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# THE COMPLEXITY OF SPHERICAL $p$-SPIN MODELS—A SECOND MOMENT APPROACH[1]

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Recently, Auffinger, Ben Arous and Černý initiated the study of critical points of the Hamiltonian in the spherical pure $p$-spin spin glass model, and established connections between those and several notions from the physics literature. Denoting the number of critical values less than $Nu$ by $\mathrm{Crt}_N(u)$, they computed the asymptotics of $\frac{1}{N}\log(\mathbb{E}\mathrm{Crt}_N(u))$, as $N$, the dimension of the sphere, goes to $\infty$. We compute the asymptotics of the corresponding second moment and show that, for $p \geq 3$ and sufficiently negative $u$, it matches the first moment:

$$\mathbb{E}\{(\mathrm{Crt}_N(u))^2\}/(\mathbb{E}\{\mathrm{Crt}_N(u)\})^2 \to 1.$$

As an immediate consequence we obtain that $\mathrm{Crt}_N(u)/\mathbb{E}\{\mathrm{Crt}_N(u)\} \to 1$, in $L^2$, and thus in probability. For any $u$ for which $\mathbb{E}\,\mathrm{Crt}_N(u)$ does not tend to 0 we prove that the moments match on an exponential scale.

## CONTENTS



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**1. Introduction.** The Hamiltonian of the spherical *pure* $p$-spin spin glass model is given by

$$H_N(\boldsymbol{\sigma}) := H_{N,p}(\boldsymbol{\sigma})$$
$$= \frac{1}{N^{(p-1)/2}} \sum_{i_1,\dots,i_p=1}^{N} J_{i_1,\dots,i_p}\sigma_{i_1}\cdots\sigma_{i_p}, \qquad \boldsymbol{\sigma} \in \mathbb{S}^{N-1}(\sqrt{N}), \tag{1.1}$$

where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$, $\mathbb{S}^{N-1}(\sqrt{N}) \triangleq \{\boldsymbol{\sigma} \in \mathbb{R}^N : \|\boldsymbol{\sigma}\|_2 = \sqrt{N}\}$, and $J_{i_1,\dots,i_p}$ are i.i.d. standard normal variables. Everywhere in the paper we shall assume that $p \geq 3$.[2] The model was introduced by Crisanti and Sommers [21] as a variant of the Ising $p$-spin spin glass model. Unlike the Ising $p$-spin model, defined on the hypercube, the spherical $p$-spin model is defined on a continuous space—a property they expected to yield a model amenable to different methods of analysis, while retaining the main features of the original model. A generalization of the model called the spherical *mixed* $p$-spin spin glass model is obtained by setting the Hamiltonian to be $H_N(\boldsymbol{\sigma}) = \sum_{p\geq 2} \beta_p H_{N,p}(\boldsymbol{\sigma})$, with $H_{N,p}(\boldsymbol{\sigma})$ being independent pure $p$-spin models and $\beta_p \geq 0$ (such that the sum is defined).

Recently, Auffinger, Ben Arous and Černý [5] suggested studying the critical points of the Hamiltonian of the spherical pure $p$-spin model in order to understand its landscape. Their work was later extended [4] to the mixed case. The main results of [5] on the complexity of the Hamiltonian for the pure $p$-spin model are as follows. Let $\mathrm{Crt}_N(B)$ denote the number of critical points of $H_N(\boldsymbol{\sigma})$ at which $H_N(\boldsymbol{\sigma})/N$ lies in a Borel set $B \subset \mathbb{R}$ [cf. (2.2)]. Use the notation $\mathrm{Crt}_{N,k}(B)$ for the number of such critical points with index $k$. It was shown in [5] that

$$\lim_{N\to\infty} \frac{1}{N} \log\bigl(\mathbb{E}\bigl\{\mathrm{Crt}_N\bigl((-\infty, u)\bigr)\bigr\}\bigr) = \Theta_p(u), \tag{1.2}$$

[2] In the case $p = 2$, the critical points of $H_N(\boldsymbol{\sigma})$ are exactly the points $\boldsymbol{\sigma} \in \mathbb{S}^{N-1}(\sqrt{N})$ which are eigenvectors of the matrix $(J_{i_1,i_2} + J_{i_2,i_1})_{i_1,i_2=1}^N$. In particular, there are exactly $2N$ such points almost surely.

$$\lim_{N\to\infty}\frac{1}{N}\log\bigl(\mathbb{E}\bigl\{\mathrm{Crt}_{N,k}\bigl((-\infty,u)\bigr)\bigr\}\bigr)=\Theta_{p,k}(u), \tag{1.3}$$

where $\Theta_p(u)$ and $\Theta_{p,k}(u)$ are known nondecreasing functions (cf. Theorem 10). Moreover, with $E_k(p)$ being equal to the unique number satisfying $\Theta_{p,k}(-E_k(p))=0$,

$$E_0(p)>E_1(p)>E_2(p)>\cdots,\quad \text{and}\quad \lim_{k\to\infty}E_k(p)=E_\infty(p)\triangleq 2\sqrt{\frac{p-1}{p}},$$

and for each $k$ and closed set $B\subset\mathbb{R}$ such that $B$ and $[-E_k(p),-E_\infty(p)]$ are disjoint, $\mathbb{P}\{\mathrm{Crt}_{N,k}(B)>0\}$ decays (at least) exponentially in $N$. In addition, they showed that for $u<-E_\infty(p)$, $\Theta_p(u)=\Theta_{p,0}(u)$, which, in particular, implies that for any $\varepsilon>0$, with high probability

$$\mathrm{Crt}_N\bigl((-\infty,-E_0(p)-\varepsilon)\bigr)=0. \tag{1.4}$$

The computation of the means is certainly a significant step in the investigation of the critical points. However, *by themselves*, the means give very limited information on the probabilistic law of the corresponding variables. Essentially, they can only be used to obtain (by appealing to Markov's inequality) the upper bounds on (1.4) stated above. A question that naturally arises is: are the corresponding variables concentrated around their means? In the general context of spherical mixed $p$-spin models, this is not necessarily the case: for a subclass of models termed by [4] *full mixture* models, there is a range of levels $u$, such that the mean number of critical points in $(-\infty,u)$ is exponentially high, while the probability of having a critical point in $(-\infty,u)$ goes to zero (see [4], Corollary 4.1).

Focusing on the pure case and on the number of critical points of general index $\mathrm{Crt}_N(\cdot)$, we establish that the answer to the above is positive. This is done, as suggested in [5], page 2, by computing the second moment in addition to the already known first moment.

THEOREM 1. *For any $p\geq 3$ and $u\in(-E_0(p),-E_\infty(p))$,*

$$\lim_{N\to\infty}\frac{\mathbb{E}\{(\mathrm{Crt}_N((-\infty,u)))^2\}}{(\mathbb{E}\{\mathrm{Crt}_N((-\infty,u))\})^2}=1. \tag{1.5}$$

As an immediate corollary, we obtain the following.

COROLLARY 2. *For any $p\geq 3$ and $u\in(-E_0(p),-E_\infty(p))$,*

$$\lim_{N\to\infty}\frac{\mathrm{Crt}_N((-\infty,u))}{\mathbb{E}\{\mathrm{Crt}_N((-\infty,u))\}}=1,$$

*in $L_2$, and thus, also in probability.*

The main motivation for the study of the Gaussian fields $H_{N,p}(\boldsymbol{\sigma})$ is their importance in the physics literature. Nevertheless, the model certainly serves as a natural setting to investigate a question of pure mathematical interest: what is the behavior of the critical points of an isotropic random function on a high dimensional manifold? To the best of our knowledge, the corollary above (combined with the computation of the first moment of [5]) is the first concentration result for the high dimensional limit.

Computations of moments of the number of critical points were done in other settings. Closest to our setting are the works of Fyodorov [32, 33] which dealt with isotropic fields on the sphere $\mathbb{S}^N$ and on $\mathbb{R}^N$ and the first moment of number of critical points and its large $N$ asymptotics. Further away, are the works of Nicolaescu [37–41], Sarnak and Wigman [44], Cammarota, Marinucci and Wigman [11, 12], Douglas, Shiffman, and Zelditch [26–28], Baugher [6], and Feng and Zelditch [30]. Those concerned Gaussian fields on a fixed space and asymptotics in parameters of different nature than the dimension, for example, ones related to roughness of the random field by adding functions of higher frequency to a random expansion. In [11, 12, 37, 39], concentration results were also derived by second moment computations. Lastly, we mention works on nodal domains of Gaussian fields. See, for example, Nazarov and Sodin [35, 36] and references therein.

For any $u$ for which $\mathbb{E}\{\mathrm{Crt}_N((-\infty,u))\}$ does not tend to 0, we show that the moments match on an exponential scale.

THEOREM 3. *For any $p \geq 3$ and $u \in (-E_0(p), \infty)$,*

$$\lim_{N\to\infty} \frac{1}{N} \log\bigl(\mathbb{E}\bigl\{\bigl(\mathrm{Crt}_N\bigl((-\infty,u)\bigr)\bigr)^2\bigr\}\bigr) = 2 \lim_{N\to\infty} \frac{1}{N} \log\bigl(\mathbb{E}\bigl\{\mathrm{Crt}_N\bigl((-\infty,u)\bigr)\bigr\}\bigr) = 2\Theta_p(u), \tag{1.6}$$

*where $\Theta_p(u)$ is given in* (3.9).

Connections between the critical points and two important notions from the physics literature were established in [4, 5]: the Thouless–Anderson–Palmer (TAP) equations and the free energy. The TAP approach suggests that "pure states" of the system can be identified with critical points of the so-called TAP functional [50]. One of the main objects of interest in the analysis using this approach is the TAP-complexity—that is, the logarithm of the number of solutions of the TAP equations. The TAP-complexity has been extensively studied in the physics literature in the context of the Sherrington–Kirkpatrick model [3, 10, 15, 18, 23], the Ising $p$-spin spin glass model [20, 34, 43] and the spherical $p$-spin spin glass model [13, 14, 19, 22]. The connection to critical points of the Hamiltonian is based on the observation of [5] (see Section 6 there for more details) that each critical point of the Hamiltonian corresponds to exactly two solutions of the TAP equations—meaning that a study of the critical points is equivalent to a study of the TAP complexity.

Another interesting link that [4, 5] found is related to the ground state

$$GS^{\infty} = \lim_{N\to\infty} GS^N \triangleq \lim_{N\to\infty} \frac{1}{N} \min_{\boldsymbol{\sigma}} H_N(\boldsymbol{\sigma}). \tag{1.7}$$

The limiting free energy $F(\beta)$ is known to exist and is given by the Parisi formula [21, 42], proved in [16, 48]. The formula expresses $F(\beta)$ through an intricate variational problem, which is greatly simplified when one-step replica symmetry breaking (1-RSB) is known to occur (see [49] for a definition of this terminology). In Section 4 of their work, [4] define the class of *pure-like* spherical $p$-spin models and prove for it that

$$E_0 \geq -GS^{\infty} = \lim_{\beta\to\infty} \frac{1}{\beta} F(\beta) \leq \lim_{\beta\to\infty} \frac{1}{\beta} F^{1\mathrm{RSB}}(\beta) = E_0, \tag{1.8}$$

where $F^{1\mathrm{RSB}}(\beta)$ is defined to be the free energy obtained from the Parisi formula under the assumption that 1-RSB occurs.

Therefore, if 1-RSB is exhibited, that is, the second inequality above holds as equality, then $GS^{\infty} = -E_0$, and the first moment computation (1.2) gives the ground state. Using the fact that pure spherical $p$-spin models are known to exhibit 1-RSB ([48], Proposition 2.2), [5] proved that $GS^{\infty} = -E_0$. Note that, since $-E_0 \leq GS^{\infty}$, in order to prove that $GS^{\infty} = -E_0$ only a corresponding reversed inequality is needed. In particular, proving that w.h.p. $\mathrm{Crt}_N((-\infty, -E_0 + \varepsilon)) \geq 1$, for any $\varepsilon > 0$, is sufficient. Corollary 2 implies this, and in fact since $H_N(\boldsymbol{\sigma})$ is a Gaussian field, using concentration inequalities even Theorem 3 is sufficient; see Appendix D. This gives an alternative derivation of the result of [5] without going through Parisi's formula.

Generally, mixed spherical $p$-spin models do not necessarily exhibit 1-RSB. But, if we are able to compute second moments and prove (1.6) for some mixture, then it would follow that $GS^{\infty} = -E_0$ and, by (1.8), that "1-RSB in the zero-temperature limit" occurs. This will be explored in future work, where we shall consider part of the mixed case regime.

We finish with a remark about two recent works which build on the concentration result for the critical points which we prove in the current paper. In the first, Zeitouni and the Subag [47] investigate the extremal point process of critical points, that is, the point process constructed from critical values in the vicinity of the global minimum of $H_N(\boldsymbol{\sigma})$—and establish its convergence to a Poisson point process of exponential density. As a corollary, they also obtain that the global minimum [without normalization, contrary to (1.7)] converges to minus a Gumbel variable. In the second work, Subag [46] relates the Gibbs measure at low temperature to the critical points and shows that the measure is supported on spherical "bands" around the deepest minima of $H_N(\boldsymbol{\sigma})$, that is, those of which the extremal process consists. This allows one to derive interesting consequences, for example, the absence of temperature chaos and precise asymptotics of the free energy.

In the next section, we introduce notation. In Section 3, we outline the proofs of Theorems 1 and 3 and state several related auxiliary results. The rest of the paper is devoted to proofs of the theorems stated above and those auxiliary results. When stating each of the latter, we will also point out where its proof is given. The proof of Theorem 3 is given is Section 7. Theorem 1 is proved in Section 8.

**2. Notation.** For any two points $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ on the sphere, define the overlap function

$$R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \triangleq \frac{\langle \boldsymbol{\sigma}, \boldsymbol{\sigma}' \rangle}{\|\boldsymbol{\sigma}\|_2 \|\boldsymbol{\sigma}'\|_2} = \frac{\sum_{i=1}^N \sigma_i \sigma_i'}{N}. \tag{2.1}$$

Adopting the notation of [5], for any Borel set $B \subset \mathbb{R}$, let $\mathscr{C}_N(B)$ denote the set of critical points of $H_N$, at which it attains a value in $NB = \{Nx : x \in B\}$, and $\mathrm{Crt}_N(B)$ denote the corresponding number of points:

$$\mathrm{Crt}_N(B) := |\mathscr{C}_N(B)| := \left|\{\boldsymbol{\sigma} \in \mathbb{S}^{N-1}(\sqrt{N}) | \nabla H_N(\boldsymbol{\sigma}) = 0, H_N(\boldsymbol{\sigma}) \in NB\}\right|, \tag{2.2}$$

where $\nabla H_N(\boldsymbol{\sigma})$ denotes the gradient of $H_N(\boldsymbol{\sigma})$ (relative to the standard differential structure on the sphere). We will also be concerned with the number of ordered pairs $(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in (\mathscr{C}_N(B))^2$ with overlap in some range. For any subset $I_R \subset [-1, 1]$, we define

$$[\mathrm{Crt}_N(B, I_R)]_2 \triangleq \#\{(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in (\mathscr{C}_N(B))^2 | R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in I_R\}.$$

Note that $\mathbb{E}[\mathrm{Crt}_N(B, I_R)]_2$ is the "contribution" of pairs with $R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in I_R$ to the second moment of $\mathrm{Crt}_N(B)$ (and that, in particular, when $I_R = [-1, 1]$, the full range of the overlap, it is equal to the second moment). In the sequel, we shall assume that each of $B$ and $I_R$ is a finite union of nondegenerate open intervals in $\mathbb{R}$. In this case, we shall say that $B$ (or $I_R$) is "nice."

A random matrix $\mathbf{X}_N$ from the (normalized) $N \times N$ Gaussian orthogonal ensemble, or an $N \times N$ GOE matrix, for short, is a real, symmetric matrix such that all elements are centered Gaussian variables which, up to symmetry, are independent with variance given by

$$\mathbb{E}\{\mathbf{X}_{N,ij}^2\} = \begin{cases} 1/N, & i \neq j, \\ 2/N, & i = j. \end{cases}$$

Denote the surface area of the $N-1$-dimensional unit sphere by

$$\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}.$$

Let $\mu^*$ denote the semicircle measure, the density of which with respect to Lebesgue measure is

$$\frac{d\mu^*}{dx} = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{|x| \leq 2}, \tag{2.3}$$

and define the function (see, e.g., [29], Proposition II.1.2)

$$\Omega(x) \triangleq \int_{\mathbb{R}} \log|\lambda - x|\, d\mu^*(\lambda)$$
$$= \begin{cases} \frac{x^2}{4} - \frac{1}{2} & \text{if } 0 \le |x| \le 2, \\ \frac{x^2}{4} - \frac{1}{2} \\ \qquad - \left[\frac{|x|}{4}\sqrt{x^2-4} - \log\left(\sqrt{\frac{x^2}{4} - 1} + \frac{|x|}{2}\right)\right] & \text{if } |x| > 2. \end{cases} \tag{2.4}$$

Lastly, set

$$\begin{aligned} \Psi_p(r, u_1, u_2) &\triangleq 1 + \log(p-1) \\ &\quad + \frac{1}{2}\log\left(\frac{1-r^2}{1-r^{2p-2}}\right) - \frac{1}{2}(u_1, u_2)\big(\Sigma_U(r)\big)^{-1}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &\quad + \Omega\left(\sqrt{\frac{p}{p-1}}u_1\right) + \Omega\left(\sqrt{\frac{p}{p-1}}u_2\right), \end{aligned} \tag{2.5}$$

where $\Sigma_U(r)$ is defined in (B.1).

**3. Outline of proofs and auxiliary results.** As in the calculation of the first moment [5], or in fact any of the moment calculations for critical points mentioned below Corollary 2, the starting point of our analysis is an application of (a variant of) the Kac–Rice formula (henceforth, K-R formula). The formula expresses the expectation of $[\mathrm{Crt}_N(B, I_R)]_2$ as an integral over $I_R$ and combined with a study of certain conditional laws, in particular those of the Hessians of the Hamiltonian at two different points $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$, yields the following lemma, proved in Section 4.

LEMMA 4. *Let* $(U_1(r), U_2(r)) \sim N(0, \Sigma_U(r))$ [*cf.* (B.1)] *be a Gaussian vector independent of* $\hat{\mathbf{M}}^{(i)}_{N-1}(r)$, $i = 1, 2$, *defined in Lemma* 13. *Let* $\mathbf{M}^{(i)}_{N-1}(r, U_1(r), U_2(r))$ *be defined by* (4.7). *Then for any nice* $B \subset \mathbb{R}$ *and* $I_R \subset (-1, 1)$,

$$\begin{aligned} \mathbb{E}\{[\mathrm{Crt}_N(B, I_R)]_2\} &= C_N \int_{I_R} dr \cdot \big(\mathcal{G}(r)\big)^N \mathcal{F}(r) \\ &\quad \times \mathbb{E}\bigg\{\prod_{i=1,2} \big|\det\big(\mathbf{M}^{(i)}_{N-1}(r, U_1(r), U_2(r))\big)\big| \\ &\quad \cdot \mathbf{1}\big\{U_1(r), U_2(r) \in \sqrt{N}B\big\}\bigg\}, \end{aligned} \tag{3.1}$$

*where*

$$C_N = \omega_N \omega_{N-1} \left( \frac{(N-1)(p-1)}{2\pi} \right)^{N-1}, \qquad \mathcal{G}(r) = \left( \frac{1-r^2}{1-r^{2p-2}} \right)^{\frac{1}{2}}, \tag{3.2}$$
$$\mathcal{F}(r) = \big(\mathcal{G}(r)\big)^{-3} \big(1-r^{2p-2}\big)^{-\frac{1}{2}} \big(1-\big(pr^p-(p-1)r^{p-2}\big)^2\big)^{-\frac{1}{2}}.$$

The analysis of the ratio of the second to first moment squared splits into two parts—analysis of the asymptotics on the exponential scale and a refinement to $O(1)$ scale. We shall now discuss the first part. Lemma 13 implies that the (correlated) random matrices $\mathbf{M}_{N-1}^{(i)}(r, U_1(r), U_2(r))$ satisfy, in distribution,

$$\begin{aligned} &\begin{pmatrix} \mathbf{M}_{N-1}^{(1)}\big(r, U_1(r), U_2(r)\big) \\ \mathbf{M}_{N-1}^{(2)}\big(r, U_1(r), U_2(r)\big) \end{pmatrix} \\ &\quad = \begin{pmatrix} \mathbf{X}_{N-1}^{(1)}(r) - \sqrt{\frac{1}{N-1}\frac{p}{p-1}} U_1(r) I + \mathbf{E}_{N-1}^{(1)}(r) \\ \mathbf{X}_{N-1}^{(2)}(r) - \sqrt{\frac{1}{N-1}\frac{p}{p-1}} U_2(r) I + \mathbf{E}_{N-1}^{(2)}(r) \end{pmatrix}, \end{aligned} \tag{3.3}$$

where $\mathbf{X}_{N-1}^{(i)}(r)$ are correlated GOE matrices independent of $(U_1(r), U_2(r))$ and $\mathbf{E}_{N-1}^{(i)}(r)$ are random matrices of rank 2 viewed as perturbations. On the exponential level, the rank 2 perturbations are easily dealt with by upper bounding their Hilbert–Schmidt norm (see Lemmas 14 and 15). We remark that in parallel to the above, in the computation of the first moment of [5] the determinant of a single shifted GOE matrix appears in the corresponding K-R formula. There, a certain algebraic identity related to the density of the eigenvalues of a GOE matrix, together with Selberg's integral formula, is key to the analysis. In our situation, explicit computations such as Selberg's formula cannot be used because of the presence of two correlated GOE matrices. Instead, the main tool we use to upper bound the product of determinants is the large deviation principle (LDP) satisfied by the empirical measure of eigenvalues proved in [8], Theorem 2.1.1 (see Theorem 28). Of course, $\frac{1}{N}$ log of the absolute value of the determinant is a linear statistic of the eigenvalues $\lambda_i$, namely, it is equal to $\frac{1}{N}\sum \log|\lambda_i|$. Combining this with the LDP, Varadhan's integral lemma ([25], Theorem 4.3.1, Exercise 4.3.11), and a truncation argument (to control extremely large or close to 0 eigenvalues), we derive the following theorem in Section 5. We stress that the fact that the LDP is at speed $N^2$ in contrast to all other quantities involved in the problem, which decay or grow exponentially with $N$, is crucial to the proof.

THEOREM 5. *For any nice $B \subset \mathbb{R}$ and nice $I_R \subset (-1, 1)$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log\big(\mathbb{E}\big\{\big[\mathrm{Crt}_N(B, I_R)\big]_2\big\}\big) \le \sup_{r\in I_R} \sup_{u_i\in B} \Psi_p(r, u_1, u_2). \tag{3.4}$$

Note that the terms involving $\Omega$ in the definition of $\Psi_p(r, u_1, u_2)$ can be identified as the contribution from $\frac{1}{N}\log$ of the absolute value of the determinants, whose asymptotic behavior is expressed in terms of the semicircle law, and that the quadratic form in $u_1$ and $u_2$ corresponds to the joint Gaussian density of $U_1(r)$ and $U_2(r)$. In order to prove Theorem 3, we need to identify the points at which the supremum above is attained. The following lemma, proved in Section 6, gives sufficient conditions allowing to restrict attention to points satisfying $u_1 = u_2$.

LEMMA 6. *Defining $\Psi_p(r, u) \triangleq \Psi_p(r, u, u)$ we have the following*:

(i) *For nice $B \subset (-\infty, -E_\infty(p))$, for any $r \in (-1, 1)$,*

$$\sup_{u_i \in B} \Psi_p(r, u_1, u_2) = \sup_{u \in B} \Psi_p(r, u).$$

(ii) *For nice $B$ that intersect $(-E_0(p), E_0(p))$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log\bigl(\mathbb{E}\bigl\{\bigl[\mathrm{Crt}_N\bigl(B, (-1, 1)\bigr)\bigr]_2\bigr\}\bigr) \le \sup_{r\in(-1,1)} \sup_{u\in B} \Psi_p(r, u).$$

We complement the above with the following lemma, also proved in Section 6, which states for which $r$ the maximum is attained [in one point of the proof we use computer for the numeric evaluation of certain expressions, see the paragraph following (6.15)].

LEMMA 7. *Setting $u_{\mathrm{th}}(p) \triangleq \sqrt{2\frac{p-1}{p-2}\log(p-1)} > E_0(p)$, for fixed $u$, $\Psi_p^u(r) \triangleq \Psi_p(r, u, u)$ can be extended to a continuous function $\bar{\Psi}_p^u(r)$ on $[-1, 1]$, such that*:

(i) *If $|u| < u_{\mathrm{th}}(p)$, then $\bar{\Psi}_p^u(r)$ attains its maximum on $[-1, 1]$, uniquely, at $r = 0$.*

(ii) *If $|u| > u_{\mathrm{th}}(p)$, then $\bar{\Psi}_p^u(r)$ is maximal on $[-1, 1]$ at any $r \in \{1, (-1)^{p+1}\}$ and only there.*

(iii) *If $|u| = u_{\mathrm{th}}(p)$, then $\bar{\Psi}_p^u(r)$ is maximal on $[-1, 1]$ at any $r \in \{0, 1, (-1)^{p+1}\}$ and only there.*

Combining Theorem 5 and Lemmas 6 and 7 (and using Theorem 10, which provides a lower bound for $[\mathrm{Crt}_N(B, (-1, 1))]_2$), we prove Theorem 3 as well as the following corollary in Section 7.

COROLLARY 8. *For any $u \in (-E_0(p), -E_\infty(p))$ and $\varepsilon > 0$,*

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \log\bigl(\mathbb{E}\bigl\{\bigl(\mathrm{Crt}_N\bigl((-\infty, u)\bigr)\bigr)^2\bigr\}\bigr) \\
&\quad = \lim_{N\to\infty} \frac{1}{N} \log\bigl(\mathbb{E}\bigl\{\bigl[\mathrm{Crt}_N\bigl((-\infty, u), (-1, 1)\bigr)\bigr]_2\bigr\}\bigr) \\
&\quad > \limsup_{N\to\infty} \frac{1}{N} \log\bigl(\mathbb{E}\bigl\{\bigl[\mathrm{Crt}_N\bigl((-\infty, u), (-1, 1) \setminus (-\varepsilon, \varepsilon)\bigr)\bigr]_2\bigr\}\bigr).
\end{aligned}$$

We now move on to discuss the refinement of the asymptotics to $O(1)$ scale, that is, the proof of Theorem 1. Corollary 8 implies that the contribution of overlaps outside $(-\varepsilon, \varepsilon)$ to the second moment of $\mathrm{Crt}_N((-\infty, u))$ is negligible, assuming $u \in (-E_0(p), -E_\infty(p))$. By the fact that $\Theta_p(u)$ [see (1.2)] is strictly increasing for $u < 0$ and the equivalence of moments on exponential scale (i.e., Theorem 3), we also have that the contribution of levels outside $(u - \varepsilon, u)$ to either the first or second moment is negligible. Thus, relying on the fact that the second moment is larger than the first squared, in order to prove Theorem 1 it is enough to show that (see Lemma 20)

$$\lim_{N\to\infty} \frac{\mathbb{E}[\mathrm{Crt}_N((u-\varepsilon_N, u), (-\rho_N, \rho_N))]_2}{(\mathbb{E}\{\mathrm{Crt}_N((u-\varepsilon_N, u))\})^2} \le 1, \tag{3.5}$$

for any sequences $\varepsilon_N, \rho_N \to 0$. Using the formula (3.1) and the corresponding formula for the first moment derived by [5], one finds that proving (3.5) boils down to showing that uniformly in $u_i \in (u - \varepsilon_N, u)$ and $r \in (-\rho_N, \rho_N)$, as $N \to \infty$,

$$\frac{\mathbb{E}\{\prod_{i=1}^2 |\det(\mathbf{M}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2))|\}}{\prod_{i=1}^2 \mathbb{E}\{\det(\mathbf{X}_{N-1} - \sqrt{\frac{N}{N-1}\frac{p}{p-1}}u_i I)\}} \le 1 + o(1), \tag{3.6}$$

where $\mathbf{X}_{N-1}$ is a GOE matrix.

