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CONVERGENCE OF THE CENTERED MAXIMUM OF LOG-CORRELATED GAUSSIAN FIELDS

BY JIAN DING[*,1] RISHIDEEP ROY[†,1] AND OFER ZEITOUNI[‡,§,2]

University of Chicago[*], Indian Institute of Manangement Bangalore[†], Weizmann Institute[‡] and New York University[§]

We show that the centered maximum of a sequence of logarithmically correlated Gaussian fields in any dimension converges in distribution, under the assumption that the covariances of the fields converge in a suitable sense. We identify the limit as a randomly shifted Gumbel distribution, and characterize the random shift as the limit in distribution of a sequence of random variables, reminiscent of the derivative martingale in the theory of branching random walk and Gaussian chaos. We also discuss applications of the main convergence theorem and discuss examples that show that for logarithmically correlated fields; some additional structural assumptions of the type we make are needed for convergence of the centered maximum.

1. Introduction. The convergence in law for the centered maximum of various log-correlated Gaussian fields, including branching Brownian motion (BBM), branching random walk (BRW), two-dimensional discrete Gaussian free field (DGFF), etc., has recently been the focus of intensive study. Of greatest relevance to the current paper are [3, 10, 11, 23, 25]. Historically, the first result showing the correct centering and the tightness of the centered maximum for BBM appears in the pioneering work [9], followed by the proof of convergence of the law of the centered maximum [10]; the latter proof relied heavily on the F-KPP equation [19, 22] describing the evolution of the distribution of the maximum. A probabilistic description of the limit was obtained in [23], using the notion of derivative martingale that they introduce. Convergence for the centered maximum of BRW with Gaussian increments was obtained in [5], while the analogous result for general BRWs under mild assumptions was only obtained quite recently in the important work [3], using the notion of derivative martingale to describe the limit; see also [12].

When no explicit tree structure is present, exact results concerning the convergence in distribution of the maximum of logarithmically correlated Gaussian fields

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are harder to establish. It was suggested in the physics literature, using a combination of recursions, moment computations and nonrigorous replica method, that a certain universality property should hold for a large class of log-correlated fields, Gaussian or not, in particular concerning the form of the centering constant (including the so-called Bramson correction) as well as the tail of the putative limiting distribution; see [14, 20, 21]. Recently, much progress has been achieved in the mathematical study of extremes of log-correlated Gaussian fields: first, the two-dimensional DGFF was treated in [11], where convergence in distribution of the centered maximum to a randomly shifted Gumbel random variable is established. The same result was obtained independently in [25] for a general class of log-correlated fields, the so-called $*$-scale invariant models, where the covariances of the fields admit a certain kernel representation which automatically yields a decomposition into independent components according to scale. (We note in passing that such a decomposition allows for an application of the modified second moment method in the form introduced by Aïdékon [3] to treat non-Gaussian branching random walks.) In the current paper, we extend in a systematic way the class of logarithmically correlated Gaussian fields for which the same results hold. Our methods are inspired by [11], which in turn relies heavily on the modified second moment method, the modified BRW introduced in [13], tail estimates proved for the DGFF in [16] and Gaussian comparisons, which allow one to forgo the assumption of the existence of an explicit hierarchical decomposition of the field.

We now introduce the class of fields considered in the paper. Fix $d \in \mathbb{N}$ and let $V_N = \mathbb{Z}_N^d$ be the d-dimensional box of side length N with the left bottom corner located at the origin. For convenience, we consider a suitably normalized version of Gaussian fields $\{\varphi_{N,v} : v \in V_N\}$ satisfying the following.

(A.0) (*Logarithmically bounded fields*) There exists a constant $\alpha_0 > 0$ such that, for all $u, v \in V_N$,

$$\operatorname{Var} \varphi_{N,v} \le \log N + \alpha_0$$

and

$$\mathbb{E}(\varphi_{N,v} - \varphi_{N,u})^2 \le 2 \log_+ |u - v| - |\operatorname{Var} \varphi_{N,v} - \operatorname{Var} \varphi_{N,u}| + 4\alpha_0,$$

where $|\cdot|$ denotes the Euclidean norm and $\log_+ x = \log x \vee 0$. Note that Assumption (A.0) is rather mild and in particular is satisfied by the two-dimensional DGFF and $*$-scale invariant models. It is, however, strong enough to provide an a priori tight estimate on the right tail of the distribution of the maximum.

Set $M_N = \max_{v \in V_N} \varphi_{N,v}$ and

$$m_N = \sqrt{2d} \log N - \frac{3}{2\sqrt{2d}} \log \log N. \tag{1}$$

PROPOSITION 1.1. *Under Assumption* (A.0), *there exists a constant* $C = C(\alpha_0) > 0$ *such that, for all* $N \in \mathbb{N}$ *and* $z \geq 1$,

$$\mathbb{P}(M_N \geq m_N + z) \leq Cze^{-\sqrt{2d}z}e^{-C^{-1}z^2/n}. \tag{2}$$

Furthermore, for all $z \geq 1$, $y \geq 0$ *and* $A \subseteq V_N$ *we have*

$$\mathbb{P}\Big(\max_{v\in A}\varphi_{N,v} \geq m_N + z - y\Big) \leq C\bigg(\frac{|A|}{|V_N|}\bigg)^{1/2} ze^{-\sqrt{2d}(z-y)}. \tag{3}$$

Here, $|A|$ *denotes the cardinality of the set* A.

The proof of Proposition 1.1 is provided in Section 2.

By Proposition 1.1, if one has a complementary lower bound showing that for a large enough constant C, $\max_{v\in V_N}\varphi_{N,v} > m_N - C$ with high probability, it follows that the maximizer of the Gaussian field is away from the boundary with high probability. Therefore, in the study of convergence of the centered maximum, it suffices to consider the Gaussian field away from the boundary (more precisely, with distance at least δN away from the boundary where $\delta \to 0$ after $N \to \infty$). In light of this, introduce the sets $V_N^\delta = \{z \in V_N : d(z, \partial V_N) \geq \delta N\}$ and $V^\delta = [\delta, 1-\delta]^d$, where $d(z, \partial V_N) = \min\{\|z - y\|_\infty : y \notin V_N\}$ at least. Then introduce the following assumption.

(A.1) (*Logarithmically correlated fields*) For any $\delta > 0$ there exists a constant $\alpha^{(\delta)} > 0$ such that, for all $u, v \in V_N^\delta$, $|\operatorname{Cov}(\varphi_{N,v}, \varphi_{N,u}) - (\log N - \log_+ |u - v|)| \leq \alpha^{(\delta)}$.

We do not assume Assumption (A.1) for $\delta = 0$ since we wish to incorporate Gaussian fields with Dirichlet boundary conditions, such as the two- dimensional DGFF.

Assumptions (A.0) and (A.1) are enough to ensure the tightness of the sequence $\{M_N - m_N\}_N$.

THEOREM 1.2. *Under Assumptions* (A.0) *and* (A.1), *we have* $\mathbb{E}M_N = m_N + O(1)$ *where the* $O(1)$ *term depends on* α_0 *and* $\alpha^{(1/10)}$. *In addition, the sequence* $M_N - \mathbb{E}M_N$ *is tight*.

(The constant $1/10$ in Theorem 1.2 could be replaced by any positive number that is less than $1/3$.) The proof of Theorem 1.2 is provided in Section 2. A similar result (in the slightly different setup of fields indexed by a continuous parameter) appears in [1].

As we will explain later, Assumptions (A.0) and (A.1) on their own cannot ensure convergence in law for the centered maximum. To ensure the latter, we introduce the following additional assumptions. First, we assume convergence of the covariance in finite scale around the diagonal.

(A.2) (*Near diagonal behavior*) There exist a continuous function $f : (0,1)^d \mapsto \mathbb{R}$ and a function $g : \mathbb{Z}^d \times \mathbb{Z}^d \mapsto \mathbb{R}$ such that the following holds. For all $L, \varepsilon, \delta > 0$, there exists $N_0 = N_0(\varepsilon, \delta, L)$ such that, for all $x \in V^\delta$, $u, v \in [0, L]^d$ and $N \geq N_0$ we have

$$\left|\mathrm{Cov}(\varphi_{N,xN+v}, \varphi_{N,xN+u}) - \log N - f(x) - g(u,v)\right| < \varepsilon.$$

Next, we introduce an assumption on convergence of the covariance off-diagonal (in a macroscopic scale). Let $\mathcal{D}^d = \{(x,y) : x, y \in (0,1)^d, x \neq y\}$.

(A.3) (*Off diagonal behavior*) There exists a continuous function $h : \mathcal{D}^d \mapsto \mathbb{R}$ such that the following holds. For all $L, \varepsilon, \delta > 0$, there exists $N_1 = N_1(\varepsilon, \delta, L) > 0$ such that, for all $x, y \in V^\delta$ with $|x - y| \geq \frac{1}{L}$ and $N \geq N_1$ we have

$$\left|\mathrm{Cov}(\varphi_{N,xN}, \varphi_{N,yN}) - h(x,y)\right| < \varepsilon.$$

Assumptions (A.2) and (A.3) control the convergence of the covariance on both the microscopic and macroscopic scale, but allows for fluctuations of order 1 in the mesoscopic scale. It is not hard to check that both the DGFF and the $*$-scale fields satisfy Assumptions (A.0)–(A.3). A further example will be discussed in Section 5.

We remark that Assumptions (A.2) and (A.3) are not necessary for the convergence of the centered maximum. Indeed, one can violate Assumptions (A.2) and (A.3) by perturbing the field at a single vertex, but this would not affect the convergence in law of the centered maximum, since with overwhelming probability, the maximizer is not at the perturbed vertex. However, if Assumptions (A.2) and (A.3) are violated "systematically," one should not expect a convergence in law for the centered maximum. We will give two examples at the end of the Introduction as a demonstration on how violating (A.2) or (A.3) could destroy convergence in law for the centered maximum.

Our main result is the following theorem.

THEOREM 1.3. *Under Assumptions* (A.0), (A.1), (A.2) *and* (A.3), *the sequence* $\{M_N - \mathbb{E}M_N\}_N$ *converges in distribution.*

As a byproduct of our proof, we also characterize the limiting law of $(M_N - m_N)$ as a Gumbel distribution with random shift, given by a positive random variable $\mathcal{Z}$ which is the weak limit of a sequence of a sequence $\mathcal{Z}_N$, defined as

$$\mathcal{Z}_N = \sum_{v \in V_N} (\sqrt{2d}\log N - \varphi_{N,v}) e^{-\sqrt{2d}(\sqrt{2d}\log N - \varphi_{N,v})}. \tag{4}$$

In the case of BBM, the corresponding sequence $\mathcal{Z}_N$ is precisely the derivative martingale, introduced in [23]. It also occurs in the case of BRW (see [3]), and

plays a similar role in the study of critical Gaussian multiplicative chaos [17]. Even though in our case the sequence $\mathcal{Z}_N$ is not necessarily a martingale, in analogy with these previous situations we keep referring to it as *the derivative martingale*. The definition naturally extends to a derivative martingale *measure* on V_N by setting, for $A \subset V_N$,

$$\mathcal{Z}_{N,A} = \sum_{v \in A} (\sqrt{2d}\log N - \varphi_{N,v}) e^{-\sqrt{2d}(\sqrt{2d}\log N - \varphi_{N,v})}.$$

THEOREM 1.4. *Suppose that Assumptions* (A.0), (A.1), (A.2) *and* (A.3) *hold. Then the derivative martingale $\mathcal{Z}_N$ converges in law to a positive random variable $\mathcal{Z}$. In addition, the limiting law μ_∞ of $M_N - m_N$ can be expressed by*

$$\mu_\infty\big((-\infty, x]\big) = \mathbb{E} e^{-\beta^* \mathcal{Z} e^{-\sqrt{2d}x}} \qquad \text{for all } x \in \mathbb{R},$$

where β^* is a positive constant.

Theorems 1.3 and 1.4 are generalizations of [11], Theorems 1.1 and 2.5, and [25], Theorem 1.1 (see Remark 1.5 below). See also the thesis [2] for an extension to the continuous setup. Theorems 1.3 and 1.4 also overlap with [6], that studied the *conformal symmetry* (in the language of [6]) of the law of the maximum of GFF in general domains—the main results in [6] were presented in terms of the intensity measure for the extremal process, but the spatial integral of the latter yields also a representation for the law of the maximum. We emphasize, however, that in terms of proof strategies, the works of [6, 11] relied heavily on the Markov field property of the DGFF, while [25] relied crucially on the integral representation for the covariances of $*$-scale invariant fields. Comparing to [6, 11, 25], our current work studies the maximum of log-correlated Gaussian fields under *minimal* assumptions, and describes a *universal* structure of the law of the centered maximum as a randomly shifted Gumbel variable. In particular, our results shows that the Markov property of the field is not relevant to the convergence of the law of the centered maximum.

REMARK 1.5. Despite the fact that the results in this paper are stated for discrete parameters log-correlated fields, it is worthwhile to note that they can be adapted to imply, for example, [25], Theorem 1.1, which deals with the convergence of the law of the centered supremum of $*$-scale invariant log-correlated fields (with continuous parameter). This can be done by applying our results to the $*$-scale invariant field over a discretized index set, and then use the smoothness in short scales of the $*$-scale invariant field in order to pass from the discrete index set to the continuous one.

REMARK 1.6. Our proof shows in particular that the random variable $\mathcal{Z}$ appearing in Theorem 1.4 depends only on the functions $f(x), h(x, y)$ appearing in

Assumptions (A.2) and (A.3), while the constant β^* depends on other parameters as well. In particular, two sequences of fields that differ only at the microscopic level will have the same limit law for their centered maxima, up to a (deterministic) shift. We provide more details at the end of Section 4.

REMARK 1.7. In the same spirit as the preceding remark, if the field $\{\varphi_{N,v}\}_{v\in V_N}$ is stationary, then Assumption (A.2) can be removed, at the cost of replacing in Theorems 1.3 and 1.4 m_N by an appropriate sequence $\tilde{m}_N$ with $|m_N - \tilde{m}_N| = O(1)$. This is proved by a diagonalization procedure similar to that used for Remark 1.6. We omit further details.

REMARK 1.8. In [6–8], the authors used the convergence of the centered maximum, a priori information on the geometric properties of the clusters of near-maxima of the DGFF and a beautiful invariance argument to derive the convergence in law of the process of extrema of the two-dimensional DGFF and its properties. A natural extension of our work would be to study the extremal process in the class of processes studied here, and tie it to properties of the derivative martingale measure.

A word on proof strategy. The convergence proof in this paper is closely related to the one [11], that dealt with 2D GFF. Related ideas appear also in [25], and both use arguments inspired by [3]. The proof in [11] consists of three main steps:

(a) Decompose the DGFF to a sum of a coarse field and a fine field (which itself is a DGFF), and further approximate the fine field as a sum of modified branching random walk (see Section 2.1 for definition) and a local DGFF. It is crucial for the proof that the different components are independent of each other, and that the approximation error is small enough so that the value of the maximum is not altered significantly. These approximations were constructed using heavily the Markov field property of DGFF, and detailed estimates for the Green function of random walk.
(b) Use a modified second moment method in order to compute the asymptotics of the right tail for the distribution of the maximum of the fine field, as well as derive a limiting distribution for the location of the maximizer in the fine field.
(c) Combine the limiting right tail estimates for the maximum of the fine field and the behavior of the coarse field to deduce the convergence in law.

(We note that the continuous parameter analogue of the MBRW, introduced in [1] under the name MBBM, is $*$-scale invariant in the language of [25]; The MBRW in the sketch above could be replaced by the MBBM or by other $*$-scale invariant processes.)

In the general setup of logarithmically correlated fields, it is not a priori clear how can one decompose the field as an (independent) sum of a coarse field, an

MBRW and a local field, as the Markov field property (or, as in [25], the decomposition into independent scales) is no longer available. A natural approach under our assumptions is to employ the self-similarity of the fields, and to approximate the coarse and local fields by an instance of $\{\varphi_{K,v} : v \in V_K\}$ for some $K \ll N$. One difficulty in this attempt is to control the error of the approximation and its influence on the law of the maximum. In order to address this issue, we partition the box V_N to sub-boxes congruent to V_L, and borrow a key idea from [7] to show that the law of the maximum of a log-correlated fields has the following invariance property: if one adds i.i.d. Gaussian variables with variance $O(1)$ to each sub-box of the field (here the same variable will be added to each vertex in the same sub-box), where the size L of the sub-box is either K or N/K (assuming K grows to infinity arbitrarily slow in N), then the law of the maximum for the perturbed field is simply a shift of the original law where the shift can be explicitly determined (see Lemma 3.1). In light of this, in Section 4.1 we approximate the field $\{\varphi_{N,v}\}$ by the sum of coarse field (which is given by $\{\varphi_{KL,v} : v \in V_{KL}\}$), an MBRW and a local field (which is given by independent copies of $\{\varphi_{K'L',v} : v \in V_{K'L'}\}$) (here the parameters satisfy $N \gg K' \gg L' \gg K \gg L$). In this construction, we make sure that the error in the covariance between two vertices is $o(1)$ if their distance is not in between L and N/L', and the error is $O(1)$ otherwise. Then we apply Lemma 3.1 (and Lemma 3.2) to argue that our approximation indeed recovers the law of the maximum for the original field. In Section 4.2, we present the proof for the convergence in law for the centered maximum of the approximated field we constructed and, as in [11], it readily also yields the convergence in distribution for the derivative martingale constructed from the original field.

As in the case of the DGFF in two dimensions, a number of properties for the log-correlated fields are needed, and are proved by adapting or modifying the arguments used in that case. Those properties are:

(a) The tightness of $M_N - m_N$, and the bounds on the right and left tails of $M_N - m_N$ as well as certain geometric properties of maxima for the log-correlated fields under consideration, follow from modifying arguments in [13, 15, 16]. This is explained in Section 2.
(b) Precise asymptotics for the right tail of the distribution of the maximum of the fine field follow from arguments similar to [11] with a number of simplifications, as our fine field has a nicer structure than its analogue in [11], whereas the coarse field employed in this paper is constant over each box; in particular, there is no need to consider the distribution for the location of the maximizer in the fine field as done in [11]. The adaption is explained in Section 6.

The role of Assumptions (A.2) *and* (A.3). We next construct two examples that demonstrate that one cannot totally dispense of Assumptions (A.2) and (A.3). Since the examples are only ancillary to our main result, we will give only give a brief sketch for the verification of the claims made concerning these examples.

