REAL SELF-SIMILAR PROCESSES STARTED FROM THE ORIGIN

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Since the seminal work of Lamperti, there is a lot of interest in the understanding of the general structure of self-similar Markov processes. Lamperti gave a representation of positive self-similar Markov processes with initial condition strictly larger than 0 which subsequently was extended to zero initial condition.

For real self-similar Markov processes (rssMps), there is a generalization of Lamperti's representation giving a one-to-one correspondence between Markov additive processes and rssMps with initial condition different from the origin.

We develop fluctuation theory for Markov additive processes and use Kuznetsov measures to construct the law of transient real self-similar Markov processes issued from the origin. The construction gives a pathwise representation through two-sided Markov additive processes extending the Lamperti–Kiu representation to the origin.

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1. Introduction. A fundamental property appearing in probabilistic models is self-similarity, also called the scaling property. In the context of stochastic processes, this is the phenomenon of scaling time and space in a carefully chosen manner such that there is distributional invariance. An example of the latter is Brownian motion for which the distribution of $(cB_{c^{-2}t})_{t\geq 0}$ and $(B_t)_{t\geq 0}$ is the same for any $c > 0$; its so-called scaling index is thus understood as 2. A natural question is if the knowledge of the scaling property alone implies structural properties for a given model and whether such can be used to deduce nontrivial implications. In this article, we focus on the case of Markov processes taking values in $\mathbb{R}$ that fulfil the same scaling relation as a Brownian motion, except the scaling index is taken more generally to be $\alpha > 0$ rather than 2. In particular, we focus on entrance laws of such processes from the origin, a problem which, although well understood in the case of Brownian motion, is more difficult to address in the general setting of real self-similar Markov processes.

Before coming to our main results, we review results and ideas for self-similar Markov processes with nonnegative sample paths.

1.1. *Positive self-similar Markov processes.* A strong Markov family $\{\mathbb{P}^z, z > 0\}$ with càdlàg paths on the state space $[0, \infty)$, with 0 being an absorbing cemetery state, is called *positive self-similar Markov process of index* $\alpha > 0$ (briefly *pssMp*) if the scaling property holds:

(1) the law of $(cZ_{c^{-\alpha}t})_{t\geq 0}$ under $\mathbb{P}^z$ is $\mathbb{P}^{cz}$

for all $z, c > 0$. Here, and in what follows, $Z = (Z_t)_{t\geq 0}$ denotes the canonical process. The analysis of positive self-similar processes is typically based on the Lamperti representation (see, for instance, Chapter 13 of [23] or [25]). It ensures the existence of a Lévy process $(\xi_t)_{t\geq 0}$, possibly killed at an exponential time with cemetery state $-\infty$, such that, under $\mathbb{P}^z$ for $z > 0$,

$$Z_t = \exp(\xi_{\varphi^{-1}(t)}), \qquad t \geq 0,$$

where $\varphi(t) = \int_0^t \exp(\alpha\xi_s)\,ds$ and the Lévy process ξ is started in $\log(z)$. We use the convention that $\exp(\xi_{\varphi^{-1}(t)})$ is equal to zero, if $t \notin \varphi([0, \infty))$.

It is a consequence of the Lamperti representation that pssMps can be split into two regimes:

(**R**) $\mathbb{P}^z(T_0 < \infty) = 1$ for all $z > 0 \Longleftrightarrow \xi$ drifts to $-\infty$ or is killed,
(**T**) $\mathbb{P}^z(T_0 < \infty) = 0$ for all $z > 0 \Longleftrightarrow \xi$ drifts to $+\infty$ or oscillates.

Two major questions remained open after Lamperti:

 (i) How to extend a pssMp after hitting 0 in the recurrent regime (**R**) with an instantaneous entrance from zero?

(ii) How to start a pssMp from the origin in the transient regime (**T**)? More precisely, one asks for extensions $\{\mathbb{P}^z, z \geq 0\}$ with the Feller property so that in particular $\mathbb{P}^0 := \text{w-}\lim_{z \downarrow 0} \mathbb{P}^z$ exists in the Skorokhod topology.

Both questions have been solved in recent years: In the recurrent regime, it was proved by Fitzsimmons [11] and Rivero [27] that there is a unique recurrent self-similar Markov extension (or equivalently a self-similar excursion measure with summable excursion lengths) that leaves zero continuously if and only if

$$(2) \qquad\qquad E\big[e^{\lambda \xi_1}\big] = 1 \qquad \text{for some } 0 < \lambda < \alpha.$$

For the transient regime, it was shown in Chaumont et al. [6] and also in Bertoin and Savov [4] that, if the ascending ladder height process of ξ is non-lattice, the weak limit $\mathbb{P}^0$ exists if and only if the weak limit of overshoots

$$(3) \qquad\qquad O := \text{w-}\lim_{x \uparrow \infty}(\xi_{\tau_x} - x) \qquad \text{exists,}$$

where $\tau_x := \inf\{t : \xi_t \geq x\}$. If (3) holds, then one says ξ has stationary overshoots.

There are several different ways of proving the results that, at their heart, involve a construction of the underlying Lévy process ξ as a stationary process (in the appropriate sense) indexed by $\mathbb{R}$. The construction that appears to be the most natural to us, in the sense that it works for (**R**) and (**T**), was carried out for the recurrent regime by Fitzsimmons and shall be developed in this article for the transient regime.

It has been known for a long time in probabilistic potential theory that excessive measures of Markov processes are closely linked to the entrance behaviour from so-called entrance boundaries. One way the relation is implemented involves Markov processes with random birth and death (Kuznetsov measures). Apart from diffusion processes not many examples are known in which the general theory yields concrete results. Self-similar Markov processes form a nice class of nontrivial examples for which the abstract theory gives explicit results. The essence is a combination of Lamperti's representation with Kaspi's theorem on time-changing Kuznetsov measures. Excursions away from 0 of a pssMp are governed by an excursion measure n corresponding to a particular excessive measure for the pssMp that itself turns out to be a transformation of an invariant measure of ξ. Invariant measures for Lévy processes are known explicitly from the Choquet–Dény theorem, hence, excursion measures for pssMp can be identified and constructed through Kuznetsov measures.

It is interesting to observe that the construction of self-similar excursion measures n as Kuznetsov measures work in the recurrent and transient regimes without using Conditions (2) and (3) (Note that we also interpret $\mathbb{P}^0$ as a normalized "excursion measure" even though an excursion starts at 0, does not return to 0 and $\mathbb{P}^0$ must be a probability measure). The necessity and sufficiency enters as follows:

(**R**) Condition (2) is necessary and sufficient to construct from n a Markov process by gluing excursions drawn according to a point process of excursions (using Blumenthal's theorem on Itô's synthesis).

(**T**) To define $\mathbb{P}^0$ as normalized "excursion measure" the Kuznetsov measure needs to be finite and this is equivalent to Condition (3); see Remark 16 below.

We present our constructions for (**T**) directly in the more general setting of real self-similar Markov processes.

REMARK 1. The argument of Fitzsimmons [12] for recurrent extensions extends readily to the real-valued setting by replacing Lévy processes through MAPs. Since our main purpose is to show how the potential theoretic approach has to be carried out in the transient case and since the article is already technical enough we do not address this topic here.

1.2. *Real self-similar Markov processes—main results.* Let $\mathbf{D}(\mathbb{R})$ be the space of càdlàg functions $w : \mathbb{R}_+ \to \mathbb{R}$ endowed with the Skorokhod topology and the corresponding Borel σ-field $\mathcal{D}(\mathbb{R})$. A family of distributions $\{\mathbb{P}^z : z \in \mathbb{R} \setminus \{0\}\}$ on $(\mathbf{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ is called *strong Markov family* on $\mathbb{R} \setminus \{0\}$ if the canonical process $(Z_t)_{t \geq 0}$ is strong Markov with respect to the canonical right continuous filtration. If additionally the process satisfies the scaling property (1) for all $z \in \mathbb{R} \setminus \{0\}$ and $c > 0$, then the process is called *real self-similar Markov process*. A result of Chaumont et al. [7], completing earlier work of Kiu [20], is that for any real self-similar Markov process, there is a Markov additive process $(\xi_t, J_t)_{t \geq 0}$ on $\mathbb{R} \times \{\pm 1\}$ such that under $\mathbb{P}^z$ the canonical process can be represented as

$$(4) \qquad Z_t = \exp(\xi_{\varphi^{-1}(t)}) J_{\varphi^{-1}(t)}, \qquad t \geq 0,$$

where $\varphi(t) = \int_0^t \exp(\alpha \xi_s)\,ds$ and $(\xi_0, J_0) = (\log|z|, [z])$ with

$$[z] = \begin{cases} 1, & \text{if } z > 0, \\ -1, & \text{if } z < 0. \end{cases}$$

Again we use the convention that $\exp(\xi_{\varphi^{-1}(t)}) J_{\varphi^{-1}(t)}$ is equal to zero if $t \notin \varphi([0, \infty))$. A *Markov additive process* (MAP) is a stochastic process $(\xi_t, J_t)_{t \geq 0}$ on $\mathbb{R} \times E$, where E is a finite set, if $(J_t)_{t \geq 0}$ is a continuous time Markov chain on E (called modulating chain) and, for any $i \in E$ and $s, t \geq 0$,

given $\{J_t = i\}$, the pair $(\xi_{t+s} - \xi_t, J_{t+s})_{s \geq 0}$ is independent of the past

and has the same distribution as $(\xi_s, J_s)_{s \geq 0}$ under $P^{0,i}$.

If the MAP is killed, then ξ is sent to the cemetery state $\{-\infty\}$. An important feature of MAPs that will be used throughout our analysis is their close mathematical proximity to Lévy processes. For a textbook treatment of standard results for MAPs see, for instance, Chapter XI of Asmussen [2].

PROPOSITION 2. *A process* (ξ, J) *is a MAP if and only if there exist sequences of*:

- *Lévy processes* $(\xi^{n,i})_{n\in\mathbb{N}_0}$, *i.i.d. for* $i \in E$ *fixed*,
- *real random variables* $(\Delta^n_{i,j})_{n\in\mathbb{N}}$, *i.i.d. for distinct and fixed* $i, j \in E$,

independent of J *and of each other such that, if* T_n *is the nth jump-time of* J, *then* ξ *can be written as*

$$\xi_t = \begin{cases} \xi_0 + \xi_t^{0,J_0}, & t < T_1, \\ \xi_{T_n-} + \Delta^n_{J_{T_n-},J_{T_n}} + \xi_{t-T_n}^{n,J_{T_n}}, & t \in [T_n, T_{n+1}), t < \mathbf{k}, \\ \xi_t = -\infty, & t \geq \mathbf{k}, \end{cases}$$

where the killing time $\mathbf{k}$ *is the first time one of the appearing Lévy processes is killed*:

$$\mathbf{k} = \inf\{t > 0 : \exists n \in \mathbb{N}_0, T_n \leq t \text{ such that } \xi^{n,J_{T_n}} \text{ is killed at time } t - T_n\}.$$

In words, the idea behind a MAP is as follows: There is a time-dependent random environment governed by the state of J and for every state there is a corresponding Lévy process ξ^i with triplet (a_i, σ_i^2, Π_i). If J is in state i, then ξ evolves according to a copy of ξ^i. Once J changes from i to j, which happens at rate $q_{i,j}$, ξ has an additional transitional jump $\Delta_{i,j}$ and until the next jump of J, ξ evolves according to a copy of ξ^j. The MAP is killed as soon as one of the Lévy processes is killed.

Consequently, the mechanism behind the Lamperti–Kiu representation is simple: J governs the sign of Z and on intervals with constant sign the Lamperti–Kiu representation simplifies to the Lamperti representation.

REMARK 3. The MAP formalism for the Lamperti–Kiu representation does not appear in [7] but has been introduced in [21].

From now on, we assume:

(**I**) J is irreducible on $\{\pm 1\}$,

that is, neither 1 nor -1 is absorbing. Assumption (**I**) carries no loss of generality: If J is not irreducible, then (4) implies that the self-similar process changes sign at most once; thus, can be treated as positive (or negative) self-similar process to which the results for pssMps apply. Note also that (**I**) ensures that the modulating chain J has a unique stationary distribution, which we denote by $\pi = (\pi_+, \pi_-)$. In keeping with this notation, we shall also write the off-diagonal elements of the transition matrix of J as $q_{+,-}$ and $q_{-,+}$.

We also assume:

(**NL**) ξ is nonlattice,

which is a standard assumption to avoid technicalities. The reader is referred to the discussion at the end of Appendix A.3 for further comments concerning this assumption.

Throughout the article, some notation for first hitting times is used: For a real-valued process

$$T_{\{0\}} = \inf\{t : Z_t = 0\} \quad \text{and} \quad T_\varepsilon = \inf\{t : |Z_t| \geq \varepsilon\},$$

where $\varepsilon > 0$, and for a bi-variate process (Z^1, Z^2)

(5) $$\tau_x^- = \inf\{t : Z_t^1 \leq x\} \quad \text{and} \quad \tau_x^+ = \inf\{t : Z_t^1 \geq x\}$$

for $x \in \mathbb{R}$.

Analogously to Lévy processes one knows that an unkilled MAP (ξ, J) either drifts to $+\infty$ (i.e., $\lim_{t\uparrow\infty} \xi_t = +\infty$), drifts to $-\infty$ (i.e., $\lim_{t\uparrow\infty} \xi_t = -\infty$) or oscillates (i.e., $\liminf_{t\uparrow\infty} \xi_t = -\infty$ and $\limsup_{t\uparrow\infty} \xi_t = +\infty$), in the almost sure sense. As for pssMps a simple 0–1 law for real self-similar Markov processes can be deduced from the Lamperti–Kiu representation.

PROPOSITION 4. *If (ξ, J) is the Markov additive process corresponding to a real self-similar Markov process through the Lamperti–Kiu representation, then one has the following dichotomy*:

(**R**) $\mathbb{P}^z(T_{\{0\}} < \infty) = 1$ *for all $z \neq 0 \iff (\xi, J)$ drifts to $-\infty$ or is killed*,
(**T**) $\mathbb{P}^z(T_{\{0\}} < \infty) = 0$ *for all $z \neq 0 \iff (\xi, J)$ drifts to $+\infty$ or oscillates*.

The proof is very close in spirit to the proof of the analogous result for pssMps (see, for instance, Chapter 13 of [23]).

For the rest of this article, we assume (**T**) and ask for the existence and a construction of a measure $\mathbb{P}^0$ on the Skorokhod space $(\mathbf{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ such that the extension $\{\mathbb{P}^z : z \in \mathbb{R}\}$ of $\{\mathbb{P}^z : z \in \mathbb{R} \setminus \{0\}\}$ is a self-similar Markov family. In other words, the aim is to extend the Lamperti–Kiu representation to transient self-similar Markov processes that do not have zero as a trap.

Let ξ^+ and ξ^- be the Lévy processes and $\Delta_{+,-}$ and $\Delta_{-,+}$ the random variables appearing in the representation of (ξ, J) from Proposition 2 when applied to the two state MAP of the Lamperti–Kiu representation (4).

(**C**) ξ_1 has finite absolute moment and either of the following holds:
  (i) (ξ, J) drifts to $+\infty$,
  (ii) (ξ, J) oscillates and

$$\int_1^\infty \frac{x \Pi([x, \infty))}{1 + \int_0^x \int_y^\infty \Pi((-\infty, -z]) \, dz \, dy} \, dx < \infty,$$

  where Π is the measure

$$\Pi = \Pi_+ + \Pi_- + q_{+,-}\mathcal{L}(\Delta_{+,-}) + q_{-,+}\mathcal{L}(\Delta_{-,+})$$

  for the Lévy measure Π_+ of ξ^+ (resp., Π_- of ξ^-) and the probability distribution $\mathcal{L}(\Delta_{+,-})$ of $\Delta_{+,-}$ [resp., $\mathcal{L}(\Delta_{-,+})$ of $\Delta_{-,+}$].

Condition (**C**) shall be called stationary overshoot condition for (ξ, J) as it is the precise condition for the corresponding MAP to have stationary overshoots in the following sense.

THEOREM 5.   *If* (**NL**) *and* (**I**) *hold, then*

$$\underset{a \to +\infty}{\text{w-}\lim} P^{0,i}\big(\xi_{\tau_a^+} - a \in dx, J_{\tau_a^+} = j\big)$$

$$= \underset{a \to +\infty}{\text{w-}\lim} P^{-a,i}\big(\xi_{\tau_0^+} \in dx, J_{\tau_0^+} = j\big)$$

exists independently of $i \in \{\pm 1\}$ and is nondegenerate if and only if Condition (**C**) *holds.*

Theorem 5 is the MAP version of an important result on the existence of stationary overshoots for Lévy processes (see, for instance, Chapter 7 of [23]) for which Π reduces to the Lévy measure only. From Theorem 28 in the Appendix, it follows that stationary overshoots are equivalent to requiring finite mean for the ladder height processes of (ξ, J), the analytic condition from (**C**) is provided in Theorem 35.

We can now state the main theorem of the present article.

THEOREM 6.   *Suppose $\{\mathbb{P}^z : z \neq 0\}$ is a real self-similar Markov process for which the corresponding MAP (ξ, J) satisfies Conditions* (**I**) *and* (**NL**). *Then Condition* (**C**) *for (ξ, J) is necessary and sufficient for the existence of an extension $\{\mathbb{P}^z : z \in \mathbb{R}\}$ on $(\mathbf{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ such that the following properties hold*:

1. *Under $\mathbb{P}^0$ the process leaves 0 instantaneously.*
2. *The corresponding transition semigroup (P_t) on $\mathbb{R}$ has the Feller property.*
3. *The family $\{\mathbb{P}^z : z \in \mathbb{R}\}$ is self-similar.*

Furthermore, $\mathbb{P}^0$ is the unique distribution satisfying either of the first two properties listed above.

The reader might have realized that Assumption (**I**) excludes the special case of positive self-similar Markov processes that occurs for the trivial case that the Markov chain J is constant and the MAP (ξ, J) reduces to a Lévy process. In fact, the proof for pssMps is a line-by-line translation of the proof given here replacing in all arguments MAPs by Lévy processes. Since the fluctuation theory for MAPs developed in the Appendix is classical for Lévy processes, the proof for pssMps only requires the main body of the article which also simplifies drastically in notation.

1.3. *Sketch of the proof.* The necessity of Condition (**C**) is straight forward. Combining the Lamperti–Kiu representation and Theorem 35, the failure of Condition (**C**) implies

$$\lim_{|z|\to 0} \mathbb{P}^z\big(|Z_{T_\varepsilon}| < c\big) = \lim_{|z|\to 0} P^{\log|z|,[z]}\big(\exp(\xi_{\tau^+_{\log(\varepsilon)}}) < c\big)$$

(6)
$$= \lim_{|z|\to 0} P^{\log|z|,[z]}\big(\xi_{\tau^+_{\log(\varepsilon)}} - \log(\varepsilon) < \log(c/\varepsilon)\big)$$

$$= 0$$

for any positive c, ε fixed. Now define

$$f(z) = \begin{cases} \mathbb{P}^z\big(|Z_{T_\varepsilon}| < c\big), & z \neq 0, \\ 0, & z = 0, \end{cases}$$

then, using the calculation from (6) and the remark following (30) in the Appendix, f is continuous. Hence, for any $\delta > 0$ we may choose $a > 0$ so that $\sup_{|z|\le a} f(z) < \delta$. Suppose $\mathbb{P}^0$ is as in Theorem 6, then, by the strong Markov property,

$$\mathbb{P}^0\big(|Z_{T_\varepsilon}| < c\big) = \lim_{\varepsilon'\to 0} \int \mathbb{P}^z\big(|Z_{T_\varepsilon}| < c\big)\mathbb{P}^0(Z_{T_{\varepsilon'}} \in dz)$$

$$= \lim_{\varepsilon'\to 0} \bigg(\int_{|z|\le a} f(z)\mathbb{P}^0(Z_{T_{\varepsilon'}} \in dz) + \int_{|z|>a} f(z)\mathbb{P}^0(Z_{T_{\varepsilon'}} \in dz) \bigg)$$

$$\le \delta + \lim_{\varepsilon'\to 0} \mathbb{P}^0\big(|Z_{T_{\varepsilon'}}| \in [a, \infty)\big).$$

By assumption, under $\mathbb{P}^0$, paths are right-continuous and start from zero so the limiting probability on the right-hand side vanishes. As δ is arbitrary, we proved that $\mathbb{P}^0(|Z_{T_\varepsilon}| < c) = 0$ for all $\varepsilon, c > 0$ which contradicts the first property listed in Theorem 6.

