*The Annals of Probability*
2017, Vol. 45, No. 5, 3266–3292
DOI: 10.1214/16-AOP1136


# A CLARK–OCONE FORMULA FOR TEMPORAL POINT PROCESSES AND APPLICATIONS

BY IAN FLINT AND GIOVANNI LUCA TORRISI[1]

*Nanyang Technological University and Istituto per le Applicazioni del Calcolo "Mauro Picone"*

We provide a Clark–Ocone formula for square-integrable functionals of a general temporal point process satisfying only a mild moment condition, generalizing known results on the Poisson space. Some classical applications are given, namely a deviation bound and the construction of a hedging portfolio in a pure-jump market model. As a more modern application, we provide a bound on the total variation distance between two temporal point processes, improving in some sense a recent result in this direction.

**1. Introduction.** Clark–Ocone formulas are powerful results in stochastic analysis with many significant applications such as deviation inequalities and portfolio replication; see, for example, [14, 15, 22, 23, 31]. On the Poisson space, the Clark–Ocone formula provides the following representation of a square-integrable random variable $F$:

$$F = \mathbb{E}[F] + \int_0^1 \mathbb{E}[D_t F \mid \mathcal{F}_t]\big(N(\mathrm{d}t) - \mathrm{d}t\big),$$

where $D_t$ is the finite difference operator [see (4)] and $\{\mathcal{F}_t\}$ is the filtration generated by the homogeneous Poisson process $N$. Similar formulas for square-integrable functionals hold also on the Wiener and Lévy spaces; see, for example, [9]. Various kinds of generalizations have been obtained recently in, for example, [1, 10].

In addition to the well-known applications cited above, Clark–Ocone formulas have been recently used to prove Gaussian and Poisson approximation of random variables; see [19, 24, 25, 28, 29]. In another direction, such formulas have been used to obtain variational representations for the Laplace transform of functionals on the Wiener and Poisson spaces; cf. [32, 33].

The main achievement of this paper is a Clark–Ocone formula for point processes on a finite interval possessing a conditional intensity; see Theorem 4.1. Specifically, we shall consider two different notions of conditional intensity. The

Received October 2015; revised May 2016.

[1]Supported by Grant Singapore MOE Tier 2 Grant MOE2014-T2-2-015.

*MSC2010 subject classifications.* 60G55, 60H07.

*Key words and phrases.* Clark–Ocone formula, point processes, Malliavin calculus, conditional intensity, deviation inequalities, option hedging.

first, denoted by $\{\lambda_t\}$, which we call classical stochastic intensity, is commonly defined for point processes on the line (see, e.g., [2, 5]). Roughly speaking, $\lambda_t(\omega)\,\mathrm{d}t$ is the probability that the point process has a point in the infinitesimal region $\mathrm{d}t$ given that it agrees with the configuration $\omega$ before time $t$. The second, denoted by $\{\pi_t\}$ and called the Papangelou conditional intensity, is defined for general point processes (see, e.g., [6, 18]). Intuitively, $\pi_t(\omega)\,\mathrm{d}t$ is the probability that the point process has a point in the infinitesimal region $\mathrm{d}t$ given that it agrees with the configuration $\omega$ outside of $\mathrm{d}t$.

The proof of the Clark–Ocone formula in Theorem 4.1 is based on the representation theorem for square-integrable martingales, and a use of an integration by parts and an isometry formula to identify the integrand.

We provide three applications of our result. The first is a deviation inequality for functionals of the point process (see Proposition 5.1), which extends to our setting the bound provided in Proposition 3.1 of [31]. The second provides a bound on the total variation distance between the laws of two point processes possessing Papangelou conditional intensities; see Proposition 5.5. Under mild assumptions on the point processes, we improve to some extent the constant appearing in the analogous bound given in Theorem 4 of [26]; see Remark 5.6. The third provides a self-financing strategy for option hedging in a market with one risky asset and dynamics according to a general pure-jump process.

All our results hold for point processes possessing a conditional intensity and such that the variance of the number of points in the interval is finite. If we further assume that the point process is locally stable, as defined in Section 2.4, then we obtain a more explicit deviation bound; see Corollary 5.3 as well as Remark 5.4. For a large class of renewal processes which is particularly suited to our setting, we prove local stability in Proposition 2.10.

In order to not overshadow the main ideas with technicalities, in this paper we stick to the simple case of a 1-dimensional point process on a finite interval. Extensions of the Clark–Ocone formula to a point process on a more general space, together with new applications (e.g., to optimal transport problems) are presently under investigation by the authors and shall be the subject of a future work.

The paper is structured as follows. In Section 2, we give some preliminaries on point processes, proving that, under natural assumptions, a point process on a finite interval has a Papangelou conditional intensity if and only if it has a classical stochastic intensity, while expressing one intensity as a function of the other. In Section 3, we recall some results related to the martingale representation theorem, namely Proposition 3.1 and Proposition 3.2. After these preparations, we give our main result in Section 4. The aforementioned applications are provided in Section 5.

**2. Point processes with conditional intensities.** Let $T > 0$ be a fixed positive constant. All the random quantities considered in this paper are assumed to be defined on the space $\Omega$ of all integer-valued measures $\omega$ on $([0, T], \mathcal{B}([0, T]))$, where $\mathcal{B}([0,T])$ is the Borel $\sigma$-field on $[0,T]$, such that $\omega(\{0\})=0$, $\omega(\{t\})\le 1$ for any $t\in[0,T]$ and $\omega([0,T])<\infty$. We define

$$N(\omega)=\omega, \qquad \omega\in\Omega$$

and for a Borel set $D\in\mathcal{B}([0,T])$ we shall consider the $\sigma$-field

$$\mathcal{F}_D:=\sigma\{N(A):A\in\mathcal{B}(D)\}.$$

For ease of notation, we also set $\mathcal{F}_t:=\mathcal{F}_{[0,t]}$ and $\mathcal{F}_{t^-}:=\mathcal{F}_{[0,t)}$ for $t\in[0,T]$.

We set $\mathcal{F}:=\mathcal{F}_T$, let $\mathbb{P}$ be a probability on $(\Omega,\mathcal{F})$, and consider the canonical probability space $(\Omega,\mathcal{F},\mathbb{P})$. Throughout the paper, we denote by $\mathbb{E}$ the expectation operator with respect to $\mathbb{P}$. For any Borel set $D\in\mathcal{B}([0,T])$, we let $\Omega_D$ denote the space of integer-valued measures $\omega\in\Omega$ such that $\operatorname{supp}(\omega)\subset D$, where $\operatorname{supp}(\omega)$ denotes the support of $\omega$. We define $\mathbb{P}_D$ as the restriction of $\mathbb{P}$ to $(\Omega_D,\mathcal{F}_D)$. The sample measure $L_D$ on $(\Omega_D,\mathcal{F}_D)$ is defined by

$$\int_{\Omega_D} f(\omega)L_D(\mathrm{d}\omega):=f(\mathbf{0})+\sum_{n\ge 1}\frac{1}{n!}\int_{D^n} f\left(\sum_{i=1}^n \varepsilon_{t_i}\right)\mathrm{d}t_1\cdots\mathrm{d}t_n,$$

for any nonnegative measurable function $f:\Omega_D\to[0,\infty)$. Here, $\varepsilon_t$ denotes the Dirac measure at $t\in[0,T]$ and $\mathbf{0}$ denotes the null measure.

A stochastic process $X:[0,T]\times\Omega\to\mathbb{R}$ is called predictable if it is measurable with respect to the predictable $\sigma$-field on $[0,T]\times\Omega$ defined by $\sigma\{(a,b]\times A: a,b\in[0,T], A\in\mathcal{F}_a\}$. We often denote $X$ by $\{X_t\}_{t\in[0,T]}$. For later purposes, we recall that if $X$ is predictable, then for a fixed $t\in[0,T]$ the random variable $X_t$ is measurable with respect to $\mathcal{F}_{t^-}$ (and, therefore, with respect to $\mathcal{F}_t$); see, for example, Lemma A3.3.I, page 425 in [5].

The point process $N$ may be described in terms of its points, that is, the random sequence $T_0:=0<T_1<T_2<\cdots<T_n<\cdots<T_{N([0,T])}$, where $N([0,t])=n$ if and only if $t\in[T_n,T_{n+1})$. Throughout this paper, for a stochastic process $\{X_t\}_{t\in[0,T]}$, we set

$$\int_0^t X_s N(\mathrm{d}s):=\sum_{n\ge 1}X_{T_n}\mathbb{1}_{[0,t]}(T_n), \qquad t\in[0,T].$$

We shall consider two different notions of conditional intensity for point processes: the classical stochastic intensity and the Papangelou conditional intensity.

2.1. *Classical stochastic intensity.* We start by recalling the notion of classical stochastic intensity (see [2, 5]). A nonnegative predictable stochastic process $\{\lambda_t\}_{t\in[0,T]}$ such that $\int_0^T \lambda_t\,\mathrm{d}t<\infty$ $\mathbb{P}$-almost surely is a classical stochastic intensity of $N$ if for any nonnegative and predictable stochastic process $\{X_t\}_{t\in[0,T]}$,

$$\mathbb{E}\left[\int_0^T X_t N(\mathrm{d}t)\right]=\mathbb{E}\left[\int_0^T X_t\lambda_t\,\mathrm{d}t\right].$$

Hereafter, given two stochastic processes $X, Y : [0, T] \times \Omega \to \mathbb{R}$ we say that $X \sim Y$ if $X$ and $Y$ are equal $\lambda_t(\omega)\mathbb{P}(\mathrm{d}\omega)\,\mathrm{d}t$-almost everywhere on $[0, T] \times \Omega$.

For $p \in [1, +\infty)$, we denote by $\mathcal{P}_p(\lambda)$ the family of equivalence classes formed by predictable stochastic processes $X \equiv \{X_t\}_{t\in[0,T]}$ such that

$$\|X\|_{\mathcal{P}_p}^p := \mathbb{E}\left[\int_0^T |X_t|^p \lambda_t \,\mathrm{d}t\right] < \infty$$

and, for ease of notation, we set $\mathcal{P}_{1,2}(\lambda) := \mathcal{P}_1(\lambda) \cap \mathcal{P}_2(\lambda)$. Note that $\|\cdot\|_{\mathcal{P}_p}$ is a norm on $\mathcal{P}_p(\lambda)$.

Under suitable integrability conditions on $X$ and $\lambda$, we shall consider the compensated stochastic integral

$$\delta(X) := \int_0^T X_t\big(N(\mathrm{d}t) - \lambda_t \,\mathrm{d}t\big), \qquad \mathbb{P}\text{-almost surely.}$$

The following proposition is proved in [3] (see Theorem 3 therein) and provides a fundamental isometry formula for point processes with classical stochastic intensity.

PROPOSITION 2.1. *The following relations hold*:

(i) $\mathbb{E}[\delta(X)] = 0$, *for any* $X \in \mathcal{P}_1(\lambda)$;
(ii) $\mathbb{E}[\delta(X)\delta(Y)] = \mathbb{E}[\int_0^T X_t Y_t \lambda_t \,\mathrm{d}t]$, *for any* $X, Y \in \mathcal{P}_{1,2}(\lambda)$.

2.2. *Papangelou conditional intensity.* We now recall the notion of Papangelou conditional intensity (see, e.g., [6, 18]). A nonnegative stochastic process $\{\pi_t\}_{t\in[0,T]}$ is a Papangelou conditional intensity of $N$ if, for any stochastic process $\{X_t\}_{t\in[0,T]}$,

$$\mathbb{E}\left[\int_0^T X_t(N - \varepsilon_t)N(\mathrm{d}t)\right] = \mathbb{E}\left[\int_0^T X_t(N)\pi_t \,\mathrm{d}t\right]. \tag{1}$$

If $N$ has a Papangelou conditional intensity $\{\pi_t\}_{t\in[0,T]}$, then we define its compound Papangelou conditional intensity $\widehat{\pi} : \Omega \times \Omega \to [0, \infty)$ as

$$\widehat{\pi}(\mathbf{0}, \omega) := 1 \quad \text{and} \quad \widehat{\pi}\left(\sum_{i=1}^n \varepsilon_{t_i}, \omega\right) := \pi_{t_1}(\omega) \prod_{i=2}^n \pi_{t_i}(\omega + \varepsilon_{t_1} + \cdots + \varepsilon_{t_{i-1}}),$$

for $\omega \in \Omega$ and $t_1, \ldots, t_n \in [0, T]$.