EXAMPLE 1.9. Fix $d = 2$ and let $\{\varphi_{N,v} : v \in V_N\}$ be the DGFF on V_N [normalized so that it satisfies Assumptions (A.0), (A.1), (A.2) and (A.3)], with $Z_N = \max_{v\in V_N} \varphi_{N,v}$. Let $V_{N,1}$ and $V_{N,2}$ be the left and right halves of the box V_N. Let $\{\varepsilon_{N,v} : v \in V_N\}$ and X be i.i.d. standard Gaussian variables. Let $\sigma'_N > 0$ be a constant, to be chosen below. Define

$$\tilde{\varphi}_{N,v} = \begin{cases} \varphi_{N,v} + \sigma X + \varepsilon_{N,v}, & v \in V_{N,1}, \\ \varphi_{N,v}, & v \in V_{N,2}, \end{cases}$$

$$\hat{\varphi}_{N,v} = \begin{cases} \varphi_{N,v} + \sigma X, & v \in V_{N,1}, \\ \varphi_{N,v} + \sigma'_N \varepsilon_{N,v}, & v \in V_{N,2}. \end{cases}$$

Set $\tilde{M}_N = \max_{v\in V_N} \tilde{\varphi}_{N,v}$ and $\hat{M}_N = \max_{v\in V_N} \hat{\varphi}_{N,v}$. We first claim that there exist σ'_N depending on (N, σ) but bounded from above by an absolute constant such that $\mathbb{E}\tilde{M}_N = \mathbb{E}\hat{M}_N$. In order to see that, note that, by Theorem 1.2,

$$\mathbb{E}\tilde{M}_N \le \mathbb{E} \max_{v\in V_{N/2}} \varphi_{N,v} + \sigma \mathbb{E} \max(0, X) + O(1),$$

where $O(1)$ is an error term independent of all parameters, while

$$\mathbb{E}\tilde{M}_N \ge \mathbb{E} \max_{v\in V_{N/2}} \varphi_{N,v} + \sigma \mathbb{E} \max(0, X).$$

In addition, by considering a $N/2$-box in the left-hand side and dividing the right half box into two copies of $N/2$-boxes, one gets that

$$\begin{aligned} \mathbb{E}\hat{M}_N &\ge \mathbb{E} \max\bigl(Z_{N/2} + \sigma X, Z'_{N/2} + \sigma'_N \varepsilon', Z''_{N/2} + \sigma'_N \varepsilon''\bigr) \\ &\ge \mathbb{E} Z_{N/2} + \frac{1}{2}\sigma'_N \mathbb{E} \max\bigl(\varepsilon', \varepsilon''\bigr) + \sigma \mathbb{E} X \mathbf{1}_{X\ge 0}, \end{aligned}$$

where $Z_{N/2}, Z'_{N/2}, Z''_{N/2}$ are three independent copies with law $\max_{v\in V_{N/2}} \varphi_{N,v}$ and $\varepsilon' = \varepsilon_{N,v_1^*}, \varepsilon'' = \varepsilon''_{N,v_2^*}$ (here v_1^* and v_2^* are the maximizers of the DGFF in the two $N/2$-boxes on the right half of V_N, respectively). The claim follows from combining the last two displays.

Now, choose σ to be a large fixed constant so that for $0 < \lambda < \log\log N$,

$$\begin{aligned} \mathbb{P}(\tilde{M}_N \ge \mathbb{E}Z_N + \lambda) &\ge \mathbb{P}\Bigl(\max_{v\in V_{N,1}} \{\varphi_{N,v} + \sigma X + \varepsilon_{N,v}\} \ge \mathbb{E}Z_N + \lambda \Bigr) \\ &\ge \mathbb{P}\Bigl((1 + 1/4\log N) \max_{v\in V_{N,1}} \{\varphi_{N,v} + \sigma X\} \ge \mathbb{E}Z_N + \lambda \Bigr) \\ &\ge \mathbb{P}\Bigl(\max_{v\in V_{N,1}} \varphi_{N,v} + \sigma X \ge \mathbb{E}Z_N + \lambda - 1/10 \Bigr). \end{aligned} \tag{5}$$

[Here, the second inequality is due to Slepian's lemma (Lemma 2.4) and the fact that σ is large, while the last inequality uses that $\frac{2}{(1+1/(4\log N))} \le 2 - (\log N)/10$

for N large.] Further,

$$\begin{aligned}\mathbb{P}(\hat{M}_N \geq \mathbb{E}Z_N + \lambda) &\leq \mathbb{P}\Big(\max_{v\in V_{N,1}} \varphi_{N,v} + \sigma X \geq \mathbb{E}Z_N + \lambda\Big) \\ &\quad + \mathbb{P}\Big(\max_{v\in V_{N,2}} \varphi_{N,v} + \varepsilon'_{N,v} \geq \mathbb{E}Z_N + \lambda\Big) \\ &\leq \mathbb{P}\Big(\max_{v\in V_{N,1}} \varphi_{N,v} + \sigma X \geq \mathbb{E}Z_N + \lambda\Big) + O(1)\lambda e^{-2\lambda},\end{aligned} \tag{6}$$

where the last inequality follows from Proposition 1.1. Combining (5) and (6) and using the form of the limiting right tail of the two-dimensional DGFF as in [11], Proposition 4.1, one obtains that for λ, σ sufficiently large but independent of N,

$$\limsup_{N\to\infty} \mathbb{P}(\tilde{M}_N \geq \mathbb{E}Z_N + \lambda) \geq (1+c) \limsup_{N\to\infty} \mathbb{P}(\hat{M}_N \geq \mathbb{E}Z_N + \lambda) \geq c(\sigma)\lambda e^{-2\lambda},$$

where $c > 0$ is an absolute constant and $c(\sigma)$ satisfies $c(\sigma) \to_{\sigma\to\infty} \infty$. This implies that the laws of $\tilde{M}_N - E\tilde{M}_N$ and $\hat{M}_N - E\hat{M}_N$ do not coincide in the limit $N \to \infty$.

Finally, set $\bar{\varphi}_{N,v} = \tilde{\varphi}_{N,v}$ for all $v \in V_N$ and odd N, and $\bar{\varphi}_{N,v} = \hat{\varphi}_{N,v}$ for all $v \in V_N$ and even N. One then sees that the sequence of Gaussian fields $\{\bar{\varphi}_{N,v} : v \in V_N\}$ satisfies Assumptions (A.0), (A.1) and (A.3) [while not satisfying (A.2)], but the law of the centered maximum does not converge.

EXAMPLE 1.10. Let $\{\varphi_{N,v} : v \in V_N\}$ be a sequence of Gaussian fields satisfying (A.0), (A.1) and (A.2), such that the law of the centered maximum converges. Consider the fields $\{\tilde{\varphi}_{N,v} : v \in V_N\}$ where $\tilde{\varphi}_{N,v} = \varphi_{N,v} + \mathbf{1}_{\{N \text{ is even}\}} X_N$ with X_N a sequence of i.i.d. standard Gaussian variables. Then the field $\{\tilde{\varphi}_{N,v} : v \in V_N\}$ satisfies (A.0), (A.1) and (A.2) (possibly increasing the values of $\alpha^{(\delta)}$ by 1 for all $0 \leq \delta \leq 1$). However, the centered law of the maximum of $\{\tilde{\varphi}_{N,v} : v \in V_N\}$ cannot converge.

2. Expectation and tightness for the maximum. This section is devoted to the proofs of Proposition 1.1 and Theorem 1.2, and to an auxiliary lower bound on the right tail of the distribution of the maximum; see Lemma 2.2. We note that Proposition 1.1 gives the correct right tail behavior of the distribution of the maximum. In contrast, given the proposition, in order to prove Theorem 1.2, one needs an upper bound on the *left* tail of that distribution. In the generality of this work, one cannot hope for a universal sharp estimate on the left tail, as witnessed by the drastically different left tails exhibited in the cases of the modified branching random walk and the two-dimensional DGFF; see [15]. We will, however, provide the following universal *upper* bound for the decay of the left tail.

LEMMA 2.1. *Under Assumption* (A.1), *there exist constants* $C, c > 0$ (*depending only on* $\alpha_{1/10}, d$) *so that for all* $n \in \mathbb{N}$ *and* $0 \leq \lambda \leq (\log n)^{2/3}$,

$$\mathbb{P}\Big(\max_{v\in V_N} \varphi_{N,v} \leq m_N - \lambda\Big) \leq Ce^{-c\lambda}.$$

Theorem 1.2 follows at once from Proposition 1.1 and Lemma 2.1.

Later, we will need the following complimentary lower bound on the right tail.

LEMMA 2.2. *Under Assumption* (A.1), *there exists a constant* $C > 0$ *depending only on* $(\alpha_0, \alpha^{(1/10)}, d)$ *such that, for all* $\lambda \in [1, \sqrt{\log N}]$,

$$\mathbb{P}(M_N > m_N + \lambda) \geq C^{-1}\lambda e^{-\sqrt{2d}\lambda}.$$

We emphasize that all arguments in this section are very similar to arguments used in the case of the DGFF in dimension two, and employ comparison with either BRW or MBRW. These ideas can be traced back to [13] and [16]. Versions of the same argument also appear in [1, 25]. For this reason, some of the proofs are only sketched, with references provided to corresponding proofs in the literature.

2.1. *Branching random walk and modified branching random walk.* The study of extrema for log-correlated Gaussian fields is possible because they exhibit an approximate tree structure and can be efficiently compared with branching random walk and the modified branching random walk introduced in [13]. In this subsection, we briefly review the definitions of BRW and MBRW in $\mathbb{Z}^d$. We remark that the MBRW can be seen as a discrete analogue of the $*$-scale invariant log-correlated fields studied in [25]; we further remark that the natural continuous parameter extension of the MBRW is not exactly a $*$-scale invariant field since the corresponding kernel function (in the language of [25]) is not continuous.

Suppose $N = 2^n$ for some $n \in \mathbb{N}$. For $j = 0, 1, \ldots, n$, define $\mathcal{B}_j$ to be the set of d-dimensional cubes of side length 2^j with corners in $\mathbb{Z}^d$. Define $\mathcal{BD}_j$ to be those elements of $\mathcal{B}_j$ which are of the form $([0, 2^j - 1] \cap \mathbb{Z})^d + (i_1 2^j, i_2 2^j, \ldots, i_d 2^j)$, where $i_1, i_2, \ldots, i_d$ are integers. For $x \in V_N$, define $\mathcal{B}_j(x)$ to be those elements of $\mathcal{B}_j$ which contains x. Define $\mathcal{BD}_j(x)$ similarly.

Let $\{a_{j,B}\}_{j\geq 0, B\in\mathcal{BD}_j}$ be a family of i.i.d. Gaussian variables of variance $\log 2$. Define the *branching random walk* (BRW) $\{\mathcal{R}_{N,z}\}_{z\in V_N}$ by

$$\mathcal{R}_{N,z} = \sum_{j=0}^{n} \sum_{B\in\mathcal{BD}_j(z)} a_{j,B}, \qquad z \in V_N.$$

Let $\mathcal{B}_j^N$ be the subset of $\mathcal{B}_j$ consisting of elements of the latter with lower left corner in V_N. Let $\{b_{j,B} : j \geq 0, B \in \mathcal{B}_j^N\}$ be a family of independent Gaussian variables such that $\operatorname{Var} b_{j,B} = \log 2 \cdot 2^{-dj}$ for all $B \in \mathcal{B}_j^N$. Write $B \sim_N B'$ if $B = B' + (i_1 N, \ldots, i_d N)$ for some integers $i_1, \ldots, i_d \in \mathbb{Z}$. Let

$$b_{j,B}^N = \begin{cases} b_{j,B}, & B \in \mathcal{B}_j^N, \\ b_{j,B'}, & B \sim_N B' \in \mathcal{B}_j^N. \end{cases}$$

Define the *modified branching random walk* (MBRW) $\{\mathcal{S}_{N,z}\}_{z\in V_N}$ by

$$\mathcal{S}_{N,z} = \sum_{j=0}^{n} \sum_{B\in\mathcal{B}_j(z)} b_{j,B}^N, \qquad z \in V_N. \tag{7}$$

The proof of the following lemma is a straightforward adaption of [13], Lemma 2.2, for dimension d, which we omit.

LEMMA 2.3. *There exists a constant C depending only on d such that, for $N = 2^n$ and $x, y \in V_N$,*

$$\left|\mathrm{Cov}(\mathcal{S}_{N,x}, \mathcal{S}_{N,y}) - \left(\log N - \log\left(|x-y|_N \vee 1\right)\right)\right| \le C,$$

where $|x-y|_N = \min_{y'\sim_N y} |x-y'|$.

In the rest of the paper, we assume that the constants $\alpha_0, \alpha^{(\delta)}$ in Assumptions (A.0) and (A.1) are taken large enough so that the MBRW satisfies the assumptions.

2.2. *Comparison of right tails.* The following Slepian's comparison lemma for Gaussian processes [28] will be useful.

LEMMA 2.4. *Let $\mathcal{A}$ be an arbitrary finite index set and let $\{X_a : a \in \mathcal{A}\}$ and $\{Y_a : a \in \mathcal{A}\}$ be two centered Gaussian processes such that: $\mathbb{E}(X_a - X_b)^2 \ge \mathbb{E}(Y_a - Y_b)^2$, for all $a, b \in \mathcal{A}$ and $\mathrm{Var}(X_a) = \mathrm{Var}(Y_a)$ for all $a \in \mathcal{A}$. Then $\mathbb{P}(\max_{a\in\mathcal{A}} X_a \ge \lambda) \ge \mathbb{P}(\max_{a\in\mathcal{A}} Y_a \ge \lambda)$ for all $\lambda \in \mathbb{R}$.*

The next lemma compares the right tail for the maximum of $\{\varphi_{N,v} : v \in V_N\}$ to that of a BRW.

LEMMA 2.5. *Under Assumption* (A.0), *there exists a positive integer $\kappa = \kappa(\alpha_0)$ such that, for all N and $\lambda \in \mathbb{R}$ and any subset $A \subseteq V_N$:*

$$\mathbb{P}\Big(\max_{v\in A} \varphi_{N,v} \ge \lambda\Big) \le 2\mathbb{P}\Big(\max_{v\in 2^\kappa A} \mathcal{R}_{2^\kappa N, v} \ge \lambda\Big). \tag{8}$$

PROOF. For $\kappa \in \mathbb{N}$, consider the map:

$$\psi_N = \psi_N^{(\kappa)} : V \mapsto 2^\kappa V \qquad \text{such that } \psi_N(v) = 2^\kappa v. \tag{9}$$

By Assumption (A.0), we can choose a sufficiently large κ depending on α_0 such that $\mathrm{Var}(\varphi_{N,v}) \le \mathrm{Var}(\mathcal{R}_{2^\kappa N, \psi_N(v)})$ for all $v \in V_N$. So, we can choose a collection of positive numbers:

$$a_v^2 = \mathrm{Var}\,\mathcal{R}_{2^\kappa N, \psi_N(v)} - \mathrm{Var}\,\varphi_{N,v},$$

such that $\operatorname{Var}(\varphi_{N,v} + a_v X) = \operatorname{Var}(\mathcal{R}_{2^\kappa N, \psi_N(v)})$ for all $v \in V_N$, where X is a standard Gaussian random variable, independent of everything else. Since the BRW has constant variance over all vertices, we get that

$$\begin{aligned}
&\mathbb{E}(\varphi_{N,u} + a_u X - \varphi_{N,v} - a_v X)^2 \\
&\quad \le \mathbb{E}(\varphi_{N,u} - \varphi_{N,v})^2 + (a_v - a_u)^2 \\
&\quad \le \mathbb{E}(\varphi_{N,u} - \varphi_{N,v})^2 + |\operatorname{Var}\varphi_{N,v} - \operatorname{Var}\varphi_{N,u}|.
\end{aligned}$$

Combined with Assumption (A.0), it yields that

$$\mathbb{E}(\varphi_{N,u} + a_u X - \varphi_{N,v} - a_v X)^2 \le 2\log_+ |u - v| + 4\alpha_0.$$

Since $\mathbb{E}(\mathcal{R}_{2^\kappa N, \psi_N(u)} - \mathcal{R}_{2^\kappa N, \psi_N(v)})^2 - 2\log_+ |u - v| \ge \log 2\kappa - C_0$ (where C_0 is an absolute constant), we can choose sufficiently large κ depending only on α_0 such that

$$\begin{aligned}
&\mathbb{E}(\varphi_{N,u} + a_u X - \varphi_{N,v} - a_v X)^2 \\
&\quad \le \mathbb{E}(\mathcal{R}_{2^\kappa N, \psi_N(u)} - \mathcal{R}_{2^\kappa N, \psi_N(v)})^2 \qquad \text{for all } u, v \in V_N.
\end{aligned}$$

Combined with Lemma 2.4, it follows that for all $\lambda \in \mathbb{R}$ and $A \subseteq V_N$:

$$\mathbb{P}\Big(\max_{v\in A} \varphi_{N,v} + a_v X \ge \lambda\Big) \le \mathbb{P}\Big(\max_{v\in A} \mathcal{R}_{2^\kappa N, \psi_N(v)} \ge \lambda\Big).$$

In addition, by independence and symmetry of X we have

$$\mathbb{P}\Big(\max_{v\in A} \varphi_{N,v} + a_v X \ge \lambda\Big) \ge \mathbb{P}\Big(\max_{v\in A} \varphi_{N,v} \ge \lambda, X \ge 0\Big) = \frac{1}{2}\mathbb{P}\Big(\max_{v\in A} \varphi_{N,v} \ge \lambda\Big).$$

This completes the proof of the desired bound. □

PROOF OF PROPOSITION 1.1. An analogous statement was proved in [11], Lemma 3.8, for the case of 2D DGFF. In the proof of [11], Lemma 3.8, the desired inequality was first proved for BRW on the 2D lattice and then deduced for 2D DGFF applying [16], Lemma 2.6, which is the analogue of Lemma 2.5 above. The argument for BRW in [11], Lemma 3.8, carries out (essentially with no change) from dimension two to dimension d. Given that, an application of Lemma 2.5 completes the proof of the proposition. □

A complementary lower bound on the right tail is also available.