The sufficiency of stationary overshoots in Theorem 6 is nontrivial. Here is the strategy of the proof, the potential theoretic terminology will be clarified in the course of the proof.

Step 1: Suppose $\{\mathbb{P}^z : z \neq 0\}$ is a Markov family that is continuous in $\mathbb{R} \setminus \{0\}$ with respect to weak convergence on the Skorokhod space—which is true for real self-similar Markov processes due to the Lamperti–Kiu representation—and $\mathbb{P}^0$ is a candidate for the weak limit $\lim_{|z|\downarrow 0} \mathbb{P}^z$. Then a natural guess, for instance from Aldous' criterion, of conditions for the weak convergence is as follows:

(a) All overshoots for given levels should converge weakly to the overshoot of $\mathbb{P}^0$ for that level. If so, then nothing has to be controlled past the overshoots due to the strong Markov property and the weak continuity of $z \mapsto \mathbb{P}^z$ away from 0.

(b) The behaviour before the overshoots should be nice in the sense that overshoots over small levels will occur quickly.

To summarize, and this is the content of our Proposition 7, to have weak convergence of $\mathbb{P}^z$ toward $\mathbb{P}^0$ one needs control on overshoots and times of overshoots under $\mathbb{P}^z$. For real self-similar Markov processes, both quantities can be expressed and analyzed through the Lamperti–Kiu representation and fluctuation theory for Markov additive processes.

Step 2: To construct the candidate $\mathbb{P}^0$ assumed in Step 1, we use potential theory: If $\mathbb{P}^0$ is the self-similar process started from zero, then it is a restriction of the Kuznetsov measure $\mathcal{Q}_\eta$ corresponding to the excessive measure $\eta(dx) = \mathbb{E}^0[\int_0^\infty 1_{(Z_s \in dx)} ds]$, Proposition 3.2 of [13]. Of course, η is not known a priori without knowing $\mathbb{P}^0$ but this Ansatz leads to a good guess for $\mathbb{P}^0$: $\mathbb{P}^0$ is necessarily the restriction of a Kuznetsov measure for some purely excessive measure. Since there are many excessive measures of which only one can be the good one, the Ansatz might be too naive.

What saves us here is the Lamperti–Kiu representation and Kaspi's time-change theorem: Combined they tell us that the excessive measure should be the Revuz measure of an invariant measure of the MAP. Since invariant measures for MAPs are easy to find, this approach works.

Potential theory is most effective when the Markov process is transient. We distinguished two cases in our proof: (ξ, J) drifting to $+\infty$ and (ξ, J) oscillating. In the latter case, the transience is artificially achieved by killing at T_1. Such a killing is by no means unnatural: Since only the entrance behaviour from 0 needs clarification, it is equivalent to explain the entrance behaviour for the entire process or the process killed at a set bounded away from 0.

1.4. *Organisation of the article*.   The main argument is relatively short but also we need to develop a fair amount of fluctuation theory for Markov additive processes, which, unfortunately, does not appear to be present in mainstream literature. In order to keep a clear focus, the proof is split into two parts: In the next section, we give the main argument containing Lamperti–Kiu based calculations for overshoots and times of overshoots (Section 2.2) and the potential theoretic construction of $\mathbb{P}^0$ (Section 2.3). The fluctuation theory is collected in an Appendix.

2. Proof.   Throughout the proof, fluctuation theory for Markov additive processes is applied as developed in the Appendix. Unless otherwise stated, we assume throughout that (**NL**), (**I**) and (**C**) are in force. An initial browse of the Appendix at this point may prove to be instructive before digesting the remainder of this section. The main items that are needed from the Appendix is the role of the occupation formula (Theorem 27), the Markov renewal theorem (Theorem 28) and the equivalent conditions for the existence of stationary overshoots (Theorem 35).

2.1. *Convergence lemma*.   The following proposition is the formalization of Step 1 in the sketch of the proof given in Section 1.3.

PROPOSITION 7.   *Suppose the following conditions hold for a strong Markov family* $\{\mathbb{P}^z : z \in \mathbb{R} \setminus \{0\}\}$ *and a candidate law* $\mathbb{P}^0$ *on* $(\mathbf{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$:

(1a) $\lim_{\varepsilon \to 0} \limsup_{|z| \to 0} \mathbb{E}^z[T_\varepsilon] = 0$,

(1b) w-$\lim_{z \to 0} \mathbb{P}^z(Z_{T_\varepsilon} \in \cdot) =: \mu_\varepsilon(\cdot)$ *exists for all* $\varepsilon > 0$,

(1c) $\mathbb{R} \setminus \{0\} \ni z \mapsto \mathbb{P}^z$ *is continuous in the weak topology on the Skorokhod space*

and

(2a) $\mathbb{P}^0$-*almost surely,* $Z_0 = 0$ *and* $Z_t \neq 0$ *for all* $t > 0$,

(2b) $\mathbb{P}^0((Z_{T_\varepsilon + t})_{t \geq 0} \in \cdot) = \mathbb{P}^{\mu_\varepsilon}(\cdot)$ *for every* $\varepsilon > 0$.

Then the mapping

$$\mathbb{R} \ni z \mapsto \mathbb{P}^z$$

is continuous in the weak topology on the Skorokhod space.

PROOF.   To show convergence in the Skorokhod topology, we work with Prokhorov's metric: for $m \in \mathbb{N}$ and two càdlàg paths $x, y : \mathbb{R}_+ \to \mathbb{R}$ define

$$d_m(x, y) = \inf\{\delta > 0 : \exists \text{ an increasing continuous function}$$

$$S : [0, m] \to [0, \infty) \text{ with } S_0 = 0, \|S - \mathrm{id}\,\|_{[0,m]} \leq \delta$$

$$\text{and } \|x \circ S - y\|_{[0,m]} \leq \delta\}$$

and set

$$d(x, y) = \sum_{m=1}^{\infty} 2^{-m}\big(d_m(x, y) + d_m(y, x)\big) \wedge 1.$$

Since d generates the Skorokhod topology it suffices to verify that, for arbitrary bounded Lipschitz functions $f : \mathbf{D}(\mathbb{R}) \to \mathbb{R}$ with Lipschitz constant κ, say, one has

$$\mathbb{E}^{z_n}[f(Z)] \to \mathbb{E}^0[f(Z)]$$

for every sequence $(z_n) \to 0$. By property (1b),

$$\underset{z_n \to 0}{\text{w-}\lim}\, \mathbb{P}^{z_n}(Z_{T_\varepsilon} \in \cdot) = \mu_\varepsilon(\cdot),$$

so that by the continuity property (1c)

$$\underset{z_n \to 0}{\text{w-}\lim} \int \mathbb{P}^x(\cdot)\mathbb{P}^{z_n}(Z_{T_\varepsilon} \in dx) = \int \mathbb{P}^x(\cdot)\mu_\varepsilon(dx) = \mathbb{P}^{\mu_\varepsilon}(\cdot).$$

In combination with the Markov property and property (2b), we get

$$\underset{z_n \to 0}{\text{w-}\lim}\, \mathbb{P}^{z_n}\big((Z_{T_\varepsilon + \cdot}) \in \cdot\big) = \underset{z_n \to 0}{\text{w-}\lim} \int \mathbb{P}^x(\cdot)\mathbb{P}^{z_n}(Z_{T_\varepsilon} \in dx)$$

$$= \mathbb{P}^{\mu_\varepsilon}(\cdot)$$

$$= \mathbb{P}^0\big((Z_{T_\varepsilon + \cdot}) \in \cdot\big).$$

Using the Skorokhod coupling, we can define càdlàg processes $Z^0, Z^1, Z^2, \ldots$ on an appropriate probability space $(\Omega, \mathcal{F}, P)$ on which:

- $\mathcal{L}(Z^n) = \mathcal{L}_{z_n}(Z)$ for $n \in \mathbb{N}$ and $\mathcal{L}(Z^0) = \mathcal{L}_0(Z)$,
- $(Z^n_{T^n_\varepsilon + \cdot}) \to (Z_{T_\varepsilon + \cdot})$, almost surely, in the Skorokhod space.

For $n \in \{0, 1, \ldots\}$ we denote by T^n_ε the first entrance time of Z^n into $(-\varepsilon, \varepsilon)^c$. We note that, for every $m \in \mathbb{N}$ and $n, n' \in \{0, 1, \ldots\}$,

$$(7) \qquad d_m(Z^n, Z^{n'}) \le 2\varepsilon + |T^n_\varepsilon - T^{n'}_\varepsilon| + d_m((Z^n_{T^n_\varepsilon + \cdot}), (Z^{n'}_{T^{n'}_\varepsilon + \cdot}))$$

which yields

$$d(Z^n, Z^0) \le 4\varepsilon + 2|T^n_\varepsilon - T^0_\varepsilon| \wedge 1 + d((Z^n_{T^n_\varepsilon + \cdot}), (Z^0_{T^0_\varepsilon + \cdot})).$$

Consequently, using Lipschitz continuity of f, we get

$$\begin{aligned} |E[f(Z^n)] &- E[f(Z)]| \\ &\le \kappa E[d(Z^n, Z)] \\ &\le 4\kappa\varepsilon + 2\kappa E[|T^n_\varepsilon - T^0_\varepsilon| \wedge 1] + \kappa E[d((Z^n_{T^n_\varepsilon + \cdot}), (Z^0_{T^0_\varepsilon + \cdot}))]. \end{aligned}$$

By dominated convergence, this gives

$$\limsup_{n \to \infty} |E[f(Z^n)] - E[f(Z)]| \le 4\kappa\varepsilon + 2\kappa \limsup_{n \to \infty} E[|T^n_\varepsilon - T^0_\varepsilon| \wedge 1]$$

and letting $\varepsilon \to 0$ yields the result since by property (1a),

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} E[T^n_\varepsilon \wedge 1] = 0$$

and, using (2a), $\lim_{\varepsilon \to 0} E[T^0_\varepsilon \wedge 1] = 0$. $\quad \square$

2.2. *Verification of conditions* (1a)–(1c).   To verify the first three conditions of Proposition 7, we use the Lamperti–Kiu representation and fluctuation theory for Markov additive processes.

LEMMA 8.   *Condition* (1a) *from Proposition* 7 *holds.*

PROOF.   Using the Lamperti–Kiu representation (4), one has

$$T_\varepsilon = \inf\{t \ge 0 : |Z_t| \ge \varepsilon\} \stackrel{(d)}{=} \inf\{t : \xi_{\varphi^{-1}(t)} \ge \log(\varepsilon)\} = \varphi(\tau^+_{\log(\varepsilon)})$$

with $\tau^+_{\log(\varepsilon)} = \inf\{t : \xi_t \ge \log(\varepsilon)\}$. Taking expectations and applying the definition of φ yields

$$\mathbb{E}^z[T_\varepsilon] = E^{\log|z|,[z]}[\varphi(\tau^+_{\log(\varepsilon)})] = E^{\log|z|,[z]}\left[\int_0^{\tau^+_{\log(\varepsilon)}} e^{\alpha\xi_s}\, ds\right].$$

In order to calculate the right-hand side, we use results from the Appendix. Let $\hat{P}$ be the law of the dual MAP introduced in Appendix A.2. It will be useful below to note that, for example, for bounded measurable functions f,

$$E^{z,i}\big[f(-\xi_t), J_t = j\big] = \frac{\pi_j}{\pi_i} \hat{E}^{-z,j}\big[f(\xi_t), J_t = i\big], \qquad z \in \mathbb{R}, i, j \in \{\pm 1\}, t \geq 0.$$

[Compare, for instance, (20) in the Appendix.] Similarly to Lévy processes, MAPs are spatially homogeneous in the first variable. Using duality in the second and homogeneity in the third equality gives

$$E^{\log|z|,[z]}\left[\int_0^{\tau^+_{\log(\varepsilon)}} e^{\alpha\xi_s}\, ds\right]$$

$$= \sum_{j=\pm 1} E^{\log|z|,[z]}\left[\int_0^{\tau^+_{\log(\varepsilon)}} e^{\alpha\xi_s}\, ds;\, J_{\tau^+_{\log(\varepsilon)}} = j\right]$$

$$= \sum_{j=\pm 1} \frac{\pi_j}{\pi_{[z]}} \hat{E}^{-\log|z|,j}\left[\int_0^{\tau^-_{-\log(\varepsilon)}} e^{-\alpha\xi_s}\, ds;\, J_{\tau^-_{-\log(\varepsilon)}} = [z]\right]$$

$$= \sum_{j=\pm 1} \frac{\pi_j}{\pi_{[z]}} \hat{E}^{\log(\varepsilon/|z|),j}\left[\int_0^{\tau^-_0} e^{-\alpha(\xi_s - \log(\varepsilon))}\, ds;\, J_{\tau^-_0} = [z]\right]$$

$$= \varepsilon^\alpha \sum_{j=\pm 1} \frac{\pi_j}{\pi_{[z]}} \hat{E}^{\log(\varepsilon/|z|),j}\left[\int_0^{\tau^-_0} e^{-\alpha\xi_s}\, ds;\, J_{\tau^-_0} = [z]\right]$$

$$\leq \varepsilon^\alpha \sum_{j,k=\pm 1} \frac{\pi_j}{\pi_{[z]}} \hat{E}^{\log(\varepsilon/|z|),j}\left[\int_0^{\tau^-_0} e^{-\alpha\xi_s} 1_{(J_s=k)}\, ds\right].$$

Appealing to Remark 25 and Theorem 27 in Appendix A.5, we can put the pieces above together and write

$$\mathbb{E}^z[T_\varepsilon] \leq \varepsilon^\alpha \sum_{j,k=\pm 1} \frac{\pi_j}{\pi_{[z]}} \sum_{\ell=\pm 1} \int_{[0,\infty)} e^{-\alpha y} \hat{U}^+_{j,\ell}(dy)$$

$$\times \int_{[0,\log(\varepsilon/|z|)]} e^{-\alpha(\log(\varepsilon/|z|)-z)} \frac{\pi_k}{\pi_\ell} U^+_{k,\ell}(dz),$$

where the measure $U^+_{k,\ell}$ (resp. $\hat{U}^+_{j,\ell}$) is the potential measure of the ascending (resp., dual ascending) Markov additive ladder height process of ξ. The reader is referred to Appendix A.5 for the precise definition. What is important to note for their use in this proof are the following two facts. First, the integrals $\int_{[0,\infty)} e^{-\alpha y} \hat{U}^+_{j,\ell}(dy)$ are all finite; see, for example, formula (28) in Appendix A.5. Second, the key renewal-type theorem given in Theorem 28(ii) of Appendix A.6

ensures that

$$\lim_{|z| \to 0} \int_{[0, \log(\varepsilon/|z|))} e^{-\alpha(\log(\varepsilon/|z|) - z)} U_{k,\ell}^+(dz) = \pi_\ell / \alpha E^{0,\pi}[H_1^+]$$

for each $k, \ell \in \{\pm 1\}$, where the exact nature of the expectation $E^{0,\pi}[H_1^+] \in (0, \infty]$ is again explained in the Appendix. All that we need to know at this point of the argument is that it is finite. This follows from Theorem 35 in the Appendix thanks to Condition (C). In conclusion, we have that there exists a constant $C \in (0, \infty)$ such that

$$\lim_{\varepsilon \to 0} \lim_{|z| \to 0} \mathbb{E}^z[T_\varepsilon] \leq \lim_{\varepsilon \to 0} C \varepsilon^\alpha = 0$$

and the proof is complete.   □

In the next lemma, we deduce the overshoot distributions for real self-similar Markov processes from the overshoot distributions of the corresponding Markov additive processes. In particular, we prove that Condition (1b) is satisfied in the setting of Theorem 6.

LEMMA 9.   *Let $\varepsilon > 0$ and $|y| \geq \varepsilon$.*

(i) *There are proper weak limits*

$$\text{w-}\lim_{|z| \to 0} \mathbb{P}^z(Z_{T_\varepsilon} \in dy) = \mu_\varepsilon(dy),$$

if and only if Condition (C) holds.

(ii) *If Condition (C) holds, then $\mathbb{P}^{\mu_\varepsilon}(Z_{T_{\varepsilon'}} \in dz) = \mu_{\varepsilon'}(dz)$ for $0 < \varepsilon < \varepsilon'$ and $|z| \geq \varepsilon'$.*

PROOF.   (i) The Lamperti–Kiu representation (4) and spatial homogeneity of Markov additive processes imply that, for $0 < a < b$,

$$\mathbb{P}^z(Z_{T_\varepsilon} \in [a, b])$$
$$= P^{\log|z|, [z]}(\exp(\xi_{\tau_{\log(\varepsilon)}^+}) \in [a, b]; J_{\tau_{\log(\varepsilon)}^+} = 1)$$
$$= P^{\log|z|, [z]}(\xi_{\tau_{\log(\varepsilon)}^+} - \log(\varepsilon) \in [\log(a/\varepsilon), \log(b/\varepsilon)]; J_{\tau_{\log(\varepsilon)}^+} = 1)$$

and, analogously,

$$\mathbb{P}^z(Z_{T_\varepsilon} \in [-b, -a])$$
$$= P^{\log|z|, [z]}(\xi_{\tau_{\log(\varepsilon)}^+} - \log(\varepsilon) \in [\log(a/\varepsilon), \log(b/\varepsilon)]; J_{\tau_{\log(\varepsilon/|z|)}^+} = -1).$$

Hence, the distributions $\mathcal{L}^z(Z_{T_\varepsilon})$ converge for $|z| \to 0$ if and only if the overshoots of the Markov additive process converge to a proper limit. This is equivalent to Condition (C) by Theorem 5.

(ii) We use the strong Markov property and (i) for an interval A:

$$\mu_{\varepsilon'}(A) = \lim_{|z| \to 0} \mathbb{P}^z(Z_{T_{\varepsilon'}} \in A)$$

$$= \lim_{|z| \to 0} \int \mathbb{P}^x(Z_{T_{\varepsilon'}} \in A)\mathbb{P}^z(Z_{T_\varepsilon} \in dx)$$

$$= \lim_{|z| \to 0} \int f_A(x)\mathbb{P}^z(Z_{T_\varepsilon} \in dx)$$

with $f_A(x) := \mathbb{P}^x(Z_{T_{\varepsilon'}} \in A)$. Using that f_A is bounded and continuous [see (30) and the remark beneath it] and the weak convergence from (i) yields

$$\mu_{\varepsilon'}(A) = \int f_A(x)\mu_\varepsilon(dx) = \mathbb{P}^{\mu_\varepsilon}(Z_{T_{\varepsilon'}} \in A)$$

as required. □

A direct consequence of the Lamperti–Kiu representation (4) is the following result.

LEMMA 10. *Condition* (1c) *from Proposition* 7 *holds.*

2.3. *Verification of conditions* (2a)–(2b) *and construction of* $\mathbb{P}^0$. In this section, we construct the measure $\mathbb{P}^0$ and verify conditions (2a)–(2b) of Proposition 7. Before doing so, we give a brief overview of some notation and results from probabilistic potential theory. For a more detailed account, the reader is referred to Dellacherie et al. [8] (available in French only).

Notation. We work in the setting of Fitzsimmons and Maisonneuve [14] that was also used by Kaspi [18].