A stochastic process is called exvisible if it is measurable with respect to the exvisible $\sigma$-field on $[0, T] \times \Omega$ defined by $\sigma\{(a, b] \times A : a, b \in [0, T], A \in \mathcal{F}_{[0,T]\setminus(a,b]}\}$. For later purposes, we note that if $X$ is predictable then $X$ is exvisible. Indeed, for any $a \in [0, T]$, we have $\mathcal{F}_a \equiv \mathcal{F}_{[0,T]\setminus(a,T]}$. If $N$ has a Papangelou conditional intensity $\{\pi_t\}_{t\in[0,T]}$, then for any nonnegative and exvisible process $\{X_t\}_{t\in[0,T]}$

$$\mathbb{E}\left[\int_0^T X_t N(\mathrm{d}t)\right] = \mathbb{E}\left[\int_0^T X_t \pi_t \,\mathrm{d}t\right].$$

Indeed, by Proposition 5.2 in [8], for any nonnegative exvisible process $\{X_t\}_{t\in[0,T]}$, we have $X_t(\omega - \varepsilon_t) = X_t(\omega)$ for any $\omega \in \Omega$ and $t \in \operatorname{supp}(\omega)$.

The next proposition may be found in [17] (see Theorem 3.3 therein) and [20] (see Theorem $2'$ therein).

PROPOSITION 2.2. *For any $D \in \mathcal{B}([0,T])$,*

$$\mathbb{P}\left(0 < \int_{\Omega_D} \widehat{\pi}(\alpha, N_{[0,T]\setminus D}) L_D(\mathrm{d}\alpha) < \infty\right) = 1,$$

*and for any nonnegative random variable $Z : \Omega \to [0,\infty)$ we have*

$$\mathbb{E}[Z] = \int_\Omega \int_{\Omega_D} \left(\int_{\Omega_D} \widehat{\pi}(\beta, \omega) L_D(\mathrm{d}\beta)\right)^{-1} \times Z(\alpha + \omega) \widehat{\pi}(\alpha, \omega) L_D(\mathrm{d}\alpha) \mathbb{P}_{[0,T]\setminus D}(\mathrm{d}\omega). \tag{2}$$

REMARK 2.3. It is worth noting that, for any $D \in \mathcal{B}([0,T])$, we have that $(\int \widehat{\pi}(\beta, \omega|_{[0,T]\setminus D}) L_D(\mathrm{d}\beta))^{-1}$ is a version of $\mathbb{P}(N(D) = 0 \mid N|_{[0,T]\setminus D} = \omega|_{[0,T]\setminus D})$, cf. p. 123 in [17] as well as Remark 2.5(c) in [11]. Consequently, (2) may be rewritten as

$$\mathbb{E}[Z] = \int_{\Omega_{[0,T]\setminus D}} \int_{\Omega_D} Z(\alpha + \omega) \widehat{\pi}(\alpha, \omega) L_D(\mathrm{d}\alpha) \mathbb{P}(\mathrm{d}\omega),$$

which is the presentation adopted in Theorem $2'$ of [20].

As a straightforward consequence of Proposition 2.2, we obtain an explicit predictable projection of a stochastic process under mild integrability conditions (see (2.10) in [11] for a similar result).

PROPOSITION 2.4. *Let $\{X_t\}_{t\in[0,T]}$ be a stochastic process which is either assumed to be nonnegative or satisfying $X_t \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for all $t \in [0,T]$. Then $\mathbb{E}[X_t \mid \mathcal{F}_{t-}](\omega)$ has the version*

$$\left(\int_{\Omega_{[t,T]}} \widehat{\pi}(\beta, \omega|_{[0,t)}) L_{[t,T]}(\mathrm{d}\beta)\right)^{-1} \times \int_{\Omega_{[t,T]}} X_t(\alpha + \omega|_{[0,t)}) \widehat{\pi}(\alpha, \omega|_{[0,t)}) L_{[t,T]}(\mathrm{d}\alpha), \tag{3}$$

*for $t \in [0,T)$ and $\omega \in \Omega$. The version (3) of $\mathbb{E}[X_t \mid \mathcal{F}_{t-}](\omega)$ depends only on $\omega|_{[0,t)}$ and is therefore predictable by Proposition 3.3 in [15]. It is denoted by $p(X)_t(\omega)$.*

PROOF. For $A \in \mathcal{F}_{t^-}$ and $t \in [0, T]$, by Proposition 2.2 we have

$$\mathbb{E}[X_t \mathbb{1}_A] = \int_\Omega \int_{\Omega_{[t,T]}} \left( \int_{\Omega_{[t,T]}} \widehat{\pi}(\beta, \omega) L_{[t,T]}(\mathrm{d}\beta) \right)^{-1} \mathbb{1}_A(\alpha + \omega) X_t(\alpha + \omega) \times \widehat{\pi}(\alpha, \omega) L_{[t,T]}(\mathrm{d}\alpha) \mathbb{P}_{[0,t)}(\mathrm{d}\omega)$$

$$= \int_\Omega \int_{\Omega_{[t,T]}} \left( \int_{\Omega_{[t,T]}} \widehat{\pi}(\beta, \omega) L_{[t,T]}(\mathrm{d}\beta) \right)^{-1} \mathbb{1}_A(\omega) X_t(\alpha + \omega) \times \widehat{\pi}(\alpha, \omega) L_{[t,T]}(\mathrm{d}\alpha) \mathbb{P}_{[0,t)}(\mathrm{d}\omega)$$

$$= \mathbb{E}\left[ \mathbb{1}_A(N) \left( \int_{\Omega_{[t,T]}} \widehat{\pi}(\beta, N|_{[0,t)}) L_{[t,T]}(\mathrm{d}\beta) \right)^{-1} \times \int_{\Omega_{[t,T]}} X_t(\alpha + N|_{[0,t)}) \widehat{\pi}(\alpha, N|_{[0,t)}) L_{[t,T]}(\mathrm{d}\alpha) \right],$$

which concludes the proof. □

We define the discrete Malliavin derivative of a random variable $F : \Omega \to \mathbb{R}$ as

$$D_t F(\omega) := F_t^+(\omega) - F(\omega), \qquad t \in [0, T], \omega \in \Omega, \tag{4}$$

where

$$F_t^+(\omega) := F(\omega + \varepsilon_t) = \begin{cases} F(\omega) & \text{if } t \in \operatorname{supp}(\omega), \\ F(\omega + \varepsilon_t) & \text{if } t \notin \operatorname{supp}(\omega). \end{cases}$$

Under suitable integrability conditions on $X$ and $\pi$, we shall consider the stochastic integral

$$\Delta(X) := \int_0^T X_t \big( N(\mathrm{d}t) - \pi_t \, \mathrm{d}t \big), \qquad \mathbb{P}\text{-almost surely.}$$

In particular, we note that if $X$ is predictable, then as remarked previously, $X_t(\omega - \varepsilon_t) = X_t(\omega)$ and so

$$\Delta(X)(\omega) := \int_0^T X_t(\omega - \varepsilon_t) \big( \omega(\mathrm{d}t) - \pi_t(\omega) \, \mathrm{d}t \big).$$

We conclude this subsection by stating the following lemma, whose proof may be found in [27] (see Corollary 3.1 therein).

LEMMA 2.5. *For any $\{X_t\}_{t \in [0,T]} \in \mathcal{P}_1(p(\pi))$ and any measurable $F : \Omega \to \mathbb{R}$ such that*

$$\mathbb{E}\left[ \int_0^T |F X_t| \pi_t \, \mathrm{d}t \right] < \infty \quad \text{and} \quad \mathbb{E}\left[ \int_0^T |X_t D_t F| \pi_t \, \mathrm{d}t \right] < \infty, \tag{5}$$

*we have*

$$\mathbb{E}\big[ F \Delta(X) \big] = \mathbb{E}\left[ \int_0^T X_t \pi_t D_t F \, \mathrm{d}t \right]. \tag{6}$$

2.3. *Relations between the two notions of conditional intensity.* In this subsection, we assume that $N$ is such that $\mathbb{E}[N([0,T])] < \infty$.

The next lemma characterizes the classical stochastic intensity of a point process $N$ which possesses a Papangelou conditional intensity.

LEMMA 2.6. *If $N$ has a Papangelou conditional intensity $\{\pi_t\}_{t\in[0,T]}$, then $N$ has a classical stochastic intensity $\{p(\pi)_t\}_{t\in[0,T]}$.*

PROOF. Since predictability implies exvisibility, for any nonnegative and predictable stochastic process $\{X_t\}_{t\in[0,T]}$, by Fubini's theorem and standard properties of the conditional expectation, we have

$$\begin{aligned}\mathbb{E}\left[\int_0^T X_t N(\mathrm{d}t)\right] &= \int_0^T \mathbb{E}[X_t\pi_t]\,\mathrm{d}t = \int_0^T \mathbb{E}\big[X_t\mathbb{E}[\pi_t|\mathcal{F}_{t^-}]\big]\,\mathrm{d}t\\ &= \int_0^T \mathbb{E}\big[X_t p(\pi)_t\big]\,\mathrm{d}t.\end{aligned}$$

Additionally, by taking $X_t = 1$ in the previous series of equalities we get that $\int_0^T p(\pi)_t\,\mathrm{d}t < \infty$ $\mathbb{P}$-almost surely, and so $\{p(\pi)_t\}_{t\in[0,T]}$ is a classical stochastic intensity of $N$. $\square$

The next lemma explicits the Papangelou conditional intensity of a point process possessing a classical stochastic intensity.

LEMMA 2.7. *If $N$ has a classical stochastic intensity $\{\lambda_t\}_{t\in[0,T]}$, then $N$ has a Papangelou conditional intensity $\{\pi_t\}_{t\in[0,T]}$ defined by*

$$\begin{aligned}\pi_t(\omega) := \exp\biggl(&\int_0^T \big(\lambda_s(\omega) - \lambda_s(\omega+\varepsilon_t)\big)\,\mathrm{d}s\\ &+ \int_0^T \big(\ln\big(\lambda_s(\omega+\varepsilon_t)\big)(\omega+\varepsilon_t)(\mathrm{d}s) - \ln\big(\lambda_s(\omega)\big)\omega(\mathrm{d}s)\big)\biggr),\end{aligned} \tag{7}$$

*and $\{\lambda_t\}_{t\in[0,T]}$ is a version of $\{p(\pi)_t\}_{t\in[0,T]}$.*

PROOF. Let $\mathbb{P}^*$ be a new probability measure on $(\Omega,\mathcal{F})$ under which $N$ is a homogeneous Poisson process on $[0,T]$ with intensity 1. Since $\mathbb{E}[N([0,T])] < \infty$, by a result in [13] (see also Theorem 19.7, page 315 in [16]) we have that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{P}^*$, and setting $\mathbb{P}_t := \mathbb{P}_{[0,t]}$ and $\mathbb{P}^*_t := \mathbb{P}^*_{[0,t]}$, we have

$$\mathbb{P}_t(\mathrm{d}\omega) = \rho_t(\omega)\mathbb{P}^*_t(\mathrm{d}\omega),$$

where

$$\rho_t(\omega) := \exp\left(\int_0^t \big(1-\lambda_s(\omega)\big)\,\mathrm{d}s + \int_0^t \ln\big(\lambda_s(\omega)\big)\omega(\mathrm{d}s)\right).$$

Since $\{\lambda_t\}_{t\in[0,T]}$ is predictable, by Theorem T12, page 31 in [2] [see formula (4.4)] we have that $\lambda_s(\omega) > 0$ for any $s \in \mathrm{supp}(\omega)$, and $\mathbb{P}$-almost all $\omega \in \Omega$. Therefore, $\rho_t > 0$ $\mathbb{P}$-almost surely and so the probability measures $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent. The claim then follows by, for example, Theorem 1.6, page 41 in [30]. By the uniqueness of the classical stochastic intensity (cf., e.g., T12, page 31 in [2]), the latter claim follows by Lemma 2.6. □

REMARK 2.8. Since the stochastic intensity characterizes the law of a point process (cf. Proposition 7.2.IV., page 233 in [5]), under the foregoing assumption, by Lemma 2.6 we have that the Papangelou conditional intensity also determines the probability structure of the point process uniquely.

