad Q_r^{n,m} = \sum_{j=1}^{\infty} \mathbb{E}[V_r^{n,m} | \mathcal{G}_{r-j}^1]. \tag{5.5}$$

According to Remark 5.1 below, the two series $R_r^{n,m}$ and $Q_r^{n,m}$ converge with probability one, and the following decomposition of $S_n - S_{n,m}$ holds with probability one:

$$S_n - S_{n,m} = \sum_{r=k}^{n} R_r^{n,m} + \sum_{r=k}^{n} Q_r^{n,m}. \tag{5.6}$$

Decompositions of the type (5.6) have been successfully used in the theory of discrete time moving averages (see, e.g., Ho and Hsing [22]), and will also play a crucial role in the proof of Theorem 1.2. Indeed, for the proof of Theorem 1.2(i) we will choose $m = 0$ in (5.6) and since $S_{n,0} = 0$ we have the following decomposition of S_n:

$$S_n = \sum_{r=k}^{n} R_r^{n,0} + \sum_{r=k}^{n} (Q_r^{n,0} - Z_r) + \sum_{r=k}^{n} Z_r, \tag{5.7}$$

where

$$Z_r = \sum_{j=1}^{\infty} \{\Phi_{\rho_j^\infty}(U_{j+r,r}^\infty) - \mathbb{E}[\Phi_{\rho_j^\infty}(U_{j+r,r}^\infty)]\}. \tag{5.8}$$

After suitable scaling we show that the first two sums on the right-hand side of (5.7) are negligible; see (5.30). To analyse the third sum, we note that the random variables $\{Z_r : r \ge k\}$ are independent and identically distributed, which follows from their definition. Hence, to complete the proof of Theorem 1.2(i), it is enough to show that the common law of $\{Z_r : r \ge k\}$ belong to the domain of attraction of a $(k - \alpha)\beta$-stable random variable, which is done in (5.33).

The main part of the proof of Theorem 1.2(ii) consists in showing that

$$\lim_{m \to \infty} \limsup_{n \to \infty} (n^{-1} \mathbb{E}[(S_n - S_{n,m})^2]) = 0; \tag{5.9}$$

see (5.47). We prove (5.9) by estimating each of the two sums on the right-hand side of (5.6) separately. We note that for each fixed $m \geq 1$ the sequences $\{S_{n,m} : n \geq 1\}$ are partial sums of m-dependent random variables, since for all $j \geq 1$ the random variables $\{Y_1^{n,m}, \dots, Y_j^{n,m}\}$ are independent of $\{Y_r^{n,m} : r \geq j+1+m\}$. Hence, using a standard result for m-dependent sequences one can deduce a central limit theorem for the sequences $\{S_{n,m} : n \geq 1\}$ and by using (5.9) transfer this result to S_n, which will prove Theorem 1.2(ii). In the next subsection, we present some estimates which play a key role in the proof of Theorem 1.2.

5.2. *Preliminary estimates*. The assumption $|g^{(k)}(x)| \leq K x^{\alpha-k}$ for all $x > 0$ implies that

$$\|\phi_j^n\|_{L^\beta([0,1])} \leq K j^{\alpha-k} \tag{5.10}$$

for some finite constant K, which does not depend on $j \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$. In the following, we will collect some estimates on the functions Φ_ρ defined in (5.3), which will be used various places in the proofs. We observe the identity, for $x \in \mathbb{R}$,

$$|x|^p = a_p^{-1} \int_{\mathbb{R}} \big(1 - \exp(iux)\big)|u|^{-1-p}\, du \qquad \text{for } p \in (0,1), \tag{5.11}$$

with $a_p = \int_{\mathbb{R}} (1 - \exp(iu))|u|^{-1-p}\, du \in \mathbb{R}_+$, which can be shown by substitution $y = ux$. Applying the identities (5.11) and (4.40), we obtain the representation

$$\Phi_\rho(x) = a_p^{-1} \int_{\mathbb{R}} \big(1 - \cos(ux)\big) e^{-\rho^\beta |u|^\beta} |u|^{-1-p}\, du. \tag{5.12}$$

From (5.12), we deduce that $\Phi_\rho \in C^3(\mathbb{R})$ and it holds that

$$\begin{aligned}
\Phi_\rho'(x) &= a_p^{-1} \int_{\mathbb{R}} \sin(ux)|u|^{-p} e^{-\rho^\beta |u|^\beta}\, du, \\
\Phi_\rho''(x) &= a_p^{-1} \int_{\mathbb{R}} \cos(ux)|u|^{1-p} e^{-\rho^\beta |u|^\beta}\, du, \\
\Phi_\rho'''(x) &= -a_p^{-1} \int_{\mathbb{R}} \sin(ux)|u|^{2-p} e^{-\rho^\beta |u|^\beta}\, du.
\end{aligned} \tag{5.13}$$

In the following, we let $\varepsilon > 0$ be a fixed number. The identities at (5.13) imply that for $v = 1, 2, 3$ there exists a finite constant K_ε such that for all $\rho \geq \varepsilon$ and all $x \in \mathbb{R}$

$$\big|\Phi_\rho^{(v)}(x)\big| \leq K_\varepsilon. \tag{5.14}$$

By (5.12), we also deduce the following estimate by several applications of the mean value theorem

$$\begin{aligned}
&\big|\Phi_\rho(x) - \Phi_\rho(y)\big| \\
&\quad \leq K_\varepsilon\big(\big(|x| \wedge 1 + |y| \wedge 1\big)|x-y|\mathbb{1}_{\{|x-y|\leq 1\}} + |x-y|^p \mathbb{1}_{\{|x-y|>1\}}\big)
\end{aligned} \tag{5.15}$$

which holds for all $\rho \geq \varepsilon$ and all $x, y \in \mathbb{R}$. Equation (5.15) used on $y = 0$ yields that

$$|\Phi_\rho(x)| \leq K_\varepsilon(|x|^p \wedge |x|^2). \tag{5.16}$$

In particular, it implies that

$$|\Phi_\rho(x)| \leq K_\varepsilon |x|^l \qquad \text{for all } l \in (p, \beta). \tag{5.17}$$

Moreover, for all $r \in [p, 2]$ and $\rho_1, \rho_2 \geq \varepsilon$ we deduce by (5.12) that

$$|\Phi_{\rho_1}(x) - \Phi_{\rho_2}(x)| \leq K_\varepsilon |\rho_1^\beta - \rho_2^\beta| \cdot |x|^r \qquad \text{for all } x \in \mathbb{R}. \tag{5.18}$$

REMARK 5.1. In the following, we will show that the three series $R_r^{n,m}$, $Q_r^{n,m}$ and Z_r defined in (5.5) and (5.8) converge almost surely, and the identity (5.6) holds almost surely. To show the above claim, we will first prove that for all $n \geq 1$ and $m \geq 0$ the two series

$$\text{(a):} \quad \sum_{j=1}^{\infty} \mathbb{E}[V_r^{n,m} | \mathcal{G}_{r-j}^1], \qquad \text{(b):} \quad \sum_{j=1}^{\infty} (\Phi_{\rho_j^\infty}(U_{j+r,r}^\infty) - \mathbb{E}[\Phi_{\rho_j^\infty}(U_{j+r,r}^\infty)]) \tag{5.19}$$

converge absolutely with probability one. The definitions of $V_r^{n,m}$, $\mathcal{G}_{r-j}^1$ and Φ_ρ yields the following representation:

$$\begin{aligned} \mathbb{E}[V_r^{n,m} | \mathcal{G}_{r-j}^1] &= \Phi_{\rho_j^n}(U_{r,r-j}^n) - \Phi_{\rho_j^{n,m}}(U_{r,r-j}^n) \mathbb{1}_{\{j \leq m\}} \\ &\quad - \mathbb{E}[(\Phi_{\rho_j^n}(U_{r,r-j}^n) - \Phi_{\rho_j^{n,m}}(U_{r,r-j}^n) \mathbb{1}_{\{j \leq m\}})]. \end{aligned} \tag{5.20}$$

We have that $\rho_j^n \to \|\phi_1^n\|_{L^\beta(\mathbb{R})} > 0$ as $j \to \infty$, and hence $\{\rho_j^n : j \geq N\}$ is bounded away from zero for N large enough. For all $j > N$ and all $\gamma \in (p, \beta)$, we have

$$\begin{aligned} \mathbb{E}[|\mathbb{E}[V_r^{n,m} | \mathcal{G}_{r-j}^1]|] &\leq 2\mathbb{E}[|\Phi_{\rho_j^n}(U_{r,r-j}^n)|] \leq K \mathbb{E}[|U_{r,r-j}^n|^\gamma] \\ &\leq K \|\phi_j^n\|_{L^\beta([0,1])}^\gamma \leq K j^{(\alpha-k)\gamma}, \end{aligned}$$

where the first inequality follows by (5.20), the second inequality follows by (5.17) and the last inequality follows by (5.10). By choosing γ close enough to β and using the assumption $(\alpha - k)\beta < -1$, it follows that the series (a) in (5.19) converges absolutely almost surely. A similar application of (5.17) and (5.10) shows that the series (b) in (5.19) converges absolutely almost surely.

Next, we note that $V_r^{n,m} = \mathbb{E}[V_r^{n,m} | \mathcal{G}_r]$ and $\mathbb{E}[V_r^{n,m} | \mathcal{G}_{-j}] \to \mathbb{E}[V_r^{n,m}] = 0$ almost surely as $j \to \infty$. The latter claim follows from Kolmogorov's 0–1 law and the backward martingale convergence theorem. From these two properties, we deduce that $V_r^{n,m}$ has the following telescoping sum representation:

$$V_r^{n,m} = \sum_{j=1}^{\infty} (\mathbb{E}[V_r^{n,m} | \mathcal{G}_{r-j+1}] - \mathbb{E}[V_r^{n,m} | \mathcal{G}_{r-j}]), \tag{5.21}$$

where the sum converges almost surely. Due to (5.21) and (5.19)(a), the series $R_r^{n,m}$ converges a.s. By (5.19)(a), it also follows that $Q_r^{n,m}$ converges a.s., and Z_r converges a.s. due to (5.19)(b). By the decomposition $S_n - S_{n,m} = \sum_{r=k}^n V_r^{n,m}$ and (5.21) the two identities (5.6) and (5.7) follow by adding and subtracting.

The following estimates will play a key role in the proof of Theorem 1.2.

PROPOSITION 5.2. *Suppose that the conditions of Theorem 1.2 hold, and hence in particular $p < \beta/2$ and $\alpha < k - 1/\beta$. For all $\varepsilon > 0$, there exists a finite constant K such that for all $n \geq 1$ and $m \geq 0$ we have the following estimates*:

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg(\sum_{r=k}^n R_r^{n,m}\Bigg)^2\Bigg] \\
&\quad \leq K\big(n\big[(m+1)^{(\alpha-k)\beta+1}\log^2(m+1) + (m+1)^{2(\alpha-k)\beta+3}\big] \\
&\qquad + n^{2(\alpha-k)\beta+4+\varepsilon} + \log(n)\big).
\end{aligned} \tag{5.22}$$

If in addition $\alpha < k - 2/\beta$, then the estimate (5.23) *holds*:

$$\mathbb{E}\Bigg[\Bigg(\sum_{r=k}^n Q_r^{n,m}\Bigg)^2\Bigg] \leq K\big(n^{(\alpha-k)\beta+3+\varepsilon} + n(m+1)^{(\alpha-k)\beta+2+\varepsilon} + 1\big). \tag{5.23}$$

On the other hand, if $\alpha > k - 2/\beta$ then there exists $\xi > 0$ such that

$$\mathbb{E}\Bigg[\Bigg|\sum_{r=k}^n (Q_r^{n,0} - Z_r)\Bigg|\Bigg] \leq K\big(n^{(\alpha-k)\beta+2+\varepsilon} + n^{1/((k-\alpha)\beta)-\xi}\big). \tag{5.24}$$

The proof of Proposition 5.2 is carried out in Sections 5.5 and 5.6. We will also need the following inequality.