Recall the equality in distribution (3.3). As we shall see (in Lemma 24), the perturbations $\mathbf{E}_{N-1}^{(i)}(r)$ are negligible when computing the expectation above, even on $O(1)$ scale. That is, it is sufficient to prove (3.6) with its numerator replaced by

$$\mathbb{E}\left\{\prod_{i=1}^2 \left|\det\left(\mathbf{X}_{N-1}^{(i)}(r) - \sqrt{\frac{N}{N-1}\frac{p}{p-1}}u_i I\right)\right|\right\}, \tag{3.7}$$

where $\mathbf{X}_{N-1}^{(i)}(r)$ are the correlated GOE matrices in (3.3). Note that in the setting of Theorem 1 we assume that $u$ is strictly less than $-E_\infty(p)$. This exactly means that the shifts $-\sqrt{\frac{N}{N-1}\frac{p}{p-1}}u_i$ are larger than 2 and, therefore, the eigenvalues of the shifted GOE matrices in (3.7) are bounded away from 0 with high probability. This will allow us to apply concentration inequalities of linear statistics of the eigenvalues to $\frac{1}{N}\log$ of the product in (3.7) (truncated) and its derivative in $u_i$. Using the latter, we will relate (3.7) to

$$w_u(r) = \mathbb{E}\left\{\prod_{i=1}^2 \det\left(\mathbf{X}_{N-1}^{(i)}(r) - \sqrt{\frac{N}{N-1}\frac{p}{p-1}}uI\right)\right\}.$$

We note that with $r = 0$, $\mathbf{X}_{N-1}^{(1)}(0)$ and $\mathbf{X}_{N-1}^{(2)}(0)$ are i.i.d., so that $w_u(0)$ coincides with the denominator of (3.6) with $u_i = u$. Combining the above, at this point what we will need to show in order to conclude (3.6) is that $w_u(r) = (1+o(1))w_u(0)$ as $N \to \infty$, uniformly in $r \in (-\rho_N, \rho_N)$. The key to proving this will be to show that $w_u(r)$ is convex in a power of $r$ and bound the ratio $|w_u(1)/w_u(0)|$ by a constant independent of $N$ (see Lemma 25).[3]

We finish with two remarks about generalizations. First, we note that parts of the current work generalize to the case of general mixed models. Specifically, by the same method, and a somewhat more tedious algebra, one can obtain an equivalent of Theorem 5. In the general case, however, the function that replaces $\Psi_p$ is more complicated (mainly due to changes in the conditional law of the Hessians of the Hamiltonian) and its analysis, albeit just "a matter of calculus," seems to be substantially more difficult. (Moreover, from the remark made in the Introduction, we know that the second moment cannot match the first squared for full mixture models, which implies that for certain mixed models the function $\Psi_p$ achieve its maximum in the interior of the interval [0, 1]. We do not have a characterization of the mixtures that allow one to carry out the analysis we performed in the pure $p$-spin case.)

In another direction, the authors of [4, 5] treat the case of critical points of any given index. To complete the analysis of the corresponding second moment, note that the effect of introducing a restriction on the index in (3.1) is simply adding there the indicator of the corresponding event. By a similar method to that used in the proof of Theorem 5, this would result in an addition to $\Psi_p(r, u_1, u_2)$ of the term

$$\limsup_{N\to\infty} \frac{1}{N} \log\bigl(\mathbb{P}\bigl\{\bigl(\mathbf{M}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2)\bigr)_{i=1,2} \text{ are of index } k\bigr\}\bigr),$$

and would require both analyzing the probability above and the modified function $\Psi_p(r, u_1, u_2)$ in order to obtain an upper bound on the logarithmic asymptotics of the second moment of the number of critical points of index $k$. We have not attempted to complete this computation. We remark, however, that for the study of the Gibbs measure at low enough temperature it is sufficient to understand the critical points with no restriction on the index; see [46]. In fact, only the critical points close to $-NE_0(p)$ play a role in [46] and those are typically local minima (e.g., as follows from bounds on critical points of positive index proved in [5]).

Lastly, we state two results of [5] that will be needed later.

*An integral formula and the logarithmic asymptotics of the first moment.* We shall need the following two results borrowed from [5].

[3]To be precise, $w_u(r)$ is convex in a power of $r$ only on [0, 1], and for negative $r$ we will use a certain relation between $w_u(r)$ and $w_u(-r)$.

LEMMA 9 ([5], Lemmas 3.1, 3.2). *For all* $p \geq 3$,

$$\begin{aligned} &\mathbb{E}\{\mathrm{Crt}_N((-\infty, u))\} \\ &\quad = \omega_N \left( \frac{p-1}{2\pi}(N-1) \right)^{\frac{N-1}{2}} \\ &\qquad \times \mathbb{E}\left\{ \left| \det\left( \mathbf{M}_{N-1} - \sqrt{\frac{p}{p-1}\frac{1}{N-1}} U I \right) \right| \mathbf{1}\{U < \sqrt{N} u\} \right\}, \end{aligned} \tag{3.8}$$

*where* $\mathbf{M}_{N-1}$ *is a GOE matrix of dimension* $N-1 \times N-1$ *independent of* $U \sim N(0,1)$.

THEOREM 10 ([5], Theorem 2.8). *For all* $p \geq 3$,

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \log(\mathbb{E}\{\mathrm{Crt}_N((-\infty, u))\}) \\ &\quad = \Theta_p(u) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{2}\log(p-1) - \dfrac{u^2}{2} + \Omega\left( \sqrt{\dfrac{p}{p-1}} u \right) & \text{if } u < 0, \\ \dfrac{1}{2}\log(p-1) & \text{if } u \geq 0. \end{cases} \end{aligned} \tag{3.9}$$

**4. Proof of Lemma 4.** This section is devoted to the proof of Lemma 4. Let $f_N(\boldsymbol{\sigma})$ be equal to $H_N(\boldsymbol{\sigma})$ reparametrized and normalized to be a Gaussian field on

$$\mathbb{S} = \mathbb{S}^{N-1} = \{\boldsymbol{\sigma} \in \mathbb{R}^N : \|\boldsymbol{\sigma}\|_2 = 1\}$$

with constant variance 1,

$$f_N(\boldsymbol{\sigma}) = f_{N,p}(\boldsymbol{\sigma}) = \frac{1}{\sqrt{N}} H_{N,p}(\sqrt{N}\boldsymbol{\sigma}). \tag{4.1}$$

The covariance of $f_N(\boldsymbol{\sigma})$ is given by

$$\mathbb{E}\{f_N(\boldsymbol{\sigma}), f_N(\boldsymbol{\sigma}')\} = \langle \boldsymbol{\sigma}, \boldsymbol{\sigma}' \rangle^p,$$

where $\langle \boldsymbol{\sigma}, \boldsymbol{\sigma}' \rangle = \sum_{i=1}^N \sigma_i \sigma_i'$ is the usual inner product.

Note that

$$\begin{aligned} \mathrm{Crt}_N(B) &= \mathrm{Crt}_N^f(B) \\ &\triangleq \#\{\boldsymbol{\sigma} \in \mathbb{S}^{N-1} | \nabla f_N(\boldsymbol{\sigma}) = 0, f_N(\boldsymbol{\sigma}) \in \sqrt{N} B\}, \\ [\mathrm{Crt}_N(B, I_R)]_2 &= [\mathrm{Crt}_N^f(B, I_R)]_2 \\ &\triangleq \#\{(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in (\mathbb{S}^{N-1})^2 | \langle \boldsymbol{\sigma}, \boldsymbol{\sigma}' \rangle \in I_R, \ldots \\ &\qquad \nabla f_N(\boldsymbol{\sigma}) = \nabla f_N(\boldsymbol{\sigma}') = 0, \\ &\qquad f_N(\boldsymbol{\sigma}) \in \sqrt{N} B, f_N(\boldsymbol{\sigma}') \in \sqrt{N} B\}. \end{aligned} \tag{4.2}$$

Endow the sphere $\mathbb{S}^{N-1}$ with the standard Riemannian structure, induced by the Euclidean Riemannian metric on $\mathbb{R}^N$. Given a (piecewise) smooth orthonormal frame field $E=(E_i)_{i=1}^{N-1}$ on $\mathbb{S}^{N-1}$, we define

$$\nabla f_N(\boldsymbol{\sigma})=\bigl(E_i f_N(\boldsymbol{\sigma})\bigr)_{i=1}^{N-1},\qquad \nabla^2 f_N(\boldsymbol{\sigma})=\bigl(E_iE_j f_N(\boldsymbol{\sigma})\bigr)_{i,j=1}^{N-1}. \tag{4.3}$$

LEMMA 11. *Let $E=(E_i)_{i=1}^{N-1}$ be an arbitrary* (*piecewise*) *smooth orthonormal frame field on $\mathbb{S}^{N-1}$ and use the notation* (4.3). *For any nice $B\subset\mathbb{R}$ and nice $I_R\subset(-1,1)$,*

$$\begin{aligned}
&\mathbb{E}\bigl\{\bigl[\mathrm{Crt}_N(B,I_R)\bigr]_2\bigr\}\\
&\quad=\omega_N\omega_{N-1}\bigl((N-1)p(p-1)\bigr)^{N-1}\int_{I_R}dr\cdot\bigl(1-r^2\bigr)^{\frac{N-3}{2}}\\
&\qquad\times\varphi_{\nabla f(\mathbf{n}),\nabla f(\boldsymbol{\sigma}(r))}(0,0)\\
&\qquad\times\mathbb{E}\biggl\{\biggl|\det\biggl(\frac{\nabla^2 f(\mathbf{n})}{\sqrt{(N-1)p(p-1)}}\biggr)\biggr|\cdot\biggl|\det\biggl(\frac{\nabla^2 f(\boldsymbol{\sigma}(r))}{\sqrt{(N-1)p(p-1)}}\biggr)\biggr|\\
&\qquad\times\mathbf{1}\bigl\{f(\mathbf{n}),f\bigl(\boldsymbol{\sigma}(r)\bigr)\in\sqrt{N}B\bigr\}\big|\nabla f(\mathbf{n})=\nabla f\bigl(\boldsymbol{\sigma}(r)\bigr)=0\biggr\},
\end{aligned}\tag{4.4}$$

*where $\varphi_{\nabla f(\boldsymbol{\sigma}),\nabla f(\boldsymbol{\sigma}')}$ is the joint density of the gradients $\nabla f(\boldsymbol{\sigma})$ and $\nabla f(\boldsymbol{\sigma}')$, and where*

$$\boldsymbol{\sigma}(r)=\bigl(0,\dots,0,\sqrt{1-r^2},r\bigr). \tag{4.5}$$

The proof of Lemma 11 is deferred to the end of the section. Clearly, the left-hand side of (4.4) is independent of the choice of the orthonormal frame $E$. Thus, as a corresponding continuous Radon–Nikodym derivative, the integrand in the right-hand side is also independent of $E$. Therefore, Lemma 4 follows from Lemma 11, combined with Lemmas 12 and 13 given below. Their computationally heavy proof is given in Appendix B.

LEMMA 12 (The density of the gradients and the conditional law of $(f(\mathbf{n}),f(\boldsymbol{\sigma}(r)))$). *For any $r\in(-1,1)$, there exists a choice of $E=(E_i)_{i=1}^{N-1}$ such that the following holds. The density of $(\nabla f(\mathbf{n}),\nabla f(\boldsymbol{\sigma}(r)))$ at $(0,0)\in\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}$ is*

$$\begin{aligned}
&\varphi_{\nabla f(\mathbf{n}),\nabla f(\boldsymbol{\sigma}(r))}(0,0)\\
&\quad=(2\pi p)^{-(N-1)}\bigl[1-r^{2p-2}\bigr]^{-\frac{N-2}{2}}\bigl[1-\bigl(pr^p-(p-1)r^{p-2}\bigr)^2\bigr]^{-\frac{1}{2}},
\end{aligned}\tag{4.6}$$

*and conditional on $(\nabla f(\mathbf{n}),\nabla f(\boldsymbol{\sigma}(r)))=(0,0)$, the vector $(f(\mathbf{n}),f(\boldsymbol{\sigma}(r)))$ is a centered Gaussian vector with covariance matrix $\Sigma_U(r)$* [*cf.* (B.1)].

LEMMA 13 (The conditional law of the Hessians). *For any $r \in (-1,1)$, with the same choice of $E = (E_i)_{i=1}^{N-1}$ as in Lemma 12, the following holds. Conditional on $f(\mathbf{n}) = u_1$, $f(\boldsymbol{\sigma}(r)) = u_2$, $\nabla f(\mathbf{n}) = \nabla f(\boldsymbol{\sigma}(r)) = 0$, the random variable*

$$\left(\frac{\nabla^2 f(\mathbf{n})}{\sqrt{(N-1)p(p-1)}}, \frac{\nabla^2 f(\boldsymbol{\sigma}(r))}{\sqrt{(N-1)p(p-1)}}\right)$$

*has the same law as*

$$\big(\mathbf{M}_{N-1}^{(1)}(r,u_1,u_2), \mathbf{M}_{N-1}^{(2)}(r,u_1,u_2)\big),$$

*where*

$$\begin{aligned}\mathbf{M}_{N-1}^{(i)}(r,u_1,u_2) = \hat{\mathbf{M}}_{N-1}^{(i)}(r) - \sqrt{\frac{1}{N-1}\frac{p}{p-1}}u_i I \\ + \frac{m_i(r,u_1,u_2)}{\sqrt{(N-1)p(p-1)}} e_{N-1,N-1},\end{aligned} \tag{4.7}$$

*$e_{N-1,N-1}$ is an $N-1 \times N-1$ matrix whose $N-1, N-1$ entry is equal to* 1 *and all other entries are* 0, *$m_i$ is given in* (B.4), *and $\hat{\mathbf{M}}_{N-1}^{(1)}(r)$ and $\hat{\mathbf{M}}_{N-1}^{(2)}(r)$ are $N-1 \times N-1$ Gaussian random matrices with block structure*

$$\hat{\mathbf{M}}_{N-1}^{(i)}(r) = \begin{pmatrix} \hat{\mathbf{G}}_{N-2}^{(i)}(r) & Z^{(i)}(r) \\ (Z^{(i)}(r))^T & Q^{(i)}(r) \end{pmatrix}, \tag{4.8}$$

*satisfying the following*:

(i) *The random elements $(\hat{\mathbf{G}}_{N-2}^{(1)}(r), \hat{\mathbf{G}}_{N-2}^{(2)}(r))$, $(Z^{(1)}(r), Z^{(2)}(r))$ and $(Q^{(1)}(r), Q^{(2)}(r))$ are independent.*

(ii) *The matrices $\hat{\mathbf{G}}^{(i)}(r) = \hat{\mathbf{G}}_{N-2}^{(i)}(r)$ are $N-2 \times N-2$ random matrices such that $\sqrt{\frac{N-1}{N-2}}\hat{\mathbf{G}}^{(i)}(r)$ is a GOE matrix and, in distribution,*

$$\begin{pmatrix} \hat{\mathbf{G}}^{(1)}(r) \\ \hat{\mathbf{G}}^{(2)}(r) \end{pmatrix} = \begin{pmatrix} \sqrt{1-|r|^{p-2}}\bar{\mathbf{G}}^{(1)} + (\operatorname{sgn}(r))^p\sqrt{|r|^{p-2}}\bar{\mathbf{G}} \\ \sqrt{1-|r|^{p-2}}\bar{\mathbf{G}}^{(2)} + \sqrt{|r|^{p-2}}\bar{\mathbf{G}} \end{pmatrix},$$

*where $\bar{\mathbf{G}} = \bar{\mathbf{G}}_{N-2}$, $\bar{\mathbf{G}}^{(1)} = \bar{\mathbf{G}}_{N-2}^{(1)}$ and $\bar{\mathbf{G}}^{(2)} = \bar{\mathbf{G}}_{N-2}^{(2)}$ are independent and have the same law as $\hat{\mathbf{G}}^{(i)}(r)$.*

(iii) *The column vectors $Z^{(i)}(r) = (Z_j^{(i)}(r))_{j=1}^{N-2}$ are Gaussian such that for any $j \le N-2$, $(Z_j^{(1)}(r), Z_j^{(2)}(r))$ is independent of all the other elements of the two vectors and*

$$\big(Z_j^{(1)}(r), Z_j^{(2)}(r)\big) \sim N\big(0, \big((N-1)p(p-1)\big)^{-1} \cdot \Sigma_Z(r)\big),$$

*where $\Sigma_Z(r)$ is given in* (B.3).

(iv) *Lastly,* $Q^{(i)}(r)$ *are Gaussian random variables with*

$$\big(Q^{(1)}(r), Q^{(2)}(r)\big) \sim N\big(0, \big((N-1)p(p-1)\big)^{-1} \cdot \Sigma_Q(r)\big),$$

*where* $\Sigma_Q(r)$ *is given in* (B.3).

4.1. *Proof of Lemma* 11. First, note that from additivity it is enough to prove the lemma under the assumption that $I_R$ is an open interval. By the monotone convergence theorem, we may also assume that the closure of $I_R$ is contained in $(-1, 1)$. Defining

$$\mathcal{S}_N^2(I_R) \triangleq \{(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in (\mathbb{S}^{N-1})^2 | \langle \boldsymbol{\sigma}, \boldsymbol{\sigma}' \rangle \in I_R\}, \tag{4.9}$$

we have

$$\begin{aligned} [\mathrm{Crt}_N(B, I_R)]_2 = \#\{(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in \mathcal{S}_N^2(I_R) | \nabla f_N(\boldsymbol{\sigma}) = \nabla f_N(\boldsymbol{\sigma}') = 0, \\ f_N(\boldsymbol{\sigma}), f_N(\boldsymbol{\sigma}') \in \sqrt{N} B\}. \end{aligned} \tag{4.10}$$

Consider the ($\mathbb{R}^{2(N-1)}$-valued) Gaussian field

$$\big(\nabla f_N(\boldsymbol{\sigma}), \nabla f_N(\boldsymbol{\sigma}')\big), \tag{4.11}$$

defined on the $[2(N-1)$-dimensional$]$ submanifold $\mathcal{S}_N^2(I_R)$ (with boundary).

We are interested in the mean number of points in $\mathcal{S}_N^2(I_R)$ for which the field (4.11) satisfies the condition in the definition of (4.10). This fits the setting of the variant of the K-R theorem given in [1], Theorem 12.1.1. The latter requires several regularity conditions to hold, which we prove in Appendix C. From [1], Theorem 12.1.1, and an argument along the lines of [1], Section 11.5, we have that

$$\begin{aligned} \mathbb{E}\{[\mathrm{Crt}_N(B, I_R)]_2\} = \int_{\mathbb{S}^{N-1}} d\boldsymbol{\sigma} \int_{\{\boldsymbol{\sigma}' \in \mathbb{S}^{N-1} : \langle \boldsymbol{\sigma}, \boldsymbol{\sigma}' \rangle \in I_R\}} & d\boldsymbol{\sigma}' \varphi_{\nabla f(\boldsymbol{\sigma}), \nabla f(\boldsymbol{\sigma}')}(0, 0) \\ & \times \mathbb{E}\{|\det \nabla^2 f(\boldsymbol{\sigma})||\det \nabla^2 f(\boldsymbol{\sigma}')| \\ & \times \mathbf{1}\{f(\boldsymbol{\sigma}), f(\boldsymbol{\sigma}') \in \sqrt{N} B\} | \nabla f(\boldsymbol{\sigma}) = \nabla f(\boldsymbol{\sigma}') = 0\}, \end{aligned}$$

where $d\boldsymbol{\sigma}$ denotes the usual surface area on $\mathbb{S}^{N-1}$.

Denote the north pole $\mathbf{n} \triangleq (0, 0, \dots, 0, 1) \in \mathbb{S}^{N-1}$. By symmetry, the inner integral is independent of $\boldsymbol{\sigma}$. Thus, above we can set $\boldsymbol{\sigma} = \mathbf{n}$, remove the integration over $\boldsymbol{\sigma}$ and multiply by a factor of $\omega_N$. Now, note that with $\boldsymbol{\sigma} = \mathbf{n}$, the integrand depends on $\boldsymbol{\sigma}'$ only through the overlap $\rho(\boldsymbol{\sigma}') = \langle \mathbf{n}, \boldsymbol{\sigma}' \rangle$. Thus, we can use the co-area formula with the function $\rho(\boldsymbol{\sigma}')$ to express the second integral as a one-dimensional integral over a parameter $r$ [the volume of the inverse-image $\rho^{-1}(r)$ and the inverse of the Jacobian are given by $\omega_{N-1}(1-r^2)^{\frac{N-2}{2}}$ and $(1-r^2)^{-\frac{1}{2}}$, resp.]. Doing so yields (4.4), and completes the proof. □

**5. Proof of Theorem 5.** This section is dedicated to the proof of Theorem 5. For this, we shall need the three lemmas below, which are proved in the following subsections. Throughout the section, we use the following notation. Let

$$\big(U_1(r), U_2(r)\big) \sim N\big(0, \Sigma_U(r)\big) \tag{5.1}$$

[cf. (B.1)] be a Gaussian vector independent of all other variables and set

$$\bar{U}_i(r) = \sqrt{\frac{1}{N-1}\frac{p}{p-1}}U_i(r). \tag{5.2}$$

Also, let $\mathbf{G}_{N-2}^{(i)}(r)$ be the upper-left $N-2 \times N-2$ submatrix of $\mathbf{M}_{N-1}^{(i)}(r) := \mathbf{M}_{N-1}^{(i)}(r, U_1(r), U_2(r))$ (cf. Lemma 13). With $\hat{\mathbf{G}}_{N-2}^{(i)}(r)$ as defined in (4.8), we have

$$\mathbf{G}_{N-2}^{(i)}(r) \triangleq \hat{\mathbf{G}}_{N-2}^{(i)}(r) - \bar{U}_i(r)I. \tag{5.3}$$

Set

$$\begin{aligned} W_i(r) &= W_{i,N}(r) \\ &\triangleq \left(2\sum_{j=1}^{N-2}\big(\big(\mathbf{M}_{N-1}^{(i)}(r)\big)_{j,N-1}\big)^2 + \big(\big(\mathbf{M}_{N-1}^{(i)}(r)\big)_{N-1,N-1}\big)^2\right)^{1/2}. \end{aligned} \tag{5.4}$$

For any $\kappa > \varepsilon > 0$, define

$$h_\varepsilon(x) = \max\{\varepsilon, x\},$$

and

$$h_\varepsilon^\kappa(x) = \begin{cases} \varepsilon & \text{if } x < \varepsilon, \\ x & \text{if } x \in [\varepsilon, \kappa], \\ 1 & \text{if } x > \kappa, \end{cases} \quad \text{and} \quad h_\kappa^\infty(x) = \begin{cases} 1 & \text{if } x \le \kappa, \\ x & \text{if } x > \kappa, \end{cases} \tag{5.5}$$

so that $h_\varepsilon^\kappa(x)h_\kappa^\infty(x) = h_\varepsilon(x)$. Lastly, define

$$\log_\varepsilon^\kappa(x) = \log\big(h_\varepsilon^\kappa(x)\big). \tag{5.6}$$

For a real symmetric matrix $\mathbf{A}$, let $\lambda_j(\mathbf{A})$ denote the eigenvalues of $\mathbf{A}$.

The following bounds the determinant of $\mathbf{M}_{N-1}^{(i)}(r)$ in terms of the eigenvalues of $\mathbf{G}_{N-2}^{(i)}(r)$, up to a multiplicative error term depending only on the last column and row of $\mathbf{M}_{N-1}^{(i)}(r)$.

LEMMA 14. *Under the notation of Lemma 13, for any $\varepsilon > 0$, $r \in (-1, 1)$, almost surely,*

$$\big|\det\big(\mathbf{M}_{N-1}^{(i)}\big(r, U_1(r), U_2(r)\big)\big)\big| \le \frac{W_i(r)(W_i(r)+\varepsilon)}{\varepsilon}\prod_{j=1}^{N-2} h_\varepsilon\big(\big|\lambda_j\big(\mathbf{G}_{N-2}^{(i)}(r)\big)\big|\big).$$

We shall need the following bound on $W_i(r)$.

LEMMA 15. *There exists a bounded function* $v(r):(-1,1)\to\mathbb{R}$ *for which*

$$\lim_{\delta\searrow 0}\frac{v(1-\delta)}{\delta}\quad\text{and}\quad\lim_{\delta\searrow 0}\frac{v(\delta-1)}{\delta}$$

*exist and are finite, such that for any natural* $m$, *the nonnegative random variables* $W_i(r)$ *satisfy for large enough* $N$

$$\mathbb{E}\{(W_i(r))^{2m}\}\le v^m(r).$$

The following bounds, which are uniform in $r$, are the last ingredient we need for proving Theorem 5.

LEMMA 16. *For any* $q>0$ *and nice set* $B$, *the following hold*:

(i) *For any* $\varepsilon>0$ *and* $\kappa>\max\{\varepsilon,1\}$, *there exists a constant* $c=c(\varepsilon,\kappa)>0$, *such that for large enough* $N$, *uniformly in* $r\in(-1,1)$,

$$\begin{aligned}
&\mathbb{E}\left\{\prod_{i=1,2}\prod_{j=1}^{N-2}\left(h_\varepsilon^\kappa\left(\left|\lambda_j\left(\mathbf{G}_{N-2}^{(i)}(r)\right)\right|\right)\right)^q\cdot\mathbf{1}\{U_1(r),U_2(r)\in\sqrt{N}B\}\right\}\\
&\quad\le\exp\{-cN^2\}\\
&\qquad+\mathbb{E}\left\{\exp\left\{\sum_{i=1,2}qN\int\log_\varepsilon^\kappa(|\lambda-\bar{U}_i|)\,d\mu^*+2q\varepsilon N\right\}\right.\\
&\qquad\left.\cdot\mathbf{1}\{U_1(r),U_2(r)\in\sqrt{N}B\}\right\},
\end{aligned}\tag{5.7}$$

*where* $\mu^*$ *is the semicircle law, given in* (2.3).

(ii) *For large enough* $\kappa>0$, *uniformly in* $r\in(-1,1)$,

$$\mathbb{E}\left\{\prod_{i=1,2}\prod_{j=1}^{N-2}\left(h_\kappa^\infty\left(\left|\lambda_j\left(\mathbf{G}_{N-2}^{(i)}(r)\right)\right|\right)\right)^q\right\}\le 2.\tag{5.8}$$

5.1. *Proof of Lemma* 14. Let $\dot{\mathbf{M}}_{N-1}^{(i)}(r)$ denote the matrix obtained from $\mathbf{M}_{N-1}^{(i)}(r)$ by replacing all entries in the last row and column by 0. The eigenvalues of $\dot{\mathbf{M}}_{N-1}^{(i)}(r)$ are the same as those of $\mathbf{G}_{N-2}^{(i)}(r)$, with an extra eigenvalue equal to 0. For a general symmetric matrix $\mathbf{A}$, $\sum_{i,j}\mathbf{A}_{i,j}^2=\sum_j\lambda_j^2(\mathbf{A})$. Thus,

$$\sum_j\lambda_j^2\left(\mathbf{M}_{N-1}^{(i)}(r)-\dot{\mathbf{M}}_{N-1}^{(i)}(r)\right)=W_i^2(r).$$

Hence, the absolute value of any eigenvalue of $\mathbf{M}_{N-1}^{(i)}(r) - \dot{\mathbf{M}}_{N-1}^{(i)}(r)$ is bounded by $W_i(r)$. Note that $\mathbf{M}_{N-1}^{(i)}(r) - \dot{\mathbf{M}}_{N-1}^{(i)}(r)$ has rank 2 at most and, therefore, has at most 2 nonzero eigenvalues. By an application of Corollary 29, we have that, almost surely,

$$\left|\det\left(\mathbf{M}_{N-1}^{(i)}(r)\right)\right| \leq \frac{W_i(r)(W_i(r)+T_i(r))}{T_i(r)} \prod_{j=1}^{N-2} \left|\lambda_j\left(\mathbf{G}_{N-2}^{(i)}(r)\right)\right|,$$

where $T_i(r)$ is the minimal absolute value of an eigenvalue of $\mathbf{G}_{N-2}^{(i)}(r)$. The lemma follows from this. $\square$

5.2. *Proof of Lemma* 15. From symmetry, it is enough to prove the lemma with $i=1$. From Lemma 13, it follows that the law of $\mathbf{M}_{N-1}^{(1)}(r)$ is the same as the law of

$$\frac{\nabla^2 f(\mathbf{n})}{\sqrt{(N-1)p(p-1)}} \tag{5.9}$$

conditional on

$$\nabla f(\mathbf{n}) = \nabla f\left(\boldsymbol{\sigma}(r)\right) = 0 \tag{5.10}$$

[where $\boldsymbol{\sigma}(r)$ is given in (4.5)]. We emphasize that here the conditioning is only on the gradient at the two points and not on the values of the Hamiltonian. The covariance structure of the Gaussian matrix $\nabla^2 f(\mathbf{n})$, conditional on (5.10), is computed in Section B.1. In particular, it is given by (B.5), in which $\mathrm{Cov}_{\nabla f}$ denotes the conditional covariance. In particular, we have that $(W_1(r))^2$ is identical in distribution to

$$2\frac{\mathrm{Cov}_{\nabla f}\{E_1E_{N-1}f(\mathbf{n}), E_1E_{N-1}f(\mathbf{n})\}}{(N-1)p(p-1)} \sum_{i=1}^{N-2} X_i^2$$
$$+ \frac{\mathrm{Cov}_{\nabla f}\{E_{N-1}E_{N-1}f(\mathbf{n}), E_{N-1}E_{N-1}f(\mathbf{n})\}}{(N-1)p(p-1)} X_{N-1}^2,$$

where the covariances are as in (B.5) and $X_i$ are i.i.d. standard Gaussian variables and where we used the fact that the conditional variance of $E_iE_{N-1}f(\mathbf{n})$ is identical for all $i \leq N-2$.