Let E be a locally compact Polish space equipped with its Borel σ-algebra $\mathcal{E}$. We extend E by an isolated cemetery state ∂ and also equip the extended space $E \cup \{\partial\}$ with its respective Borel σ-algebra. Let W be the space of functions $w : \mathbb{R} \to E \cup \{\partial\}$ that are E-valued and càdlàg on a nonempty interval $(\alpha(w), \beta(w))$ and are equal to ∂ on the complement of $(\alpha(w), \beta(w))$. One calls $\alpha(w) = \inf\{t : w_t \in E\}$ the time of birth, $\beta(w) = \sup\{t : w_t \in E\}$ the time of death and $\zeta(w) := \beta(w) - \alpha(w)$ the life-time. We denote by $(Y_t(w))_{t \in \mathbb{R}} = (w_t)_{t \in \mathbb{R}}$ the canonical process on W and by $\mathcal{G} = \sigma(Y_s : s \in \mathbb{R})$ the canonical σ-algebra on W. Assume that $P = (P_t)_{t \geq 0}$ is the transition semigroup of a Feller process on E. A family $(\eta_t)_{t \in \mathbb{R}}$ of measures on $(E, \mathcal{E})$ is called an *entrance rule* for P if $\eta_t P_{s-t} \leq \eta_s$ for $s > t$, and an *entrance law* (*at time zero*) if $\eta_t = 0$ for $t \leq 0$ and $\eta_t P_{s-t} = \eta_s$ for $s \geq t > 0$. In the stationary case where $\eta_t \equiv m$, m is called *excessive measure*. Write $\mathcal{Q}_\eta$ for the Kuznetsov measure corresponding to (η, P) and $\mathcal{Q}_m$ for the stationary

case. That is to say, $\mathcal{Q}_\eta$ is the unique measure on $(W, \mathcal{G})$ with one-dimensional marginals η_t and transition semigroup (P_t). More precisely,

$$\mathcal{Q}_\eta\big(\alpha(Y) < t_1, Y_{t_1} \in dx_1, \ldots, Y_{t_n} \in dx_n, t_n < \beta(Y)\big)$$

$$= \eta_{t_1}(dx_1) P_{t_2-t_1}(x_1, dx_2) \cdots P_{t_n-t_{n-1}}(x_{n-1}, dx_n)$$

for $-\infty < t_1 < \cdots < t_n < +\infty$. Under a Kuznetsov measure, the canonical process is a strong Markov process with random birth and death, that is, if τ is a stopping time with respect to the canonical right continuous filtration $(\mathcal{G}_t)$ one has

$$\mathcal{Q}_\eta\big((Y_{\tau+t})_{t\geq 0} \in \cdot | \mathcal{G}_\tau\big) = P^{Y_\tau}(\cdot), \qquad \text{on } \{\alpha < \tau < \beta\}.$$

Conditions under which Kuznetsov measures $\mathcal{Q}_\eta$ exist (which are necessarily satisfied in the stationary case) can be found in Kuznetsov's work [22].

For the stationary case $\eta_t = m$, a particularly simple construction of Kuznetsov measures was given by Mitro [26] for a Markov process in duality, with respect to m, to a second Markov process $(\hat{X}_t)_{t\geq 0}$ with transition semigroup $(\hat{P}_t)_{t\geq 0}$, that is,

$$(8) \qquad\qquad P_t(x, dy)m(dx) = \hat{P}_t(y, dx)m(dy).$$

In the dual setting $\mathcal{Q}_m$ is the unique measure on $(W, \mathcal{G})$ that is translation invariant and has finite dimensional marginals

$$\mathcal{Q}\big(\alpha(Y) < s_l, Y_{s_l} \in dy_l, \ldots, Y_{s_1} \in dy_1, Y_{t_1} \in dx_1, \ldots, Y_{t_k} \in dx_k, \beta(Y) > t_k\big)$$

$$= \int_E m(dx) \hat{P}^x[\hat{X}_{s_1} \in dy_1, \ldots, \hat{X}_{s_l} \in dy_l] P^x[X_{t_1} \in dx_1, \ldots, X_{t_k} \in dx_k]$$

at the times $s_l < \cdots < s_1 < 0 \leq t_1 < \cdots < t_k$. In words, to build $\mathcal{Q}_m|_{\{\alpha<0<\beta\}}$ one samples the invariant measure m at time 0, and from the outcome starts an independent copy of X to the right and an independent copy of the dual $\hat{X}$ to the left. An important consequence is that time-reversing the Kuznetsov measure for (η, P) yields the Kuznetsov measure for $(\eta, \hat{P})$. We should also recall the fact:

- $\mathcal{Q}_m(\alpha = -\infty) = 0$ if m is purely excessive (i.e., $mP_t \to 0$ as $t \to \infty$),
- $\mathcal{Q}_m(\alpha > -\infty) = 0$ if m is invariant (i.e., $mP_t = m$ for all $t > 0$).

Later on we will use an entrance law at time zero for the real self-similar Markov process to construct $\mathcal{Q}_\eta$—recall that automatically $\alpha = 0$ for almost all trajectories—and via $\mathcal{Q}_\eta$ extend the Markov family $\{\mathbb{P}^z : z \in \mathbb{R} \setminus \{0\}\}$ in the following way.

LEMMA 11. *Let $E \cup \{\theta\}$ be a Polish space and let $\{P^x : x \in E\}$ denote a (killed) Markov family on the space E. Suppose that (η_t) is an entrance law for the Markov family on E for which the corresponding Kuznetsov measure $\mathcal{Q}_\eta$ fulfills*:

(i) *$\mathcal{Q}_\eta$ is a finite nontrivial measure,*

(ii) $\lim_{t \to 0} Y_t = \theta$, $\mathcal{Q}_\eta$-a.e., in the space $E \cup \{\theta\}$

and define the restriction mapping

$$\pi : W \to \mathbf{D}(E \cup \{\theta, \partial\}), \qquad \pi(w)_t = \begin{cases} \theta, & t = 0, \\ w_t, & t > 0, \end{cases}$$

where $E \cup \{\theta, \partial\}$ denotes the extension of the Polish space $E \cup \{\theta\}$ by an isolated point ∂ and $\mathbf{D}(E \cup \{\theta, \partial\})$ denotes the corresponding space of càdlàg functions. For the normalized measure,

$$P^\theta(A) := \frac{\mathcal{Q}_\eta(\pi^{-1}(A))}{\mathcal{Q}_\eta(W)}, \qquad A \in \mathcal{D}(E \cup \{\theta \cup \partial\}),$$

the extended family $\{P^x : x \in E \cup \{\theta\}\}$ is a (killed) Markov family on $E \cup \{\theta\}$ so that under P^θ the canonical process leaves the initial value θ instantaneously and satisfies the strong Markov property for strictly positive stopping times.

The lemma is an immediate consequence of the strong Markov property of Kuznetsov measures.

In order to construct a good entrance law at zero for the real self-similar Markov process, we use the theory of random time-changes for Kuznetsov measures as developed by Kaspi.

Random time-change. Let us recall Theorems (2.3) and (2.10) of Kaspi [18] in the simplest form: Given a (killed) Markov process on E with transition semigroup (P_t) and a locally bounded measurable function $h : E \cup \{\partial\} \to (0, \infty)$ that defines a time-changed Markov transition semigroup via

$$\tilde{P}_t f(x) := E^x[f(Z_{S_t})], \qquad \text{where } S_t = \inf\left\{s > 0 : \int_0^s h(Z_u)\, du > t\right\}.$$

Let $\mathcal{Q}_m$ be the Kuznetsov measure for (m, P) and suppose

$$B_t := \int_{(\alpha, t]} h(Y_s)\, ds < \infty$$

for almost all realizations (by time homogeneity of $\mathcal{Q}_m$ it suffices to check the property only for time $t = 0$). Then there is an entrance law (η_t) at time zero for $(\tilde{P}_t)$ such that the corresponding Kuznetsov measure $\tilde{\mathcal{Q}}_\eta$ satisfies

$$\tilde{\mathcal{Q}}_\eta(A, \beta > t) = \mathcal{Q}_m(\pi^{-1}(A), 0 < B_t^{-1} \leq 1), \qquad A \in \mathcal{G}, t > 0,$$

where

$$\pi(Y)_t = \begin{cases} Y_{B_t^{-1}}, & t > 0, \\ \partial, & t \leq 0. \end{cases}$$

In what follows, we fix the MAP (ξ, J) on $\mathbb{R} \times \{\pm 1\}$ obtained from the given real self-similar Markov process through the Lamperti–Kiu representation and consider the time-change

(9)
$$\tilde{P}_t f(x, i) := E^{x,i}[f(\xi_{S_t}, J_{S_t})],$$

$$\text{where } S_t = \inf\left\{s > 0 : \int_0^s e^{\alpha \xi_u} \, du > t\right\}.$$

We use the knowledge of invariant measures for MAPs to construct an entrance law at zero for $(\tilde{P}_t)$; thus, through concatenation with $h(x, i) = \exp(x)i$, for the real self-similar Markov process.

LEMMA 12. *If (ξ, J) drifts to $+\infty$, then there exists a distribution $\mathbb{P}^0$ on $(\mathbf{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ for which Conditions (2a) and (2b) of Proposition 7 hold.*

PROOF. We construct an entrance law (η_t) at time zero for $(\tilde{P}_t)$ such that the associated Kuznetsov measure $\tilde{\mathcal{Q}}_\eta$ satisfies, for $Y = (Y^1, Y^2)$:

  (i) $\lim_{t \downarrow 0} Y_t^1 = -\infty$ and $\beta(Y) = \infty$, $\tilde{\mathcal{Q}}_\eta$-a.e.,
 (ii) $\tilde{\mathcal{Q}}_\eta$ is a finite measure,
(iii) if $\tau_z^+ = \inf\{t : Y_t^1 \geq z\}$ for $z \in \mathbb{R}$ then

$$\tilde{\mathcal{Q}}_\eta\big((Y_{\tau_z^+}^1 - z, Y_{\tau_z^+}^2) \in (dx, \{i\})\big) = \tilde{\mathcal{Q}}_\eta(W)\nu(dx, \{i\}),$$

where ν is the stationary overshoot distribution appearing in Theorem 28 for the MAP (ξ, J).

If such a measure $\tilde{\mathcal{Q}}_\eta$ can be constructed, then by the Lamperti–Kiu representation (4) and through Lemma 11, we obtain $\mathbb{P}^0$ from $\tilde{\mathcal{Q}}_\eta$ by taking its image under the pathwise transformation $h(\xi, J) = \exp(\xi)J$ and normalizing to a probability measure. The claimed properties (2a) and (2b) follow from the construction.

Lemma 22 in the Appendix shows that (ξ, J) and $(\hat{\xi}, \hat{J})$ are in duality on $E = \mathbb{R} \times \{\pm 1\}$ with respect to the invariant measure $m(dx, \{i\}) = dx \, \pi(i)$. By assumption, (ξ, J) drifts to $+\infty$ and the dual $(\hat{\xi}, \hat{J})$ drifts to $-\infty$. We use Mitro's construction for $\mathcal{Q}_m$: Sample (x, i) from m and start independently copies of $P^{x,i}$ in the positive time-direction and $\hat{P}^{x,i}$ in the negative time-direction. We conclude that, $\mathcal{Q}_m$-a.e., $\alpha(Y) = -\infty$ and $\beta(Y) = +\infty$ as well as

(10)
$$\lim_{t \to -\infty} Y_t^1 = -\infty \quad \text{and} \quad \lim_{t \to +\infty} Y_t^1 = +\infty.$$

We now apply Kaspi's time-change as discussed above the lemma to $\mathcal{Q}_m$ with $B_t = \int_{-\infty}^t \exp(\alpha Y_r^1) \, dr$. In order to use Kaspi's result, we need to check that $B_0 < \infty$ for $\mathcal{Q}_m$-almost all realizations. From the two-sided construction of $\mathcal{Q}_m$, it is enough to show that $\int_0^\infty \exp(\alpha \xi_r) \, dr < \infty$ for $\hat{P}^{x,i}$-almost all (ξ, J). This holds due to

the law of large numbers for the dual Markov additive process that drifts to $-\infty$. Hence, there is an entrance law (η_t) at time zero for $(\tilde{P}_t)$ and the corresponding Kuznetsov measure $\tilde{\mathcal{Q}}_\eta$ satisfies

$$(11) \qquad \tilde{\mathcal{Q}}_\eta(A, \beta > t) = \mathcal{Q}_m\big(\pi^{-1}(A), 0 < B_t^{-1} \le 1\big), \qquad A \in \mathcal{F},$$

with $\pi(Y)_t = Y_{B_t^{-1}}$ for $t > 0$ and $\pi(Y)_t = \partial$ for $t \le 0$. Formula (11) combined with (10) entail property (i).

Next, we show that the measure $\tilde{\mathcal{Q}}_\eta$ is finite. We combine convergence of the overshoots of the MAP with Theorem (2.3) of Kaspi. By Theorem 28 in the Appendix, there exists a limiting overshoot distribution for the MAP, say ν. We choose $c > 0$ such that $\nu((0, c) \times \{\pm 1\}) > 0$ and set $A = (0, c) \times \{\pm 1\}$. Note that the map

$$\mathbb{R} \ni x \mapsto E^{x,i}\left[\int_0^\infty \mathbf{1}_A(\xi_{S_t}, J_{S_t)}) \, dt\right]$$

is lower semi-continuous so that by the Markov property and weak convergence of the overshoot distribution

$$\liminf_{x \downarrow -\infty} E^{x,i}\left[\int_0^\infty \mathbf{1}_A(\xi_{S(t)}, J_{S(t)}) \, dt\right] \ge E^\nu\left[\int_0^\infty \mathbf{1}_A(\xi_{S(t)}, J_{S(t)}) \, dt\right] =: \kappa > 0.$$

Hence, by Fatou's inequality and the strong Markov property for $\tilde{\mathcal{Q}}_\eta$,

$$(12) \qquad \tilde{\mathcal{Q}}_\eta\left(\int_0^\infty \mathbf{1}_A(Y_s) \, ds\right) \ge \liminf_{\varepsilon \downarrow 0} \tilde{\mathcal{Q}}_\eta\left(E^{Y_\varepsilon}\left[\int_0^\infty \mathbf{1}_A(\xi_{S(t)}, J_{S(t)}) \, dt\right]\right)$$

$$\ge \kappa \tilde{\mathcal{Q}}_\eta(W),$$

where we have used that $\lim_{\varepsilon \downarrow 0} Y_\varepsilon^1 = -\infty$ $\tilde{\mathcal{Q}}_\eta$-a.e. Furthermore, Theorem (2.3) of Kaspi relates the occupation time of the set A under the measures $\tilde{\mathcal{Q}}_\eta$ and $\mathcal{Q}_m$ as follows:

$$(13) \qquad \tilde{\mathcal{Q}}_\eta\left(\int_0^\infty \mathbf{1}_A(Y_s^1) \, ds\right) = \mathcal{Q}_m\left(\int_{[0,1)} \mathbf{1}_A(Y_t^1) e^{\alpha Y_t^1} \, dt\right)$$

$$= \int_A e^{\alpha y_1} m(dy) < \infty.$$

Here, we used that we can interchange the order of integration by Fubini's theorem since $\mathcal{Q}_m$ is σ-finite by construction. Combining the two display formulas (12) and (13) gives that $\tilde{\mathcal{Q}}_\eta(W)$ is finite and nonzero. Thus, we proved property (ii).

To prove property (iii), we note that the overshoot distribution is not affected by a time change, and hence agrees for (P_t) and $(\tilde{P}_t)$. Consequently, using the

Markov property under the measure $\tilde{\mathcal{Q}}_\eta$ we get that

$$
\tilde{\mathcal{Q}}_\eta\big((Y^1_{\tau^+_z} - z, Y^2_{\tau^+_z}) \in \cdot\big) = \underset{k\downarrow-\infty}{\text{w-}\lim}\, \tilde{\mathcal{Q}}_\eta\big[\tilde{P}^{Y_{\tau^+_k}}\big((\xi_{\tau^+_z} - z, J_{\tau^+_z}) \in \cdot\big)\big]
$$

$$
= \underset{k\downarrow-\infty}{\text{w-}\lim}\, \tilde{\mathcal{Q}}_\eta\big[P^{Y_{\tau^+_k}}\big((\xi_{\tau^+_z} - z, J_{\tau^+_z}) \in \cdot\big)\big]
$$

$$
= \tilde{\mathcal{Q}}_\eta(W)\nu(\cdot).
$$

This shows (iii) and the proof is complete. $\square$

The same proof cannot be carried out if (ξ, J) oscillates. Choosing the same invariant measure η leads to a Kuznetsov measure $\mathcal{Q}_\eta$ under which trajectories oscillate in both directions of time. Hence, there is no way this construction yields a law $\mathbb{P}^0$ satisfying (2a) of Proposition 7. Essentially, the problem is that Z is not transient. To circumvent this issue, Z is killed at T_1 and then we proceed similarly as before. This is captured in the lemma below.

REMARK 13. Before turning to the aforesaid lemma, let us note that the cases that (ξ, J) drifts to $+\infty$ or oscillates can of course be treated both with killing as in the proof of Lemma 14. In order to work out clearly the main ideas we preferred to give the more transparent case without killing separately in Lemma 12. In particular, the curious reader will find it easier to compare our proof to Fitzsimmons' [12] construction of excursion measures in the recurrent case.

LEMMA 14. *If (ξ, J) oscillates, then there exists a distribution $\mathbb{P}^0$ on the measurable space $(\mathbf{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ for which Conditions* (2a) *and* (2b) *of Proposition 7 hold.*

PROOF. We mimic the proof of Lemma 12 with additional killing.

Recall from Remark 32 in the Appendix that there exists a harmonic function $(x, i) \mapsto U_i^+(x)$ related to the MAP killed when its first component reaches the positive half-line, henceforth denoted by $(\xi^\dagger, J^\dagger)$. The corresponding h-transformed process is indicated with the superscript $\downarrow$. We shall also write their respective transition kernels as $P_t^\dagger((x, i), (dy, \{j\}))$ and $P_t^\downarrow((x, i), (dy, \{j\}))$, with the addition of a hat to mean the dual map as defined in Appendix A.2.

Next, we show duality in the sense of (8) for $(\hat{\xi}^\downarrow, \hat{J}^\downarrow)$ and $(\xi^\dagger, J^\dagger)$ with respect to the duality measure $m(dx, \{i\}) = \pi_i \hat{U}_i^+(x)\,dx$ on $(-\infty, 0) \times \{\pm 1\}$. The duality comes from the short calculation

$$
\hat{P}_t^\downarrow((x, i), (dy, \{j\}))m(dx, \{i\}) = \frac{\hat{U}_j^+(y)}{\hat{U}_i^+(x)}\hat{P}_t^\dagger((x, i), (dy, \{j\}))\pi_i \hat{U}_i^+(x)\,dx
$$

$$
= \pi_j \hat{U}_j^+(y)P_t^\dagger((y, j), (dx, \{i\}))\,dy
$$

$$
= P_t^\dagger((y, j), (dx, \{i\}))m(dy, \{j\}),
$$

where we used the definition of m twice, the generic h-transform formula

$$(14) \qquad P_t^h(x, dy) = \frac{h(y)}{h(x)} P_t(x, dy)$$

for transition probabilities of h-transformed processes, and the ordinary MAP duality formula

$$\hat{P}_t^\dagger((x, i), (dy, \{j\}))\pi_i \, dx = P_t^\dagger((y, j), (dx, \{i\}))\pi_j \, dy$$

from Lemma 22 in the Appendix.

Mitro's construction of the Kuznetsov measure $\mathcal{Q}_m^\dagger$ for the killed MAP with respect to m works as follows: Sample $(x, i) \in (-\infty, 0) \times \{\pm 1\}$ according to m at time zero and start independently a copy of the killed process $P^{x,i,\dagger}$ in positive time-direction and a copy of the conditioned process $\hat{P}^{x,i,\downarrow}$ in negative time-direction. Since the MAP was assumed to oscillate, the killing time of the former is finite almost surely. Furthermore, the conditioned process drifts to $-\infty$ almost surely by Proposition 33 in the Appendix. Hence, almost all trajectories $Y = (Y^1, Y^2)$ under $\mathcal{Q}_m^\dagger$ are born at time $\alpha(Y) = -\infty$, die at a finite time $\beta(Y) < +\infty$ and satisfy $\lim_{t \downarrow -\infty} Y_t^1 = -\infty$.

