CONDITIONS FOR PERMANENTAL PROCESSES TO BE UNBOUNDED[1]

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An α-permanental process $\{X_t, t \in T\}$ is a stochastic process determined by a kernel $K = \{K(s, t), s, t \in T\}$, with the property that for all $t_1, \ldots, t_n \in T$, $|I + K(t_1, \ldots, t_n)S|^{-\alpha}$ is the Laplace transform of $(X_{t_1}, \ldots, X_{t_n})$, where $K(t_1, \ldots, t_n)$ denotes the matrix $\{K(t_i, t_j)\}_{i,j=1}^n$ and S is the diagonal matrix with entries $s_1, \ldots, s_n$. $(X_{t_1}, \ldots, X_{t_n})$ is called a permanental vector.

Under the condition that K is the potential density of a transient Markov process, $(X_{t_1}, \ldots, X_{t_n})$ is represented as a random mixture of n-dimensional random variables with components that are independent gamma random variables. This representation leads to a Sudakov-type inequality for the sup-norm of $(X_{t_1}, \ldots, X_{t_n})$ that is used to obtain sufficient conditions for a large class of permanental processes to be unbounded almost surely. These results are used to obtain conditions for permanental processes associated with certain Lévy processes to be unbounded.

Because K is the potential density of a transient Markov process, for all $t_1, \ldots, t_n \in T$, $A(t_1, \ldots, t_n) := (K(t_1, \ldots, t_n))^{-1}$ are M-matrices. The results in this paper are obtained by working with these M-matrices.

1. Introduction. An R^n valued α-permanental random variable $X = (X_1, \ldots, X_n)$ is a random variable with Laplace transform

$$\text{(1.1)} \qquad E\big(e^{-\sum_{i=1}^n s_i X_i}\big) = \frac{1}{|I + KS|^\alpha}$$

for some $n \times n$ matrix K and diagonal matrix S with entries s_i, $1 \le i \le n$, and $\alpha > 0$. Permanental random variables were introduced by Vere-Jones, [13], who referred to them as random variables with multivariate gamma distributions. (Actually he considered the moment generating function.)

An α-permanental process $\{X_t, t \in T\}$ is a stochastic process which has finite dimensional distributions that are α-permanental vectors. The permanental process is determined by a kernel $\{K(s, t), s, t \in T\}$, with the property that for all $t_1, \ldots, t_n$ in T, $\{K(t_i, t_j), i, j \in [0, n]\}$ determines an α-permanental random variable by (1.1). (Sometimes we refer to these processes simply as permanental processes.) Vere-Jones briefly considers permanental processes in [13]. Note that when (1.1)

holds for a kernel $K(s, t)$ for all $\alpha > 0$, the family of permanental processes obtained are infinitely divisible. The permanental processes considered in this paper have this property.

Local times of Markov processes with symmetric potential densities are related by isomorphism theorems to the squares of Gaussian processes. Note that when K is symmetric and positive definite and $\alpha = 1/2$, $(\eta_1^2/2, \ldots, \eta_n^2/2)$, where $(\eta_1, \ldots, \eta_n)$ is an n-dimensional normal random variable with mean zero and covariance matrix K, is a $1/2$-permanental process. When $\alpha \neq 1/2$ or K is not symmetric, the isomorphism theorems can be generalized, by replacing the squares of the Gaussian processes by other permanental processes, so that they also hold for Markov processes with potential densities that are not symmetric. To apply these isomorphism theorems it is important to know sample path properties of permanental processes.

In this paper, we give a concrete representation of permanental vectors that is used to obtain a Sudakov type inequality that gives lower bounds for permanental processes that only requires that the inverses of the matrices $\{K(t_i, t_j), i, j \in [0, n]\}$ are M-matrices. It does not require that the matrices are symmetric. This work supplements [10] in which we give sufficient conditions for the continuity of permanental processes.

Since the definition of permanental processes requires that their finite dimensional distributions are permanental random variables, a fundamental question is: For which matrices K do there exist random variables X satisfying (1.1)? Vere-Jones answers this question but with criteria that are, in general, very difficult to verify. On the other hand, as we just pointed out, when K is symmetric and positive definite and $\alpha = 1/2$ then $X = (\eta_1^2/2, \ldots, \eta_n^2/2)$, where $(\eta_1, \ldots, \eta_n)$ is an n-dimensional normal random variable with mean zero and covariance matrix K.

There are other cases in which it is easy to see that the right-hand side of (1.1) is the Laplace transform of an R^n valued random variable. Recall that a gamma random variable is one with probability density function

$$(1.2) \qquad f(u, v; x) = \frac{v^u x^{u-1} e^{-vx}}{\Gamma(u)} \qquad \text{for } x \geq 0 \text{ and } u, v > 0,$$

and equal to 0 for $x \leq 0$, where $\Gamma(u) = \int_0^\infty x^{u-1} e^{-x} \, dx$ is the gamma function. The parameter u is called the shape of the gamma distribution and the parameter v is called the scale of the gamma distribution.