We also remark that if $N$ has a classical stochastic intensity then, under the foregoing assumption, the moment formulas in [4] and [8] apply with the Papangelou conditional intensity given by (7).

2.4. *Locally stable point processes and renewal processes.* In this subsection, we begin by recalling the definition of locally stable point processes, a class of processes for which, as already mentioned in the Introduction, we are able to provide a more explicit deviation bound (see Corollary 5.3 and Remark 5.4). We introduce afterwards a large class of locally stable renewal processes.

Let $N$ be a point process on $[0, T]$ and assume that $\mathbb{E}[N([0, T])] < \infty$ and that $N$ has a Papangelou conditional intensity $\{\pi_t\}_{t\in[0,T]}$. We remark that, due to Lemma 2.6 and Lemma 2.7, this is equivalent to assuming that $N$ has a classical stochastic intensity. We say that $N$ is locally stable with dominating function $\beta$ (see, e.g., [18]) if there exists $\beta \in L^1([0, T], \mathcal{B}([0, T]), \mathrm{d}t)$ such that

$$\pi_t \le \beta(t), \qquad t \in [0, T], \mathbb{P}\text{-almost surely.}$$

For classical examples of locally stable point processes whose dynamics are described via a Papangelou conditional intensity, we refer the reader to [18, 30]. If instead the dynamics of $N$ are given by a classical stochastic intensity, then one may try to check the local stability of the point process with the aid of Lemma 2.7. As an illustration, we consider a large class of renewal processes on $[0, T]$ which is particularly suited to the 1-dimensional setting.

Specifically, we say that $N$ is a renewal process on $[0, T]$ with spacing density $f$ if, conditional on $N([0, T]) = n$, the realization $T_0 = 0 < T_1 < \cdots < T_n$ is such that $T_{i+1} - T_i$, $1 \le i \le n - 1$ are independent and identically distributed random variables with density $f$ with respect to the Lebesgue measure on $[0, \infty)$. We start by providing a Papangelou conditional intensity of $N$ as a corollary of Lemma 2.7.

COROLLARY 2.9. *Let $N$ be a renewal process on $[0, T]$ with spacing density $f$ which is continuous on $[0, +\infty)$ and such that $f > 0$ on $(a, C)$ for some $a \in$*

$[0, T]$ *and* $C \in (T, +\infty]$. *Then* $N$ *has a Papangelou conditional intensity*

$$\pi_t = \begin{cases} f(T_i - t) f(t - T_{i-1}) \,/\, f(T_i - T_{i-1}) \\ \quad \text{if } T_{i-1} \le t < T_i, \\ f(t - T_{N([0,T])}) \int_{T-t}^{\infty} f(s)\,\mathrm{d}s \,/\, \int_{T-T_{N([0,T])}}^{\infty} f(s)\,\mathrm{d}s \\ \quad \text{if } T_{N([0,T])} \le t \le T. \end{cases} \tag{8}$$

*In particular, note that the above assumption on* $f$ *covers many significant examples, for example, exponential, gamma, Weibull and Pareto distributions.*

PROOF. It is well known that the classical stochastic intensity of the renewal process is given by

$$\lambda_t(\omega) = \frac{f(t - \omega_t^-)}{\int_{t-\omega_t^-}^{\infty} f(s)\,\mathrm{d}s}, \tag{9}$$

where $t \in [0, T]$, and $\omega_t^- := \max\{\omega_i \in \operatorname{supp}(\omega) : \omega_i < t\}$ with the convention $\max\{\varnothing\} = 0$ (see, e.g., Exercise 7.2.3 in [5]). Note that the classical stochastic intensity is integrable on $[0, T]$ because

$$\lambda_t(\omega) \le \frac{\sup_{s \in [0,T]} f(s)}{\int_T^C f(s)\,\mathrm{d}s}, \tag{10}$$

and that $\{\lambda_t\}_{0 \le t \le T}$ is predictable by Proposition 3.3 in [15] since $\lambda_t(\omega) = \lambda_t(\omega|_{[0,t)})$. Letting $\omega_t^+ := \min\{\omega_i \in \operatorname{supp}(\omega) : \omega_i \ge t\}$ (with the convention $\min\{\varnothing\} = +\infty$), we have

$$\begin{aligned} \pi_t(\omega) &= \exp\biggl(\int_t^{\omega_t^+} \bigl(\lambda_s(\omega) - \lambda_s(\omega + \varepsilon_t)\bigr)\,\mathrm{d}s + \ln\bigl(\lambda_t(\omega + \varepsilon_t)\bigr) \\ &\qquad + \ln\biggl(\frac{\lambda_{\omega_t^+}(\omega + \varepsilon_t)}{\lambda_{\omega_t^+}(\omega)}\biggr)\mathbb{1}_{\{\omega_t^+ < T\}}\biggr) \\ &= \exp\biggl(\int_t^{\omega_t^+} \biggl(\frac{f(s - \omega_t^-)}{\int_{s-\omega_t^-}^{\infty} f(u)\,\mathrm{d}u} - \frac{f(s-t)}{\int_{s-t}^{\infty} f(u)\,\mathrm{d}u}\biggr)\mathrm{d}s + \ln\bigl(\lambda_t(\omega + \varepsilon_t)\bigr) \\ &\qquad + \ln\biggl(\frac{\lambda_{\omega_t^+}(\omega + \varepsilon_t)}{\lambda_{\omega_t^+}(\omega)}\biggr)\mathbb{1}_{\{\omega_t^+ < T\}}\biggr) \\ &= \exp\biggl(\int_{t-\omega_t^-}^{\omega_t^+ - \omega_t^-} \frac{f(u)}{\int_u^{\infty} f(v)\,\mathrm{d}v}\,\mathrm{d}u - \int_0^{\omega_t^+ - t} \frac{f(u)}{\int_u^{\infty} f(v)\,\mathrm{d}v}\,\mathrm{d}u \\ &\qquad + \ln\bigl(\lambda_t(\omega + \varepsilon_t)\bigr) + \ln\biggl(\frac{\lambda_{\omega_t^+}(\omega + \varepsilon_t)}{\lambda_{\omega_t^+}(\omega)}\biggr)\mathbb{1}_{\{\omega_t^+ < T\}}\biggr) \end{aligned} \tag{11}$$

$$= \exp\Big(\ln\Big(\int_{t-\omega_t^-}^{\infty} f(s)\,\mathrm{d}s\Big) - \ln\Big(\int_{\omega_t^+-\omega_t^-}^{\infty} f(s)\,\mathrm{d}s\Big) + \ln\big(\lambda_t(\omega+\varepsilon_t)\big)$$

$$+ \ln\Big(\int_{\omega_t^+-t}^{\infty} f(s)\,\mathrm{d}s\Big) + \ln\Big(\frac{\lambda_{\omega_t^+}(\omega+\varepsilon_t)}{\lambda_{\omega_t^+}(\omega)}\Big)\mathbb{1}_{\{\omega_t^+<T\}}\Big).$$

Here, the first equality follows by Lemma 2.7 noticing that $\lambda_s(\omega) = \lambda_s(\omega+\varepsilon_t)$ for all $s \in [0,t] \cup (\omega_t^+, T]$ and the last equality follows by

$$\int_0^x \frac{f(u)}{\int_u^\infty f(s)\,\mathrm{d}s}\,\mathrm{d}u = -\ln\Big(\int_x^\infty f(s)\,\mathrm{d}s\Big), \qquad x \in [0,\infty).$$

Combining (11) with (9), we conclude the proof. □

Finally, we show that the class of renewal processes considered above is locally stable.

PROPOSITION 2.10. *Let the notation and the assumptions of Corollary 2.9 prevail. Then the renewal process with spacing density $f$ is locally stable.*

PROOF. It suffices to prove that the two terms in (8) are bounded uniformly in $\omega$ by a Lebesgue-integrable function on $[0,T]$; let us thus fix an $\omega \in \Omega$, denote its support by $\omega_0 = 0 < \omega_1 < \cdots < \omega_n < T$ and let $t \in [0,T]$. If $\omega_i \le t < \omega_{i+1}$ for some $0 \le i \le n-1$, then we separate two cases: $f(a) > 0$ and $f(a) = 0$. In the first case, we have $f > 0$ on $[a,T]$, and thus

$$\pi_t(\omega) \le \frac{(\sup_{s\in[a,T]} f(s))^2}{\min_{s\in[a,T]} f(s)} < \infty, \qquad \mathbb{P}\text{-almost every } \omega \in \Omega.$$

Next, if $f(a) = 0$ then since $f > 0$ on $(a,C)$, there exists an $\varepsilon > 0$ such that $f$ is increasing on $[a,\varepsilon]$. It follows that

$$\pi_t(\omega) \le f(t-\omega_i)\mathbb{1}_{\{\omega_{i+1}-\omega_i \le \varepsilon\}} + \frac{(\sup_{s\in[0,T]} f(s))^2}{\min_{s\in[\varepsilon,T]} f(s)}\mathbb{1}_{\{\omega_{i+1}-\omega_i > \varepsilon\}}$$

$$\le \sup_{s\in[0,\varepsilon]} f(s) + \frac{(\sup_{s\in[0,T]} f(s))^2}{\min_{s\in[\varepsilon,T]} f(s)} < \infty.$$

Lastly, if $t \ge \omega_n$,

$$\pi_t(\omega) \le \frac{\sup_{s\in[0,T]} f(s)}{\int_T^C f(s)\,\mathrm{d}s} < \infty,$$

which concludes the proof. □

**3. Predictable representation of square-integrable functionals.** Throughout this section, it is assumed that $N$ has a classical stochastic intensity $\{\lambda_t\}_{t\in[0,T]}$.

By the martingale representation theorem (see, e.g., Theorem T11, page 68 in [2]), we have that any right-continuous $\{\mathcal{F}_t\}_{t\in[0,T]}$-martingale $\{M_t\}_{t\in[0,T]}$ such that $\sup_{t\in[0,T]}\mathbb{E}[M_t^2]<\infty$ admits the representation

$$M_t = M_0 + \int_0^t u_s^{(M)}\big(N(\mathrm{d}s)-\lambda_s\,\mathrm{d}s\big), \qquad \mathbb{P}\text{-almost surely} \tag{12}$$

for some $u^{(M)}\equiv\{u_t^{(M)}\}_{t\in[0,T]}\in\mathcal{P}_2(\lambda)$. Note that $M_0$ is a constant since $\mathcal{F}_0$ is the trivial $\sigma$-field. Note also that if $\widetilde{u}^{(M)}\equiv\{\widetilde{u}_t^{(M)}\}_{t\in[0,T]}\in\mathcal{P}_2(\lambda)$ is such that

$$M_t = M_0 + \int_0^t \widetilde{u}_s^{(M)}\big(N(\mathrm{d}s)-\lambda_s\,\mathrm{d}s\big), \qquad \mathbb{P}\text{-almost surely}$$

then

$$u_t^{(M)}(\omega)=\widetilde{u}_t^{(M)}(\omega), \qquad \mathbb{P}(\mathrm{d}\omega)\lambda_t(\omega)\,\mathrm{d}t\text{-almost everywhere;}$$

see, for example, Theorem T10, pages 67-68 in [2]. In other words, $u^{(M)}\sim\widetilde{u}^{(M)}$ (using the notation introduced in Section 2.1).