We now apply Kaspi's time-change to $\mathcal{Q}_m^\dagger$ with $B_t = \int_{-\infty}^t \exp(\alpha Y_r^1) \, dr$. In order to use Kaspi's result, we need to check that $B_0 < \infty$ for $\mathcal{Q}_m^\dagger$-almost all realizations. From the two-sided construction of $\mathcal{Q}_m^\dagger$, it is clearly enough to show that $\hat{P}^{x,i,\downarrow}$-almost surely $\int_0^\infty \exp(\alpha \xi_r) \, dr < \infty$ for all $(x, i) \in (-\infty, 0) \times \{\pm 1\}$. To do so, we show finiteness of the expectation:

$$\hat{E}^{x,i,\downarrow}\left[\int_0^\infty e^{\alpha \xi_s} \, ds\right] = \int_0^\infty \hat{E}^{x,i,\downarrow}[e^{\alpha \xi_s}] \, ds$$

$$= \int_0^\infty \sum_{j=1,2} \int_{\mathbb{R}} e^{\alpha y} \hat{P}^{x,i,\downarrow}(\xi_s \in dy, J_s = j) \, ds$$

$$= \sum_{j=1,2} \int_{\mathbb{R}} e^{\alpha y} \int_0^\infty \hat{P}^{x,i,\downarrow}(\xi_s \in dy, J_s = j) \, ds$$

$$=: \sum_{j=1,2} \int_{\mathbb{R}} e^{\alpha y} \hat{U}^{\downarrow}((x, i), dy, \{j\})$$

$$(15) \qquad = \frac{1}{\hat{U}_i^+(x)} \sum_{j=1,2} \int_0^\infty e^{\alpha y} \hat{U}_j^+(y) \hat{U}^\dagger((x, i), (dy, \{j\}))$$

$$\leq \frac{C}{\hat{U}_i^+(x)} \sum_{j=1,2} \int_0^\infty e^{2\alpha y} \hat{U}^\dagger((x, i), (dy, \{j\}))$$

$$= \frac{C}{\hat{U}_i^+(x)} \hat{E}^{x,i,\dagger}\left[\int_0^\infty e^{2\alpha \xi_s} \, ds\right]$$

$$= \frac{C}{\hat{U}_i^+(x)} \hat{E}^{x,i}\left[\int_0^{\tau_0^+} e^{2\alpha \xi_s} \, ds\right],$$

where we used Fubini's theorem and the relation

$$\hat{U}^{\downarrow}\big((x,i),(dy,\{j\})\big) = \frac{\hat{U}_j^+(y)}{\hat{U}_i^+(x)} \hat{U}^{\dagger}\big((x,i),(dy,\{j\})\big),$$

with $\hat{U}^{\dagger}((x,i),(dy,\{j\}))$ being the potential measure of $(\xi^{\dagger}, J^{\dagger})$, [a consequence of (14)] and that the potentials $y \mapsto \hat{U}_j^+(y)$ grow at most linearly (see Theorem 28 of the Appendix). The right-hand side of (15) was already shown to be finite in the proof of Lemma 8.

Theorems (2.3) and (2.10) of Kaspi [18] thus give us an entrance law (η_t) at zero and a corresponding Kuznetsov measure $\tilde{\mathcal{Q}}_\eta^{\dagger}$ for the time-changed killed process

$$(16) \qquad \tilde{P}_t^{\dagger} f(x,i) := E^{x,i,\dagger}\big[f(\xi_{S(t)}, J_{S(t)})\big],$$

with $S_t = \inf\{s > 0 : \int_0^s \exp(\alpha\xi_u)\,du > t\}$ and furthermore

$$(17) \qquad \tilde{\mathcal{Q}}_\eta^{\dagger}(A, \beta > t) = \mathcal{Q}_m^{\dagger}\big(\pi^{-1}(A), 0 < B_t^{-1} \le 1\big), \qquad A \in \mathcal{F},$$

with $\pi(Y)_t = Y_{B_t^{-1}}$. As in the previous proof, (17) and the almost sure behaviour under $\mathcal{Q}_m^{\dagger}$ imply the following claims:

Claim: $\tilde{\mathcal{Q}}_\eta^{\dagger}$-almost all trajectories satisfy $\lim_{t\downarrow 0} Y_t^1 = -\infty$ and $\beta(Y) < +\infty$.

Claim: $\tilde{\mathcal{Q}}_\eta^{\dagger}(W) < \infty$.

The proof is exactly as in the proof of Lemma 12.

Claim: $\mathcal{Q}_\eta^{\dagger}\big((Y_{\tau_z^+}^1 - z, Y_{\tau_z^+}^2) \in (dx, \{i\})\big) = \mathcal{Q}_\eta(W)\nu(dx, \{i\})$ for all $z < 0$.

The proof is exactly as in the proof of Lemma 12 using only $z < 0$.

Normalizing $\tilde{\mathcal{Q}}_\eta^{\dagger}$ to a probability measure and taking its image under the pathwise transformation $h(\xi, J) = \exp(\xi)J$ and normalising yields a probability law $\mathbb{P}^{0,\dagger}$, which is a Kuznetsov measure for the transition semigroup $(P_t^{\dagger})$ killed at T_1. The overshoot distribution under $\mathbb{P}^{0,\dagger}$ at levels $\varepsilon < 1$ have distributions μ_ε (see the proof of Lemma 9). Concatenating $\mathbb{P}^{0,\dagger}$ with an independent copy of $\mathbb{P}^{\mu_1}$, that is, running a trajectory under $\mathbb{P}^{0,\dagger}$ until T_1 and then continuing with an independent copy of $\mathbb{P}^{\mu_1}$, yields $\mathbb{P}^0$. From the above and Lemma 9, $\mathbb{P}^0$ has the claimed properties. $\square$

2.4. *Proof of Theorem* 6. The argument for the necessity of Condition (**C**) was given in Section 1.3.

Now suppose (**C**) holds and let $\mathbb{P}^0$ as in Lemma 12 or Lemma 14, respectively. Then the first property of Theorem 6 is satisfied and the canonical process under $\mathbb{P}^0$ is strongly Markov for strictly positive stopping times as it is a Kuznetsov measure. In particular, properties (2a) and (2b) of Proposition 7 are true. As shown in Section 2.2, properties (1a) to (1c) are also fulfilled, thus,

$$\text{w-}\lim_{|z| \to 0} \mathbb{P}^z = \mathbb{P}^0.$$

We will use these properties to conclude the remaining assertions of Theorem 6.

Step 1: We show that the extension $\{\mathbb{P}^z : z \in \mathbb{R}\}$ is Feller. First, we show that for arbitrary $t > 0$ and continuous and bounded functions $f : \mathbb{R} \to \mathbb{R}$ the semigroup $P_t f(x) = \mathbb{E}^x[f(X_t)]$ is continuous on $\mathbb{R}$. Suppose that the sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in \mathbb{R}$. We know already that w-$\lim_{n \to \infty} \mathbb{P}^{x_n} = \mathbb{P}^x$ on the Skorokhod space and it follows that

$$P_t f(x_n) = \mathbb{E}^{x_n}[f(Z_t)] \to \mathbb{E}^x[f(Z_t)] = P_t f(x),$$

once we ensured that under $\mathbb{P}^x$ the canonical process Z is almost surely continuous in t since point evaluations on the Skorokhod space are continuous on the set of functions being continuous in the respective point. To show this, we recall that the paths of real self-similar Markov processes are quasi-left-continuous because the same is true of MAPs, in particular, when they are time changed by the sequence of stopping times that appear in the Lamperti–Kiu transform. This means that Z is continuous in t, almost surely, under $\mathbb{P}^x$ if $x \neq 0$. In the case where $x = 0$, we use the Markov property to conclude that

$$\mathbb{P}^0(Z \text{ has jump at } t) = \mathbb{E}^0\big[\mathbb{P}^{Z_{t/2}}(Z \text{ has jump at } t/2)\big] = 0.$$

Next, we show that, if additionally f vanishes at infinity, then this is also the case for $P_t f$. This is a consequence of the fact that for every $C > 0$

$$\lim_{|x| \to \infty} \mathbb{P}^x\Big(\min_{s \in [0,t]} |Z_s| < C\Big) = 0,$$

which itself follows easily from the Lamperti–Kiu representation. Indeed, this estimate implies that

$$
\begin{aligned}
(18) \qquad \big|P_t f(x)\big| &\leq \max_{y:|y| \geq C} \big|f(y)\big| + \mathbb{P}^x\Big(\min_{s \in [0,t]} |Z_s| < C\Big) \max_{y \in \mathbb{R}} \big|f(y)\big| \\
&\to \max_{y:|y| \geq C} \big|f(y)\big|
\end{aligned}
$$

for $|x| \to \infty$. Thus, $P_t f$ is vanishing at infinity since $C > 0$ is arbitrary.

It remains to show the strong continuity for a continuous function $f : \mathbb{R} \to \mathbb{R}$ vanishing at infinity. Let (t_n) be a decreasing sequence with $t_n \to 0$ and (x_n) a sequence in $\mathbb{R}$ with either $|x_n| \to \infty$ or $x_n \to x$ for an $x \in \mathbb{R}$. In the case where $|x_n| \to \infty$, with the same estimate as in (18), we find

$$\big|P_{t_n} f(x_n) - f(x_n)\big| \leq \big|P_{t_n} f(x_n)\big| + \big|f(x_n)\big| \to 0.$$

Moreover, if $x_n \to x$, we get that

$$P_{t_n} f(x_n) = \mathbb{E}^{x_n}[f(Z_{t_n})] \to \mathbb{E}^x[f(Z_0)] = f(x)$$

since the functional

$$\mathbf{D}(\mathbb{R}) \times [0, \infty) \ni (w, t) \mapsto w_t \in \mathbb{R}$$

is continuous in $\mathbb{P}^x \otimes \delta_0$-almost all entries. Consequently, one has

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}} |P_t f(x) - f(x)| = 0,$$

since we could otherwise construct sequences (t_n) and (x_n) as above contradicting the above properties (based on the compactness of the one point compactification of $\mathbb{R}$).

Step 2: Next, we show that $\mathbb{P}^0$ provides the law of a self-similar process. For a continuous and bounded functional $f : \mathbf{D}(\mathbb{R}) \to \mathbb{R}$, we have

$$\mathbb{E}^0 \big[f(c Z_{c^{-\alpha}.}) \big] = \lim_{z \to 0} \mathbb{E}^z \big[f(c Z_{c^{-\alpha}.}) \big] = \lim_{z \to 0} \mathbb{E}^{cz} \big[f(Z) \big] = \mathbb{E}^0 \big[f(Z) \big].$$

Step 3: Finally, we show that $\mathbb{P}^0$ is the unique Markovian extension satisfying either the first or second property in the statement of the theorem. Suppose there exists another Markovian extension satisfying the first property in the statement of the theorem and denote it by $\bar{\mathbb{P}}^0$. Then, for $t > 0$,

$$\bar{\mathbb{P}}^0(Z_t \in \cdot) = \text{w-} \lim_{\varepsilon \downarrow 0} \bar{\mathbb{P}}^0(Z_{t+\varepsilon} \in \cdot)$$

$$= \text{w-} \lim_{\varepsilon \downarrow 0} \bar{\mathbb{P}}^0 \big(\mathbb{P}^{Z_\varepsilon}(Z_t \in \cdot) \big)$$

$$= \mathbb{P}^0(Z_t \in \cdot),$$

where we used in the first step that (Z_t) is right-continuous, in the second step the Markov property of $\bar{\mathbb{P}}^0$ and in the third step that $Z_\varepsilon \Rightarrow \delta_0$ under $\bar{\mathbb{P}}^0$ and $\text{w-}\lim_{z \to 0} \mathbb{P}^z(Z_t \in \cdot) = \mathbb{P}^0(Z_t \in \cdot)$ by the Feller property for $\mathbb{P}^0$. By using the Markov property, one easily sees that the distributions $\bar{\mathbb{P}}^0$ and $\mathbb{P}^0$ coincide.

Suppose now that, instead, that $\bar{\mathbb{P}}^0$ satisfies the second (Feller) property in the statement of the theorem. Then using the Feller property twice we get

$$\bar{\mathbb{P}}^0(Z_t \in \cdot) = \text{w-} \lim_{x \to 0} \mathbb{P}^x(Z_t \in \cdot) = \mathbb{P}^0(Z_t \in \cdot)$$

so that $\bar{\mathbb{P}}^0$ and $\mathbb{P}^0$ coincide again by the Markov property.

2.5. *Remarks on the proof.*

REMARK 15.   The way the limiting law $\mathbb{P}^0$ is constructed one can say that the Lamperti–Kiu representation extends in a slightly unhandy way to initial condition 0. Due to the explicit construction of the Kuznetsov measure from two-sided MAPs one can for instance deduce from almost sure results for MAPs almost sure results for self-similar Markov processes started from zero.

REMARK 16 (Proof of Theorem 6 fails if (**C**) fails).   Calculations similar to those from Lemma 12 (resp., Lemma 14) can be used in order to show that the divergence of overshoots implies $\tilde{\mathcal{Q}}_\eta(W) = \infty$ [resp. $\tilde{\mathcal{Q}}_\eta^\dagger(W) = \infty$]. Hence, if Condition (**C**) fails, then necessarily $\tilde{\mathcal{Q}}_\eta$ (resp. $\tilde{\mathcal{Q}}_\eta^\dagger$) is an infinite measure and as such cannot be normalized to a probability measure $\mathbb{P}^0$.

REMARK 17. The previous remark has an interesting consequence: in contrast to other known constructions of $\mathbb{P}^0$ in the setting of pssMps, our construction works irrespectively of Condition (**C**). When (**C**) fails, then the infinite Kuznetsov measure can still be used to study conditional limits, such as $\lim_{|x|\to 0} \mathbb{P}^z(\cdot|$the interval $[a, b]$ is hit).

REMARK 18 (Relation to Bertoin, Savov [4]). For pssMps Bertoin and Savov constructed $\mathbb{P}^0$ by hand without appealing to the probabilistic potential theory centred around Kuznetsov's measure. Their construction is in the spirit of the Fitzsimmons and Taksar [15] construction of stationary regenerative sets as range of stationary subordinators. In essence, we first constructed a Kuznetsov measure and then produced the so-called quasi-process by taking Palm measures in (11) [resp., in (17)]. Bertoin and Savov directly wrote down the quasi-process—their construction only works under Condition (**C**); see Remark 17.

REMARK 19. The advantage of undertaking the detour through Kuznetsov measures is mostly technical. It allowed us to write down, with a minimal use of fluctuation theory, the limiting object $\mathbb{P}^0$. For instance, there was no need to use the nontrivial existence of $\hat{P}^\downarrow$ issued from the origin. Since fluctuation theory is delicate, a proof with minimal use is desirable, in particular, for possible future generalizations to more general domains. One direction for which our construction works but fluctuation theory is not available are multi-self-similar Markov processes introduced in Yor, Jacobson [16].

REMARK 20. For real self-similar Markov processes, with jumps only toward the origin, a construction of $\mathbb{P}_0$ was already given in [10] through jump-type stochastic differential equations. That approach lacks generality since the weak uniqueness argument does not extend to full generality. It might be an interesting question to ask if the potential theory of the present article can be used to prove the weak uniqueness of the differential equations.

APPENDIX: RESULTS FOR MARKOV ADDITIVE PROCESSES

Unlike the case of Lévy processes, general fluctuation theory for Markov additive processes (MAPs) appears to be relatively incomplete in the literature. Accordingly, in this Appendix, we address those parts of the fluctuation theory that are needed in the main body of the text above.

The contents of the Appendix is as follows:

A.1. Basics.
A.2. Duality.
A.3. Local time and Cox process of excursions.
A.4. Splitting at the maximum.

Unfortunately, a complete treatment would require a whole book's worth of text. Therefore, as a compromise and with an apology to the reader, the presentation of A.1 to A.8 mostly highlights selected results and the main steps to prove them. Almost all fluctuation theory can be constructed by analogy with fluctuation theory of Lévy processes. The selected computations we dwell on below pertain largely to the peculiarities that are specific to the case of MAPs. Results in A.9 and A.10 are not in analogy to Lévy processes and nontrivial so full proofs are given.

A.1. Basics. Recall that $(\xi_t, J_t)_{t \geq 0}$ denotes a MAP on $\mathbb{R} \times E$, where E is a finite set. Recall also that its natural filtration is denoted by $(\mathcal{F}_t)_{t \geq 0}$ and its probabilities by $(P^{x,i})_{x \in \mathbb{R}, i \in E}$. We shall also assume that E is irreducible and aperiodic, and hence ergodic. Denote the intensity matrix of J by $Q = (q_{i,j})_{i,j \in E}$. Its stationary distribution is denoted by $\pi = (\pi_1, \ldots, \pi_{|E|})$.

Unless otherwise stated, we assume throughout that ξ is nonlattice, that is, (**NL**) *is in force.*

Referring to Proposition 2, the characteristic exponents of the "pure-state" Lévy processes appearing in Proposition 2 will be denoted by $\psi_i(z) = \log \mathbb{E}[\exp(z \xi_1^i)]$, $z \in \mathbb{C}$, whenever the right-hand side exists. It suffices for us to deal with the case that $\psi_i(0) = 0$ for all $i \in E$, that is, none of the Lévy processes are killed. Furthermore, whenever it exists, define the matrix $G(z) = (G_{i,j}(z))_{i,j \in E}$, where $G_{i,j}(z) = \mathbb{E}[\exp(z \Delta_{i,j})]$, $i, j \in E$. For each $i, j \in E$ such that $i \neq j$, the random variables $\Delta_{i,j}$ have law $F_{i,j}$ corresponding to the distribution of the additional jump that is inserted into the path of the MAP when J undergoes a transition from i to j. For convenience, we assume that $\Delta_{i,j} = 0$ whenever $q_{i,j} = 0$ and also set $\Delta_{i,i} = 0$ for each $i \in E$. According to Proposition 2, this assumption is without loss of generality since those transitional jumps never occur.

A crucial role will be played by the matrices

$$(19) \qquad F(z) := \mathrm{diag}\big(\psi_1(z), \ldots, \psi_{|E|}(z)\big) + \big(q_{i,j} G_{i,j}(z)\big)_{i,j \in E},$$

which are defined on $\mathbb{C}$ whenever the right-hand side exists. The matrix F is called the matrix exponent of the MAP (ξ, J) because

$$E^{0,i}\big[e^{z \xi_t}, J_t = j\big] = \big(e^{F(z)t}\big)_{i,j}, \qquad i, j \in E,$$

for all $z \in \mathbb{C}$ for which one of the sides is defined.

A.2. Duality. Given the MAP ξ with probabilities $P^{x,i}$, $x \in \mathbb{R}$, $i \in E$, we can introduce the dual process; that is, the MAP with probabilities $\hat{P}^{x,i}$, $x \in \mathbb{R}$, $i \in E$, whose matrix exponent, when it is defined, is given by

$$\hat{E}^{0,i}[e^{z\xi_t}, J_t = j] = (e^{\hat{F}(z)t})_{i,j}, \qquad i, j \in E,$$

where

$$\hat{F}(z) := \operatorname{diag}(\psi_1(-z), \ldots, \psi_{|E|}(-z)) + \hat{Q} \circ G(-z)^{\mathrm{T}}$$

and $\hat{Q}$ is the intensity matrix of the modulating Markov chain on E with entries given by

$$\hat{q}_{i,j} = \frac{\pi_j}{\pi_i} q_{j,i}, \qquad i, j \in E.$$

Note that the latter can also be written $\hat{Q} = \Delta_\pi^{-1} Q^{\mathrm{T}} \Delta_\pi$, where $\Delta_\pi = \operatorname{diag}(\pi_1, \ldots, \pi_{|E|})$, and hence, when it exists,

$$\hat{F}(z) = \Delta_\pi^{-1} F(-z)^{\mathrm{T}} \Delta_\pi,$$

showing that

(20) $$\pi_i \hat{E}^{0,i}[e^{z\xi_t}, J_t = j] = \pi_j E^{0,j}[e^{-z\xi_t}, J_t = i].$$

At the level of processes, one can understand (20) as changing time-directions.

LEMMA 21. *We have that* $\{(\xi_{(t-s)-} - \xi_t, J_{(t-s)-}) : s \le t\}$ *under* $P^{0,\pi} = \sum_{i=1}^{|E|} \pi_i P^{0,i}$ *is equal in law to* $\{(\xi_s, J_s) : s \le t\}$ *under* $\hat{P}^{0,\pi}$.

Additionally to the ordinary duality (20), we will use duality in the general sense of (8) for the killed MAP

$$P_t^\dagger((x,i), (dy, \{j\})) = P^{x,i}[\xi_t \in dy, \bar{\xi}_t \le 0; J_t = j], \qquad x, y \le 0, t \ge 0, i, j \in E,$$

where $\bar{\xi}_t = \sup_{s \le t} \xi$. The next two duality formulas are called switching identities.

LEMMA 22. *If* $x, y \in \mathbb{R}$ *and* $i, j \in E$, *then*

$$\hat{P}^{x,i}(\xi_t \in dy; J_t = j)\pi_i \, dx = P^{y,j}(\xi_t \in dx; J_t = i)\pi_j \, dy$$

and, for $x, y \le 0$,

$$\hat{P}_t^\dagger((x,i), (dy, \{j\}))\pi_i \, dx = P_t^\dagger((y, j), (dx, \{i\}))\pi_j \, dy.$$

The proofs of the previous two lemmas are standard, especially in light of the straightforward nature of the analogous proofs for Lévy processes (see, e.g., Chapter II of [3]), and we leave them to the reader.