Setting

$$\begin{aligned} \bar{v}(r) &= 2(N-1)p(p-1) \\ &\quad \cdot \max_{i\in\{1,N-1\}} \left\{\mathrm{Cov}_{\nabla f}\left\{E_iE_{N-1}f(\mathbf{n}), E_iE_{N-1}f(\mathbf{n})\right\}\right\}, \end{aligned} \tag{5.11}$$

by straightforward algebra, using (B.5), we have that

$$\lim_{\varepsilon\searrow 0} \frac{\bar{v}(1-\varepsilon)}{\varepsilon} \quad \text{and} \quad \lim_{\varepsilon\searrow 0} \frac{\bar{v}(\varepsilon-1)}{\varepsilon}$$

exist and are finite, and that $\bar{v}(r)$ is a bounded function on $(-1,1)$.

Since $W_1(r)$ is stochastically dominated by

$$\sqrt{\frac{\bar{v}(r)}{p(p-1)}\frac{1}{N-1}\sum_{i=1}^{N-1}X_i^2},$$

we conclude that

$$\mathbb{E}\{(W_1(r))^{2m}\} \le \left(\frac{\bar{v}(r)}{(N-1)p(p-1)}\right)^m \mathbb{E}\left\{\left(\sum_{i=1}^{N-1}X_i^2\right)^m\right\}.$$

Since $\sum_{i=1}^{N-1} X_i^2$ is a chi-squared variable of $N-1$ degrees of freedom (cf. [45], page 13),

$$\mathbb{E}\left\{\left(\sum_{i=1}^{N-1}X_i^2\right)^m\right\} = (N-1)(N+1)\cdots(N-3+2m).$$

The lemma follows from this. $\square$

5.3. *Proof of Lemma* 16. Note that

$$\begin{aligned}
&\mathbb{E}\left\{\prod_{i=1,2}\prod_{j=1}^{N-2}(h_\varepsilon^\kappa(|\lambda_j(\mathbf{G}_{N-2}^{(i)}(r))|))^q \cdot \mathbf{1}\{U_i(r)\in\sqrt{N}B\}\right\}\\
&\quad= \mathbb{E}\left\{\prod_{i=1,2}\exp\left\{q\sum_{j=1}^{N-2}\log_\varepsilon^\kappa(|\lambda_j(\hat{\mathbf{G}}_{N-2}^{(i)}(r)) - \bar{U}_i(r)|)\right\}\right.\\
&\qquad\qquad \left.\cdot\mathbf{1}\{U_i(r)\in\sqrt{N}B\}\right\}\\
&\quad= \mathbb{E}\left\{\prod_{i=1,2}\exp\left\{q(N-2)\int\log_\varepsilon^\kappa(|\lambda-\bar{U}_i(r)|)\,dL_{N-2}^{(i)}(\lambda)\right\}\right.\\
&\qquad\qquad \left.\cdot\mathbf{1}\{U_i(r)\in\sqrt{N}B\}\right\},
\end{aligned}\tag{5.12}$$

where $L_{r,N-2}^{(i)}$ is the empirical measure of eigenvalues of $\hat{\mathbf{G}}_{N-2}^{(i)}(r)$ [cf. (A.1)].

The function $\log_\varepsilon^\kappa(|\cdot - x|)$ is bounded and Lipschitz continuous, with the same bound and Lipschitz constant for all $x\in\mathbb{R}$. Thus, there exists $c_{\varepsilon,\kappa}>0$ such that (cf. Appendix A)

$$\begin{aligned}
A_\varepsilon &\triangleq \bigcup_{i=1,2}\bigcup_{x\in\mathbb{R}}\left\{\int\log_\varepsilon^\kappa(|\lambda-x|)\,d(L_{r,N-2}^{(i)}-\mu^*)>\varepsilon\right\}\\
&\subset \bigcup_{i=1,2}\{d_{\mathrm{LU}}(\mu^*, L_{r,N-2}^{(i)})>c_{\varepsilon,\kappa}\}.
\end{aligned}\tag{5.13}$$

Since $\log_\varepsilon^\kappa$ is bounded from above by $\log(\kappa)$ and since on $A_\varepsilon^c$,

$$\int \log_\varepsilon^\kappa(|\lambda - x|)\, dL_{r,N-2}^{(i)}(\lambda) \le \int \log_\varepsilon^\kappa(|\lambda - x|)\, d\mu^*(\lambda) + \varepsilon,$$

with

$$S(r, \mu_1, \mu_2) \triangleq \exp\left\{q(N-2)\sum_{i=1,2}\int \log_\varepsilon^\kappa(|\lambda - \bar{U}_i(r)|)\, d\mu_i\right\},$$

$$F_N(r) \triangleq \{U_1(r), U_2(r) \in \sqrt{N}B\},$$

we have

$$\begin{aligned}\mathbb{E}\{S(r, L_{r,N-2}^{(1)}, L_{r,N-2}^{(2)})\mathbf{1}_{F_N(r)}\} &= \mathbb{E}\{S(r, L_{r,N-2}^{(1)}, L_{r,N-2}^{(2)}) \cdot \mathbf{1}_{A_\varepsilon^c}\mathbf{1}_{F_N(r)}\} \\ &\quad + \mathbb{E}\{S(r, L_{r,N-2}^{(1)}, L_{r,N-2}^{(2)}) \cdot \mathbf{1}_{A_\varepsilon}\mathbf{1}_{F_N(r)}\} \\ &\le \exp\{2q\varepsilon N\} \cdot \mathbb{E}\{S(r, \mu^*, \mu^*)\mathbf{1}_{F_N(r)}\} \\ &\quad + \exp\{2q\log(\kappa)N\} \cdot \mathbb{P}\{A_\varepsilon\}.\end{aligned} \tag{5.14}$$

From Theorem 28 and (5.13), setting

$$c'_{\varepsilon,\kappa} = \frac{1}{2}\inf_{\mu \in (B(\mu^*, c_{\varepsilon,\kappa}\varepsilon))^c} J(\mu) > 0$$

(where positivity follows from the fact that $J$ is a good rate function with unique minimizer), one obtains for large enough $N$,

$$\mathbb{P}\{A_\varepsilon\} \le 2\exp\{-c'_{\varepsilon,\kappa}N^2\}. \tag{5.15}$$

Combining (5.12), (5.14) and (5.15), we obtain, for large enough $N$,

$$\begin{aligned}&\mathbb{E}\left\{\prod_{i=1,2}\prod_{j=1}^{N-2}(h_\varepsilon^\kappa(|\lambda_j(\mathbf{G}_{N-2}^{(i)}(r))|))^q \cdot \mathbf{1}\{U_1(r), U_2(r) \in \sqrt{N}B\}\right\} \\ &\quad \le \exp\{2q\varepsilon N\} \\ &\qquad \times \mathbb{E}\left\{\prod_{i=1,2}\exp\left\{qN\int \log_\varepsilon^\kappa(|\lambda - \bar{U}_i(r)|)\, d\mu^*\right\} \cdot \mathbf{1}\{U_i(r) \in \sqrt{N}B\}\right\} \\ &\qquad + 2\exp\{2q\log(\kappa)N\}\exp\{-c'_{\varepsilon,\kappa}N^2\},\end{aligned}$$

from which part (i) follows.

Define

$$\Lambda(r) = \Lambda_N(r) \triangleq \max_{\substack{i=1,2 \\ j \le N-2}} |\lambda_j(\mathbf{G}_{N-2}^{(i)}(r))|.$$

From a union bound and (5.3),

$$\mathbb{P}\{\Lambda(r) > t\} \le \sum_{i=1,2}\left(\mathbb{P}\left\{\max_{j \le N-2}|\lambda_j(\hat{\mathbf{G}}_{N-2}^{(i)}(r))| > t/2\right\} + \mathbb{P}\{\bar{U}_i(r) > t/2\}\right). \tag{5.16}$$

It is easy to verify that the variance of $U_i(r)$ is bounded by 1, uniformly in $r \in (-1, 1)$. Recall that $\sqrt{\frac{N-1}{N-2}}\hat{\mathbf{G}}^{(i)}_{N-2}(r)$ is a GOE matrix. Thus, from (5.16) and Lemma 26, there exists a constant $\tilde{c} > 0$ such that for large enough $t$ and any $N$,

$$\mathbb{P}\{\Lambda(r) > t\} \leq \sqrt{\frac{\tilde{c}N}{2\pi}} e^{-\frac{1}{2}\tilde{c}t^2 N}.$$

Let $\Lambda_0 \sim N(0, (\tilde{c}N)^{-1})$. For large enough $\kappa > 0$ and any $N$,

$$\begin{aligned}
&\mathbb{E}\left\{\prod_{i=1,2}\prod_{j=1}^{N-2}(h_\kappa^\infty(|\lambda_j(\mathbf{G}^{(i)}_{N-2}(r))|))^q\right\} \\
&\quad\leq \mathbb{P}\{\Lambda(r) \leq \kappa\} + \mathbb{E}\{(\Lambda(r))^{2qN}\mathbf{1}\{\Lambda(r) > \kappa\}\} \\
&\quad\leq 1 + \mathbb{E}\{\Lambda_0^{2qN}\mathbf{1}\{\Lambda_0 > \kappa\}\}.
\end{aligned} \tag{5.17}$$

From the Cauchy–Schwarz inequality,

$$\begin{aligned}
\mathbb{E}\{\Lambda_0^{2qN}\mathbf{1}\{\Lambda_0 > \kappa\}\} &\leq [\mathbb{E}\{\Lambda_0^{4qN}\}\mathbb{P}\{\Lambda_0 > \kappa\}]^{1/2} \\
&\leq \exp\left\{-N\left(\frac{\tilde{c}\kappa^2}{4} - c_q\right)\right\},
\end{aligned}$$

for some $c_q$. Finally, taking $\kappa$ to be large enough, this together with (5.17) yields (5.8). $\square$

5.4. *Proof of Theorem 5.* Let $\kappa > \varepsilon > 0$, let $2 \leq m \in \mathbb{N}$ and set $q = q(m) = m/(m-1)$. From Lemma 14, the fact that $h_\varepsilon^\kappa(x)h_\kappa^\infty(x) = h_\varepsilon(x)$ and Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\left\{\prod_{i=1,2}|\det(\mathbf{M}^{(i)}_{N-1}(r))| \cdot \mathbf{1}\{U_i(r) \in \sqrt{N}B\}\right\} \\
&\quad\leq (\mathcal{E}^{(1)}_{\varepsilon,\kappa}(r))^{1/q}(\mathcal{E}^{(2)}_{\varepsilon,\kappa}(r))^{1/2m}(\mathcal{E}^{(3)}_{\varepsilon,\kappa}(r))^{1/4m},
\end{aligned} \tag{5.18}$$

where

$$\begin{aligned}
\mathcal{E}^{(1)}_{\varepsilon,\kappa}(r) &= \mathbb{E}\left\{\prod_{i=1,2}\prod_{j=1}^{N-2}(h_\varepsilon^\kappa(|\lambda_j(\mathbf{G}^{(i)}_{N-2}(r))|))^q \cdot \mathbf{1}\{U_i(r) \in \sqrt{N}B\}\right\}, \\
\mathcal{E}^{(2)}_{\varepsilon,\kappa}(r) &= \mathbb{E}\left\{\prod_{i=1,2}\prod_{j=1}^{N-2}(h_\kappa^\infty(|\lambda_j(\mathbf{G}^{(i)}_{N-2}(r))|))^{2m}\right\}, \\
\mathcal{E}^{(3)}_{\varepsilon,\kappa}(r) &= \mathbb{E}\left\{\left(\frac{W_1(r)(W_1(r)+\varepsilon)}{\varepsilon}\right)^{4m}\right\}\mathbb{E}\left\{\left(\frac{W_2(r)(W_2(r)+\varepsilon)}{\varepsilon}\right)^{4m}\right\}.
\end{aligned} \tag{5.19}$$

Substituting this in (3.1) and using Hölder's inequality yields

$$\mathbb{E}\{[\mathrm{Crt}_N(B,I_R)]_2\} \le C_N\left[\int_{I_R} (\mathcal{G}(r))^{qN}\mathcal{E}^{(1)}_{\varepsilon,\kappa}(r)\,dr\right]^{1/q} \times \left[\int_{I_R} (\mathcal{F}(r))^{m}\big(\mathcal{E}^{(2)}_{\varepsilon,\kappa}(r)\big)^{1/2}\big(\mathcal{E}^{(3)}_{\varepsilon,\kappa}(r)\big)^{1/4}\,dr\right]^{1/m},$$

where $C_N$, $\mathcal{F}(r)$, and $\mathcal{G}(r)$ are given in (3.2).

Therefore,

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{N}\log\big(\mathbb{E}\{[\mathrm{Crt}_N(B,I_R)]_2\}\big)\\
&\quad\le \limsup_{N\to\infty}\frac{1}{N}\log(C_N) + \limsup_{N\to\infty}\frac{1}{qN}\log\left(\int_{I_R}(\mathcal{G}(r))^{qN}\mathcal{E}^{(1)}_{\varepsilon,\kappa}(r)\,dr\right)\\
&\qquad + \limsup_{N\to\infty}\frac{1}{mN}\log\left(\int_{I_R}(\mathcal{F}(r))^{m}\big(\mathcal{E}^{(2)}_{\varepsilon,\kappa}(r)\big)^{1/2}\big(\mathcal{E}^{(3)}_{\varepsilon,\kappa}(r)\big)^{1/4}\,dr\right).
\end{aligned} \tag{5.20}$$

The first summand is equal to

$$1+\log(p-1).$$

One has that $\mathcal{F}(r)$ is bounded on any interval $(-r_0, r_0)$ with $0<r_0<1$, and that the limits

$$\lim_{\delta\searrow 0}\delta\mathcal{F}(1-\delta) \quad\text{and}\quad \lim_{\delta\searrow 0}\delta\mathcal{F}(\delta-1)$$

exist and are finite. Using Lemma 15, we therefore have that

$$(\mathcal{F}(r))^{m}\big(\mathcal{E}^{(3)}_{\varepsilon,\kappa}(r)\big)^{1/4}$$

is a bounded function of $r$ on $(-1,1)$. Thus, from part (ii) of Lemma 16, for $\kappa$ large enough the third summand of (5.20) is equal to 0.

Lastly, we need to analyze the second summand. To do so, we use part (i) of Lemma 16 and Varadhan's integral lemma ([25], Theorem 4.3.1, Exercise 4.3.11). Define

$$\Omega^{\kappa}_{\varepsilon}(x) \triangleq \int_{\mathbb{R}} \log^{\kappa}_{\varepsilon}\big(|\lambda-x|\big)\,d\mu^*(\lambda), \qquad \gamma_p \triangleq \sqrt{\frac{p}{p-1}}.$$

Note that, for $(\tilde{U}_1,\tilde{U}_2)\sim N(0, I_{2\times 2})$,

$$\big(U_1(r),U_2(r)\big) \overset{d}{=} (\tilde{U}_1,\tilde{U}_2)\cdot\big(\Sigma_U(r)\big)^{1/2}.$$

Let $e_i$, $i=1,2$, denote the standard basis of $\mathbb{R}^2$, taken as $2\times 1$ column vectors; so that $(t_1,t_2)e_i = t_i$. Lastly, define

$$T(B) \triangleq \big\{(r,\tilde{u}_1,\tilde{u}_2) : r\in(-r_0,r_0), (\tilde{u}_1,\tilde{u}_2)\cdot\big(\Sigma_U(r)\big)^{1/2}\in B\times B\big\}.$$

Using part (i) of Lemma 16, we obtain that, for large $N$, assuming $\kappa > 1$, for some constant $c > 0$,

$$\begin{aligned}
&\int_{-r_0}^{r_0} (\mathcal{G}(r))^{qN} \mathcal{E}_{\varepsilon,\kappa}^{(1)}(r)\,dr - \exp\{-cN^2\} \\
&\quad \le e^{2q\varepsilon N} \int_{-r_0}^{r_0} (\mathcal{G}(r))^{qN} \mathbb{E}\biggl\{ \prod_{i=1,2} \exp\{qN\Omega_\varepsilon^\kappa(\bar{U}_i(r))\} \\
&\qquad \cdot \mathbf{1}\biggl\{ \frac{U_i(r)}{\sqrt{N}} \in B \biggr\}\biggr\}\,dr \\
&\quad = 2r_0 e^{2q\varepsilon N} \mathbb{E}\biggl\{ \exp\biggl\{ qN \cdot \phi_\varepsilon^\kappa\biggl(R, \frac{\tilde{U}_1}{\sqrt{N}}, \frac{\tilde{U}_2}{\sqrt{N}}\biggr)\biggr\} \\
&\qquad \cdot \mathbf{1}\biggl\{\biggl(R, \frac{\tilde{U}_1}{\sqrt{N}}, \frac{\tilde{U}_2}{\sqrt{N}}\biggr) \in T(B)\biggr\}\biggr\} \\
&\quad \triangleq 2r_0 e^{2q\varepsilon N} \zeta_{\varepsilon,\kappa,N},
\end{aligned} \tag{5.21}$$

where $R$ is independent of $\tilde{U}_1, \tilde{U}_1$ and is uniformly distributed in $(-r_0, r_0)$, and where

$$\phi_\varepsilon^\kappa(r, \bar{u}_1, \bar{u}_2) \triangleq \log(\mathcal{G}(r)) + \sum_{i=1,2} \Omega_\varepsilon^\kappa\bigl(\gamma_p(\tilde{u}_1, \tilde{u}_2) \cdot (\Sigma_U(R))^{1/2} \cdot e_i\bigr).$$

Note that $\phi_\varepsilon^\kappa$ is a continuous function on $(-1, 1) \times \mathbb{R} \times \mathbb{R}$. Since $\mathcal{G}(r) \in (0, 1)$ and $\Omega_\varepsilon^\kappa$ is bounded from above by $\log \kappa$, for any $q' > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \log\biggl(\mathbb{E}\biggl\{\exp\biggl\{q'N \cdot \phi_\varepsilon^\kappa\biggl(R, \frac{\tilde{U}_1}{\sqrt{N}}, \frac{\tilde{U}_2}{\sqrt{N}}\biggr)\biggr\}\biggr\}\biggr) \le 2q' \log \kappa.$$

The random variable $(R, \frac{\tilde{U}_1}{\sqrt{N}}, \frac{\tilde{U}_2}{\sqrt{N}})$ satisfies the LDP with the good rate function

$$J_0(r, \tilde{u}_1, \tilde{u}_2) = \frac{\tilde{u}_1^2}{2} + \frac{\tilde{u}_2^2}{2}.$$

Therefore, from Varadhan's integral lemma ([25], Theorem 4.3.1, Exercise 4.3.11) combined with (5.21),

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N} \log\biggl(\int_{-r_0}^{r_0} (\mathcal{G}(r))^{qN} \mathcal{E}_{\varepsilon,\kappa}^{(1)}(r)\,dr\biggr) \\
&\quad \le \limsup_{N\to\infty} \frac{1}{N} \log\bigl(2r_0 e^{2q\varepsilon N} \zeta_{\varepsilon,\kappa,N}\bigr) \\
&\quad \le 2q\varepsilon + \sup_{(r,\tilde{u}_1,\tilde{u}_2)\in T(B)} \biggl\{ q\phi_\varepsilon^\kappa(r, \tilde{u}_1, \tilde{u}_2) - \frac{\tilde{u}_1^2}{2} - \frac{\tilde{u}_2^2}{2}\biggr\}.
\end{aligned}$$

Together with our analysis of the two other summands in (5.20), this yields, for large enough $\kappa$,

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{N}\log\bigl(\mathbb{E}\{[\mathrm{Crt}_N(B)]_2^{r_0}\}\bigr)\\
&\qquad\le 1+\log(p-1)+2\varepsilon\\
&\qquad\quad+\frac{1}{q}\sup_{(r,\tilde{u}_1,\tilde{u}_2)\in T(B)}\biggl\{q\phi_\varepsilon^\kappa(r,\tilde{u}_1,\tilde{u}_2)-\frac{\tilde{u}_1^2}{2}-\frac{\tilde{u}_2^2}{2}\biggr\}.
\end{aligned}\tag{5.22}$$

Letting $m\to\infty$, which implies that $q=q(m)\to 1$, we obtain (5.22) with $q=1$.

By a change of variables,

$$\begin{aligned}
&\sup_{(r,\tilde{u}_1,\tilde{u}_2)\in T(B)}\biggl\{\phi_\varepsilon^\kappa(r,\tilde{u}_1,\tilde{u}_2)-\frac{\tilde{u}_1^2}{2}-\frac{\tilde{u}_2^2}{2}\biggr\}\\
&\quad=\sup_{r\in(-r_0,r_0)}\sup_{u_1,u_2\in B}\biggl\{\log\bigl(\mathcal{G}(r)\bigr)+\sum_{i=1,2}\Omega_\varepsilon^\kappa(\gamma_p u_i)\\
&\qquad-\frac{1}{2}(u_1,u_1)\bigl(\Sigma_U(r)\bigr)^{-1}(u_1,u_1)^T\biggr\}.
\end{aligned}$$

Letting $\kappa\to\infty$ and then $\varepsilon\to 0$ completes the proof. $\square$

**6. Proofs of Lemmas 6 and 7.** The bound of Theorem 5 is given in terms of the supremum of $\Psi_p(r,u_1,u_2)$ on the region $I_R\times B\times B$. In order to complete the proof of Theorem 3, we need to identify the points at which the supremum is attained. This is the content of Lemmas 6 and 7, which we prove in this section. The following simple remark is related to the proof of Lemma 6, and will also be used in the sequel.

REMARK 17. The bound of Theorem 5 holds for any nice $I_R\subset(-1,1)$. We are particularly interested in the case where $I_R=[-1,1]$,

$$\bigl[\mathrm{Crt}_N\bigl(B,[-1,1]\bigr)\bigr]_2=\bigl(\mathrm{Crt}_N(B)\bigr)^2.$$

The difference

$$\bigl[\mathrm{Crt}_N\bigl(B,[-1,1]\bigr)\bigr]_2-\bigl[\mathrm{Crt}_N\bigl(B,(-1,1)\bigr)\bigr]_2$$

is simply the number of ordered pairs of points $\boldsymbol{\sigma}=\pm\boldsymbol{\sigma}'$ with $H_N(\boldsymbol{\sigma}),H_N(\boldsymbol{\sigma}')\in NB$. Thus, it is bounded from above by $2\,\mathrm{Crt}_N(B)$.

Therefore, assuming $\lim_{N\to\infty}\mathbb{E}\,\mathrm{Crt}_N(B)=\infty$,

$$\frac{\mathbb{E}\{[\mathrm{Crt}_N(B,(-1,1))]_2\}}{\mathbb{E}\{(\mathrm{Crt}_N(B))^2\}}\xrightarrow{N\to\infty}1.\tag{6.1}$$

6.1. *Proof of Lemma* 6. We begin with part (i). Fix $r \in (-1, 1)$. Note that $\log(x)$ is a concave function on $(0, \infty)$, and thus $\Omega(x)$ [defined in (2.4)] is concave on $(-\infty, -2)$. Since $\Sigma_U^{-1}(r)$ is positive definite for any $r \in (-1, 1)$, we conclude that, for $u_1, u_2 < -2\sqrt{\frac{p-1}{p}} = -E_\infty(p)$, the function

$$(u_1, u_2) \mapsto -\frac{1}{2}(u_1, u_2)\big(\Sigma_U(r)\big)^{-1}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \Omega\bigg(\sqrt{\frac{p}{p-1}}u_1\bigg) + \Omega\bigg(\sqrt{\frac{p}{p-1}}u_2\bigg) \tag{6.2}$$

is concave.

Let $u \in \mathbb{R}$ and define

$$\begin{aligned}\Psi_u^*(v) &= \Psi_p(r, u+v, u-v) \\ &= \tau_{p,r} - \frac{1}{2}(u+v, u-v)\big(\Sigma_U(r)\big)^{-1}\begin{pmatrix} u+v \\ u-v \end{pmatrix} \\ &\quad + \Omega\bigg(\sqrt{\frac{p}{p-1}}(u+v)\bigg) + \Omega\bigg(\sqrt{\frac{p}{p-1}}(u-v)\bigg),\end{aligned}$$

where $\tau_{p,r}$ is a constant depending on $p, r$.

If $u \in (-\infty, -E_\infty(p))$, then for

$$v \in \big(E_\infty(p) + u, -E_\infty(p) - u\big) \triangleq D(u),$$

the function $\Psi_u^*(v)$ is concave in $v$ [as a restriction of (6.2) to a line in $\mathbb{R}^2$, up to adding the constant $\tau_{p,r}$]. Moreover, by symmetry,

$$\frac{\partial}{\partial v}\Psi_u^*(0) = 0$$

and, therefore,

$$\sup_{v \in D(u)} \Psi_u^*(v) = \Psi_u^*(0) = \Psi_p(r, u, u).$$

Hence, for nice $B \subset (-\infty, -E_\infty(p))$, since

$$B \times B \subset \big\{(u+v, u-v) : u \in B, v \in D(u)\big\},$$

we conclude that

$$\sup_{u_i \in B} \Psi_p(r, u_1, u_2) \le \sup_{u \in B} \sup_{v \in D(u)} \Psi_u^*(v) = \sup_{u \in B} \Psi_p(r, u, u).$$

This completes the proof of part (i) of Lemma 6.

Now, assume that $B \subset \mathbb{R}$ is nice. Let $B_1$ and $B_2$ be nice disjoint sets whose union is $B$. Note that, since $x^2 + y^2 \geq 2xy$, for any $x, y \in \mathbb{R}$,

$$\begin{aligned}\big[\mathrm{Crt}_N\big(B, (-1, 1)\big)\big]_2 &\leq \big(\mathrm{Crt}_N(B_1) + \mathrm{Crt}_N(B_2)\big)^2 \\ &\leq 2\big(\big(\mathrm{Crt}_N(B_1)\big)^2 + \big(\mathrm{Crt}_N(B_2)\big)^2\big).\end{aligned}$$

Note that (see Remark 17)

$$\begin{aligned}\big(\mathrm{Crt}_N(B_i)\big)^2 &= \big[\mathrm{Crt}_N\big(B_i, [-1, 1]\big)\big]_2 \\ &\leq \big[\mathrm{Crt}_N\big(B_i, (-1, 1)\big)\big]_2 + 2\mathrm{Crt}_N(B_i).\end{aligned}$$

Thus, by Theorem 5,

$$\begin{aligned}&\limsup_{N\to\infty} \frac{1}{N} \log\big(\mathbb{E}\big\{\big[\mathrm{Crt}_N\big(B, (-1, 1)\big)\big]_2\big\}\big) \\ &\quad \leq \max_{i=1,2}\Big\{\sup_{r\in(-1,1)} \sup_{u_1,u_2\in B_i} \Psi_p(r, u_1, u_2)\Big\} \vee \limsup_{N\to\infty} \frac{1}{N} \log\big(\mathbb{E}\big\{\mathrm{Crt}_N(B)\big\}\big),\end{aligned} \tag{6.3}$$

where $x \vee y = \max\{x, y\}$, for any two numbers $x, y$.