LEMMA 2.6. *Under Assumption* (A.1), *there exists a positive integer* $\kappa = \kappa(\alpha^{(1/10)})$ *such that, for all* N *and* $\lambda \in \mathbb{R}$:

$$\mathbb{P}\Big(\max_{v\in V_N} \varphi_v^N \ge \lambda\Big) \ge \frac{1}{2}\mathbb{P}\Big(\max_{v\in V_{2^{-\kappa}N}} \mathcal{S}_{2^{-\kappa}N,v} \ge \lambda\Big). \tag{10}$$

PROOF. It suffices to consider $M_N^{(1/10)} = \max_{v \in V_N^{1/10}} \varphi_{N,v}$. By Assumption (A.1) and an argument analogous to that used in the proof of Lemma 2.5 (which can be traced back to the proof of [16], Lemma 2.6), one deduces that for $\kappa = \kappa(\alpha^{(1/10)})$,

$$\mathbb{P}\big(M_N^{(1/10)} \geq \lambda\big) \geq \frac{1}{2}\mathbb{P}\Big(\max_{v \in V_{2^{-\kappa}N}} \mathcal{S}_{2^{-\kappa}N,v} \geq \lambda\Big) \qquad \text{for all } \lambda \in \mathbb{R}.$$

This completes the proof of the lemma. □

We also need the following estimate on the right tail for MBRW in d-dimension. The proof is a routine adaption of the proof of [16], Lemma 3.7, to arbitrary dimension, and is omitted.

LEMMA 2.7. *There exists an absolute constant $C > 0$ such that, for all $\lambda \in [1, \sqrt{\log n}]$, we have*

$$C^{-1}\lambda e^{-\sqrt{2d}\lambda} \leq \mathbb{P}\Big(\max_{v \in V_N} \mathcal{S}_{N,v} > m_N + \lambda\Big) \leq C\lambda e^{-\sqrt{2d}\lambda}.$$

PROOF OF LEMMA 2.2. Combine Lemma 2.6 and Lemma 2.7. □

2.3. *An upper bound on the left tail.* This subsection is devoted to the proof of Lemma 2.1. The proof consists of two steps: (1) a derivation of an exponential upper bound on the left tail for the MBRW; (2) a comparison of the left tail for general log-correlated Gaussian field to that of the MBRW.

LEMMA 2.8. *There exist constants $C, c > 0$ so that for all $n \in \mathbb{N}$ and $0 \leq \lambda \leq (\log n)^{2/3}$:*

$$\mathbb{P}\Big(\max_{v \in V_N} \mathcal{S}_{N,v} \leq m_N - \lambda\Big) \leq Ce^{-c\lambda}.$$

PROOF. A trivial extension of the arguments in [13] (for the MBRW in dimension two) yields the tightness for the maximum of the MBRW in dimension d around its expectation, with the latter given by (1). Therefore, there exist constants $\kappa, \beta > 0$ such that, for all $N \geq 4$,

$$\mathbb{P}\Big(\max_{v \in V_N} \mathcal{S}_{N,v} \geq m_N - \beta\Big) \geq 1/2. \tag{11}$$

In addition, a simple calculation gives that for all $N \geq N' \geq 4$ (adjusting the value of κ if necessary),

$$\begin{aligned} \sqrt{2d}\log(N/N') - \frac{3}{4d}\log(\log N/\log N') - \kappa \leq m_N - m_{N'} \\ \leq \sqrt{2d}\log(N/N') + \kappa. \end{aligned} \tag{12}$$

Let $\lambda' = \lambda/2$ and $N' = N\exp(-\frac{1}{\sqrt{2d}}(\lambda' - \beta - \kappa - 4))$. By (12), one has $m_N - m_{N'} \le \lambda' - \beta$. Divide V_N into disjoint boxes of side length N', and consider a maximal collection $\mathcal{B}$ of N'-boxes such that all the pairwise distances are at least $2N'$, implying that $|\mathcal{B}| \ge \exp(\frac{\sqrt{d}}{\sqrt{2}}(\lambda' - \beta - \kappa - 8 - 4\sqrt{d}))$. Now consider the modified MBRW:

$$\tilde{\mathcal{S}}_{N,v} = g_{N',v} + \phi \qquad \forall v \in B \in \mathcal{B},$$

where ϕ is a zero mean Gaussian variable with variance $\log(N/N')$ and $\{g_{N',v} : v \in B\}_B$ are the MBRWs defined on the boxes B, independently of each other and of ϕ. It is straightforward to check that

$$\operatorname{Var}\mathcal{S}_{N,v} = \operatorname{Var}\tilde{\mathcal{S}}_{N,v} \quad \text{and} \quad \mathbb{E}\mathcal{S}_{N,v}\mathcal{S}_{N,u} \le \mathbb{E}\tilde{\mathcal{S}}_{N,v}\tilde{\mathcal{S}}_{N,u} \qquad \text{for all } u, v \in \bigcup_{B\in\mathcal{B}} B.$$

Combined with Lemma 2.4, it follows that

$$\begin{aligned}\mathbb{P}\Big(\max_{v\in V_N}\mathcal{S}_{N,v} \le t\Big) &\le \mathbb{P}\Big(\max_{v\in\bigcup_{B\in\mathcal{B}}B}\mathcal{S}_{N,v} \le t\Big)\\ &\le \mathbb{P}\Big(\max_{v\in\bigcup_{B\in\mathcal{B}}B}\tilde{\mathcal{S}}_{N,v} \le t\Big) \qquad \text{for all } t \in \mathbb{R}.\end{aligned} \tag{13}$$

By (11), one has that for each $B \in \mathcal{B}$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{v\in B} g_{N',v} \ge m_N - \lambda'\Big) &= \mathbb{P}\Big(\sup_{v\in B} g_{N',v} \ge m_{N'} + m_N - m_{N'} - \lambda'\Big)\\ &\ge \mathbb{P}\Big(\sup_{v\in B} g_{N',v} \ge m_{N'} - \beta\Big) \ge \frac{1}{2},\end{aligned}$$

and, therefore,

$$\mathbb{P}\Big(\sup_{v\in\bigcup_{B\in\mathcal{B}}B} g_{N',v} < m_N - \lambda'\Big) \le \left(\frac{1}{2}\right)^{|\mathcal{B}|}.$$

Thus,

$$\begin{aligned}\mathbb{P}\Big(\max_{v\in\bigcup_{B\in\mathcal{B}}B}\tilde{\mathcal{S}}_{N,v} \le m_N - \lambda\Big) &\le \mathbb{P}\Big(\sup_{v\in\bigcup_{B\in\mathcal{B}}B} g_{N',v} < m_N - \lambda'\Big) + \mathbb{P}(\phi \le -\lambda')\\ &\le Ce^{-c\lambda'}\end{aligned}$$

for some constants $C, c > 0$. Combined with (13), this completes the proof of the lemma. □

PROOF OF LEMMA 2.1. In order to prove Lemma 2.1, we will compare the maximum of a sparsified version of the log-correlated field to that of a modified version of MBRW. By Assumption (A.1) and Lemma 2.3, there exists a $\kappa_0 = \kappa_0(\alpha^{(1/10)})$ such that, for all $\kappa \ge \kappa_0$,

$$\operatorname{Var}(\varphi_{2^\kappa N, 2^\kappa v}) \le \operatorname{Var}(\mathcal{S}_{2^{2\kappa}N,v}) \qquad \text{for all } v \in V_N^{1/10}.$$

Therefore, one can choose a collection of positive numbers $\{a_v : v \in V_N^{1/10}\}$ such that

$$\mathrm{Var}(\varphi_{2^\kappa N,2^\kappa v} + a_v X) = \mathrm{Var}(\mathcal{S}_{2^{2\kappa}N,v}),$$

where X is a standard Gaussian variable. Since the MBRW has constant variance, we have that $|a_v - a_u| \le C_1$ for some constant $C_1 = C_1(\alpha^{(1/10)}) > 0$. By Lemma 2.3 again, one has

$$\mathbb{E}(\mathcal{S}_{2^{2\kappa}N,v} - \mathcal{S}_{2^{2\kappa}N,u})^2 \le 2\log_+ |u - v| + O(1),$$

where the $O(1)$ term is bounded by a absolute constant. On the other hand, for all $u, v \in V_N^{1/10}$,

$$\mathbb{E}(\varphi_{2^\kappa N,2^\kappa v} + a_v X - \varphi_{2^\kappa N,2^\kappa u} - a_u X)^2 \ge \log 2 \cdot \kappa + 2\log_+ |u - v| - O_{\alpha^{(1/10)}}(1),$$

where $O_{\alpha^{(1/10)}}(1)$ is a term that is bounded by a constant depending only on $\alpha^{(1/10)}$. Therefore, there exists a $\kappa = \kappa(\alpha^{(1/10)})$ such that, for all $u, v \in V_N^{1/10}$,

$$\mathbb{E}(\varphi_{2^\kappa N,2^\kappa v} + a_v X - \varphi_{2^\kappa N,2^\kappa u} - a_u X)^2 \ge \mathbb{E}(\mathcal{S}_{2^{2\kappa}N,v} - \mathcal{S}_{2^{2\kappa}N,u})^2.$$

Combined with Lemma 2.4, this implies that for a suitable C_κ depending on κ,

$$\begin{aligned}
&\mathbb{P}\Big(\max_{v\in V_N} \varphi_{2^\kappa N,2^\kappa v} \le m_N - \lambda\Big) \\
&\quad\le \mathbb{P}\Big(\max_{v\in V_N^{1/10}} (\varphi_{2^\kappa N,2^\kappa v} + a_v X) \le m_N - \lambda/2\Big) + \mathbb{P}(X \le -\lambda/C_\kappa) \\
&\quad\le \mathbb{P}\Big(\max_{v\in V_N^{1/10}} \mathcal{S}_{2^{2\kappa}N,v} \le m_N - \lambda/2\Big) + \mathbb{P}(X \le -\lambda/C_\kappa).
\end{aligned} \tag{14}$$

There are number of ways to bound $\mathbb{P}(\max_{v\in V_N^{1/10}} \mathcal{S}_{2^{2\kappa}N,v} \le m_N - \lambda/2)$, and we choose not to optimize the bound, but instead simply apply the FKG inequality [27]. More precisely, we note that there exists a collection of boxes $\mathcal{V}$ with $|\mathcal{V}| \le 2^{4d\kappa}$ where each box is a translated copy of $V_N^{1/10}$ such that $V_{2^{2\kappa}N} \subseteq \bigcup_{V\in\mathcal{V}} V$. Since $\{\max_{v\in V_{2^{2\kappa}N}} \mathcal{S}_{2^{2\kappa}N,v} \le m_N - \lambda/2\} = \bigcap_{V\in\mathcal{V}}\{\max_{v\in V} \mathcal{S}_{2^{2\kappa}N,v} \le m_N - \lambda/2\}$, the FKG inequality gives that

$$\mathbb{P}\Big(\max_{v\in V_{2^{2\kappa}N}} \mathcal{S}_{2^{2\kappa}N,v} \le m_N - \lambda/2\Big) \ge \Big(\mathbb{P}\Big(\max_{v\in V_N^{1/10}} \mathcal{S}_{2^{2\kappa}N,v} \le m_N - \lambda/2\Big)\Big)^{2^{4d\kappa}}.$$

Combined with (14) and Lemma 2.8, this completes the proof of the lemma. □

3. Robustness of the maximum under perturbations. The main goal of this section is to establish that the law of the maximum for a log-correlated Gaussian field is robust under certain perturbations. These invariance properties will be crucial in Section 4.1 when constructing a new field that approximates our target field.

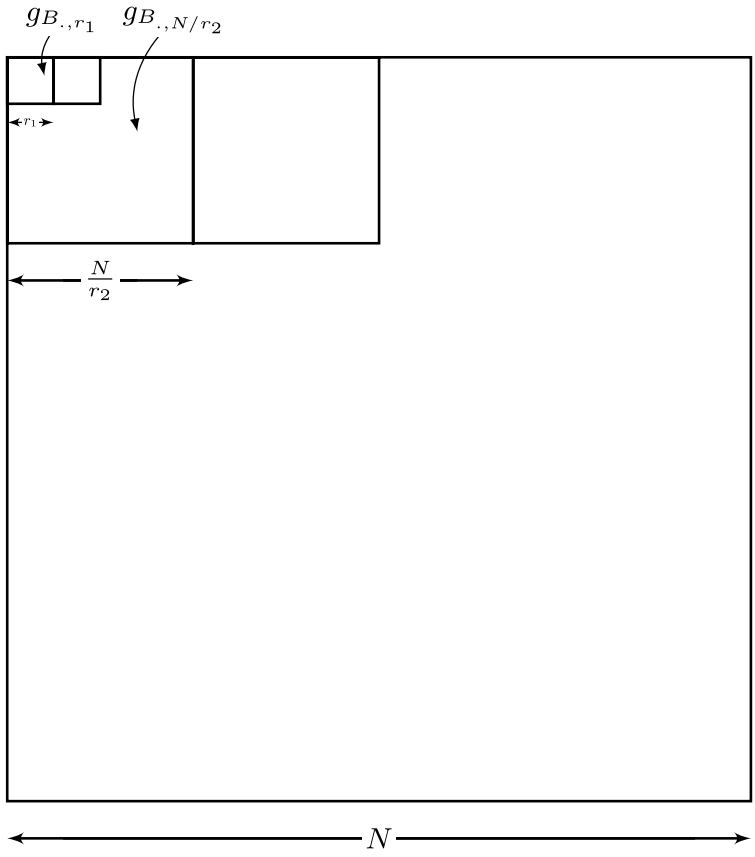

FIG. 1. *Perturbation levels of the Gaussian field.*

For a positive integer r, let $\mathcal{B}_r$ be a collection of sub-boxes of side length r which forms a partition of $V_{\lfloor N/r \rfloor r}$. Write $\mathcal{B} = \bigcup_{r \in [N]} \mathcal{B}_r$. Let $\{g_B : B \in \mathcal{B}\}$ be a collection of i.i.d. standard Gaussian variables. For $v \in V_N$, denote by $B_{v,r} \in \mathcal{B}_r$ the box that contains v (see Figure 1). For $\sigma = (\sigma_1, \sigma_2)$ with $\|\sigma\|_2^2 = \sigma_1^2 + \sigma_2^2$ and r_1, r_2, define

$$\tilde{\varphi}_{N,r_1,r_2,\sigma,v} = \varphi_{N,v} + \sigma_1 g_{B_{v,r_1}} + \sigma_2 g_{B_{v,N/r_2}}, \tag{15}$$

and set $\tilde{M}_{N,r_1,r_2,\sigma} = \max_{v \in V_N} \tilde{\varphi}_{N,r_1,r_2,\sigma,v}$.

For probability measures ν_1, ν_2 on $\mathbb{R}$, let $d(\nu_1, \nu_2)$ denote the Lévy distance between ν_1, ν_2, that is,

$$d(\nu_1, \nu_2) = \inf\{\delta > 0 : \nu_1(B) \le \nu_2(B^\delta) + \delta \text{ for all open sets } B\},$$

where $B^\delta = \{y : |x - y| < \delta \text{ for some } x \in B\}$. In addition, define

$$\tilde{d}(\nu_1, \nu_2) = \inf\{\delta > 0 : \nu_1((x, \infty)) \le \nu_2((x - \delta, \infty)) + \delta \text{ for all } x \in \mathbb{R}\}.$$

If $\tilde{d}(\nu_1, \nu_2) = 0$, then ν_1 is stochastically dominated by ν_2. Thus, $\tilde{d}(\nu_1, \nu_2)$ measures the approximate stochastic domination of ν_1 by ν_2; in particular, unlike $d(\cdot, \cdot)$, the function $\tilde{d}(\cdot, \cdot)$ is not symmetric.

With a slight abuse of notation, if X, Y are random variables with laws μ_X, μ_Y respectively, we also write $d(X, Y)$ for $d(\mu_X, \mu_Y)$ and $\tilde{d}(X, Y)$ for $\tilde{d}(\mu_X, \mu_Y)$.

A notation convention: By Proposition 1.1, one has that

$$\limsup_{\delta \to 0} \limsup_{N} d\Big(\max_{v \in V_N^\delta} \varphi_{N,v}, \max_{v \in V_N} \varphi_{N,v} \Big) = 0.$$

Therefore, in order to prove Theorem 1.3, it suffices to show that for each fixed $\delta > 0$, the law of $\max_{v \in V_N^\delta} \varphi_{N,v} - m_N$ converges. To this end, one only needs to consider the Gaussian field restricted to V_N^δ. For convenience of notation, we will treat V_N^δ as the whole box that is under consideration. Equivalently, throughout the rest of the paper when assuming (A.1), (A.2) or (A.3) holds, we assume these assumptions hold with $\delta = 0$, and we set $\alpha := \max(\alpha_0, \alpha^{(0)})$.

The following lemma, which is one of the main results of this section, relates the laws of M_N and $\tilde{M}_{N,r_1,r_2,\sigma}$.

LEMMA 3.1. *The following holds uniformly for all Gaussian fields* $\{\varphi_{N,v} : v \in V_N\}$ *satisfying Assumption* (A.1):

$$\limsup_{r_1,r_2 \to \infty} \limsup_{N \to \infty} d\big(M_N - m_N, \tilde{M}_{N,r_1,r_2,\sigma} - m_N - \|\sigma\|_2^2 \sqrt{d/2}\big) = 0. \tag{16}$$

The next lemma states that under Assumption (A.1), the law of the maximum is robust under small perturbations (in the sense of ℓ_∞ norm) of the covariance matrix.

LEMMA 3.2. *Let* $\{\varphi_{N,v} : v \in V_N\}$ *be a sequence of Gaussian fields satisfying Assumption* (A.1), *and let* σ *be fixed. Let* $\{\bar{\varphi}_{N,v} : v \in V_N\}$ *be Gaussian fields such that, for all* $u, v \in V_N$

$$|\operatorname{Var} \varphi_{N,v} - \operatorname{Var} \bar{\varphi}_{N,v}| \le \varepsilon \quad \text{and} \quad \mathbb{E}\bar{\varphi}_{N,v}\bar{\varphi}_{N,u} \le \mathbb{E}\varphi_{N,v}\varphi_{N,u} + \varepsilon.$$

Then there exists $\iota = \iota(\varepsilon)$ *with* $\iota \to_{\varepsilon \to 0} 0$ *such that*

$$\limsup_{N \to \infty} \tilde{d}\Big(M_N - m_N, \max_{v \in V_N} \bar{\varphi}_{N,v} - m_N\Big) \le \iota.$$

A key step in the proof of Lemma 3.1 is the following characterization of the geometry of vertices achieving large values in the fields, an extension of [16], Theorem 1.1; it states that near maxima are either at microscopic or macroscopic distance from each other. This may be of independent interest.