A.3. Local time and Cox process of excursions. Let $Y_t^{(x)} = (x \vee \bar{\xi}_t) - \xi_t$, $t \geq 0$, where we recall that $\bar{\xi}_t = \sup_{s \leq t} \xi_s$. Following ideas that are well known from the theory of Lévy processes, it is straightforward to show that, as a pair, the process $(Y^{(x)}, J)$ is a strong Markov process. For convenience, write Y in place of $Y^{(0)}$. Since (Y, J) is a strong Markov process, by the general theory (cf. Chapter IV of [3]) there exists a local time at the point $(0, i)$, which we henceforth denote by $\{\bar{L}_t^{(i)} : t \geq 0\}$. Now consider the process

$$\bar{L}_t := \sum_{i \in E} \bar{L}_t^{(i)}, \qquad t \geq 0,$$

where each of the local time processes $L^{(i)}$ may be chosen up to an arbitrary scaling constant. Since, almost surely, for each $i \neq j$ in E, the points of increase of $\bar{L}^{(i)}$ and $\bar{L}^{(j)}$ are disjoint, it follows that $(\bar{L}^{-1}, H^+, J^+) := \{(\bar{L}_t^{-1}, H_t^+, J_t^+) : t \geq 0\}$ is a (possibly killed) Markov additive bivariate subordinator, where

$$H_t^+ := \xi_{\bar{L}_t^{-1}} \quad \text{and} \quad J_t^+ := J_{\bar{L}_t^{-1}}, \qquad \text{if } \bar{L}_t^{-1} < \infty,$$

and $H_t^+ := \infty$ and $J_t^+ := \infty$ otherwise. Note that the rate at which the process $(\bar{L}^{-1}, H^+, J^+)$ is killed depends on the state of the chain J^+ when killing occurs. This will be addressed in more detail shortly. We also note that $\{\epsilon_t : t \geq 0\}$ is a (killed) Cox process, where

$$\epsilon_t = \{\xi_{\bar{L}_{t-}^{-1}+s} - \xi_{\bar{L}_{t-}^{-1}} : s \leq \Delta \bar{L}_t^{-1}\}, \qquad \text{if } \Delta \bar{L}_t^{-1} > 0,$$

and $\epsilon_t = \partial$, some isolated state, otherwise. Henceforth, write n_i for the intensity measure of this Cox process when the underlying modulating chain J^+ is in state $i \in E$. As a bivariate Markov additive subordinator, the process $(\bar{L}^{-1}, H^+, J^+)$ has a matrix Laplace exponent given by

$$E^{0,i}[e^{-\alpha \bar{L}_t^{-1} - \beta H_t^+}, J_t^+ = j] = (e^{-\kappa^+(\alpha,\beta)t})_{i,j}, \qquad \alpha, \beta \geq 0,$$

where the matrix $\kappa^+(\alpha, \beta)$ has the structure

$$\kappa^+(\alpha, \beta) = \operatorname{diag}(\Phi_1^+(\alpha, \beta), \ldots, \Phi_k^+(\alpha, \beta)) - Q^+ \circ G^+(\alpha, \beta), \qquad \alpha, \beta \geq 0$$

such that, for $i \in E$, $\Phi_i^+(\alpha, \beta)$ is the subordinator exponent that describes the movement of $(\bar{L}^{-1}, H^+)$ when the modulating chain J^+ is in state i. Moreover, Q^+ is the intensity of J^+ and the matrix $G^+(\alpha, \beta) = (G^+(\alpha, \beta))_{i,j}$ is such that, for $i \neq j$ in E, its (i, j)th entry is the Laplace transform of $F_{i,j}^+(dy, dx)$, the joint distribution of the additional jump incurred by (L^{-1}, H) when the modulating chain changes state from i to j. The diagonal elements of $G^+(\beta)$ are set to unity. Note later on, we will abuse notation and write $F_{i,j}^+(dx)$ in place of $F_{i,j}^+(\mathbb{R}^+, dx)$. For $i \in E$, we can now identify the exponents

$$\Phi_i^+(\alpha, \beta) = n_i(\zeta = \infty) + a_i \alpha + b_i \beta$$
$$+ \int_0^\infty \int_0^\infty (1 - e^{-\alpha x - \beta y}) n_i(\zeta \in dx, \epsilon_\zeta \in dy, J_\zeta = i),$$

for $\alpha, \beta \geq 0$, where $a_i, b_i \geq 0$ and $\zeta = \inf\{s \geq 0 : \epsilon > 0\}$ the excursion length. Note in particular that the matrix

$$\kappa^+(0,0) = \operatorname{diag}(n_1(\zeta = \infty), \dots, n_k(\zeta = \infty)),$$

encodes the respective killing rates of $(\bar{L}^{-1}, H^+, J^+)$ when J^+ is in each state of E.

On a final note, since the local times $L^{(i)}$ may be chosen up to an arbitrary scaling constant, the matrix exponent $\kappa^+(\alpha, \beta)$ can only be defined up to pre-multiplication of a diagonal matrix with strictly positive entries (henceforth referred to as a *strictly positive diagonal matrix*). The reader will note that, in the forthcoming identities that concern the MAP, the choice of local time normalisation is (obviously) irrelevant.

The assumption that ξ is nonlattice implies that the jump measures associated to H^+, namely $n_i(\epsilon_\zeta \in dx, J^+_\zeta = i), i \in E$, and $F^+_{i,j}, i \neq j, i, j \in E$, are diffuse on $(0, \infty)$. For the sake of brevity, we give no proof of this fact here. Instead, we refer to the proof of the analogous result for the case of Lévy processes. In that case, one may draw the desired conclusion out of, for example, Vigon's identity for the jump measure of the ascending ladder height process; see Theorem 7.8 in [23]. As one sees from the proof there, this identity is derived using the so-called quintuple law of the first passage problem, which itself follows from a straightforward application of the compensation formula for the Poisson point process of jumps. A quintuple law can also be derived in the MAP setting using the same technique as in the Lévy setting, where one appeals to an analogue of the compensation formula for the Cox process of jumps. This would also form the basis of the proof that the jump measures associated to H^+ are diffuse in the MAP case.

A.4. Splitting at the maximum. Now suppose that $\mathbf{e}_q$ is an exponentially distributed random variable with rate $q > 0$. Consider a marked version of the Cox process described above in which each excursion $\epsilon_t \neq \partial$ is marked with an independent copy of $\mathbf{e}_q$, denoted by $\mathbf{e}_q^{(t)}$, for $t \geq 0$. Let $\overline{m}_t = \sup\{s \leq t : \overline{\xi}_t = \xi_s\}$. Poisson thinning dictates that $(\overline{m}_{\mathbf{e}_q}, \overline{\xi}_{\mathbf{e}_q})$ is equal in law to the process $(\bar{L}^{-1}, H^+)$ conditioned on $\{\Delta \bar{L}^{-1}_t < \mathbf{e}_q^{(t)}$ for all $t \geq 0\}$ and stopped with rate matrix

$$\operatorname{diag}(a_1 q + n_1(\zeta > \mathbf{e}_q), \dots, a_{|E|} q + n_{|E|}(\zeta > \mathbf{e}_q))$$

$$= \operatorname{diag}(a_1 q + n_1(1 - e^{-q\zeta}), \dots, a_{|E|} q + n_{|E|}(1 - e^{-q\zeta}))$$

$$= \operatorname{diag}(\Phi_1^+(q, 0), \dots, \Phi_{|E|}^+(q, 0)).$$

In particular, the conditioned process is stopped at a random time θ_q with the property that

$$P^{0,i}(\theta_q > t | \sigma\{J^+_s : s \leq t\}) = \exp\left(-\int_0^t \Phi^+_{J_s}(q, 0) \, ds\right).$$

The aforementioned conditioned process has matrix exponent which can be derived from the matrix exponent $\kappa^+(\alpha, \beta)$. Indeed, whereas in $\kappa^+(\alpha, \beta)$ the pure states are represented as $\Phi_i^+(\alpha, \beta)$ in the conditioned process. This is replaced by

$$n_i(\zeta = \infty) + a_i\alpha + b_i\beta + \int_0^\infty \int_0^\infty (1 - e^{-\alpha x - \beta y}) e^{-qx} n_i(\zeta \in dx, \epsilon_\zeta \in dy, J_\zeta = i),$$

for $\alpha, \beta \geq 0$, which is also equal to $\Phi_i^+(q + \alpha, \beta) - \Phi_i^+(q, 0)$. Hence, the conditioned process has matrix exponent given by

$$\tilde{\kappa}^+(\alpha, \beta)$$

$$(21) \qquad := \operatorname{diag}\big(\Phi_1^+(q + \alpha, \beta) - \Phi_1^+(q, 0), \ldots, \Phi_{|E|}^+(q + \alpha, \beta) - \Phi_{|E|}^+(q, 0)\big)$$

$$\qquad\qquad - Q^+ \circ G^+(\alpha, \beta),$$

for $\alpha, \beta \geq 0$.

For convenience, denote by $(\mathcal{L}^{-1}, \mathcal{H}, J^+)$ the process corresponding to the pair $(\bar{L}^{-1}, H^+)$ conditioned on $\{\Delta \bar{L}_t^{-1} < \mathbf{e}_q^{(t)}$ for all $t \geq 0\}$, that is, the Markov additive process with joint Laplace exponent give by (21). Define $J_{\overline{m}_{\mathbf{e}_q}} = \lim_{s \uparrow \overline{m}_{\mathbf{e}_q}} J_s \mathbf{1}_{(\overline{\xi}_{\overline{m}_{\mathbf{e}_q}} > \xi_{\overline{m}_{\mathbf{e}_q}})} + \lim_{s \downarrow \overline{m}_{\mathbf{e}_q}} J_s \mathbf{1}_{(\overline{\xi}_{\overline{m}_{\mathbf{e}_q}} = \xi_{\overline{m}_{\mathbf{e}_q}})}$; that is, $J_{\overline{m}_{\mathbf{e}_q}}$ takes the value of the chain approaching the supremum (if it is not attained by right-continuity) or at the time of the supremum (if it is attained). It now follows that $(\overline{\xi}_{\mathbf{e}_q}, \overline{m}_{\mathbf{e}_q})$ has matrix Laplace transform given by

$$E^{0,i}\big(e^{-\alpha \overline{m}_{\mathbf{e}_q} - \beta \overline{\xi}_{\mathbf{e}_q}}, J_{\overline{m}_{\mathbf{e}_q}} = j\big)$$

$$= E^{0,i}\big[e^{-\alpha \mathcal{L}_{\theta_q}^{-1} - \beta \mathcal{H}_{\theta_q}} \mathbf{1}_{(J_{\theta_q}^+ = j)}\big]$$

$$(22)$$

$$= E^{0,i}\bigg[\int_0^\infty du \, \mathbf{1}_{(J_u^+ = j)} \Phi_{J_u^+}^+(q, 0) e^{-\int_0^u \Phi_{J_s^+}^+(q, 0)\, ds} e^{-\alpha \mathcal{L}_u^{-1} - \beta \mathcal{H}_u}\bigg]$$

$$= \int_0^\infty du \, \Phi_j^+(q, 0) E^{0,i}\big[e^{-\int_0^u \Phi_{J_s^+}^+(q, 0)\, ds} e^{-\alpha \mathcal{L}_u^{-1} - \beta \mathcal{H}_u} \mathbf{1}_{(J_u^+ = j)}\big],$$

for $\alpha, \beta \geq 0$. Note that the final expectation above can be written in terms of the matrix Laplace exponent of $(\mathcal{L}^{-1}, \mathcal{H}, J^+)$ with a potential corresponding to $\operatorname{diag}(\Phi_1^+(q, 0), \ldots, \Phi_{|E|}^+(q, 0))$, that is,

$$\kappa^+(q + \alpha, \beta) = \operatorname{diag}\big(\Phi_1^+(q + \alpha, \beta), \ldots, \Phi_{|E|}^+(q + \alpha, \beta)\big) - Q^+ \circ G^+(\beta),$$

$$\alpha, \beta \geq 0.$$

Indeed, one has

$$E^{0,i}\big[e^{-\int_0^u \Phi_{J_s^+}^+(q, 0)\, ds} e^{-\alpha \mathcal{L}_u^{-1} - \beta \mathcal{H}_u} \mathbf{1}_{(J_u^+ = j)}\big] = \big[e^{-\kappa^+(q + \alpha, \beta)}\big]_{i,j}.$$

Continuing the computation in (22), we now have the following result.

THEOREM 23. *For $i, j \in E$, $\alpha, \beta \geq 0$ and $q > 0$,*

$$(23) \qquad E^{0,i}\big[e^{-\alpha \overline{m}_{\mathbf{e}_q} - \beta \overline{\xi}_{\mathbf{e}_q}}, J_{\overline{m}_{\mathbf{e}_q}} = j\big] = \Phi_j^+(q,0)\big[\kappa^+(q+\alpha,\beta)^{-1}\big]_{i,j}.$$

We can go a little further in our analysis of the previous section and note that, on the event $\{J_{\theta_q}^+ = j\}$, the excursion $\epsilon_{J_{\theta_q}^+}$ is independent of

$$\{(\bar{L}_t^{-1}, H_t^+, J_t^+) : t < \theta_q\}.$$

In particular, on $\{J_{\theta_q}^+ = j\}$, we have that $(\overline{\xi}_{\mathbf{e}}, \overline{m}_{\mathbf{e}_q})$ is independent of $(\xi_{\mathbf{e}_q} - \overline{\xi}_{\mathbf{e}_q}, \mathbf{e}_q - \overline{m}_{\mathbf{e}_q})$. Moreover, duality allows us to conclude that, when the modulating chain is in the stationary state, on the event $\{J_{\theta_q}^+ = j, J_{\mathbf{e}_q} = k\} = \{J_{\overline{m}_{\mathbf{e}_q}} = j, J_{\mathbf{e}_q} = k\}$ the pair $(\overline{\xi}_{\mathbf{e}_q} - \xi_{\mathbf{e}_q}, \mathbf{e}_q - \overline{m}_{\mathbf{e}_q})$ is equal in law to the pair $(\hat{\overline{\xi}}_{\mathbf{e}_q}, \hat{\overline{m}}_{\mathbf{e}_q})$ on $\{\hat{J}_0 = k, \hat{J}_{\hat{\overline{m}}_{\mathbf{e}_q}} = j\}$, where $\{(\hat{\xi}_s, \hat{J}_s) : s \leq t\} := \{(\xi_{(t-s)-} - \xi_t, J_{(t-s)-}) : s \leq t\}$, $t \geq 0$, is equal in law to the dual of ξ, $\hat{\overline{\xi}}_t = \sup_{s \leq t} \hat{\xi}_s$ and $\hat{\overline{m}} = \sup\{s \leq t : \hat{\overline{\xi}}_s = \hat{\xi}_t\}$.

From the previous discussion, we may now deduce, for example, that, for $i, j, k \in E$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$,

$$(24) \quad \begin{aligned} & E^{0,i}\big[e^{-\alpha_1 \overline{m}_{\mathbf{e}_q} - \beta_1 \overline{\xi}_{\mathbf{e}_q}} e^{-\alpha_2(\mathbf{e}_q - \overline{m}_{\mathbf{e}_q}) - \beta_2(\overline{\xi}_{\mathbf{e}_q} - \xi_{\mathbf{e}_q})}, J_{\overline{m}_{\mathbf{e}_q}} = j, J_{\mathbf{e}_q} = k\big] \\ & = E^{0,i}\big[e^{-\alpha_1 \overline{m}_{\mathbf{e}_q} - \beta_1 \overline{\xi}_{\mathbf{e}_q}}, J_{\overline{m}_{\mathbf{e}_q}} = j\big] \frac{\pi_k}{\pi_j} \hat{E}^{0,k}\big[e^{-\alpha_2 \overline{m}_{\mathbf{e}_q} - \beta_2 \overline{\xi}_{\mathbf{e}_q}}, J_{\overline{m}_{\mathbf{e}_q}} = j\big]. \end{aligned}$$

We can also use the ideas above to prove the following technical lemma which will be of use later on.

LEMMA 24. *For all $j \in E$,*

$$c := \sum_{j \in E} \lim_{q \downarrow 0} \frac{\Phi_j^+(q,0)\hat{\Phi}_j^+(q,0)}{q}$$

exists in $(0, \infty)$ and, for each $j \in E$,

$$(25) \qquad c_j := \lim_{q \downarrow 0} \frac{\Phi_j^+(q,0)\hat{\Phi}_j^+(q,0)}{q}$$

exists in $[0, \infty)$.

PROOF. Write $\hat{\kappa}^+(\alpha,\beta)$ for the dual matrix exponent, that is, to $\hat{F}(z)$ what $\kappa^+(\alpha,\beta)$ is to $F(z)$. On the one hand, for all $i, k \in E$ and $\alpha > 0$,

$$\begin{aligned} & E^{0,i}\big[e^{-\alpha \mathbf{e}_q}, J_{\mathbf{e}_q} = k\big] \\ & = \bigg[\int_0^\infty q e^{-(\alpha+q)t} e^{Qt}\, dt\bigg]_{i,k} \\ & = q\big[((q+\alpha)I - Q)^{-1}\big]_{i,k}. \end{aligned}$$

On the other hand, from (24), for all $i, k \in E$ and $\alpha > 0$,

$$
\begin{aligned}
E^{0,i}&\big[e^{-\alpha \mathbf{e}_q}, J_{\mathbf{e}_q} = k\big] \\
&= \sum_{j \in E} E^{0,i}\big[e^{-\alpha(\overline{m}_{\mathbf{e}_q} + \mathbf{e}_q - \overline{m}_{\mathbf{e}_q})}, J_{\overline{m}_{\mathbf{e}_q}} = j, J_{\mathbf{e}_q} = k\big] \\
&= \sum_{j \in E} \Phi_j^+(q, 0)\big[\kappa^+(q+\alpha, 0)^{-1}\big]_{i,j} \hat{\Phi}_j^+(q, 0)\big[\hat{\kappa}^+(q+\alpha, 0)^{-1}\big]_{k,j} \frac{\pi_k}{\pi_j}.
\end{aligned}
$$

Taking limits as $q \downarrow 0$ it follows from continuity that

$$
\big[(\alpha I - Q)^{-1}\big]_{i,k} = \sum_{j \in E} \lim_{q \downarrow 0} \frac{\Phi_j^+(q, 0)\hat{\Phi}_j^+(q, 0)}{q}\big[\kappa^+(\alpha, 0)^{-1}\big]_{i,j}\big[\hat{\kappa}^+(\alpha, 0)^{-1}\big]_{k,j} \frac{\pi_k}{\pi_j},
$$

where the limit on the right-hand side exists because the limit exits on the left-hand side. The statement of the theorem now follows. □

The next theorem below gives the Wiener–Hopf factorisation for MAPs. It is a natural consequence of Theorem 23 and a well-established method of splitting stochastic processes at their maximum. Some results already exist in the literature in this direction (see, e.g., Chapter XI of [2] and [17]); however, none of them are in an appropriate form for our purposes.

REMARK 25.   As a consequence of the Wiener–Hopf factorisation, it will turn out that the constants c_j, $j \in E$, are all strictly positive and may be taken to be equal to unity without loss of generality.