By applying the same argument iteratively, we obtain that if $B_i$, $i = 1, \dots, n$, is an $N$-independent partition of $B$ to nice sets, then (6.3) holds with the maximum taken over all $i \leq n$.

Let $\varepsilon > 0$ and choose a partition $B_1, \dots, B_{n+1}, B_{n+2}$ of $B$ such that $B_1, \dots, B_n$ are intervals that form a partition of $B' = B \cap [-E_0(p), E_0(p)]$ such that the diameter of $B_i$ is less then $\varepsilon$ and such that

$$\begin{aligned}B_{n+1} &= B \cap \big(-\infty, -E_0(p)\big), \\ B_{n+2} &= B \cap \big(E_0(p), \infty\big).\end{aligned}$$

Then,

$$\begin{aligned}&\limsup_{N\to\infty} \frac{1}{N} \log\big(\mathbb{E}\big\{\big[\mathrm{Crt}_N\big(B, (-1, 1)\big)\big]_2\big\}\big) \\ &\quad \leq \limsup_{N\to\infty} \frac{1}{N} \log\big(\mathbb{E}\big\{\mathrm{Crt}_N(B)\big\}\big) \vee \sup_{r\in(-1,1)} \sup_{\substack{u_1,u_2\in B' \\ |u_1-u_2|<\varepsilon}} \Psi_p(r, u_1, u_2) \\ &\qquad \vee \sup_{r\in(-1,1)} \sup_{u_1,u_2\in B_{n+1}} \Psi_p(r, u_1, u_2) \\ &\qquad \vee \sup_{r\in(-1,1)} \sup_{u_1,u_2\in B_{n+2}} \Psi_p(r, u_1, u_2).\end{aligned} \tag{6.4}$$

Since $B_{n+1} \subset (-\infty, -E_\infty(p))$, by the first part of the lemma,

$$\sup_{u_1,u_2\in B_{n+1}} \Psi_p(r, u_1, u_2) = \sup_{u\in B_{n+1}} \Psi_p(r, u, u). \tag{6.5}$$

By symmetry of $\Psi_p(r, u_1, u_2)$ in $(u_1, u_2)$, the same holds with $B_{n+2}$.

By concavity considerations similar to those used in the proof of part (i), for any $u_1, u_2 \in \mathbb{R}$, setting $u = (u_1 + u_2)/2$,

$$-\frac{1}{2}(u_1, u_2)\big(\Sigma_U(r)\big)^{-1}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \le -\frac{1}{2}(u, u)\big(\Sigma_U(r)\big)^{-1}\begin{pmatrix} u \\ u \end{pmatrix}.$$

Therefore,

$$\Psi_p(r, u_1, u_2) \le \Psi_p(r, u, u) + \left|2\Omega\left(\sqrt{\frac{p}{p-1}}u\right) - \Omega\left(\sqrt{\frac{p}{p-1}}u_1\right) - \Omega\left(\sqrt{\frac{p}{p-1}}u_2\right)\right|.$$

The function $\Omega$ is uniformly continuous on $[-E_0(p), E_0(p)]$. Therefore, for any $u_1$, $u_2$ such that $|u_1 - u_2| < \varepsilon$,

$$\Psi_p(r, u_1, u_2) \le \Psi_p(r, u, u) + O(\varepsilon) \qquad \text{as } \varepsilon \to 0.$$

Therefore,

$$\sup_{r \in (-1,1)} \sup_{\substack{u_1, u_2 \in B' \\ |u_1 - u_2| < \varepsilon}} \Psi_p(r, u_1, u_2) \le \sup_{r \in (-1,1)} \sup_{u \in B'} \Psi_p(r, u, u) + O(\varepsilon).$$

By letting $\varepsilon \to 0$, combining the above with (6.5) and the similar equality for $B_{n+2}$, we obtain from (6.4),

$$\begin{aligned} &\limsup_{N \to \infty} \frac{1}{N} \log\big(\mathbb{E}\big\{\big[\mathrm{Crt}_N\big(B, (-1, 1)\big)\big]_2\big\}\big) \\ &\qquad \le \limsup_{N \to \infty} \frac{1}{N} \log\big(\mathbb{E}\big\{\mathrm{Crt}_N(B)\big\}\big) \vee \sup_{r \in (-1,1)} \sup_{u \in B} \Psi_p(r, u, u). \end{aligned} \tag{6.6}$$

Now, assume that $B$ intersects $(-E_0(p), E_0(p))$. Since it is nice, the intersection contains an open interval and by Theorem 10,

$$\lim_{N \to \infty} \frac{1}{N} \log\big(\mathbb{E}\big\{\mathrm{Crt}_N(B)\big\}\big) > 0.$$

By Remark 17, it follows that

$$\limsup_{N \to \infty} \frac{1}{N} \log\big(\mathbb{E}\big\{\big[\mathrm{Crt}_N\big(B, (-1, 1)\big)\big]_2\big\}\big) > \lim_{N \to \infty} \frac{1}{N} \log\big(\mathbb{E}\big\{\big(\mathrm{Crt}_N(B)\big)\big\}\big), \tag{6.7}$$

meaning that (6.6) is equal to $\sup_{r \in (-1,1)} \sup_{u \in B} \Psi_p(r, u, u)$. This completes the proof of part (ii). $\square$

6.2. *Proof of Lemma* 7. By straightforward algebra,

$$\tag{6.8} \begin{aligned} \Psi_p^u(r) = \zeta_{p,u} + \frac{1}{2}\log\left(\frac{1-r^2}{1-r^{2p-2}}\right) \\ - u^2 \frac{1-r^p+(p-1)r^{p-2}(1-r^2)}{1-r^{2p-2}+(p-1)r^{p-2}(1-r^2)}, \end{aligned}$$

where $\zeta_{p,u}$ depends only on $p$ and $u$.

Note that

$$\tag{6.9} \begin{aligned} &\frac{1-r^p+(p-1)r^{p-2}(1-r^2)}{1-r^{2p-2}+(p-1)r^{p-2}(1-r^2)} \\ &\quad = 1 - \frac{r^p-r^{2p-2}}{1-r^{2p-2}+(p-1)r^{p-2}(1-r^2)} \end{aligned}$$

and

$$\tag{6.10} \begin{aligned} &1-r^{2p-2}+(p-1)r^{p-2}(1-r^2) \\ &\quad = (1-r^2)(p-1)\left(\frac{1+r^2+\cdots+r^{2p-4}}{p-1} + r^{p-2}\right). \end{aligned}$$

For any $r \in (-1,1)$,

$$\tag{6.11} \begin{aligned} &\frac{1+r^2+\cdots+r^{2p-4}}{p-1} > |r^{p-2}|, \quad \text{and thus} \\ &\frac{1+r^2+\cdots+r^{2p-4}}{p-1} + r^{p-2} > 0, \end{aligned}$$

since these are the arithmetic and geometric means of the same nondegenerate, nonnegative sequence.

That is, the denominator in (6.8) above is positive for $r \in (-1,1)$. Hence, in order to see that $\Psi_p^u(r)$ can be continuously extended to $[-1,1]$ all that is need is to check that the limits at $r = \pm 1$ exist. This can be verified using L'Hôpital's rule.

Moreover, for odd $p$, (6.9) is less then 1 for $r \in (0,1)$ and is greater then 1 for $r \in (-1,0)$. For even $p$, of course, the expression is symmetric in $r$. Thus, the maximum of $\bar{\Psi}_p^u(r)$ is achieved on $[0,1]$, and if and only if $p$ is even, then the maximum can be attained at some $r^* < 0$. In that case, it is also attained at $-r^*$.

Set, for $r \in [0,1)$,

$$\tag{6.12} Q_p^u(r) \triangleq \frac{1}{2}\log\left(\frac{1-r^2}{1-r^{2p-2}}\right) + u^2 \frac{r^p - r^{2p-2}}{1-r^{2p-2}+(p-1)r^{p-2}(1-r^2)}$$

and

$$Q_p^u(1) \triangleq \lim_{r \nearrow 1} Q_p^u(r) = \frac{1}{2}\log\left(\frac{1}{p-1}\right) + u^2 \frac{p-2}{4(p-1)}.$$

We conclude that in order to prove the lemma, it is enough to prove it with $\Psi_p(r,u)$ replaced by $Q_p^u(r)$, with $[-1,1]$ replaced by $[0,1]$, and with the term $(-1)^{p+1}$ removed.

Setting, for $r \in [0,1)$,

$$g_0(r) \triangleq \frac{r^p - r^{2p-2}}{1 - r^{2p-2} + (p-1)r^{p-2}(1-r^2)} \tag{6.13}$$

and

$$g_0(1) \triangleq \lim_{r \nearrow 1} g_0(r) = \frac{p-2}{4(p-1)},$$

we have, for $r \in (0,1)$,

$$\frac{d}{dr} g_0(r) = \frac{pr^{p-1} + [p(p-2)]r^{3p-3} - (p-1)(p-2)r^{3p-5}}{(1 - r^{2p-2} + (p-1)r^{p-2}(1-r^2))^2} > 0. \tag{6.14}$$

That is, $g_0(r)$ is strictly increasing in $r$.

We now show that if part (iii) of the lemma holds, the other two follow. Assume that part (iii) holds. Let $u \in \mathbb{R}$ such that $|u| < u_{\mathrm{th}}(p)$. For any $r \in (0,1]$, $g_0(r) > 0$ and

$$Q_p^u(r) < Q_p^{u_{\mathrm{th}}(p)}(r) \le Q_p^{u_{\mathrm{th}}(p)}(0) = Q_p^u(0).$$

Similarly, let $u \in \mathbb{R}$ such that $|u| > u_{\mathrm{th}}(p)$. For any $r \in [0,1)$,

$$\begin{aligned} Q_p^u(1) &= Q_p^{u_{\mathrm{th}}(p)}(1) + \big(u^2 - u_{\mathrm{th}}^2(p)\big)g_0(1) \ge Q_p^{u_{\mathrm{th}}(p)}(r) + \big(u^2 - u_{\mathrm{th}}^2(p)\big)g_0(1) \\ &= Q_p^u(r) + \big(u^2 - u_{\mathrm{th}}^2(p)\big)\big(g_0(1) - g_0(r)\big) > Q_p^u(r). \end{aligned}$$

All that remains is to prove part (iii). First, we note that

$$\begin{aligned} Q_p^{u_{\mathrm{th}}(p)}(1) &= \frac{1}{2}\log\left(\frac{1}{p-1}\right) + 2\frac{p-1}{p-2}\log(p-1)\frac{p-2}{4(p-1)} \\ &= 0 = Q_p^{u_{\mathrm{th}}(p)}(0). \end{aligned} \tag{6.15}$$

We need to show that for any $r \in (0,1)$, $Q_p^{u_{\mathrm{th}}(p)}(r) < 0$. First, we assume that $p \le 10$. We have that $\frac{d}{dr}Q_p^{u_{\mathrm{th}}(p)}(0) = 0$ and $\frac{d}{dr}Q_p^{u_{\mathrm{th}}(p)}(1), -\frac{d^2}{dr^2}Q_p^{u_{\mathrm{th}}(p)}(0) > c_0$ for some $c_0 > 0$ ($c_0$ and $t_0$, $\varepsilon_0$, to be defined soon, can be computed explicitly). By a Taylor expansion combined with bounds on higher order derivatives, for some $t_0 > 0$, for any $r \in (0,t_0) \cup (1-t_0,1)$, $Q_p^{u_{\mathrm{th}}(p)}(r) < 0$. By bounding the absolute value of the derivative $\frac{d}{dr}Q_p^{u_{\mathrm{th}}(p)}(r)$ on the interval $(t_0, 1-t_0)$, we have that for some $\varepsilon_0 > 0$, in order to prove that $Q_p^{u_{\mathrm{th}}(p)}(r) < 0$ for any $r \in (t_0, 1-t_0)$ it is enough to verify the same only for a finite mesh $t_0 = r_1 < \cdots < r_k = 1 - t_0$, with differences $r_{i+1} - r_i$ that are bounded from above by $\varepsilon_0$. We verified the latter numerically using computer (see also Figure 1).

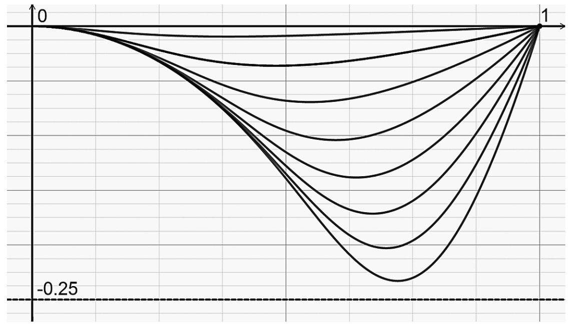


FIG. 1. *The functions* $Q_p^{u_{\mathrm{th}}(p)}(r)$ *in the interval* $[0,1]$, *for* $3 \le p \le 10$. *For any* $r$, $Q_p^{u_{\mathrm{th}}(p)}(r)$ *decreases in* $p$: $Q_p^{u_{\mathrm{th}}(p)}(r) \ge Q_{p+1}^{u_{\mathrm{th}}(p+1)}(r)$.

We now assume that $p > 10$. First, suppose also that $r \in (0, 0.65]$. By (6.11),

$$\begin{aligned}\frac{1-r^{2p-2}}{1-r^2} &= 1+(p-2)r^2\frac{1+r^2+\cdots+r^{2p-6}}{p-2}\\ &\ge 1+(p-2)r^{p-1}.\end{aligned}$$

From the inequality $\log(1+x) \ge \frac{x}{1+x}$, valid for $x > 0$, we then have, for $r \in (0, 0.65]$, $p \ge 10$,

$$\log\left(\frac{1-r^{2p-2}}{1-r^2}\right) \ge \frac{(p-2)r^{p-1}}{1+(p-2)r^{p-1}} \ge \frac{(p-2)r^{p-1}}{1+8\cdot 0.65^{(-9)}},$$

where the last inequality follows since $(p-2)\cdot 0.65^{p-1}$ is decreasing in $p$, for $p \ge 10$. In addition, for $r \in (0,1)$,

$$\frac{r^p - r^{2p-2}}{1-r^{2p-2}+(p-1)r^{p-2}(1-r^2)} \le r^p\frac{1-r^{p-2}}{1-r^{2p-2}} \le r^p.$$

Thus, for $r \in (0, 0.65]$, $p \ge 10$,

$$\begin{aligned}Q_p^{u_{\mathrm{th}}(p)}(r) &= \frac{1}{2}\log\left(\frac{1-r^2}{1-r^{2p-2}}\right) + \big(u_{\mathrm{th}}(p)\big)^2\frac{r^p - r^{2p-2}}{1-r^{2p-2}+(p-1)r^{p-2}(1-r^2)}\\ &\le -\frac{1}{2}\frac{(p-2)r^{p-1}}{1+8\cdot 0.65^{(-9)}} + \big(u_{\mathrm{th}}(p)\big)^2 r^p\\ &\le r^{p-1}\left\{0.65\cdot u_{\mathrm{th}}^2(p) - \frac{(p-2)}{2(1+8\cdot 0.65^{(-9)})}\right\} \triangleq \tau_p r^{p-1} \triangleq \bar{Q}_p(r).\end{aligned}$$

We have that $\tau_{10} < 0$ and $\tau_p$ decreases in $p$, for $p \ge 10$. Hence, for $r \in (0, 0.65]$, $p \ge 10$,

$$Q_p^{u_{\mathrm{th}}(p)}(r) < 0 = Q_p^{u_{\mathrm{th}}(p)}(0).$$

Now, assume that $r \in [0.65, 1)$. From (6.11) and (6.10),

$$Q_p^{u_{\mathrm{th}}(p)}(r) \leq \frac{1}{2}\log\left(\frac{1-r^2}{1-r^{2p-2}}\right) + u_{\mathrm{th}}^2(p)\frac{r^p - r^{2p-2}}{2(p-1)r^{p-2}(1-r^2)}$$
$$= \frac{1}{2}\log\left(\frac{1-r^2}{1-r^{2p-2}}\right) + \frac{\log(p-1)}{p-2}\frac{1-r^{p-2}}{1-r^2}r^2 \triangleq \widetilde{Q}_p(r).$$

The derivative of $\widetilde{Q}_p(r)$ by $p$ is given, for $r \in (0,1)$, by

$$\begin{aligned}\frac{d}{dp}\widetilde{Q}_p(r) &= \frac{r^{2p-2}\log r}{1-r^{2p-2}} + \frac{\frac{p-2}{p-1} - \log(p-1)}{p-2} \cdot \frac{1-r^{p-2}}{(1-r^2)} \cdot r^2 \\ &\quad + \frac{\log(p-1)}{p-2} \cdot \frac{-r^p \log r}{(1-r^2)} \\ &\leq \frac{r^2}{(p-2)(1-r^2)} \\ &\quad \times \left[(1-\log(p-1))(1-r^{p-2}) - \log r \cdot \log(p-1) r^{p-2}\right].\end{aligned}$$

Therefore, for $r \in (0,1)$, $\frac{d}{dp}\widetilde{Q}_p(r) < 0$ if

$$\frac{1-\log(p-1)}{\log(p-1)}\left(1-r^{p-2}\right) - \log r < 0.$$

Since for any $r \in [0.6, 1)$ and any $p \geq 10$, $\frac{1-\log(p-1)}{\log(p-1)}$ decreases in $p$, $(1-r^{p-2})$ increases in $p$, and

$$\frac{1-\log(10-1)}{\log(10-1)}\left(1-r^{10-2}\right) - \log r < 0,$$

it follows that $\frac{d}{dp}\widetilde{Q}_p(r) < 0$, for any $r \in [0.6, 1)$ and any $p \geq 10$. Thus, if $\widetilde{Q}_{10}(r) < 0$ for all $r \in [0.6, 1)$, then the same holds for $Q_p^{u_{\mathrm{th}}(p)}(r)$, for any $p \geq 10$. For $\widetilde{Q}_{10}(r)$, this was verified numerically using a computer using a similar method to one described above (see also Figure 2). $\square$

**7. Proofs of Theorem 3 and Corollary 8.** The content of this section is in its title. Our starting point is the bound of Theorem 3 and the main tools we shall use are Lemmas 6 and 7.

7.1. *Proof of Theorem* 3. By Theorem 10, denoting $u_- = u \wedge 0 = \min\{u, 0\}$,

$$\begin{aligned}\frac{1}{N}\log\left(\mathbb{E}\left\{\left(\mathrm{Crt}_N((-\infty,u))\right)^2\right\}\right) \quad &\geq \quad \frac{1}{N}\log\left(\left(\mathbb{E}\{\mathrm{Crt}_N((-\infty,u))\}\right)^2\right) \\ &\xrightarrow{N\to\infty} 2\Theta_N(u_-) = \Psi_p(0, u_-, u_-).\end{aligned}$$

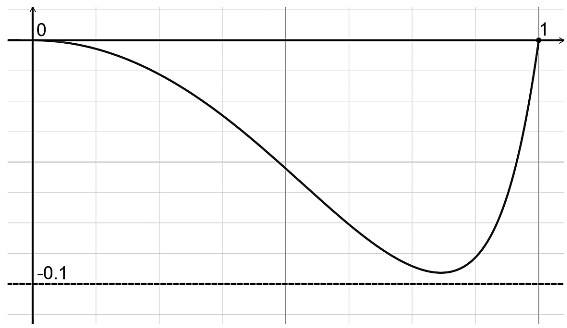


FIG. 2. *The function* $\widetilde{Q}_{10}(r)$ *in the interval* $[0, 1]$.

Combining this with (6.1), it follows that what remains to show in order to prove the theorem is that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\{[\mathrm{Crt}_N((-\infty, u), (-1, 1))]_2\} \leq \Psi_p(0, u_-, u_-). \tag{7.1}$$

Theorem 5, part (ii) of Lemma 6, Lemma 7, and the fact that $\bar{\Psi}_p^v(0)$ is symmetric in $v$, yield

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\{[\mathrm{Crt}_N((-\infty, u), (-1, 1))]_2\} \\ &\quad \leq \Big(\sup_{v\in(-\infty,-u_{\mathrm{th}}(p))} \bar{\Psi}_p^v(1)\Big) \vee \Big(\sup_{v\in[-u_{\mathrm{th}}(p),u_-]} \bar{\Psi}_p^v(0)\Big). \end{aligned} \tag{7.2}$$

We note that, for $v \leq 0$, $\bar{\Psi}_p^v(0) = 2\Theta_p(v)$ (cf. Theorem 10). Also, the monotonicity of the left-hand side of (3.9) implies that $\Theta_p(v)$ is nondecreasing for $v \leq 0$. Since $u \in (-E_0, \infty)$, the supremum on the right-hand side of (7.2) is positive. Hence, (7.1) holds if we are able to show that

$$\sup_{v\in(-\infty,-u_{\mathrm{th}}(p))} \bar{\Psi}_p^v(1) \leq 0. \tag{7.3}$$

By a straightforward calculation,

$$\frac{\partial}{\partial v}\bar{\Psi}_p^v(1) = -\frac{v(3p-2)}{2(p-1)} + 2\sqrt{\frac{p}{p-1}}\,\Omega'\Big(\sqrt{\frac{p}{p-1}}v\Big). \tag{7.4}$$

We note that for $x < -2$,

$$\Omega'(x) = \int_{(-2,2)} \frac{d}{dx}\log(\lambda - x)\,d\mu^*(\lambda) \geq \inf_{\lambda\in(-2,2)} \frac{1}{x-\lambda} = \frac{1}{x+2}. \tag{7.5}$$

From the above, one can verify that $\frac{\partial}{\partial v}\bar{\Psi}_p^v(1) \geq 0$ for $v \in (-\infty, -u_{\mathrm{th}}(p))$.

With $v = -u_{\mathrm{th}}(p) < -E_0(p)$, by Lemma 7,

$$\bar{\Psi}_p^v(1) = \bar{\Psi}_p^v(0) = 2\Theta_p(v) < 0.$$

This proves (7.3) and completes the proof. $\square$

7.2. *Proof of Corollary* 8. The equality follows from Remark 17 and the fact that $u > -E_0(p)$.

Let $u \in (-E_0(p), -E_\infty(p))$, let $\varepsilon > 0$ and set $I_\varepsilon = (-1,1) \setminus (-\varepsilon, \varepsilon)$. For arbitrary $\tilde{u} \in (-u_{\text{th}}(p), u)$, Theorem 5, Lemma 6 and Lemma 7 yield

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\{[\mathrm{Crt}_N((-\infty,u), I_\varepsilon)]_2\} \\ (7.6) \qquad &\le \Big(\sup_{r\in(-1,1)} \sup_{u_1,u_2\in(-\infty,\tilde{u})} \Psi_p(r,u_1,u_2)\Big) \vee \Big(\sup_{r\in I_\varepsilon} \sup_{u_1,u_2\in[\tilde{u},u)} \Psi_p(r,u_1,u_2)\Big) \\ &\le \Big(\sup_{v\in(-\infty,\tilde{u})} \bar{\Psi}_p^v(1)\Big) \vee \Big(\sup_{r\in I_\varepsilon} \sup_{v\in[\tilde{u},u)} \Psi_p(r,v)\Big). \end{aligned}$$

We note that $\Psi_p(r,v)$ is continuous as a function of $r$ at $(0,v)$. In the proof of Lemma 7, we saw that $\Psi_p(|r|,v) \ge \Psi_p(r,v)$, thus

$$\sup_{r\in I_\varepsilon} \Psi_p(r,v) = \sup_{0<r\in I_\varepsilon} \Psi_p(r,v).$$

From (6.8), (6.9), with $g_0(r)$ as defined in (6.13),

$$\Psi_p(0,v) - \Psi_p(r,v) = T_r - g_0(r)v^2,$$

where $T_r$ depends only on $r$. From this and since $g_0(r)$ strictly increases in $r > 0$ [see (6.14)] and $g_0(0) = 0$, we have that, uniformly in $v \in [\tilde{u}, u)$,

$$\begin{aligned} &\Psi_p(0,v) - \sup_{0<r\in I_\varepsilon} \Psi_p(r,v) \\ &\qquad = \Psi_p(0,\tilde{u}) - \sup_{0<r\in I_\varepsilon} \big(\Psi_p(r,\tilde{u}) - (\tilde{u}^2 - v^2) g_0(r)\big) \\ (7.7) \qquad &\qquad \ge \Psi_p(0,\tilde{u}) - \sup_{0<r\in I_\varepsilon} \Psi_p(r,\tilde{u}) + (\tilde{u}^2 - v^2) \inf_{r\in I_R} g_0(r) \\ &\qquad \ge \Psi_p(0,\tilde{u}) - \sup_{0<r\in I_\varepsilon} \Psi_p(r,\tilde{u}) \\ &\qquad \triangleq c_\varepsilon > 0, \end{aligned}$$

where the last inequality follows from Lemma 7.

Therefore,

$$(7.8) \qquad \sup_{r\in I_\varepsilon} \sup_{v\in[\tilde{u},u)} \Psi_p(r,v) \le \sup_{v\in[\tilde{u},u)} \Psi_p(0,v) - c_\varepsilon < \Psi_p(0,u) = 2\Theta_p(u).$$

Recall that (7.3) holds. Thus, since $\Theta_p(u) > 0$ and $\bar{\Psi}_p^v(1)$ is continuous in $v$, assuming $\tilde{u}$ is close enough to $-u_{\text{th}}(p)$,

$$(7.9) \qquad \sup_{v\in(-\infty,\tilde{u})} \bar{\Psi}_p^v(1) < 2\Theta_p(u).$$

Equations (7.6), (7.8) and (7.9) give

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}\{[\mathrm{Crt}_N((-\infty,u),I_\varepsilon)]_2\}$$
$$<2\Theta_p(u)=\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\{[\mathrm{Crt}_N((-\infty,u),(-1,1))]_2\},$$

where the equality follows from Theorems 3 and 10. $\square$

**8. Proof of Theorem 1.** The following notation will be used throughout the section. With $\mathbf{X}:=\mathbf{X}_{N-1}$ being a GOE matrix of dimension $N-1$, setting $\bar{u}:=\bar{u}_N=\sqrt{\frac{1}{N-1}\frac{p}{p-1}}u$, we define for any $u<-E_\infty(p)$,

$$\mathfrak{S}(u)=\int\frac{1}{\sqrt{\frac{p-1}{p}}\lambda-u}\,d\mu^*(\lambda),\tag{8.1}$$

$$\mathfrak{C}_N(u)=\omega_N\left(\frac{p-1}{2\pi}(N-1)\right)^{\frac{N-1}{2}}\sqrt{\frac{N}{2\pi}}e^{-N\frac{u^2}{2}}\mathbb{E}\{\det(\mathbf{X}-\sqrt{N}\bar{u}I)\},\tag{8.2}$$

where $\mu^*$ denotes the semicircle law (2.3). We note that, as follows from a direct computation, for $u<-E_\infty(p)$,

$$\frac{d}{du}\Theta_p(u)=-(\mathfrak{S}(u)+u)>0.\tag{8.3}$$

Below we use the standard big- and little-O notation to describe asymptotic behavior as $N\to\infty$. Often, equations will contain several $o(a_N^{(i)})$ terms and will be said to hold uniformly in some variable (or several), say $x\in B_N$. Such statements are to be understood as follows. The equation holds as an equality with each of the $o(a_N^{(i)})$ terms replaced by a function $h_N^{(i)}(x)$ satisfying $\sup_{x\in B_N}|h_N^{(i)}(x)|/|a_N^{(i)}|\to 0$ as $N\to\infty$.

LEMMA 18. *Let $u<-E_\infty(p)$ and suppose $J_N=(a_N,b_N)$ is an interval such that $a_N,b_N\to u$ as $N\to\infty$. Then, as $N\to\infty$,*

$$\mathbb{E}\{\mathrm{Crt}_N(J_N)\}=(1+o(1))\mathfrak{C}_N(b_N)\int_{J_N}\exp\{-N(u+\mathfrak{S}(u))(v-b_N)\}\,dv.\tag{8.4}$$

For brevity, we shall use the notation $[\mathrm{Crt}_N(B)]_2^\rho\triangleq[\mathrm{Crt}_N(B,(-\rho,\rho))]_2$ in the sequel.