LEMMA 3.3. *There exists a constant $c > 0$ such that, uniformly for all Gaussian fields satisfying Assumption* (A.1), *we have*

$$\lim_{r\to\infty}\lim_{N\to\infty}\mathbb{P}\Big(\exists u, v : |u - v| \in \Big(r, \frac{N}{r}\Big), \varphi_{N,v}, \varphi_{N,u} \geq m_N - c\log\log r\Big) = 0.$$

3.1. *Maximal sum over restricted pairs.* As in the case of 2D DGFF discussed in [16], in order to prove Lemma 3.3, we will study the maximum of the sum over restricted pairs. For any Gaussian field $\{\eta_{N,v} : v \in V_N\}$ and $r > 1$, define

$$\eta^{\diamond}_{N,r} = \max\{\eta_{N,u} + \eta_{N,v} : u, v \in V_N, r \leq |u - v| \leq N/r\}.$$

LEMMA 3.4. *There exist constants c_1, c_2 depending only on d and $C > 0$ depending only on (α, d) such that, for all r, n with $N = 2^n$ and all Gaussian fields satisfying Assumption* (A.1), *we have*

$$2m_N - c_2\log\log r - C \leq \mathbb{E}\varphi^{\diamond}_{N,r} \leq 2m_N - c_1\log\log r + C. \tag{17}$$

PROOF. In order to prove Lemma 3.4, we will show that

$$\mathbb{E}\mathcal{S}^{\diamond}_{2^{-\kappa}N,r} \leq \mathbb{E}\varphi^{\diamond}_{N,r} \leq \mathbb{E}\mathcal{S}^{\diamond}_{2^{\kappa}N,r}. \tag{18}$$

To this end, we recall the following Sudakov–Fernique inequality [18] which compares the first moments for maxima of two Gaussian processes.

LEMMA 3.5. *Let $\mathcal{A}$ be an arbitrary finite index set and let $\{X_a : a \in \mathcal{A}\}$ and $\{Y_a : a \in \mathcal{A}\}$ be two centered Gaussian processes such that*

$$\mathbb{E}(X_a - X_b)^2 \geq \mathbb{E}(Y_a - Y_b)^2 \qquad \text{for all } a, b \in \mathcal{A}.$$

Then $\mathbb{E}(\max_{a\in\mathcal{A}} X_a) \leq \mathbb{E}(\max_{a\in\mathcal{A}} Y_a)$.

We will give a proof for the upper bound in (17). The proof of the lower bound follows using similar arguments. For $\kappa \in \mathbb{N}$, recall the definition of the restriction map ψ_N as in (9). By Lemma 2.3, there exists a $\kappa > 0$ [depending only on (α, d)] such that, for all $u, v, u', v' \in V_N$,

$$\mathbb{E}(\varphi_{N,u} + \varphi_{N,v} - \varphi_{N,u'} - \varphi_{N,v'})^2 \leq \mathbb{E}\big(\mathcal{S}^{2^\kappa N}_{\psi_N(u)} + \mathcal{S}^{2^\kappa N}_{\psi_N(v)} - \mathcal{S}^{2^\kappa N}_{\psi_N(u')} - \mathcal{S}^{2^\kappa N}_{\psi_N(v')}\big)^2.$$

(To see this, note that the variance of $\mathcal{S}^{2^\kappa N}_{\psi_N(u)}$ increases with κ but the covariance between $\mathcal{S}^{2^\kappa N}_{\psi_N(u)}$ and $\mathcal{S}^{2^\kappa N}_{\psi_N(v)}$ does not.) In addition, note that for $r \leq |u - v| \leq N/r$ one has $r \leq |\psi_N(u) - \psi_N(v)| \leq 2^\kappa N/r$. Combined with Lemma 3.5, this yields $\mathbb{E}\varphi^{\diamond}_{N,r} \leq \mathbb{E}\mathcal{S}^{\diamond}_{2^\kappa N,r}$, completing the proof of the upper bound in (18).

To complete the proof of Lemma 3.5, note that [16], Lemma 3.1, readily extends to MBRW in d-dimension, and thus

$$2m_N - c_2\log\log r - C \leq \mathbb{E}\mathcal{S}^{\diamond}_{N,r} \leq 2m_N - c_1\log\log r + C,$$

where c_1, c_2 are constants depending only on d and C is a constant depending on (α, d). Combined with (18), this completes the proof of the lemma. □

We will also need the following tightness result.

LEMMA 3.6. *Under Assumption* (A.1), *the sequence* $\{\frac{(\varphi^{\diamond}_{N,r} - \mathbb{E}\varphi^{\diamond}_{N,r})}{\log\log r} : N \in \mathbb{N}, r \geq 100\}$ *is tight. Further, there exists a constant* $C > 0$ *depending only on* d *such that, for all* $r \geq 100$ *and* $N \in \mathbb{N}$,

$$\big|(\varphi^{\diamond}_{N,r} - \mathbb{E}\varphi^{\diamond}_{N,r})\big| \leq C \log\log r.$$

PROOF. Take $N' = 2N$ and partition $V_{N'}$ into 2^d copies of V_N, denoted by $V_N^{(1)}, \ldots, V_N^{(2^d)}$. For each $i \in [2^d]$, let $\{\varphi^{(i)}_{N,v} : v \in V_N^{(i)}\}$ be an independent copy of $\{\varphi_{N,v} : v \in V_n\}$ where we identify V_N and $V_N^{(i)}$ by the suitable translation such that the two boxes coincide. Denote by

$$\hat{\varphi}_{N',v} = \varphi^{(i)}_{N,v} \qquad \text{for } v \in V_N^{(i)} \text{ and } i \in [2^d]. \tag{19}$$

Clearly, $\{\varphi_{N',v}\}$ is a Gaussian field that satisfies Assumption (A.1) (with α increased by an absolute constant). Therefore, by Lemma 3.4, we have

$$2m_N - c_2 \log\log r - C \leq \mathbb{E}\hat{\varphi}^{\diamond}_{N,r} \leq 2m_N - c_1 \log\log r + C, \tag{20}$$

where $c_1, c_2, C > 0$ are constants depending only on (d, α). In addition, we have

$$\mathbb{E}(\hat{\varphi}^{\diamond}_{N',r}) \geq \mathbb{E}\max\{\varphi^{(1),\diamond}_{N,r}, \varphi^{(2),\diamond}_{N,r}\}.$$

Combined with Lemma 3.4 and (20), and the simple algebraic fact that $|a - b| = 2(a \vee b) - a - b$, it yields that

$$\mathbb{E}|\varphi^{(1),\diamond}_{N,r} - \varphi^{(2),\diamond}_{N,r}| \leq 2\big(\mathbb{E}\hat{\varphi}^{\diamond}_{N',r} - \mathbb{E}\varphi^{\diamond}_{N,r}\big) \leq C' \log\log r \qquad \text{for all } r \geq 100,$$

where $C' > 0$ is a constant depending only on d. This completes the proof of the lemma. □

3.2. *Proof of Lemma* 3.3. In this subsection, we will prove Lemma 3.3, by contradiction. Suppose otherwise that Lemma 3.3 does not hold. Then for any constant $c > 0$, there exists $\varepsilon > 0$ and a subsequence $\{r_k\}$ such that, for all $k \in \mathbb{N}$:

$$\lim_{N\to\infty} \mathbb{P}\left(\exists u, v : |u - v| \in \left(r_k, \frac{N}{r_k}\right), \varphi_{N,v}, \varphi_{N,u} \geq m_N - c \log\log r_k\right) > \varepsilon. \tag{21}$$

Now fix $\delta > 0$ and consider $N' = 2^\kappa N$ where κ is an integer to be selected. Partition $V_{N'}$ into $2^{\kappa d}$ disjoint boxes of side length N, denoted by $V_N^{(1)}, \ldots, V_N^{(2^{\kappa d})}$. Define $\{\hat{\varphi}_{N',v} : v \in V_{N'}\}$ in the same manner as in (19) except that now we take $2^{\kappa d}$ copies of $\{\varphi_{N,v} : v \in V_N\}$ (one for each $V_N^{(i)}$ with $i \in [2^{\kappa d}]$). Clearly,

$\{\hat{\varphi}_{N',v} : v \in V_{N'}\}$ is a Gaussian field satisfies Assumption (A.1) with α replaced by a constant α' depending only on (α, d, κ). Therefore, by Lemma 3.4,

$$2m_N - c_2 \log\log r - C \le \mathbb{E}\hat{\varphi}^{\diamond}_{N',r} \le 2m_N - c_1 \log\log r + C, \tag{22}$$

where $c_1, c_2 > 0$ are two constants depending only on d and $C > 0$ is a constant depending only on (α, d, κ).

Next, we derive a contradiction to (22). Set $z_{N,r} = 2m_N - c\log\log r$, $Z_{N,r} = (\hat{\varphi}^{\diamond}_{N',r} - z_{N,r})_-$ and $Y^{(i)}_{N,r} = (\varphi^{(i),\diamond}_{N,r_k} - z_{N,r})_-$. Then (21) implies that

$$\lim_{N\to\infty} \mathbb{P}\big(Y^{(1)}_{N,r_k} > 0\big) \le 1 - \varepsilon \qquad \text{for all } k \in \mathbb{N}. \tag{23}$$

In addition, by Lemmas 3.4 and 3.6, there exists a constant $C' > 0$ depending only on d such that, for all $r \ge 100$ and $N \in \mathbb{N}$, we have

$$\mathbb{E}Y^{(1)}_{N,r} \le C' \log\log r. \tag{24}$$

Clearly, $Z_{N,r} \le \min_{i\in[2^{\kappa d}]} Y^{(i)}_{N,r}$. Combined with the fact that $Y^{(i)}_{N,r}$ are i.i.d. random variables, one obtains

$$\begin{aligned}\mathbb{E}Z_{N,r_k} &\le \int_0^\infty \big(\mathbb{P}\big(Y^{(1)}_{N,r_k} > y\big)\big)^{2^{\kappa d}}\, dy \le (1-\varepsilon)^{2^{\kappa d}-1} \int_0^\infty \big(\mathbb{P}\big(Y^{(1)}_{N,r_k} > y\big)\big)\, dy \\ &\le (1-\varepsilon)^{2^{\kappa d}-1} \mathbb{E}Y^{(1)}_{N,r_k},\end{aligned}$$

where (23) was used in the second inequality. Combined with (24), one concludes that for all $r \ge 100$ and N

$$\mathbb{E}Z_{N,r_k} \le (1-\varepsilon)^{2^{\kappa d}-1} C' \log\log r_k.$$

Now set $c = c_1/4$ and choose κ depending on $(\varepsilon, d, C', c_1)$ such that $(1-\varepsilon)^{2^{\kappa d}-1} C' \le c_1/4$. Then

$$\mathbb{E}\hat{\varphi}^{\diamond}_{N',r_k} \ge 2m_N - c_1 \log\log r_k/2$$

for all $k \in \mathbb{N}$ and sufficiently large $N \ge N_k$ where N_k is a number depending only on k. Sending $N \to \infty$ first and then $k \to \infty$ contradicts (22), thereby completing the proof of the lemma. □

3.3. *Proof of Lemmas 3.1 and 3.2.* The next lemma, which extends [11], Lemma 3.9, to the current setup, will be useful for the proof of Lemma 3.1 and later in the paper.

LEMMA 3.7. *Let Assumptions* (A.0) *and* (A.1) *hold. Let* $\{\phi^N_u : u \in V_N\}$ *be a collection of random variables independent of* $\{\varphi_{N,u} : u \in V_N\}$ *such that*

$$\mathbb{P}\big(\phi^N_u \ge 1 + y\big) \le e^{-y^2} \qquad \textit{for all } u \in V_N. \tag{25}$$

Then there exists $C = C(\alpha, d) > 0$ *such that, for any* $\varepsilon > 0, N \in \mathbb{N}$ *and* $x \geq -\varepsilon^{-1/2}$,

$$\begin{aligned}(26)\qquad &\mathbb{P}\Big(\max_{u\in V_N}(\varphi_{N,u} + \varepsilon\phi_u^N) \geq m_N + x\Big)\\ &\qquad \leq \mathbb{P}\Big(\max_{u\in V_N}\varphi_{N,u} \geq m_N + x - \sqrt{\varepsilon}\Big)\big(1 + C\big(e^{-C^{-1}\varepsilon^{-1}}\big)\big).\end{aligned}$$

PROOF. We first give the proof for $\varepsilon \leq 1$. Define $\Gamma_y = \{u \in V_N : y/2 \leq \varepsilon\phi_u^N \leq y\}$. Then

$$\begin{aligned}&\mathbb{P}\Big(\max_{u\in V_N}(\varphi_{N,u} + \varepsilon\phi_u^N) \geq m_N + x\Big)\\ &\quad\leq \mathbb{P}(M_N \geq m_N + x - \sqrt{\varepsilon})\\ &\qquad + \sum_{i=0}^{\infty}\mathbb{E}\Big(\mathbb{P}\Big(\max_{u\in\Gamma_{2^i\sqrt{\varepsilon}}}\varphi_{N,u} \geq m_N + x - 2^i\sqrt{\varepsilon}\,|\Gamma_{2^i\sqrt{\varepsilon}}\Big)\Big).\end{aligned}$$

By Proposition 1.1, one can bound the second term on the right-hand side above by

$$\begin{aligned}&\sum_{i=0}^{\infty}\mathbb{E}\Big(\mathbb{P}\Big(\max_{u\in V_N}\varphi_{N,u} \geq m_N + x - 2^i\sqrt{\varepsilon}\,|\Gamma_{2^i\sqrt{\varepsilon}}\Big)\Big)\\ &\qquad\lesssim \frac{x\vee 1}{e^{\sqrt{2d}x}}\sum_{i=0}^{\infty}\mathbb{E}\big(|\Gamma_{2^i\sqrt{\varepsilon}}|/N^d\big)^{1/2}e^{\sqrt{2d}2^i\sqrt{\varepsilon}}.\end{aligned}$$

By (25), one has $\mathbb{E}(|\Gamma_{2^i\sqrt{\varepsilon}}|/N^d)^{1/2} \leq e^{-4^i(C\varepsilon)^{-1}}$. Altogether, one gets

$$\sum_{i=0}^{\infty}\mathbb{E}\Big(\mathbb{P}\Big(\max_{u\in V_N}\varphi_{N,u} \geq m_N + x - 2^i\sqrt{\varepsilon}\,|\Gamma_{2^i\sqrt{\varepsilon}}\Big)\Big) \lesssim \frac{x\vee 1}{e^{\sqrt{2d}x}}e^{-(C\varepsilon)^{-1}},$$

completing the proof of the lemma when $\varepsilon \leq 1$. The case $\varepsilon > 1$ is simpler and follows by repeating the same argument with $\Gamma_{2^i\varepsilon}$ replacing $\Gamma_{2^i\sqrt{\varepsilon}}$. We omit further details. □

We next consider a combination of two independent copies of $\{\varphi_{N,v}\}$. For $\sigma > 0$, define

$$\begin{aligned}(27)\qquad &\varphi^*_{N,\sigma,v} = \varphi_{N,v} + \sqrt{\frac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,v} \qquad \text{for } v \in V_N, \quad \text{and}\\ &M^*_{N,\sigma} = \max_{v\in V_N}\varphi^*_{N,\sigma,v},\end{aligned}$$

where $\{\varphi'_{N,v} : v \in V_N\}$ is an independent copy of $\{\varphi_{N,v} : v \in V_N\}$. Note that the field $\{\varphi^*_{N,\sigma,v}\}$ is distributed like the field $\{a_N\varphi_{N,v}\}$ where $a_N = \sqrt{1 + \|\sigma\|_2^2/\log N}$.

REMARK 3.8. The idea of writing a Gaussian field as a sum of two independent Gaussian fields has been extensively employed in the study of Gaussian processes. In the context of the study of extrema of the 2D DGFF, this idea was first used in [7], where (combined with an invariance result from [24] as well as the geometry of the maxima of DGFF [16], see Lemma 3.4) it led to a complete description of the extremal process of 2D DGFF. The definition (27) is inspired by [7].

The following is the key to the proof of Lemma 3.1.

PROPOSITION 3.9. *Let Assumption* (A.1) *hold. Let* $\{\tilde{\varphi}_{N,r,\sigma,v} : v \in V_N\}$ *and* $\{\varphi^*_{N,\sigma,v} : v \in V_N\}$ *be defined as in* (15) *and* (27), *respectively. Then for any fixed* σ,

$$\lim_{r_1,r_2\to\infty} \limsup_{N\to\infty} d\big(\tilde{M}_{N,r_1,r_2,\sigma} - m_N, M^*_{N,\sigma} - m_N\big) = 0. \tag{28}$$

PROOF. Partition V_N into boxes of side length N/r_2 and denote by $\mathcal{B}$ the collection of boxes. Fix an arbitrary small $\delta > 0$, and let B_δ denote the box in the center of B with side length $(1-\delta)N/r_2$ for each $B \in \mathcal{B}$. Write $V_{N,\delta} = \bigcup_{B\in\mathcal{B}} B_\delta$. Set $\tilde{M}_{N,r_1,r_2,\sigma,\delta} = \max_{v\in V_{N,\delta}} \tilde{\varphi}_{N,r_1,r_2,\sigma,v}$ and $M^*_{N,\sigma,\delta} = \max_{v\in V_{N,\delta}} \varphi^*_{N,\sigma,v}$. By (3), one has

$$\lim_{\delta\to 0}\lim_{N\to\infty} \mathbb{P}(\tilde{M}_{N,r_1,r_2,\sigma,\delta} \neq \tilde{M}_{N,r_1,r_2,\sigma}) = \lim_{\delta\to 0}\lim_{N\to\infty} \mathbb{P}(M^*_{N,\sigma,\delta} \neq M^*_{N,\sigma}) = 0.$$

Therefore, it suffices to prove (28) with $\tilde{M}_{N,r_1,r_2,\sigma,\delta}$ and $M^*_{N,\sigma,\delta}$ replacing $\tilde{M}_{N,r_1,r_2,\sigma}$ and $M^*_{N,\sigma}$. To this end, let z_B be such that

$$\max_{v\in B_\delta} \varphi_{N,v} = \varphi_{N,z_B} \qquad \text{for every } B \in \mathcal{B}.$$

We will show below that

$$\begin{aligned} &\lim_{r_1,r_2\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big(\Big|\tilde{M}_{N,r_1,r_2,\sigma,\delta} - \max_{B\in\mathcal{B}} \tilde{\varphi}_{N,r_1,r_2,\sigma,z_B}\Big| \geq 1/\log\log N\Big) \\ &\quad = \limsup_{N\to\infty} \mathbb{P}\Big(\Big|M^*_{N,\sigma,\delta} - \max_{B\in\mathcal{B}} \varphi^*_{N,\sigma,z_B}\Big| \geq 1/\log\log N\Big) = 0. \end{aligned} \tag{29}$$

Note that the fields $\{\varphi_{N,v} : v \in V_N\}$ and $\{\sqrt{\|\sigma\|_2^2/\log N}\,\varphi'_{N,v} : v \in V_N\}$ are independent of each other. Thus, conditioning on the field $\{\varphi_{N,v} : v \in V_N\}$, the field $\{\sqrt{\|\sigma\|_2^2/\log N}\,\varphi'_{N,z_B} : B \in \mathcal{B}\}$ is a centered Gaussian field with pairwise correlation bounded by $O(1/\log N)$. Therefore, the conditional covariance matrix of $\{\sqrt{\frac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,z_B} : B \in \mathcal{B}\}$ and that of $\{\sigma_1 g_{B_{z_B,r_1}} + \sigma_2 g_{B_{z_B,N/r_2}} : B \in \mathcal{B}\}$ are within additive $O(1/\log N)$ of each other entrywise. In addition, $|\mathcal{B}| \leq (2r_2)^d$. Therefore, it

is clear that there exists a coupling between the two fields such that

$$\lim_{N\to\infty} \mathbb{P}\Big(\max_{B\in\mathcal{B}} \big|\sqrt{\|\sigma\|_2^2/\log N}\,\varphi'_{N,z_B} - (\sigma_1 g_{B_{z_B,r_1}} + \sigma_2 g_{B_{z_B,N/r_2}})\big| \\ \geq 1/\log\log N \,\big|\, \{\varphi_{N,v} : v \in V_N\}\Big) = 0$$

[here the term $1/\log\log N$ is somewhat arbitrary, any negative power larger than $1/2$ of $(\log N)$ would work]. Note that the preceding equality holds for almost all realizations of $\{\varphi_{N,v} : v \in V_N\}$. Combined with (29), it then yields the proposition.