For square-integrable functionals, the following predictable representation holds.

PROPOSITION 3.1. *For any $G\in L^2(\Omega,\mathcal{F},\mathbb{P})$, there exists $\{u_t^{(G)}\}_{t\in[0,T]}\in\mathcal{P}_2(\lambda)$ such that*

$$G=\mathbb{E}[G]+\int_0^T u_t^{(G)}\big(N(\mathrm{d}t)-\lambda_t\,\mathrm{d}t\big), \qquad \mathbb{P}\text{-almost surely.} \tag{13}$$

We remark that the stochastic process $\{u_t^{(G)}\}_{t\in[0,T]}\in\mathcal{P}_2(\lambda)$ is not made explicit. In the next section, we shall give an explicit expression for this process under some additional assumptions.

PROOF. Define the martingale $G_t:=\mathbb{E}[G|\mathcal{F}_t]$, $t\in[0,T]$. Since $G$ is square integrable, a simple application of Jensen's inequality ensures that we have $\sup_{t\in[0,T]}\mathbb{E}[G_t^2]<\infty$. Since the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is right-continuous (see, e.g., Theorem T25, page 304 in [2]), the martingale $\{G_t\}_{t\in[0,T]}$ is right-continuous, and by (12) we have

$$G_T=G_0+\int_0^T u_t^{(G)}\big(N(\mathrm{d}t)-\lambda_t\,\mathrm{d}t\big), \qquad \mathbb{P}\text{-almost surely}$$

for some $\{u_t^{(G)}\}_{t\in[0,T]}\in\mathcal{P}_2(\lambda)$. The claim follows noticing that by the $\mathcal{F}$-measurability of $G$ we have $G_T=\mathbb{E}[G|\mathcal{F}_T]=\mathbb{E}[G|\mathcal{F}]=G$ and by $\mathcal{F}_0\equiv\{\varnothing,\Omega\}$ we have $G_0=\mathbb{E}[G|\mathcal{F}_0]=\mathbb{E}[G]$. $\square$

Using the predictable representation (13), it is possible to derive a formula, called hereafter smoothing formula, which allows to transform the expectation of the product between a random variable and an integral with respect to a compensated random point measure into the expectation of an integral with respect to the Lebesgue measure.

We start with some preliminaries. Note that if $\mathbb{E}[N([0,T])] < \infty$, then $\mathcal{P}_2(\lambda) \equiv \mathcal{P}_{1,2}(\lambda)$ and so, for any random variable $G \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, the process $\{u_t^{(G)}\}_{t\in[0,T]}$ in the representation (13) belongs to $\mathcal{P}_{1,2}(\lambda)$. Indeed, for any $\{u_t\}_{t\in[0,T]} \in \mathcal{P}_2(\lambda)$, applying the Cauchy–Schwarz inequality we have

$$\mathbb{E}\left[\int_0^T |u_t|\lambda_t \, \mathrm{d}t\right] \le \mathbb{E}\left[\int_0^T \lambda_t \, \mathrm{d}t\right]^{1/2} \|u\|_{\mathcal{P}_2(\lambda)} = \mathbb{E}\big[N\big([0,T]\big)\big]^{1/2} \|u\|_{\mathcal{P}_2(\lambda)} < \infty,$$

and so $u \in \mathcal{P}_{1,2}(\lambda)$.

The following smoothing formula holds.

PROPOSITION 3.2. *Let $G \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a random variable whose stochastic process $\{u_t^{(G)}\}_{t\in[0,T]}$ in the representation* (13) *belongs to $\mathcal{P}_{1,2}(\lambda)$ and let $\{u_t\}_{t\in[0,T]} \in \mathcal{P}_{1,2}(\lambda)$. Then*

$$\mathbb{E}\big[G\delta(u)\big] = \mathbb{E}\left[\int_0^T u_t^{(G)} u_t \lambda_t \, \mathrm{d}t\right]. \tag{14}$$

PROOF. By Proposition 2.1(i), we have $\mathbb{E}[\delta(u)] = 0$. By Proposition 3.1, we have $G - \mathbb{E}[G] = \delta(u^{(G)})$, for some $\{u_t^{(G)}\}_{t\in[0,T]} \in \mathcal{P}_{1,2}(\lambda)$. Therefore,

$$\mathbb{E}\big[G\delta(u)\big] = \mathbb{E}\big[\big(G - \mathbb{E}[G]\big)\delta(u)\big] = \mathbb{E}\big[\delta\big(u^{(G)}\big)\delta(u)\big] = \mathbb{E}\left[\int_0^T u_t^{(G)} u_t \lambda_t \, \mathrm{d}t\right],$$

where the latter equality follows by Proposition 2.1(ii). □

**4. Clark–Ocone formula.** Throughout this section, we assume that $\mathbb{E}[N([0,T])^2] < \infty$. Moreover, we assume that $N$ has a Papangelou conditional intensity which, due to Lemma 2.6 and Lemma 2.7, is equivalent to assuming that it has a classical stochastic intensity. Theorem 4.1 below provides a Clark–Ocone formula. Note that, in contrast with the predictable representation (13), the integrand appearing in (15) is explicit [and given by (16)]. Hereafter, we work under the convention $0/0 := 0$.

THEOREM 4.1. *Let $G$ be a random variable which is either*:

(i) *nonnegative and in $L^1(\Omega, \mathcal{F}, \mathbb{P})$; or*
(ii) *in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.*

*Then*

$$G = \mathbb{E}[G] + \int_0^T \varphi_t^{(G)}\big(N(\mathrm{d}t) - p(\pi)_t\,\mathrm{d}t\big), \qquad \mathbb{P}\text{-almost surely,} \tag{15}$$

*where*

$$\varphi_t^{(G)} := \frac{p(\pi G^+)_t - p(G)_t p(\pi)_t}{p(\pi)_t}, \tag{16}$$

*for* $t \in [0, T]$.

REMARK 4.2. It should be noticed that under either (i) or (ii), $p(G)_t$ is well defined by Proposition 2.4. Additionally, under (ii) we have $\pi_t G_t^+ \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for $\mathrm{d}t$-almost all $t \in [0, T]$ since

$$\begin{aligned}\int_0^T \mathbb{E}\big[|\pi_t G_t^+|\big]\,\mathrm{d}t &= \mathbb{E}\bigg[\int_0^T |G| N(\mathrm{d}t)\bigg] \\ &\le \|G\|_{L^2(\Omega,\mathcal{F},\mathbb{P})} \big\|N\big([0,T]\big)\big\|_{L^2(\Omega,\mathcal{F},\mathbb{P})} < \infty,\end{aligned} \tag{17}$$

where the first equality follows from (1). This implies that $p(\pi G^+)_t$ is also well defined. We also note that under (i) or (ii) the difference $p(\pi G^+)_t - p(G)_t p(\pi)_t$ is well defined since the second term is actually finite $\mathrm{d}t\,\mathrm{d}\mathbb{P}$-almost everywhere [which can be seen, e.g., by checking that $p(G)_t$ and $p(\pi)_t$ are integrable]. Note also that by the Cauchy–Schwarz inequality,

$$p\big(\pi G^+\big)_t^2 \le p(\pi)_t\, p\big(\pi \big(G^+\big)^2\big)_t, \qquad t \in [0, T], \mathbb{P}\text{-almost surely,}$$

and so if $p(\pi)_t = 0$ then, by the convention $0/0 = 0$, $\varphi_t^{(G)} = 0$. Thus, $\varphi^{(G)}$ is welldefined by (16).

REMARK 4.3. For later purposes, in the following we provide sufficient conditions to ensure that $\varphi^{(G)} \in \mathcal{P}_{1,2}(p(\pi)) \equiv \mathcal{P}_2(p(\pi))$. More precisely:

(i) If $G \in L^4(\Omega, \mathcal{F}, \mathbb{P})$ and $N$ is locally stable with dominating function $\beta$, as defined in Section 2.4, then $\varphi^{(G)} \in \mathcal{P}_{1,2}(p(\pi))$. Indeed,

$$\begin{aligned}\mathbb{E}\bigg[\int_0^T |\varphi_t^{(G)}|^2 p(\pi)_t\,\mathrm{d}t\bigg] &= \mathbb{E}\bigg[\int_0^T \frac{(p(\pi G^+)_t - p(\pi)_t p(G)_t)^2}{p(\pi)_t}\,\mathrm{d}t\bigg] \\ &\le 2\mathbb{E}\bigg[\int_0^T \frac{p(\pi G^+)_t^2}{p(\pi)_t}\,\mathrm{d}t\bigg] \\ &\quad + 2\mathbb{E}\bigg[\int_0^T p(\pi)_t p(G)_t^2\,\mathrm{d}t\bigg].\end{aligned} \tag{18}$$

The first term in (18) is bounded, indeed

$$\mathbb{E}\left[\int_0^T \frac{p(\pi G^+)_t^2}{p(\pi)_t}\,\mathrm{d}t\right] \le \mathbb{E}\left[\int_0^T \mathbb{E}[\pi_t (G_t^+)^2 \mid \mathcal{F}_{t-}]\,\mathrm{d}t\right] \tag{19}$$

$$= \mathbb{E}\big[G^2 N([0,T])\big] \tag{20}$$

$$\le \|G\|_{L^4(\Omega,\mathcal{F},\mathbb{P})} \big\|N([0,T])\big\|_{L^2(\Omega,\mathcal{F},\mathbb{P})}, \tag{21}$$

where (19) and (21) follow by the Cauchy–Schwarz inequality and (20) follows by (1). The second term in (18) is also finite since

$$\mathbb{E}\left[\int_0^T p(\pi)_t p(G)_t^2\,\mathrm{d}t\right] \le \mathbb{E}\left[\int_0^T \beta(t)\mathbb{E}[G^2 \mid \mathcal{F}_{t-}]\,\mathrm{d}t\right] = \mathbb{E}[G^2]\int_0^T \beta(t)\,\mathrm{d}t.$$

(ii) If $G \in L^\infty(\Omega,\mathcal{F},\mathbb{P})$ then $\varphi^{(G)} \in \mathcal{P}_2(p(\pi))$, cf. part 1 of the proof of Theorem 4.1.

REMARK 4.4. Under either (i) or (ii) from Remark 4.3, by the considerations preceding Proposition 3.1 we have that $\varphi^{(G)}$ given by (16) is unique in the sense that any $u^{(G)} \in \mathcal{P}_2(p(\pi))$ which satisfies a representation formula for $G$ of the type (15) verifies $\varphi^{(G)} \sim u^{(G)}$ (where the equivalence relation is defined in Section 2.1).

PROOF OF THEOREM 4.1. We divide the proof in two steps: in the first step, we prove (15) under the stronger condition $G \in L^\infty(\Omega,\mathcal{F},\mathbb{P})$, and in the second step we prove the general case.