LEMMA 5.3. *Assume that the conditions of Theorem 1.2 hold. Then there exists a finite constant K such that for all $j, n \geq 1$ we have*

$$\int_{\mathbb{R}} \big||\phi_j^n(x)|^\beta - |\phi_j^\infty(x)|^\beta\big|\,dx \leq K \begin{cases} n^{-1}, & \text{when } \alpha \in (0, k-2/\beta), \\ n^{(\alpha-k)\beta+1}, & \text{when } \alpha \in (k-2/\beta, k-1/\beta), \end{cases}$$

where the functions ϕ_j^n and ϕ_j^∞ have been introduced at (4.37).

The proof of Lemma 5.3 is postponed to Section 5.7. We are now ready to show Theorem 1.2(i).

5.3. *Proof of Theorem* 1.2(i). To prove Theorem 1.2(i), we will first state and prove the following lemma.

LEMMA 5.4. *For any $q \geq 1$, there exists $\delta > 0$ and a finite $K > 0$ such that for all $\varepsilon \in (0, \delta)$, $\rho > \delta$, $\kappa, \tau \in L^\beta([0,1])$ with $\|\kappa\|_{L^\beta([0,1])}, \|\tau\|_{L^\beta([0,1])} \leq 1$ we have*

$$\left\| \Phi_\rho\left(\int_0^1 \kappa(s)\, dL_s\right) - \Phi_\rho\left(\int_0^1 \tau(s)\, dL_s\right)\right\|_{L^q} \leq K \begin{cases} \|\kappa - \tau\|_{L^\beta([0,1])}^{\beta/q}, & \beta < q < \beta/p, \\ \left(\|\kappa\|_{L^\beta([0,1])}^{(\beta-q)/q-\varepsilon} + \|\tau\|_{L^\beta([0,1])}^{(\beta-q)/q-\varepsilon}\right)\|\kappa-\tau\|_{L^\beta([0,1])}^{1-\varepsilon} & \\ \quad + \|\kappa - \tau\|_{L^\beta([0,1])}^{\beta/q}, & \beta > q. \end{cases}$$

To prove Lemma 5.4, we will among others use the following simple estimates.

LEMMA 5.5. *There exists a finite constant K such that for all symmetric β-stable random variables W with scale parameter $\rho \in (0,1]$ we have the estimates*

$$\mathbb{E}\big[|W|^\gamma \mathbb{1}_{\{|W|\geq 1\}}\big] \leq K\rho^\beta \qquad \text{for } \gamma < \beta,$$
$$\mathbb{E}\big[(|W| \wedge 1)^\gamma\big] \leq K\rho^\beta \qquad \text{for } \gamma > \beta.$$

PROOF. Let η be the density of a standard symmetric β-stable random variable. According to [41], Theorem 1.1, we have that $\eta(x) \leq K(1+|x|)^{-1-\beta}$, $x \in \mathbb{R}$. To prove the first inequality, we use substitution to get

$$\begin{aligned} \mathbb{E}\big[|W|^\gamma \mathbb{1}_{\{|W|\geq 1\}}\big] &= \int_{\mathbb{R}} |\rho x|^\gamma \mathbb{1}_{\{|\rho x|\geq 1\}} \eta(x)\, dx \\ &\leq K\rho^{-1} \int_{\mathbb{R}} |x|^\gamma \mathbb{1}_{\{|x|\geq 1\}} \big|\rho^{-1}x\big|^{-1-\beta}\, dx \leq K\rho^\beta, \end{aligned}$$

where we use that $\gamma < \beta$ in the last inequality. To show the second inequality, we note that the assumption $\gamma > \beta$ implies that

$$\begin{aligned} \mathbb{E}\big[|W|^\gamma \mathbb{1}_{\{|W|\leq 1\}}\big] &= \int_{\mathbb{R}} |\rho x|^\gamma \mathbb{1}_{\{|\rho x|\leq 1\}} \eta(x)\, dx \\ &\leq K\rho^{-1} \int_{\mathbb{R}} |x|^\gamma \mathbb{1}_{\{|x|\leq 1\}} \big|\rho^{-1}x\big|^{-1-\beta}\, dx \leq K\rho^\beta. \end{aligned} \tag{5.25}$$

Moreover, if W_0 denotes symmetric β-stable random variable with scale parameter 1 then

$$\mathbb{E}[\mathbb{1}_{\{|W|\geq 1\}}] = \mathbb{P}\big(|W_0| \geq \rho^{-1}\big) \leq K\rho^\beta,$$

which together with (5.25) completes the proof of Lemma 5.5. □

PROOF OF LEMMA 5.4. For notation simplicity, set $U = \int_0^1 \kappa(s)\,dL_s$ and $V = \int_0^1 \tau(s)\,dL_s$. To prove the lemma, we apply (5.15) to get

$$\begin{aligned}
&\|\Phi_\rho(U) - \Phi_\rho(V)\|_{L^q} \\
&\quad \le K\big(\big\|(|U| \wedge 1 + |V| \wedge 1)|U - V|\mathbb{1}_{\{|U-V|<1\}}\big\|_{L^q} \\
&\qquad + \big\||U - V|^p \mathbb{1}_{\{|U-V|\ge 1\}}\big\|_{L^q}\big).
\end{aligned} \tag{5.26}$$

For all $q < \beta/p$, we have

$$\big\||U - V|^p \mathbb{1}_{\{|U-V|\ge 1\}}\big\|_{L^q} = \mathbb{E}\big[|U - V|^{pq} \mathbb{1}_{\{|U-V|\ge 1\}}\big]^{1/q} \le K\|\kappa - \tau\|_{L^\beta([0,1])}^{\beta/q}$$

according to Lemma 5.5(i). To estimate the first term in (5.26), suppose first that $q > \beta$. Then

$$\begin{aligned}
&\big\|(|U| \wedge 1 + |V| \wedge 1)|U - V|\mathbb{1}_{\{|U-V|<1\}}\big\|_{L^q} \\
&\quad \le 2\mathbb{E}\big[|U - V|^q \mathbb{1}_{\{|U-V|<1\}}\big]^{1/q} \le \|\kappa - \tau\|_{L^\beta([0,1])}^{\beta/q}
\end{aligned}$$

according to Lemma 5.5. On the other hand, suppose that $q < \beta$. Let $\tilde{\beta} \in (0, \beta)$ be any positive number such that $\gamma := \tilde{\beta}/q$ is strictly greater than one, and let $\gamma' = \tilde{\beta}/(\tilde{\beta} - q)$ denote the conjugated number to γ. From Hölder's inequality used for γ and γ', we obtain that

$$\begin{aligned}
&\big\|(|U| \wedge 1 + |V| \wedge 1)|U - V|\mathbb{1}_{\{|U-V|<1\}}\big\|_{L^q} \\
&\quad \le 2^{\gamma'}\big(\mathbb{E}\big[|U|^{q\gamma'} \wedge 1\big]^{1/(q\gamma')} + \mathbb{E}\big[|V|^{q\gamma'} \wedge 1\big]^{1/(q\gamma')}\big) \\
&\qquad \times \mathbb{E}\big[|U - V|^{q\gamma} \mathbb{1}_{\{|U-V|<1\}}\big]^{1/(q\gamma)}.
\end{aligned} \tag{5.27}$$

We note that $q\gamma = \tilde{\beta} < \beta$. Furthermore, since $\gamma < 2$ it follows that $\gamma' > 2$, and hence $\gamma' q > \beta$. Therefore, by (5.27) and Lemma 5.5(i)–(ii) we have that

$$\begin{aligned}
&\big\|(|U| \wedge 1 + |V| \wedge 1)|U - V|\mathbb{1}_{\{|U-V|<1\}}\big\|_{L^q} \\
&\quad \le K\big(\|\kappa\|_{L^\beta([0,1])}^{\beta/(q\gamma')} + \|\tau\|_{L^\beta([0,1])}^{\beta/(q\gamma')}\big)\|\kappa - \tau\|_{L^\beta([0,1])}^{\beta/(q\gamma)}
\end{aligned}$$

and choosing $\tilde{\beta}$ close enough to β yields the lemma. □

To prove Theorem 1.2(i), we use (5.1) to obtain the decomposition

$$\begin{aligned}
&n^{1-\frac{1}{(k-\alpha)\beta}}\big(n^{-1+p(\alpha+1/\beta)} V(p;k)_n - m_p\big) \\
&\quad \overset{d}{=} n^{\frac{1}{(\alpha-k)\beta}} S_n + n^{1-\frac{1}{(k-\alpha)\beta}}\bigg(\frac{n-k+1}{n}\mathbb{E}\big[|Y_1^n|^p\big] - m_p\bigg).
\end{aligned} \tag{5.28}$$

First, we will prove that

$$n^{\frac{1}{(\alpha-k)\beta}} S_n \xrightarrow{d} S \qquad \text{as } n \to \infty, \tag{5.29}$$

where the random variable S is defined in Theorem 1.2(i). Afterwards, we show that the second term on the right-hand side of (5.28) converges to zero. To show (5.29), we will use the decomposition (5.7), which shows that it suffices to prove that

$$n^{\frac{1}{(\alpha-k)\beta}} \sum_{r=k}^{n} R_r^{n,0} \xrightarrow{\mathbb{P}} 0, \qquad n^{\frac{1}{(\alpha-k)\beta}} \sum_{r=k}^{n} \big(Q_r^{n,0} - Z_r\big) \xrightarrow{\mathbb{P}} 0, \tag{5.30}$$

$$n^{\frac{1}{(\alpha-k)\beta}} \sum_{r=k}^{n} Z_r \xrightarrow{d} S \tag{5.31}$$

as $n \to \infty$. For all $\varepsilon > 0$, we have according to (5.22) of Proposition 5.2 that

$$\mathbb{E}\Bigg[\bigg(n^{\frac{1}{(\alpha-k)\beta}} \sum_{r=k}^{n} R_r^{n,0}\bigg)^2\Bigg]$$
$$\leq K\big(n^{\frac{2}{(\alpha-k)\beta}+1} + n^{2(\frac{1}{(\alpha-k)\beta}+(\alpha-k)\beta+2)+\varepsilon} + n^{\frac{2}{(\alpha-k)\beta}} \log(n)\big) \to 0$$

as $n \to \infty$ for ε small enough, where we have used the inequality $2 < x + 1/x$ for all $x > 1$ and the fact that $(k-\alpha)\beta > 1$ by assumption. Furthermore, for all $\varepsilon > 0$ we have, according to (5.24) of Proposition 5.2 and the assumption $\alpha > k - 2/\beta$, that as $n \to \infty$

$$\mathbb{E}\Bigg[\bigg|n^{\frac{1}{(\alpha-k)\beta}} \sum_{r=k}^{n} \big(Q_r^{n,0} - Z_r\big)\bigg|\Bigg] \leq K\big(n^{\frac{1}{(\alpha-k)\beta}+(\alpha-k)\beta+2+\varepsilon} + n^{-\xi}\big) \to 0, \tag{5.32}$$

where the first term on the right-hand side of (5.32) converges to zero for all $\varepsilon > 0$ small enough by the inequality $2 < x + 1/x$ for all $x > 1$ and the assumption $(k-\alpha)\beta > 1$.