It remains to prove (29). Write $r = r_1 \wedge r_2$ and let C be a constant which we will send to infinity after sending first $N \to \infty$ and then $r \to \infty$, and let c be the constant from Lemma 3.3. Suppose that either of the events that are considered in (29) occurs. In this case, one of the following events has to occur:

- The event $E_1 = \{\tilde{M}_{N,r_1,r_2,\sigma,\delta} \notin (m_N - C, m_N + C)\} \cup \{M^*_{N,\sigma,\delta} \notin (m_N - C, m_N + C)\}$.
- The event E_2 that there exists $u, v \in (r, N/r)$ such that $\varphi_{N,u} \wedge \varphi_{N,v} > m_N - c\log\log r$.
- The event $E_3 = \tilde{E}_3 \cup E_3^*$ where $\tilde{E}_3$ (E_3^*) is the event that $\tilde{M}_{N,r_1,r_2,\sigma}$ ($M^*_{N,\sigma,\delta}$) is achieved at a vertex v such that $\varphi_{N,v} \leq m_N - c\log\log r$.
- The event E_4 that there exists $v \in B \in \mathcal{B}$ with $\varphi_{N,v} \geq m_N - c\log\log r$ and

$$\sqrt{\frac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,v} - \sqrt{\frac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,z_B} \geq \frac{1}{\log\log N}.$$

By Theorem 1.2, $\lim_{C\to\infty} \limsup_{N\to\infty} \mathbb{P}(E_1) = 0$.

By Lemma 3.3, $\lim_{r\to\infty} \limsup_{N\to\infty} \mathbb{P}(E_2) = 0$. In addition, writing $\Gamma_x = \{v \in V_N : \tilde{\varphi}_{N,r_1,r_2,\sigma,v} - \varphi_{N,v} \in (x, x+1)\}$, one has

$$\begin{aligned}
\mathbb{P}(E_1^c \cap \tilde{E}_3) &\leq \mathbb{P}\Big(\max_{x\geq c\log\log r - C} \max_{v\in\Gamma_x} \tilde{\varphi}_{N,r_1,r_2,\sigma,v} \geq m_N - C\Big) \\
&\leq \sum_{x\geq c\log\log r - C} \mathbb{P}\Big(\max_{v\in\Gamma_x} \tilde{\varphi}_{N,r_1,r_2,\sigma,v} \geq m_N - C\Big) \\
&\leq \sum_{x\geq c\log\log r - C} \mathbb{E}\Big(\mathbb{P}\Big(\max_{v\in\Gamma_x} \varphi_{N,v} \geq m_N - x - C \,\big|\, \Gamma_x\Big)\Big) \\
&\lesssim_C \sum_{x\geq c\log\log r - C} \mathbb{E}\big(|\Gamma_x|/N^d\big)^{1/2} x e^{\sqrt{2d}x},
\end{aligned}$$

where the last inequality follows from (3). From simple estimates using the Gaussian distribution, one has $\mathbb{E}(|\Gamma_x|/N^d)^{1/2} \leq e^{-c'x^2}/c'$ where $c' = c'(\sigma) > 0$. Therefore, one concludes that

$$\limsup_{C\to\infty} \limsup_{r\to\infty} \limsup_{N\to\infty} \mathbb{P}(E_1^c \cap \tilde{E}_3) = 0.$$

A similar argument leads to the same estimate with E_3^* replacing E_3. Thus,

$$\limsup_{C\to\infty}\limsup_{r\to\infty}\limsup_{N\to\infty}\mathbb{P}\big(E_1^c\cap E_3\big)=0.$$

Finally, let $\Gamma_r'=\{v:\varphi_{N,v}\ge m_N-c\log\log r\}$. On the event E_2^c, one has $|\Gamma_r'|\le r^4$. Further, for each $v\in B\cap\Gamma_r'$, on E_2^c one has $|v-z_B|\le r$, and thus (by the independence between $\{\varphi_{N,v}\}$ and $\{\varphi'_{N,v}\}$),

$$\mathbb{P}\bigg(\sqrt{\frac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,v}-\sqrt{\frac{\|\sigma\|_2^2}{\log N}}\varphi'_{N,z_B}\ge 1/\log\log N\bigg)=o_N(1).$$

Therefore, a union bound gives that

$$\limsup_{r\to\infty}\limsup_{N\to\infty}\mathbb{P}\big(E_4\cap E_2^c\big)\le\limsup_{r\to\infty}\limsup_{N\to\infty}r^4o_N(1)=0.$$

Altogether, this completes the proof of (29), and hence of the proposition. □

PROOF OF LEMMA 3.1. Define

$$\bar\varphi_{N,\sigma,v}=\bigg(1+\frac{\|\sigma\|_2^2}{2\log N}\bigg)\varphi_{N,v}\qquad\text{for } v\in V_N,\quad\text{and}\quad \bar M_{N,\sigma}=\max_{v\in V_N}\bar\varphi_{N,v}.$$

Clearly, we have $\bar M_{N,\sigma}=(1+\frac{\|\sigma\|_2^2}{2\log N})M_N$. Combined with (1), it follows that $\mathbb{E}\bar M_{N,\sigma}=\mathbb{E}M_N+\sigma^2\sqrt{d/2}+o(1)$ and that $d(M_N-\mathbb{E}M_N,\bar M_{N,\sigma}-\mathbb{E}\bar M_{N,\sigma})\to 0$ as $N\to\infty$. Further, define $\{\varphi^*_{N,\sigma,v}:v\in V_N\}$ as in (27). By the fact that the field $\{\bar\varphi_{N,\sigma,v}\}$ can be seen as a sum of $\{\varphi^*_{N,\sigma,v}\}$ and an independent field whose variances are $O((1/\log N)^3)$ across the field, we see that $\mathbb{E}\bar M_{N,\sigma}=\mathbb{E}M^*_{N,\sigma}+o(1)$ and that

$$d\big(\bar M_{N,\sigma}-\mathbb{E}\bar M_N,M^*_{N,\sigma}-\mathbb{E}M^*_N\big)\to 0. \tag{30}$$

Combined with Proposition 3.9, this completes the proof of the lemma. □

PROOF OF LEMMA 3.2. Let ϕ and $\phi_{N,v}$ be i.i.d. standard Gaussian variables, and for $\varepsilon^*>0$ let

$$\varphi_{\mathrm{lw},N,\varepsilon^*,v}=\big(1-\varepsilon^*/\log N\big)\varphi_{N,v}+\varepsilon'_{N,v}\phi\quad\text{and}$$
$$\bar\varphi_{\mathrm{up},N,\varepsilon^*,v}=\big(1-\varepsilon^*/\log N\big)\bar\varphi_{N,v}+\varepsilon''_{N,v}\phi_{N,v},$$

where $\varepsilon'_{N,v},\varepsilon''_{N,v}$ are chosen so that $\operatorname{Var}\varphi_{\mathrm{lw},N,\varepsilon^*,v}=\operatorname{Var}\bar\varphi_{\mathrm{up},N,\varepsilon^*,v}=\operatorname{Var}\varphi_{N,v}+\varepsilon$. We can choose $\varepsilon^*=\varepsilon^*(\varepsilon)$ with $\varepsilon^*\to_{\varepsilon\to 0}0$ so that $\mathbb{E}\varphi_{\mathrm{lw},N,\varepsilon^*,v}\varphi_{\mathrm{lw},N,\varepsilon^*,u}\ge\mathbb{E}\bar\varphi_{\mathrm{up},N,\varepsilon^*,v}\bar\varphi_{\mathrm{up},N,\varepsilon^*,u}$ for all $u,v\in V_N$. By Lemma 2.4, one has

$$\tilde d\Big(\max_{v\in V_N}\varphi_{\mathrm{lw},N,\varepsilon^*,v}-m_N,\max_{v\in V_N}\bar\varphi_{\mathrm{up},N,\varepsilon^*,v}-m_N\Big)=0.$$

Combined with Lemma 3.7, this completes the proof of the lemma. □

4. Proofs of Theorems 1.3 and 1.4. In this section, we assume (A.0)–(A.3) and prove Theorem 1.3. Toward this end, in Section 4.1 we will approximate the field $\{\varphi_{N,v} : v \in V_N\}$ by a simpler to analyze field, in such a way that the results of Section 3 apply and yield the asymptotic equivalence of their respective laws of the centered maximum. In Section 4.2, we prove the convergence in law for the centered maximum of the new field. Our method of proof yields Theorem 1.4 as a byproduct.

4.1. *An approximation of the log-correlated Gaussian field.* In this subsection, we approximate the log-correlated Gaussian field. Let $R_N(u, v) = E(\varphi_{N,u}\varphi_{N,v})$. We consider three scales for the approximation of the field $\{\varphi_{N,v}\}$:

(a) The top (macroscopic) scale, dealing with $R_N(u, v)$ for $|u - v| \asymp N$.
(b) The bottom (microscopic) scale, dealing with $R_N(u, v)$ for $|u - v| \asymp 1$.
(c) The middle (mesoscopic) scale, dealing with $R_N(u, v)$ for $1 \ll |u - v| \ll N$.

By Assumptions (A.2) and (A.3), $R_N(u, v)$, properly centered, converges in the top and bottom scale. So in those scales, we approximate $\{\varphi_{N,u}\}$ by the corresponding "limiting" fields. In the middle scale, we simply approximate $\{\varphi_{N,u}\}$ by the MBRW. One then expects that this approximation gives an additive $o(1)$ error for $R_N(u, v)$ in the top and bottom scale, and an additive $O(1)$ error in the middle scale. It turns out that this guarantees that the limiting laws of the centered maxima coincide.

In what follows, for any integer t we refer to a box of side length t as an t-box. Take two large integers $L = 2^\ell$ and $K = 2^k$. Consider first $\{\varphi_{KL,u} : u \in V_{KL}\}$ in a KL-box whose left-bottom corner is identified as the origin, and let Σ denote its covariance matrix.

Recall that by Proposition 1.1, with probability tending to 1 as $N \to \infty$, the maximum of $\varphi_{N,v}$ over V_N occurs in a sub-box of V_N with side length $\lfloor N/KL \rfloor \cdot KL$. Therefore, one may neglect the maximization over the indices in $V_N \setminus V_{\lfloor N/KL \rfloor \cdot KL}$. For notational convenience, we will assume throughout that KL divides N in what follows.

We use Σ to approximate the macroscopic scale of $R_N(u, v)$, as follows. Partition V_N into a disjoint union of boxes of side length N/KL, denoted $\mathcal{B}_{N/KL} = \{B_{N/KL,i} : i = 1, \ldots, (KL)^d\}$. Let $v_{N/KL,i}$ be the left bottom corner of box $B_{N/KL,i}$ and write $w_i = \frac{v_{N/KL,i}}{N/KL}$. Let Ξ^c be a matrix of dimension $N^d \times N^d$ such that $\Xi^c_{u,v} = \Sigma_{w_i,w_j}$ for $u \in B_{N/KL,i}$ and $v \in B_{N/KL,j}$. Note that Ξ^c is a positive definite matrix with diagonal terms $\log(KL) + O_{KL}(1)$.

Next, take two other integers $K' = 2^{k'}$ and $L' = 2^{\ell'}$. As above, we assume that $K'L'$ divides N. Consider $\{\varphi_{K'L',u} : u \in V_{K'L'}\}$ in a $K'L'$-box whose left-bottom corner is identified as the origin, and denote by Σ' the covariance matrix for $\{\varphi_{K'L',u} : u \in V_{K'L'}\}$. As above, assume for notational convenience that $K'L'$ divides N. Partition V_N into a disjoint union of boxes of side length $K'L'$, denoted $\mathcal{B}_{K'L'} = \{B_{K'L',i} : i = 1, \ldots, (N/K'L')^d\}$. Let $v_{K'L',i}$ be the left bottom corner of

$B_{K'L',i}$. Let Ξ^b be a matrix of dimension $N^d \times N^d$ so that

$$\Xi^b_{u,v} = \begin{cases} \Sigma'_{u-v_{K'L',i},v-v_{K'L',i}}, & u,v \in B_{K'L',i}, \\ 0, & u \in B_{K'L',i}, v \in B_{K'L',j}, i \neq j. \end{cases}$$

Note that Ξ^b is a positive definite matrix with diagonal terms $\log(K'L') + O_{K'L'}(1)$.

Let $\{\xi^c_{N,v} : v \in V_N\}$ be a Gaussian field with covariance matrix Ξ^c, which we occasionally refer to as the coarse field, and let $\{\xi^b_{N,v} : v \in V_N\}$ be a Gaussian field with covariance matrix Ξ^b, which we occasionally refer to as the bottom field. Note that the coarse field is constant in each box $B_{N/KL,i}$, and the bottom fields in different boxes $B_{K'L',i}$ are independent of each other.

We will consider the limits when L, K, L', K' are sent to infinity in that order. In what follows, we denote by $(L, K, L', K') \Rightarrow \infty$ sending these parameters to infinity in the order of K', L', K, L (so $K' \gg L' \gg K \gg L$).

Finally, we give the MBRW approximation for the mesoscopic scales. Recall the definitions of $\mathcal{B}^N_j$ and $\mathcal{B}_j(v)$ in Section 2.1, and recall that $\{b_{i,k,B} : k \geq 0, 1 \leq i \leq (KL)^d, B \in \mathcal{B}^N_k\}$ is a family of independent Gaussian variables such that $\operatorname{Var} b_{i,j,B} = \log 2 \cdot 2^{-dj}$ for all $B \in \mathcal{B}^N_j$ and $1 \leq i \leq (KL)^d$. For $v \in B_{N/KL,i} \cap B_{K'L',i'}$ [where $i = 1, \ldots, (KL)^d$ and $i' = 1, \ldots, (N/K'L')^d$], define

$$\xi_{N,v,\mathrm{MBRW}} = \sum_{j=\ell'+k'}^{n-k-\ell} \sum_{B \in \mathcal{B}_j(v_{K'L',i'})} b^N_{i,j,B}. \tag{31}$$

Note that by our construction $\{\xi_{N,v,\mathrm{MBRW}} : v \in B_{N/KL,i}\}$ are independent of each other for $i = 1, \ldots, (KL)^d$, and in addition $\xi_{N,\cdot,\mathrm{MBRW}}$ is constant over each $K'L'$-box. Further, let $\{\xi^c_{N,v} : v \in V_N\}$, $\{\xi^b_{N,v} : v \in V_N\}$ and $\{\xi_{N,v,\mathrm{MBRW}} : v \in V_N\}$ be independent of each other, see Figure 2 for an illustration. One has by Assumption (A.1) that

$$\left|\operatorname{Var}\left(\xi^c_{N,v} + \xi^b_{N,v} + \xi_{N,v,\mathrm{MBRW}}\right) - \operatorname{Var}\varphi_{N,v}\right| \leq 4\alpha.$$

Let $a_{N,v}$ be a sequence of numbers such that, for all $v \in B_{N/KL,i}$ and all $1 \leq i \leq (KL)^d$,

$$\operatorname{Var}\left(\xi^c_{N,v} + \xi^b_{N,v} + \xi_{N,v,\mathrm{MBRW}}\right) + a^2_{N,v} = \operatorname{Var}\varphi_{N,v} + 4\alpha. \tag{32}$$

[Here, the sequence $a_{N,v}$ implicitly depends on (KL).] It is clear that

$$\max_{v \in V_N} a_{N,v} \leq \sqrt{8\alpha}. \tag{33}$$

For $v \in B_{N/KL,i}$ and $v \equiv \bar{v} \mod K'L'$, one has

$$\begin{aligned} a^2_{N,v} &= \operatorname{Var}\varphi_{N,v} + 4\alpha - \operatorname{Var}\varphi_{KL,w_i} - \operatorname{Var}\varphi_{K'L',\bar{v}} - \log\left(\frac{N}{KLK'L'}\right) \\ &= \log N - \log(KL) + \varepsilon_{N,KL,K'L'} + 4\alpha - \operatorname{Var}\varphi_{K'L',\bar{v}} - \log\left(\frac{N}{KLK'L'}\right) \\ &\geq 0, \end{aligned}$$