THEOREM 26.   *For $z \in \mathbb{R} \setminus \{0\}$ and $\alpha \geq 0$, up to the identification of κ^+ (or equivalently $\hat{\kappa}^+$) by a pre-multiplicative strictly positive diagonal matrix,*

$$
\alpha I - F(\mathrm{i}z) = \Delta_\pi^{-1}\big[\hat{\kappa}^+(\alpha, \mathrm{i}z)^{\mathrm{T}}\big]\Delta_\pi \kappa^+(\alpha, -\mathrm{i}z).
$$

PROOF.   We start by sampling ξ over an independent and exponentially distributed time horizon denoted, as usual, by $\mathbf{e}_q$. By splitting at the maximum, applying duality and appealing to the identity (23), we have for $\alpha \geq 0$

$$
\begin{aligned}
E^{0,i}&\big[e^{-\alpha \mathbf{e}_q + \mathrm{i}z\xi_{\mathbf{e}_q}}, J_{\mathbf{e}_q} = j\big] \\
&= \sum_{k \in E} E^{0,i}\big[e^{-\alpha(\mathbf{e}_q - \overline{m}_{\mathbf{e}_q} + \overline{m}_{\mathbf{e}_q}) + \mathrm{i}z\overline{\xi}_{\mathbf{e}_q}} e^{\mathrm{i}z(\xi_{\mathbf{e}_q} - \overline{\xi}_{\mathbf{e}_q})}, J_{\overline{m}_{\mathbf{e}_q}} = k, J_{\mathbf{e}_q} = j\big] \\
&= \sum_{k \in E} E^{0,i}\big[e^{-\alpha \overline{m}_{\mathbf{e}_q} + \mathrm{i}z\overline{\xi}_{\mathbf{e}_q}}, J_{\overline{m}_{\mathbf{e}_q}} = k\big] \frac{\pi_j}{\pi_k} \hat{E}^{0,j}\big[e^{-\alpha \overline{m}_{\mathbf{e}_q} - \mathrm{i}z\overline{\xi}_{\mathbf{e}_q}}, J_{\overline{m}_{\mathbf{e}_q}} = k\big] \\
&= \sum_{k \in E} \Phi_k^+(q, 0)\big[\kappa^+(q+\alpha, -\mathrm{i}z)^{-1}\big]_{i,k} \frac{\pi_j}{\pi_k} \hat{\Phi}_k^+(q, 0)\big[\hat{\kappa}^+(q+\alpha, \mathrm{i}z)^{-1}\big]_{j,k}.
\end{aligned}
$$

Noting that we can write the left-hand side above as $q[((q + \alpha)I - F(\mathrm{i}z))^{-1}]_{i,j}$, we can divide by q and take limits as $q \downarrow 0$ to find that

$$[(\alpha I - F(\mathrm{i}z))^{-1}]_{i,j} = \sum_{k \in E} c_k [\kappa^+(\alpha, -\mathrm{i}z)^{-1}]_{i,k} \frac{\pi_j}{\pi_k} [[\hat{\kappa}^+(\alpha, \mathrm{i}z)^\mathrm{T}]^{-1}]_{k,j},$$

where we recall that the constants c_k, $k \in E$ were introduced in (25). In matrix form, the above equality can be rewritten as

$$(26) \qquad (\alpha I - F(\mathrm{i}z))^{-1} = \kappa^+(\alpha, -\mathrm{i}z)^{-1} \Delta_{c/\pi} [\hat{\kappa}^+(\alpha, \mathrm{i}z)^\mathrm{T}]^{-1} \Delta_\pi,$$

where $\Delta_{c/\pi} = \mathrm{diag}(c_1/\pi_1, \ldots c_{|E|}/\pi_{|E|})$. Since all matrices are invertible except possibly $\Delta_{c/\pi}$ (on account of the fact that some of the constants c_k may be zero), it follows that necessarily $c_k > 0$ for all $k \in E$, and hence the matrix $\Delta_{c/\pi}$ is indeed invertible and is its inverse equal to $\Delta_{\pi/c}^{-1}$ (using obvious notation). The proof is now completed by inverting the matrices on both left- and right-hand sides of (26) and noting that, without loss of generality, the constants c_k may be taken as unity by choosing an appropriate normalisation of local times [which in turn means that, for each $j \in E$, the equality in (25) can be determined up to a multiplicative constant]. This is equivalent to the statement that κ^+, or equivalently $\hat{\kappa}^+$, is identified up to a pre-multiplicative strictly positive diagonal matrix. $\square$

A.5. Occupation formula. The objective in this section is to use the preceding constructions to establish a key identity which is central to the analysis of real self-similar Markov processes in the main body of the text. In order to state the main result, some more notation is needed. For $i, j \in E$, the potential measure $U_{i,j}^+$ on $[0, \infty)$ is defined to be

$$(27) \qquad U_{i,j}^+(dx) = E^{0,i} \left[\int_0^\infty \mathbf{1}_{(H_t^+ \in dx, J_t^+ = j)} \, dt \right], \qquad x \geq 0.$$

Note that, for $\lambda > 0$,

$$(28) \qquad \int_0^\infty e^{-\lambda x} U_{i,j}^+(dx) = \int_0^\infty E^{0,i} [e^{-\lambda H_t^+}, J_t^+ = j] \, dt = [\kappa^+(0, \lambda)^{-1}]_{i,j}.$$

Moreover, it should also be noted that the nonlattice assumption on the process ξ ensures that the measure $U_{i,j}^+$ is diffuse on $(0, \infty)$; see the discussion at the end of A.3 as well as the proof of Theorem 5.4 in [23] in the Lévy case for guidance. We can define by analogy the measures $\hat{U}_{i,j}^+$, $i, j \in E$, for to the dual process $\hat{\xi}$. The reader might also want to recall the definitions of τ_0^- and τ_0^+ from (5).

THEOREM 27. *There exist nonnegative constants c_j, $j \in E$, satisfying $\sum_{j \in E} c_j > 0$ such that for all bounded measurable $f : \mathbb{R} \to [0, \infty)$ and $x > 0$,*

$$E^{x,i} \left[\int_0^{\tau_0^-} f(\xi_t) \mathbf{1}_{(J_t = k)} \, dt \right]$$
$$= \sum_{j \in E} c_j \frac{\pi_k}{\pi_j} \int_{y \in [0,\infty)} \int_{z \in [0,x]} U_{i,j}^+(dy) \hat{U}_{k,j}^+(dz) f(x + y - z).$$

PROOF.   Start by noting that

$$
E^{x,i}\left[\int_0^{\tau_0^-} e^{-qt} f(\xi_t)\mathbf{1}_{(J_t=k)}\,dt\right]
$$

$$
= \frac{1}{q} E^{x,i}\left[f(\xi_{\mathbf{e}_q})\mathbf{1}_{(J_{\mathbf{e}_q}=k)}, \mathbf{e}_q < \tau_0^-\right]
$$

(29)

$$
= \frac{1}{q}\sum_{j\in E} E^{x,i}\left[f(\overline{\xi}_{\mathbf{e}_q} - (\overline{\xi}_{\mathbf{e}_q} - \xi_{\mathbf{e}_q}))\mathbf{1}_{(J_{\overline{m}_{\mathbf{e}_q}}=j)}\mathbf{1}_{(J_{\mathbf{e}_q}=k)}, \mathbf{e}_q < \tau_0^-\right]
$$

$$
= \int_{y\in[0,\infty)} \int_{z\in[0,x]} f(x+y-z)\sum_{j\in E}\frac{1}{q} P^{0,i}(\overline{\xi}_{\mathbf{e}_q}\in dy, J_{\overline{m}_{\mathbf{e}_q}}=j)\frac{\pi_k}{\pi_j}
$$

$$
\times\, P^{0,k}(\overline{\hat{\xi}}_{\mathbf{e}_q}\in dz, J_{\overline{\hat{m}}_{\mathbf{e}_q}}=j),
$$

where we have used duality in the final equality. Next, with the help of (23),

$$
\int_{[0,\infty)}\int_{[0,\infty)} e^{-\lambda y - \mu z}\sum_{j\in E}\frac{1}{q} P^{0,i}(\overline{\xi}_{\mathbf{e}_q}\in dy, J_{\overline{m}_{\mathbf{e}_q}}=j)\frac{\pi_k}{\pi_j} P^{0,k}(\overline{\hat{\xi}}_{\mathbf{e}_q}\in dz, J_{\overline{\hat{m}}_{\mathbf{e}_q}}=j)
$$

$$
= \sum_{j\in E}\frac{\Phi_j^+(q,0)\hat{\Phi}_j^+(q,0)}{q}\frac{\pi_k}{\pi_j}[\kappa^+(q,\lambda)^{-1}]_{i,j}[\hat{\kappa}^+(q,\mu)^{-1}]_{k,j},
$$

for $\lambda, \mu > 0$. Taking account of (28), it follows with the help of Lebesgue's continuity theorem for Laplace transforms that, in the vague sense, the product measure on the right-hand side of (29) satisfies

$$
\lim_{q\downarrow 0}\sum_{j\in E}\frac{1}{q} P^{0,i}(\overline{\xi}_{\mathbf{e}_q}\in dy, J_{\overline{m}_{\mathbf{e}_q}}=j) P^{0,k}(\overline{\hat{\xi}}_{\mathbf{e}_q}\in dz, J_{\overline{\hat{m}}_{\mathbf{e}_q}}=j)
$$

$$
= \sum_{j\in E} c_j\frac{\pi_k}{\pi_j} U_{i,j}^+(dy)\hat{U}_{k,j}^+(dz).
$$

The result now follows for nonnegative compactly supported, bounded measurable $f \geq 0$, and hence, appealing to standard monotonicity arguments, one can upgrade the result to deal with bounded measurable $f \geq 0$. □

A.6. Markov Renewal theory. The measures $U_{i,j}^+$ play an analogous role to the potential measure U of the ascending ladder process for a Lévy process, which can also be seen as a renewal measure. For example, using an analogue of the compensation formula for Cox processes, it is straightforward to deduce that,

for $a, x > 0$,

$$P^{0,i}\left(\xi_{\tau_a^+} - a > x, J_{\tau_a^+} = j\right)$$

$$(30) \qquad = \int_{[0,a)} U_{i,j}^+(dy) n_j(\epsilon_\zeta > a - y + x, J_\zeta = j)$$

$$+ \sum_{k \neq j} \int_{[0,a)} q_{k,j}^+ U_{i,k}^+(dy)\left(1 - F_{k,j}^+(a - y + x)\right).$$

It is worth noting here that the fact that $U_{i,j}^+$ is diffuse on $(0, \infty)$ ensures that the right-hand side above is continuous in x.

There is a relatively wide body of literature concerning Markov additive renewal theory; see, for example, [19, 24] and [1]. Although mostly dealt with for the case of discrete-time, we can nonetheless identify the following renewal-type theorem for the nonlattice measures $U_{i,j}^+$.

THEOREM 28. *The family $\{\xi_{\tau_a^+} - a : a > 0\}$ of overshoots converges in distribution under $P^{0,i}$ for every $i \in E$ if and only if*

$$E^{0,\pi}\left[H_1^+\right] := \sum_{i \in E} \pi_i E^{0,i}\left[H_1^+\right] < \infty$$

and in that case the following hold:

(i) *For all $i, j \in E$,*

$$\lim_{x \to \infty} \frac{U_{i,j}^+(x)}{x} = \frac{\pi_j}{E^{0,\pi}[H_1^+]}.$$

(ii) *In the spirit of the key renewal theorem, for $\alpha > 0$ and $i, j \in E$,*

$$\lim_{y \to \infty} \int_{[0,y]} e^{-\alpha(y-z)} U_{i,j}^+(dz) = \frac{\pi_j}{\alpha E^{0,\pi}[H_1^+]}.$$

(iii) *For $x > 0$ and $i, j \in E$,*

$$\nu(dx, \{j\})$$

$$:= \text{w-}\lim_{a \to \infty} P^{0,i}\left(\xi_{\tau_a^+} - a \in dx, J_{\tau_a^+}^+ = j\right)$$

$$= \frac{1}{E^{0,\pi}[H_1^+]}\left[\pi_j n_j(\epsilon_\zeta > x, J_\zeta = j) + \sum_{k \neq j} \pi_k q_{k,j}^+\left(1 - F_{k,j}^+(x)\right)\right] dx,$$

where $F_{k,j}^+$ is the distribution whose Laplace transform is $G_{k,j}^+$.

For all limits above, we interpret the right-hand side as zero when $E^{0,\pi}[H_1^+] = \infty$. In particular, this means that the overshoot distributions diverge to an atom at $+\infty$ and are not tight.

Parts (i) and (iii) are the continuous-time analogue of the Markov additive renewal theorem in [24], whereas part (ii) is the continuous time analogue of the version of the Markov additive renewal theorem in [19].

A.7. Harmonic functions.   The main objective of this section is to prove a result which identifies a harmonic function for the process (ξ, J) when killed on entering $(-\infty, 0) \times E$. In the forthcoming analysis, we use $\underline{L}_t$ to denote $\sum_{k \in E} \underline{L}_t^{(k)}$, the sum of local times (each up to an arbitrarily scaling of a strictly positive constant) of $\xi - \underline{\xi}$ at $(0, k)$, $k \in E$, where $\underline{\xi}_t = \inf_{s \le t} \xi_s$, $t \ge 0$. Moreover, similarly to previous sections in this Appendix, we work with $H_t^- := -\xi_{\underline{L}_t^{-1}}$ and $J_t^- = J_{\underline{L}_t^{-1}}$, for all t such that $\underline{L}_t^{-1} < \infty$, and otherwise the pair H_t^- and J_t^- are both assigned the value ∞.

THEOREM 29.   *For all $i \in E$ and $x > 0$,*

$$U_i^-(x) := \sum_{j \in E} U_{i,j}^-(x)\pi_j.$$

There exists an appropriate normalisation of local times $\underline{L}^{(i)}$, $i \in E$, such that

$$U_{J_t}^-(\xi_t)\mathbf{1}_{(t < \tau_0^-)}, \qquad t \ge 0,$$

is a $P^{x,i}$-martingale if and only if $\overline{\xi} - \xi$ is recurrent at zero; that is to say the Markov process $(\overline{\xi} - \xi, J)$ is recurrent at $(0, k)$ for some (and hence all) $k \in E$.

We start by proving a preliminary lemma giving us an important fluctuation identity. To this end, define for $q > 0$ the measure,

$$^q U_{i,j}^-(dx) = E^{0,i}\left[\int_0^\infty e^{-q\underline{L}_t^{-1}}\mathbf{1}_{(H_t^- \in dx, J_t^- = j)}\, dt\right], \qquad x \ge 0.$$

Recall that $\mathbf{e}_q$ denotes an independent exponentially distributed random variable with rate $q > 0$ and $\tau_0^- := \inf\{t > 0 : \xi_t < 0\}$. Let $\underline{n}_i$ be the excursion measures of $\xi - \underline{\xi}$ from the point $(0, i)$, $i \in E$. Write $\underline{m}_t = \sup\{s \le t : \xi_s = \underline{\xi}_t\}$. For convenience, let us assume that each of the subordinators $[\underline{L}^{(k)}]_t^{-1}$, $k \in E$, have no drift component. The corresponding forthcoming computation when this is not the case is a straightforward modification, for example, in the spirit of, for example, the proof of Lemma VI.8 of [3].

If we mark the excursion from the minimum indexed by local time $t > 0$ with an independent exponentially distributed random variables, say $\mathbf{e}_q^{(t)}$, then using the compensation formula for the Cox process of excursions of $\underline{\xi} - \xi$ from 0, we

have

$$P^{x,i}\big(\tau_0^- > \mathbf{e}_q, J_{\underline{m}_{\mathbf{e}_q}} = j\big)$$

$$= E^{0,i}\bigg[E^{0,i}\bigg[\sum_{t \geq 0} \mathbf{1}_{(H_{t-}^- \leq x, \Delta \underline{L}_s^{-1} < \mathbf{e}^{(s)} \ \forall s < t)} \mathbf{1}_{(\Delta \underline{L}_t^{-1} > \mathbf{e}^{(t)}, J_t^- = j)} | \sigma(J_u : u \geq 0)\bigg]\bigg]$$

$$= E^{i,0}\bigg[\int_0^\infty dt \cdot \mathbf{1}_{(H_{t-}^- \leq x, J_{t-}^- = j, \Delta \underline{L}_s^{-1} < \mathbf{e}^{(s)} \ \forall s < t)}\bigg] \underline{n}_j(\zeta > \mathbf{e}_q)$$

$$= E^{0,i}\bigg[\int_0^\infty dt \cdot e^{-q\underline{L}_t^{-1}} \mathbf{1}_{(H_{t-}^- \leq x, J_{t-}^- = j)}\bigg] \underline{n}_j(1 - e^{-q\zeta})$$

$$= {}^q U_{i,j}^-(x) \Phi_j^-(q,0),$$

where $\Phi_j^-(q,0) := \underline{n}_j(1 - e^{-q\zeta})$ is a notational choice that, by analogy, respects the definition of $\Phi_j^+(\alpha, \beta)$ given in Appendix A.2. In the second and third equalities above, the letter ζ denotes the canonical excursion length. In conclusion, we have established the following lemma.

LEMMA 30. *For all $i, j \in E$ and $x > 0$,*

$$(31) \qquad P^{x,i}\big(\tau_0^- > \mathbf{e}_q, J_{\underline{m}_{\mathbf{e}_q}} = j\big) = {}^q U_{i,j}^-(x) \Phi_j^-(q,0).$$

We now return to the proof of Theorem 29.

PROOF OF THEOREM 29. Thanks to the Markov property, it suffices to prove that, for all $i \in E$ and $x > 0$,

$$E^{x,i}\big[U_{J_t}^-(\xi_t) \mathbf{1}_{(t < \tau_0^-)}\big] = U_i^-(x).$$

As a first step, we show that

$$(32) \qquad \lim_{q \downarrow 0} \frac{P^{x,i}(\tau_0^- > \mathbf{e}_q)}{\sum_{j \in E} \Phi_j^-(q,0)} = U_i^-(x), \qquad x > 0.$$

From (23), we have that, for $i, k \in E$ and $\alpha > 0$,

$$E^{0,i}\big[e^{-\alpha \underline{m}_{\mathbf{e}_q}}, J_{\underline{m}_{\mathbf{e}_q}} = k\big] = \Phi_k^-(q,0)\big[\kappa^-(q+\alpha,0)^{-1}\big]_{i,k}.$$

Duality dictates that

$$E^{0,i}\big[e^{-\alpha \underline{m}_{\mathbf{e}_q}}, J_{\underline{m}_{\mathbf{e}_q}} = k\big] = \frac{\pi_k}{\pi_i} \hat{E}^{0,k}\big[e^{-\alpha \overline{m}_{\mathbf{e}_q}}, J_{\overline{m}_{\mathbf{e}_q}} = i\big],$$

which tells us that

$$(33) \qquad \Phi_k^-(q,0)\big[\kappa^-(q+\alpha,0)^{-1}\big]_{i,k} = \frac{\pi_k}{\pi_i} \hat{\Phi}_i^+(q,0)\big[\hat{\kappa}^+(q+\alpha,0)^{-1}\big]_{k,i}.$$

Fixing i and considering (33) for different values of k and taking limits as $q \downarrow 0$, tells us that

$$\lim_{q \downarrow 0} \frac{\Phi_k^-(q, 0)}{\Phi_j^-(q, 0)} = \frac{\pi_k}{\pi_j} C_{j,k}, \qquad j, k \in E,$$

for some constant $C_{j,k} \in (0, \infty)$. On account of the fact that each of the local times $\underline{L}^{(i)}$ can be defined up to arbitrarily scaling of a strictly positive constant, without loss of generality, we may take $C_{j,k} = 1$. Note that there can be no dependency on i or α in the limit on the right-hand side as the left-hand side has no dependency on α. From the above limit and (31), we have that (32) holds.

To complete the proof, we use ideas from [5] and Chapter 13 of [23]. With the help of monotone convergence, we have that

$$E^{x,i}\big[U_{J_t}^-(\xi_t), t < \tau_0^-\big]$$

$$= \lim_{q \downarrow 0} E^{x,i}\bigg[\mathbf{1}_{(t < \tau_0^-)} \frac{P^{\xi_t, J_t}(\tau_0^- > \mathbf{e}_q)}{\sum_{j \in E} \Phi_j^-(q, 0)}\bigg]$$

$$= \lim_{q \downarrow 0} \frac{1}{\sum_{j \in E} \Phi_j^-(q, 0)} P^{x,i}\big[\tau_0^- > \mathbf{e}_q | \mathbf{e}_q > t\big]$$

$$= \lim_{q \downarrow 0}\bigg[e^{qt} \frac{P^{x,i}(\tau_0^- > \mathbf{e}_q)}{\sum_{j \in E} \Phi_j^-(q, 0)} - e^{qt} \int_0^t q e^{-qs} \frac{P^{x,i}(\tau_0^- > s)}{\sum_{j \in E} \Phi_j^-(q, 0)} \, ds\bigg]$$

$$= U_i^-(x) - \lim_{q \downarrow 0} \frac{q}{\sum_{j \in E} \Phi_j^-(q, 0)} \int_0^t P^{x,i}(\tau_0^- > s) \, ds.$$

The proof is complete as soon as we can show that the limit preceding the integral term is equal to zero. To this end, note that for each $j \in E$,

$$\lim_{q \downarrow 0} \frac{1}{q} \Phi_j^-(q, 0) = \Phi_j^{-\prime}(0, 0) = E^{0,k}\big[[\underline{L}^{(k)}]_1^{-1}\big] \in (0, \infty].$$

We want to show that the expectation on the right-hand side above to be $+\infty$ as a consequence of the fact that 0 is recurrent for $\overline{\xi} - \xi$. Referring again to (33), we see that $\lim_{q \downarrow 0} \Phi_k^-(q, 0)/q$ and $\lim_{q \downarrow 0} \hat{\Phi}_k^+(q, 0)/q$ are simultaneously (in)finite. Note that both have limits because they are Bernstein functions.