1. Assume that $G \in L^\infty(\Omega,\mathcal{F},\mathbb{P})$. By Lemma 2.6, $\{p(\pi)_t\}_{t\in[0,T]}$ is a classical stochastic intensity of $N$. Since $\mathbb{E}[N([0,T])] < \infty$, we have $\mathcal{P}_2(p(\pi)) \equiv \mathcal{P}_{1,2}(p(\pi))$ and so, for any random variable $Z \in L^2(\Omega,\mathcal{F},\mathbb{P})$, the stochastic process $\{u_t^{(Z)}\}_{t\in[0,T]}$ in the representation (13) belongs to $\mathcal{P}_{1,2}(\lambda)$. Let $\{u_t\}_{t\in[0,T]}$ be a predictable stochastic process and, for any $n \ge 0$, define $u^{(n)} \equiv \{u_t^{(n)}\}_{t\in[0,T]}$ by $u_t^{(n)} := u_t \mathbb{1}_{\{u_t\in[-n,n]\}}$. We have $u^{(n)} \in \mathcal{P}_1(p(\pi))$, for any $n \ge 0$, and so by Lemma 2.5, Proposition 3.2 and Proposition 2.1

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T u_t^{(n)} \pi_t D_t G\,\mathrm{d}t\right]\\
&\quad = \mathbb{E}\big[G\Delta\big(u^{(n)}\big)\big]\\
&\quad = \mathbb{E}\left[G\int_0^T u_t^{(n)}\big(N(\mathrm{d}t) - p(\pi)_t\,\mathrm{d}t\big)\right] + \mathbb{E}\left[G\int_0^T u_t^{(n)}\big(p(\pi)_t - \pi_t\big)\,\mathrm{d}t\right] \qquad (22)\\
&\quad = \mathbb{E}\left[\int_0^T u_t^{(n)} u_t^{(G)} p(\pi)_t\,\mathrm{d}t\right]\\
&\qquad + \mathbb{E}\left[G\int_0^T u_t^{(n)}\big(p(\pi)_t - \pi_t\big)\,\mathrm{d}t\right] \qquad \forall n \ge 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}0 &= \mathbb{E}\left[\int_0^T u_t^{(n)}\left(\pi_t D_t G - u_t^{(G)} p(\pi)_t - G p(\pi)_t + G\pi_t\right) \mathrm{d}t\right] \\ &= \mathbb{E}\left[\int_0^T u_t^{(n)}\left(\mathbb{E}[\pi_t G_t^{+} | \mathcal{F}_{t-}] - u_t^{(G)} p(\pi)_t - \mathbb{E}[G|\mathcal{F}_{t-}] p(\pi)_t\right) \mathrm{d}t\right] \qquad \forall n \geq 0.\end{aligned}$$

As discussed in Remark 4.2, there exist the predictable projections $p(\pi G^+)$ and $p(G)$. Consequently,

$$\mathbb{E}\left[\int_0^T u_t^{(n)}\left(p\left(\pi G^+\right)_t - u_t^{(G)} p(\pi)_t - p(G)_t p(\pi)_t\right) \mathrm{d}t\right] = 0 \qquad \forall n \geq 0. \tag{23}$$

The stochastic process

$$v \equiv \{v_t\}_{t\in[0,T]} \equiv \left\{p(\pi G^+)_t - u_t^{(G)} p(\pi)_t - p(G)_t p(\pi)_t\right\}_{t\in[0,T]}$$

is clearly predictable. Since $u \equiv \{u_t\}_{t\in[0,T]}$ is an arbitrary predictable stochastic process, choosing $u = v$, equation (23) reads as

$$\mathbb{E}\left[\int_0^T v_t^2 \mathbb{1}_{\{v_t\in[-n,n]\}} \,\mathrm{d}t\right] = 0 \qquad \forall n \geq 0.$$

By the monotone convergence theorem, letting $n$ tend to infinity, we deduce $\mathbb{E}[\int_0^T v_t^2 \,\mathrm{d}t] = 0$ which implies

$$u_t^{(G)} p(\pi)_t = p\left(\pi G^+\right)_t - p(G)_t p(\pi)_t, \qquad \mathrm{d}t\mathbb{P}(\mathrm{d}\omega)\text{-almost everywhere,}$$

and thus

$$u_t^{(G)} = \varphi_t^{(G)}, \qquad p(\pi)_t \,\mathrm{d}t\mathbb{P}(\mathrm{d}\omega)\text{-almost everywhere.}$$

2. Now let $G$ be a random variable verifying the assumptions of the theorem. For a positive integer $m > 0$, set $G^{(m)} := \sup(\inf(G, m), -m)$, and for $t \in [0, T]$ define

$$\varphi_t^m := \frac{p(\pi(G^{(m)})^+)_t - p(G^{(m)})_t p(\pi)_t}{p(\pi)_t}.$$

By the previous step,

$$G^{(m)} = \mathbb{E}\left[G^{(m)}\right] + \int_0^T \varphi_t^m\left(N(\mathrm{d}t) - p(\pi)_t \,\mathrm{d}t\right), \qquad \mathbb{P}\text{-almost surely.} \tag{24}$$

Note that $G^{(m)}$ converges to $G$ $\mathbb{P}$-almost surely and that, by the dominated convergence theorem, $\mathbb{E}[G^{(m)}]$ converges to $\mathbb{E}[G]$ as $m$ goes to infinity. So (15) follows from (24) if we show that

$$\begin{aligned}&\int_0^T \varphi_t^m\left(N(\mathrm{d}t) - p(\pi)_t \,\mathrm{d}t\right) \\ &\qquad \xrightarrow[m\to\infty]{} \int_0^T \varphi_t^{(G)}\left(N(\mathrm{d}t) - p(\pi)_t \,\mathrm{d}t\right), \qquad \mathbb{P}\text{-almost surely.}\end{aligned} \tag{25}$$

We begin by proving that

$$\varphi_t^m \xrightarrow[m\to\infty]{} \varphi_t^{(G)}, \qquad \mathrm{d}t\,\mathrm{d}\mathbb{P}\text{-almost surely.} \tag{26}$$

For all $t \in [0, T]$, by Proposition 2.4 and the dominated convergence theorem, we have $p(G^{(m)})_t \xrightarrow[m\to\infty]{} p(G)_t$ $\mathbb{P}$-almost surely. Indeed, $|G^{(m)}(\omega)| \le |G(\omega)|$ for all $\omega \in \Omega$ and

$$\left(\int \widehat{\pi}(\beta, N|_{[0,t)})L_{[t,T]}(\mathrm{d}\beta)\right)^{-1} \int \big|G(\alpha \cup N|_{[0,t)})\big|\widehat{\pi}(\alpha, N|_{[0,t)})L_{[t,T]}(\mathrm{d}\alpha) < \infty,$$

$\mathbb{P}$-almost surely, since its mean equals $\mathbb{E}[|G|]$ which is finite. In order to prove (26), it remains to show

$$p\big(\pi\big(G^{(m)}\big)^+\big)_t \xrightarrow[m\to\infty]{} p\big(\pi G^+\big)_t, \qquad \mathrm{d}t\,\mathrm{d}\mathbb{P}\text{-almost surely.} \tag{27}$$

Under (i), (27) follows by Proposition 2.4 and the monotone convergence theorem. Under (ii), (27) follows by the dominated convergence theorem. Indeed, $\pi_t(\omega)|(G_t^{(m)})^+(\omega)| \le \pi_t(\omega)|G^+(\omega)|$ for all $\omega \in \Omega$, $t \in [0, T]$, and

$$\begin{aligned}\int_0^T \mathbb{E}\bigg[\int \bigg(\int \widehat{\pi}(\beta, N|_{[0,t)})L_{[t,T]}(\mathrm{d}\beta)\bigg)^{-1} \pi_t(\alpha \cup N|_{[0,t)})& \\ \times \big|G_t^+(\alpha \cup N|_{[0,t)})\big|\widehat{\pi}(\alpha, N|_{[0,t)})L_{[t,T]}(\mathrm{d}\alpha)\bigg]\mathrm{d}t &= \int_0^T \mathbb{E}\big[\pi_t\big|G_t^+\big|\big]\,\mathrm{d}t < \infty,\end{aligned}$$

where the inequality follows from (17). We conclude the proof of (25) by a proper application of the dominated convergence theorem. For $t \in [0, T]$, let us set

$$\overline{\varphi}_t := \frac{p(\pi|G^+| + p(\pi)|G|)_t}{p(\pi)_t}.$$

Letting $\{Y_t\}_{t\in[0,T]}$ be any nonnegative predictable process, we have

$$\begin{aligned}\mathbb{E}\bigg[\int_0^T Y_t\overline{\varphi}_t p(\pi)_t\,\mathrm{d}t\bigg] &= \mathbb{E}\bigg[\int_0^T Y_t\big(\pi_t\big|G_t^+\big| + p(\pi)_t|G|\big)\,\mathrm{d}t\bigg] \\ &= \mathbb{E}\bigg[\int_0^T Y_t|G|\big(N(\mathrm{d}t) + p(\pi)_t\,\mathrm{d}t\big)\bigg],\end{aligned} \tag{28}$$

where the last equality follows by (1), the exvisibility of $\{Y_t\}_{t\in[0,T]}$ and Proposition 5.2 in [8]. For $n > 0$, we define

$$\zeta_n := \inf\bigg\{s \in [0, T] : \int_0^s \big(N(\mathrm{d}t) + p(\pi)_t\,\mathrm{d}t\big) \ge n - 1\bigg\},$$

and take $Y_t := \mathbb{1}_{[0,\zeta_n]}(t)$ thereby obtaining by (28) the inequality

$$\mathbb{E}\bigg[\int_0^{\zeta_n} \overline{\varphi}_t p(\pi)_t\,\mathrm{d}t\bigg] \le n\mathbb{E}[|G|].$$

Additionally, since $\overline{\varphi}_t \mathbb{1}_{[0,\zeta_n]}(t)$ is predictable and, by Lemma 2.6, $\{p(\pi)_t\}_{t\in[0,T]}$ is a classical stochastic intensity of $N$, we have

$$\mathbb{E}\left[\int_0^{\zeta_n} \overline{\varphi}_t \big(N(\mathrm{d}t) + p(\pi)_t \,\mathrm{d}t\big)\right] = 2\mathbb{E}\left[\int_0^{\zeta_n} \overline{\varphi}_t \, p(\pi)_t \,\mathrm{d}t\right] \le 2n\mathbb{E}\big[|G|\big] < \infty.$$

Hence, for all $n > 0$, setting

$$A_n := \left\{\int_0^{\zeta_n} \overline{\varphi}_t \big(N(\mathrm{d}t) + p(\pi)_t \,\mathrm{d}t\big) < \infty\right\},$$

we have $\mathbb{P}(A_n) = 1$. Moreover,

$$\int_0^T \big(N(\mathrm{d}t) + p(\pi)_t \,\mathrm{d}t\big) < \infty, \qquad \mathbb{P}\text{-almost surely,}$$

thus $\zeta_n = T$ for $n$ sufficiently large, $\mathbb{P}$-almost surely. Consequently,

$$\mathbb{P}\left(\int_0^T \overline{\varphi}_t \big(N(\mathrm{d}t) + p(\pi)_t \,\mathrm{d}t\big) < \infty\right) = \mathbb{P}\left(\bigcap_{n>0} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n) = 1,$$

where the second equality follows since the sequence $\{A_n\}_{n>0}$ is nonincreasing. To conclude the proof by the dominated convergence theorem, it suffices to notice that $\mathrm{d}t\,\mathrm{d}\mathbb{P}$-almost surely, we have

$$|\varphi_t^m| \le \frac{\mathbb{E}[\pi_t |G_t^+| \mid \mathcal{F}_{t-}] + \mathbb{E}[|G| \mid \mathcal{F}_{t-}] p(\pi)_t}{p(\pi)_t} = \overline{\varphi}_t. \qquad \square$$

We now give a bound on $\varphi^{(G)}$, which will be important for some of our applications.

LEMMA 4.5. *For any* $G \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$,

$$\|\varphi_t^{(G)}\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})} \le 2\|G\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})} \qquad \forall t \in [0, T].$$

*If* $G$ *is further assumed to be nonnegative, then*

$$\|\varphi_t^{(G)}\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})} \le \|G\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})} \qquad \forall t \in [0, T].$$

PROOF. Note that for $\mathbb{P}$-almost all $\omega \in \Omega$ and any $t \in [0, T]$, we have

$$\left|\frac{p(\pi G^+)_t - p(G)_t p(\pi)_t}{p(\pi)_t}\right| = \left|\frac{\mathbb{E}[G_t^+ \pi_t \mid \mathcal{F}_{t-}]}{\mathbb{E}[\pi_t \mid \mathcal{F}_{t-}]} - \mathbb{E}[G \mid \mathcal{F}_{t-}]\right| \le 2\|G\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})}.$$

Note also that in the latter inequality one can take $\|G\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})}$ in place of $2\|G\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})}$ if $G \ge 0$. $\square$

We conclude this section with a remark on two different extensions of the classical integration by parts formula on the Poisson space; cf. [21].