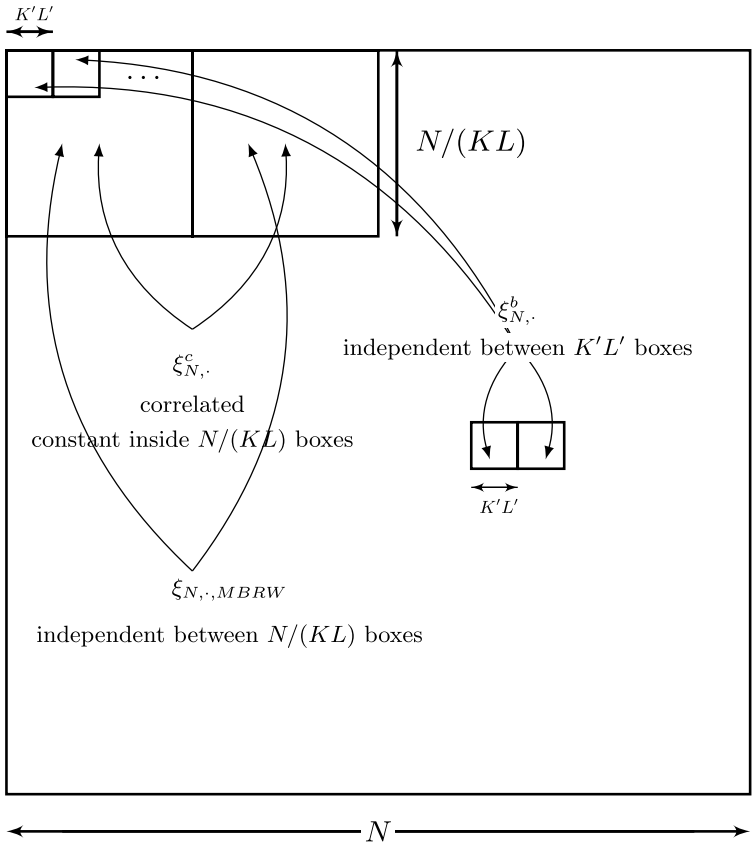

FIG. 2. *Hierarchy of construction of the approximating Gaussian field.*

where, by Assumptions (A.2),

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \varepsilon_{N,KL,K'L'} = 0. \tag{34}$$

Therefore, one can write

$$a^2_{N,v} = a^2_{K',L',\bar{v}} + \varepsilon_{N,KL,K'L'}, \tag{35}$$

where $a_{K'L',\bar{v}}$ depends on $(K'L', \bar{v})$. By Assumption (A.2) and the continuity of f, one has

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \sup_{u,v:\|u-v\|_\infty\le L'} \limsup_{N\to\infty} \left|\operatorname{Var}\xi^b_{N,v} - \operatorname{Var}\xi^b_{N,u}\right| = 0.$$

Therefore, we can further require that

$$|a_{K'L',\bar{v}} - a_{K'L',\bar{u}}| \le \varepsilon_{N,KL,K'L'} \qquad \text{for all } \|\bar{v}-\bar{u}\|_\infty \le L'. \tag{36}$$

Let ϕ_j be i.i.d. standard Gaussian variables. For $v \in B_{K'L',j}$ and $v \equiv \bar{v} \mod K'L'$, define

$$\xi_{N,v} = \xi^c_{N,v} + \xi^b_{N,v} + \xi_{N,v,\mathrm{MBRW}} + a_{K'L',\bar{v}}\phi_j. \tag{37}$$

It follows from (32) and (35) that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} |\operatorname{Var}\xi_{N,v} - \operatorname{Var}\varphi_{N,v} - 4\alpha| = 0. \tag{38}$$

Finally, we partition V_N into a disjoint union of boxes of side length N/L which we denote by $\mathcal{B}_{N/L} = \{B_{N/L,i} : 1 \le i \le L^d\}$, as well as a disjoint union of boxes of side length L which we denote by $\mathcal{B}_L = \{B_{L,i} : 1 \le i \le (N/L)^d\}$. Again, we denote by $v_{N/L,i}$ and $v_{L,i}$ the left bottom corner of the boxes $B_{N/L,i}$ and $B_{L,i}$, respectively.

For $\delta > 0$ and any box B, denote by $B^\delta \subseteq B$ the collection of all vertices in B that are $\delta\ell_B$ away from its boundary ∂B (here ℓ_B is the side length of B). Let

$$V^*_{N,\delta} = \Big(\bigcup_i B^\delta_{N/L,i}\Big) \cap \Big(\bigcup_i B^\delta_{N/KL,i}\Big) \cap \Big(\bigcup_i B^\delta_{L,i}\Big) \cap \Big(\bigcup_i B_{KL,i}\Big).$$

One has $|V^*_{N,\delta}| \ge (1 - 100d\delta)|V_N|$.

The following lemma suggests that $\{\xi_{N,v} : v \in V_N\}$ is a good approximation of $\{\varphi_{N,v} : v \in V_N\}$.

LEMMA 4.1. *Let Assumptions* (A.1), (A.2) *and* (A.3) *hold. Then there exist* $\varepsilon'_{N,K,L,K',L'} > 0$ *with* $\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \varepsilon'_{N,K,L,K',L'} = 0$, *such that the following hold for all* $u, v \in V^*_{N,\delta}$:

(a) *If* $u, v \in B_{L',i}$ *for some* $1 \le i \le (N/L')^d$, *then* $|\mathbb{E}(\xi_{N,u} - \xi_{N,v})^2 - \mathbb{E}(\varphi_{N,u} - \varphi_{N,v})^2| \le \varepsilon'_{N,K,L,K',L'}$.
(b) *If* $u \in B_{N/L,i}$, $v \in B_{N/L,j}$ *with* $i \ne j$, *then* $|\mathbb{E}\xi_{N,u}\xi_{N,v} - \mathbb{E}\varphi_{N,v}\varphi_{N,u}| \le \varepsilon'_{N,K,L,K',L'}$.
(c) *Otherwise*, $|\mathbb{E}\xi_{N,u}\xi_{N,v} - \mathbb{E}\varphi_{N,v}\varphi_{N,u}| \le 4\log(1/\delta) + 40\alpha$.

PROOF. (a) Let i' be such that $B_{L',i} \subseteq B_{K'L',i'}$. By (36) and (37), one has

$$\big|\mathbb{E}(\xi_{N,u} - \xi_{N,v})^2 - \mathbb{E}(\varphi_{KL,u-v_{KL,i'}} - \varphi_{KL,v-v_{KL,i'}})^2\big| \le 4\varepsilon_{N,KL,K'L'},$$

where $\varepsilon_{N,KL,K'L'}$ satisfies (34) (and was defined therein). By Assumption (A.2), one has

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \big|\mathbb{E}(\varphi_{KL,u-v_{KL,i'}} - \varphi_{KL,v-v_{KL,i'}})^2 - \mathbb{E}(\varphi_{N,u} - \varphi_{N,v})^2\big| = 0.$$

Altogether, this completes the proof for (a).

(b) Let i', j' be such that $u \in B_{N/KL,i'}$ and $v \in B_{N/KL,j'}$, and assume w.l.o.g. that $K' \gg L' \gg K \gg L \gg 1/\delta$. The definition of $\{\xi_{N,v}\}$ gives

$$\mathbb{E}\xi_{N,v}\xi_{N,u} = \mathbb{E}\varphi_{KL,w_{i'}}\varphi_{KL,w_{j'}},$$

where $w_{i'} = \frac{v_{N/KL,i'}}{N/KL}$ and $w_{j'} = \frac{v_{N/KL,j'}}{N/KL}$. In this case, we have $|w_{i'} - w_{j'}| \geq \delta K$. Writing $x_u = u/N$, $x_v = v/N$ and $y_u = w_{i'}/KL$, $y_v = w_{j'}/KL$, one obtains

$$|y_u - y_v| \geq \delta/L, \qquad |x_u - x_v| \geq \delta/L,$$
$$|x_u - y_u| \leq 1/K, \qquad |x_v - y_v| \leq 1/K.$$

Therefore, Assumption (A.3) yields $\overline{\lim}_{(L,K,L',K')\Rightarrow\infty}\overline{\lim}_{N\to\infty}|\mathbb{E}\xi_{N,u}\xi_{N,v} - \mathbb{E}\varphi_{N,v}\varphi_{N,u}| = 0$, completing the proof of (b).

(c) In this case, one has

$$\begin{aligned}\mathbb{E}\xi_{N,v}\xi_{N,u} &= \mathbb{E}\xi^c_{N,v}\xi^c_{N,u} + \mathbb{E}\xi^b_{N,v}\xi^b_{N,u} + \mathbb{E}\xi_{N,u,\mathrm{MBRW}}\xi_{N,v,\mathrm{MBRW}} + \mathrm{err}_1\\ &= \log KL - \log_+\left(\frac{|u-v|}{N/KL}\right)\\ &\quad + \mathbf{1}_{|u-v|\leq N/KL}\left(\log\frac{N}{(KLK'L')} - \log_+\frac{|u-v|}{K'L'}\right) + \mathrm{err}_2\\ &= \log N - \log_+|u-v| + \mathrm{err}_2,\end{aligned}$$

where $|\mathrm{err}_1| \leq 8\alpha$ and $|\mathrm{err}_2| \leq 2\log 1/\delta + 20\alpha$. Combined with Assumption (A.1), this completes the proof of (c), and hence of the lemma. □

LEMMA 4.2. *Let Assumptions* (A.0), (A.1), (A.2) *and* (A.3) *hold. Then*

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty} d\Big(M_N - m_N,\ \max_{v\in V_N}\xi_{N,v} - m_N - 2\alpha\sqrt{2d}\Big) = 0.$$

PROOF. By Proposition 1.1, it suffices to show that for all $\delta > 0$:

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty}\ \limsup_{N\to\infty} d\Big(\max_{v\in V^*_{N,\delta}}\varphi_{N,v} - m_N,\ \max_{v\in V^*_{N,\delta}}\xi_{N,v} - m_N - 2\alpha\sqrt{2d}\Big) = 0.$$

Consider a fixed $\delta > 0$. Let $\sigma_*^2 = 4\log(1/\delta) + 60\alpha$. Let $\sigma_{\mathrm{lw}} = (0, \sqrt{\sigma_*^2 + 4\alpha})$ and $\sigma_{\mathrm{up}} = (\sigma^*, 0)$. Define $\{\tilde\varphi_{N,L',L,\sigma_{\mathrm{lw}},v} : v \in V_N\}$ as in (15) with $r_1 = L'$, $r_2 = L$ and $\sigma = \sigma_{\mathrm{lw}}$. Analogously, define $\{\tilde\xi_{N,L',L,\sigma_{\mathrm{up}},v} : v \in V_N\}$. By (37) and Lemma 4.1, one has for all $u, v \in V^*_{N,\delta}$,

$$|\operatorname{Var}\tilde\varphi_{N,L',L,\sigma_{\mathrm{lw}},v} - \operatorname{Var}\tilde\xi_{N,L',L,\sigma_{\mathrm{up}},v}| \leq \bar\varepsilon_{N,K,L,K',L'},$$
$$\mathbb{E}\tilde\xi_{N,L,\sigma_{\mathrm{up}},v}\tilde\xi_{N,L,\sigma_{\mathrm{up}},u} \leq \mathbb{E}\tilde\varphi_{N,L,\sigma_{\mathrm{lw}},v}\tilde\varphi_{N,L,\sigma_{\mathrm{lw}},u} + \bar\varepsilon_{N,K,L,K',L'},$$

where $\limsup_{(L,K,L',K')\Rightarrow\infty}\limsup_{N\to\infty}\bar{\varepsilon}_{N,K,L,K',L'}=0$. Since $\{\tilde{\varphi}_{N,L',L,\sigma_{\mathrm{lw}},v}: v\in V^*_{N,\delta}\}$ satisfies Assumption (A.1) with α being replaced by $10\alpha+\sigma_*^2$, one may apply Lemma 3.2 and obtain that

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\limsup_{N\to\infty}\tilde{d}\Big(\max_{v\in V^*_{N,\delta}}\tilde{\varphi}_{N,L',L,\sigma_{\mathrm{lw}},v}-m_N,\ \max_{v\in V^*_{N,\delta}}\tilde{\xi}_{N,L',L,\sigma_{\mathrm{up}},v}-m_N\Big)=0.$$

By Lemma 3.1 (it is clear that the same statement holds for maximum over $V^*_{N,\delta}$), one gets

$$\begin{aligned}&\limsup_{(L,K,L',K')\Rightarrow\infty}\limsup_{N\to\infty}d\Big(\max_{v\in V^*_{N,\delta}}\tilde{\varphi}_{N,L',L,\sigma_{\mathrm{lw}},v}-m_N-(\sigma_*^2+4\alpha)\sqrt{d/2},\\&\quad\max_{v\in V_{N,\delta^*}}\varphi_{N,v}-m_N\Big)\\&\quad=0,\\&\limsup_{(L,K,L',K')\Rightarrow\infty}\limsup_{N\to\infty}d\Big(\max_{v\in V^*_{N,\delta}}\tilde{\xi}_{N,L',L,\sigma_{\mathrm{up}},v}-m_N-(\sigma_*^2)\sqrt{d/2},\\&\quad\max_{v\in V_{N,\delta^*}}\xi_{N,v}-m_N\Big)\\&\quad=0.\end{aligned}$$

Altogether, this gives that

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\limsup_{N\to\infty}\tilde{d}\Big(\max_{v\in V^*_{N,\delta}}\varphi_{N,v}-m_N,\ \max_{v\in V^*_{N,\delta}}\xi_{N,v}-m_N-2\alpha\sqrt{2d}\Big)=0.$$

The other direction of stochastic domination follows in the same manner. Altogether, this completes the proof of the lemma. □

4.2. *Convergence in law for the centered maximum.* In light of Lemma 4.2, in order to prove Theorem 1.3 it remains to show the convergence in law for the centered maximum of $\{\xi_{N,v}: v\in V_N\}$. To this end, we will follow the proof of the convergence in law in the case of the 2D DGFF given in [11]. Let the *fine field* be defined as $\xi^f_{N,v}=\xi_{N,v}-\xi^c_{N,v}$, and note that it implicitly depends on $K'L'$. As in [11], a key step in the proof of convergence of the centered maximum is the following sharp tail estimate on the right tail of the distribution of $\max_{v\in B}\xi^f_{N,v}$ for $B\in\mathcal{B}_{N/KL}$. The proof of this estimate is postponed to Section 6.

PROPOSITION 4.3. *Let Assumptions* (A.1), (A.2) *and* (A.3) *hold. Then there exist constants $C_\alpha, c_\alpha>0$ depending only on α and constants $c_\alpha\le\beta^*_{K',L'}\le C_\alpha$ such that as $z\to\infty$ we have*

$$\limsup_{L'\to\infty}\limsup_{K'\to\infty}\limsup_{N\to\infty}\left|\frac{e^{\sqrt{2d}z}}{z}\mathbb{P}\Big(\max_{v\in B_{N/KL,i}}\xi^f_{N,v}\ge m_{N/KL}+z\Big)-\beta^*_{K',L'}\right|\to 0. \tag{39}$$

REMARK 4.4. Proposition 4.3 is analogous to [11], Proposition 4.1, but there are two important differences:

(a) In Proposition 4.3, the convergence is to a constant $\beta^*_{K',L'}$ which depends on K', L', while in [11], Proposition 4.1, the convergence is to an absolute constant α^*. This is because the fine field $\xi_{N,v}$ here implicitly depends on K', L', and thus a priori one is not able to eliminate the dependence on (K', L') from the limit. However, in the same spirit as in [11], the dependence on (K', L') is not an issue for deducing a convergence in law—the crucial requirement is the independence of N. Eventually, we will deduce the convergence of $\beta^*_{K',L'}$ as $K', L' \to \infty$ in that order from the convergence in law of the centered maximum.

(b) In [11], Proposition 4.1, one also controls the limiting distribution of the location of the maximizer while in Proposition 4.3 this is not mentioned. This is because in the current situation and unlike the construction in [11], the coarse field $\{\xi^c_{N,v}\}$ is constant over each box $B_{N/KL,i}$, and thus the location of the maximizer of the fine field in each of the boxes $B_{N/KL,i}$ is irrelevant to the value of the maximum for $\{\xi_{N,v}\}$.

Next, we construct the limiting law of the centered maximum of $\{\xi_{N,v} : v \in V_N\}$. We partition $[0,1]^d$ into $R = (KL)^d$ disjoint boxes of equal sizes. Let $\beta^*_{K',L'}$ be as defined in the statement of Proposition 4.3. By that proposition, there exists a function $\gamma : \mathbb{R} \mapsto \mathbb{R}$ that grows to infinity arbitrarily slowly [in particular, we may assume that $\gamma(x) \le \log\log\log x$] such that

$$\lim_{z'\to\infty} \limsup_{L'\to\infty} \limsup_{K'\to\infty} \limsup_{N\to\infty} \sup_{z'\le z\le\gamma(K'L')} \Big| z^{-1} e^{\sqrt{2d}z} \times \mathbb{P}\Big(\max_{v\in B_{N/KL,i}} \xi^f_{N,v} \ge m_{N/KL} + z \Big) - \beta^*_{K',L'} \Big| = 0.$$

Let $\{\varrho_{R,i}\}_{i=1}^R$ be independent Bernoulli random variables with

$$\mathbb{P}(\varrho_{R,i} = 1) = \beta^*_{K',L'} \gamma(KL) e^{-\sqrt{2d}\gamma(KL)}.$$

In addition, consider independent random variables $\{Y_{R,i}\}_{i=1}^R$ such that

$$\mathbb{P}(Y_{R,i} \ge x) = \frac{\gamma(KL) + x}{\gamma(KL)} e^{-\sqrt{2d}x} x \ge 0. \tag{40}$$

Let $\{Z_{R,i} : 1 \le i \le R\}$ be an independent Gaussian field with covariance matrix Σ (recall that Σ is of dimension $R \times R$). We then define $G^*_{K,L,K',L'} = \max_{1\le i\le R, \varrho_{R,i}=1} G_{R,i}$, where $G_{R,i} = \varrho_{R,i}(Y_{R,i} + \gamma(KL)) + Z_{R,i} - \sqrt{2d}\log(KL)$ (here we use the convention that $\max \varnothing = 0$). Let $\bar{\mu}_{K,L,K',L'}$ be the distribution of $G^*_{K,L,K',L'}$. We note that $\bar{\mu}_{K,L,K',L'}$ does not depend on N.