In the following, we will show (5.31). Since $(Z_r)_{r \geq k}$ are i.i.d. with mean zero, it is enough to show that

$$\lim_{x\to\infty} x^{(k-\alpha)\beta} \mathbb{P}(Z > x) = \gamma \quad \text{and} \quad \lim_{x\to\infty} x^{(k-\alpha)\beta} \mathbb{P}(Z < -x) = 0 \tag{5.33}$$

with $Z := Z_k$; cf. [34], Theorem 1.8.1. The constant γ is defined in (5.37) below. To show (5.33), let us define the function $\overline{\Phi} : \mathbb{R} \to \mathbb{R}_+$ via

$$\overline{\Phi}(x) := \sum_{j=1}^{\infty} \Phi_{\rho_j^\infty}\big(\phi_j^\infty(0)x\big).$$

Note that (5.12) implies that $\Phi_{\rho_j^\infty}(x) \geq 0$ and hence $\overline{\Phi}$ is positive. Note that as $j \to \infty$, $\rho_j^\infty \to \rho_\infty^\infty := \|h_k\|_{L^\beta(\mathbb{R})} > 0$ which implies that $(\rho_j^\infty)_{j\geq 1}$ is bounded away from 0, and hence by (5.17) and for $l \in (p, \beta)$ with $(\alpha - k)l < -1$ we have

$$\big|\overline{\Phi}(x)\big| \leq K|x|^l \sum_{j=1}^{\infty} \phi_j^\infty(0)^l \leq K|x|^l \sum_{j=1}^{\infty} j^{l(\alpha-k)} < \infty, \tag{5.34}$$

which shows that $\overline{\Phi}$ is well defined. Equation (5.34) shows moreover that $\mathbb{E}[\overline{\Phi}(L_{k+1}-L_k)]<\infty$, and hence we can define a random variable Q via

$$\begin{aligned} Q &= \overline{\Phi}(L_{k+1}-L_k)-\mathbb{E}[\overline{\Phi}(L_{k+1}-L_k)] \\ &= \sum_{j=1}^{\infty}\bigl(\Phi_{\rho_j^\infty}\bigl(\phi_j^\infty(0)(L_{k+1}-L_k)\bigr)-\mathbb{E}\bigl[\Phi_{\rho_j^\infty}\bigl(\phi_j^\infty(0)(L_{k+1}-L_k)\bigr)\bigr]\bigr), \end{aligned}$$

where the last sum converges absolutely almost surely. Due to the lower bound $Q\geq -\mathbb{E}[\overline{\Phi}(L_{k+1}-L_k)]$, we have that

$$\lim_{x\to\infty} x^{(k-\alpha)\beta}\mathbb{P}(Q<-x)=0. \tag{5.35}$$

By the substitution $t=(x/u)^{1/(k-\alpha)}$, we have that

$$\begin{aligned} & x^{1/(\alpha-k)}\overline{\Phi}(x) \\ &\quad = x^{1/(\alpha-k)}\int_0^\infty \Phi_{\rho^\infty_{1+[t]}}\bigl(\phi^\infty_{1+[t]}(0)x\bigr)\,dt \\ &\quad = (k-\alpha)^{-1}\int_0^\infty \Phi_{\rho^\infty_{1+[(x/u)^{1/(k-\alpha)}]}}\bigl(\phi^\infty_{1+[(x/u)^{1/(k-\alpha)}]}(0)x\bigr)u^{-1+1/(\alpha-k)}\,du \\ &\quad \to (k-\alpha)^{-1}\int_0^\infty \Phi_{\rho^\infty_\infty}(k_\alpha u)u^{-1+1/(\alpha-k)}\,du =: \kappa \qquad \text{as } x\to\infty, \end{aligned} \tag{5.36}$$

where $k_\alpha=\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)$. Here, we have used that $(\rho_j^\infty)_{j\geq 1}$ are bounded away from zero together with the estimate (5.16) on $\Phi_{\rho_j^\infty}$ and Lebesgue's dominated convergence theorem. Note that the constant κ defined in (5.36) coincides with the κ defined in Remark 2.3. The connection between the tail behaviour of a symmetric ρ-stable random variable S_ρ, $\rho\in(0,2)$, and its scale parameter $\bar{\sigma}$ is given via

$$\mathbb{P}(S_\rho>x)\sim \tau_\rho\bar{\sigma}^\rho x^{-\rho}/2 \qquad \text{as } x\to\infty,$$

where the function τ_ρ has been defined in Remark 2.3 (see [34], Equation (1.2.10)). Hence, $\mathbb{P}(|L_{k+1}-L_k|>x)\sim\tau_\beta x^{-\beta}$ as $x\to\infty$, and by (5.36) we readily deduce that as $x\to\infty$

$$\mathbb{P}(Q>x)\sim \gamma x^{-(k-\alpha)\beta} \qquad \text{with } \gamma=\tau_\beta\kappa^{(k-\alpha)\beta}. \tag{5.37}$$

Next, we will show that for some $r>(k-\alpha)\beta$ we have

$$\mathbb{P}\bigl(|Z-Q|>x\bigr)\leq Kx^{-r} \qquad \text{for all } x\geq 1, \tag{5.38}$$

which implies (5.33); cf. (5.35) and (5.37). To show (5.38), it is sufficient to find $r>(k-\alpha)\beta$ such that $\mathbb{E}[|Z-Q|^r]<\infty$ by Markov's inequality. Furthermore, by Minkowski inequality and the definitions of Q and Z it suffices to show that

$$\sum_{j=1}^{\infty}\bigl\|\Phi_{\rho_j^\infty}\bigl(U^\infty_{j+k,k}\bigr)-\Phi_{\rho_j^\infty}\bigl(\phi_j^\infty(0)(L_{k+1}-L_k)\bigr)\bigr\|_{L^r}<\infty \tag{5.39}$$

[recall that $(k-\alpha)\beta > 1$]. To show (5.39), we note that for all $x \in [0,1]$ and $j \in \mathbb{N}$ there exists $\theta_{j,x} \in [j-x, j]$ such that

$$|\phi_j^\infty(x) - \phi_j^\infty(0)| = |h_k(j-x) - h_k(j)| \le |h_k'(\theta_{j,x})| \le Kj^{\alpha-k-1}. \tag{5.40}$$

Choose $\delta > 0$ according to Lemma 5.4 and let $r_\varepsilon = (k-\alpha)\beta + \varepsilon$ for all $\varepsilon \in (0,\delta)$. By Lemma 5.4 and (5.40), we have that

$$\begin{aligned} &\|\Phi_{\rho_j^\infty}(U_{j+k,k}^\infty) - \Phi_{\rho_j^\infty}(\phi_j^\infty(0)(L_{k+1} - L_k))\|_{L^{r_\varepsilon}} \\ &\quad \le K\big(\|\phi_j^\infty - \phi_j^\infty(0)\|_{L^\beta([0,1])} + \|\phi_j^\infty - \phi_j^\infty(0)\|_{L^\beta([0,1])}^{\frac{1}{k-\alpha+\varepsilon/\beta}}\big) \\ &\quad \le K\big(j^{\alpha-k-1} + j^{\frac{\alpha-k-1}{k-\alpha+\varepsilon/\beta}}\big). \end{aligned} \tag{5.41}$$

Our assumption $\alpha < k - 1/\beta$ implies that $\alpha - k < 0$. Furthermore, since

$$\frac{\alpha-k-1}{k-\alpha+\varepsilon/\beta} \to -1 - 1/(k-\alpha) < -1 \qquad \text{as } \varepsilon \to 0,$$

we may, according to (5.41), choose $\varepsilon > 0$ such that (5.39) holds for $r = r_\varepsilon$ which satisfies the condition $r > (k-\alpha)\beta$. This completes the proof of (5.38), and hence also of (5.30).

To complete the proof of Theorem 1.2(i), we show that the second term in (5.28) converges to zero. For this purpose, it is enough to show that

$$n^{1-\frac{1}{(k-\alpha)\beta}}\big(\mathbb{E}[|Y_1^n|^p] - m_p\big) \to 0 \qquad \text{as } n \to \infty, \tag{5.42}$$

since $1 - \frac{1}{(k-\alpha)\beta} < 1$. Recall that $m_p = \|h_k\|_{L^\beta(\mathbb{R})}^p \mathbb{E}[|Z|^p]$, where Z is a standard symmetric β-stable random variable and $\|h_k\|_{L^\beta(\mathbb{R})} = \|\phi_1^\infty\|_{L^\beta(\mathbb{R})}$. By Lemma 5.3, we have that

$$\big|\|\phi_j^n\|_{L^\beta(\mathbb{R})}^\beta - \|\phi_j^\infty\|_{L^\beta(\mathbb{R})}^\beta\big| \le Kn^{(\alpha-k)\beta+1} \to 0, \tag{5.43}$$

where the convergence to zero is due to the fact that $(k-\alpha)\beta > 1$ under our assumptions. Since the function $x \mapsto x^{p/\beta}$ is continuously differentiable on $(0,\infty)$ and $\|h_k\|_{L^\beta(\mathbb{R})}^\beta > 0$, it follows by the mean value theorem that

$$\big|\|\phi_1^n\|_{L^\beta(\mathbb{R})}^p - \|h_k\|_{L^\beta(\mathbb{R})}^p\big| \le K\big|\|\phi_1^n\|_{L^\beta(\mathbb{R})}^\beta - \|h_k\|_{L^\beta(\mathbb{R})}^\beta\big|,$$

which together with (5.43) and the definition of Y_1^n in (4.37) shows that

$$\begin{aligned} &n^{1-\frac{1}{(k-\alpha)\beta}}\big|\mathbb{E}[|Y_1^n|^p] - m_p\big| \\ &\quad = n^{1-\frac{1}{(k-\alpha)\beta}}\mathbb{E}[|Z|^p]\big|\|\phi_1^n\|_{L^\beta(\mathbb{R})}^p - \|h_k\|_{L^\beta(\mathbb{R})}^p\big| \\ &\quad \le Kn^{1-\frac{1}{(k-\alpha)\beta}}\big|\|\phi_1^n\|_{L^\beta(\mathbb{R})}^\beta - \|h_k\|_{L^\beta(\mathbb{R})}^\beta\big| \le Kn^{2-\frac{1}{(k-\alpha)\beta}-(k-\alpha)\beta}. \end{aligned} \tag{5.44}$$

By (5.44) and the assumption $(k-\alpha)\beta > 1$, we obtain (5.42), and the proof of Theorem 1.2(i) is complete.

5.4. *Proof of Theorem* 1.2(ii). To prove Theorem 1.2(ii), we start by noticing that

$$\sqrt{n}\big(n^{-1+p(\alpha+1/\beta)}V(p;k)_n - m_p\big) \stackrel{d}{=} \frac{1}{\sqrt{n}}S_n + \sqrt{n}\bigg(\frac{n-k+1}{n}\mathbb{E}[|Y_1^n|^p] - m_p\bigg) \tag{5.45}$$

due to (5.1). First, we will show that

$$\frac{1}{\sqrt{n}}S_n \xrightarrow{d} \mathcal{N}(0,\eta^2) \qquad \text{as } n\to\infty, \tag{5.46}$$

for some $\eta^2 \in [0,\infty)$. Afterwards, we will show that the second term on the right-hand side of (5.45) converges to zero, which will complete the proof of Theorem 1.2(ii). To prove (5.46), it is according to a standard result (see, e.g., [12], Theorem 3.2) enough to show the following statements:

$$\lim_{m\to\infty}\limsup_{n\to\infty}\big(n^{-1}\mathbb{E}[(S_n - S_{n,m})^2]\big) = 0, \tag{5.47}$$

$$\frac{1}{\sqrt{n}}S_{n,m} \xrightarrow{d} \mathcal{N}(0,\eta_m^2) \qquad \text{as } n\to\infty \text{ for some } \eta_m^2 \in [0,\infty), \tag{5.48}$$

$$\eta_m^2 \to \eta^2 \qquad \text{as } m\to\infty. \tag{5.49}$$

To prove (5.47), we use Proposition 5.2 and the assumption $\alpha < k - 2/\beta$ to obtain that

$$\frac{1}{n}\mathbb{E}\Bigg[\bigg(\sum_{r=k}^{n} R_r^{n,m}\bigg)^2\Bigg] \le K\big((m+1)^{(\alpha-k)\beta/4+1/2} + n^{2(\alpha-k)\beta+3+\varepsilon} + n^{-1}\log n\big), \tag{5.50}$$

$$\frac{1}{n}\mathbb{E}\Bigg[\bigg(\sum_{r=k}^{n} Q_r^{n,m}\bigg)^2\Bigg] \le K\big(n^{(\alpha-k)\beta+2+\varepsilon} + (m+1)^{(\alpha-k)\beta+2+\varepsilon} + n^{-1}\big), \tag{5.51}$$

for all $\varepsilon > 0$. Thus, by the decomposition (5.7) of $S_n - S_{n,m}$, (5.50), (5.51) and the assumption $\alpha < k - 2/\beta$ we deduce (5.46), which completes the proof of (5.47).