In this paper, we describe a large class of infinitely divisible permanental random variables. We use $\xi_{u,v}$ to denote a random variable with probability density function $f(u, v; x)$. The Laplace transform of $\xi_{u,v}$ is

$$(1.3) \qquad \int_0^\infty \frac{v^u x^{u-1} e^{-(v+s)x}}{\Gamma(u)} \, dx = \frac{1}{(1 + \frac{s}{v})^u} = \frac{v^u}{(v+s)^u}.$$

Therefore, if K is a diagonal matrix with entries $1/v_i$, (1.1) is the Laplace transform of $(\xi_{\alpha,v_1}, \ldots, \xi_{\alpha,v_n})$, in which all the components are independent. Consequently, when the right-hand side of (1.1) is the Laplace transform of an R^n valued random variable X, it is reasonable to say that X has an n-dimensional gamma distribution.

We assume that $|K| > 0$. Therefore, $A = K^{-1}$ exists and we can also define X by

$$(1.4) \qquad E\big(e^{-\sum_{i=1}^{n} s_i X_i}\big) = \frac{|A|^{\alpha}}{|A + S|^{\alpha}}.$$

It turns out that it is simpler to describe the random variables X that are defined by matrices K as in (1.1), by focusing on A, and describing the random variables X that are defined by matrices A as in (1.4).

The results in this paper all depend on a concrete representation of permanental random variables which we can obtain when the matrix A in (1.4) is a nonsingular M-matrix.

Let $C = \{c_{i,j}\}_{1 \le i,j \le n}$ be an $n \times n$ matrix. We call C a positive matrix and write $C \ge 0$ if $c_{i,j} \ge 0$ for all i, j.

The matrix A is said to be a nonsingular M-matrix if:

(1) $a_{i,j} \le 0$ for all $i \ne j$.
(2) A is nonsingular and $A^{-1} \ge 0$.

Theorem 2.1 gives a representation of α permanental vectors. It is rather technical and requires some preparation so we hold off presenting it until Section 2. The following consequence of Theorem 2.1 is our key to obtaining conditions for the paths of permanental processes to be unbounded.

THEOREM 1.1. *Let $X = (X_1, \ldots, X_n)$ be an α-permanental vector with nonsingular kernel K. Assume that $A = K^{-1}$ is an M-matrix with diagonal entries $(a_1, \ldots, a_n)$. Then there exists a coupling between X and an n-tuple $(\xi_{\alpha,1}^{(1)}, \ldots, \xi_{\alpha,1}^{(n)})$ of independent identically distributed copies of $\xi_{\alpha,1}$ such that*

$$(1.5) \qquad X \ge \big(a_1^{-1} \xi_{\alpha,1}^{(1)}, \ldots, a_n^{-1} \xi_{\alpha,1}^{(n)}\big) \qquad a.s.$$

This immediately implies the next theorem.

THEOREM 1.2. *Let X be as in Theorem 1.1. Then if f is an increasing function on R_+^n*

$$(1.6) \qquad E\big(f(X)\big) \ge E\big(f\big(a_1^{-1} \xi_{\alpha,1}^{(1)}, \ldots, a_n^{-1} \xi_{\alpha,1}^{(n)}\big)\big).$$

Equivalently,

$$(1.7) \qquad Ef\big((a_1 X_1, \ldots, a_n X_n)\big) \ge E\big(f(\xi_{\alpha,1}^{(1)}, \ldots, \xi_{\alpha,1}^{(n)})\big).$$

We call (1.6) the permanental inequality. We explain in Section 4 that it is a generalization, in a certain sense, of the Sudakov inequality.

It is shown in [5] that when $\{u(s,t), s, t \in T\}$ is the potential density of a transient Markov process with state space T, then for any $\alpha > 0$, there exists an α-permanental process with kernel $\{K(s,t), s, t \in T\} = \{u(s,t), s, t \in T\}$. In this case, we refer to the permanental process as an associated α-permanental process. (It is associated with the transient Markov process.) We use this terminology in what follows.

We can use Theorem 1.2 to give conditions for a permanental process to be unbounded in terms of the diagonals of the M-matrices of its finite dimensional distributions. Let $X = \{X_t, t \in T\}$, T a countable set, be an α-permanental process with kernel $\{u(s,t), s, t \in T\}$. Since, in Theorem 1.2, we require that A is an M-matrix, the α-permanental processes that we can consider must have a kernel with the property that for all $(t_1, \ldots, t_n)$ in T, the matrix with elements $\{u(t_i, t_j)\}_{i,j=1}^n$ is invertible and its inverse $A(t_1, \ldots, t_n)$ is a nonsingular M-matrix. This is the case if (and only if) X is an associated α-permanental process. (This result is part of [9], Theorem 13.1.2. This theorem it is stated for symmetric kernels but symmetry is not used in the proof. For the convenience of the reader, in Section A.1, we repeat the proof of the portion of [9], Theorem 13.1.2, that we use in this paper.)

Suppose that X is an associated α-permanental process. Let $a_i(t_1, \ldots, t_n)$, $i = 1, \ldots, n$, denote the diagonal elements of $A(t_1, \ldots, t_n)$. We use Theorem 1.2 in the following lemma which is proved in Section 3. It is a useful generalization of Theorem 1.2 that enables us to only consider a fraction of the diagonal elements of A.