LEMMA 19. *Let $u<-E_\infty(p)$ and suppose $J_N=(a_N,b_N)$ is an interval such that $a_N,b_N\to u$ as $N\to\infty$. Let $0<\rho_N$ be a sequence such that $\rho_N\to 0$ as*

$N \to \infty$. *Then*, *as* $N \to \infty$,

$$\mathbb{E}\{[\mathrm{Crt}_N(J_N)]_2^{\rho_N}\} \le (1+o(1))\Big(\mathfrak{C}_N(b_N)\int_{J_N} \exp\{-N(u+\mathfrak{S}(u))(v-b_N)\}\,dv\Big)^2. \tag{8.5}$$

LEMMA 20. *Let* $u \in (-E_0(p), -E_\infty(p))$, $\rho \in (0,1)$ *and* $\varepsilon > 0$. *Then*

$$\lim_{N\to\infty} \mathbb{E}\{[\mathrm{Crt}_N(u-\varepsilon,u)]_2^{\rho}\}/\mathbb{E}\{(\mathrm{Crt}_N((-\infty,u)))^2\} = 1.$$

In Section 8.1, we prove Theorem 1 assuming Lemmas 18, 19 and 20. The proof of Lemma 20 only requires bounds on the exponential scale we have already proved and will be given in Section 8.2. Lemmas 18 and 19 will be proved in Sections 8.4 and 8.5 after we prove several auxiliary results in Section 8.3.

8.1. *Proof of Theorem* 1 *assuming Lemmas* 18, 19 *and* 20. From Theorem 10, Lemma 20 and the fact that $\Theta_p(u)$ is strictly increasing for $u < -E_\infty(p)$ [see (8.3)], there exist positive sequences $\varepsilon_N$, $\rho_N$ such that as $N \to \infty$, $\varepsilon_N, \rho_N \to 0$ and

$$\lim_{N\to\infty} \frac{\mathbb{E}\{[\mathrm{Crt}_N(u-\varepsilon_N,u)]_2^{\rho_N}\}}{\mathbb{E}\{(\mathrm{Crt}_N((-\infty,u)))^2\}} = \lim_{N\to\infty} \frac{\mathbb{E}\{\mathrm{Crt}_N((u-\varepsilon_N,u))\}}{\mathbb{E}\{\mathrm{Crt}_N((-\infty,u))\}} = 1.$$

By Lemmas 18 and 19,

$$\lim_{N\to\infty} \frac{\mathbb{E}\{[\mathrm{Crt}_N(u-\varepsilon_N,u)]_2^{\rho_N}\}}{(\mathbb{E}\{\mathrm{Crt}_N((u-\varepsilon_N,u))\})^2} \le 1.$$

For any $N$,

$$\frac{\mathbb{E}\{(\mathrm{Crt}_N((-\infty,u)))^2\}}{(\mathbb{E}\{\mathrm{Crt}_N((-\infty,u))\})^2} \ge 1.$$

Theorem 1 follows from the above. $\square$

8.2. *Proof of Lemma* 20. Note that

$$\begin{aligned}&(\mathrm{Crt}_N((-\infty,u)))^2 - (\mathrm{Crt}_N((u-\varepsilon,u)))^2\\ &\quad = (\mathrm{Crt}_N((-\infty,u-\varepsilon]))^2 + 2\,\mathrm{Crt}_N((-\infty,u-\varepsilon])\mathrm{Crt}_N((u-\varepsilon,u)).\end{aligned}$$

By Theorem 3 and the Cauchy–Schwarz inequality,

$$\lim_{N\to\infty} \frac{1}{N}\log(\mathbb{E}\{(\mathrm{Crt}_N((-\infty,u)))^2\}) = 2\Theta_p(u),$$

$$\lim_{N\to\infty} \frac{1}{N}\log(\mathbb{E}\{(\mathrm{Crt}_N((-\infty,u-\varepsilon]))^2\}) = 2\Theta_p(u-\varepsilon),$$

$$\limsup_{N\to\infty} \frac{1}{N}\log(\mathbb{E}\{2\mathrm{Crt}_N((-\infty,u-\varepsilon])\,\mathrm{Crt}_N((u-\varepsilon,u))\}) = \Theta_p(u)+\Theta_p(u-\varepsilon).$$

For any $u < -E_\infty(p)$, by (8.3), $\Theta_p(u)$ is strictly increasing and, therefore, the expressions in the last two lines above are strictly less than $2\Theta_p(u)$. It follows that

$$\lim_{N\to\infty} \mathbb{E}\{(\mathrm{Crt}_N((-\infty,u)))^2\}/\mathbb{E}\{(\mathrm{Crt}_N((u-\varepsilon,u)))^2\} = 1.$$

By Remark 17 and the fact that $u > -E_0(p)$, also

$$\lim_{N\to\infty} \mathbb{E}\{[\mathrm{Crt}_N(u-\varepsilon,u)]_2^1\}/\mathbb{E}\{(\mathrm{Crt}_N((u-\varepsilon,u)))^2\} = 1.$$

Since

$$[\mathrm{Crt}_N((-\infty,u),(-1,1)\setminus(-\rho,\rho))]_2 \ge [\mathrm{Crt}_N((u-\varepsilon,u),(-1,1)\setminus(-\rho,\rho))]_2,$$

Corollary 8 implies that

$$\lim_{N\to\infty} \mathbb{E}\{[\mathrm{Crt}_N((u-\varepsilon,u),(-1,1)\setminus(-\rho,\rho))]_2\}/\mathbb{E}\{[\mathrm{Crt}_N(u-\varepsilon,u)]_2^1\} = 0,$$

and completes the proof. $\square$

8.3. *Auxiliary results.* The expectations in Lemmas 18 and 19 are expressed by the integral formulas of Lemmas 9 and 4, which by further conditioning on the value of $U$ and $U_1(r), U_2(r)$, respectively, can be written as integrals over $J_N$ and $J_N \times J_N$. In this section, we prove several auxiliary results that are concerned with the corresponding integrands.

We now discuss elements in the proofs related to the more involved Lemma 19. We note that the random matrices $\mathbf{M}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2)$ which appear in Lemma 4 satisfy, in distribution,

$$\begin{pmatrix} \mathbf{M}_{N-1}^{(1)}(r,\sqrt{N}u_1,\sqrt{N}u_2) \\ \mathbf{M}_{N-1}^{(2)}(r,\sqrt{N}u_1,\sqrt{N}u_2) \end{pmatrix} = \begin{pmatrix} \mathbf{X}_{N-1}^{(1)}(r) - \sqrt{N}\bar{u}_1 I + \mathbf{E}_{N-1}^{(1)} \\ \mathbf{X}_{N-1}^{(2)}(r) - \sqrt{N}\bar{u}_2 I + \mathbf{E}_{N-1}^{(2)} \end{pmatrix},$$

where $\bar{u}_i = \sqrt{\frac{1}{N-1}\frac{p}{p-1}}u_i$, $\mathbf{X}_{N-1}^{(i)}(r)$ are correlated GOE matrices and $\mathbf{E}_{N-1}^{(i)} := \mathbf{E}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2)$ are random matrices of rank 2, viewed as perturbations. We are interested in values of $u_1$ and $u_2$ that are approximately equal to some fixed $u$ and values of $r$ which are close to 0. In order to prove Lemma 19, we will need to compute the asymptotics of the ratio of

$$\mathbb{E}\Big\{\prod_{i=1,2} |\det(\mathbf{M}_{N-1}^{(i)}(r,\sqrt{N}u_1,\sqrt{N}u_2))|\Big\} \quad \text{and} \tag{8.6}$$

$$(\mathbb{E}\{\det(\mathbf{X}_{N-1} - \sqrt{N}\bar{b}_N I)\})^2,$$

where $\bar{b}_N = \sqrt{\frac{p}{p-1}\frac{1}{N-1}}b_N$ and $\mathbf{X}_{N-1}$ is a GOE matrix. This will be done in three steps: (1) We will show that the perturbations $\mathbf{E}_{N-1}^{(i)}$ are negligible, that is, the expectation on the left-hand side of (8.6) is asymptotically equivalent to

$\mathbb{E}\prod_{i=1,2}|\det(\mathbf{X}_{N-1}^{(i)}(r)-\sqrt{N}\bar{u}_i I)|$; (2) relate the latter expectation to the same without the absolute value and with $\bar{u}_i=\bar{b}_N$; and (3) prove that taking $\mathbf{X}_{N-1}^{(i)}(r)$ to be independent in the expectation with $\bar{u}_i=\bar{b}_N$ asymptotically does not affect the expectation.

The first step is dealt with in Lemma 24 where we bound the Hilbert–Schmidt norms of the perturbations $\mathbf{E}_{N-1}^{(i)}$ and relate them to the ratio of the perturbed and unperturbed determinants. The importance of the assumption in Lemma 19 that $u<-E_\infty(p)$, is that for large $N$, we have that $-\sqrt{N}\bar{u}>2$, as in the setting of Lemma 21 below. The fact that the shifts are greater than 2, and thus the corresponding spectra of the shifted GOE matrices are strictly positive, is crucial to the proof of Lemma 21 since it allows us to use concentration results for linear statistics of the eigenvalues. The latter will be applied to (uniformly) control the fluctuation of the corresponding determinants and their derivatives in the shifts ($v_i$ in Lemma 21, which correspond to $-\sqrt{N}\bar{u}_i$ above). Other arguments in the proof of Lemma 21 are related to large deviations and similar to ones we already used, for example, in the proof of Lemma 16. Once the bound on the fluctuations of the derivative in $\bar{u}_i$ is obtained, step (2) above can be completed. Finally, in Lemma 25 we shall exploit certain Gaussian identities to analyze the expectation of a product related to two shifted GOE matrices, assuming a particular correlation structure. In the case where the product is of the determinants of the two matrices, the lemma asserts that the corresponding expectation is convex in a parameter controlling the correlation. This allows us to relate the situation of low correlation to that where the matrices are completely independent and complete step (3) above. We now proceed to state and prove the auxiliary results.

With $\mathbf{X}_i=\mathbf{X}_{i,N-1}$, $i\le k$, being random $N-1\times N-1$ matrices, denote by $\mathcal{L}_{k,N-1}^{\mathrm{GOE}}$ the space of probability measures on $(\mathbb{R}^{N-1\times N-1})^k$ such that

$$\mathbb{P}\{(\mathbf{X}_i)_{i\le k}\in\cdot\}\in\mathcal{L}_{k,N-1}^{\mathrm{GOE}}\quad\Longleftrightarrow\quad\forall i\le k,\text{ under }\mathbb{P}\{\mathbf{X}_i\in\cdot\}\text{ is a GOE matrix.}$$

That is, the collection of probability laws such that marginally each $\mathbf{X}_i$ is a GOE matrix, but with no further assumptions on the joint law. For a measure $\nu\in\mathcal{L}_{k,N-1}^{\mathrm{GOE}}$, we will use $(\mathbf{X}_i)_{i\le k}\sim\nu$ to denote $\mathbb{P}\{(\mathbf{X}_i)_{i\le k}\in\cdot\}=\nu(\cdot)$.

LEMMA 21. *Assume $(\mathbf{X}_i)_{i\le k}\sim\nu$ with $\nu\in\mathcal{L}_{k,N-1}^{\mathrm{GOE}}$ and denote by $\lambda_j^{(i)}$ the eigenvalues of $\mathbf{X}_i:=\mathbf{X}_{i,N-1}$. Let $t_2>t_1>2$ be real numbers. Then*:

(i) *For any $\delta>0$, there exists $c>0$ such that, for large enough $N$, uniformly in $v_i:=v_{i,N}\in(-t_2,-t_1)$ and $\nu\in\mathcal{L}_{k,N-1}^{\mathrm{GOE}}$,*

$$\begin{aligned}&\mathbb{E}\left\{\prod_{i=1}^k|\det(\mathbf{X}_i-v_iI)|\mathbf{1}\left\{\min_{i,j}\lambda_j^{(i)}\le-2-\delta\right\}\right\}\\&\qquad\le e^{-cN}\mathbb{E}\left\{\prod_{i=1}^k|\det(\mathbf{X}_i-v_iI)|\right\},\end{aligned}\tag{8.7}$$

$$\mathbb{E}\left\{\prod_{i=1}^{k}\left|\det(\mathbf{X}_i - v_i I)\right|\mathbf{1}\left\{\max_{i,j}\lambda_j^{(i)} \ge 2+\delta\right\}\right\} \\ \le e^{-cN}\mathbb{E}\left\{\prod_{i=1}^{k}\left|\det(\mathbf{X}_i - v_i I)\right|\right\}. \tag{8.8}$$

(ii) *With* $\mu^*$ *denoting the semicircle law* (2.3), *as* $N \to \infty$, *uniformly in* $v_i := v_{i,N} \in (-t_2, -t_1)$ *and* $\nu := \nu_N \in \mathcal{L}_{k,N-1}^{\mathrm{GOE}}$,

$$\frac{d}{dv_1}\log\left(\mathbb{E}\left\{\prod_{i=1}^{k}\det(\mathbf{X}_i - v_i I)\right\}\right) = -\big(1+o(1)\big)N\int\frac{1}{\lambda - v_1}\,d\mu^*(\lambda). \tag{8.9}$$

PROOF. All the equalities, inequalities and limits in the proof should be understood to hold uniformly in $v_i \in (-t_2, -t_1)$ and $\nu \in \mathcal{L}_{k,N-1}^{\mathrm{GOE}}$. First, we show that

$$\limsup_{N\to\infty}\frac{1}{N}\log\left(\mathbb{E}\left\{\prod_{i=1}^{k}\left|\det(\mathbf{X}_i - v_i I)\right|\right\}\right) \le \sum_{i=1}^{k}\Omega(v_i). \tag{8.10}$$

Recall the definition (5.5) of the truncation functions $h_\varepsilon^\kappa(x)$ and $h_\kappa^\infty(x)$. Fix some $\bar\kappa > \bar\varepsilon > 0$. By the Cauchy–Schwarz inequality,

$$\mathbb{E}\left\{\prod_{i=1}^{k}\left|\det(\mathbf{X}_i - v_i I)\right|\right\} \le \left(\mathbb{E}\left\{\prod_{i=1}^{k}\prod_{j=1}^{N-1}\big(h_{\bar\varepsilon}^{\bar\kappa}\big(\big|\lambda_j^{(i)} - v_i\big|\big)\big)^2\right\}\right)^{1/2} \\ \times\left(\mathbb{E}\left\{\prod_{i=1}^{k}\prod_{j=1}^{N-1}\big(h_{\bar\kappa}^{\infty}(x)\big(\big|\lambda_j^{(i)} - v_i\big|\big)\big)^2\right\}\right)^{1/2}.$$

Similarly to part (ii) of Lemma 16, using Lemma 26 and a union bound (over $i \le k$), one can show that the second expectation above is smaller than 2, assuming $\bar\kappa$ is larger than some appropriate constant $\bar\kappa_0$. From the LDP for the empirical measure of eigenvalues of Theorem 28 [similar to the proof of part (i) of Lemma 16], we therefore have that[4]

$$\limsup_{N\to\infty}\frac{1}{N}\log\left(\mathbb{E}\left\{\prod_{i=1}^{k}\left|\det(\mathbf{X}_i - v_i I)\right|\right\}\right) \le \sum_{i=1}^{k}\int\log_{\bar\varepsilon}^{\bar\kappa}\big(|\lambda - v_i|\big)\,d\mu^*(\lambda), \tag{8.11}$$

where $\log_{\bar\varepsilon}^{\bar\kappa}(x) = \log(h_{\bar\varepsilon}^{\bar\kappa}(x))$. By choosing small enough $\bar\varepsilon$ and large enough $\bar\kappa$ so that $\bar\varepsilon < t_1 - 2 < -v_i - 2$ and $\bar\kappa > t_2 + 2 > -v_i + 2$, (8.10) follows.

[4]We remark that uniformity in $v_i$ relies on the fact that the LDP for the empirical measure of the eigenvalues is phrased in terms of the Lipschitz bounded metric and we use the functions $\log_{\bar\varepsilon}^{\bar\kappa}(|\cdot - v_i|)$ which have the same bound and Lipschitz constant for all $v_i$.

Suppose $\delta, \varepsilon, \kappa > 0$ satisfy $0 < \varepsilon < t_1 - 2 - \delta$ and $\kappa > t_2 + 2 + \delta$. Then on the event

$$A(\delta) = \Big\{ -2 - \delta < \min_{i,j} \lambda_j^{(i)} \leq \max_{i,j} \lambda_j^{(i)} < 2 + \delta \Big\} \tag{8.12}$$

all the eigenvalues of $\mathbf{X}_i - v_i I$, $i \leq k$, are in $(\varepsilon, \kappa)$ and

$$\prod_{i=1}^{k} \det(\mathbf{X}_i - v_i I) = e^{V_N} \quad \text{and} \quad \frac{1}{N} \sum_{j=1}^{N-1} \frac{1}{\lambda_j^{(1)} - v_1} = V_N',$$

with

$$V_N \triangleq \sum_{i=1}^{k} \sum_{j=1}^{N-1} \log\big(h_\varepsilon^\kappa(\lambda_j^{(i)} - v_i)\big), \qquad V_N' \triangleq \frac{1}{N} \sum_{j=1}^{N-1} \frac{1}{h_\varepsilon^\kappa(\lambda_j^{(1)} - v_1)}. \tag{8.13}$$

From the LDP of Theorem 28, as $N \to \infty$,[5]

$$\mathbb{E}\{V_N'\} \to \int \frac{1}{\lambda - v_1} \, d\mu^*(\lambda) \quad \text{and} \quad \frac{1}{N} \log\big(\mathbb{E}\{e^{V_N}\}\big) \to \sum_{i=1}^{k} \Omega(v_i), \tag{8.14}$$

where we used the fact that for $\lambda$ in the support of $\mu^*$, $\lambda - v_i \in (\varepsilon, \kappa)$.

For large enough $L = L(\varepsilon, \kappa) > 0$, $\log(h_\varepsilon^\kappa(x))$ and $\frac{1}{N}(h_\varepsilon^\kappa(x))^{-1}$ are Lipschitz continuous with Lipschitz constant $L$ and $\frac{1}{N}L$, respectively. Thus, by the concentration of linear statistics of Wigner matrices as in [2], Theorem 2.3.5, and the union bound, we have that

$$\mathbb{P}\big\{|V_N - \mathbb{E}V_N| > s\big\} \leq 2ke^{-Cs^2}, \qquad \mathbb{P}\big\{\big|V_N' - \mathbb{E}V_N'\big| > s\big\} \leq 2e^{-N^2Cs^2}, \tag{8.15}$$

for some constant $C > 0$. By the LDP for the maximal (and by symmetry, minimal) eigenvalue of $\mathbf{X}_i$ (see Theorem 27),

$$\limsup_{N\to\infty} \frac{1}{N} \log\big(\mathbb{P}\big\{\big(A(\delta)\big)^c\big\}\big) < 0. \tag{8.16}$$

Therefore, using (8.15) and the Cauchy–Schwarz inequality we have that, as $N \to \infty$,

$$\begin{aligned} &\mathbb{E}\{V_N' e^{V_N}\} = \mathbb{E}\{V_N'\}\mathbb{E}\{e^{V_N}\}\big(1 + o(1)\big), \\ &\mathbb{E}\{V_N' e^{V_N} \mathbf{1}_{(A(\delta))^c}\} \leq \big(\mathbb{E}\big\{\big(V_N' e^{V_N}\big)^2\big\}\mathbb{P}\big\{\big(A(\delta)\big)^c\big\}\big)^{1/2} = o\big(\mathbb{E}\{V_N' e^{V_N}\}\big) \end{aligned} \tag{8.17}$$

and similarly

$$\mathbb{E}\{V_N' \mathbf{1}_{(A(\delta))^c}\} = o\big(\mathbb{E}\{V_N'\}\big), \qquad \mathbb{E}\{e^{V_N} \mathbf{1}_{(A(\delta))^c}\} = o\big(\mathbb{E}\{e^{V_N}\}\big). \tag{8.18}$$

[5] See Footnote 4.

Since $\prod_{i=1}^k |\det(\mathbf{X}_i - v_i I)| \geq e^{V_N}\mathbf{1}_{A(\delta)}$, from (8.10), (8.14) and (8.18) we have

$$\lim_{N\to\infty}\frac{1}{N}\log\left(\mathbb{E}\left\{\prod_{i=1}^k\left|\det(\mathbf{X}_i - v_i I)\right|\right\}\right) = \sum_{i=1}^k \Omega(v_i). \tag{8.19}$$

Since $k \geq 1$ was general, by taking two copies of each of the matrices in (8.19), we also have

$$\lim_{N\to\infty}\frac{1}{N}\log\left(\mathbb{E}\left\{\prod_{i=1}^k\left|\det(\mathbf{X}_i - v_i I)\right|^2\right\}\right) = 2\sum_{i=1}^k \Omega(v_i), \tag{8.20}$$

and by (8.16) and the Cauchy–Schwarz inequality, the first part of Lemma 21 follows.

Since $\det(\mathbf{X}_i - v_i I)$ is a polynomial function of the Gaussian entries of $\mathbf{X}_i - v_i I$, the left-hand side of (8.9) is equal to

$$\frac{d}{dv_1}\log(\mathbb{E}\{Y_N\}) = \frac{\mathbb{E}\{\frac{d}{dv_1}Y_N\}}{\mathbb{E}\{Y_N\}} = -\frac{N\mathbb{E}\{Y_N Z_N\}}{\mathbb{E}\{Y_N\}},$$

where we denote

$$Y_N = \prod_{i=1}^k \det(\mathbf{X}_i - v_i I) \quad \text{and} \quad Z_N = \frac{1}{N}\sum_{j=1}^{N-1}\frac{1}{\lambda_j^{(1)} - v_1}.$$

By (8.17), (8.18) and (8.14), as $N\to\infty$,

$$\begin{aligned}\frac{\mathbb{E}\{Y_N Z_N \mathbf{1}_{A(\delta)}\}}{\mathbb{E}\{Y_N \mathbf{1}_{A(\delta)}\}} &= \frac{\mathbb{E}\{V_N' e^{V_N}\mathbf{1}_{A(\delta)}\}}{\mathbb{E}\{e^{V_N}\mathbf{1}_{A(\delta)}\}} \\ &= (1+o(1))\mathbb{E}\{V_N'\} = (1+o(1))\int\frac{1}{\lambda - v_1}\,d\mu^*(\lambda),\end{aligned}$$

where the first equality follows since on $A(\delta)$ all the eigenvalues of $\mathbf{X}_i - v_i I$, $i \leq k$, are in $(\varepsilon, \kappa)$. By the first part of Lemma 21, since $Y_N = |Y_N|$ on $A(\delta)$,

$$\frac{\mathbb{E}\{|Y_N|\mathbf{1}_{(A(\delta))^c}\}}{\mathbb{E}\{|Y_N|\}} \xrightarrow{N\to\infty} 0 \quad \text{and} \quad \frac{\mathbb{E}\{Y_N\mathbf{1}_{A(\delta)}\}}{\mathbb{E}\{Y_N\}} \xrightarrow{N\to\infty} 1. \tag{8.21}$$

What remains to show in order to complete the proof of (8.9) is that

$$\frac{\mathbb{E}\{Y_N Z_N\mathbf{1}_{A(\delta)}\}}{\mathbb{E}\{Y_N Z_N\}} \xrightarrow{N\to\infty} 1. \tag{8.22}$$

Note that, for any $\bar{\varepsilon} > 0$,

$$|Y_N Z_N| \leq \frac{1}{\bar{\varepsilon}}\prod_{i=1}^k\prod_{j=1}^{N-1} h_{\bar{\varepsilon}}(|\lambda_j^{(i)} - v_i|)$$

and similarly to the proof of (8.10) and (8.11), by letting $\bar{\varepsilon} \to 0$, it can be shown that

$$\limsup_{N\to\infty} \frac{1}{2N} \log\bigl(\mathbb{E}\{(Y_N Z_N)^2\}\bigr) \le \sum_{i=1}^{k} \Omega(v_i).$$

On $A(\delta)$, $Z_N \in (c_1, c_2)$ for appropriate constants $0 < c_1 < c_2$. Thus, from (8.21) and (8.19),

$$\lim_{N\to\infty} \frac{1}{N} \log\bigl(\mathbb{E}\{Y_N Z_N \mathbf{1}_{A(\delta)}\}\bigr) = \lim_{N\to\infty} \frac{1}{N} \log\bigl(\mathbb{E}\{Y_N \mathbf{1}_{A(\delta)}\}\bigr) = \sum_{i=1}^{k} \Omega(v_i).$$

From the Cauchy–Schwarz inequality and (8.16),

$$\frac{\mathbb{E}\{|Y_N Z_N|\mathbf{1}_{(A(\delta))^c}\}}{\mathbb{E}\{Y_N Z_N \mathbf{1}_{A(\delta)}\}} \overset{N\to\infty}{\longrightarrow} 0.$$

This implies (8.22) and the proof is complete. $\square$

COROLLARY 22. *Let $u < -E_\infty(p)$ and suppose $J_N = (a_N, b_N)$ is an interval such that $a_N, b_N \to u$ as $N \to \infty$. Assume $(\mathbf{X}_i)_{i\le k} \sim \nu$ with $\nu \in \mathcal{L}^{\mathrm{GOE}}_{k,N-1}$. Then, uniformly in $u_i := u_{i,N} \in J_N$ and $\nu := \nu_N \in \mathcal{L}^{\mathrm{GOE}}_{k,N-1}$, as $N \to \infty$,*

$$\begin{aligned}
\log\Biggl(\mathbb{E}\Biggl\{\prod_{i=1}^{k}\bigl|\det(\mathbf{X}_i - \sqrt{N}\bar{u}_i I)\bigr|\Biggr\}\Biggr) &= \log\Biggl(\mathbb{E}\Biggl\{\prod_{i=1}^{k}\det(\mathbf{X}_i - \sqrt{N}\bar{u}_i I)\Biggr\}\Biggr) + o(1) \\
&= \log\Biggl(\mathbb{E}\Biggl\{\prod_{i=1}^{k}\det(\mathbf{X}_i - \sqrt{N}\bar{b}_N I)\Biggr\}\Biggr) + o(1) \qquad (8.23)\\
&\quad + N\mathfrak{S}(u)\sum_{i=1}^{k}\bigl(1+o(1)\bigr)(b_N - u_i),
\end{aligned}$$

*where $\bar{b}_N = \sqrt{\frac{p}{p-1}\frac{1}{N-1}}b_N$, $\bar{u}_i = \sqrt{\frac{p}{p-1}\frac{1}{N-1}}u_i$ and $\mathfrak{S}(u)$ is given by (8.1).*

PROOF. From our assumption on $u$, for some $t_2 > t_1 > 2$, for large $N$, $\sqrt{N}\bar{b}_N, \sqrt{N}\bar{u}_i \in (-t_2, -t_1)$ for any $u_i \in J_N$. On the event $A(\delta)$ defined in (8.12), for small enough $\delta$,

$$\prod_{i=1}^{k}\bigl|\det(\mathbf{X}_i - \sqrt{N}\bar{u}_i I)\bigr| = \prod_{i=1}^{k}\det(\mathbf{X}_i - \sqrt{N}\bar{u}_i I).$$

Therefore, the first equality in (8.23) follows from the first part of Lemma 21 which asserts that, as $N \to \infty$,

$$\mathbb{E}\Biggl\{\prod_{i=1}^{k}\bigl|\det(\mathbf{X}_i - \sqrt{N}\bar{u}_i I)\bigr|\mathbf{1}_{(A(\delta))^c}\Biggr\} = o(1)\mathbb{E}\Biggl\{\prod_{i=1}^{k}\bigl|\det(\mathbf{X}_i - \sqrt{N}\bar{u}_i I)\bigr|\Biggr\}.$$