To prove (5.48), we note that for fixed $n, m \ge 1$, $\{|Y_i^{n,m}|^p : i = k,\dots,n\}$ is a stationary m-dependent sequence, and hence

$$n^{-1}\operatorname{var}(S_{n,m}) = n^{-1}(n-k)\theta_0^{n,m} + 2n^{-1}\sum_{i=1}^{m}(n-k-i)\theta_i^{n,m}, \tag{5.52}$$

where we set $\theta_i^{n,m} = \operatorname{cov}(|Y_k^{n,m}|^p, |Y_{k+i}^{n,m}|^p)$ for all $n \in \mathbb{N}\cup\{\infty\}$, $m, i \ge 1$. By the symmetrisation inequality, we have that $\mathbb{P}(|Y_i^{n,m} - Y_i^{\infty,m}| > u) \le 2\mathbb{P}(|Y_i^n - Y_i^\infty| > u)$ for all $u > 0$, where the quantities Y_i^n and Y_i^∞ have been introduced in (4.38).

By the equivalence of moments of stable random variables, we have for all $q < \beta$ that

$$\begin{aligned}\mathbb{E}\big[\big|Y_i^{n,m} - Y_i^{\infty,m}\big|^q\big] &\le K_q \mathbb{E}\big[\big|Y_i^{n,m} - Y_i^{\infty,m}\big|^p\big]^{q/p} \\ &\le K_q 2^{q/p} \mathbb{E}\big[\big|Y_k^n - Y_k^{\infty}\big|^p\big]^{q/p} \to 0\end{aligned} \tag{5.53}$$

as $n \to \infty$, where the convergence to zero follows by (4.44). Since $p < \beta/2$, (5.53) implies that $\theta_i^{n,m} \to \theta_i^{\infty,m}$ as $n \to \infty$, and by (5.52) we deduce that

$$n^{-1} \operatorname{var}(S_{n,m}) \to \theta_0^{\infty,m} + 2\sum_{i=1}^{m} \theta_i^{\infty,m} =: \eta_m^2 \qquad \text{as } n \to \infty. \tag{5.54}$$

By (5.53), (5.54) and since for all $n \ge 1$, the sequences $\{|Y_i^{n,m}|^p : i = k, \dots, n\}$ are m-dependent, the convergence (5.48) follows by the main theorem of [11], and the proof of (5.48) is complete.

The proof of (5.49) uses a Cauchy sequence argument. For all $m, j \ge 1$, we have by the triangle inequality that

$$\begin{aligned}\big||\eta_m| - |\eta_j|\big| &= \lim_{n\to\infty} \big(n^{-1/2}\big|\|S_{n,m}\|_{L^2} - \|S_{n,j}\|_{L^2}\big|\big) \\ &\le \limsup_{n\to\infty} \big(n^{-1/2}\|S_{n,m} - S_{n,j}\|_{L^2}\big) \\ &\le \limsup_{n\to\infty} \big(n^{-1/2}\|S_{n,m} - S_n\|_{L^2}\big) + \limsup_{n\to\infty} \big(n^{-1/2}\|S_n - S_{n,j}\|_{L^2}\big),\end{aligned}$$

which according to (5.47) shows that $(|\eta_m|)_{m\ge1}$ is a Cauchy sequence in $\mathbb{R}_+$. Hence, $(\eta_m^2)_{m\ge1}$ is convergent.

To show that the second term on the right-hand side of (5.45) converges to zero it suffices to prove that $\sqrt{n}(\mathbb{E}[|Y_1^n|^p] - m_p) \to 0$ as $n \to \infty$. By Lemma 5.3, we have that

$$\big|\|\phi_1^n\|_{L^\beta(\mathbb{R})}^\beta - \|\phi_1^\infty\|_{L^\beta(\mathbb{R})}^\beta\big| \le K n^{-1} \to 0. \tag{5.55}$$

Since the function $x \mapsto x^{p/\beta}$ is continuously differentiable on $(0,\infty)$ and $\|\phi_1^\infty\|_{L^\beta(\mathbb{R})}^\beta > 0$, it follows by the mean value theorem that

$$\big|\|\phi_1^n\|_{L^\beta(\mathbb{R})}^p - \|\phi_1^\infty\|_{L^\beta(\mathbb{R})}^p\big| \le K\big|\|\phi_1^n\|_{L^\beta(\mathbb{R})}^\beta - \|h_k\|_{L^\beta(\mathbb{R})}^\beta\big|.$$

Together with (5.55) and the definition of Y_1^n in (4.37), it shows that

$$\begin{aligned}\sqrt{n}\big|\mathbb{E}[|Y_1^n|^p] - m_p\big| &= \sqrt{n}\mathbb{E}[|Z|^p]\big|\|\phi_1^n\|_{L^\beta(\mathbb{R})}^p - \|\phi_1^\infty\|_{L^\beta(\mathbb{R})}^p\big| \\ &\le K\sqrt{n}\big|\|\phi_1^n\|_{L^\beta(\mathbb{R})}^\beta - \|\phi_1^\infty\|_{L^\beta(\mathbb{R})}^\beta\big| \le K n^{-1/2} \to 0\end{aligned} \tag{5.56}$$

as $n \to \infty$. Hence, (5.56) completes the proof of Theorem 1.2(ii).

5.5. *An estimate.* This subsection is devoted to proving the following lemma, which is used in the proof of (5.22) of Proposition 5.2.

LEMMA 5.6. *Let $\zeta_{r,j}^{n,m}$ be defined in (5.4). Then there exists a finite constant K such that for all $n \geq 1$, $r = k, \dots, n$, $m \geq 0$ and $j \geq 1$ we have*

$$\mathbb{E}[|\zeta_{r,j}^{n,m}|^2] \leq K \begin{cases} (m+1)^{(\alpha-k)\beta+1} j^{(\alpha-k)\beta}, & j = 1, \dots, m, \\ j^{2(\alpha-k)\beta+1}, & j > m. \end{cases}$$

To show Lemma 5.6, we will use the following telescoping sum decomposition of $\zeta_{r,j}^{n,m}$:

$$\begin{aligned} \zeta_{r,j}^{n,m} &= \sum_{l=j}^{\infty} \vartheta_{r,j,l}^{n,m}, \\ \vartheta_{r,j,l}^{n,m} &:= \mathbb{E}[\zeta_{r,j}^{n,m} | \mathcal{G}_{r-j}^1 \vee \mathcal{G}_{r-l}] - \mathbb{E}[\zeta_{r,j}^{n,m} | \mathcal{G}_{r-j}^1 \vee \mathcal{G}_{r-l-1}]. \end{aligned} \tag{5.57}$$

The series (5.57) converges almost surely and the representation follows from the fact that $\lim_{l\to\infty} \mathbb{E}[\zeta_{r,j}^{n,m} | \mathcal{G}_{r-j}^1 \vee \mathcal{G}_{r-l}] = \mathbb{E}[\zeta_{r,j}^{n,m} | \mathcal{G}_{r-j}^1] = 0$ almost surely, similar to the argument used in Remark 5.1. The next lemma gives a moment estimate for $\vartheta_{r,j,l}^{n,m}$.

LEMMA 5.7. *Let $\vartheta_{r,j,l}^{n,m}$ be defined in (5.57) and suppose that $\beta < \gamma < \beta/p$. Then there exists $N \geq 1$ such that for all $n \geq N$, $r = k, \dots, n$, $j \geq 1$ and $m \geq 0$ we have that*

$$\mathbb{E}[|\vartheta_{r,j,l}^{n,m}|^\gamma] \leq K \begin{cases} j^{(\alpha-k)\beta} l^{(\alpha-k)\beta}, & l \geq m, \\ (m+1)^{(\alpha-k)\beta+1} j^{(\alpha-k)\beta} l^{(\alpha-k)\beta}, & l = j, \dots, m-1. \end{cases} \tag{5.58}$$

To prove Lemma 5.7, we use the following estimate on Φ_ρ defined in (5.3).

LEMMA 5.8. *For all $\varepsilon > 0$, there exists a finite constant K such that for all $\rho \in [\varepsilon, \varepsilon^{-1}]$, all $x, y, z \geq 0$ and all $a \in \mathbb{R}$ we have that*

$$\begin{aligned} &\int_0^z \int_0^y \int_0^x |\Phi_\rho'''(a + u_1 + u_2 + u_3)| \, du_1 \, du_2 \, du_3 \\ &\qquad \leq K\big((x \wedge 1)(y \wedge 1)\big(z \mathbb{1}_{\{z \leq 1\}} + z^p \mathbb{1}_{\{z > 1\}}\big)\big), \\ &\int_0^y \int_0^x |\Phi_\rho''(a + u_1 + u_2)| \, du_1 \, du_2 \leq K\big((x \wedge 1)\big(y \mathbb{1}_{\{y \leq 1\}} + y^p \mathbb{1}_{\{y > 1\}}\big)\big). \end{aligned}$$

PROOF. First, we will show that for all $v = 1, 2, 3$, all $a \in \mathbb{R}$ and all $z > 0$ we have that

$$\int_0^z |\Phi_\rho^{(v)}(a + u)| \, du \leq K\big(\mathbb{1}_{\{z \leq 1\}} z + \mathbb{1}_{\{z > 1\}} z^p\big), \tag{5.59}$$

where $\Phi_\rho^{(v)}$ denotes the vth derivative of Φ_ρ. To this aim, we first show that for $v = 1, 2, 3$ we have that

$$|\Phi_\rho^{(v)}(x)| \le K(1 \wedge |x|^{p-v}) \qquad \text{for all } x \in \mathbb{R}, \tag{5.60}$$

which, in particular, yields that

$$|\Phi_\rho^{(v)}(x)| \le K(1 \wedge |x|^{p-1}) \qquad \text{for all } x \in \mathbb{R}. \tag{5.61}$$

For all $u > 0$, we define $q(u) = u^{v-1-p}e^{-\rho^\beta u}$ and $\psi(u) = u^{v-1-p}(e^{-\rho^\beta u^\beta} - e^{-\rho^\beta u})$. By recalling (5.13), we have by the triangle inequality that

$$|\Phi_\rho^{(v)}(x)| \le 2a_p^{-1}\left(\left|\int_0^\infty \cos(xu)\psi(u)\,du\right| + \left|\int_0^\infty \cos(xu)q(u)\,du\right|\right). \tag{5.62}$$

To estimate the second integral on the right-hand side of (5.62), we note that $u \mapsto q(u)\rho^{\beta(v-p)}/\Gamma(v-p)$ is the density of a gamma distribution with shape parameter $v - p$ and rate parameter ρ^β. Hence, using the expression for the characteristic function for the gamma distribution we get for all $x \neq 0$ that

$$\begin{aligned}\left|\int_0^\infty \cos(xu)q(u)\,du\right| &\le \left|\int_0^\infty e^{ixu}q(u)\,du\right| \\ &= \frac{\Gamma(v-p)}{\rho^{\beta(v-p)}}\left|(1 - ix\rho^{-\beta})^{p-v}\right| \\ &= \frac{\Gamma(v-p)}{\rho^{\beta(v-p)}}(1 + x^2\rho^{-2\beta})^{\frac{p-v}{2}} \le \Gamma(v-p)|x|^{p-v}.\end{aligned} \tag{5.63}$$

To estimate the first integral on the right-hand side of (5.62), we set $\zeta(u) = e^{-\rho^\beta u^\beta} - e^{-\rho^\beta u}$ for $u \ge 0$ such that $\psi(u) = u^{v-1-p}\zeta(u)$. For all $j = 0, 1, 2, 3$, we obtain the estimates

$$|\zeta^{(j)}(u)| \le \begin{cases} Ku^{\beta\wedge 1-j}, & u \in (0,1), \\ Ku^2e^{-\varepsilon^\beta u^{\beta\wedge 1}}, & u \ge 1, \end{cases}$$

which imply that

$$|\psi^{(j)}(u)| \le \begin{cases} Ku^{\beta\wedge 1+v-1-p-j}, & u \in (0,1), \\ Ku^3e^{-\varepsilon^\beta u^{\beta\wedge 1}}, & u \ge 1. \end{cases} \tag{5.64}$$

Hence, by (5.64) and integration by parts, we have for all $x > 0$ that

$$\begin{aligned}\left|\int_0^\infty \cos(xu)\psi(u)\,du\right| &= \begin{cases} x^{-v}\left|\int_0^\infty \cos(xu)\psi^{(v)}(u)\,du\right|, & v \text{ even}, \\ x^{-v}\left|\int_0^\infty \sin(xu)\psi^{(v)}(u)\,du\right|, & v \text{ odd}, \end{cases} \\ &\le x^{-v}\int_0^\infty |\psi^{(v)}(u)|\,du \le Kx^{-v},\end{aligned} \tag{5.65}$$

where the last inequality follows from (5.64) used on $j = v$. The estimates (5.62), (5.63) and (5.65) imply (5.60).