LEMMA 1.1.   *Let $a_i^*(t_1, \ldots, t_n)$ denote a nondecreasing rearrangement of $a_i(t_1, \ldots, t_n)$. For any integer $p \geq 1$, let*

(1.8)
$$\psi_{[n/p]}^* = \inf_{(t_1, \ldots, t_n) \in T^n} a_{[n/p]}^*(t_1, \ldots, t_n).$$

Then

(1.9)
$$P\left(\sup_{t \in T} X_t \geq \lambda/\psi_{[n/p]}^*\right) \geq P\left(\max_{1 \leq i \leq [n/p]} \xi_{\alpha,1}^{(i)} \geq \lambda\right).$$

Therefore, if

(1.10)
$$\limsup_{n \to \infty} P\left(\max_{1 \leq i \leq [n/p]} \xi_{\alpha,1}^{(i)} \geq \lambda_n\right) = 1$$

we have

(1.11)
$$P\left(\sup_{t \in T} X_t \geq \lambda_n/\psi_{[n/p]}^*, i.o.\right) = 1.$$

In Section 3, we show that (1.10) holds with $\lambda_n = \log n$. Therefore, we can use (1.11) to obtain the following sufficient condition for permanental processes to be unbounded.

THEOREM 1.3. *Let $\{X_t, t \in T\}$ be an associated α-permanental process. If*

$$\limsup_{n \to \infty} \frac{\log n}{\psi^*_{[n/p]}} = \infty, \tag{1.12}$$

then $\sup_{t \in T} X_t = \infty$ almost surely.

The next corollary is an immediate consequence of Theorem 1.3.

COROLLARY 1.1. *Let $d := d_{s,t}$ be a function on $T \times T$. Set*

$$d_n^*(t_1, \ldots, t_n) = \inf_{1 \le i, j \le n, i \ne j} d_{t_i, t_j}. \tag{1.13}$$

Suppose that a fraction of the entries

$$a_i(t_1, \ldots, t_n) \le \frac{C'}{(d_n^*)^2(t_1 \ldots, t_n)} \tag{1.14}$$

for some constant C'. Then

$$\limsup_{n \to \infty} \left(\sup_{(t_1, \ldots, t_n)} (d_n^*)^2(t_1, \ldots, t_n) \right) \log n = \infty \tag{1.15}$$

implies that $\sup_{t \in T} X_t = \infty$ almost surely.

The condition in (1.14) is not very useful because, in general one does not know the inverse of the matrices $\{u(t_i, t_j)\}_{i,j=1}^n$. In Lemma 5.2, we give conditions on the kernel $u(x, y)$ so that (1.14) holds with the function

$$\sigma_{s,t} = \left(u(s, s) + u(t, t) - \left(u(s, t) + u(t, s) \right) \right)^{1/2} \tag{1.16}$$

replacing $d_{s,t}$. This enables us to obtain the following theorem.

THEOREM 1.4. *Let u be the potential density of a transient Markov process in R^1 and assume that $u(s, s)$ is constant for all $|s| \le \varepsilon$, for some $\varepsilon > 0$. Set*

$$\sigma_{s,t}^2 = 2u(0, 0) - u(s, t) - u(t, s) \tag{1.17}$$

and assume that

$$\left| u(s, t) - u(t, s) \right| \le C\sigma_{s,t}^2, \qquad C < 1, \tag{1.18}$$

for all $|s|, |t| \le \varepsilon$. Then

$$\limsup_{\substack{n \to \infty}} \left(\sup_{\substack{(t_1, \ldots, t_n) \\ \forall |t_i| \le \varepsilon}} (\sigma_n^*)^2(t_1, \ldots, t_n) \right) \log n = \infty \tag{1.19}$$

implies that the α-permanental process with kernel u is unbounded almost surely.

It follows from Lemma 5.4 and that fact that $u(s, s)$ is constant for all $|s| \leq \varepsilon$, for some $\varepsilon > 0$, that (1.18) always holds for $C = 1$.

In Theorem 5.1, we remove the hypothesis that $u(s, s)$ is constant for all $|s|$ sufficiently small. We do not consider this here because the result is not as easy to state as Theorem 1.4.

If $u(s, t)$ is symmetric and positive definite it is the covariance of a Gaussian process. Let $\{\widetilde{X}_t, t \in R^1\}$ be a mean zero Gaussian process with covariance $u(s, t)$. In this case,

$$(1.20) \qquad \sigma_{s,t}^2 = E(\widetilde{X}_t - \widetilde{X}_s)^2 = u(s, s) + u(t, t) - 2u(s, t).$$

(In particular, this shows that $\sigma_{s,t}$ is a metric on R^1.)