From the second part of Lemma 21,

$$\log\left(\mathbb{E}\left\{\prod_{i=1}^{k}\det(\mathbf{X}_i-\sqrt{N}\bar{u}_i I)\right\}\right)$$
$$=\log\left(\mathbb{E}\left\{\prod_{i=1}^{k}\det(\mathbf{X}_i-\sqrt{N}\bar{b}_N I)\right\}\right)$$
$$+N^{3/2}\int\frac{1}{\lambda-\sqrt{N}\bar{u}}\,d\mu^*(\lambda)\sum_{i=1}^{k}(1+o(1))(\bar{b}_N-\bar{u}_i),$$

as $N\to\infty$, uniformly in $u_i\in J_N$ and $\nu\in\mathcal{L}_{k,N-1}^{\mathrm{GOE}}$. This completes the proof. □

COROLLARY 23. *Let $u<-E_\infty(p)$ and suppose $J_N=(a_N,b_N)$ is an interval such that $a_N,b_N\to u$ as $N\to\infty$. Assume $(\mathbf{X}_i)_{i\le k}\sim\nu$ with $\nu\in\mathcal{L}_{k,N-1}^{\mathrm{GOE}}$. Then, uniformly in $u_i:=u_{i,N}\in J_N$ and $\nu:=\nu_N\in\mathcal{L}_{k,N-1}^{\mathrm{GOE}}$,*

$$\mathbb{E}\left\{\prod_{i=1}^{k}\det(\mathbf{X}_i-\sqrt{N}\bar{u}_i I)\right\}\le c_k\prod_{i=1}^{k}\mathbb{E}\{\det(\mathbf{X}_i-\sqrt{N}\bar{u}_i I)\},\tag{8.24}$$

*for appropriate constants $c_k>0$ independent of $N$, where $\bar{u}_i=\sqrt{\frac{p}{p-1}\frac{1}{N-1}}u_i$.*

PROOF. From our assumption on $u$ for some $t_2>t_1>2$, for large $N$, $\sqrt{N}\bar{u},\sqrt{N}\bar{u}_i\in(-t_2,-t_1)$ for any $u_i\in J_N$. Let $\lambda_j^{(i)}$ denote the eigenvalues of $\mathbf{X}_i$ and recall the definition of $A(\delta)$ given in (8.12). From the first part of Lemma 21 for small $\delta>0$, uniformly in $u_i:=u_{i,N}\in J_N$ and $\nu:=\nu_N\in\mathcal{L}_{k,N-1}^{\mathrm{GOE}}$, as $N\to\infty$,

$$\mathbb{E}\left\{\prod_{i=1}^{k}\det(\mathbf{X}_i-\sqrt{N}\bar{u}_i I)\right\}=(1+o(1))\mathbb{E}\left\{\prod_{i=1}^{k}\det(\mathbf{X}_i-\sqrt{N}\bar{u}_i I)\mathbf{1}_{A(\delta)}\right\}.$$

For small enough $\delta,\varepsilon>0$ and large enough $\kappa>0$, on $A(\delta)$ we have that $\prod_{i=1}^{k}\det(\mathbf{X}_i-\sqrt{N}\bar{u}_i I)=e^{\bar{V}_N}$, where

$$\bar{V}_N\triangleq\sum_{i=1}^{k}\sum_{j=1}^{N-1}\log\bigl(h_\varepsilon^\kappa(\lambda_j^{(i)}-\sqrt{N}\bar{u}_i)\bigr)$$

is defined similarly to $V_N$ [see (8.13)]. Similar to (8.15), by the concentration of linear statistics of Wigner matrices as in [2], Theorem 2.3.5, defining $\bar{V}_{N,i}=\sum_{j=1}^{N-1}\log(h_\varepsilon^\kappa(\lambda_j^{(i)}-\sqrt{N}\bar{u}_i))$, we have for all $i\le k$,

$$\mathbb{P}\{|\bar{V}_{N,i}-\mathbb{E}\bar{V}_{N,i}|>s\}\le 2ke^{-Cs^2},\tag{8.25}$$

with some constant $C=C(\varepsilon,\kappa)>0$ that depends on the Lipschitz constant of $\log(h_\varepsilon^\kappa(x))$. From the above, (8.24) follows. □

LEMMA 24. *Let $u < -E_\infty(p)$ and suppose $J_N = (a_N, b_N)$ is an interval such that $a_N, b_N \to u$ as $N \to \infty$. Let $0 < \rho_N = o(1)$, let $\mathbf{X}_{\mathrm{iid}}^{(i)} = \mathbf{X}_{\mathrm{iid},N-1}^{(i)}$, $i = 1, 2$, and $\mathbf{X}_{\mathrm{iid}} = \mathbf{X}_{\mathrm{iid},N-1}$ be three i.i.d. GOE matrices of dimension $N - 1$, and set*

$$\mathbf{X}_{N-1}^{(i)}(r) = \sqrt{1 - |r|^{p-2}}\mathbf{X}_{\mathrm{iid}}^{(i)} + \big(\mathrm{sgn}(r)\big)^{ip}\sqrt{|r|^{p-2}}\mathbf{X}_{\mathrm{iid}}. \tag{8.26}$$

*Let $\mathbf{M}_{N-1}^{(i)}(r, u_1, u_2)$ be as defined in Lemma 13 and set $\bar{u}_i = \sqrt{\frac{p}{p-1}\frac{1}{N-1}}u_i$. Then, as $N \to \infty$, uniformly in $u_i := u_{i.N} \in J_N$ and $r := r_N \in (-\rho_N, \rho_N)$,*

$$\begin{aligned}\mathbb{E}\Big\{\prod_{i=1,2}\big|\det\big(\mathbf{M}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2)\big)\big|\Big\}& \\ \le \big(1 + o(1)\big)\mathbb{E}&\Big\{\prod_{i=1,2}\big|\det\big(\mathbf{X}_{N-1}^{(i)}(r) - \sqrt{N}\bar{u}_i I\big)\big|\Big\}.\end{aligned}$$

PROOF. We start from the representation of Lemma 13. Conditional on $f(\mathbf{n}) = \sqrt{N}u_1$, $f(\boldsymbol{\sigma}(r)) = \sqrt{N}u_2$ and $\nabla f(\mathbf{n}) = \nabla f(\boldsymbol{\sigma}(r)) = 0$ we have that, in distribution,

$$\begin{pmatrix}\dfrac{\nabla^2 f(\mathbf{n})}{\sqrt{(N-1)p(p-1)}} \\ \dfrac{\nabla^2 f(\boldsymbol{\sigma}(r))}{\sqrt{(N-1)p(p-1)}}\end{pmatrix} = \begin{pmatrix}\mathbf{M}_{N-1}^{(1)}(r, \sqrt{N}u_1, \sqrt{N}u_2) \\ \mathbf{M}_{N-1}^{(2)}(r, \sqrt{N}u_1, \sqrt{N}u_2)\end{pmatrix},$$

with

$$\begin{aligned}\mathbf{M}_{N-1}^{(i)}(r, u_1, u_2) &= \hat{\mathbf{M}}_{N-1}^{(i)}(r) - \sqrt{N}\bar{u}_i I + \frac{m_i(r, \sqrt{N}u_1, \sqrt{N}u_2)}{\sqrt{(N-1)p(p-1)}}e_{N-1,N-1}, \\ \hat{\mathbf{M}}_{N-1}^{(i)}(r) &= \begin{pmatrix}\hat{\mathbf{G}}_{N-2}^{(i)}(r) & Z^{(i)}(r) \\ \big(Z^{(i)}(r)\big)^T & Q^{(i)}(r)\end{pmatrix}, \\ \hat{\mathbf{G}}^{(i)} &= \sqrt{1 - |r|^{p-2}}\bar{\mathbf{G}}^{(i)} + \big(\mathrm{sgn}(r)\big)^{ip}\sqrt{|r|^{p-2}}\bar{\mathbf{G}},\end{aligned}$$

where all the variables are as described in Lemma 13.

Denote by $\tilde{\mathbf{X}}_{N-1}^{(i)}(r)$ the matrix obtained from $\mathbf{X}_{N-1}^{(i)}(r)$ [defined in (8.26)] by replacing every element not in the last row or column by 0 and denote by $\bar{\mathbf{X}}_{N-2}^{(i)}(r)$ the upper-left $N - 2 \times N - 2$ submatrix of $\mathbf{X}_{N-1}^{(i)}(r)$. Couple the variables so that, almost surely,

$$\bar{\mathbf{X}}_{N-2}^{(i)}(r) = \hat{\mathbf{G}}_{N-2}^{(i)}(r), \tag{8.27}$$

and, denoting by $(\mathbf{A})_{i,j}$ the $i$, $j$ element of a general matrix $\mathbf{A}$,

$$
\begin{aligned}
Z_j^{(i)}(r) &= \sqrt{\frac{\Sigma_{Z,11}(r) - |\Sigma_{Z,12}(r)|}{p(p-1)}}\big(\mathbf{X}_{\mathrm{iid}}^{(i)}\big)_{j,N-1} \\
&\quad + \big(\operatorname{sgn}\big(\Sigma_{Z,12}(r)\big)\big)^i \sqrt{\frac{|\Sigma_{Z,12}(r)|}{p(p-1)}}(\mathbf{X}_{\mathrm{iid}})_{j,N-1}, \\
Q_i(r) &= \sqrt{\frac{\Sigma_{Q,11}(r) - |\Sigma_{Q,12}(r)|}{2p(p-1)}}\big(\mathbf{X}_{\mathrm{iid}}^{(i)}\big)_{N-1,N-1} \\
&\quad + \big(\operatorname{sgn}\big(\Sigma_{Q,12}(r)\big)\big)^i \sqrt{\frac{|\Sigma_{Q,12}(r)|}{2p(p-1)}}(\mathbf{X}_{\mathrm{iid}})_{N-1,N-1}.
\end{aligned}
\tag{8.28}
$$

Define

$$
\mathbf{T}_{N-1}^{(i)}(r) \triangleq \begin{pmatrix} 0 & Z^{(i)}(r) \\ \big(Z^{(i)}(r)\big)^T & Q_i(r) \end{pmatrix} - \tilde{\mathbf{X}}_{N-1}^{(i)}(r),
$$

and note that

$$
\hat{\mathbf{M}}_{N-1}^{(i)}(r) = \mathbf{X}_{N-1}^{(i)}(r) + \mathbf{T}_{N-1}^{(i)}(r).
$$

For a general matrix $\mathbf{A}$ with eigenvalues $\lambda_i(\mathbf{A})$, denote $\lambda_*(\mathbf{A}) = \max_i |\lambda_i(\mathbf{A})|$. Define the event

$$
E_N(\delta) = \bigcap_{r\in(-\rho_N,\rho_N)} \bigg(\bigcap_{i=1,2} \big\{\lambda_*\big(\mathbf{X}_{N-1}^{(i)}(r)\big) < 2+\eta\big\} \cap \big\{\lambda_*\big(\mathbf{T}_{N-1}^{(i)}(r)\big) < \delta\big\}\bigg),
$$

where $\eta > 0$, which will be fixed from now on, is such that

$$
\lambda_*\big(\mathbf{X}_{N-1}^{(i)}(r)\big) < 2+\eta \quad \Longrightarrow \quad \min_j \lambda_j\big(\mathbf{X}_{N-1}^{(i)} - \sqrt{N}\bar{u}_i I\big) > \eta,
$$

for large $N$, uniformly in $u_i \in J_N$ [which is possible to choose since $u < -E_\infty(p)$]. Note that

$$
\begin{aligned}
\mathbf{M}_{N-1}^{(i)}(r,u_1,u_2) &= \underbrace{\mathbf{X}_{N-1}^{(i)}(r) - \sqrt{N}\bar{u}_i I}_{\triangleq \mathbf{D}_{N-1}^{(i)}(r,\sqrt{N}u_1,\sqrt{N}u_2)} \\
&\quad \underbrace{+\,\mathbf{T}_{N-1}^{(i)}(r) + \frac{m_i(r,\sqrt{N}u_1,\sqrt{N}u_2)}{\sqrt{(N-1)p(p-1)}} e_{N-1,N-1}}_{\triangleq \mathbf{E}_{N-1}^{(i)}(r,\sqrt{N}u_1,\sqrt{N}u_2)}.
\end{aligned}
$$

The rank of $\mathbf{E}_{N-1}^{(i)}(r,\sqrt{N}u_1,\sqrt{N}u_2)$ and, therefore, the number of nonzero eigenvalues, is 2 at most. On $E_N(\delta)$, the eigenvalues of $\mathbf{E}_{N-1}^{(i)}(r,\sqrt{N}u_1,\sqrt{N}u_2)$

are bounded in absolute value by

$$\delta + 2\frac{\sup_{u_i \in J_N} |m_i(r, \sqrt{N}u_1, \sqrt{N}u_2)|}{\sqrt{p(p-1)}} \overset{N\to\infty}{\longrightarrow} \delta,$$

uniformly in $u_i \in J_N$ and $r \in (-\rho_N, \rho_N)$. From the bound (A.2) of Corollary 29 with $\mathbf{C}_1 = \mathbf{D}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2)$ and $\mathbf{C}_2 = \mathbf{E}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2)$ we obtain that on $E_N(\delta)$, for large enough $N$, for any $u_i \in J_N$ and $r \in (-\rho_N, \rho_N)$,

$$\text{(8.29)}\quad \left|\det\left(\mathbf{M}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2)\right)\right| \le \left|\det\left(\mathbf{X}_{N-1}^{(i)}(r) - \sqrt{N}\bar{u}_i I\right)\right| \cdot \left(1 + 2\frac{\delta}{\eta}\right)^2.$$

In order to conclude the proof of Lemma 24, it will be enough to show that for any $\delta$, uniformly in $u_i \in J_N$ and $r \in (-\rho_N, \rho_N)$,

$$\text{(8.30)}\qquad \lim_{N\to\infty} \frac{\mathbb{E}\{\prod_{i=1,2} |\det(\mathbf{M}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2))| \mathbf{1}\{(E_N(\delta))^c\}\}}{\mathbb{E}\{\prod_{i=1,2} |\det(\mathbf{X}_{N-1}^{(i)}(r) - \sqrt{N}\bar{u}_i I)| \mathbf{1}\{E_N(\delta)\}\}} = 0.$$

By (8.19) [which holds uniformly in $v_i \in (-t_2, -t_1)$ as in the statement of Lemma 21], uniformly in $u_i \in J_N$ and $r \in (-\rho_N, \rho_N)$,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \log\left(\mathbb{E}\left\{\prod_{i=1,2} \left|\det\left(\mathbf{X}_{N-1}^{(i)}(r) - \sqrt{N}\bar{u}_i I\right)\right|\right\}\right) \\
\text{(8.31)}\qquad &= \frac{1}{2} \cdot \lim_{N\to\infty} \frac{1}{N} \log\left(\mathbb{E}\left\{\prod_{i=1,2} \left|\det\left(\mathbf{X}_{N-1}^{(i)}(r) - \sqrt{N}\bar{u}_i I\right)\right|^2\right\}\right) \\
&= \sum_{i=1,2} \Omega\left(\sqrt{\frac{p}{p-1}} u_i\right).
\end{aligned}$$

By Lemmas 14 and 15 and the Cauchy–Schwarz inequality, for any $\varepsilon > 0$, uniformly in $u_i \in J_N$ and $r \in (-\rho_N, \rho_N)$,

$$\begin{aligned}
&\frac{1}{2} \cdot \limsup_{N\to\infty} \frac{1}{N} \log\left(\mathbb{E}\left\{\prod_{i=1,2} \left|\det\left(\mathbf{M}_{N-1}^{(i)}(r, u_1\sqrt{N}, u_2\sqrt{N})\right)\right|^2\right\}\right) \\
\text{(8.32)}\qquad &\le \frac{1}{4} \cdot \limsup_{N\to\infty} \frac{1}{N} \log\left(\mathbb{E}\left\{\prod_{i=1,2} \prod_{j=1}^{N-2} \left(h_\varepsilon\left(\left|\lambda_j\left(\hat{\mathbf{G}}_{N-2}^{(i)}(r) - \sqrt{N}\bar{u}_i I\right)\right|\right)\right)^4\right\}\right),
\end{aligned}$$

where $h_\varepsilon(x) = \max\{\varepsilon, x\}$. By the same arguments used to derive (8.11) and by letting $\varepsilon \to 0$, we obtain that (8.32) is bounded from above by $\sum_{i=1,2} \Omega(\sqrt{\frac{p}{p-1}} u_i)$.

If we prove that for large $N$,

$$\text{(8.33)}\qquad \mathbb{P}\left\{\left(E_N(\delta)\right)^c\right\} < e^{-C_0 N}$$

for some $C_0 = C_0(\delta) > 0$, then from the above and the Cauchy–Schwarz inequality we would have that the limit supremum of $\frac{1}{N}$ log of the numerator of (8.30) and the limit supremum of $\frac{1}{N}$ log of

$$\mathbb{E}\Big\{\prod_{i=1,2}\big|\det\big(\mathbf{X}_{N-1}^{(i)}(r)-\sqrt{N}\bar{u}_i I\big)\big|\mathbf{1}\big\{\big(E_N(\delta)\big)^c\big\}\Big\}$$

are both asymptotically strictly smaller than $\sum_{i=1,2}\Omega(\sqrt{\frac{p}{p-1}}u_i)$, which together with (8.31), would imply (8.30).

From the LDP of the maximal eigenvalue of GOE matrices (see Theorem 27) and (8.26),

$$\mathbb{P}\Big\{\sup_{r\in(-r_N,r_N)}\lambda_*\big(\mathbf{X}_{N-1}^{(i)}(r)\big)\geq 2+\eta\Big\}<e^{-C_1N}$$

for some $C_1 > 0$, for large $N$. Thus, in order to prove (8.33) it is enough to show that, for large $N$,

$$\mathbb{P}\Big\{\sup_{r\in(-r_N,r_N)}\lambda_*\big(\mathbf{T}_{N-1}^{(i)}(r)\big)\geq \delta\Big\}<e^{-C_2N} \tag{8.34}$$

for some $C_2 = C_2(\delta) > 0$. From (8.28) and the expressions for $\Sigma_Z$ and $\Sigma_Q$ (B.3), it follows that any element of $\mathbf{T}_{N-1}^{(i)}(r)$ in the last row or column can be written as

$$\alpha_1(r)\big(\mathbf{X}_{\mathrm{iid}}^{(i)}\big)_{j,N-1}+\alpha_2(r)(\mathbf{X}_{\mathrm{iid}})_{j,N-1},$$

for some $j \leq N-1$, such that $\sup_{i\in\{1,2\},r\in(-r_N,r_N)}|\alpha_i(r)|\to 0$, as $N\to\infty$. The variance of the Gaussian elements of $\mathbf{X}_{\mathrm{iid}}^{(i)}$ and $\mathbf{X}_{\mathrm{iid}}$ is bounded from above by $2/(N-1)$. Also,

$$2\sum_{m=1}^{N-1}\big(\big(\mathbf{T}_{N-1}^{(i)}(r)\big)_{N-1,m}\big)^2\geq\big(\lambda_*\big(\mathbf{T}_{N-1}^{(i)}(r)\big)\big)^2.$$

Using, for example, Cramér's theorem ([25], Theorem 2.2.3), (8.34) follows and the proof is complete. $\square$

LEMMA 25. *For any $\rho\in[-1,1]$, let $\mathbf{W}_N^{(1)}(\rho)$ and $\mathbf{W}_N^{(2)}(\rho)$ be $N\times N$ centered jointly Gaussian Wigner matrices with*

$$\mathrm{Cov}\big(\mathbf{W}_{ij}^{(m)}(\rho),\mathbf{W}_{kl}^{(n)}(\rho)\big)=\delta_{\{i,j\}=\{k,l\}}(1+\delta_{i=j})\big(\rho+(1-\rho)\delta_{m=n}\big). \tag{8.35}$$

*Let $g:\mathbb{R}^{N\times N}\to\mathbb{R}$ be a smooth function and assume all its derivatives have a $O(|x|^n)$ growth rate at infinity. If we define*

$$\hat{g}(\rho)\triangleq\mathbb{E}\big\{g\big(\mathbf{W}_N^{(1)}(\rho)\big)g\big(\mathbf{W}_N^{(2)}(\rho)\big)\big\},$$

*then* $\frac{d^k}{d\rho^k}\hat{g}(0) \geq 0$ *for all* $k \geq 1$. *In particular, if* $g(\mathbf{A})$ *is a polynomial function of the elements of* $\mathbf{A}$, *then* $\hat{g} : [-1, 1] \to \mathbb{R}$ *is a polynomial function, it is convex on* $[0, 1]$, *and for any* $\rho \in [0, 1]$ *it satisfies*

$$|\hat{g}(-\rho) - \hat{g}(0)| \leq \hat{g}(\rho) - \hat{g}(0) \leq \rho\big(\hat{g}(1) - \hat{g}(0)\big). \tag{8.36}$$

PROOF. In the current proof for any function $h(\mathbf{A})$ of a symmetric matrix $\mathbf{A}$, we denote

$$\frac{\partial}{\partial \mathbf{A}_{ij}} h(\mathbf{A}) := \lim_{t\to 0}\big(h\big(\mathbf{A} + t\big(e_{ij} + (1 - \delta_{ij})e_{ji}\big)\big) - h(\mathbf{A})\big)/t,$$

where $e_{ij}$ is the matrix whose only nonzero entry is the $(i, j)$ entry, which is equal to 1. We will also use the notation

$$\partial_{i_1, j_1, \dots, i_k, j_k} h(\mathbf{A}) = \frac{\partial}{\partial \mathbf{A}_{i_1 j_1}} \cdots \frac{\partial}{\partial \mathbf{A}_{i_k j_k}} h(\mathbf{A}).$$

Suppose that $X_{\mathbf{C}} \sim N(0, \mathbf{C})$ is a general Gaussian vector of length $k$ with density $\varphi_{\mathbf{C}}(x)$, where $\mathbf{C} = (\mathbf{C}_{ij})$ is a nonsingular covariance matrix. From integration by parts and the well-known fact that for $i \neq j$,

$$\frac{\partial}{\partial \mathbf{C}_{ij}} \varphi_{\mathbf{C}}(x) = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \varphi_{\mathbf{C}}(x),$$

one has that, for any function $w : \mathbb{R}^k \to \mathbb{R}$ with $O(|x|^n)$ growth rate at infinity,

$$\frac{\partial}{\partial \mathbf{C}_{ij}} \mathbb{E}\{w(X_{\mathbf{C}})\} = \int w(x) \frac{\partial}{\partial \mathbf{C}_{ij}} \varphi_{\mathbf{C}}(x)\, dx = \int \left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} w(x)\right) \varphi_{\mathbf{C}}(x)\, dx.$$

Therefore, by applying the above with the function $(\mathbf{A}, \mathbf{B}) \mapsto g(\mathbf{A})g(\mathbf{B})$ and $(\mathbf{W}_N^{(1)}(\rho), \mathbf{W}_N^{(2)}(\rho))$, treated as a vector of the on-and-above elements, we obtain

$$\begin{aligned} \frac{d^k}{d\rho^k}\hat{g}(\rho) = \sum_{\forall l \leq k: 1 \leq i_l \leq j_l \leq N} \prod_{l=1}^{k} (1 + \delta_{i_l = j_l}) \\ \times \mathbb{E}\big\{\big(\partial_{i_1, j_1, \dots, i_k, j_k} g\big(\mathbf{W}_N^{(1)}(\rho)\big)\big)\big(\partial_{i_1, j_1, \dots, i_k, j_k} g\big(\mathbf{W}_N^{(2)}(\rho)\big)\big)\big\}. \end{aligned} \tag{8.37}$$

For $\rho = 0$, $\mathbf{W}_N^{(1)}(0)$ and $\mathbf{W}_N^{(2)}(0)$ are i.i.d. and the expectation in (8.37) is equal to

$$\big(\mathbb{E}\big\{\big(\partial_{i_1, j_1, \dots, i_k, j_k} g\big(\mathbf{W}_N^{(1)}(0)\big)\big)\big\}\big)^2,$$

which proves that $\frac{d^k}{d\rho^k}\hat{g}(0) \geq 0$. Lastly, the fact that $\hat{g}(\rho)$ is a polynomial function whenever $g$ is, follows from the fact that $\mathbf{W}_N^{(i)}(\rho)$ are jointly Gaussian and (8.35). Convexity on $[0, 1]$ and (8.36) are direct consequences since the coefficients of the polynomial function are equal to $\frac{d^k}{d\rho^k}\hat{g}(0)/k!$. $\square$

8.4. *Proof of Lemma* 18. Lemma 9 expresses $\mathbb{E}\{\mathrm{Crt}_N(J_N)\}$. By further conditioning on $U$, substituting (8.23), and using the fact that uniformly in $v \in J_N$, as $N \to \infty$,

$$v^2 = b_N^2 + 2u(v - b_N)\big(1 + o(1)\big)$$

(where $u$ and $b_N$ are related to $J_N$ as in the statement of Lemma 18), we obtain that, as $N \to \infty$,

$$\begin{aligned}\mathbb{E}\{\mathrm{Crt}_N(J_N)\} = \omega_N \left(\frac{p-1}{2\pi}(N-1)\right)^{\frac{N-1}{2}} \sqrt{\frac{N}{2\pi}} e^{-\frac{Nb_N^2}{2}} \\ \times \mathbb{E}\{\det(\mathbf{X}_{N-1} - \sqrt{N}\bar{b}_N I)\} \int_{J_N} g(v)\, dv,\end{aligned}$$

where uniformly in $v \in J_N$,

$$g(v) = \exp\{-\big(1 + o(1)\big)N\big(\mathfrak{S}(u) + u\big)(v - b_N) + o(1)\}.$$

Recall that $\mathfrak{S}(u) + u < 0$ [see (8.3)]. Thus,

$$\int_{J_N} g(v)\, dv = \big(1 + o(1)\big) \int_{J_N} \exp\{-N\big(\mathfrak{S}(u) + u\big)(v - b_N)\}\, dv,$$

which completes the proof. $\square$

8.5. *Proof of Lemma* 19. By Lemma 4 and with the definitions in its statement, by conditioning on $U_1(r)$, $U_2(r)$,

$$\begin{aligned}\mathbb{E}\{[\mathrm{Crt}_N(J_N)]_2^{\rho_N}\} = C_N N \int_{-\rho_N}^{\rho_N} dr \cdot \big(\mathcal{G}(r)\big)^N \mathcal{F}(r) \int_{J_N \times J_N} du_1\, du_2 \\ \times \varphi_{\Sigma_U(r)}(\sqrt{N}u_1, \sqrt{N}u_2) \\ \times \mathbb{E}\left\{\prod_{i=1,2} \big|\det\big(\mathbf{M}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2)\big)\big|\right\},\end{aligned}$$

where by straightforward analysis, as $r \to 0$,

$$\begin{aligned}\varphi_{\Sigma_U(r)}(u_1, u_2) &\triangleq \frac{1}{2\pi}\big(\det\big(\Sigma_U(r)\big)\big)^{-1/2} \exp\left\{-\frac{1}{2}(u_1, u_2)\big(\Sigma_U(r)\big)^{-1}(u_1, u_2)^T\right\} \\ &= \big(1 + O(r^p)\big)\frac{1}{2\pi} \exp\left\{-\frac{1}{2}(u_1^2 + u_2^2) + (u_1 + u_2)^2 O(r^p)\right\},\end{aligned}$$

$\mathcal{F}(r) = 1 + O(r)$ and $\mathcal{G}(r) = e^{-\frac{1}{2}r^2 + O(r^4)}$. Also note that

$$\frac{\omega_{N-1}}{\omega_N} \Big/ \sqrt{\frac{N}{2\pi}} \xrightarrow{N\to\infty} 1$$

and that, as $N \to \infty$, uniformly in $u_i \in J_N$ (with $u$ and $b_N$ related to $J_N$ as in the statement of Lemma 19),

$$u_i^2 = b_N^2 + 2u(u_i - b_N)\big(1 + o(1)\big).$$

Combining all of the above, we arrive at

$$\begin{aligned}\mathbb{E}\{[\mathrm{Crt}_N(J_N)]_2^{\rho_N}\} = \big(\mathfrak{C}_N(b_N)\big)^2 \sqrt{\frac{N}{2\pi}} \int_{-\rho_N}^{\rho_N} dr \cdot e^{-\frac{1}{2}Nr^2 + N \cdot O(r^3)} \\ \times \int_{J_N \times J_N} du_1\, du_2 g(u_1, u_2) \\ \times \frac{\mathbb{E}\{\prod_{i=1,2} |\det(\mathbf{M}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2))|\}}{(\mathbb{E}\{\det(\mathbf{X}_{N-1} - \sqrt{N}\bar{b}_N I)\})^2},\end{aligned}$$

where $\mathfrak{C}_N(x)$ is defined in (8.2), $\mathbf{X}_{N-1}$ is a GOE matrix and as $N \to \infty$, uniformly in $u_i \in J_N$,

$$g(u_1, u_2) = \big(1 + o(1)\big) \exp\left\{-N \sum_{i=1}^{2} u(u_i - b_N)\big(1 + o(1)\big)\right\}.$$

Note that from our assumption that $\rho_N \to 0$ as $N \to \infty$,

$$\lim_{N\to\infty} \sqrt{\frac{N}{2\pi}} \int_{-\rho_N}^{\rho_N} dr \cdot e^{-\frac{1}{2}Nr^2 + N \cdot O(r^3)} \le 1.$$

Therefore, since $\mathfrak{S}(u) + u < 0$ [see (8.3)], Lemma 19 follows if we can show that as $N \to \infty$, uniformly in $u_i \in J_N$ and $r \in (-\rho_N, \rho_N)$,

$$\begin{aligned}&\mathbb{E}\left\{\prod_{i=1,2} \big|\det\big(\mathbf{M}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2)\big)\big|\right\} \\ &\quad \le \big(1 + o(1)\big)\big(\mathbb{E}\big\{\det(\mathbf{X}_{N-1} - \sqrt{N}\bar{b}_N I)\big\}\big)^2 \\ &\quad\quad \times \exp\left\{N\mathfrak{S}(u) \sum_{i=1}^{2} \big(1 + o(1)\big)(b_N - u_i)\right\}.\end{aligned} \tag{8.38}$$

By Lemma 24 and Corollary 22, as $N \to \infty$, uniformly in $u_i \in J_N$ and $r \in (-\rho_N, \rho_N)$,

$$\begin{aligned}&\mathbb{E}\left\{\prod_{i=1,2} \big|\det\big(\mathbf{M}_{N-1}^{(i)}(r, \sqrt{N}u_1, \sqrt{N}u_2)\big)\big|\right\} \\ &\quad \le \big(1 + o(1)\big)\mathbb{E}\left\{\prod_{i=1}^{2} \det\big(\mathbf{X}_{N-1}^{(i)}(r) - \sqrt{N}\bar{b}_N I\big)\right\} \\ &\quad\quad \times \exp\left\{N\mathfrak{S}(u) \sum_{i=1}^{2} \big(1 + o(1)\big)(b_N - u_i)\right\},\end{aligned} \tag{8.39}$$

with $\mathbf{X}_{N-1}^{(i)}(r)$ as defined in Lemma 24.