To show (5.59), it suffices [cf. (5.61)] to show that there exists a finite constant K such that for all $z > 0$ and $a \in \mathbb{R}$

$$\int_0^z \big(1 \wedge |a+u|^{p-1}\big)\,du \le K\big(\mathbb{1}_{\{z\le 1\}} z + \mathbb{1}_{\{z>1\}} z^p\big). \tag{5.66}$$

To show (5.66), we may and do assume that $z > 1$ since the estimate (5.66) holds for $z \le 1$ by dominating the integrand by 1. We split the integral in three parts:

$$\begin{aligned}
&\int_0^z \big(1 \wedge |a+u|^{p-1}\big)\,du \\
&\quad = \int_{(-a-1,1-a)\cap[0,z]} 1\,du \\
&\qquad + \int_{(1-a,\infty)\cap[0,z]} (a+u)^{p-1}\,du + \int_{(-\infty,-a-1)\cap[0,z]} (-a-u)^{p-1}\,du.
\end{aligned} \tag{5.67}$$

Since $p \in (0,1]$, we have by subadditivity that $x^p - y^p \le (x-y)^p$ for all $0 \le y \le x$. Hence,

$$\begin{aligned}
\int_{(1-a,\infty)\cap[0,z]} (a+u)^{p-1}\,du &= \mathbb{1}_{\{z\ge 1-a\}} \frac{1}{p} \begin{cases} (a+z)^p - a^p, & a \ge 1, \\ (a+z)^p - 1, & a < 1, \end{cases} \\
&\le \mathbb{1}_{\{z\ge 1-a\}} \frac{1}{p} z^p,
\end{aligned}$$

$$\begin{aligned}
&\int_{(-\infty,-a-1)\cap[0,z]} (-a-u)^{p-1}\,du \\
&\quad = \mathbb{1}_{\{-a-1\ge 0\}} \frac{1}{p} \begin{cases} (-a)^p - 1, & -a-1 \le z, \\ (-a)^p - (-a-z)^p, & z \le -a-1, \end{cases} \qquad \le \mathbb{1}_{\{-a-1\ge 0\}} \frac{1}{p} z^p.
\end{aligned}$$

Thus, by (5.67) we obtain for $z \ge 1$ that

$$\int_0^z \big(1 \wedge |a+u|^{p-1}\big)\,du \le 2 + \frac{2}{p} z^p \le 2\Big(1 + \frac{1}{p}\Big) z^p,$$

which implies (5.66), and completes the proof of (5.59). We will now deduce the first inequality of Lemma 5.8 from (5.59). For $x \ge 1$ we have that, with $\bar{a} = a + x$,

$$\begin{aligned}
&\int_0^z \int_0^y \int_0^x \big|\Phi_\rho'''(a+u_1+u_2+u_3)\big|\,du_1\,du_2\,du_3 \\
&\quad \le \int_0^z \int_0^y \big|\Phi_\rho''(\bar{a}+u_2+u_3)\big|\,du_2\,du_3 \\
&\qquad + \int_0^z \int_0^y \big|\Phi_\rho''(a+u_2+u_3)\big|\,du_2\,du_3.
\end{aligned}$$

For $x \in (0,1)$, there exists an $\tilde{a} \in \mathbb{R}$ such that

$$\int_0^z \int_0^y \int_0^x \left|\Phi_\rho'''(a+u_1+u_2+u_3)\right| du_1\, du_2\, du_3 \\ = x \int_0^z \int_0^y \left|\Phi_\rho'''(\tilde{a}+u_2+u_3)\right| du_2\, du_3.$$

Repeating this argument shows that for any $\tilde{a} \in \mathbb{R}$ and $v = 2, 3$ we have for $y \geq 1$ that with $\bar{a} = \tilde{a} + y$

$$\int_0^z \int_0^y \left|\Phi_\rho^{(v)}(\tilde{a}+u_2+u_3)\right| du_2\, du_3 \\ \leq \int_0^z \left|\Phi_\rho^{(v-1)}(\bar{a}+u_3)\right| du_3 + \int_0^z \left|\Phi_\rho^{(v-1)}(\tilde{a}+u_3)\right| du_3,$$

and for $y < 1$ there exists $\bar{a} \in \mathbb{R}$ such that

$$\int_0^z \int_0^y \left|\Phi_\rho^{(v)}(\tilde{a}+u_2+u_3)\right| du_2\, du_3 \leq y \int_0^z \left|\Phi_\rho^{(v)}(\bar{a}+u_3)\right| du_3.$$

By collecting all the terms and using (5.59), we obtain the first inequality of Lemma 5.8. The second inequality of Lemma 5.8 follows by similar arguments. □

We are now ready to prove Lemma 5.7.

PROOF OF LEMMA 5.7. For fixed n, m, j, l, $\{\vartheta_{r,j,l}^{n,m} : r \geq 1\}$ is a stationary sequence, and hence we may and do assume that $r = 1$. Furthermore, we may assume that $l \geq j \vee 2$, since the case $l = j = 1$ can be covered by choosing a new constant K. By definition of $\vartheta_{1,j,l}^{n,m}$, we obtain the representation

$$\vartheta_{1,j,l}^{n,m} = \mathbb{E}\big[V_r^{n,m} | \mathcal{G}_{1-j}^1 \vee \mathcal{G}_{1-l}\big] - \mathbb{E}\big[V_r^{n,m} | \mathcal{G}_{1-l}\big] \\ - \mathbb{E}\big[V_r^{n,m} | \mathcal{G}_{1-j}^1 \vee \mathcal{G}_{-l}\big] + \mathbb{E}\big[V_r^{n,m} | \mathcal{G}_{-l}\big]. \tag{5.68}$$

Set $\rho_{j,l}^n = \|\phi_1^n\|_{L^\beta([1-l,1-j]\cup[2-j,1])}$. For large enough $N \geq 1$, there exists $\varepsilon > 0$ such that $\rho_{j,l}^n \geq \varepsilon$ for all $n \geq N$, $j \geq 1$, $l \geq j \vee 2$ (we have $\rho_{j,l}^n = 0$ for $l = 1$). Hence, by (5.14), there exists a finite constant K such that

$$\left|\Phi''_{\rho_{j,l}^n}(x)\right| \leq K \qquad \text{for all } n \geq N,\ j \geq 1,\ l \geq j \vee 2,\ x \in \mathbb{R}.$$

Let

$$A_l^n = \int_{-\infty}^{-l} \phi_1^n(s)\, dL_s \quad \text{and} \quad A_l^{n,m} = \int_{1-m}^{-l} \phi_1^n(s)\, dL_s$$

and $(\tilde{U}_{1,-l}^n, \tilde{U}_{1,1-j}^n)$ denote a random vector, which is independent of L, and which equals $(U_{1,-l}^n, U_{1,1-j}^n)$ in law [cf. definition (5.2)]. Let moreover $\tilde{\mathbb{E}}$ denote the expectation with respect to $(\tilde{U}^n_{1,-l}, \tilde{U}^n_{1,1-j})$ only. For all $j = 1, \dots, m$ and $l = j, \dots, m-1$, we deduce from (5.68) that

$$
\begin{aligned}
\vartheta^{n,m}_{1,j,l} &= \tilde{\mathbb{E}}\big[\Phi_{\rho^n_{j,l}}\big(A^n_l + U^n_{1,-l} + U^n_{1,1-j}\big) - \Phi_{\rho^n_{j,l}}\big(A^n_l + \tilde{U}^n_{1,-l} + U^n_{1,1-j}\big) \\
&\quad - \Phi_{\rho^n_{j,l}}\big(A^n_l + U^n_{1,-l} + \tilde{U}^n_{1,1-j}\big) + \Phi_{\rho^n_{j,l}}\big(A^n_l + \tilde{U}^n_{1,-l} + \tilde{U}^n_{1,1-j}\big) \\
&\quad - \big(\Phi_{\rho^n_{j,l}}\big(A^{n,m}_l + U^n_{1,-l} + U^n_{1,1-j}\big) - \Phi_{\rho^n_{j,l}}\big(A^{n,m}_l + \tilde{U}^n_{1,-l} + U^n_{1,1-j}\big) \\
&\quad - \Phi_{\rho^n_{j,l}}\big(A^{n,m}_l + U^n_{1,-l} + \tilde{U}^n_{1,1-j}\big) + \Phi_{\rho^n_{j,l}}\big(A^{n,m}_l + \tilde{U}^n_{1,-l} + \tilde{U}^n_{1,1-j}\big)\big)\big] \\
&= \tilde{\mathbb{E}}\bigg[\int_{A^{n,m}_l}^{A^n_l} \int_{\tilde{U}^n_{1,1-j}}^{U^n_{1,1-j}} \int_{\tilde{U}^n_{1,-l}}^{U^n_{1,-l}} \Phi'''_{\rho^n_{j,l}}(u_1 + u_2 + u_3)\, du_1\, du_2\, du_3\bigg],
\end{aligned} \tag{5.69}
$$

where $\int_y^x$ denotes $-\int_x^y$ if $x < y$. For $l \geq m$, we have that

$$
\begin{aligned}
\vartheta^{n,m}_{1,j,l} &= \tilde{\mathbb{E}}\big[\Phi_{\rho^n_{j,l}}\big(A^n_l + U^n_{1,-l} + U^n_{1,1-j}\big) - \Phi_{\rho^n_{j,l}}\big(A^n_l + \tilde{U}^n_{1,-l} + U^n_{1,1-j}\big) \\
&\quad - \Phi_{\rho^n_{j,l}}\big(A^n_l + U^n_{1,-l} + \tilde{U}^n_{1,1-j}\big) + \Phi_{\rho^n_{j,l}}\big(A^n_l + \tilde{U}^n_{1,-l} + \tilde{U}^n_{1,1-j}\big)\big] \\
&= \tilde{\mathbb{E}}\bigg[\int_{\tilde{U}^n_{1,1-j}}^{U^n_{1,1-j}} \int_{\tilde{U}^n_{1,-l}}^{U^n_{1,-l}} \Phi''_{\rho^n_{j,l}}\big(A^n_l + u_1 + u_2\big)\, du_1\, du_2\bigg].
\end{aligned}
$$

Let $l = j, \dots, m-1$. By (5.69), substitution and the first inequality of Lemma 5.8 we have that

$$
\begin{aligned}
&\mathbb{E}\big[\big|\vartheta^{n,m}_{1,j,l}\big|^\gamma\big] \\
&\quad \leq K\big(\mathbb{E}\big[\big|A^n_l - A^{n,m}_l\big|^{p\gamma} \mathbb{1}_{\{|A^n_l - A^{n,m}_l| \geq 1\}}\big] + \mathbb{E}\big[\big|A^n_l - A^{n,m}_l\big|^{\gamma} \mathbb{1}_{\{|A^n_l - A^{n,m}_l| \leq 1\}}\big]\big) \\
&\qquad \times \mathbb{E}\big[\tilde{\mathbb{E}}\big[\big(\big|\tilde{U}^n_{1,1-j} - U^n_{1,1-j}\big| \wedge 1\big)^\gamma\big]\big] \mathbb{E}\big[\tilde{\mathbb{E}}\big[\big(\big|\tilde{U}^n_{1,-l} - U^n_{1,-l}\big| \wedge 1\big)^\gamma\big]\big] \\
&\quad \leq K \|\phi^n_1\|^\beta_{L^\beta((-\infty,1-m])} \|\phi^n_1\|^\beta_{L^\beta([1-j,2-j])} \|\phi^n_1\|^\beta_{L^\beta([-l,1-l])} \\
&\qquad \leq K m^{(\alpha-k)\beta+1} j^{(\alpha-k)\beta} l^{(\alpha-k)\beta}.
\end{aligned}
$$