Since $\{\widetilde{X}_t, t \in R^1\}$ is a mean zero Gaussian process, we can use Slepian's lemma to show that (1.19) implies that $\sup_{t \in R^1} \widetilde{X}_t = \infty$ almost surely. This also follows from Theorem 1.4, when $E\widetilde{X}_t^2$ is constant, since in this case the left-hand side of (1.18) is equal to 0. (What Theorem 1.4 shows is that the $1/2$-permanental process $\sup_{t \in R^1} \widetilde{X}_t^2 = \infty$ almost surely. Of course, we also require that the inverse of $\{u(x_j, x_j)\}_{i,j=1}^n$ is an M-matrix for all $x_{i_1}, \ldots, x_{i_n} \in R^1$.)

Even when $u(s, t)$ is not symmetric, $u(s, t) + u(t, s)$ is symmetric, and if it is also positive definite it is the covariance of a Gaussian process. In this case, we can still associate a permanental process with a Gaussian process. We plan to take this up in a subsequent paper.

We can use Theorem 1.4 to study the boundedness of permanental processes with kernels that are the potential densities of transient Lévy processes in R^1. Let $Y = \{Y_t, t \in R_+\}$ be a Lévy process and consider the transient Lévy process $\overline{Y} = \{\overline{Y}_t, t \in R_+\}$ that is Y killed at $\xi_{1,1/\beta}$, an independent exponential time with mean $\beta > 0$. If $u^\beta(x, y)$ is the β-potential density of Y it is the zero potential of $\overline{Y}$, and thus is also the kernel of a permanental process. In this example, $u^\beta(x, y) = u^\beta(0, y - x) =: u^\beta(y - x)$.

As we have mentioned above, since $u^\beta(x, y)$ is the 0-potential density of a transient Lévy process, for every finite collection $x_1, \ldots, x_n \in R^1$, the $n \times n$ matrix $U = \{u(x_i, x_j)\}_{1 \leq i, j \leq n}$ is invertible and its inverse is a nonsingular M-matrix. We use Theorem 1.4 to find conditions under which the α-permanental process with kernel u^β is unbounded.

We write the characteristic function of Y as

$$(1.21) \qquad\qquad\qquad E e^{i\lambda Y_t} = e^{-t\psi(\lambda)}.$$

When $u^\beta(y - x)$ is not symmetric, $\psi(\lambda)$ is complex. Set

$$(1.22) \qquad \mathcal{R}_\beta(\lambda) = \mathcal{R}e\big(1/(\beta + \psi(\lambda))\big) \quad \text{and} \quad \mathcal{I}_\beta(\lambda) = \mathcal{I}m\big(1/(\beta + \psi(\lambda))\big).$$

LEMMA 1.2 ([7], Lemma 5.2).   *For $\beta > 0$, assume that $\mathcal{R}_\beta(\lambda) \in L^1(R_+)$. Then the the β-potential density of X is*

$$(1.23) \qquad u^\beta(z) = R_\beta(z) + H_\beta(z) \quad and \quad u^\beta(-z) = R_\beta(z) - H_\beta(z),$$

where

$$(1.24) \qquad R_\beta(z) = \frac{1}{\pi} \int_0^\infty \cos(\lambda z) \mathcal{R}_\beta(\lambda) \, d\lambda$$

and

$$(1.25) \qquad H_\beta(z) = \frac{1}{\pi} \int_0^\infty \sin(\lambda z) \mathcal{I}_\beta(\lambda) \, d\lambda.$$

As a special case of (1.17), we consider the metric

$$
\begin{aligned}
\sigma_\beta(z) &= \left(2u^\beta(0) - u^\beta(z) - u^\beta(-z) \right)^{1/2} \\
&= \left(\frac{2}{\pi} \int_0^\infty \left(1 - \cos(\lambda z) \right) \mathcal{R}_\beta(\lambda) \, d\lambda \right)^{1/2}.
\end{aligned}
$$
(1.26)

(Note that because $\mathcal{R}_\beta(\lambda)$ is positive and in $L^1(R_+)$, $\Gamma(x, y) = R_\beta(y - x)$ is the covariance function of a stationary Gaussian process, say $\{G(z), z \in R^1\}$. Therefore, $\sigma^2(z) = E(G(z) - G(0))^2$.)

The following condition for the α-permanental process with kernel u^β to be unbounded is an immediate application of Theorem 1.4.

THEOREM 1.5. *Suppose that $\mathcal{R}_\beta(\lambda) \in L^1(R_+)$ and*

$$(1.27) \qquad |H_\beta(z)| \le C\sigma_\beta^2(z) \qquad \text{for some } C < 1/2$$

and all $|z|$ sufficiently small. Suppose, in addition, that $\sigma_\beta^2(z) \ge f(|z|)$ for some increasing function f for all $|z|$ sufficiently small. Then

$$(1.28) \qquad \limsup_{n \to \infty} f(1/n) \log n = \infty$$

implies that the α-permanental process with kernel u^β is unbounded almost surely.