Since for $r = 0$, $\mathbf{X}_{N-1}^{(1)}(0)$ and $\mathbf{X}_{N-1}^{(2)}(0)$ are i.i.d., defining

$$\Phi_X(r) := \Phi_{X,N}(r) = \mathbb{E}\left\{\prod_{i=1}^{2} \det\left(\mathbf{X}_{N-1}^{(i)}(r) - \sqrt{N}\bar{b}_N I\right)\right\},$$

what remains to show is that $\Phi_X(r) = (1 + o(1))\Phi_X(0)$ as $N \to \infty$, uniformly in $r \in (-\rho_N, \rho_N)$. We show this by appealing to Lemma 25. First, suppose that $\mathbf{W}_{N-1}^{(i)}(r)$ are defined as in this lemma and set

$$\Phi_W(r) := \Phi_{W,N}(r) = \mathbb{E}\left\{\prod_{i=1}^{2} \det\left(\frac{1}{\sqrt{N-1}}\mathbf{W}_{N-1}^{(i)}(r) - \sqrt{N}\bar{b}_N I\right)\right\}.$$

Since, in distribution,

$$\left(\mathbf{X}_{N-1}^{(1)}(r), \mathbf{X}_{N-1}^{(2)}(r)\right) = \frac{1}{\sqrt{N-1}}\left(\mathbf{W}_{N-1}^{(1)}\left(s(r)\right), \mathbf{W}_{N-1}^{(2)}\left(s(r)\right)\right),$$

with $s(r) = (\mathrm{sgn}(r))^p\sqrt{|r|^{p-2}}$, it follows that

$$\Phi_X(r) = \Phi_W\left(s(r)\right).$$

Thus, it is enough to show that for any $\rho_N' > 0$ such that $\rho_N' \to 0$, as $N \to \infty$,

$$\Phi_W(r) = \left(1 + o(1)\right)\Phi_W(0) \qquad \text{uniformly in } r \in \left(-\rho_N', \rho_N'\right). \tag{8.40}$$

Assume $\rho_N' > 0$ is such an arbitrary sequence. By Corollary 23,

$$\Phi_W(1) \le C\Phi_W(0) = C\left(\mathbb{E}\left\{\det(\mathbf{X}_{N-1} - \sqrt{N}\bar{b}_N I)\right\}\right)^2, \tag{8.41}$$

where $\mathbf{X}_{N-1}$ is a GOE matrix of dimension $N - 1$ and $C > 0$ is an appropriate constant.

In the notation of Lemma 25, $\hat{g}(r) = \Phi_W(r)$ where $g(\mathbf{A}) = \det(\frac{1}{\sqrt{N-1}}\mathbf{A} - \sqrt{N}\bar{b}_N I)$ is a polynomial function of the elements of the matrix $\mathbf{A}$. Thus by Lemma 25, uniformly in $r \in (-\rho_N', \rho_N')$, as $N \to \infty$,

$$\begin{aligned}\left|\Phi_W(r) - \Phi_W(0)\right| &\le \rho_N'\left(\Phi_W(1) - \Phi_W(0)\right)\\ &\le \rho_N'(C-1)\Phi_W(0) = o(1)\Phi_W(0)\end{aligned}$$

and, therefore, (8.40) follows. This completes the proof of Lemma 19. □

## APPENDIX A: EIGENVALUES

Let $\lambda_i = \lambda_i^N$, $i = 1, \dots, N$ denote the eigenvalues of an $N \times N$ GOE matrix and denote by

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i^N} \tag{A.1}$$

the empirical measure of eigenvalues. The following two bounds on the maximal eigenvalue, both proved in [7], are useful to us.

LEMMA 26 ([7], Lemma 6.3). *For large enough $m$ and all $N$,*

$$\mathbb{P}\Big\{\max_{i=1}^{N} |\lambda_i| \geq m\Big\} \leq e^{-Nm^2/9}.$$

THEOREM 27 ([7], Theorem 6.2). *The maximal eigenvalues $\lambda_+^N = \max_{i=1}^N \lambda_i^N$ satisfy the large deviation principle in $\mathbb{R}$ with speed $N$ and the good rate function*

$$I^+(x) = \begin{cases} \int_2^x \sqrt{(z/2)^2 - 1}\, dz, & x \geq 2, \\ \infty, & \textit{otherwise}. \end{cases}$$

Next, we state the LDP satisfied by $L_N$ proved in [8]. Let $M_1(\mathbb{R})$ be the space of Borel probability measures on $\mathbb{R}$, and endow it with the weak topology, which is compatible with the Lipschitz bounded metric $d_{\mathrm{LU}}(\cdot, \cdot)$, defined by

$$d_{\mathrm{LU}}(\mu, \mu') = \sup_{f \in \mathcal{F}_{\mathrm{LU}}} \left| \int_{\mathbb{R}} f\, d\mu - \int_{\mathbb{R}} f\, d\mu' \right|,$$

where $\mathcal{F}_{\mathrm{LU}}$ is the class of Lipschitz continuous functions $f : \mathbb{R} \to \mathbb{R}$, with Lipschitz constant 1 and uniform bound 1. The specific form of the rate function in the LDP is of no importance to us and will therefore not be included in the statement below.

THEOREM 28 ([8], Theorem 2.1.1). *There exists a good rate function $J(\mu)$, for which $J(\mu) = 0$ if and only if $\mu = \mu^*$, where $\mu^*$ is the semicircle law* [*see* (2.3)], *and such that the empirical measure $L_N$ satisfies the large deviation principle on $M_1(\mathbb{R})$ with speed $N^2$ and the rate function $J(\mu)$.*

We finish with a corollary of the main theorem of [31].

COROLLARY 29 ([31]). *Let $\mathbf{C}_1$, $\mathbf{C}_2$ be two* (*deterministic*) *real, symmetric $N \times N$ matrices and let $\lambda_j(\mathbf{C}_i)$ denote the eigenvalues of $\mathbf{C}_i$, ordered with nondecreasing absolute value. Suppose that the number of nonzero eigenvalues of $\mathbf{C}_2$ is $d$ at most. Then*

$$\left|\det(\mathbf{C}_1 + \mathbf{C}_2)\right| \leq \prod_{i=1}^{N} \left( |\lambda_i(\mathbf{C}_1)| + |\lambda_i(\mathbf{C}_2)| \right),$$

*and if* $|\lambda_1(\mathbf{C}_1)| > 0$,

$$\left|\det(\mathbf{C}_1 + \mathbf{C}_2)\right| \leq \left|\det(\mathbf{C}_1)\right| \left( 1 + \frac{|\lambda_N(\mathbf{C}_2)|}{|\lambda_1(\mathbf{C}_1)|} \right)^d. \tag{A.2}$$

## APPENDIX B: COVARIANCES, DENSITIES AND CONDITIONAL LAWS

In this appendix we study the covariance structure of

$$\{f(\mathbf{n}), \nabla f(\mathbf{n}), \nabla^2 f(\mathbf{n}), f(\boldsymbol{\sigma}(r)), \nabla f(\boldsymbol{\sigma}(r)), \nabla^2 f(\boldsymbol{\sigma}(r))\},$$

where

$$\boldsymbol{\sigma}(r) = (0, \dots, 0, \sqrt{1-r^2}, r),$$

and prove Lemmas 12 and 13.

With the standard notation,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

in the lemma below we denote $\delta_{i=j} = \delta_{ij}$, $\delta_{i=j=k} = \delta_{ij}\delta_{jk}$, $\delta_{i=j\neq k} = \delta_{ij}(1-\delta_{jk})$, etc.

LEMMA 30. *For any $r \in [-1, 1]$, there exists an orthonormal frame field $E = (E_i)$ such that*

$$\begin{aligned}
\mathbb{E}\{f(\mathbf{n}) f(\boldsymbol{\sigma}(r))\} &= r^p, \\
\mathbb{E}\{f(\mathbf{n}) E_l f(\boldsymbol{\sigma}(r))\} &= -\mathbb{E}\{E_l f(\mathbf{n}) f(\boldsymbol{\sigma}(r))\} \\
&= -pr^{p-1}(1-r^2)^{1/2}\delta_{l=N-1}, \\
\mathbb{E}\{f(\mathbf{n}) E_k E_l f(\boldsymbol{\sigma}(r))\} &= \mathbb{E}\{E_k E_l f(\mathbf{n}) f(\boldsymbol{\sigma}(r))\} \\
&= p(p-1)r^{p-2}(1-r^2)\delta_{l=k=N-1} - pr^p\delta_{k=l}, \\
\mathbb{E}\{E_j f(\mathbf{n}) E_l f(\boldsymbol{\sigma}(r))\} &= [pr^p - p(p-1)r^{p-2}(1-r^2)]\delta_{l=j=N-1} \\
&\quad + pr^{p-1}\delta_{l=j\neq N-1}, \\
\mathbb{E}\{E_j f(\mathbf{n}) E_k E_l f(\boldsymbol{\sigma}(r))\} &= -\mathbb{E}\{E_k E_l f(\mathbf{n}) E_j f(\boldsymbol{\sigma}(r))\} \\
&= p(p-1)(p-2)r^{p-3}(1-r^2)^{3/2}\delta_{j=k=l=N-1} \\
&\quad - p(p-1)r^{p-2}(1-r^2)^{1/2} \\
&\quad \times [(\delta_{j=k\neq N-1} + r\delta_{j=k=N-1})\delta_{l=N-1} \\
&\quad + (\delta_{j=l<N-1} + r\delta_{j=l=N-1})\delta_{k=N-1}] \\
&\quad - p^2 r^{p-1}(1-r^2)^{1/2}\delta_{k=l}\delta_{j=N-1}, \\
\mathbb{E}\{E_i E_j f(\mathbf{n}) E_k E_l f(\boldsymbol{\sigma}(r))\} &= p(p-1)(p-2)(p-3)r^{p-4}(1-r^2)^2 \\
&\quad \times \delta_{i=j=k=l=N-1} \\
&\quad - p(p-1)(p-2)r^{p-3}(1-r^2)[4r\delta_{i=j=k=l=N-1}
\end{aligned}$$

$$
\begin{aligned}
&+ r\delta_{i=j}\delta_{k=l=N-1} + r\delta_{i=j=N-1}\delta_{k=l}\\
&+ \delta_{j=l=N-1}\delta_{i=k\neq N-1} + \delta_{i=k=N-1}\delta_{j=l\neq N-1}\\
&+ \delta_{i=l=N-1}\delta_{j=k\neq N-1} + \delta_{j=k=N-1}\delta_{i=l\neq N-1}]\\
&+ p(p-1)r^{p-2}\big[-2(1-r^2)\delta_{i=j=N-1}\delta_{k=l}\\
&+ (\delta_{j=l\neq N-1} + r\delta_{j=l=N-1})\\
&\times (\delta_{i=k\neq N-1} + r\delta_{i=k=N-1})\\
&+ (\delta_{i=l\neq N-1} + r\delta_{i=l=N-1})\\
&\times (\delta_{j=k\neq N-1} + r\delta_{j=k=N-1})\big]\\
&+ p(p-1)r^{p-2}\\
&\times \big[-(1-r^2)\delta_{i=j}\delta_{l=k=N-1} + r^2\delta_{i=j}\delta_{k=l}\big]\\
&- p(p-1)r^{p-2}(1-r^2)\delta_{i=j}\delta_{k=l=N-1}\\
&+ pr^p\delta_{i=j}\delta_{k=l}.
\end{aligned}
$$

Note that $r=1$ corresponds to the case $\boldsymbol{\sigma}(r)=\mathbf{n}$. (This is the case considered in [5], Lemma 3.2.)

PROOF. We begin by defining the orthonormal frame field $E$. Let $r\in[-1,1]$ and let $P_{\mathbf{n}}:\mathbb{S}^{N-1}\to\mathbb{R}^{N-1}$ be the projection to $\mathbb{R}^{N-1}$,

$$P_{\mathbf{n}}(x_1,\dots,x_N)=(x_1,\dots,x_{N-1}),$$

set $\theta\in[-\pi/2,\pi/2]$ to be the angle such that $\sin\theta=r$, and let $R_\theta$ be the rotation mapping

$$
\begin{aligned}
&R_\theta(x_1,\dots,x_N)\\
&\quad=(x_1,\dots,x_{N-2},\sin\theta\cdot x_{N-1}+\cos\theta\cdot x_N,-\cos\theta\cdot x_{N-1}+\sin\theta\cdot x_N).
\end{aligned}
$$

Let $U$ and $V$ be neighborhoods of $\mathbf{n}$ and $\boldsymbol{\sigma}(r)$, respectively. Assuming $U$ and $V$ are small enough, the restrictions of $P_{\mathbf{n}}$ and $P_{\mathbf{n}}\circ R_{-\theta}$ to $U$ and $V$, respectively, are coordinate systems.

On $\mathrm{Im}(P_{\mathbf{n}})$ and $\mathrm{Im}(P_{\mathbf{n}}\circ R_{-\theta})$, the images of the charts above, define

$$\bar{f}_1=f\circ P_{\mathbf{n}}^{-1}\quad\text{and}\quad \bar{f}_2=f\circ(P_{\mathbf{n}}\circ R_{-\theta})^{-1}.$$

We let $E=(E_i)$ be an orthonormal frame field on the sphere such that [under the notation (4.3)][6]

$$
\begin{aligned}
\{f(\mathbf{n}),\nabla f(\mathbf{n}),\nabla^2 f(\mathbf{n})\}&=\{\bar{f}_1(0),\nabla\bar{f}_1(0),\nabla^2\bar{f}_1(0)\},\\
\{f(\boldsymbol{\sigma}(r)),\nabla f(\boldsymbol{\sigma}(r)),\nabla^2 f(\boldsymbol{\sigma}(r))\}&=\{\bar{f}_2(0),\nabla\bar{f}_2(0),\nabla^2\bar{f}_2(0)\},
\end{aligned}
$$

[6]The fact that such frame field exists can be seen from the following. If we let $\{\frac{\partial}{\partial x_i}\}_{i=1}^{N-1}$ be the pull-back of $\{\frac{d}{dx_i}\}_{i=1}^{N-1}$ by $P_{\mathbf{n}}$, then $\{\frac{\partial}{\partial x_i}(\mathbf{n})\}_{i=1}^{N-1}$ is an orthonormal frame at the north pole.

where in $\mathbb{R}^{N-1}$, $\nabla \bar{f}_i$ and $\nabla^2 \bar{f}_i$ are the usual gradient and Hessian.

Define $C(x,y) = \mathrm{Cov}\{\bar{f}_1(x), \bar{f}_2(y)\}$ on $\mathrm{Im}(P_{\mathbf{n}}) \times \mathrm{Im}(P_{\mathbf{n}} \circ R_{-\theta})$, and note that

$$\begin{aligned} C(x,y) &= \big(\rho(x,y)\big)^p \triangleq \big\langle P_{\mathbf{n}}^{-1}(x), (P_{\mathbf{n}} \circ R_{-\theta})^{-1}(y)\big\rangle^p \\ &= \Bigg(\sum_{i=1}^{N-2} x_i y_i + r x_{N-1} y_{N-1} + \sqrt{1-r^2}\, x_{N-1}\sqrt{1-\langle y,y\rangle} \\ &\qquad + r\sqrt{1-\langle x,x\rangle}\sqrt{1-\langle y,y\rangle} - \sqrt{1-r^2}\, y_{N-1}\sqrt{1-\langle x,x\rangle}\Bigg)^p. \end{aligned}$$

The lemma follows by a (straightforward, but long) computation of the corresponding derivatives, using the well-known formula [cf. [1], equation (5.5.4)]

$$\mathrm{Cov}\bigg\{\frac{d^k}{dx_{i_1}\cdots dx_{i_k}} \bar{f}_1(x), \frac{d^l}{dy_{i_1}\cdots dy_{i_l}} \bar{f}_2(y)\bigg\} = \frac{d^k}{dx_{i_1}\cdots dx_{i_k}} \frac{d^l}{dy_{i_1}\cdots dy_{i_l}} C(x,y).$$

□

The variables in Lemma 30 are jointly Gaussian. Now that we have their covariances, the required conditional laws can be computed using the well-known formulas for the Gaussian conditional distribution (see [1], pages 10–11). We shall need the following notation.

Define, for any $r \in (-1,1)$,

$$\begin{aligned} a_1(r) &= \frac{1}{p(1-r^{2p-2})}, \qquad a_2(r) = \frac{1}{p[1-(r^p-(p-1)r^{p-2}(1-r^2))^2]}, \\ a_3(r) &= \frac{-r^{p-1}}{p(1-r^{2p-2})}, \qquad a_4(r) = \frac{-r^p+(p-1)r^{p-2}(1-r^2)}{p[1-(r^p-(p-1)r^{p-2}(1-r^2))^2]}, \\ b_1(r) &= -p + a_2(r)p^3 r^{2p-2}(1-r^2), \\ b_2(r) &= -pr^p - a_4(r)p^3 r^{2p-2}(1-r^2), \\ b_3(r) &= a_2(r)p^2(p-1)r^{2p-4}(1-r^2)\big[-(p-2)+pr^2\big], \\ b_4(r) &= p(p-1)r^{p-2}(1-r^2) \\ &\quad - a_4(r)p^2(p-1)r^{2p-4}(1-r^2)\big[-(p-2)+pr^2\big]. \end{aligned}$$

---

For any point in $U$, we can define an orthonormal frame as the parallel transport of $\{\frac{\partial}{\partial x_i}(\mathbf{n})\}_{i=1}^{N-1}$ along a geodesic from $\mathbf{n}$ to that point. This yields an orthonormal frame field on $U$, say $E_i(\boldsymbol{\sigma}) = \sum_{j=1}^{N-1} a_{ij}(\boldsymbol{\sigma})\frac{\partial}{\partial x_j}(\boldsymbol{\sigma})$, $i=1,\dots,N-1$. Working with the coordinate system $P_{\mathbf{n}}$, one can verify that at $x=0$ the Christoffel symbols $\Gamma_{ij}^k$ are equal to 0, and, therefore (see, e.g., [24], equation (2), page 53), the derivatives $\frac{d}{dx_k}a_{ij}(P_{\mathbf{n}}^{-1}(x))$ at $x=0$ are also equal to 0. If $r=1$, that is, $\boldsymbol{\sigma}(r)=\mathbf{n}$, extend the orthonormal frame field $E_i(\boldsymbol{\sigma})$ to the sphere arbitrarily. Otherwise, assume $U$ and $V$ are disjoint and construct the frame field on $V$ similarly to $U$ and then extend it to the sphere.

Define $\Sigma_U(r) = (\Sigma_{U,ij}(r))_{i,j=1}^{2,2}$ by

$$\Sigma_U(r) = -\frac{1}{p}\begin{pmatrix} b_1(r) & b_2(r) \\ b_2(r) & b_1(r) \end{pmatrix}, \tag{B.1}$$

and define $\Sigma_Z(r) = (\Sigma_{Z,ij}(r))_{i,j=1}^{2,2}$ and $\Sigma_Q(r) = (\Sigma_{Q,ij}(r))_{i,j=1}^{2,2}$ by

$$\begin{aligned} \Sigma_{Z,11}(r) = \Sigma_{Z,22}(r) &= p(p-1) - a_1(r)p^2(p-1)^2 r^{2p-4}(1-r^2), \\ \Sigma_{Z,12}(r) = \Sigma_{Z,21}(r)& \\ &= p(p-1)^2 r^{p-1} - p(p-1)(p-2)r^{p-3} \\ &\quad + a_3(r)p^2(p-1)^2 r^{2p-4}(1-r^2), \end{aligned} \tag{B.2}$$

$$\begin{aligned} \Sigma_{Q,11}(r) = \Sigma_{Q,22}(r) &= 2p(p-1) \\ &\quad - a_2(r)(1-r^2)\big[p(p-1)r^{p-3}(pr^2-(p-2))\big]^2 \\ &\quad - (b_3(r), b_4(r))(\Sigma_U(r))^{-1}\begin{pmatrix} b_3(r) \\ b_4(r) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \Sigma_{Q,12}(r) = \Sigma_{Q,21}(r) &= p^4 r^p - 2p(p-1)(p^2-2p+2)r^{p-2} \\ &\quad + p(p-1)(p-2)(p-3)r^{p-4} \\ &\quad + a_4(r)p^2 r^{2p-6}(1-r^2)(p^2r^2-(p-1)(p-2))^2 \\ &\quad - (b_1(r)+b_3(r), b_2(r)+b_4(r))(\Sigma_U(r))^{-1}\begin{pmatrix} b_2(r)+b_4(r) \\ b_1(r)+b_3(r) \end{pmatrix}. \end{aligned} \tag{B.3}$$

Lastly, define

$$\begin{aligned} m_1(r,u_1,u_2) &= (b_3(r), b_4(r))(\Sigma_U(r))^{-1}(u_1,u_2)^T, \\ m_2(r,u_1,u_2) &= m_1(r,u_2,u_1). \end{aligned} \tag{B.4}$$

REMARK 31. By standard analysis $1 \pm (pr^p - (p-1)r^{p-2})$, and thus the denominators of $a_i(r)$ above, are positive for any $r \in (-1,1)$. It is straightforward to verify that

$$\begin{aligned} &(\Sigma_{U,11}(r) \pm \Sigma_{U,12}(r))(1 \mp (pr^p - (p-1)r^{p-2})) \\ &\quad = (1-r^2)(p-1)\left[\frac{1+r^2+\cdots+r^{2p-4}}{p-1} \pm r^{p-2}\right]. \end{aligned}$$

Thus, from (6.11), $\Sigma_{U,11}(r) \pm \Sigma_{U,12}(r) > 0$ for any $r \in (-1,1)$. Since these are the two eigenvalues of $\Sigma_U(r)$, it is strictly positive definite for $r \in (-1,1)$. In Lemma 32, we shall prove that $\Sigma_Z(r)$ is strictly positive definite for $r \in (-1,1)$. In the proof of Lemmas 12 and 13, we show that $\Sigma_Q(r)$ is positive semi-definite.

Finally, we turn to the proof of Lemmas 12 and 13.

**B.1. Proof of Lemmas 12 and 13.** Fix $r \in (-1, 1)$ and let $E$ be the orthonormal frame field defined in the proof of Lemma 30. We remind the reader that

$$\nabla f_N(\boldsymbol{\sigma}) = \big(E_i f_N(\boldsymbol{\sigma})\big)_{i=1}^{N-1}, \qquad \nabla^2 f_N(\boldsymbol{\sigma}) = \big(E_i E_j f_N(\boldsymbol{\sigma})\big)_{i,j=1}^{N-1}.$$

Assume all vectors in the proof are column vectors and denote the concatenation of any two vectors $v_1, v_2$ by $(v_1; v_2)$. The covariance matrix of the vector $(\nabla f(\mathbf{n}); \nabla f(\boldsymbol{\sigma}(r)))$ can be extracted from Lemma 30. By standard calculations, one can prove (4.6) and show that the inverse of the covariance matrix is the block matrix

$$G(r) = \begin{pmatrix} a_1(r)I_{N-1} + (a_2(r) - a_1(r))e_{N-1,N-1} & a_3(r)I_{N-1} + (a_4(r) - a_3(r))e_{N-1,N-1} \\ a_3(r)I_{N-1} + (a_4(r) - a_3(r))e_{N-1,N-1} & a_1(r)I_{N-1} + (a_2(r) - a_1(r))e_{N-1,N-1} \end{pmatrix},$$

where $I_{N-1}$ is the $N-1 \times N-1$ identity matrix and where $e_{N-1,N-1}$ is the $N-1 \times N-1$ matrix whose $N-1 \times N-1$ element is 1 and all others are 0.