REMARK 4.6. Suppose that $N$ has a classical stochastic intensity $\{\lambda_t\}_{t\in[0,T]}$. Then, under the assumptions of Lemma 2.5, the integration by parts formula (6) holds with $\{\pi_t\}_{t\in[0,T]}$ defined as in (7). Also, under the assumptions (i) or (ii) of Remark 4.3, the smoothing formula (14) holds with $\varphi^{(G)}$ in place of $u^{(G)}$. Note that these formulas are not contained in [7] where a different gradient is considered.

**5. Applications.** Throughout this section, we assume that $\mathbb{E}[N([0,T])]<\infty$ and that $N$ has a Papangelou conditional intensity $\{\pi_t\}_{t\in[0,T]}$ which, due to Lemma 2.6 and Lemma 2.7, is equivalent to assuming that it has a classical stochastic intensity.

5.1. *Deviation bound.* The following deviation bound holds.

PROPOSITION 5.1. *Assume $\mathbb{E}[N([0,T])^2]<\infty$. Let $G$ be a random variable satisfying the assumptions of Theorem 4.1 and such that there exists $k>0$ verifying*

$$\|\varphi^{(G)}\|_{L^\infty(\Omega\times[0,T],\mathcal{F}\otimes\mathcal{B}([0,T]),\mathrm{d}\mathbb{P}\,\mathrm{d}t)}\le k, \tag{29}$$

*and*

$$\alpha^2:=\left\|\int_0^T\frac{|p(\pi G^+)_t-p(G)_tp(\pi)_t|^2}{p(\pi)_t}\,\mathrm{d}t\right\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})}<\infty, \tag{30}$$

*then*

$$\begin{aligned}\mathbb{P}(G-\mathbb{E}[G]\ge x)&\le\exp\left(\frac{x}{k}-\left(\frac{\alpha^2}{k^2}+\frac{x}{k}\right)\log\left(1+\frac{xk}{\alpha^2}\right)\right)\\&\le\exp\left(-\frac{x}{2k}\log\left(1+\frac{xk}{\alpha^2}\right)\right),\end{aligned} \tag{31}$$

*for any $x>0$.*

The proof of Proposition 5.1 is based on the following lemma.

LEMMA 5.2. *Under the assumptions of Proposition 5.1, for all convex functions $\phi:\mathbb{R}\to\mathbb{R}$ such that $\phi'$ is convex,*

$$\mathbb{E}[\phi(G-\mathbb{E}[G])]\le\mathbb{E}[\phi(k\widetilde{\mathrm{Po}}(\alpha^2/k^2))], \tag{32}$$

*where $k$ is the constant in (29), $\alpha^2$ is defined by (30) and $\widetilde{\mathrm{Po}}(\alpha^2/k^2)$ is a centered Poisson distributed random variable with parameter $\alpha^2/k^2$.*

PROOF OF LEMMA 5.2. The claim easily follows combining Theorem 4.1 with Theorem 4.1 in [14] [formula (4.3)]. $\square$

PROOF OF PROPOSITION 5.1. Taking $\phi(u) := \mathrm{e}^{\lambda u}$, $\lambda > 0$, in (32), for any $G$ as in the statement and $x > 0$, we deduce

$$
\begin{aligned}
\mathbb{P}\big(G - \mathbb{E}[G] \geq x\big) &\leq \inf_{\lambda>0} \mathbb{E}\big[\mathrm{e}^{\lambda(G-\mathbb{E}[G]-x)} \mathbb{1}_{\{G-\mathbb{E}[G]\geq x\}}\big] \\
&\leq \inf_{\lambda>0} \mathbb{E}\big[\mathrm{e}^{\lambda(G-\mathbb{E}[G]-x)}\big] \\
&\leq \inf_{\lambda>0} \mathbb{E}\big[\mathrm{e}^{\lambda(k\widetilde{\mathrm{Po}}(\alpha^2/k^2)-x)}\big] \\
&= \inf_{\lambda>0} \exp\biggl(\frac{\alpha^2}{k^2}(\mathrm{e}^{\lambda k} - \lambda k - 1) - \lambda x\biggr).
\end{aligned} \tag{33}
$$

Let $C > 0$ be a positive constant. One may easily see that the function $\lambda \mapsto \frac{C}{k^2}(\mathrm{e}^{\lambda k} - \lambda k - 1) - \lambda x$ attains its minimum at $\lambda_0(C) := k^{-1}\log(1 + C^{-1}kx)$ and so

$$
\inf_{\lambda>0} \exp\biggl(\frac{\alpha^2}{k^2}(\mathrm{e}^{\lambda k} - \lambda k - 1) - \lambda x\biggr) = \exp\biggl(\frac{x}{k} - \biggl(\frac{x}{k} + \frac{\alpha^2}{k^2}\biggr)\log\big(1 + \alpha^{-2}kx\big)\biggr).
$$

The proof is complete. $\square$

The next corollary provides a further deviation bound for functionals of locally stable point processes. In Remark 5.4, which follows the corollary, we provide a worse but more explicit deviation bound.

COROLLARY 5.3. *Assume that $N$ is locally stable with dominating function $\beta$. Then, for all $G$ satisfying the assumptions of Proposition 5.1, inequality (31) holds with $\alpha^2$ replaced by*

$$
\int_0^T \beta(t) \big\| \mathbb{E}\big[\big|G_t^+ - \mathbb{E}[G \mid \mathcal{F}_{t-}]\big|^2 \mid \mathcal{F}_{t-}\big] \big\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})} \,\mathrm{d}t. \tag{34}
$$

PROOF. We have, for example, by [8],

$$
\begin{aligned}
\big\|N\big([0, T]\big)\big\|^2_{L^2(\Omega,\mathcal{F},\mathbb{P})} &= \mathbb{E}\biggl[\int_0^T \pi_t(N)\,\mathrm{d}t\biggr] + \mathbb{E}\biggl[\int_0^T \int_0^T \widehat{\pi}\big(\{s, t\}, N\big)\,\mathrm{d}s\,\mathrm{d}t\biggr] \\
&\leq \int_0^T \beta(t)\,\mathrm{d}t + \biggl(\int_0^T \beta(t)\,\mathrm{d}t\biggr)^2 < \infty,
\end{aligned}
$$

and so, by Proposition 5.1, inequality (31) holds. Therefore, since the right-hand side of (31) is nondecreasing in $\alpha^2$, the claim follows if we check that $\alpha^2$ is less than or equal to the term in (34). To this aim, we note that

$$
\begin{aligned}
&\big|\mathbb{E}\big[\pi_t G_t^+ \mid \mathcal{F}_{t-}\big] - \mathbb{E}[\pi_t \mid \mathcal{F}_{t-}]\mathbb{E}[G \mid \mathcal{F}_{t-}]\big| \\
&\quad \leq \mathbb{E}\big[\pi_t\big|G_t^+ - \mathbb{E}[G \mid \mathcal{F}_{t-}]\big| \mid \mathcal{F}_{t-}\big] \\
&\quad \leq \mathbb{E}[\pi_t \mid \mathcal{F}_{t-}]^{1/2}\mathbb{E}\big[\pi_t\big|G_t^+ - \mathbb{E}[G \mid \mathcal{F}_{t-}]\big|^2 \mid \mathcal{F}_{t-}\big]^{1/2},
\end{aligned}
$$

where the last relation follows by the Cauchy–Schwarz inequality. Therefore, by the local stability we have

$$\alpha^2 \leq \left\| \int_0^T \mathbb{E}\left[\pi_t \left|G_t^+ - \mathbb{E}[G \mid \mathcal{F}_{t^-}]\right|^2 \mid \mathcal{F}_{t^-}\right] \mathrm{d}t \right\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})}$$
$$\leq \int_0^T \beta(t) \left\| \mathbb{E}\left[|G_t^+ - \mathbb{E}[G \mid \mathcal{F}_{t^-}]|^2 \mid \mathcal{F}_{t^-}\right] \right\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})} \mathrm{d}t,$$

which concludes the proof. $\square$

REMARK 5.4. Let $G \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Then by Lemma 4.5, one may choose

$$k = \begin{cases} \|G\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})} & \text{if } G \geq 0, \\ 2\|G\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})} & \text{otherwise,} \end{cases}$$

in Proposition 5.1. Moreover, under the assumptions of Corollary 5.3 we have that the term in (34) is less than or equal to $k^2 \|\beta\|_{L^1([0,T],\mathcal{B}([0,T]),\mathrm{d}t)}$. Thus,

$$\mathbb{P}\big(G - \mathbb{E}[G] \geq x\big) \leq \exp\left(\frac{x}{k} - \left(\|\beta\|_{L^1([0,T],\mathcal{B}([0,T]),\mathrm{d}t)} + \frac{x}{k}\right)\right.$$
$$\left.\times \log\left(1 + \frac{x}{k\|\beta\|_{L^1([0,T],\mathcal{B}([0,T]),\mathrm{d}t)}}\right)\right),$$

for $x > 0$.

5.2. *Bound for the total variation distance between finite point processes on the line*. Let $\widetilde{N} : \Omega \to \Omega$ be a measurable map and consider the filtration

$$\widetilde{\mathcal{F}}_t := \sigma\big\{\widetilde{N}(A) : A \in \mathcal{B}\big([0,t]\big)\big\}, \qquad t \in [0,T].$$

Setting $\widetilde{\mathcal{F}} := \widetilde{\mathcal{F}}_T$, we denote by $\mathbf{P}$ a probability measure on $(\Omega, \mathcal{F} \vee \widetilde{\mathcal{F}})$ under which $N$ has a Papangelou conditional intensity $\{\pi_t\}_{t\in[0,T]}$ and $\widetilde{N}$ has a Papangelou conditional intensity $\{\widetilde{\pi}_t(\widetilde{N})\}_{t\in[0,T]}$. We set $\widetilde{\mathbb{P}} := \mathbf{P}|_{\widetilde{\mathcal{F}}}$ and note that since an event $A \in \mathcal{F}$ depends only on $N$, by Remark 2.8 we have $\mathbb{P} = \mathbf{P}|_{\mathcal{F}}$.

We recall that the total variation distance between (the laws of) $N$ and $\widetilde{N}$ is by definition

$$d_{TV}(N, \widetilde{N}) := \sup_{\substack{G:\Omega\to[0,1],\\ G \text{ measurable}}} \big|\mathbf{E}\big[G(N)\big] - \mathbf{E}\big[G(\widetilde{N})\big]\big|,$$

where $\mathbf{E}$ denotes the expectation operator with respect to $\mathbf{P}$. The following bound on $d_{TV}(N, \widetilde{N})$ holds.

PROPOSITION 5.5. *If* $\mathbf{E}[\widetilde{N}([0,T])^2] < \infty$, *then*

$$d_{TV}(N, \widetilde{N}) \leq \int_0^T \mathbf{E}\big[\big|\pi_t(N) - \widetilde{\pi}_t(N)\big|\big] \mathrm{d}t. \tag{35}$$

REMARK 5.6. Assume:

(i) $\sup_{\omega\in\Omega}\int_0^T \widetilde{\pi}_t(\omega)\,\mathrm{d}t < \infty$ and
(ii) $\mathbb{P}$ absolutely continuous with respect to the law of a Poisson process with intensity 1 on $[0, T]$.