We use Lemma 5.5(i) and (ii), $p\gamma < \beta < \gamma$ and $|x-y| \wedge 1 \leq |x| \wedge 1 + |y| \wedge 1$. For $l \geq m$, we have by the second inequality of Lemma 5.8 that

$$
\begin{aligned}
\mathbb{E}\big[\big|\vartheta^{n,m}_{1,j,l}\big|^\gamma\big] &\leq K \mathbb{E}\big[\tilde{\mathbb{E}}\big[\big(\big|U^n_{1,1-j} - \tilde{U}^n_{1,1-j}\big| \wedge 1\big)^\gamma\big]\big] \\
&\qquad \times \big(\mathbb{E}\big[\tilde{\mathbb{E}}\big[\big|U^n_{1,1-j} - \tilde{U}^n_{1,1-j}\big|^{p\gamma} \mathbb{1}_{\{|U^n_{1,1-j} - \tilde{U}^n_{1,1-j}| \geq 1\}}\big]\big] \\
&\qquad + \mathbb{E}\big[\tilde{\mathbb{E}}\big[\big|U^n_{1,1-j} - \tilde{U}^n_{1,1-j}\big|^{\gamma} \mathbb{1}_{\{|U^n_{1,1-j} - \tilde{U}^n_{1,1-j}| \leq 1\}}\big]\big]\big) \\
&\qquad \leq K \|\phi^n_1\|^\beta_{L^\beta([1-j,2-j])} \|\phi^n_1\|^\beta_{L^\beta([1-j,2-j])} \leq K j^{(\alpha-k)\beta} l^{(\alpha-k)\beta}
\end{aligned}
$$

again using Lemma 5.5(i) and (ii), $p\gamma < \beta < \gamma$ and $|x-y| \wedge 1 \leq |x| \wedge 1 + |y| \wedge 1$. This completes the proof of (5.58). $\square$

We are now ready to prove Lemma 5.6.

PROOF OF LEMMA 5.6. We will use Lemma 5.7 for $\gamma = 2$, which satisfies $\beta < \gamma < \beta/p$. Suppose that $j = 1, \dots, m$. By orthogonality of $\{\vartheta^{n,m}_{r,j,l} : l = 1, 2, \dots\}$ in L^2, we have that

$$\begin{aligned}\mathbb{E}[|\zeta^{n,m}_{r,j}|^2] &= \sum_{l=j}^{\infty} \mathbb{E}[|\vartheta^{n,m}_{r,j,l}|^2] \\ &\le K\left(\sum_{l=j}^{m-1} m^{(\alpha-k)\beta+1} l^{(\alpha-k)\beta} j^{(\alpha-k)\beta} + \sum_{l=m}^{\infty} l^{(\alpha-k)\beta} j^{(\alpha-k)\beta}\right) \\ &\le K\big((m+1)^{(\alpha-k)\beta+1} j^{2(\alpha-k)\beta+1} + j^{(\alpha-k)\beta}(m+1)^{(\alpha-k)\beta+1}\big) \\ &\le K(m+1)^{(\alpha-k)\beta+1} j^{(\alpha-k)\beta}\end{aligned}$$

since $2(\alpha-k)\beta + 1 < (\alpha-k)\beta < -1$. Similarly, for $j > m$ we have that

$$\mathbb{E}[|\zeta^{n,m}_{r,j}|^2] = \sum_{l=j}^{\infty} \mathbb{E}[|\zeta^{n,m}_{r,j}|^2] \le K j^{(\alpha-k)\beta} \sum_{l=j}^{\infty} l^{(\alpha-k)\beta} \le K j^{2(\alpha-k)\beta+1},$$

which completes the proof. □

5.6. *Proof of Proposition* 5.2. We start by proving (5.22). By rearranging the terms using the substitution $s = r - j$, we have

$$\sum_{r=k}^{n} R^{n,m}_r = \sum_{s=-\infty}^{n-1} M^{n,m}_s \qquad \text{with } M^{n,m}_s := \sum_{r=1\vee(s+1)}^{n} \zeta^{n,m}_{r,r-s}.$$

Recalling the definition of $\zeta^{n,m}_{r,j}$ in (5.4), we note that $\mathbb{E}[\zeta^{n,m}_{r,r-s}|\mathcal{G}_s] = 0$ for all s and r, showing that $\{M^{n,m}_s : s \in (-\infty, n) \cap \mathbb{Z}\}$ are martingale differences. By orthogonality we have that

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{r=k}^{n} R^{n,m}_r\right)^2\right] &= \sum_{s=-\infty}^{n-1} \mathbb{E}[|M^{n,m}_s|^2] \\ &\le \sum_{s=-\infty}^{n-1} \left(\sum_{r=1\vee(s+1)}^{n} \mathbb{E}[|\zeta^{n,m}_{r,r-s}|^2]^{1/2}\right)^2 =: A_{n,m}.\end{aligned} \tag{5.70}$$

We split $A_{n,m} = \sum_{s=1}^{n-1} + \sum_{s=-n}^{0} + \sum_{s=-\infty}^{-n} = A'_{n,m} + A''_{n,m} + A'''_{n,m}$. By the substitution $\tilde{s} = n - s$ and $\tilde{r} = r - s$, we obtain

$$A'_{n,m} = \sum_{s=1}^{n-1} \left(\sum_{r=1}^{s} \mathbb{E}[|\zeta^{n,m}_{r+n-s,r}|^2]^{1/2}\right)^2.$$

For $s = 1, \dots, n$, we have (cf. Lemma 5.6)

$$\begin{aligned}
&\sum_{r=k}^{s} \mathbb{E}\big[\big|\zeta^{n,m}_{r+n-s,r}\big|^2\big]^{1/2} \\
&\quad \le K\left((m+1)^{((\alpha-k)\beta+1)/2} \sum_{r=k}^{m} r^{(\alpha-k)\beta/2} + \sum_{r=m}^{s} r^{2(\alpha-k)\beta+1}\right) \\
&\quad \le K\big(m^{((\alpha-k)\beta+1)/2} \log(m+1) + (m+1)^{(\alpha-k)\beta+3/2}\big),
\end{aligned} \tag{5.71}$$

where we have used the assumption $(\alpha - k)\beta < -1$ in the second inequality. Equation (5.71) shows that

$$A'_{n,m} \le Kn\big((m+1)^{(\alpha-k)\beta+1}\big(\log(m+1)\big)^2 + (m+1)^{2(\alpha-k)\beta+3}\big). \tag{5.72}$$

The substitution $\tilde{s} = -s$ and $\tilde{r} = r - s$ together with Lemma 5.6 yields that

$$A''_{n,m} = \sum_{s=0}^{n}\left(\sum_{r=s+1}^{n+s} \mathbb{E}\big[\big|\zeta^{n,m}_{r-s,r}\big|^2\big]^{1/2}\right)^2 \le K\sum_{s=0}^{n}\left(\sum_{r=s+1}^{n+s} r^{(\alpha-k)\beta+1/2}\right)^2. \tag{5.73}$$

Let $\varepsilon > 0$. For $\alpha < k - \frac{3}{2\beta}$, the inner sum on the right-hand side of (5.73) is summable. Thus, we deduce

$$A''_{n,m} \le K\sum_{s=0}^{n} s^{2(\alpha-k)\beta+3} \le K\big(n^{2(\alpha-k)\beta+4} + \log(n)\big). \tag{5.74}$$

On the other hand, for $\alpha \ge k - \frac{3}{2\beta}$ we have by Jensen's inequality that

$$\begin{aligned}
A''_{n,m} &\le Kn\sum_{s=0}^{n}\left(\sum_{r=s+1}^{n+s} r^{2(\alpha-k)\beta+1}\right) \\
&\le Kn\sum_{s=0}^{n} s^{2(\alpha-k)\beta+2} \le Kn^{2(\alpha-k)\beta+4+\varepsilon},
\end{aligned} \tag{5.75}$$

where we have used the assumption $(\alpha - k)\beta < -1$ in the second inequality and the fact that $\alpha \ge k - \frac{3}{2\beta}$ in the third inequality. Again by the substitution $\tilde{s} = -s$ and $\tilde{r} = r - s$ and Lemma 5.6, we have

$$\begin{aligned}
A'''_{n,m} &= \sum_{s=n}^{\infty}\left(\sum_{r=s+1}^{n+s} \mathbb{E}\big[\big|\zeta^{n,m}_{r+s,r}\big|^2\big]^{1/2}\right)^2 \le K\sum_{s=n}^{\infty}\left(\sum_{r=s+1}^{n+s} r^{(\alpha-k)\beta+1/2}\right)^2 \\
&\le K\sum_{s=n}^{\infty}\big(ns^{(\alpha-k)\beta+1/2}\big)^2 \le Kn^{2(\alpha-k)\beta+4},
\end{aligned} \tag{5.76}$$

where we have used the assumption $(\alpha - k)\beta < -1$ in the last inequality. Combining the estimates (5.70)–(5.76) yields (5.22).

In the proof of (5.23) and (5.24), we will use the following decomposition:

$$\sum_{r=k}^{n} Q_r^{n,m} = \sum_{s=-\infty}^{n-1} \sum_{j=(k-s)\vee 1}^{n-s} \mathbb{E}[V_{s+j}^{n,m} | \mathcal{G}_s^1] \tag{5.77}$$

which follows by the substitution $s = r - j$. To prove (5.23), we assume that $\alpha < k - 2/\beta$ and let $\varepsilon > 0$. By (5.18), we have for all $p \le \gamma < \beta/2$ that

$$\begin{aligned}
\mathbb{E}[|\Phi_{\rho_j^n}(U_{s+j,s}^n) - \Phi_{\rho_j^{n,m}}(U_{s+j,s}^n)|^2] &\le ||\rho_j^n|^\beta - |\rho_j^{n,m}|^\beta|^2 \mathbb{E}[|U_{s+j,s}^n|^{2\gamma}] \\
&\le K||\rho_j^n|^\beta - |\rho_j^{n,m}|^\beta|^2 j^{(\alpha-k)2\gamma} \\
&\le K||\rho_j^n|^\beta - |\rho_j^{n,m}|^\beta|^2 j^{(\alpha-k)\beta+2\varepsilon},
\end{aligned} \tag{5.78}$$

where the last inequality holds for γ close enough to $\beta/2$. We have that

$$\begin{aligned}
&||\rho_j^n|^\beta - |\rho_j^{n,m}|^\beta| \\
&= \left| \int_{(-\infty,s+j]\setminus[s,s+1]} |\phi_{s+j}^n(u)|^\beta \, du - \int_{(-s+j-m,s+j]\setminus[s,s+1]} |\phi_{s+j}^n(u)|^\beta \, du \right| \\
&\le \int_{-\infty}^{-m} |\phi_0^n(u)|^\beta \, du \le m^{(\alpha-k)\beta+1}.
\end{aligned} \tag{5.79}$$

By recalling the identity (5.20), we have

$$\begin{aligned}
\|\mathbb{E}[V_{s+j}^{n,m} | \mathcal{G}_s^1]\|_{L^2} &\le 2\|\Phi_{\rho_j^n}(U_{s+j,s}^n) - \Phi_{\rho_j^{n,m}}(U_{s+j,s}^n)\|_{L^2} \\
&\le K \begin{cases} m^{(\alpha-k)\beta+1} j^{(\alpha-k)\beta/2+\varepsilon}, & j = 1, \dots, m, \\ j^{(\alpha-k)\beta/2+\varepsilon}, & j > m, \end{cases}
\end{aligned} \tag{5.80}$$

where the last inequality follows from (5.78) and (5.79). By orthogonality in L^2 of the inner sums on the right-hand side of (5.77), we have that