Now recall from (25) that, since

$$(34) \qquad \lim_{q \downarrow 0} \frac{\Phi_k^+(q, 0) \hat{\Phi}_k^+(q, 0)}{q} = c_k \in (0, \infty),$$

it follows that $\lim_{q \downarrow 0} \Phi_k^-(q, 0)/q = \infty$ if and only if $\Phi_k^+(0, 0) = 0$. However, the assumption that $\overline{\xi} - \xi$ is recurrent at 0 ensures that $\Phi_k^+(0, 0) = 0$ for all $k \in E$.

In conclusion, we have that, under the assumption that $\overline{\xi} - \xi$ is recurrent at 0, the term

$$\lim_{q \downarrow 0} \frac{q}{\sum_{j \in E} \Phi_j^-(q, 0)} = 0,$$

and subsequently the claim of the theorem is proved. $\quad \square$

A.8. Conditioning to stay positive. It turns out that the harmonic function $U_j^-(x)$, $j \in E$, $x > 0$, corresponds to the h-function that appears in the Doob h-transform corresponding to the process ξ conditioned to stay positive.

Let $A \in \mathcal{F}_t := \sigma((\xi_s, J_s) : s \le t)$ and assume that 0 is recurrent for $\overline{\xi} - \xi$. Appealing to the Markov and lack of memory properties, we have

$$\lim_{q \downarrow 0} P^{x,i}\big(A, t < \mathbf{e}_q | \tau_0^- > \mathbf{e}_q\big) = \lim_{q \downarrow 0} E^{x,i}\left[\mathbf{1}_{(A, t < \tau_0^- < \mathbf{e}_q)} \frac{P^{\xi_t, J_t}(\tau_0^- > \mathbf{e}_q)}{P^{x,i}(\tau_0^- > \mathbf{e}_q)} \right].$$

Next, note that, for all $q < q_0$,

$$\frac{P^{\xi_t, J_t}(\tau_0^- > \mathbf{e}_q)}{P^{x,i}(\tau_0^- > \mathbf{e}_q)} = \frac{q U_{J_t}^-(\xi_t)}{q U_i^-(x)} \le \frac{q_0 U_{J_t}^-(\xi_t)}{U_i^-(x)}.$$

Hence, by dominated convergence, we have that

$$\lim_{q \downarrow 0} P^{x,i}\big(A, t < \mathbf{e}_q | \tau_0^- > \mathbf{e}_q\big) = E^{x,i}\left[\mathbf{1}_{(A, t < \tau_0^-)} \frac{U_{J_t}^-(\xi_t)}{U_i^-(x)} \right].$$

In conclusion, we have the following theorem which confirms the existence of the law of (ξ, J) with ξ conditioned to stay positive.

THEOREM 31. *Suppose that 0 is recurrent for $\overline{\xi} - \xi$. Then there exists a family of probability measures on the Skorokhod space, say $\mathbb{P}_{x,i}^{\uparrow}$, defined via the Doob h-transform*

$$\left. \frac{dP^{x,i,\uparrow}}{dP^{x,i}} \right|_{\mathcal{F}_t} = \frac{U_{J_t}^-(\xi_t)}{U_i^-(x)} \mathbf{1}_{(t < \tau_0^-)}, \qquad t \ge 0, i \in E, x > 0,$$

such that, for all A in $\mathcal{F}_t$,

$$P^{x,i,\uparrow}(A) = \lim_{q \downarrow 0} P^{x,i}\big(A, t < \mathbf{e}_q | \tau_0^- > \mathbf{e}_q\big).$$

REMARK 32. The above discussion applied to the MAP $(-\xi, J)$ implies the existence of a positive function $U_i^+(x)$, $i \in E$ such that $U_{J_t}^+(\xi_t) \mathbf{1}_{(t < \tau_0^+)}$ is a martingale and the h-transformed law

$$\left. \frac{dP^{x,i,\downarrow}}{dP^{x,i}} \right|_{\mathcal{F}_t} = \frac{U_{J_t}^+(\xi_t)}{U_i^+(x)} \mathbf{1}_{(t < \tau_0^+)}, \qquad t \ge 0, i \in E, x < 0,$$

is the MAP conditioned to be negative.

For the proof of Lemma 14 below, we shall need that conditioned MAPs tend to infinity. In the context of Lévy processes, many proofs exist for the analogue of the next lemma. Those proofs are consequences of complicated pathwise constructions for the conditioned processes that we do not want to repeat for the setting of MAPs. Instead we give a simple argument based on potential calculations only. The argument is inspired by more explicit calculations for spectrally negative Lévy processes in Lemma VII.12 of [3].

PROPOSITION 33.   *For each $x < 0$ and $i \in E$, we have that $P^{x,i,\downarrow}(\lim_{t \to \infty} \xi_t = -\infty) = 1$.*

PROOF.   First, note that, for all $z < x < 0$ and $i \in E$, with the help of (14),

$$
E^{x,i,\downarrow}\left[\int_0^\infty \mathbf{1}_{(\xi_t \geq z)}\, dt\right]
$$

(35)
$$
= U^\downarrow\big((x,i), ([z,0], E)\big)
$$

$$
= \sum_{j \in E} \int_{[z,0]} \frac{U_j^+(y)}{U_i^+(x)} U^\dagger\big((x,i),(dy,\{j\})\big),
$$

where $U^\dagger((x,i),(dy,\{j\}))$ is the potential measure of the process (ξ, J) killed when ξ first enters $(0,\infty)$. Since U^+ is locally bounded, the right-hand side can be estimated from above by $\frac{C}{U^+(x)} U^\dagger((x,i), ([0,z], E))$ which is finite by Theorem 27 applied with $f \equiv 1$ and using the local boundedness of the appearing potential measures of the ladder processes. This implies that

(36)
$$
P^{x,i,\downarrow}\big(\tau_z^- < \infty\big) = 1, \qquad \text{for all } z < x < 0, i \in E.
$$

Otherwise, the trajectory of ξ is bounded from below by z with positive probability under $P^{x,i,\downarrow}$, and hence, $\int_0^\infty \mathbf{1}_{(\xi_t \geq z)}\, dt = \infty$ with positive probability. But then the left-hand side of (35) would be infinite, giving a contradiction.

Next, we show that

(37)
$$
\lim_{z \to -\infty} P^{z,i,\downarrow}(\xi_t < a \text{ for all } t \geq 0) = 1, \qquad \text{for all } a < 0, i \in E.
$$

To see this, define $\tau_{[a,0]} = \inf\{t > 0 : \xi_t \in [a,0]\}$. Use the change of measure in Remark 32 to note that, for $z < a$,

$$
P^{z,i,\downarrow}(\text{there is } t \geq 0 \text{ such that } \xi_t \geq a)
$$

$$
= E^{z,i,\downarrow}[\mathbf{1}_{(\tau_{[a,0]} < \infty)}]
$$

$$
= E^{z,i,\dagger}\left[\frac{U_{J_{\tau_{[a,0]}}}^+(\xi_{\tau_{[a,0]}})}{U_i^+(z)} \mathbf{1}_{(\tau_{[0,a]} < \tau_0^+)} \mathbf{1}_{(\tau_{[0,a]} < \infty)}\right].
$$

Using the monotonicity of $z \mapsto U_i^+(z)$, the right-hand side can be bounded from above by

$$\frac{\max_{j \in E} U_j^+(a)}{U_i^+(z)} P^{z,i,\dagger}(\tau_{[0,a]} < \tau_0^+).$$

Finally, since $\lim_{z \to -\infty} U_i^+(z) = +\infty$, (37) is proved.

The claim of the proposition now follows from the strong Markov property applied to τ_z^-, which is finite almost surely by (36) and (37). $\quad\square$

A.9. Laws of large numbers. Similarly to the case of Lévy process, it is known that a MAP (ξ, J) grows linearly, meaning that

$$\text{(38)} \qquad \lim_{t \to \infty} \frac{\xi_t}{t} = E^{0,\pi}[\xi_1]$$

provided

$$E^{0,\pi}[\xi_1] = \sum_{i \in E} \pi_i E^{0,i}[\xi_1]$$

is defined. Moreover, when $E^{0,\pi}[\xi_1]$ is defined there is a trichotomy which dictates whether (ξ, J) drifts to $+\infty$, $-\infty$ or oscillates accordingly as $E^{0,\pi}[\xi_1] > 0$, < 0 or $= 0$, respectively. See, for example, Chapter XI of [2].

We fix a state $k \in E$ and consider the MAP at the discrete set of return times of J to k. Let $\sigma_0 = \inf\{t \geq 0 : J_t = k\}$ and inductively define, for $n \in \mathbb{N}$,

$$\text{(39)} \qquad \sigma_{n+1} = \inf\{t > \sigma_n : J_t = k \text{ and } \exists s \in (\sigma_{n-1}, t) \text{ with } J_s \neq k\}.$$

The skeleton $(\xi_{\sigma_n})_{n \in \mathbb{N}_0}$ is a Markov chain. The following theorem relates the law of large numbers to moments of the underlying Lévy processes and transition jumps appearing in Proposition 2 and gives an identity that is crucial for the next section.

THEOREM 34. *The following statements are equivalent for a MAP (ξ, J):*

(i) *ξ_1 has finite absolute mean for one (any) starting distribution with $\xi_0 = 0$.*
(ii) *ξ_{σ_1} has finite absolute mean when started in $(0, k)$.*
(iii) *The Lévy processes ξ^i have finite absolute moment and any $\Delta_{i,j}$ with $q_{i,j} > 0$ has finite absolute moment.*
(iv) *$\lim_{t \to \infty} \frac{\xi_t}{t}$ exists almost surely for one (any) starting distribution.*

Under (i) *to* (iv), *we have*

$$\text{(40)} \qquad \lim_{t \to \infty} \frac{\xi_t}{t} = E^{0,\pi}[\xi_1] = \frac{E^{0,k}[\xi_{\sigma_1}]}{E^{0,k}[\sigma_1]}, \qquad k \in E.$$

PROOF.   Throughout the proof, we shall use the fact that for any Lévy process $\{\eta_t : t \geq 0\}$

$$E[|\eta_s|] < \infty \text{ for some } s > 0 \quad \Longleftrightarrow \quad E[|\eta_t|] < \infty \text{ for all } t \geq 0$$

$$\Longleftrightarrow \quad E\Big[\sup_{s \leq t} |\eta_s|\Big] < \infty \text{ for all } t \geq 0.$$

See Theorem 25.18 of Sato [28] for a proof.

(iii) $\Rightarrow$ (i): Note that for a fixed distribution $P^{0,\mu}$, by Proposition 2, the distribution of ξ_1 is identical to the law of

$$(41) \qquad \sum_{i \in E} \xi_{t_i(1)}^i + \sum_{i \neq j} \sum_{\ell=1}^{n_{i,j}(1)} \Delta_{i,j}^{\ell},$$

where $t_i(1)$ denotes the time J spends in state i, and $n_{i,j}(1)$ the number of jumps of J from i to j over the time interval $[0, 1]$ and, for each $i, j \in E$ such that $i \neq j$, $\{\Delta_{i,j}^{\ell} : \ell \geq 1\}$ are i.i.d. copies of $\Delta_{i,j}$. Since the expected number of total jumps is finite, the triangle inequality shows that (iii) implies (i).

(i) $\Rightarrow$ (iii): By considering the event that the first jump away from the initial state $i \in E$ occurs after time 1, we have that $E^{0,i}[|\xi_1|] \geq E^{0,i}[|\xi_1^i|]\exp\{-|q_{i,i}|\}$, thereby showing that each of the pure-state Lévy processes ξ^i, $i \in E$, have finite absolute moment. Now consider the event that the first jump of the Markov chain J occurs before time 1 and the second jump occurs after time 1. In that case, we have

$$\int_0^1 |q_{i,i}| e^{-|q_{i,i}|t} \sum_{j \neq i} \frac{q_{i,j}}{|q_{i,i}|} e^{-|q_{j,j}|(1-t)} E^{0,i}\big[|\xi_t^i + \Delta_{i,j} + \xi_{1-t}^j|\big]\,dt < E^{0,i}[|\xi_1|] < \infty.$$

This tells us that for each $j \in E$, Lebesgue almost everywhere in $[0, 1]$,

$$(42) \qquad E^{0,i}\big[|\xi_t^i + \Delta_{i,j} + \xi_{1-t}^j|\big] < \infty.$$

For a given $j \in E$ with $j \neq i$, fix such a $t \in [0, 1]$ and note that

$$E^{0,i}\big[|\Delta_{i,j}|\big] = E^{0,i}\big[|\xi_t^i + \Delta_{i,j} + \xi_{1-t}^j - \xi_t^i - \xi_{1-t}^j|\big]$$

$$\leq E^{0,i}\big[|\xi_t^i + \Delta_{i,j} + \xi_{1-t}^j|\big] + E^{0,i}\big[|\xi_t^i|\big] + E^{0,i}\big[|\xi_{1-t}^j|\big]$$

$$< \infty,$$

where the final inequality follows by (42), the previously established fact that $E^{0,i}[|\xi_1^i|] < \infty$ for $i \in E$ and the opening remark at the beginning of this proof.

(i) $\Rightarrow$ (ii): We can identify the distribution of ξ_{σ_1} with that of

$$(43) \qquad \sum_{i \in E} \xi_{t_i(\sigma_1)}^i + \sum_{i \neq j} \sum_{\ell=1}^{n_{i,j}(\sigma_1)} \Delta_{i,j}^{\ell},$$

where $t_i(\sigma_1)$ denotes the time J spends in state i, and $n_{i,j}(\sigma_1)$ the number of jumps of J from i to j over the time interval $[0, \sigma_1]$ and, for each $i, j \in E$ such that $i \neq j$, $\{\Delta_{i,j}^\ell : \ell \geq 1\}$ are i.i.d. copies of $\Delta_{i,j}$ [also independent of $n_{i,j}(\sigma_1)$, which depends only on the chain J]. Note that $t_i(\sigma_1)$ is a random sum of an independent, geometrically distributed number of independent exponential random variables that depend only on J, so that $E^{0,k}[|\xi_{t_i(\sigma_1)}^i|] < \infty$ whenever (iii) holds. Having already shown the equivalence of (i) and (iii), it follows from the triangle inequality and the distributional equivalence in (43) that (i) implies (ii).

(ii) $\Rightarrow$ (iii): On the event that the sojourn of J from k consists of a first jump from k to $j \neq k$, followed by a jump back to k, written $\{k \to j \to k\}$, we can write

$$\xi_{\sigma_1} = \xi_{\mathbf{e}_{|q_{k,k}|}}^k + \Delta_{k,j} + \xi_{\mathbf{e}_{|q_{j,j}|}}^j + \Delta_{j,k},$$

where, for $i \in E$, $\mathbf{e}_{|q_{i,i}|}$ is an independent exponentially distributed random variable with rate $|q_{i,i}|$ and the sum on the right-hand side above consists of four independent random variables. This means that

$$(44) \qquad \begin{aligned} \infty > E^{0,k}\big[|\xi_{\sigma_1}|\big] &\geq E^{0,k}\big[|\xi_{\sigma_1}|\mathbf{1}_{\{k \to j \to k\}}\big] \\ &= \mathbb{E}\big[|\xi_{\mathbf{e}_{|q_{k,k}|}}^k + \Delta_{k,j} + \xi_{\mathbf{e}_{|q_{j,j}|}}^j + \Delta_{j,k}|\big] \end{aligned}$$

if we denote by $\mathbb{E}$ the product space of the two Lévy processes, two transition jumps and two exponential variables.

From the aforesaid independence, we can deduce (iii). As a first step, integrate out the final three summands on the right-hand side of (44):

$$E^{0,k}\big[|\xi_{\sigma_1}|\mathbf{1}_{\{k \to j \to k\}}\big] = \int_{\mathbb{R}} \int_0^\infty \int_0^\infty \mathbb{E}\big[|\xi_{\mathbf{e}_{|q_{k,k}|}}^k + a + b + c|\big]$$
$$\times \mathbb{P}\big(\Delta_{k,j} \in da, \xi_{\mathbf{e}_{|q_{j,j}|}}^j \in db, \Delta_{j,k} \in dc\big).$$

The left-hand side is finite and nonzero, so there is some $x \in \mathbb{R}$ with $\mathbb{E}[|\xi_{\mathbf{e}_{|q_{k,k}|}}^k + x|] < \infty$. Integrating out the independent exponential time and using that ξ^k is a Lévy process implies that $\mathbb{E}[|\xi_1^k|] < \infty$ and $\mathbb{E}[|\xi_{\mathbf{e}_{|q_{k,k}|}}^k|] < \infty$ (compare the remark at the beginning of the proof and also note that $\mathbb{E}[|\xi_1^k|] < \infty$ if and only if $\mathbb{E}[|\xi_1^k + x|] < \infty$ for any $x \in \mathbb{R}$).

Similarly, we find that $\mathbb{E}[|\xi_1^j|] < \infty$ and $\mathbb{E}[|\xi_{\mathbf{e}_{|q_{j,j}|}}^j|] < \infty$. Using the triangle inequality implies

$$\mathbb{E}\big[|\Delta_{k,j} + \Delta_{k,j}|\big] \leq \mathbb{E}\big[|(\Delta_{k,j} + \Delta_{j,k}) + \big(\xi_{\mathbf{e}_{|q_{k,k}|}}^k + \xi_{\mathbf{e}_{|q_{j,j}|}}^j\big)|\big]$$
$$+ \mathbb{E}\big[|\xi_{\mathbf{e}_{|q_{k,k}|}}^k|\big] + \mathbb{E}\big[|\xi_{\mathbf{e}_{|q_{j,j}|}}^j|\big]$$

and the right-hand side is finite by (44) and the above. Hence, by positivity of the transition jumps we obtain

$$\mathbb{E}\big[|\Delta_{j,k}|\big] \leq \mathbb{E}\big[|\Delta_{k,j} + \Delta_{k,j}|\big] < \infty \quad \text{and} \quad \mathbb{E}\big[|\Delta_{k,j}|\big] \leq \mathbb{E}\big[|\Delta_{k,j} + \Delta_{k,j}|\big] < \infty.$$

In total, we proved that $\Delta_{j,k}$, $\Delta_{k,j}$, ξ_1^j and ξ_1^k all have finite absolute mean which confirms (iii).

(iv) $\Leftrightarrow$ (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii): First, note that under $P^{0,k}$, σ_1 has finite first moment so that

$$\lim_{n \to \infty} \frac{\sigma_n - \sigma_0}{n} = \lim_{n \to \infty} \frac{\sum_{i=1}^n \sigma_i}{n} = E^{0,k}[\sigma_1].$$

Assume that the limit $\lim_{t \to \infty} \xi_t / t$ exists almost surely. In this case, the limit is equal to

$$\lim_{n \to \infty} \frac{1}{n E^{0,k}[\sigma_1]} \sum_{l=1}^n (\xi_{\sigma_l} - \xi_{\sigma_{l-1}}).$$

However, considering the case of strong laws of large numbers for random walks (cf. Theorem 7.2 of [23]), the latter limit exists and is finite if and only if $E^{0,k}[|\xi_{\sigma_1}|] < \infty$, in which case the limit above must equal $E^{0,k}[\xi_{\sigma_1}]/E^{0,k}[\sigma_1]$. It follows that (iv) implies (i)–(iii) and also that $\lim_{t \to \infty} \xi_t / t$ has the second claimed limit in (40).

Conversely, now assuming the equivalent statements (i), (ii) and (iii), in particular (ii), we can conclude that

$$\lim_{n \to \infty} \frac{\xi_{\sigma_n}}{\sigma_n} = E^{0,k}[\xi_{\sigma_1}] \tag{45}$$

almost surely by the strong law of large numbers for random walks. Next, we need that

$$E^{0,k}\left[\sup_{t \in [0, \sigma_1]} |\xi_t| \right] < \infty \tag{46}$$

which can be seen as follows: By the triangle inequality and (43),

$$\sup_{t \in [0, \sigma_1]} |\xi_t| \le \sum_{i \in E} \sup_{t \in [0, \sigma_1]} |\xi_t^i| + \sum_{i \neq j} \sum_{\ell=1}^{n_{i,j}(\sigma_1)} |\Delta_{i,j}^\ell|.$$

The expectation of the right-hand side is finite thanks to the independence of ξ^i, $i \in E$ and J, the assumption (iii) and the remark at the very beginning of this proof. Now we use (46) to deduce

$$\lim_{n \to \infty} \frac{\sup_{t \in [\sigma_{n-1}, \sigma_n]} |\xi_t - \xi_{\sigma_{n-1}}|}{n} = 0,$$

which then implies in combination with (45) almost sure convergence of ξ_t / t to a finite constant which is (iv).