Then by Theorem 4 in [26] one has

$$d_{TV}(N, \widetilde{N}) \le c_1(\widetilde{\pi}) \int_0^T \mathbf{E}\big[\big|\pi_t(N) - \widetilde{\pi}_t(N)\big|\big]\,\mathrm{d}t, \tag{36}$$

where $c_1(\widetilde{\pi})$ is a finite constant such that, for any $n^* \in \mathbb{N} \cup \{\infty\}$,

$$\begin{aligned} c_1(\widetilde{\pi}) \le{}& \big(n^* - 1\big)!\left(\frac{\varepsilon}{c}\right)^{n^*-1}\left(\frac{1}{c}\sum_{i\ge n^*}\frac{c^i}{i!} + \int_0^c \frac{1}{s}\sum_{i\ge n^*}\frac{s^i}{i!}\,\mathrm{d}s\right) \\ &+ \frac{1+\varepsilon}{\varepsilon}\sum_{i=1}^{n^*-1}\frac{\varepsilon^i}{i}, \end{aligned} \tag{37}$$

where, letting $\|\cdot\|$ denote the total variation norm for signed measures on $[0, T]$,

$$\varepsilon := \sup_{\omega_1,\omega_2\in\Omega:\|\omega_1-\omega_2\|=1} \int_0^T \big|\pi_t(\omega_1) - \pi_t(\omega_2)\big|\,\mathrm{d}t < \infty$$

and

$$c = c\big(n^*\big) := \sup_{\omega_1,\omega_2\in\Omega:\|\omega_1-\omega_2\|\ge n^*} \int_0^T \big|\pi_t(\omega_1) - \pi_t(\omega_2)\big|\,\mathrm{d}t < \infty.$$

If $n^* = \infty$, we interpret the long first summand in the upper bound as 0. For $\varepsilon = 0$ and/or $c = 0$, the upper bound is to be understood in the limit sense. We also recall that by the definition of the total variation norm $\|\cdot\|$ the quantity $\|\omega_1 - \omega_2\|$, $\omega_1, \omega_2 \in \Omega$, is the total number of points appearing in the support of one of the counting measures $\omega_i$, $i = 1, 2$, but not in the other.

We remark that the proof of (36) in [26] is based on the generator approach to Stein's method, and the result indeed holds for finite spatial point processes.

Note that, if we replace condition (i) with the significantly weaker condition $\mathbf{E}[\widetilde{N}([0, T])^2] < \infty$ and condition (ii) with the slightly stronger condition $\mathbf{E}[N([0, T])] < \infty$, by Proposition 5.5 we have the bound (36) with $c_1(\widetilde{\pi}) = 1$.

It is in general a difficult task to further bound the constants $\varepsilon$ and $c$ in order to obtain an explicit bound on $c_1(\widetilde{\pi})$. In Remark 6 of [26], the authors show that if $\varepsilon < 1$, then

$$c_1(\widetilde{\pi}) \le \frac{1+\varepsilon}{\varepsilon}\log\left(\frac{1}{1-\epsilon}\right). \tag{38}$$

Note that the term on the right-hand side of (38) is greater than or equal to 1. In this sense, (35) is certainly a relevant improvement of (36).

PROOF OF PROPOSITION 5.5. Let $\mathbb{P}'$ be a probability measure on $(\Omega, \mathcal{F})$ under which $N$ has a Papangelou conditional intensity $\{\widetilde{\pi}_t\}_{t\in[0,T]}$. We verify first that $\mathbb{P}$ and $\mathbb{P}'$ are equivalent, that is, they have the same null sets. As in the proof of Lemma 2.7, we use the absolute continuity of the law of a point process with a classical stochastic intensity with respect to the law of a Poisson process with intensity 1. Let $\mathbb{P}^*$ be a probability measure on $(\Omega, \mathcal{F})$ under which $N$ is a Poisson process with intensity 1. Since, under $\mathbb{P}'$, $N$ has the same law as $\widetilde{N}$ under $\widetilde{\mathbb{P}}$ (see Remark 2.8) and, under $\widetilde{\mathbb{P}}$, $\widetilde{N}([0,T])$ has a finite mean, we have $\mathbb{E}'[N([0,T])] < \infty$, where $\mathbb{E}'$ denotes the expectation under $\mathbb{P}'$. Consequently, by Lemma 2.6 and a result in [13] (see also Theorem 19.7, page 315 in [16]) we have that $\mathbb{P}$ and $\mathbb{P}'$ are absolutely continuous with respect to $\mathbb{P}^*$ and letting $\mathbb{P}_t$, $\mathbb{P}'_t$ and $\mathbb{P}^*_t$ denote the restrictions of $\mathbb{P}$, $\mathbb{P}'$ and $\mathbb{P}^*$ on $(\Omega, \mathcal{F}_t)$, respectively, we have

$$\mathbb{P}_t(\mathrm{d}\omega) = \rho_t(\omega)\mathbb{P}^*_t(\mathrm{d}\omega) \quad \text{and} \quad \mathbb{P}'_t(\mathrm{d}\omega) = \rho'_t(\omega)\mathbb{P}^*_t(\mathrm{d}\omega),$$

where

$$\rho_t(\omega) := \exp\left(\int_0^t \bigl(1 - p(\pi)_s(\omega)\bigr)\,\mathrm{d}s + \int_0^t \ln\bigl(p(\pi)_s(\omega)\bigr)\omega(\mathrm{d}s)\right)$$

and

$$\rho'_t(\omega) := \exp\left(\int_0^t \bigl(1 - p'(\widetilde{\pi})_s(\omega)\bigr)\,\mathrm{d}s + \int_0^t \ln\bigl(p'(\widetilde{\pi})_s(\omega)\bigr)\omega(\mathrm{d}s)\right).$$

Here, for a stochastic process $\{X_t\}_{0\le t\le T}$, $\{p'(X)_t\}_{t\in[0,T]}$ denotes the predictable version of $\{\mathbb{E}'[X_t \mid \mathcal{F}_{t-}]\}_{t\in[0,T]}$ (see Proposition 2.4). By Theorem T12, page 31 in [2] [see formula (4.4)], we have $p(\pi)_s(\omega) > 0$ for any $s \in \operatorname{supp}(\omega)$, $\mathbb{P}$-almost surely, and $p'(\widetilde{\pi})_s(\omega) > 0$ for any $s \in \operatorname{supp}(\omega)$, $\mathbb{P}'$-almost surely. Therefore, for any $t \in [0,T]$, $\rho_t > 0$ $\mathbb{P}_t$-almost everywhere and $\rho'_t > 0$ $\mathbb{P}'_t$-almost everywhere, and so the laws $\mathbb{P}$, $\mathbb{P}^*$ and $\mathbb{P}'$ are equivalent.

By Theorem 4.1, for any $G : \Omega \to [0,1]$ measurable, we have

$$\begin{aligned} G(\omega) = \mathbb{E}'[G] + \int_0^T &\left(\frac{p'(\widetilde{\pi}G^+)_t(\omega) - p'(\widetilde{\pi})_t(\omega)p'(G)_t(\omega)}{p'(\widetilde{\pi})_t(\omega)}\right) \\ &\times \bigl(\omega(\mathrm{d}t) - p'(\widetilde{\pi})_t(\omega)\,\mathrm{d}t\bigr), \end{aligned} \tag{39}$$

for $\mathbb{P}'$-almost every $\omega \in \Omega$. Thus, by the equivalence of $\mathbb{P}$ and $\mathbb{P}'$, taking the expectation with respect to $\mathbb{P}$ in (39) yields

$$\begin{aligned} &\bigl|\mathbb{E}'[G] - \mathbb{E}[G]\bigr| \\ &\quad \le \left|\mathbb{E}\left[\int_0^T \left(\frac{p'(\widetilde{\pi}G^+)_t - p'(\widetilde{\pi})_t p'(G)_t}{p'(\widetilde{\pi})_t}\right)\bigl(N(\mathrm{d}t) - p(\pi)_t\,\mathrm{d}t\bigr)\right]\right| \\ &\qquad + \mathbb{E}\left[\int_0^T \left|\frac{p'(\widetilde{\pi}G^+)_t - p'(\widetilde{\pi})_t p'(G)_t}{p'(\widetilde{\pi})_t}\right| \bigl|p(\pi)_t - p'(\widetilde{\pi})_t\bigr|\,\mathrm{d}t\right], \end{aligned} \tag{40}$$

by the triangle inequality. Note that the term

$$\left|\frac{p'(\widetilde{\pi}G^+)_t - p'(\widetilde{\pi})_t p'(G)_t}{p'(\widetilde{\pi})_t}\right|$$

is bounded by 1 due to Lemma 4.5 (indeed, $0 \le G \le 1$). Note also that

$$\mathbb{E}\left[\int_0^T \left(\frac{p'(\widetilde{\pi}G^+)_t - p'(\widetilde{\pi})_t p'(G)_t}{p'(\widetilde{\pi})_t}\right)\bigl(N(\mathrm{d}t) - p(\pi)_t\,\mathrm{d}t\bigr)\right] = 0$$

since $\{p(\pi)_t\}_{t\in[0,T]}$ is a classical stochastic intensity of $N$ under $\mathbb{P}$ and the integrand is bounded and, therefore, belongs to $\mathcal{P}_{1,2}(p(\pi))$. Combining these considerations with (40), we get

$$\begin{aligned}
\bigl|\mathbb{E}'[G] - \mathbb{E}[G]\bigr| &\le \mathbb{E}\left[\int_0^T \bigl|p(\pi)_t - p'(\widetilde{\pi})_t\bigr|\,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \bigl|\mathbb{E}[\pi_t \mid \mathcal{F}_{t-}] - \mathbb{E}'[\widetilde{\pi}_t \mid \mathcal{F}_{t-}]\bigr|\,\mathrm{d}t\right].
\end{aligned} \tag{41}$$

We note that, for any $A \in \mathcal{F}_{t-}$,

$$\mathbb{E}\left[\frac{\rho_t'}{\rho_t}\widetilde{\pi}_t \mathbb{1}_A\right] = \mathbb{E}'[\widetilde{\pi}_t \mathbb{1}_A] = \mathbb{E}'\bigl[\mathbb{E}'[\widetilde{\pi}_t|\mathcal{F}_{t-}]\mathbb{1}_A\bigr] = \mathbb{E}\left[\frac{\rho_t'}{\rho_t}\mathbb{E}'[\widetilde{\pi}_t|\mathcal{F}_{t-}]\mathbb{1}_A\right]$$

and so, using the $\mathcal{F}_{t-}$-measurability of $\{\rho_t'/\rho_t\}_{t\in[0,T]}$, we have

$$\frac{\rho_t'}{\rho_t}\mathbb{E}[\widetilde{\pi}_t|\mathcal{F}_{t-}] = \mathbb{E}\left[\frac{\rho_t'}{\rho_t}\widetilde{\pi}_t|\mathcal{F}_{t-}\right] = \frac{\rho_t'}{\rho_t}\mathbb{E}'[\widetilde{\pi}_t|\mathcal{F}_{t-}], \qquad \mathbb{P}\text{-almost surely}$$

which gives $\mathbb{E}[\widetilde{\pi}_t|\mathcal{F}_{t-}] = \mathbb{E}'[\widetilde{\pi}_t|\mathcal{F}_{t-}]$, $\mathbb{P}$-almost surely. Combining this with (41) and using the definition of the probability measure $\mathbf{P}$, we deduce

$$\bigl|\mathbf{E}\bigl[G(\widetilde{N})\bigr] - \mathbf{E}\bigl[G(N)\bigr]\bigr| \le \int_0^T \mathbf{E}\bigl[\bigl|\pi_t(N) - \widetilde{\pi}_t(N)\bigr|\bigr]\,\mathrm{d}t,$$

and the claim follows by taking the supremum over all the functionals $G$. □

5.3. *Option hedging in a market model with jumps governed by a classical stochastic intensity.* In this subsection, we apply the Clark–Ocone formula to option hedging in a pure jump market model. In the following, we shall use the same notation as in the previous sections.

The application of the Clark–Ocone formula to mathematical finance is well documented in both continuous and discontinuous models. It is a central part of Ocone's original paper; see Sections 3-5 in [22] and also [9, 15] and Chapter 8 in [23] for additional details on the topic. We refer to these texts for the common terminology adopted in mathematical finance.