THEOREM 4.5. *Let Assumptions* (A.0), (A.1), (A.2) *and* (A.3) *hold. Then*

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty} d(\mu_N, \bar{\mu}_{K,L,K',L'}) = 0, \tag{41}$$

where μ_N *is the law of* $\max_{v\in V_N}\xi_{N,v} - m_N$.

(Note that μ_N does depend on KL, $K'L'$.)

PROOF OF THEOREM 1.3. Theorem 1.3 follows from Lemma 4.2 and Theorem 4.5. □

Next, we give the proof of Theorem 4.5. Our proof is conceptually simpler than that of its analogue [11], Theorem 2.4, since our coarse field is constant over a box of size N/KL (and thus no consideration of the location for the maximizer in the fine field is needed).

PROOF OF THEOREM 4.5. Denote by $\tau = \arg\max_{v\in V_N}\xi_{N,v}$. Applying Theorem 1.2 to the Gaussian fields $\{\xi_{N,v} : v \in V_N\}$ and $\{\xi^c_{N,v} : v \in V_N\}$ (where the maximum of $\{\xi^c_{N,v} : v \in V_N\}$ is equivalent to the maximum of a log-correlated Gaussian field in a KL-box), we deduce that

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty}\mathbb{P}\big(\xi^f_{N,\tau} \ge m_{N/KL} + \gamma(KL) + 1\big) = 1. \tag{42}$$

Therefore, in what follows, we assume w.l.o.g. the occurrence of the event

$$\left\{\xi^f_{N,\tau} \ge \sqrt{2d}\log\frac{N}{KL} - \frac{3}{2\sqrt{2d}}\log\log\frac{N}{KL} + \gamma(KL) + 1\right\}.$$

Let $\mathcal{E} = \bigcup_{1\le i\le R}\{\max_{v\in B_{N/KL,i}}\xi^f_{N,v} \ge m_{N/KL} + KL + 1\}$. A simple union bound over i gives that

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty}\mathbb{P}(\mathcal{E}) = 0. \tag{43}$$

Thus, in what follows we assume without loss that $\mathcal{E}$ does not occur. Analogously, we let $\mathcal{E}' = \bigcup_{1\le i\le R}\{Y_{R,i} \ge KL + 1 - \gamma(KL)\}$. We see from the union bound that

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty}\mathbb{P}(\mathcal{E}') = 0. \tag{44}$$

In what follows, we assume without loss that $\mathcal{E}'$ does not occur.

For convenience of notation, we denote by

$$M^f_{N,i} = \max_{v\in B_{N/KL,i}}\xi^f_{N,v} - \big(m_{N/KL} + \gamma(KL)\big).$$

By Proposition 4.3, there exists $\varepsilon^* = \varepsilon^*(N, K, L, K', L')$ with

$$\limsup_{(L,K,L',K')\Rightarrow\infty}\ \limsup_{N\to\infty}\varepsilon^*\big(N, K, L, K', L'\big) = 0,$$

such that, for some $|\varepsilon^\diamond| \le \varepsilon^*/4$

$$\mathbb{P}\big(\varepsilon^\diamond \le M^f_{N,i} \le KL - \gamma(KL) + 1\big) = \mathbb{P}\big(\varrho_{R,i} = 1, Y_{R,i} \le KL - \gamma(KL) + 1\big),$$

and that for all $-1 \le t \le KL - \gamma(KL) + 1$

$$\mathbb{P}\big(\varrho_{R,i=1}, Y_{R,i} \le t - \varepsilon^*/2\big) \le \mathbb{P}\big(\varepsilon^\diamond \le M^f_{N,i} \le t\big) \le \mathbb{P}\big(\varrho_{R,i=1}, Y_{R,i} \le t + \varepsilon^*/2\big).$$

Therefore, there exists a coupling between $\{M^f_{N,i} : 1 \le i \le R\}$ and $\{\varrho_i, Y_{R,i} : 1 \le i \le R\}$ such that on the event $(\mathcal{E} \cup \mathcal{E}')^c$,

$$\begin{aligned} &\varrho_{R,i} = 1, \qquad \big|Y_{R,i} - M^f_{N,i}\big| \le \varepsilon^* \qquad \text{if } M^f_{N,i} \ge \varepsilon^*, \quad \text{and} \\ &\big|Y_{R,i} - M^f_{N,i}\big| \le \varepsilon^* \qquad \text{if } \varrho_{R,i} = 1. \end{aligned} \tag{45}$$

In addition, it is trivial to couple such that $\xi^c_{N,v} = Z_{R,i}$ for all $v \in B_{N/KL,i}$ and $1 \le i \le R$. Also, notice the following simple fact:

$$\limsup_{L\to\infty} \limsup_{K\to\infty} \limsup_{N\to\infty} \big(m_N - m_{N/KL} - \sqrt{2d}\log(KL)\big) = 0.$$

Altogether, we conclude that there exists a coupling such that outside an event of probability tending to 0 as $N \to \infty$ and then $(L, K, L', K') \Rightarrow \infty$ [cf. (42), (43), (44)] we have

$$\max_{v\in V_N} (\xi_{N,v} - m_N) - G^*_{K,L,K',L'} \le 2\varepsilon^*.$$

Now, let $\tau' = \arg\max_{1\le i\le R} G_{R,i}$. Applying Theorem 1.2 to the Gaussian field $\{Z_{R,i}\}$ and using the preceding inequality, we see that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \mathbb{P}(\varrho_{R,\tau'} = 1) = 1. \tag{46}$$

Combined with (45), this yields that there exists a coupling such that except with probability tending to 0 as $N \to \infty$ and then $(L, K, L', K') \Rightarrow \infty$ we have

$$\Big|\max_{v\in V_N} (\xi_{N,v} - m_N) - G^*_{K,L,K',L'}\Big| \le 2\varepsilon^*,$$

thereby completing the proof of Theorem 4.5. □

PROOF OF THEOREM 1.4. Recall that $G^*_{K,L,K',L'}$ is a random variable with law $\bar\mu_{K,L,K',L'}$. We will construct random variables $\mathcal{Z}_{K,L}$, measurable with respect to $\mathcal{F}^c := \sigma(\{Z_{R,i}\})$, so that for all x

$$\begin{aligned} &\limsup_{(L,K,L',K')\Rightarrow\infty} \frac{\bar\mu_{K,L,K',L'}((-\infty, x])}{\mathbb{E}(e^{-\beta^*_{K',L'}\mathcal{Z}_{K,L}e^{-\sqrt{2d}x}})} \\ &\quad = \liminf_{(L,K,L',K')\Rightarrow\infty} \frac{\bar\mu_{K,L,K',L'}((-\infty, x])}{\mathbb{E}(e^{-\beta^*_{K',L'}\mathcal{Z}_{K,L}e^{-\sqrt{2d}x}})} = 1. \end{aligned} \tag{47}$$

To demonstrate (47), due to (46), we may and will assume without loss that $\varrho_{R,\tau'} = 1$. Define $S_{R,i} := \sqrt{2d}\log(KL) - Z_{R,i}$. Then, for any real x,

$$\mathbb{P}\big(G^*_{K,L,K',L'} \le x\big) = \mathbb{E}\bigg(\prod_{i=1}^{R}\big(1 - \mathbb{P}\big(\varrho_{R,i}Y_{R,i} > S_{R,i} + x - \gamma(KL)|\mathcal{F}^c\big)\big)\bigg). \tag{48}$$

In addition, the union bound gives that

$$\limsup_{KL\to\infty} \mathbb{P}(\mathcal{D}) = 1, \qquad \text{where } \mathcal{D} = \Big\{\min_{1\le i\le R} S_{R,i} \ge 2\gamma(KL)\Big\}.$$

So in the sequel we assume that $\mathcal{D}$ occurs. By the definition of $\varrho_{R,i}$ and $Y_{R,i}$, we get that

$$\mathbb{P}\big(\varrho_{R,i}Y_{R,i} > S_{R,i} + x - \gamma(KL)|\mathcal{F}^c\big) = \beta^*_{K',L'}(S_{R,i} + x)e^{-\sqrt{2d}(S_{R,i}+x)},$$

which converges to 0 as $KL \to \infty$. Therefore,

$$\begin{aligned}
&\exp\big(-(1+\varepsilon_{K,L})\beta^*_{K',L'}S_{R,i}e^{-\sqrt{2d}(x+S_{R,i})}\big)\\
&\quad\le \mathbb{P}\big(\varrho_{R,i}Y_{R,i} \le S_{R,i} + x - \gamma(KL)|\mathcal{F}^c\big)\\
&\quad\le \exp\big(-(1-\varepsilon_{K,L})\beta^*_{K',L'}S_{R,i}e^{-\sqrt{2d}(x+S_{R,i})}\big)
\end{aligned} \tag{49}$$

for $\varepsilon_{K,L} > 0$ with

$$\limsup_{KL\to\infty} \varepsilon_{K,L} = 0.$$

Define $\mathcal{Z}_{K,L} = \sum_{i=1}^{R} S_{R,i}e^{-\sqrt{2d}S_{R,i}}$ [this is the analogue of a derivative martingale, see (4)]. Substituting (49) into (48) completes the proof of (47). Now, combining (47) and Theorem 4.5, we see that we necessarily have

$$\limsup_{K'\to\infty}\limsup_{L'\to\infty}|\beta^*_{K',L'} - \beta^*| = 0$$

for a number β^* that does not depend on (K', L'). Plugging the preceding inequality into (47), we deduce that

$$\begin{aligned}
&\limsup_{(L,K,L',K')\Rightarrow\infty} \frac{\bar{\mu}_{K,L,K',L'}((-\infty,x])}{\mathbb{E}(e^{-\beta^*\mathcal{Z}_{K,L}e^{-\sqrt{2d}x}})}\\
&\quad= \liminf_{(L,K,L',K')\Rightarrow\infty} \frac{\bar{\mu}_{K,L,K',L'}((-\infty,x])}{\mathbb{E}(e^{-\beta^*\mathcal{Z}_{K,L}e^{-\sqrt{2d}x}})} = 1.
\end{aligned} \tag{50}$$

Combining (50) with Theorem 4.5 again, we see that $\mathcal{Z}_{K,L}$ converges weakly to a random variable $\mathcal{Z}$ as $K \to \infty$ and then $L \to \infty$. Also note that $\mathcal{Z}_{K,L}$ depends only on the product KL. Therefore, this implies that $\mathcal{Z}_N$ converges weakly to a random variable $\mathcal{Z}$. From the tightness of the laws $\bar{\mu}_{K,L,K',L'}$, it follows that $\mathcal{Z} > 0$ a.s. This completes the proof of Theorem 1.4. $\square$

PROOF OF REMARK 1.6. Consider two sequences $\{\varphi_{N,v}\}$ and $\{\tilde{\varphi}_{N,v}\}$ that satisfy assumptions (A.0)–(A.3) with the same functions $h(x,y)$ and $f(x)$ but possibly different functions $g(u,v)$, $\tilde{g}(u,v)$ and different constants $\alpha^{(\delta)}$, $\alpha^{(\delta),\prime}$ and α_0, α_0'. Define the corresponding (see Section 4.1) fields:

$$\xi_{N,KL,K'L'} = \xi^c_{N,KL,K'L'} + \xi^f_{N,KL,K'L'},$$
$$\tilde{\xi}_{N,KL,K'L'} = \tilde{\xi}^c_{N,KL,K'L'} + \tilde{\xi}^f_{N,KL,K'L'}.$$

Set also

$$\hat{\xi}_{N,KL,K'L'} = \tilde{\xi}^c_{N,KL,K'L'} + \xi^f_{N,KL,K'L'}.$$

Let $\nu_N, \tilde{\nu}_N$ denote the laws of the centered maxima $\max_{v\in V_N}\varphi_{N,v} - m_N$, $\max_{v\in V_N}\tilde{\varphi}_{N,v} - \tilde{m}_N$, and let $\mu_N, \tilde{\mu}_N, \hat{\mu}_N$ denote the laws of the centered maxima of the $\xi_N, \tilde{\xi}_N, \hat{\xi}_N$ fields. (Recall that the latter depend also on $KL, K'L'$ but we drop that fact from the notation.) By Lemma 4.2, we have

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \big(d(\mu_N,\nu_N) + d(\tilde{\mu}_N,\tilde{\nu}_N)\big) = 0. \tag{51}$$

For $s\in\mathbb{R}$, let $\theta_s\mu$ denote the shift of a probability measure μ on $\mathbb{R}$, that is, $\theta_s\mu(A) = \mu(A+s)$ for any measurable set A. Recall the construction of $\bar{\mu}_{K,L,K',L'}$ (see Theorem 4.5), and construct similarly $\tilde{\mu}_{K,L,K',L'}$ and $\hat{\mu}_{K,L,K',L'}$. Note that, by construction, there exists $s = s(KL)$, bounded uniformly in KL, so that $\theta_s\hat{\mu}_{K,L,K',L'} = \tilde{\mu}_{K,L,K',L'}$. In particular, from Theorem 4.5 we get that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \big(d(\mu_N,\bar{\mu}_{K,L,K',L'}) + d(\tilde{\mu}_N,\theta_s\hat{\mu}_{K,L,K',L'})\big) = 0. \tag{52}$$

From (51) and (52), one can find a sequence $L(N), K(N), K'(N), L'(N)$ along which the convergence still holds (as $N\to\infty$). Let $\{\eta_{v,N}\}$ and $\{\hat{\eta}_{v,N}\}$ denote the fields $\{\xi_{v,N}\}$ and $\{\hat{\xi}_{v,N}\}$ with this choice of parameters, and let $\bar{\mu}_N$ and $\hat{\mu}_N$ denote the corresponding laws of the maximum. Let μ_∞, $\tilde{\mu}_\infty$ denote the limits of μ_N and $\tilde{\mu}_N$, which exist by Theorem 1.3. From the above considerations, we have that $\bar{\mu}_N\to\mu_\infty$ and $\theta_{s(N)}\hat{\mu}_N\to\tilde{\mu}_\infty$. On the other hand, the fields $\eta_{N,\cdot}$ and $\hat{\eta}_{N,\cdot}$ both satisfy assumptions (A.0)–(A.3) with the same functions f, g, h, and thus, interleaving between then one deduces that the laws of their centered maxima converge to the same limit, denoted Θ_∞. It follows that necessarily, $s(N)$ converges and $\mu_\infty = \theta_s\tilde{\mu}_\infty = \Theta_\infty$. Using the characterization in Theorem 1.4, this yields the claim in the remark. □

5. An example: The circular logarithmic REM. In the important paper [20], the authors introduce a one dimensional log-correlated Gaussian field, which they call the *circular logarithmic REM* (CLREM). Fyodorov and Bouchaud

consider the CLREM as a prototype for Gaussian fields exhibiting Carpentier–LeDoussal freezing. (We do not discuss in this paper the notion of freezing, referring instead to [20] and to [4, 26, 29] and [8].) Explicitly, fix an integer N, set $\theta_k = 2\pi k/N$, and introduce the matrix:

$$R_{k,\ell} = -\frac{1}{2}\log\left(4\sin^2\left(\frac{\theta_k - \theta_\ell}{2}\right)\right)\mathbf{1}_{k\neq\ell} + (\log N + W)\mathbf{1}_{k=\ell},$$

where W is a constant independent on N. It is not hard to verify (and this is done explicitly in [20]) that one can choose W so that the matrix R is positive definite for all N; the resulting Gaussian field $\varphi_{N,v}$ with correlation matrix R is the CLERM. One may think of the CLREM as indexed by V_N in dimension $d = 1$, or (as the name indicates) by an equally spaced collection of N points on the unit circle in the complex plane.

Let $M_N = \max_{v\in V_N}\varphi_{N,v}$. The following is a corollary of Theorems 1.2 and 1.4.

COROLLARY 5.1. $\mathbb{E}M_N = \sqrt{2}\log N - (3/2\sqrt{2})\log\log N + O(1)$ *and there exist a constant* β^* *and a random variable* $\mathcal{Z}$ *so that*

$$\lim_{N\to\infty}\mathbb{P}(M_N - \mathbb{E}M_N \leq x) = \mathbb{E}\big(e^{-\beta^*\mathcal{Z}e^{-\sqrt{2}x}}\big). \tag{53}$$

PROOF. Assumptions (A.0) and (A.1) are immediate to check. An explicit computation reveals that Assumption (A.2) holds with $f(x) = 0$ and

$$g(u,v) = \begin{cases} -W, & u = v, \\ \log(4\pi) + |u - v|, & u \neq v. \end{cases}$$

Finally, note that Assumption (A.3) holds with $h(x,y) = \log(4\sin^2(2\pi|x-y|))$. Thus, Theorems 1.3 and 1.4 apply and yields (53). □

REMARK 5.2. Remarkably, in [20] the authors compute explicitly, albeit non-rigorously, the law of the maximum of the CLREM, up to a deterministic shift that they do not compute. It was observed in [29] that the law computed in [20] is in fact the law of a convolutions of two Gumbel random variables. In the notation of Corollary 5.1, this means that one expects that $2^{-1/2}\log(\beta^*\mathcal{Z})$ is Gumbel distributed. We do not have a rigorous proof for this claim.

6. Proof of Proposition 4.3. Our proof of Proposition 4.3 is highly similar to the proof in [11], Proposition 4.1, but simpler in a number of places. We will sketch the outline of the arguments, and refer to [11] extensively (it is helpful to recall Remark 4.4). To start, we note that by Lemmas 2.2 and 2.4, there exists $c_\alpha > 0$ depending only on α such that

$$\mathbb{P}\Big(\max_{v\in B_{N/KL,i}}\xi^f_{N,v} \geq m_{N/KL} + z\Big) \geq c_\alpha z e^{-\sqrt{2d}z} \tag{54}$$

for all $1 \le z \le \sqrt{\log N/KL}$, $1 \le i \le (KL)^d$. In addition, adapting the proof of (2), we deduce that there exists $C_\alpha > 0$ depending only on α such that

$$\mathbb{P}\Big(\max_{v \in B_{N/KL,i}} \xi^f_{N,v} \ge m_{N/KL} + z\Big) \le C_\alpha z e^{-\sqrt{2d}z} \tag{55}$$

for all $z \ge 1$, $1 \le i \le (KL)^d$.