THEOREM 1.6. *Let $X = \{X(t), t \in R_+\}$ be the α-permanental process with a kernel that is the the β potential density of a Lévy process with Lévy measure*

$$(1.29) \quad v(dx) = \left(x^{-2} g(1/|x|)(pI_{x>0} + qI_{x<0}) \right) dx, \qquad p, q > 0, \, p + q = 1,$$

in which g is a positive, quasi-monotonic slowly varying function at infinity. Suppose $p \ne q$ and

$$(1.30) \qquad \lim_{n \to \infty} \int_1^n \frac{g(s)}{s} \, ds = \infty.$$

Then X is unbounded almost surely if

$$(1.31) \qquad \int_1^n \frac{g(s)}{s} \, ds = o(\log n)$$

as $n \to \infty$.

If $p = q$ and

$$\lim_{n \to \infty} \int_n^\infty \frac{1}{sg(s)} \, ds = 0, \tag{1.32}$$

then X is unbounded almost surely if

$$\left(\int_n^\infty \frac{1}{sg(s)} \, ds \right)^{-1} = o(\log n). \tag{1.33}$$

It is interesting to note that the β potential density determined by (1.29) has the property that for $z > 0$

$$u^\beta(z) \sim u^\beta(0) - \frac{\sigma^2(z)}{2}(1 - |p - q|),$$

(1.34)

$$u^\beta(-z) \sim u^\beta(0) - \frac{\sigma^2(z)}{2}(1 + |p - q|)$$

as $z \to 0$. Here, we write $f \sim g$ as $z \to 0$ if $\lim_{z \to 0} f(z)/g(z) = 1$, with a similar meaning for $f \sim g$ as $z \to \infty$. The derivation of (1.34) is given in Section 6 following the proof of Theorem 1.6.

EXAMPLE 1.1. We consider Barlow's example ([1], page 1393), slightly modified, of a Lévy process with Lévy measure given by (1.29) with $g(y)$ replaced by $g_{\gamma\delta}(y)$ where

$$g_{\gamma\delta}(y) = (\log y)^\gamma (\log \log y)^\delta 1_{\{y > \varepsilon\}} \tag{1.35}$$

with $\gamma > -1$. Let $Y_{\gamma\delta}$ be the Lévy process determined by this Lévy measure and denote its β potential density by u^β. It follows from (1.31) that when $p \neq q$ the permanental process with kernel u^β is unbounded if $\gamma < 0$ or $\gamma = 0$ and $\delta < 0$.

When $p = q$,

$$\left(\int_n^\infty \frac{1}{s(\log s)^\gamma (\log \log s)^\delta} \, ds \right)^{-1} \sim C(\log n)^{\gamma - 1}(\log \log n)^\delta, \tag{1.36}$$

and we now require that $\gamma > 1$. In this case the permanental process with kernel u^β is unbounded if $\gamma < 2$ or $\gamma = 2$ and $\delta < 0$.

Let $u^\beta(s, t) = u^\beta(t - s)$ be the β-potential of a Lévy process. Using Barlow's [1] necessary and sufficient condition for the boundedness of local times of Lévy processes and an isomorphism theorem of Eisenbaum and Kaspi [4] that relates local times and permanental processes, we can show that the associated α-permanental process is unbounded almost surely if the Gaussian process with covariance $\gamma(s, t) = u^\beta(s - t) + u^\beta(t - s))$ is unbounded almost surely. (See the

comment following Lemma 1.2.) For the processes considered in Example 1.1, this occurs if and only if

$$(1.37) \qquad \int_1^\infty \frac{(\int_\lambda^\infty \mathcal{R}_\beta(u)\, du)^{1/2}}{\lambda (\log \lambda)^{1/2}}\, d\lambda = \infty.$$

Consequently, when $p \neq q$ the permanental process with kernel u^β in Example 1.1 is unbounded almost surely if $\gamma < 0$ or $\gamma = 0$ and $\delta \leq 2$ and bounded almost surely when $\gamma = 0$ and $\delta > 2$. When $p = q$, it is unbounded almost surely if $\gamma < 2$ or $\gamma = 2$ and $\delta \leq 2$ and bounded almost surely when $\gamma = 2$ and $\delta > 2$. This gives a little more than we obtain in Example 1.1.

Even though the results in Theorems 1.5 and 1.6 are not best possible, the theorems are interesting for at least two reasons. The first is that their proofs are much simpler than the proof in [1]. The second is that the proofs involving [1] and [4] are indirect and give no insight into why permanental processes have sample path properties similar to the squares of Gaussian processes. Our proofs of Theorems 1.5 and 1.6 are classical and relatively simple and show that permanental processes have sample path properties similar to the squares of Gaussian processes because the permanental inequality is a generalization, in many respects, of the Sudakov inequality.

With some restrictions and a simplification, and slight weakening, of (1.27) we get a corollary of Theorem 1.5 that is easier to use and imposes weaker conditions on the behavior of $|\mathcal{I}_\beta(\lambda)|$ and $\mathcal{R}_\beta(\lambda)$ as $\lambda \to \infty$.