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{r=k}^{n} Q_r^{n,m}\right)^2\right] \\
&= \sum_{s=-\infty}^{n-1} \mathbb{E}\left[\left(\sum_{j=(k-s)\vee 1}^{n-s} \mathbb{E}[V_{s+j}^{n,m} | \mathcal{G}_s^1]\right)^2\right] \\
&\le \sum_{s=-\infty}^{n-1} \left(\sum_{j=(k-s)\vee 1}^{n-s} \|\mathbb{E}[V_{s+j}^{n,m} | \mathcal{G}_s^1]\|_{L^2}\right)^2 \\
&= K\left[\sum_{s=-\infty}^{k-1} \left(\sum_{j=k-s}^{n-s} \|\mathbb{E}[V_{s+j}^{n,m} | \mathcal{G}_s^1]\|_{L^2}\right)^2 + \sum_{s=k}^{n-1} \left(\sum_{j=1}^{n-s} \|\mathbb{E}[V_{s+j}^{n,m} | \mathcal{G}_s^1]\|_{L^2}\right)^2\right] \\
&=: K[A'_{n,m} + A''_{n,m}].
\end{aligned} \tag{5.81}$$

By (5.80), we obtain the following estimate on $A'_{n,m}$:

$$
\begin{aligned}
A'_{n,m} &\le K \sum_{s=-\infty}^{k-1} \Bigg(\sum_{j=k-s}^{n-s} j^{(\alpha-k)\beta/2+\varepsilon} \Bigg)^2 \\
&= K \Bigg(\sum_{s=-k+1}^{n} \Bigg(\sum_{j=k+s}^{n+s} j^{(\alpha-k)\beta/2+\varepsilon} \Bigg)^2 + \sum_{s=n+1}^{\infty} \Bigg(\sum_{j=k+s}^{n+s} j^{(\alpha-k)\beta/2+\varepsilon} \Bigg)^2 \Bigg) \qquad (5.82)\\
&=: K(B_n + C_n).
\end{aligned}
$$

Since $(\alpha-k)\beta < -2$, we obtain the estimate

$$
B_n \le K \sum_{s=-k+1}^{n} s^{(\alpha-k)\beta+2+2\varepsilon} \le K\big(n^{(\alpha-k)\beta+3+2\varepsilon}+1\big). \qquad (5.83)
$$

By using $(\alpha-k)\beta < -1$, we get

$$
C_n \le K \sum_{s=n+1}^{\infty} n^2 s^{(\alpha-k)\beta+2\varepsilon} \le K n^{(\alpha-k)\beta+3+2\varepsilon}. \qquad (5.84)
$$

Moreover, the substitution $\tilde{s} = n-s$ and (5.80) show that

$$
\begin{aligned}
A''_{n,m} &= \sum_{s=1}^{n-k-1} \Bigg(\sum_{j=1}^{s} \big\| \mathbb{E}[V^{n,m}_{n+s+j} | \mathcal{G}^1_{n+s}] \big\|_{L^2} \Bigg)^2 \\
&\le \sum_{s=1}^{n-1} \Bigg(m^{(\alpha-k)\beta+1} \sum_{j=1}^{m} j^{(\alpha-k)\beta/2+\varepsilon} + \sum_{j=m+1}^{s} j^{(\alpha-k)\beta/2+\varepsilon} \Bigg)^2 \qquad (5.85)\\
&\le n\big(m^{2((\alpha-k)\beta+1)} + m^{(\alpha-k)\beta+2+2\varepsilon}\big) \le n m^{(\alpha-k)\beta+2+2\varepsilon},
\end{aligned}
$$

where the last inequality follows by the assumption $(\alpha-k)\beta < -2$. The above estimates (5.81)–(5.85) yield (5.23).

To prove (5.24), we suppose that $\alpha > k - 2/\beta$. We will again use the decomposition (5.86), which by the decomposition $\sum_{s=-\infty}^{n-1} = \sum_{s=-\infty}^{k-1} + \sum_{s=k}^{n-1}$ gives

$$
\sum_{r=k}^{n} (Q^{n,0}_r - Z_r) = H^{(1)}_n - H^{(2)}_n + H^{(3)}_n, \qquad (5.86)
$$

where

$$
\begin{aligned}
H^{(1)}_n &= \sum_{s=-\infty}^{k-1} \sum_{j=k-s}^{n-s} \mathbb{E}[V^{n,0}_{s+j} | \mathcal{G}^1_s], \\
H^{(2)}_n &= \sum_{s=k}^{n} \sum_{j=n-s+1}^{\infty} \{\Phi_{\rho^\infty_j}(U^\infty_{j+s,s}) - \mathbb{E}[\Phi_{\rho^\infty_j}(U^\infty_{j+s,s})]\}, \qquad (5.87)\\
H^{(3)}_n &= \sum_{s=k}^{n-1} \sum_{j=1}^{n-s} \big(\mathbb{E}[V^{n,0}_{s+j} | \mathcal{G}^1_s] - \{\Phi_{\rho^\infty_j}(U^\infty_{j+s,s}) - \mathbb{E}[\Phi_{\rho^\infty_j}(U^\infty_{j+s,s})]\}\big).
\end{aligned}
$$

In the following, we will estimate the sequences $H_n^{(i)}$ for $i = 1, 2, 3$ separately. To estimate $H_n^{(1)}$, we recall that according to (5.20) we have

$$\mathbb{E}[V_{s+j}^{n,0}|\mathcal{G}_s^1] = \Phi_{\rho_j^n}(U_{s+j,s}^n) - \mathbb{E}[\Phi_{\rho_j^n}(U_{s+j,s}^n)]. \tag{5.88}$$

For all $\gamma \in (p, \beta)$, such that $-2 < (\alpha - k)\gamma < -1$ we have by (5.88) that

$$\begin{aligned} \mathbb{E}[|H_n^{(1)}|] &\le 2 \sum_{s=-\infty}^{k-1} \sum_{j=k-s}^{n-s} \mathbb{E}[|\Phi_{\rho_j^n}(U_{s+j,s}^n)|] \le K \sum_{s=-\infty}^{k-1} \sum_{j=k-s}^{n-s} \mathbb{E}[|U_{s+j,s}^n|^\gamma] \\ &\le K \sum_{s=-k+1}^{\infty} \sum_{j=k+s}^{n+s} j^{(\alpha-k)\gamma} \\ &= K\left(\sum_{s=-k+1}^{n} \sum_{j=k+s}^{n+s} j^{(\alpha-k)\gamma} + \sum_{s=n+1}^{\infty} \sum_{j=k+s}^{n+s} j^{(\alpha-k)\gamma}\right) \\ &\le K\left(\sum_{s=-k+1}^{n} s^{(\alpha-k)\gamma+1} + \sum_{s=n+1}^{\infty} ns^{(\alpha-k)\gamma}\right) \le Kn^{(\alpha-k)\gamma+2}, \end{aligned} \tag{5.89}$$

where the second inequality follows by (5.17), the third inequality follows by (5.10), the fourth inequality follows by $(\alpha - k)\gamma < -1$ and the last inequality follows by $(\alpha - k)\gamma + 1 > -1$. Similarly, we have for all $\gamma \in (p, \beta)$ with $-2 < (\alpha - k)\gamma < -1$ that

$$\begin{aligned} \mathbb{E}[|H_n^{(2)}|] &\le 2 \sum_{s=k}^{n} \sum_{j=n-s+1}^{\infty} \mathbb{E}[|\Phi_{\rho_j^\infty}(U_{j+s,s}^\infty)|] \le K \sum_{s=0}^{n-k} \sum_{j=s+1}^{\infty} \mathbb{E}[|U_{j+n-s,n-s}^\infty|^\gamma] \\ &\le K \sum_{s=0}^{n-k} \sum_{j=s+1}^{\infty} j^{(\alpha-k)\gamma} \le K \sum_{s=1}^{n-1} s^{(\alpha-k)\gamma+1} \le Kn^{(\alpha-k)\gamma+2}. \end{aligned} \tag{5.90}$$

We will need more involved estimates on $H_n^{(3)}$. We note that $H_n^{(3)}$ is of the form $H_n^{(3)} = \sum_{s=k}^{n-1} Z_s^{(n)}$, where for each fixed $n \ge 1$, $\{Z_s^{(n)} : s = k, \dots, n-1\}$ are martingale differences; see (5.87). By the von Bahr–Esseen inequality, [40], Theorem 1, we have for any $q \in [1, 2]$

$$\begin{aligned} &\mathbb{E}[|H_n^{(3)}|^q] \\ &\quad \le K \sum_{s=k}^{n-1} \mathbb{E}[|Z_s^{(n)}|^q] \\ &\quad \le K \sum_{s=k}^{n-1} \left(\sum_{j=1}^{n-s} \|\mathbb{E}[V_{s+j}^{n,0}|\mathcal{G}_s^1] - \{\Phi_{\rho_j^\infty}(U_{j+s,s}^\infty) - \mathbb{E}[\Phi_{\rho_j^\infty}(U_{j+s,s}^\infty)]\}\|_{L^q}\right)^q \\ &\quad \le Kn\left(\sum_{j=1}^{n} \|\Phi_{\rho_j^n}(U_{j,0}^n) - \Phi_{\rho_j^\infty}(U_{j,0}^\infty)\|_{L^q}\right)^q, \end{aligned} \tag{5.91}$$

where we have used the representation (5.20) in the last inequality. By adding and subtracting $\Phi_{\rho_j^n}(U_{j,0}^{\infty})$, we get the decomposition $\Phi_{\rho_j^n}(U_{j,0}^n) - \Phi_{\rho_j^{\infty}}(U_{j,0}^{\infty}) = C_j^n + D_j^n$, where

$$C_j^n = \Phi_{\rho_j^n}(U_{j,0}^n) - \Phi_{\rho_j^n}(U_{j,0}^{\infty}) \quad \text{and} \quad D_j^n = \Phi_{\rho_j^n}(U_{j,0}^{\infty}) - \Phi_{\rho_j^{\infty}}(U_{j,0}^{\infty}).$$

To estimate the C_j^n-term, we will use the following inequality:

$$\|\phi_j^n - \phi_j^{\infty}\|_{L^\beta([0,1])} \le K n^{-1} j^{\alpha-k+1}, \qquad j = 1, \dots, n, \tag{5.92}$$

which will be proved in the following. For all $s \ge 0$, we have that $g_n(s) = n^\alpha g(s/n)$ and $g(s) = s^\alpha f(s)$. Thus, $\eta_n(s) := g_n(s) - s^\alpha = n^\alpha \psi_1(s/n)\psi_2(s/n)$, where $\psi_1(s) = s^\alpha$ and $\psi_2(s) = f(s) - f(0)$ for $s \ge 0$. For all $s > k$, there exists, as a consequence of the mean value theorem, a $\xi_s^n \in [s-k, s]$ such that

$$(D^k \eta_n)(s) = \eta_n^{(k)}(\xi_s^n) = n^{\alpha-k} \sum_{l=0}^{k} \binom{k}{l} \psi_1^{(l)}(\xi_s^n/n)\psi_2^{(k-l)}(\xi_s^n/n). \tag{5.93}$$