We consider a market with two assets, one nonrisky and one risky, we assume that the assets are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that the market is arbitrage-free. We also assume that the nonrisky asset $\{S_t^{(0)}\}_{t\in[0,T]}$ has dynamics

$$\mathrm{d}S_t^{(0)} = r(t)S_t^{(0)}\,\mathrm{d}t, \qquad S_0^{(0)} = 1$$

and the risky asset $\{S_t\}_{t\in[0,T]}$ has risk-neutral dynamics

$$\mathrm{d}S_t = S_{t^-}\big(r(t)\,\mathrm{d}t + \sigma(t)\big(N(\mathrm{d}t) - p(\pi)_t\,\mathrm{d}t\big)\big), \qquad S_0 = x > 0.$$

Here, $r$ and $\sigma$ are two continuous functions and it is assumed that $\sigma(t) > 0$ for $t \in [0, T]$. We also assume that the market is frictionless and that agents may buy and sell fractions of both assets. By, for example, Theorem 5.1 in [12], one has $S_t > 0$ for all $0 \le t \le T$, $\mathbb{P}$-almost surely.

Let $\{\xi_t^{(0)}\}_{t\in[0,T]}$ and $\{\xi_t\}_{t\in[0,T]}$ be two predictable processes. We say that $(\xi^{(0)}, \xi)$ is a portfolio if $\xi_t^{(0)}$ represents the number of units of investment of an agent in the risk-free asset at a time $t$, and $\xi_t$ represents the number of units of investment of an agent in the risky asset at time $t$. The value $V_t$ of a portfolio $(\xi^{(0)}, \xi)$ at time $t$ is defined by $V_t := \xi_t^{(0)} S_t^{(0)} + \xi_t S_t$, $t \in [0, T]$. We say that a portfolio $(\xi^{(0)}, \xi)$ is self-financing if

$$\mathrm{d}V_t = \xi_t^{(0)}\,\mathrm{d}S_t^{(0)} + \xi_t\,\mathrm{d}S_t, \qquad t \in [0, T].$$

In the next proposition, we compute a self-financing hedging strategy replicating an arbitrary option with payoff $G$ at time $T$.

PROPOSITION 5.7. *Let the notation and assumptions of Theorem 4.1 prevail and set*

$$\xi_t^{(0)} := \frac{p(G)_t \mathrm{e}^{-\int_t^T r(s)\,\mathrm{d}s} - \xi_t S_{t^-}}{S_t^{(0)}}, \qquad \xi_t := \frac{\varphi_t^{(G)} \mathrm{e}^{-\int_t^T r(s)\,\mathrm{d}s}}{\sigma(t) S_{t^-}}, \qquad t \in [0, T]. \tag{42}$$

*Then $(\xi^{(0)}, \xi)$ is a self-financing portfolio such that its value at time* 0 *is that of the option price at time* 0 *under the nonarbitrage assumption, that is,*

$$V_0 = \mathbb{E}[G]\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s}, \qquad \mathbb{P}\text{-almost surely} \tag{43}$$

*and its value at time $T$ is equal to $G$, that is,*

$$V_T = G, \qquad \mathbb{P}\text{-almost surely.} \tag{44}$$

PROOF. Since the arguments used in the proof are standard, we omit some details. The processes $\xi^{(0)}$ and $\xi$ defined in (42) are clearly predictable. According to the definition of $\xi^{(0)}$, $\xi$ and $V$, we easily have

$$V_t := \xi_t^{(0)} S_t^{(0)} + \xi_t S_t = p(G)_t \mathrm{e}^{-\int_t^T r(s)\,\mathrm{d}s}$$

and so (43) and (44) are verified. It remains to check that the portfolio $(\xi^{(0)}, \xi)$ is self-financing. By Theorem 4.1, we have

$$\mathbb{E}[G|\mathcal{F}_t] = \mathbb{E}[G] + \int_0^t \varphi_s^{(G)}\big(N(\mathrm{d}s) - p(\pi)_s\,\mathrm{d}s\big) \qquad \forall t \in [0,T], \mathbb{P}\text{-almost surely.}$$

Therefore, since $\mathbb{E}[G|\mathcal{F}_t] = p(G)_t$, for any $t \in [0,T]$, $\mathbb{P}$-almost surely, by a standard computation we deduce

$$V_t = V_0 \mathrm{e}^{\int_0^t r(s)\,\mathrm{d}s} + \int_0^t \mathrm{e}^{\int_s^t r(u)\,\mathrm{d}u} \sigma(s)\xi_s S_{s^-}\big(N(\mathrm{d}s) - p(\pi)_s\,\mathrm{d}s\big).$$

Consequently, setting

$$\widetilde{V}_t := V_t \mathrm{e}^{-\int_0^t r(s)\,\mathrm{d}s} \quad \text{and} \quad \widetilde{S}_t := S_t \mathrm{e}^{-\int_0^t r(s)\,\mathrm{d}s}, \qquad t \in [0,T],$$

we have

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \sigma(s)\xi_s \widetilde{S}_{s^-}\big(N(\mathrm{d}s) - p(\pi)_s\big)\,\mathrm{d}s, \qquad t \in [0,T]$$

and the claim follows by the same argument as in the proof of Lemma 8.1.2. in [23] [the implication (ii) $\Rightarrow$ (i)]. $\square$

## REFERENCES


[1] BAUDOIN, F. and FENG, Q. (2015). Log-Sobolev inequalities on the horizontal path space of a totally geodesic foliation. Available at arXiv:1503.08180.

[2] BRÉMAUD, P. (1981). *Point Processes and Queues*: *Martingale Dynamics*. Springer, New York. MR0636252

[3] BRÉMAUD, P. and MASSOULIÉ, L. (2002). Power spectra of general shot noises and Hawkes point processes with a random excitation. *Adv. in Appl. Probab.* **34** 205–222. MR1895338

[4] BRETON, J.-C. and PRIVAULT, N. (2014). Factorial moments of point processes. *Stochastic Process. Appl.* **124** 3412–3428. MR3231625

[5] DALEY, D. J. and VERE-JONES, D. (2003). *An Introduction to the Theory of Point Processes. Vol. I*: *Elementary Theory and Methods*, 2nd ed. Springer, New York. MR1950431

[6] DALEY, D. J. and VERE-JONES, D. (2008). *An Introduction to the Theory of Point Processes. Vol. II*: *General Theory and Structure*, 2nd ed. Springer, New York. MR2371524

[7] DECREUSEFOND, L. (1998). Perturbation analysis and Malliavin calculus. *Ann. Appl. Probab.* **8** 496–523. MR1624953

[8] DECREUSEFOND, L. and FLINT, I. (2014). Moment formulae for general point processes. *J. Funct. Anal.* **267** 452–476. MR3210036

[9] DI NUNNO, G., ØKSENDAL, B. and PROSKE, F. (2009). *Malliavin Calculus for Lévy Processes with Applications to Finance*. Springer, Berlin. MR2460554

[10] DI NUNNO, G. and VIVES, J. (2017). A Malliavin–Skorohod calculus in $L^0$ and $L^1$ for additive and Volterra-type processes. *Stochastics* **89** 142–170. MR3574698

[11] GEORGII, H.-O. and YOO, H. J. (2005). Conditional intensity and Gibbsianness of determinantal point processes. *J. Stat. Phys.* **118** 55–84. MR2122549

[12] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam. MR1011252

[13] JACOD, J. and MÉMIN, J. (1975). Caractéristiques locales et conditions de continuité absolue pour les semimartingales. *Prepublication, Dép. Math. Inform. Univ. de Rennes*.

[14] KLEIN, T., MA, Y. and PRIVAULT, N. (2006). Convex concentration inequalities and forward-backward stochastic calculus. *Electron. J. Probab.* **11** 486–512. MR2242653

[15] LAST, G. and PENROSE, M. D. (2011). Martingale representation for Poisson processes with applications to minimal variance hedging. *Stochastic Process. Appl.* **121** 1588–1606. MR2802467

[16] LIPTSER, R. S. and SHIRYAYEV, A. N. (1978). *Statistics of Random Processes. II: Applications*. Springer, New York. MR0488267

[17] MATTHES, K., WARMUTH, W. and MECKE, J. (1979). Bemerkungen zu einer Arbeit: "Integral and differential characterizations of the Gibbs process" [Math. Nachr. **88** (1979), 105–115; MR 80i:60081a] von Nguyen Xuan Xanh und Hans Zessin. *Math. Nachr.* **88** 117–127. MR0543397

[18] MØLLER, J. and WAAGEPETERSEN, R. P. (2004). *Statistical Inference and Simulation for Spatial Point Processes. Monographs on Statistics and Applied Probability* **100**. Chapman & Hall, Boca Raton. MR2004226

[19] NGUYEN, T. D., PRIVAULT, N. and TORRISI, G. L. (2015). Gaussian estimates for the solutions of some one-dimensional stochastic equations. *Potential Anal.* **43** 289–311. MR3374113

[20] NGUYEN, X.-X. and ZESSIN, H. (1979). Integral and differential characterizations of the Gibbs process. *Math. Nachr.* **88** 105–115. MR0543396

[21] NUALART, D. and VIVES, J. (1990). Anticipative calculus for the Poisson process based on the Fock space. In *Séminaire de Probabilités, XXIV*, 1988/89. *Lecture Notes in Math.* **1426** 154–165. Springer, Berlin. MR1071538

[22] OCONE, D. L. and KARATZAS, I. (1991). A generalized Clark representation formula, with application to optimal portfolios. *Stoch. Stoch. Rep.* **34** 187–220. MR1124835

[23] PRIVAULT, N. (2009). *Stochastic Analysis in Discrete and Continuous Settings with Normal Martingales. Lecture Notes in Math.* **1982**. Springer, Berlin. MR2531026

[24] PRIVAULT, N. and TORRISI, G. L. (2013). Probability approximation by Clark–Ocone covariance representation. *Electron. J. Probab.* **18** no. 91, 25. MR3126574

[25] PRIVAULT, N. and TORRISI, G. L. (2015). The Stein and Chen–Stein methods for functionals of non-symmetric Bernoulli processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **12** 309–356. MR3357130

[26] SCHUHMACHER, D. and STUCKI, K. (2014). Gibbs point process approximation: Total variation bounds using Stein's method. *Ann. Probab.* **42** 1911–1951. MR3262495

[27] TORRISI, G. L. (2013). Point processes with Papangelou conditional intensity: From the Skorohod integral to the Dirichlet form. *Markov Process. Related Fields* **19** 195–248. MR3113944

[28] TORRISI, G. L. (2016). Gaussian approximation of nonlinear Hawkes processes. *Ann. Appl. Probab.* **26** 2106–2140. MR3543891

[29] TORRISI, G. L. (2017). Poisson approximation of point processes with stochastic intensity, and application to nonlinear Hawkes processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 679–700. MR3634270

[30] VAN LIESHOUT, M. N. M. (2000). *Markov Point Processes and Their Applications*. Imperial College Press, London. MR1789230

[31] WU, L. (2000). A new modified logarithmic Sobolev inequality for Poisson point processes and several applications. *Probab. Theory Related Fields* **118** 427–438. MR1800540

[32] ZHANG, X. (2009). Clark–Ocone formula and variational representation for Poisson functionals. *Ann. Probab.* **37** 506–529. MR2510015

[33] ZHANG, X. (2009). A variational representation for random functionals on abstract Wiener spaces. *J. Math. Kyoto Univ.* **49** 475–490. MR2583599

DIVISION OF MATHEMATICAL SCIENCES
NANYANG TECHNOLOGICAL UNIVERSITY
SPMS-MAS-04-02, 21 NANYANG LINK
SINGAPORE 637371
SINGAPORE
E-MAIL: iflint@ntu.edu.sg

ISTITUTO PER LE APPLICAZIONI
DEL CALCOLO "MAURO PICONE"
CNR
VIA DEI TAURINI 19
00185 ROMA
ITALY
E-MAIL: torrisi@iac.rm.cnr.it