COROLLARY 1.2. *Suppose that $\mathcal{R}_\beta(\lambda) \in L^1(R_+)$ and that $|\mathcal{I}_\beta(\lambda)|$ and $\mathcal{R}_\beta(\lambda)$ are asymptotic to nonincreasing functions as $\lambda \to \infty$ and*

$$(1.38) \qquad |z| \int_0^{\pi/|z|} \lambda |\mathcal{I}_\beta(\lambda)|\, d\lambda \leq \frac{C}{2} \int_{\pi/(2|z|)}^\infty \mathcal{R}_\beta(\lambda)\, d\lambda$$

for some $C < 1$, for all $|z|$ sufficiently small. Then

$$(1.39) \qquad \limsup_{n \to \infty} \left(\int_n^\infty \mathcal{R}_\beta(\lambda)\, d\lambda \right) \log n = \infty$$

implies that the α-permanental process with kernel u^β is unbounded almost surely.

The proof of Theorem 1.1 is given in Section 2 and that of Theorem 1.3 in Section 3. In Section 4, we examine the implications of (1.6), the permanental inequality and explain why we refer to it as a Sudakov-type inequality. Theorem 1.4 is proved in Section 5. In Section 6, we prove Theorem 1.5, Corollary 1.2 and fill in the details for Example 1.1.

2. Representation of permanental processes. For any $n \times n$ matrix M, we define the α-permanent

$$(2.1) \qquad |M|_\alpha = \begin{vmatrix} m_{1,1} & \cdots & m_{1,n} \\ \cdots & & \cdots \\ m_{n,1} & \cdots & m_{n,n} \end{vmatrix}_\alpha = \sum_\pi \alpha^{c(\pi)} m_{1,\pi(1)} m_{2,\pi(1)} \cdots m_{n,\pi(n)}.$$

Here, the sum runs over all permutations π on $[1, n]$ and $c(\pi)$ is the number of cycles in π. We make the trivial observation that if all entries of M are nonnegative, then $|M|_\alpha \geq 0$.

We use boldface, such as $\mathbf{x}$, to denote vectors. Let $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $|\mathbf{k}| = \sum_{l=1}^n k_l$. For $1 \leq p \leq |\mathbf{k}|$, set $i_p = j$, where

$$(2.2) \qquad \sum_{l=1}^{j-1} k_l < p \leq \sum_{l=1}^j k_l.$$

For any $n \times n$ matrix $C = \{c_{i,j}\}_{1 \leq i,j \leq n}$ we define

$$(2.3) \qquad C(\mathbf{k}) = \begin{bmatrix} c_{i_1,i_1} & c_{i_1,i_2} & \cdots & c_{i_1,i_{|\mathbf{k}|}} \\ c_{i_2,i_1} & c_{i_2,i_2} & \cdots & c_{i_2,i_{|\mathbf{k}|}} \\ \cdots & & \cdots & \\ c_{i_{|\mathbf{k}|},i_1} & c_{i_{|\mathbf{k}|},i_2} & \cdots & c_{i_{|\mathbf{k}|},i_{|\mathbf{k}|}} \end{bmatrix},$$

and $C(\mathbf{0}) = 1$. For example, if $n = 3$ and $\mathbf{k} = (0, 2, 3)$, then $|\mathbf{k}| = 5$ and $i_1 = i_2 = 2$ and $i_3 = i_4 = i_5 = 3$:

$$(2.4) \qquad C(0, 2, 3) = \begin{bmatrix} c_{2,2} & c_{2,2} & c_{2,3} & c_{2,3} & c_{2,3} \\ c_{2,2} & c_{2,2} & c_{2,3} & c_{2,3} & c_{2,3} \\ c_{3,2} & c_{3,2} & c_{3,3} & c_{3,3} & c_{3,3} \\ c_{3,2} & c_{3,2} & c_{3,3} & c_{3,3} & c_{3,3} \\ c_{3,2} & c_{3,2} & c_{3,3} & c_{3,3} & c_{3,3} \end{bmatrix}.$$

LEMMA 2.1. *Let A be an $n \times n$ nonsingular M-matrix with diagonal entries $a_1, \ldots, a_n$ and S be an $n \times n$ diagonal matrix with entries $(s_1, \ldots, s_n)$ and set*

$$(2.5) \qquad A = D - B,$$

where $D = \mathrm{diag}(a_1, \ldots, a_n)$ and all the elements of B are nonnegative (so that all the diagonal elements of B are equal to zero). Then

$$
\frac{|A|^\alpha}{|A + S|^\alpha} = |A|^\alpha \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|B(\mathbf{k})|_\alpha}{k_1! \cdots k_n!} \frac{1}{(a_1 + s_1)^{\alpha+k_1} \cdots (a_n + s_n)^{\alpha+k_n}}
$$
$$(2.6)$$
$$
= \frac{|A|^\alpha}{\prod_{i=1}^n a_i^\alpha} \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|B(\mathbf{k})|_\alpha}{\prod_{i=1}^n a_i^{k_i} k_i!} \prod_{i=1}^n \left(\frac{a_i}{a_i + s_i} \right)^{\alpha+k_i},
$$

where the sum is over all $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$. (The series converges for all $s_1, \ldots, s_n \in R_+^n$ for all $\alpha > 0$.)