Equation (5.93) implies that

$$|(D^k \eta_n)(s)| \le K\left[\left(\sum_{l=0}^{k-1} n^{l-k}|\xi_s^n|^{\alpha-l}\right) + |\xi_s^n|^{\alpha-k+1} n^{-1}\right], \tag{5.94}$$

where we have used that $\psi_1^{(l)}(t) = \alpha(\alpha-1)\cdots(\alpha-l+1)t^{\alpha-l}$ for $t > 0$, $\psi_2^{(l)}$ is bounded on $(0,\infty)$ for $l = 1, \dots, k$, and that $|\psi_2(t)| \le Kt$ for all $t > 0$. Since $\phi_j^n(s) - \phi_j^{\infty}(s) = D^k \eta_n(j-s)$, we obtain by (5.94) the estimate

$$\|\phi_j^n - \phi_j^{\infty}\|_{L^\beta([0,1])} \le K \sum_{l=0}^{k} a_{l,j,n}, \tag{5.95}$$

where $a_{l,j,n} = n^{l-k} j^{\alpha-l}$ for $l = 0, \dots, k-1$, and $a_{k,j,n} = n^{-1} j^{\alpha-k+1}$. We note that $a_{k-1,j,n} = a_{k,j,n}$, and for all $l = 0, \dots, k-1$ and $j = 1, \dots, n$ we have $a_{l,j,n} = (n/j)^l n^{-k} j^\alpha \le (n/j)^{k-1} n^{-k} j^\alpha = n^{-1} j^{\alpha-k+1}$, which by (5.95) shows (5.92). Choose $q \in [1,2] \setminus \{\beta\}$ such that $q > (k-\alpha-1)\beta$. Set $r_0 = \max\{2(k-\alpha), 1\}$. We recall that $(k-\alpha)\beta \in (1,2)$ by our assumptions. Lemma 5.4 yields for all $\varepsilon > 0$ small enough

$$\begin{aligned}
&\sum_{j=1}^{n} \|C_j^n\|_{L^q} \\
&\quad \le K \sum_{j=1}^{n} \{(\|\phi_j^n\|_{L^\beta([0,1])}^{(\beta-q)/q-\varepsilon} + \|\phi_j^{\infty}\|_{L^\beta([0,1])}^{(\beta-q)/q-\varepsilon})\|\phi_j^n - \phi_j^{\infty}\|_{L^\beta([0,1])}^{1-\varepsilon} \mathbb{1}_{\{\beta>q\}} \\
&\qquad + \|\phi_j^n - \phi_j^{\infty}\|_{L^\beta([0,1])}^{\beta/q}\} \\
&\quad \le K \sum_{j=1}^{n} \{(j^{(\alpha-k)((\beta-q)/q-\varepsilon)}(n^{-1} j^{\alpha-k+1})^{1-\varepsilon} \mathbb{1}_{\{\beta>q\}} + (n^{-1} j^{\alpha-k+1})^{\beta/q}\} \qquad (5.96)
\end{aligned}$$

$$\leq K\left(n^{-1+\varepsilon}\sum_{j=1}^{n} j^{(\alpha-k)\beta/q+1+r_0\varepsilon}\mathbb{1}_{\{\beta>q\}}+n^{-\beta/q}\sum_{j=1}^{n} j^{(\alpha-k+1)\beta/q}\right)$$
$$\leq Kn^{(\alpha-k)\beta/q+1+\varepsilon(r_0+1)},$$

where we have used (5.10) and (5.92) in the second inequality, and $(\alpha-k)\beta/q+1>-1$ and $(\alpha-k+1)\beta/q>-1$ in the last inequality. To treat the D_j^n-term, we first apply Lemma 5.3 to obtain

$$(5.97)\qquad \big||\rho_j^n|^\beta-|\rho_j^\infty|^\beta\big|\leq\int_{\mathbb{R}}\big||\phi_j^n(s)|^\beta-|\phi_j^\infty(s)|^\beta\big|\,ds\leq Kn^{(\alpha-k)\beta+1}.$$

For any $\gamma\in(p,\beta/q)$, we have

$$\begin{aligned}\|D_j^n\|_{L^q}&=\mathbb{E}\big[|\Phi_{\rho_j^n}(U_{j,0}^n)-\Phi_{\rho_j^\infty}(U_{j,0}^n)|^q\big]^{1/q}\\ (5.98)\qquad&\leq K\big||\rho_j^n|^\beta-|\rho_j^\infty|^\beta\big|\mathbb{E}\big[|U_{j,0}^n|^{q\gamma}\big]^{1/q}\\ &\leq K\big||\rho_j^n|^\beta-|\rho_j^\infty|^\beta\big|\|\phi_j^n\|_{L^\beta([0,1])}^\gamma\leq Kn^{(\alpha-k)\beta+1}j^{(\alpha-k)\gamma},\end{aligned}$$

where the first inequality follows by (5.18) and the third inequality follows by (5.10) and (5.97). Hence, for any $q<(k-\alpha)\beta$, (5.98) yields that

$$(5.99)\qquad \sum_{j=1}^{n}\|D_j^n\|_{L^q}\leq Kn^{(\alpha-k)\beta+1}$$

since $(\alpha-k)\gamma<-1$ for γ chosen close enough to β/q. Combining the above estimates (5.91), (5.96) and (5.99) shows that, for any $q\in[1,2]\setminus\{\beta\}$ with $(k-\alpha-1)\beta<q<(k-\alpha)\beta$, we have

$$\begin{aligned}\mathbb{E}\big[|H_n^{(3)}|\big]\leq\mathbb{E}\big[|H_n^{(3)}|^q\big]^{1/q}&\leq K\big(n\big(n^{(\alpha-k)\beta/q+1}\big)^q\big)^{1/q}\\ &=Kn^{1/q+1+(\alpha-k)\beta/q+\varepsilon(r_0+1)}.\end{aligned}$$

We note that $1/q+1+x/q+1/x<0$ for all $x<-q$. Applying this observation for $x=(\alpha-k)\beta$, which satisfies $x<-q$ by the assumption $q<(k-\alpha)\beta$ above, it follows that $1/q+1+(\alpha-k)\beta/q<1/((k-\alpha)\beta)$. Hence, by choosing ε small enough, we find $\xi>0$ such that

$$(5.100)\qquad \mathbb{E}\big[|H_n^{(3)}|\big]\leq n^{1/((k-\alpha)\beta)-\xi}.$$

The three estimates (5.89), (5.90) and (5.100) complete the proof of the proposition.

5.7. *Proof of Lemma* 5.3. We have that $f(x)=g(x)x^{-\alpha}$ for $x>0$. By our assumptions, we may and do extend f to a k-times continuous differentiable function from $\mathbb{R}$ which also will be denoted f. We recall the notation from (4.37). By substitution, we have that

$$\int_{\mathbb{R}}\big||\phi_j^n(x)|^\beta-|\phi_j^\infty(x)|^\beta\big|\,dx=\int_0^\infty\big||D^kg_n(x)|^\beta-|h_k(x)|^\beta\big|\,dx$$

From Lemma 3.1 and condition $\alpha < k - 1/\beta$, we obtain for all $n \geq 1$ that

$$A_n := \int_n^\infty |h_k(x)|^\beta\, dx \leq K \int_n^\infty x^{(\alpha-k)\beta}\, dx \leq K n^{(\alpha-k)\beta+1}. \tag{5.101}$$

The same estimate holds for the quantity $\int_n^\infty |D^k g_n(x)|^\beta\, dx$. On the other hand, we have that

$$B_n := \left| \int_0^k |D^k g_n(x)|^\beta\, dx - \int_0^k |h_k(x)|^\beta\, dx \right| \leq K n^{-1}. \tag{5.102}$$

This follows by the estimate $||x|^\beta - |y|^\beta| \leq K \max\{|x|^{\beta-1}, |y|^{\beta-1}\}|x-y|$ for all $x, y > 0$, and that for all $x \in [0,k]$ we have by differentiability of f at zero that $|D^k g_n(x) - h_k(x)| \leq K n^{-1} x^\alpha$. Recalling that $g(x) = x_+^\alpha f(x)$ and using kth-order Taylor expansion of f at x, we deduce the following identity:

$$\begin{aligned} D^k g_n(x) &= n^\alpha \sum_{j=0}^k (-1)^j \binom{k}{j} g\big((x-j)/n\big) \\ &= \sum_{l=0}^{k-1} \frac{f^{(l)}(x/n)}{l!} \left(\sum_{j=0}^k (-1)^j \binom{k}{j} (-j/n)^l (x-j)_+^\alpha \right) \\ &\quad + \left(\sum_{j=0}^k \frac{f^{(k)}(\xi_{j,x})}{k!} (-1)^j \binom{k}{j} (-j/n)^k (x-j)_+^\alpha \right), \end{aligned}$$

where $\xi_{j,x}$ is a certain intermediate point. Now, by rearranging terms we can find coefficients $\lambda_0^n, \dots, \lambda_k^n : [k,n] \to \mathbb{R}$ and $\tilde{\lambda}_0^n, \dots, \tilde{\lambda}_k^n : [k,n] \to \mathbb{R}$ (which are in fact bounded functions in x uniformly in n) such that

$$\begin{aligned} D^k g_n(x) &= \sum_{l=0}^k \lambda_l^n(x) n^{-l} \left(\sum_{j=l}^k (-1)^j \binom{k}{j} j(j-1)\cdots(j-l+1)(x-j)_+^\alpha \right) \\ &= \sum_{l=0}^k \tilde{\lambda}_l^n(x) n^{-l} \left(\sum_{j=l}^k (-1)^j \binom{k-l}{j-l} (x-j)_+^\alpha \right) \\ &=: \sum_{l=0}^k r_{l,n}(x). \end{aligned}$$

At this stage, we remark that the term $r_{l,n}(x)$ involves $(k-l)$th order differences of the function x_+^α and $\lambda_0^n(x) = \tilde{\lambda}_0^n(x) = f(x/n)$. Now, observe that

$$\begin{aligned} C_n &:= \int_k^n \big| |D^k g_n(x)|^\beta - |h_k(x)|^\beta \big|\, dx \\ &\leq K \int_k^n \max\{|D^k g_n(x)|^{\beta-1}, |h_k(x)|^{\beta-1}\} |D^k g_n(x) - h_k(x)|\, dx. \end{aligned}$$

Since $r_{0,n}(x) = f(x/n)h_k(x)$ and $f(0) = 1$, it holds that $|r_{0,n}(x) - h_k(x)| \le K(x/n)|h_k(x)|$. We deduce that

$$\begin{aligned} &\int_k^n \max\{|D^k g_n(x)|^{\beta-1}, |h_k(x)|^{\beta-1}\}|r_{0,n}(x) - h_k(x)|\,dx \\ &\quad \le Kn^{-1}\int_k^n x^{(\alpha-k)\beta+1}\,dx \qquad (5.103) \\ &\quad \le K\begin{cases} n^{-1}, & \text{when } \alpha \in (0, k-2/\beta), \\ n^{(\alpha-k)\beta+1}, & \text{when } \alpha \in (k-2/\beta, k-1/\beta). \end{cases} \end{aligned}$$

For $1 \le l \le k$, we readily obtain the approximation

$$\int_k^n \max\{|D^k g_n(x)|^{\beta-1}, |h_k(x)|^{\beta-1}\}|r_{l,n}(x)|\,dx \le Kn^{-l}\int_k^n x^{(\alpha-k)\beta+l}\,dx.$$

If $\alpha \in (k-2/\beta, k-1/\beta)$, then $(\alpha-k)\beta + l > -1$ and we have

$$\int_k^n x^{(\alpha-k)\beta+l}\,dx \le Kn^{(\alpha-k)\beta+l+1}. \qquad (5.104)$$

When $\alpha \in (0, k-2/\beta)$, it holds that

$$\int_k^n x^{(\alpha-k)\beta+l}\,dx \le \begin{cases} K, & (\alpha-k)\beta+l < -1, \\ K\log(n)n^{(\alpha-k)\beta+l+1}, & (\alpha-k)\beta+l \ge -1. \end{cases} \qquad (5.105)$$

By (5.103), (5.104) and (5.105), we conclude that

$$C_n \le K\begin{cases} n^{-1}, & \text{when } \alpha \in (0, k-2/\beta), \\ n^{(\alpha-k)\beta+1}, & \text{when } \alpha \in (k-2/\beta, k-1/\beta). \end{cases}$$

Since $(\alpha-k)\beta + 1 < -1$ if and only if $\alpha < k - 2/\beta$, the result readily follows from (5.101) and (5.102).

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