Recall the definition of $\{\xi_{N,v}\}$ as in (37). In what follows, we consider a fixed i and a box $B_{N/KL,i}$. We note that the law of the fine field $\{\xi^f_{N,v} : v \in B_{N/KL,i}\}$ does not depend on K, L, i, and hence $\beta^*_{K',L'}$ does not depend on K, L, i. Write $\bar{N} = N/KL = 2^{\bar{n}}$ and $\bar{L} = K'L' = 2^{\bar{\ell}}$. For convenience of notation, we will refer to the box $B_{N/KL,i}$ as $V_{\bar{N}}$ and let $\Xi_{\bar{N}}$ be the collection of all left bottom corners of $\bar{L}$-boxes of form $B_{\bar{L},j}$ in $B_{N/KL,i}$. In addition, write $n^* = \frac{\operatorname{Var} X_{v,N}}{\log 2} = \bar{n} - \bar{\ell}$, where we denote $X_{v,N} = \xi_{N,v,\mathrm{MBRW}}$.

For convenience, we now view each $X_{v,N}$ as the value at time n^* of a Brownian motion with variance rate $\log 2$. More precisely, we assign to each Gaussian variable $b^N_{j,B}$ in (31) an independent Brownian motion, with variance rate $\log 2$, that runs for 2^{-2j} time units and ends at the value $b^N_{j,B}$. We now define a Brownian motion $\{X_{v,N}(t) : 0 \le t \le n^*\}$ by concatenating each of the previous Brownian motions associated with $v \in \Xi_{\bar{N}}$, with earlier times corresponding to larger boxes. From our construction, we see that $X_{v,N}(n^*) = X_{v,N}$. We partition $V_{\bar{N}}$ into disjoint $\bar{L}$-boxes, for which we denote $\mathcal{B}_{\bar{L}}$. Further, denote by B_v the $\bar{L}$-box in $\mathcal{B}_{\bar{L}}$ that contains v. Define

$$\begin{aligned}
E_{v,N}(z) &= \Big\{ X_{v,N}(t) \le z + \frac{m_{\bar{N}}}{\bar{n}} t \\
&\qquad \text{for all } 0 \le t \le n^*, \text{and} \max_{u \in B_v} \xi^f_{u,N} \ge m_{\bar{N}} + z \Big\}, \\
F_{v,N}(z) &= \Big\{ X_{v,N}(t) \le z + \frac{m_{\bar{N}}}{\bar{n}} t + 10\big(\log\big(t \wedge (n^* - t)\big)\big)_+ + z^{1/20} \\
&\qquad \text{for all } 0 \le t \le n^*, \text{and} \max_{u \in B_v} \xi^f_{u,N} \ge m_{\bar{N}} + z \Big\}, \\
G_N(z) &= \bigcup_{v \in \Xi_{\bar{N}}} \bigcup_{0 \le t \le n^*} \Big\{ X_{v,N}(t) > z + \frac{m_{\bar{N}}}{\bar{n}} t \\
&\qquad + 10\big(\log\big(t \wedge (n^* - t)\big)\big)_+ + z^{1/20} \Big\}.
\end{aligned} \tag{56}$$

Also define

$$\Lambda_{\bar{N},z} = \sum_{v \in \Xi_{\bar{N}}} \mathbf{1}_{E_{v,N}(z)} \quad \text{and} \quad \Gamma_{\bar{N},z} = \sum_{v \in \Xi_{\bar{N}}} \mathbf{1}_{F_{v,N}(z)}.$$

In words, the random variable $\Lambda_{N,z}$ counts the number of boxes in $\mathcal{B}_{\bar{L}}$ whose "backbone" path $X_{v,N}(\cdot)$ stays below a linear path connecting z to roughly $m_{\bar{N}}+z$, so that one of its "neighbors" achieves a terminal value that is at least $m_{\bar{N}}+z$; the random variable $\Gamma_{N,z}$ similarly counts boxes in $\mathcal{B}_{\bar{L}}$ whose backbone is constrained to stay below a slightly "upward bent" curve. Clearly, $E_{v,N}(z) \subseteq F_{v,N}(z)$ always holds, as does $\Lambda_{\bar{N},z} \le \Gamma_{\bar{N},z}$.

By (37), for each $v \in \Xi_{\bar{N}}$ we can write that

$$\max_{u\in B_v} \xi^f_{N,v} = X_{v,N} + Y_{v,N}, \tag{57}$$

where $\{Y_{v,N}\}$ are i.i.d. random variables with the same law as $\max_{u\in V_{\bar{L}}} \varphi_{\bar{L},u} + a_{K,L,K',L',u}\phi$ where ϕ is a standard Gaussian variable. Crucially, the law of $Y_{v,N}$ does not depend on N. In addition, by Proposition 1.1 and Lemma 2.2, there exist C_α depending only on α such that

$$\mathbb{P}(Y_{v,N} \ge m_{\bar{L}} + \lambda) \le C_\alpha \lambda e^{-\sqrt{2d}\lambda} e^{-C_\alpha^{-1}\lambda^2/\bar{\ell}} \qquad \text{for all } \lambda \ge 1. \tag{58}$$

When estimating the ratio $\frac{\Lambda_{\bar{N},z}}{\Gamma_{\bar{N},z}}$, it is clear that $\frac{\Lambda_{\bar{N},z}}{\Gamma_{\bar{N},z}} = \frac{\mathbb{P}(E_{v,N}(z))}{\mathbb{P}(F_{v,N}(z))}$ for any fixed $v \in \Xi_{\bar{N}}$, where the latter concerns only the associated Brownian motion to $X_{v,N}$ and the random variable $Y_{v,N}$. As such, the arguments in [11], Lemma 4.10, carry out with merely notation change and give that

$$\lim_{z\to\infty} \limsup_{\bar{L}\to\infty} \limsup_{N\to\infty} \frac{\Lambda_{\bar{N},z}}{\Gamma_{\bar{N},z}} = 1. \tag{59}$$

Analogous to the proof of [11], Equation (100), we can compare the field $\{X_{v,N}\}$ to a BRW and apply [11], Lemma 3.7, to obtain that

$$\mathbb{P}\big(G_N(z)\big) \le C_\alpha e^{-\sqrt{2d}z}. \tag{60}$$

Note that the dimension does not play a significant role in these estimates, as [11], Lemma 3.7, follows from a union bound calculation. The dimension changes the volume of the box, but the probability

$$\mathbb{P}\bigg(X_{v,N}(t) > z + \frac{m_{\bar{N}}}{\bar{n}}t + 10\big(\log\big(t \wedge (n^* - t)\big)\big)_+ + z^{1/20}\bigg)$$

scales in the dimension (recall that m_N depends on d) which exactly cancels the growth of the volume in d.

The next desired ingredient is the second moment computation for $\Lambda_{\bar{N},z}$. Note that (i) our field $\{X_{v,N} : v \in \Xi_{\bar{N}}\}$ is simply an MBRW (so $\{X_{v,N}\}$ is nicer than its analog in [11], which is a sum of an MBRW and a field with uniformly bounded variance); (ii) our $\{Y_{v,N}\}$ are i.i.d. random variable with desired tail bounds as in (57) (so also nicer than its analog in [11], which has weak correlation for two

neighboring local boxes). Therefore, the second moment computation in [11], Lemma 4.11, carries out with minimal notation change and gives

$$\lim_{z\to\infty} \limsup_{\bar{L}\to\infty} \limsup_{N\to\infty} \frac{\mathbb{E}(\Lambda_{\bar{N},z})^2}{\mathbb{E}\Lambda_{\bar{N},z}} = 1. \tag{61}$$

Note that in [11], Equation (90), there is no analog of $\limsup_{\bar{L}\to\infty}$ as in the preceding inequality. That is because we have assumed in [11] that $L \geq 2^{2^{z^4}}$. Our statement as in (61) is weaker as it does not give a quantitative dependance on how $\bar{L}$ should grow in z. But this detailed quantitative dependence is not needed for the proof of convergence in law.

Combining (54), (59), (60) and (61), we deduce that

$$\lim_{z\to\infty} \limsup_{\bar{L}\to\infty} \limsup_{N\to\infty} \left| \frac{\mathbb{P}(\max_{v\in V_{\bar{N}}} \xi^f_{N,v} \geq m_{\bar{N}} + z)}{\mathbb{E}\Lambda_{\bar{N},z}} - 1 \right| = 0. \tag{62}$$

Therefore, it remains to estimate $\mathbb{E}\Lambda_{\bar{N},z}$. To this end, we will follow [11], Section 4.3. We first note that by (54) and (62), we have

$$\lim_{z\to\infty} \limsup_{\bar{L}\to\infty} \limsup_{N\to\infty} \frac{\mathbb{E}\Lambda_{\bar{N},z}}{ze^{-\sqrt{2d}z}} \geq c_\alpha, \tag{63}$$

where $c_\alpha > 0$ is a constant depending on α.

The main goal is to derive the asymptotics for $\mathbb{E}\Lambda_{\bar{N},z}$. For $v \in \Xi_{\bar{N}}$, let $\nu_{v,\bar{N}}(\cdot)$ be the density function (of a subprobability measure on $\mathbb{R}$) such that, for all $I \subseteq \mathbb{R}$, the integral $\int_I \nu_{v,\bar{N}}(y)\,dy$ is equal to

$$\mathbb{P}\Big(X_{v,N}(t) \leq z + \frac{m_{\bar{N}}}{\bar{n}} t \text{ for all } 0 \leq t \leq n^*;\ X_{v,N}(n^*) - (\bar{n} - \bar{\ell}) m_{\bar{N}}/\bar{n} \in I \Big).$$

Clearly, by (57),

$$\mathbb{P}\big(E_{v,N}(z)\big) = \int_{-\infty}^{z} \nu_{v,\bar{N}}(y) \mathbb{P}(Y_{v,N} \geq \bar{\ell} m_{\bar{N}}/\bar{n} + z - y)\,dy.$$

For a given interval J, define

$$\lambda_{v,N,z,J} = \int_J \nu_{v,\bar{N}}(y) \mathbb{P}(Y_{v,N} \geq \bar{\ell} m_{\bar{N}}/\bar{n} + z - y)\,dy. \tag{64}$$

Set $J_{\bar{\ell}} = [-\bar{\ell}, -\bar{\ell}^{2/5}]$. For convenience of notation, we denote by $A \lesssim B$ that there exists a constant $C_\alpha > 0$ that depends only on α such that $A \leq C_\alpha B$ for two functions/sequences A and B. As in [11], Lemma 4.13, we claim that for any any sequences $x_{v,N}$ such that $|x_{v,N}| \lesssim \bar{\ell}^{1/5}$,

$$\lim_{z\to\infty} \liminf_{\bar{\ell}\to\infty} \liminf_{N\to\infty} \frac{\sum_{v\in\Xi_N} \lambda_{v,N,z,x_{v,N}+J_{\bar{\ell}}}}{\mathbb{E}\Lambda_{N,z}} = 1. \tag{65}$$

Note that, by containment, the above ratio is always at most 1. We prove (65) for the case when $x_{v,N} = 0$; the general case follows in the same manner. Application of the reflection principle (cf. [11], Equation (28)) to the Brownian motion with drift, $\bar{X}_{v,N}(\cdot) = X_{v,N}(\cdot) - m_{\bar{N}} t/\bar{n}$, together with the change of measure that removes the drift $m_{\bar{N}} t/\bar{n}$, implies that

$$\nu_{v,\bar{N}}(y) \lesssim e^{-\sqrt{2d}y} 2^{-dn^*} z|y|$$

for $y \le -\bar{\ell}$, over the given range $z \in (0, \bar{\ell})$ (which implies $z - y \asymp |y|$). Together with (58) and independence among $Y_{v,N}$ for $v \in \Xi_{\bar{N}}$, this implies the crude bound:

$$\int_{-\infty}^{-\ell} \nu_{v,\bar{N}}(y) \mathbb{P}(Y_{v,N} \ge \bar{\ell} m_{\bar{N}}/\bar{n} + z - y)\, dy \lesssim 2^{-dn^*} e^{-C_\alpha^{-1}\bar{\ell}}$$

for a constant $C_\alpha > 0$ depending on α. Similarly, for $y \le z$ (and, therefore, for $z - y \ge 0$), application of the reflection principle and (58) again implies that

$$\int_{-\bar{\ell}^{2/5}}^{z} \nu_{v,\bar{N}}(y) \mathbb{P}(Y_{v,N} \ge \bar{\ell} m_{\bar{N}}/\bar{n} + z - y)\, dy \lesssim 2^{-dn^*} \bar{\ell}^{-3/10} z e^{-\sqrt{2d}z}.$$

Together with (63), this completes the verification of (65).

Next, we claim that there exists $\Lambda^*_{K',L',z} > 0$ that does not depend on N such that

$$\lim_{z\to\infty} \limsup_{\bar{L}\to\infty} \limsup_{N\to\infty} \frac{\mathbb{E}\Lambda_{N,z}}{\Lambda^*_{K',L',z}} = \lim_{z\to\infty} \liminf_{\bar{L}\to\infty} \liminf_{N\to\infty} \frac{\mathbb{E}\Lambda_{N,z}}{\Lambda^*_{K',L',z}} = 1. \tag{66}$$

By the reflection principle and change of measure, we get that for all $y \in [-\bar{\ell}, z]$ (see the derivation of [11], Equation (107))

$$\nu_{v,\bar{N}}(y) = 2^{-dn^*} e^{-\sqrt{2d}y} \frac{z(z-y)}{\sqrt{2\pi \log 2}} \big(1 + O(\bar{\ell}^3/\bar{n})\big). \tag{67}$$

Therefore,

$$\begin{aligned}
\sum_{v\in\Xi_{\bar{N}}} \lambda_{v,N,z,J_{\bar{\ell}}} &= \left(\frac{\bar{N}}{\bar{L}}\right)^d \int_{J_{\bar{\ell}}} \nu_{v_0,\bar{N}}\left(y + O\left(\frac{\bar{\ell}}{\sqrt{\bar{n}}}\right)\right) \\
&\quad \times \mathbb{P}(Y_{v_0,N} \ge \sqrt{2d}\bar{\ell}\log 2 + z - y)\, dy \\
&= \big(1 + O(\bar{\ell}^3/\sqrt{\bar{n}})\big) \int_{J_{\bar{\ell}}} \frac{z(z-y)}{\sqrt{2\pi\log 2} e^{\sqrt{2d}y}} \\
&\quad \times \mathbb{P}\big(Y_{v_0,N} \ge (\log 2)\sqrt{2d}\bar{\ell} + z - y\big)\, dy,
\end{aligned}$$

where v_0 is any fixed vertex in $\Xi_{\bar{N}}$ and in the last step we have used the fact that $n^* = \bar{n} - \bar{\ell}$. Recall that the law of $Y_{v_0,N}$ is the same as $\max_{u\in V_{\bar{L}}} \varphi_{\bar{L},u} + a_{K',L',u}\phi$, which does not depend on N. Combined with (65), this completes the proof of (66).

Finally, we analyze how $\mathbb{E}\Lambda_{N,z}$ scales with z. To this end, consider $z_1 < z_2$. For $v \in \Xi_N$ and $j = 1, 2$, recall that

$$\lambda_{v,N,z_j,z_j+J_{\bar\ell}} = \int_{J_{\bar\ell}+z_j} \nu_{v,\bar N}(y)\mathbb{P}(Y_{v,N} \ge \ell m_{\bar N}/\bar n + z_i - y)\,dy.$$

By (67), for any $y \in J_{\bar\ell}$ and $z_1, z_2 \ll \log \bar\ell$,

$$\begin{aligned}\frac{\nu_{v,\bar N}(y+z_1)\mathbb{P}(Y_{v,N} \ge \bar\ell m_{\bar N}/\bar n - y)}{\nu_{v,\bar N}(y+z_2)\mathbb{P}(Y_{v,N} \ge \bar\ell m_{\bar N}/\bar n - y)} &= \frac{\nu_{v,\bar N}(y+z_1)}{\nu_{v,\bar N}(y+z_2)} \\ &= \Big(1 + O\Big(\frac{\bar\ell^3}{\bar n}\Big)\Big)\frac{z_1(z_1-y)}{z_2(z_2-y)}e^{-\sqrt{2d}(z_1-z_2)} \\ &= \Big(1 + O\Big(\frac{\bar\ell^3}{\bar n}\Big)\Big)\frac{z_1}{z_2}e^{-\sqrt{2d}(z_1-z_2)}(1+z_2^{-3/5}).\end{aligned}$$

This implies that

$$\frac{\lambda_{v,N,z_1,z_1+J_{\bar\ell}}}{\lambda_{v,N,z_2,z_2+J_{\bar\ell}}} = (1 + O(\bar\ell^3/\bar n))\frac{z_1}{z_2}e^{-\sqrt{2d}(z_1-z_2)}(1+z_2^{-3/5}).$$

Together with (65), the above display implies that

$$\begin{aligned}&\lim_{z_1,z_2\to\infty} \limsup_{\bar L\to\infty} \limsup_{N\to\infty} \frac{z_2 e^{-\sqrt{2d}z_2}\mathbb{E}\Lambda_{N,z_1}}{z_1 e^{-\sqrt{2d}z_1}\mathbb{E}\Lambda_{N,z_2}} \\ &\quad= \lim_{z_1,z_2\to\infty} \liminf_{\bar L\to\infty} \liminf_{N\to\infty} \frac{z_2 e^{-\sqrt{2d}z_2}\mathbb{E}\Lambda_{N,z_1}}{z_1 e^{-\sqrt{2d}z_1}\mathbb{E}\Lambda_{N,z_2}} = 1.\end{aligned}$$

Along with (66), this completes the proof of (39) for some $\beta^*_{K',L'}$. From (54) and (55), we see that $c_\alpha \le \beta^*_{K',L'} \le C_\alpha$ for all K', L'. This completes the proof of the proposition. □

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J. DING
DEPARTMENT OF STATISTICS
UNIVERSITY OF CHICAGO
5747 S. ELLIS AVENUE
JONES 322
CHICAGO, ILLINOIS 60637
USA
E-MAIL: jianding@galton.uchicago.edu

R. ROY
DECISION SCIENCES AND INFORMATION SYSTEMS
INDIAN INSTITUTE OF MANAGEMENT BANGALORE
BANNERGHATTA ROAD
BANGALORE 560076
INDIA
E-MAIL: rishideeproy@gmail.com

O. ZEITOUNI
DEPARTMENT OF MATHEMATICS
WEIZMANN INSTITUTE OF SCIENCE
POB 26
REHOVOT 76100
ISRAEL
AND
COURANT INSTITUTE OF
MATHEMATICAL SCIENCES
NEW YORK UNIVERSITY
251 MERCER STREET
NEW YORK, NEW YORK 10012
USA
E-MAIL: ofer.zeitouni@weizmann.ac.il