PROOF. For B as given in (2.5), consider

$$(2.7) \qquad \mathcal{H}(z_1, \ldots, z_n) = |I - ZB|^{-\alpha} = \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \left(\prod_{i=1}^{n} \frac{z_i^{k_i}}{k_i!} \right) |B(\mathbf{k})|_\alpha,$$

where Z is a diagonal matrix with entries $z_1, \ldots, z_n$ and the second equality is given in [13], (6). By [12], Theorem, page 120, the series (2.7) converges for $(z_1, \ldots, z_n)$, when the modulus of the maximum eigenvalue of ZB is less than 1.

We write

$$(2.8) \qquad |A + S| = |(D + S) - B| = |(D + S)||I - (D + S)^{-1}B|,$$

so that

$$(2.9) \qquad |A + S|^{-\alpha} = \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|B(\mathbf{k})|_\alpha}{k_1! \cdots k_n!} \frac{1}{(a_1 + s_1)^{\alpha+k_1} \cdots (a_n + s_n)^{\alpha+k_n}}.$$

By the statements in the first paragraph of this proof, this series converges when the modulus of the maximum eigenvalue of $(D + S)^{-1}B$ is less than 1.

We complete the proof by referring to several results in the valuable book [2]. Note that the definition of M-matrix in [2], page 133, is different from the one that we give. However, it follows by [2], N_{38}, page 137, that they are equivalent. We now write $A + S = D + S - B$ to see by [2], Chapter 7, Theorem 5.2, that the maximum eigenvalue of $(D + S)^{-1}B$ is less than 1 if and only if $(A + S)^{-1}B \geq 0$. Since A is a nonsingular M-matrix, we have $B \geq 0$ and by [2], Chapter 6, Theorem 2.4, $(A + S)^{-1} \geq 0$ as well. This completes the proof of this lemma. $\square$

In the next theorem, we give an explicit description of random variables with Laplace transforms given in (2.6).

THEOREM 2.1. *Let A be an $n \times n$ nonsingular M-matrix as in Lemma* 2.1. *Set $Z = (Z_1, \ldots, Z_n)$ with*

$$(2.10) \qquad P\big(Z = (k_1, \ldots, k_n)\big) = \frac{|A|^\alpha}{\prod_{i=1}^{n} a_i^\alpha} \frac{|B(\mathbf{k})|_\alpha}{\prod_{i=1}^{n} a_i^{k_i} k_i!},$$

and $X = (X_1, \ldots, X_n)$ with

$$(2.11) \qquad \begin{aligned} X &= \big(\xi^{(Z,1)}_{\alpha+Z_1,a_1}, \ldots, \xi^{(Z,n)}_{\alpha+Z_n,a_n}\big) \\ &= \sum_{\mathbf{k}=(k_1,\ldots,k_n)} 1_{k_1,\ldots,k_n}(Z)\big(\xi^{(\mathbf{k},1)}_{\alpha+k_1,a_1}, \ldots, \xi^{(\mathbf{k},n)}_{\alpha+k_n,a_n}\big), \end{aligned}$$

where Z and all the gamma distributed random variables, $\xi^{(\mathbf{k},i)}_{\cdot,\cdot}$, $\mathbf{k} \in \mathbb{N}^n$, $i \in 1, \ldots, n$ are independent and $\{a_i\}_{i=1}^{n}$ are the diagonal elements of A. Then

$$(2.12) \qquad E\big(e^{-\sum_{i=1}^{k} s_i X_i}\big) = \frac{|A|^\alpha}{|A + S|^\alpha}.$$

PROOF. Taking $S = 0$ in (2.6), we see that

$$(2.13) \qquad \frac{|A|^\alpha}{\prod_{i=1}^n a_i^\alpha} \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|B(\mathbf{k})|_\alpha}{\prod_{i=1}^n a_i^{k_i} k_i!} = 1.$$

Therefore, we can define an $\mathbb{N}^n$ valued random variable $Z = (Z_1, \ldots, Z_n)$ with

$$(2.14) \qquad P\big(Z = (k_1, \ldots, k_n)\big) = \frac{|A|^\alpha}{\prod_{i=1}^n a_i^\alpha} \frac{|B(\mathbf{k})|_\alpha}{\prod_{i=1}^n a_i^{k_i} k_i!}.$$

We write (2.6) in the form

$$(2.15) \qquad \frac{|A|^\alpha}{|A + S|^\alpha} = \sum_{\mathbf{k}=(k_1,\ldots,k_n)} P(Z = (k_1, \ldots, k_j)) \prod_{i=1}^n \left(\frac{a_i}{a_i + s_i}\right)^{\alpha+k_i}.$$