It remains to verify (40) under any of the equivalent conditions (i) to (iv). The first equality is the law of large numbers (38) under finite mean and the second equality was already derived in the argument for (iv) implies (i)–(iii). $\square$

A.10. Tightness of the overshoots. We now characterise tightness of overshoots for general MAPs. That is to say, taking account of the conclusion in Theorem 28, we provide necessary and sufficient conditions for $E^{0,\pi}[H_1^+] < \infty$, thereby giving a proof of Theorem 5.

Although we have assumed the nonlattice condition in this Appendix, *the results given below do not need it.*

THEOREM 35. *The MAP (ξ, J) has tight overshoots if and only if ξ_1 has finite absolute moment and*:

(i) (ξ, J) *drifts to $+\infty$; or*
(ii) (ξ, J) *oscillates and satisfies*

$$\textbf{(TO)} \quad \int_\kappa^\infty \frac{x\,\Pi([x, \infty))}{1 + \int_0^x \int_y^\infty \Pi((-\infty, -z])\,dz\,dy}\,dx < \infty$$

for one (any) $\kappa > 0$ and

$$(47) \qquad \Pi := \sum_{i \neq j, i, j \in E} q_{i,j} \mathcal{L}(\Delta_{i,j}) + \sum_{i \in E} \Pi_i,$$

where Π_i is the Lévy measure of the ith Lévy process and $\mathcal{L}(\Delta_{i,j})$ is the probability distribution of the transition jump from i to j in Proposition 2.

In order to prove the theorem, it suffices to analyze tightness of the overshoots on the discrete time skeleton embedded in (ξ, J) at the return times of the Markov chain to a fixed state $k \in E$. As in Appendix A.9, we let $\sigma_0 = \inf\{t \geq 0 : J_t = k\}$ and inductively define, for $n \in \mathbb{N}$,

$$\sigma_{n+1} = \inf\{t > \sigma_n : J_t = k \text{ and } \exists s \in (\sigma_{n-1}, t) \text{ with } J_s \neq k\}$$

so that $(\xi_{\sigma_n})_{n \in \mathbb{N}_0}$ is a Markov chain.

LEMMA 36. *The MAP (ξ, J) has tight overshoots if and only if the Markov chain $(\xi_{\sigma_n})_{n \in \mathbb{N}_0}$ has tight overshoots under $P^{0,k}$.*

PROOF. For $x, s \geq 0$, consider the stopping times

$$\rho_x = \inf\{t : \xi_t \geq x, J_t = k, J_{t-} \neq k\} \quad \text{and} \quad \sigma(s) = \inf\{t > s : J_t = k, J_{t-} \neq k\}.$$

For $c \geq 0$, one has

$$\{\xi_{\rho_x} - x \geq 3c\} \subset \left\{\xi_{\tau_{x+c}^+} - (x+c) \geq c\right\} \cup \left\{\sup_{s \in [\tau_{x+c}^+, \sigma(\tau_{x+c}^+)]} |\xi_s - \xi_{\tau_{x+c}^+}| \geq c\right\}.$$

Indeed, in the case where the overshoot of the discrete time process (ξ_{σ_n}) is larger than $3c$ and the overshoot of the continuous time process over $x + c$ is smaller than c, one has $\xi_{\tau_{x+c}^+} \in [x+c, x+2c]$ so that the process has to oscillate between

time τ^+_{x+c} and the next entry of J into k at time $\sigma(\tau^+_{x+c})$ by at least c. For every $i, j \in E$ and $x \geq 0$, one has

$$P^{0,i}\left(\sup_{s \in [\tau^+_{x+c}, \sigma(\tau^+_{x+c})]} |\xi_s - \xi_{\tau^+_{x+c}}| \geq c \,\Big|\, J_{\tau^+_{x+c}} = j\right) = P^{0,j}\left(\sup_{s \in [0, \sigma_1]} |\xi_s| \geq c\right)$$

and since finite families and mixtures thereof are always tight, there exists a decreasing function $g_2 : [0, \infty) \to [0, 1]$ with limit 0 such that

$$P^{0,i}\left(\sup_{s \in [\tau^+_{x+c}, \sigma(\tau^+_{x+c})]} |\xi_s - \xi_{\tau^+_{x+c}}| \geq c\right) \leq g_2(c), \qquad \text{for } i \in E, x, c \geq 0.$$

If the continuous time process has tight overshoots, then there is a function $g_1 : [0, \infty) \to [0, 1]$ with limit 0 such that

$$P^{0,i}\left(\xi_{\tau^+_{x+c}} - (x + c) \geq c\right) \leq g_1(c), \qquad \text{for } i \in E, x, c \geq 0,$$

so that altogether

$$\sup_{i \in E, x \geq 0} P^{0,i}(\xi_{\rho_x} - x \geq 3c) \leq g_1(c) + g_2(c)$$

and the overshoots of the discrete time process are tight.

The converse direction follows analogously. Using that

$$\{\xi_{\tau^+_x} - x \geq 2c\} \subset \{\xi_{\rho_x} - x \geq c\} \cup \left\{\sup_{s \in [\tau^+_x, \sigma(\tau^+_x)]} |\xi_s - \xi_{\tau^+_{x+c}}| \geq c\right\}$$

one deduces that the continuous time process has tight overshoots if the discrete time process has tight overshoots under any of the laws $P^{0,i}$. Further using that for $i \in E$ and $x, c \geq 0$, one has

$$P^{0,i}(\xi_{\rho_x} - x \geq c) \leq P^{0,i}\left(\sup_{s \in [0, \sigma(0)]} \xi_s \geq c\right) + E^{0,i}\left[P^{\xi_{\sigma(0)} \wedge x, k}(\xi_{\rho_x} - x \geq c)\right]$$

one deduces that tightness of the overshoots of the discrete time process under the law $P^{0,k}$ induces tightness under any law $P^{0,i}$ with $i \in E$. $\square$

The following lemma is a consequence of Theorem 8 of [9].

LEMMA 37. *A random walk has tight overshoots if and only if the distribution of its increments has finite absolute moment, it drifts to infinity or oscillates and the distribution Π of its increments satisfies the integrability condition* (**TO**).

The next result will be helpful later to separate big jumps from small jumps in the Lévy processes corresponding through Proposition 2 to the MAP (ξ, J).

LEMMA 38. *Let X, Y be real random variables with Y being square integrable, then the distribution of X satisfies* (**TO**) *if and only if the distribution of $X + Y$ satisfies* (**TO**).

PROOF. It suffices to show that $X + Y$ satisfies (**TO**), if X satisfies (**TO**). For the reverse statement, the same argument applies with the use of $-Y$ instead of Y. We use that for $z \geq 0$

$$\mathbb{P}(X + Y \geq z) \leq \mathbb{P}(X \geq z/2) + \mathbb{P}(Y \geq z/2)$$

and

$$\mathbb{P}(X + Y \leq -z) \geq \mathbb{P}(X \leq -2z) - \mathbb{P}(Y \geq z)$$

to deduce that

$$\int_\kappa^\infty \frac{x\mathbb{P}(X + Y \geq x)}{1 + \int_0^x \int_y^\infty \mathbb{P}(X + Y \leq -z)\,dz\,dy}\,dx$$

$$\leq \int_\kappa^\infty \frac{x\mathbb{P}(X \geq x/2)}{1 + \int_0^x \int_y^\infty (\mathbb{P}(X \leq -2z) - \mathbb{P}(Y \geq z))_+\,dz\,dy}\,dx$$

$$+ \int_\kappa^\infty x\mathbb{P}(Y \geq x/2)\,dx.$$

The latter integral is finite since Y has finite second moment and the proof is complete once we showed that the former integral is finite. One has

$$\int_0^\infty \int_y^\infty \mathbb{P}(Y \geq z)\,dz\,dy = \frac{1}{2}\mathbb{E}[Y_+^2] < \infty$$

and taking $c \geq 1$ with $c > \mathbb{E}[Y_+^2]$ we conclude with substitution that

$$\int_\kappa^\infty \frac{x\mathbb{P}(X \geq x/2)}{1 + \int_0^x \int_y^\infty (\mathbb{P}(X \leq -2z) - \mathbb{P}(Y \geq z))_+\,dz\,dy}\,dx$$

$$\leq 4c \int_{\kappa/2}^\infty \frac{x\mathbb{P}(X \geq x)}{c + \int_0^{2x} \int_y^\infty (\mathbb{P}(X \leq -2z) - \mathbb{P}(Y \geq z))_+\,dz\,dy}\,dx$$

$$\leq 4c \int_{\kappa/2}^\infty \frac{x\mathbb{P}(X \geq x)}{c/2 + \int_0^{2x} \int_y^\infty \mathbb{P}(X \leq -2z)\,dz\,dy}\,dx$$

$$= 4c \int_{\kappa/2}^\infty \frac{x\mathbb{P}(X \geq x)}{c/2 + \frac{1}{4}\int_0^{4x} \int_y^\infty \mathbb{P}(X \leq -z)\,dz\,dy}\,dx$$

$$\leq c' \int_{\kappa/2}^\infty \frac{x\mathbb{P}(X \geq x)}{1 + \int_0^x \int_y^\infty \mathbb{P}(X \leq -z)\,dz\,dy}\,dx < \infty,$$

where $c' = \max\{8, 16c\}$. $\quad\square$

LEMMA 39. *Let Π_i, $i \in E$, be probability distributions on $\mathbb{R}$ and let $\{X^{i,n} : i \in E, n \in \mathbb{N}\}$ be a family of independent random variables with $X^{i,n} \sim \Pi_i$. Define*

$$Z = \sum_{n=1}^N X^{Y_n,n}$$

with $(Y_n)_{n\in\mathbb{N}}$ being an E-valued process and N an $\mathbb{N}_0$-valued random variable both being jointly independent of $(X^{i,n})$. If furthermore we suppose $\mathbb{E}[N^3] < \infty$ and $\mathbb{P}(i \in \{Y_1, \ldots, Y_N\}) > 0$ for $i \in E$, then the following properties are equivalent:

(i) *The distribution of Z satisfies* (**TO**).
(ii) *For one (any) sequence $(\rho_i)_{i\in E}$ of strictly positive numbers*
$$\Pi^{\text{sum}}(\cdot) := \sum_{i\in E} \rho_i \Pi_i(\cdot)$$
satisfies (**TO**).
(iii) *The measure $\Pi^{\max}$ on $\mathbb{R} \setminus \{0\}$ defined by*
$$\Pi^{\max}([t, \infty)) = \max_{i\in E} \Pi_i([t, \infty)),$$
$$\Pi^{\max}((-\infty, -t]) = \max_{i\in E} \Pi_i((-\infty, -t])$$
for $t > 0$ satisfies (**TO**).

PROOF.   The equivalence of (ii) and (iii) follows immediately from the definition of (**TO**), the estimate
$$\min_{i\in E} \rho_i \Pi^{\max}([x, \infty)) \le \Pi^{\text{sum}}([x, \infty)) \le \sum_{i\in E} \rho_i \Pi^{\max}([x, \infty)), \qquad \text{for } x \ge 0,$$

and its analogues version for the set $(-\infty, -x]$. It remains to show that property (i) is equivalent to properties (ii) and (iii).

We start with proving that (iii) implies (i). Note that, for $x \ge 0$,

(48) $$\mathbb{P}(Z \ge x | N) \le N \Pi^{\max}([x/N, \infty)).$$

Furthermore, for any $i \in E$ there exists $1 \le n_i' \le n_i$ such that
$$\mathbb{P}(Y_{n_i'} = i, N = n_i) > 0.$$
Hence, for all $\kappa_i \in [0, \infty)$, one finds

$$\mathbb{P}(Z \le -z) \ge \mathbb{P}(Y_{n_i'} = i, N = n_i) \mathbb{P}\left(\sum_{n=1}^{n_i} \mathbf{1}_{\{n\ne n_i'\}} X^{Y_n, n} \le \kappa_i | Y_{n_i'} = i, N = n_i\right)$$
$$\times \mathbb{P}(X^{i,1} \le -z - \kappa_i).$$

Now we fix κ_i such that

$$q_i := \mathbb{P}(Y_{n_i'} = i, N = n_i) \mathbb{P}\left(\sum_{n=1}^{n_i} \mathbf{1}_{\{n\ne n_i'\}} X^{Y_n, n} \le \kappa_i | Y_{n_i'} = i, N = n_i\right) > 0.$$

We set $\kappa = \max\{\kappa_i : i \in E\}$ and $q = \min\{q_i : i \in E\}$ and get, for $z \ge 0$,

(49) $$\mathbb{P}(Z \le -z) \ge q \max_{i\in E} \mathbb{P}(X^{i,1} \le -z - \kappa) = q \Pi^{\max}((-\infty, -z - \kappa]).$$

Combining this estimate with (48), we get that

$$\int_{[\kappa,\infty)} \frac{x\mathbb{P}(Z \geq x)}{1 + \int_0^x \int_y^\infty \mathbb{P}(Z \leq -z)\, dz\, dy}\, dx$$

$$\leq \int_{[\kappa,\infty)} \frac{x\mathbb{E}[N\Pi^{\max}([x/N, \infty)]}{1 + q\int_\kappa^x \int_y^\infty \Pi^{\max}((-\infty, -2z])\, dz\, dy}\, dx.$$

Since $\int_0^\kappa \int_y^\infty \mathbb{P}(Z \leq -z)\, dz\, dy$ is finite, we conclude that there exists a constant $c > 0$ such that for $x \geq \kappa$

$$1 + q\int_\kappa^x \int_y^\infty \Pi^{\max}((-\infty, -2z])\, dz\, dy$$

$$\geq c\left(1 + \int_0^x \int_y^\infty \Pi^{\max}((-\infty, -2z])\, dz\, dy\right).$$

Hence, we have

$$\int_{[\kappa,\infty)} \frac{x\mathbb{P}(Z \geq x)}{1 + \int_0^x \int_y^\infty \mathbb{P}(Z \leq -z)\, dz\, dy}\, dx$$

$$\leq c^{-1}\sum_{n=1}^\infty \mathbb{P}(N = n)n \int_{(0,\infty)} \frac{x\Pi^{\max}([x/n, \infty)]}{1 + \int_0^x \int_y^\infty \Pi^{\max}((-\infty, -2z])\, dz\, dy}\, dx$$

$$= c^{-1}\sum_{n=1}^\infty \mathbb{P}(N = n)n^3 \int_{(0,\infty)} \frac{x\Pi^{\max}([x, \infty)]}{1 + 4^{-1}\int_0^{2nx} \int_y^\infty \Pi^{\max}((-\infty, -z])\, dz\, dy}\, dx$$

$$\leq 4c^{-1}\mathbb{E}[N^3] \int_{(0,\infty)} \frac{x\Pi^{\max}([x, \infty)]}{1 + \int_0^x \int_y^\infty \Pi^{\max}((-\infty, -z])\, dz\, dy}\, dx.$$

Next, we consider the converse direction. In analogy to the derivation of (49), one sees that there are constants $\kappa, q > 0$ such that

$$\mathbb{P}(Z \geq z) \geq q\Pi^{\max}([z + \kappa, \infty))z,$$

for all $z \geq 0$. Further, it is also the case that

$$\int_0^x \int_y^\infty \mathbb{P}(Z \leq -z)\, dz\, dy$$

$$\leq \sum_{n=1}^\infty \mathbb{P}(N = n) \int_0^x \int_y^\infty n\Pi^{\max}((-\infty, -z/n])\, dz\, dy$$

$$= \sum_{n=1}^\infty \mathbb{P}(N = n)n^3 \int_0^{x/n} \int_y^\infty \Pi^{\max}((-\infty, -z])\, dz\, dy$$

$$\leq \mathbb{E}[N^3] \int_0^x \int_y^\infty \Pi^{\max}((-\infty, -z])\, dz\, dy,$$

so that we have

$$\int_{[2\kappa,\infty)} \frac{x \, \Pi^{\max}([x,\infty))}{1 + \int_0^x \int_y^\infty \Pi^{\max}((-\infty,-z]) \, dz \, dy} \, dx$$

$$\leq q^{-1} \big(\mathbb{E}[N^3] \vee 1\big) \int_{[2\kappa,\infty)} \frac{x \mathbb{P}(Z \geq x/2)}{1 + \int_0^x \int_y^\infty \mathbb{P}(Z \leq -z) \, dz \, dy} \, dx$$

$$= 4q^{-1} \big(\mathbb{E}[N^3] \vee 1\big) \int_{[\kappa,\infty)} \frac{x \mathbb{P}(Z \geq x)}{1 + \int_0^x \int_y^\infty \mathbb{P}(Z \leq -z) \, dz \, dy} \, dx.$$

The proof is now complete. □

PROOF OF THEOREM 35.   We again consider the process ξ at the discrete set of return times to the state k. By Lemma 36, tightness of the overshoots of the MAP is equivalent to tightness of the overshoots of the discrete time process $(\xi_{\sigma_n})_{n\in\mathbb{N}_0}$ under the law $P^{0,k}$ which is the underlying measure in the following considerations. By the Markov property and the translation invariance of the MAP, the process (ξ_{σ_n}) has i.i.d. increments and starts in 0, and thus is a random walk. By Lemma 37, (ξ_{σ_n}) has tight overshoots if and only if ξ_{σ_1} has finite absolute moment and either:

- (ξ_{σ_n}) drifts to infinity, or
- (ξ_{σ_n}) oscillates and the distribution of ξ_{σ_1} satisfies (**TO**).

By Theorem 34, Formula (40), the latter properties are equivalent to the ones obtained when replacing the discrete time process (ξ_{σ_n}) by the continuous time process (ξ_t) and keeping the (**TO**) property for ξ_{σ_1}.

To complete the proof, it remains to show that in the oscillating case with finite absolute moment one has the equivalence

$$\mathcal{L}(\xi_{\sigma_1}) \text{ satisfies (\textbf{TO})} \quad \Longleftrightarrow \quad \Pi \text{ from (47) satisfies (\textbf{TO})}.$$

In order to do so, let us identify the distribution of ξ_{σ_1}. Enumerate the times at which either ξ has jumps with modulus larger than 1 or J changes its state in increasing order $0 \leq \tau_1 < \tau_2 < \cdots$ and represent ξ_{σ_1} as telescopic sum

$$(50) \qquad \xi_{\sigma_1} = \sum_{j:\tau_j \leq \sigma_1} (\xi_{\tau_j} - \xi_{\tau_j-}) + \sum_{j:\tau_j \leq \sigma_1} (\xi_{\tau_j-} - \xi_{\tau_{j-1}})$$

with $\tau_0 := 0$. Using the representation from Proposition 2, we can identify the conditional distributions of the terms appearing in the former sum when conditioning on J and the set of times $\{\tau_1, \tau_2, \dots\}$: If τ_j is triggered by a large jump of the Lévy process (meaning that the process J does not switch states at that time), the conditional distribution of $\xi_{\tau_j} - \xi_{\tau_j-}$ is the normalised Lévy measure restricted to jumps larger than one of the Lévy process that is switched on by the modulating chain. If τ_j is triggered by a change of J, then the conditional distribution of $\xi_{\tau_j} - \xi_{\tau_j-}$

is $\mathcal{L}(\Delta_{J(\tau_j-),J(\tau_j)})$ with U as in Proposition 2. The random number of j's with $\tau_j \leq \sigma_1$ has finite third moment, and applying Lemma 39 we get that

$$\mathcal{L}\left(\sum_{j:\tau_j \leq \sigma_1} (\xi_{\tau_j} - \xi_{\tau_j-})\right) \text{ satisfies } (\mathbf{TO})$$

$$\iff \sum_{i \neq j} q_{i,j} \mathcal{L}(\Delta_{i,j}) + \sum_{i \in E} \Pi_i|_{B(0,1)^c} \text{ satisfies } (\mathbf{TO}).$$

An elementary calculation furthermore shows that

$$\sum_{i \neq j} q_{i,j} \mathcal{L}(\Delta_{i,j}) + \sum_{i \in E} \Pi_i|_{B(0,1)^c} \text{ satisfies } (\mathbf{TO})$$

$$\iff \Pi \text{ from (47) satisfies } (\mathbf{TO}).$$

Combining the two equivalences with (50) the theorem is proved (compare Lemma 38) if the remainder $\sum_{j:\tau_j \leq \sigma_1} (\xi_{\tau_j-} - \xi_{\tau_{j-1}})$ has finite second moment.

However, the latter term is just the value of a MAP starting in $(0, k)$ evaluated at the time of the first return of J to k with an appropriately modified evolution: the Lévy measures need to be replaced by the old ones restricted to the unit ball and the process has no discontinuity when J switches states. Such a MAP obviously has finite second moment. $\square$

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