For any random vector $V$, let $\mathbb{E}V$ denote the corresponding vector of expectations. From Lemma 30, denoting by $e_i$ the $1 \times (2N-2)$ vector with the $i$th entry equal to 1 and all others equal to 0, we obtain

$$\mathbb{E}\{f(\mathbf{n}) \cdot (\nabla f(\mathbf{n}); \nabla f(\boldsymbol{\sigma}(r)))\} = -pr^{p-1}(1-r^2)^{1/2} e_{2N-2},$$

$$\mathbb{E}\{f(\boldsymbol{\sigma}(r)) \cdot (\nabla f(\mathbf{n}); \nabla f(\boldsymbol{\sigma}(r)))\} = pr^{p-1}(1-r^2)^{1/2} e_{N-1},$$

$$\mathbb{E}\{E_i E_j f(\mathbf{n}) \cdot (\nabla f(\mathbf{n}); \nabla f(\boldsymbol{\sigma}(r)))\}$$

$$= \begin{cases} 0, & |\{i, j, N-1\}| = 3, \\ p^2 r^{p-1}(1-r^2)^{1/2} e_{2N-2}, & i = j \neq N-1, \\ p(p-1)r^{p-2}(1-r^2)^{1/2} e_{N-1+i}, & i \neq j = N-1, \\ p(p-1)r^{p-2}(1-r^2)^{1/2} e_{N-1+j}, & j \neq i = N-1, \\ (1-r^2)^{1/2}(p^3 r^{p-1} - p(p-1)(p-2)r^{p-3}) e_{2N-2}, & \\ & i = j = N-1, \end{cases}$$

$$\mathbb{E}\{E_i E_j f(\boldsymbol{\sigma}(r)) \cdot (\nabla f(\mathbf{n}); \nabla f(\boldsymbol{\sigma}(r)))\}$$

$$= \begin{cases} 0, & |\{i, j, N-1\}| = 3, \\ -p^2 r^{p-1}(1-r^2)^{1/2} e_{N-1}, & i = j \neq N-1, \\ -p(p-1)r^{p-2}(1-r^2)^{1/2} e_i, & i \neq j = N-1, \\ -p(p-1)r^{p-2}(1-r^2)^{1/2} e_j, & j \neq i = N-1, \\ -(1-r^2)^{1/2}(p^3 r^{p-1} - p(p-1)(p-2)r^{p-3}) e_{N-1}, & \\ & i = j = N-1. \end{cases}$$

Denoting by $\mathrm{Cov}_{\nabla f}\{X, Y\}$ the covariance of two random variables $X, Y$ conditional on $\nabla f(\mathbf{n}) = \nabla f(\boldsymbol{\sigma}(r)) = 0$ (and the covariance with no conditioning by

$\mathrm{Cov}\{X,Y\}$), we have (cf. [1], pages 10–11)

$$\begin{aligned}
&\mathrm{Cov}_{\nabla f}\{X,Y\}\\
&\quad=\mathrm{Cov}\{X,Y\}\\
&\qquad-\big(\mathbb{E}\big\{X\cdot\big(\nabla f(\mathbf{n});\nabla f\big(\boldsymbol{\sigma}(r)\big)\big)\big\}\big)^T G(r)\mathbb{E}\big\{Y\cdot\big(\nabla f(\mathbf{n});\nabla f\big(\boldsymbol{\sigma}(r)\big)\big)\big\}.
\end{aligned}$$

Thus, under the conditioning, $f(\mathbf{n})$, $f(\boldsymbol{\sigma}(r))$, $\nabla^2 f(\mathbf{n})$, and $\nabla^2 f(\boldsymbol{\sigma}(r))$ are jointly Gaussian and centered and, by straightforward calculations,

$$\begin{aligned}
\mathrm{Cov}_{\nabla f}\big\{f(\mathbf{n}),f(\mathbf{n})\big\}&=\mathrm{Cov}_{\nabla f}\big\{f\big(\boldsymbol{\sigma}(r)\big),f\big(\boldsymbol{\sigma}(r)\big)\big\}\\
&=\Sigma_{U,11}(r),\\
\mathrm{Cov}_{\nabla f}\big\{f(\mathbf{n}),f\big(\boldsymbol{\sigma}(r)\big)\big\}&=\Sigma_{U,12}(r),\\
\mathrm{Cov}_{\nabla f}\big\{f(\mathbf{n}),E_iE_jf(\mathbf{n})\big\}&=\mathrm{Cov}_{\nabla f}\big\{f\big(\boldsymbol{\sigma}(r)\big),E_iE_jf\big(\boldsymbol{\sigma}(r)\big)\big\}\\
&=\delta_{ij}\big(b_1(r)+\delta_{i,N-1}b_3(r)\big),\\
\mathrm{Cov}_{\nabla f}\big\{f(\mathbf{n}),E_iE_jf\big(\boldsymbol{\sigma}(r)\big)\big\}&=\mathrm{Cov}_{\nabla f}\big\{f\big(\boldsymbol{\sigma}(r)\big),E_iE_jf(\mathbf{n})\big\}\\
&=\delta_{ij}\big(b_2(r)+\delta_{i,N-1}b_4(r)\big),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Cov}_{\nabla f}\big\{E_iE_jf(\mathbf{n}),E_kE_lf(\mathbf{n})\big\}\\
&\quad=\mathrm{Cov}_{\nabla f}\big\{E_iE_jf\big(\boldsymbol{\sigma}(r)\big),E_kE_lf\big(\boldsymbol{\sigma}(r)\big)\big\}\\
&\quad=\begin{cases}
2\delta_{ik}p(p-1)-pb_1(r)-pb_3(r)(\delta_{i,N-1}+\delta_{k,N-1})\\
\qquad-\delta_{i,N-1}\delta_{k,N-1}a_2(r)\big(1-r^2\big)\\
\qquad\times\big[p(p-1)r^{p-3}\big(pr^2-(p-2)\big)\big]^2,\\
\qquad\qquad\qquad i=j,k=l,\\
p(p-1), & i=k\neq j=l, N-1\notin\{i,j\},\\
\Sigma_{Z,11}(r), & i=k\neq j=l, N-1\in\{i,j\},\\
0, & \text{if } \big|\{i,j,k,l\}\big|\geq 3,
\end{cases}
\end{aligned}\tag{B.5}$$

$$\begin{aligned}
&\mathrm{Cov}_{\nabla f}\big\{E_iE_if(\mathbf{n}),E_jE_jf\big(\boldsymbol{\sigma}(r)\big)\big\}\\
&\quad=\begin{cases}
-pb_2(r)-pb_4(r)(\delta_{i,N-1}+\delta_{j,N-1}), & i\neq j,\\
-pb_2(r)+2p(p-1)r^{p-2}, & i=j\neq N-1,\\
p^4r^p-2p(p-1)\big(p^2-2p+2\big)r^{p-2}\\
\qquad+p(p-1)(p-2)(p-3)r^{p-4}\\
\qquad+a_4(r)p^2r^{2p-6}\big(1-r^2\big)\big(p^2r^2-(p-1)(p-2)\big)^2,\\
 & i=j=N-1,
\end{cases}
\end{aligned}$$

$$\mathrm{Cov}_{\nabla f}\{E_iE_jf(\mathbf{n}),E_iE_jf(\boldsymbol{\sigma}(r))\}$$
$$=\begin{cases} p(p-1)r^{p-2}, & |\{i,j,N-1\}|=3,\\ \Sigma_{Z,12}(r), & |\{i,j,N-1\}|=2, i\neq j,\end{cases}$$
$$\mathrm{Cov}_{\nabla f}\{E_iE_jf(\mathbf{n}),E_kE_lf(\boldsymbol{\sigma}(r))\}=0,\qquad \text{if } |\{i,j,k,l\}|\geq 3.$$

Note that, in particular, this shows that the law of $(f(\mathbf{n}), f(\boldsymbol{\sigma}(r)))$ under the conditioning is as stated in the lemma. Also, from the above it follows that $\Sigma_Z(r)$ is positive definite for any $r\in(-1,1)$.

Let $\mathrm{Cov}_{f,\nabla f}\{X,Y\}$ denote the covariance of two random variables $X$, $Y$ conditional on

$$\nabla f(\mathbf{n})=\nabla f(\boldsymbol{\sigma}(r))=0,\qquad f(\mathbf{n})=u_1,\ f(\boldsymbol{\sigma}(r))=u_2 \tag{B.6}$$

(which is independent of the values $u_i$). Note that

$$\begin{aligned}\mathrm{Cov}_{f,\nabla f}\{X,Y\}=\mathrm{Cov}_{\nabla f}\{X,Y\}&\\ &-(\mathrm{Cov}_{\nabla f}\{X,f(\mathbf{n})\},\mathrm{Cov}_{\nabla f}\{X,f(\boldsymbol{\sigma}(r))\})(\Sigma_U(r))^{-1}\\ &\times\begin{pmatrix}\mathrm{Cov}_{\nabla f}\{X,f(\mathbf{n})\}\\ \mathrm{Cov}_{\nabla f}\{X,f(\boldsymbol{\sigma}(r))\}\end{pmatrix}.\end{aligned}$$

Clearly,

$$(b_1(r),b_2(r))(\Sigma_U(r))^{-1}=-p(1,0),$$
$$(b_2(r),b_1(r))(\Sigma_U(r))^{-1}=-p(0,1).$$

Thus,

$$\begin{aligned}&\mathrm{Cov}_{f,\nabla f}\{E_iE_jf(\mathbf{n}),E_kE_lf(\mathbf{n})\}-\mathrm{Cov}_{\nabla f}\{E_iE_jf(\mathbf{n}),E_kE_lf(\mathbf{n})\}\\ &\quad=\mathrm{Cov}_{f,\nabla f}\{E_iE_jf(\boldsymbol{\sigma}(r)),E_kE_lf(\boldsymbol{\sigma}(r))\}\\ &\qquad-\mathrm{Cov}_{\nabla f}\{E_iE_jf(\boldsymbol{\sigma}(r)),E_kE_lf(\boldsymbol{\sigma}(r))\}\\ &\quad=-\delta_{ij}\delta_{kl}(b_1(r)+\delta_{i,N-1}b_3(r),b_2(r)+\delta_{i,N-1}b_4(r))(\Sigma_U(r))^{-1}\\ &\qquad\times\begin{pmatrix}b_1(r)+\delta_{k,N-1}b_3(r)\\ b_2(r)+\delta_{k,N-1}b_4(r)\end{pmatrix}\\ &\quad=\delta_{ij}\delta_{kl}\cdot p[b_1(r)+(\delta_{i,N-1}+\delta_{k,N-1})b_3(r)]\\ &\qquad-\delta_{ij}\delta_{kl}\delta_{i,N-1}\delta_{k,N-1}(b_3(r),b_4(r))(\Sigma_U(r))^{-1}\\ &\qquad\times\begin{pmatrix}b_3(r)\\ b_4(r)\end{pmatrix},\end{aligned}$$

$$
\begin{aligned}
&\mathrm{Cov}_{f,\nabla f}\{E_iE_jf(\mathbf{n}),E_kE_lf(\boldsymbol{\sigma}(r))\}-\mathrm{Cov}_{\nabla f}\{E_iE_jf(\mathbf{n}),E_kE_lf(\boldsymbol{\sigma}(r))\}\\
&\quad=-\delta_{ij}\delta_{kl}\big(b_1(r)+\delta_{i,N-1}b_3(r),b_2(r)+\delta_{i,N-1}b_4(r)\big)\big(\Sigma_U(r)\big)^{-1}\\
&\qquad\times\begin{pmatrix}b_2(r)+\delta_{k,N-1}b_4(r)\\ b_1(r)+\delta_{k,N-1}b_3(r)\end{pmatrix}\\
&\quad=\delta_{ij}\delta_{kl}\cdot p\big[b_2(r)+(\delta_{i,N-1}+\delta_{k,N-1})b_4(r)\big]\\
&\qquad-\delta_{ij}\delta_{kl}\delta_{i,N-1}\delta_{k,N-1}\big(b_3(r),b_4(r)\big)\big(\Sigma_U(r)\big)^{-1}\begin{pmatrix}b_4(r)\\ b_3(r)\end{pmatrix}.
\end{aligned}
$$

Combining the previous calculations, we arrive at

$$
\begin{aligned}
\mathrm{Cov}_{f,\nabla f}\{E_iE_if(\mathbf{n}),E_jE_jf(\mathbf{n})\}&=\mathrm{Cov}_{f,\nabla f}\{E_iE_if(\boldsymbol{\sigma}(r)),E_jE_jf(\boldsymbol{\sigma}(r))\}\\
&=\begin{cases}0, & i\neq j,\\ 2p(p-1), & i=j\neq N-1,\\ \Sigma_{Q,11}(r), & i=j=N-1,\end{cases}
\end{aligned}
$$

$$
\mathrm{Cov}_{f,\nabla f}\{E_iE_if(\mathbf{n}),E_jE_jf(\boldsymbol{\sigma}(r))\}=\begin{cases}0, & i\neq j,\\ 2p(p-1)r^{p-2}, & i=j\neq N-1,\\ \Sigma_{Q,12}(r), & i=j=N-1.\end{cases}
$$

For the cases of indices that do not appear above, we have

$$
\begin{aligned}
\mathrm{Cov}_{f,\nabla f}\{E_iE_jf(\mathbf{n}),E_kE_lf(\mathbf{n})\}&=\mathrm{Cov}_{\nabla f}\{E_iE_jf(\mathbf{n}),E_kE_lf(\mathbf{n})\},\\
\mathrm{Cov}_{f,\nabla f}\{E_iE_jf(\boldsymbol{\sigma}(r)),E_kE_lf(\boldsymbol{\sigma}(r))\}&=\mathrm{Cov}_{\nabla f}\{E_iE_jf(\boldsymbol{\sigma}(r)),E_kE_lf(\boldsymbol{\sigma}(r))\},\\
\mathrm{Cov}_{f,\nabla f}\{E_iE_jf(\mathbf{n}),E_kE_lf(\boldsymbol{\sigma}(r))\}&=\mathrm{Cov}_{\nabla f}\{E_iE_jf(\mathbf{n}),E_kE_lf(\boldsymbol{\sigma}(r))\}.
\end{aligned}
$$

From the above, it follows that $\Sigma_Q(r)$ is positive semi-definite for any $r\in(-1,1)$.

It is now easy to compare covariances and see that, conditional on (B.6), the law of

$$
\left(\frac{\nabla^2 f(\mathbf{n})-\mathbb{E}\{\nabla^2 f(\mathbf{n})\}}{\sqrt{Np(p-1)}},\frac{\nabla^2 f(\boldsymbol{\sigma}(r))-\mathbb{E}\{\nabla^2 f(\boldsymbol{\sigma}(r))\}}{\sqrt{Np(p-1)}}\right)
$$

is the same as that of

$$
\big(\hat{\mathbf{M}}^{(1)}_{N-1}(r),\hat{\mathbf{M}}^{(2)}_{N-1}(r)\big).
$$

What remains is to show that the conditional expectation of $\nabla^2 f(\mathbf{n})$ and $\nabla^2 f(\boldsymbol{\sigma}(r))$ under (B.6) are equal to

$$
\begin{aligned}
&-pu_1I+m_1(r,u_1,u_2)e_{N-1,N-1}\quad\text{and}\\
&-pu_2I+m_2(r,u_1,u_2)e_{N-1,N-1},
\end{aligned}
\tag{B.7}
$$

respectively. Denoting expectation conditional on (B.6) by $\mathbb{E}^{u_1,u_2}_{f,\nabla f}\{\cdot\}$,

$$\begin{aligned}
&\mathbb{E}^{u_1,u_2}_{f,\nabla f}\{E_i E_j f(\mathbf{n})\} \\
&\quad = \big(\mathrm{Cov}_{\nabla f}\{E_i E_j f(\mathbf{n}), f(\mathbf{n})\}, \mathrm{Cov}_{\nabla f}\{E_i E_j f(\mathbf{n}), f\big(\boldsymbol{\sigma}(r)\big)\}\big)\big(\Sigma_U(r)\big)^{-1} \\
&\qquad \times (u_1, u_2)^T \\
&\quad = \delta_{ij}\big(b_1(r) + \delta_{i,N-1} b_3(r), b_2(r) + \delta_{i,N-1} b_4(r)\big)\big(\Sigma_U(r)\big)^{-1}(u_1, u_2)^T \\
&\quad = -\delta_{ij} p u_1 + \delta_{ij}\delta_{i,N-1}\big(b_3(r), b_4(r)\big)\big(\Sigma_U(r)\big)^{-1}(u_1, u_2)^T.
\end{aligned}$$

Similarly,

$$\mathbb{E}^{u_1,u_2}_{f,\nabla f}\{E_i E_j f\big(\boldsymbol{\sigma}(r)\big)\} = -\delta_{ij} p u_2 + \delta_{ij}\delta_{i,N-1}\big(b_3(r), b_4(r)\big)\big(\Sigma_U(r)\big)^{-1}(u_2, u_1)^T.$$

Which gives the required expectation (B.7). This completes the proof. □

## APPENDIX C: REGULARITY CONDITIONS FOR THE K-R FORMULA

In Section 4, we needed to apply the K-R theorem to "count" pairs of different points $(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in \mathbb{S}^{N-1} \times \mathbb{S}^{N-1}$ at which $\nabla f_N(\boldsymbol{\sigma}) = \nabla f_N(\boldsymbol{\sigma}') = 0$ and $f_N(\boldsymbol{\sigma}), f_N(\boldsymbol{\sigma}') \in \sqrt{N}B$. The variant of the K-R theorem we used is [1], Theorem 12.1.1, which in particular accounts for the case where the parameter space is a (Riemannian) manifold. It requires a long list of technical conditions to be met [conditions (a)–(g) in the statement of the theorem] which we discuss in this section. We start by relating our notation to that of [1], Theorem 12.1.1.

In [1], Theorem 12.1.1, $f(t) = (f^1(t), \dots, f^N(t))$ is a random field on an $N$-dimensional manifold $M$ taking values in $\mathbb{R}^N$, $\nabla f(t) = (E_j f^i(t))_{i,j=1}^N$ is its Jacobian matrix (where $E$ is a fixed orthonormal frame field), and $h(t) = (h^1(t), \dots, h^K(t))$ is an additional random field from $M$ to $\mathbb{R}^K$. Those $f$, $\nabla f$ and $h$ correspond to our $(\nabla f_N(\boldsymbol{\sigma}), \nabla f_N(\boldsymbol{\sigma}'))$, $J(\boldsymbol{\sigma}, \boldsymbol{\sigma}')$, and $(f_N(\boldsymbol{\sigma}), f_N(\boldsymbol{\sigma}'))$, respectively, where $J(\boldsymbol{\sigma}, \boldsymbol{\sigma}')$ is defined as the Jacobian matrix of $(\nabla f_N(\boldsymbol{\sigma}), \nabla f_N(\boldsymbol{\sigma}'))$ with respect to the orthonormal frame field $E$. That is, if $E_i(\boldsymbol{\sigma})$ [resp., $E_j(\boldsymbol{\sigma}')$] is considered as a derivation with respect to the first (resp., second) coordinate of $f_N(\boldsymbol{\sigma}, \boldsymbol{\sigma}')$, then $J(\boldsymbol{\sigma}, \boldsymbol{\sigma}')$ is the block matrix

$$J\big(\boldsymbol{\sigma}, \boldsymbol{\sigma}'\big) \triangleq \big(E_{i'}(\boldsymbol{\sigma}_i) E_{j'}(\boldsymbol{\sigma}_j) f_N\big(\boldsymbol{\sigma}, \boldsymbol{\sigma}'\big)\big)_{i,j=1}^{2N-2} = \begin{pmatrix} \nabla^2 f_N(\boldsymbol{\sigma}) & 0 \\ 0 & \nabla^2 f_N\big(\boldsymbol{\sigma}'\big) \end{pmatrix},$$

where $i' = i \bmod N-1$ and similarly for $j'$, and

$$\boldsymbol{\sigma}_i = \begin{cases} \boldsymbol{\sigma} & \text{if } i < N-1, \\ \boldsymbol{\sigma}' & \text{if } i \geq N-2. \end{cases}$$

The manifold $M$ in our case is $\mathcal{S}_N^2(I_R)$ of (4.9) where $I_R$ is an open interval whose closure is contained in $(-1,1)$.[7] Conditions (a), (f) and (g) of [1], Theorem 12.1.1, regarding the continuity, moduli of continuity and moments of the involved random fields are trivial consequences of the representation (1.1) of the Hamiltonian $H_N(\boldsymbol{\sigma})$, Gaussianity and stationarity. The remaining conditions concern the continuity of certain conditional densities.[8] Below we will prove the following lemma.

LEMMA 32. *For any $r \in (-1,1)$, the Gaussian array*

$$\{\nabla f(\mathbf{n}), \nabla f(\boldsymbol{\sigma}(r)), \nabla^2 f(\mathbf{n}), \nabla^2 f(\boldsymbol{\sigma}(r))\}, \tag{C.1}$$

*is nondegenerate, up to symmetry of the Hessians. That is, if we replace the Hessians in (C.1) by only their on-and-above elements, then the support of the Gaussian density corresponding to (C.1) is $\mathbb{R}^{2+(N-1)(N-2)}$.*

We wish to apply the K-R formula with $\sqrt{N}B$, the target set of $f_N(\boldsymbol{\sigma})$, $f_N(\boldsymbol{\sigma}')$, being equal to an open interval or a finite union of such. Suppose that instead of considering critical points $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'$ with $f_N(\boldsymbol{\sigma}), f_N(\boldsymbol{\sigma}') \in \sqrt{N}B$, we consider critical points such that $f_N(\boldsymbol{\sigma}) + \varepsilon g_N(\boldsymbol{\sigma})$, $f_N(\boldsymbol{\sigma}') + \varepsilon g_N(\boldsymbol{\sigma}') \in \sqrt{N}B$ with $g_N(\boldsymbol{\sigma})$ being a continuous Gaussian field on $\mathbb{S}^{N-1}$ independent of $f_N(\boldsymbol{\sigma})$ such that $(g_N(\boldsymbol{\sigma}), g_N(\boldsymbol{\sigma}'))$ forms a nondegenerate Gaussian vector for any $\boldsymbol{\sigma}' \neq \pm\boldsymbol{\sigma}$. In the latter case with $\varepsilon > 0$, the additional regularity conditions, conditions (b)-(e) can be verified provided that Lemma 32 holds. Then, by letting $\varepsilon \to 0$ we obtain that the K-R formula holds for case $\varepsilon = 0$, which is what we wish to prove. Thus, what remains is to prove the lemma.

**Proof of Lemma 32.** For $r = 0$, the lemma can be verified from the covariance computations of Lemma 30. Fix $r \in (-1,1) \setminus \{0\}$. It will be enough to show that:

(i) $(\nabla f(\mathbf{n}), \nabla f(\boldsymbol{\sigma}(r)))$ is nondegenerate and that conditional on $(\nabla f(\mathbf{n}), \nabla f(\boldsymbol{\sigma}(r))) = 0$, and

(ii) $(\nabla^2 f(\mathbf{n}), \nabla^2 f(\boldsymbol{\sigma}(r)))$ is nondegenerate (in the sense as in the statement of the lemma).

[7] In [1], Theorem 12.1.1, it is required that $M$ is compact but going the proof of the theorem it can be seen that since in our case $M = \mathcal{S}_N^2(I_R)$ has a finite atlas, this requirement can be replaced by requiring conditions (a)–(g) to hold on the closure of $\mathcal{S}_N^2(I_R)$.

[8] Though this is not explicit in the statement of [1], Theorem 12.1.1, from its proof it can be seen that the support of the density of $\nabla f$ [which in our setting is $J(\boldsymbol{\sigma}, \boldsymbol{\sigma}')$] can be any subspace $L \subset \mathbb{R}^{N^2}$ such that is $\det(\nabla f)$ has density whose support is $\mathbb{R}$. For example, in our case $J(\boldsymbol{\sigma}, \boldsymbol{\sigma}')$ has entries which are identically 0.

The first of the two follows directly from the covariance computations of Lemma 30. From Lemma 13, we have that second condition follows if we are able to show that $\Sigma_Z(r)$ is invertible and that

$$\{(m_1(r,u_1,u_2), m_2(r,u_1,u_2)) : u_1, u_2 \in \mathbb{R}\} = \mathbb{R}^2.$$

It can verified that

$$\frac{(\Sigma_{Z,11}(r) \pm \Sigma_{Z,12}(r))(1 \mp r^{p-1})}{p(p-1)} = 1 - r^{2p-4} \pm (p-2)r^{p-1} \mp (p-2)r^{p-3}.$$

If $r \geq 0$ or $p$ is odd, then

$$\varpi(r) \triangleq 1 - r^{2p-4} - (p-2)r^{p-1} + (p-2)r^{p-3} > 0.$$

If $p$ is even, it can be verified that the derivative of $\varpi(r)$ has constant sign on $(-1,0)$, from which it follows, by the fact that $\varpi(0) = 1$ and $\varpi(-1) = 0$, that $\varpi(r) > 0$ for any $r \in (-1,0)$. A similar analysis shows that

$$1 - r^{2p-4} + (p-2)r^{p-1} - (p-2)r^{p-3} > 0.$$

This proves that $\Sigma_Z(r)$ is strictly positive definite for $r \in (-1,1)$.

By definition [see (B.4)],

$$\begin{pmatrix} m_1(r,u_1,u_2) \\ m_2(r,u_1,u_2) \end{pmatrix} = \begin{pmatrix} b_3(r) & b_4(r) \\ b_4(r) & b_3(r) \end{pmatrix} (\Sigma_U(r))^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where we recall that $\Sigma_U(r)$ invertible as shown in Remark 31. Thus, it is enough to show that $b_3(r) \pm b_4(r) \neq 0$ (and, therefore, the matrix above is invertible). From straightforward algebra,

$$b_3(r) \pm b_4(r) = p(p-1)r^{p-2}(1-r^2)\frac{r^{p-2} \pm 1}{1 \mp (r^p - (p-1)r^{p-2}(1-r^2))}.$$

As mentioned in Remark 31, $1 \pm (pr^p - (p-1)r^{p-2}) > 0$ and, therefore, the denominator above is positive. This completes the proof. $\square$

## APPENDIX D: UPPER BOUND ON THE GROUND STATE FROM MOMENTS EQUIVALENCE ON EXPONENTIAL SCALE

In this appendix we show how Theorem 3 can be used to prove that

$$\lim_{N\to\infty} GS^N = -E_0, \quad \text{almost surely.} \tag{D.1}$$

The fact that (D.1) holds was already proved in [5] based on fact that pure models are 1-RSB. The proof below is based on the equivalence of second and first moment squared only on the exponential level—a fact which may be useful when investigating general mixed models which are not known to exhibit 1-RSB.

The Borell-TIS inequality [9, 17] (see also [1], Theorem 2.1.1) gives, for $\varepsilon > 0$,

$$\mathbb{P}\{|GS^N - \mathbb{E}\{GS^N\}| > \varepsilon\} \leq \exp\{-\varepsilon^2 N/2\}. \tag{D.2}$$

From the Borel–Cantelli lemma that in order to prove (D.1), it is sufficient to show that

$$\lim_{N\to\infty} \mathbb{E}\{GS^N\} = -E_0. \tag{D.3}$$

Note that

$$GS^N < u \quad \Longleftrightarrow \quad \mathrm{Crt}_N\big((-\infty, u)\big) \geq 1. \tag{D.4}$$

Thus, by Markov's inequality, Theorem 10, and the definition of $E_0$,

$$\begin{aligned}\limsup_{N\to\infty} \mathbb{P}\{GS^N < -E_0 - \varepsilon\} &= \limsup_{N\to\infty} \mathbb{P}\{\mathrm{Crt}_N\big((-\infty, -E_0 - \varepsilon)\big) \geq 1\} \\ &\leq \lim_{N\to\infty} e^{-NC_\varepsilon} = 0,\end{aligned} \tag{D.5}$$

for any $\varepsilon > 0$, where $C_\varepsilon > 0$ is a constant depending on $\varepsilon$.

Now, assume toward contradiction that, for some $\delta > 0$, $N_k \to \infty$,

$$\liminf_{N\to\infty} \mathbb{E}\{GS^N\} = \lim_{k\to\infty} \mathbb{E}\{GS^{N_k}\} \leq -E_0 - \delta.$$

Then, from (D.2),

$$\lim_{k\to\infty} \mathbb{P}\{GS^{N_k} < -E_0 - \delta/2\} \geq \lim_{k\to\infty} \mathbb{P}\{|GS^{N_k} - \mathbb{E}\{GS^{N_k}\}| \leq \delta/4\} = 1,$$

which contradicts (D.5).

Next, assume toward contradiction that, for some $\delta > 0$, $N_k \to \infty$,

$$\limsup_{N\to\infty} \mathbb{E}\{GS^N\} = \lim_{k\to\infty} \mathbb{E}\{GS^{N_k}\} \geq -E_0 + \delta.$$

Then, from (D.2),

$$\begin{aligned}&\limsup_{k\to\infty} \frac{1}{N_k} \log\big(\mathbb{P}\{GS^{N_k} < -E_0(p) + \delta/2\}\big) \\ &\qquad \leq \lim_{k\to\infty} \frac{1}{N_k} \log\big(\mathbb{P}\{|GS^{N_k} - \mathbb{E}\{GS^{N_k}\}| > \delta/4\}\big) \leq -\delta^2/32.\end{aligned}$$

On the other hand, from the Paley–Zygmund inequality and (D.4),

$$\begin{aligned}&\liminf_{k\to\infty} \frac{1}{N_k} \log\big(\mathbb{P}\{GS^{N_k} < -E_0(p) + \delta/2\}\big) \\ &\qquad = \liminf_{k\to\infty} \frac{1}{N_k} \log\big(\mathbb{P}\{\mathrm{Crt}_{N_k}\big((-\infty, -E_0 + \delta/2)\big) \geq 1\}\big) \\ &\qquad = \liminf_{k\to\infty} \frac{1}{N_k} \log\left(\frac{(\mathbb{E}\{\mathrm{Crt}_{N_k}((-\infty, -E_0(p) + \delta])\})^2}{\mathbb{E}\{(\mathrm{Crt}_{N_k}((-\infty, -E_0(p) + \delta]))^2\}}\right) = 0,\end{aligned}$$

which, of course, contradicts the previous inequality. Hence, (D.3) and, therefore, (D.1) follow.

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