This is the Laplace transform of the R_+^n valued random variable

$$(2.16) \qquad \begin{aligned} X &= \big(\xi_{\alpha+Z_1,a_1}^{(\mathbf{Z},1)}, \ldots, \xi_{\alpha+Z_n,a_n}^{(\mathbf{Z},n)}\big) \\ &= \sum_{\mathbf{k}=(k_1,\ldots,k_n)} I_{k_1,\ldots,k_n}(Z)\big(\xi_{\alpha+k_1,a_1}^{(\mathbf{k},1)}, \ldots, \xi_{\alpha+k_n,a_n}^{(\mathbf{k},n)}\big), \end{aligned}$$

where all the random variables are independent. $\quad\square$

PROOF OF THEOREM 1.1. Theorem 1.1 follows from (2.11) and the facts that

$$(2.17) \qquad \xi_{\alpha+k_i,a_i} \overset{law}{=} \xi_{\alpha,a_i} + \xi_{k_i,a_i}$$

and

$$(2.18) \qquad \xi_{\alpha,a_i} \overset{law}{=} a_i^{-1} \xi_{\alpha,1},$$

which allow us to write

$$(2.19) \qquad \begin{aligned} X &= \big(\xi_{\alpha+Z_1,a_1}^{(\mathbf{Z},1)}, \ldots, \xi_{\alpha+Z_n,a_n}^{(\mathbf{Z},n)}\big) \\ &\overset{law}{=} \big(a_1^{-1}\xi_{\alpha,1}^{(1)}, \ldots, a_n^{-1}\xi_{\alpha,1}^{(n)}\big) + \big(\xi_{Z_1,a_1}^{(\mathbf{Z},1)}, \ldots, \xi_{Z_n,a_n}^{(\mathbf{Z},n)}\big), \end{aligned}$$

where $\xi_{\alpha,1}^{(i)}$ are i.i.d. copies of $\xi_{\alpha,1}$ and we set $(\xi_{0,a_1}, \ldots, \xi_{0,a_n}) = 0$. $\quad\square$

We get the following immediate corollary of Theorem 2.1.

COROLLARY 2.1. *Let A be an $n \times n$ nonsingular M-matrix. Then for each $\alpha > 0$, (1.4) defines an n-dimensional infinitely divisible random variable.*

Actually Eisenbaum and Kaspi [5], Lemma 4.2, show that the condition in Corollary 2.1 is both necessary and sufficient. They do this by extending the proof of this result by Bapat, Griffiths and Milne in the case when K is symmetric (see

[5] for references) to the case when K is not symmetric. The proof of sufficiency in Corollary 2.1 is completely different from their proof.

It follows from (2.10) and (2.11) that, for measurable functions f on R_+^n,

$$E(f(X))$$

(2.20)
$$= \frac{|A|^\alpha}{\prod_{i=1}^n a_i^\alpha} \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|B(\mathbf{k})|_\alpha}{\prod_{i=1}^n a_i^{k_i} k_i!} E(f(\xi_{\alpha+k_1,a_1}^{(\mathbf{k},1)},\ldots,\xi_{\alpha+k_n,a_n}^{(\mathbf{k},n)})).$$

Obviously, (2.20) gives us more than (1.6). Even though it is difficult to compute $B(\mathbf{k})$ for all $\mathbf{k}$ it is not difficult to obtain it for some $\mathbf{k}$ and to improve (1.6).

All the results in this paper follow from the representation in Lemma 2.1. A different form of this representation, under different hypotheses, is given in [8]. It seems to be more useful than Lemma 2.1 in obtaining explicit probability density functions of low dimensional multivariate gamma distributions. Lemma 2.1 is more useful in describing multivariate gamma distributions in high dimensions. (Multivariate gamma random variables and α-permanental random variables are synonyms.)

3. Proof of Theorem 1.3. The next three lemmas are used in the proof of Theorem 1.3.

LEMMA 3.1. *For* $\lambda > 2(u-1) \vee 0$ *and all* $u, v > 0$,

(3.1)
$$P(\xi_{u,v} \geq \lambda/v) \leq \frac{2\lambda^{u-1}e^{-\lambda}}{\Gamma(u)}$$

and for $\lambda \geq 2$ *and all* $u, v > 0$

(3.2)
$$\frac{2\lambda^{u-1}e^{-\lambda}}{3\Gamma(u)} \leq P(\xi_{u,v} \geq \lambda/v).$$

PROOF. Using the fact that $P(\xi_{u,v} \geq \lambda/v) = P(\xi_{u,1} \geq \lambda)$, it suffices to get the bounds in (3.1) for $P(\xi_{u,1} \geq \lambda)$. By an integration by parts,

(3.3)
$$\int_\lambda^\infty x^{u-1}e^{-x}\,dx = \lambda^{u-1}e^{-\lambda} + (u-1)\int_\lambda^\infty x^{u-2}e^{-x}\,dx.$$

The upper bound in (3.1) follows immediately if $u \leq 1$. If $u > 1$ and $\lambda > 2(u-1)$,

(3.4)
$$(u-1)\int_\lambda^\infty x^{u-2}e^{-x}\,dx \leq \frac{\lambda}{2}\int_\lambda^\infty x^{u-2}e^{-x}\,dx$$
$$\leq \frac{1}{2}\int_\lambda^\infty x^{u-1}e^{-x}\,dx.$$

Using this in (3.3), we see that

(3.5)
$$\int_\lambda^\infty x^{u-1}e^{-x}\,dx \leq 2\lambda^{u-1}e^{-\lambda}.$$

This gives the upper bound in (3.1).

To obtain the lower bound, we first note that for $u \geq 1$ it follows from (3.3) that for any $\lambda > 0$ we have

$$(3.6) \qquad \int_\lambda^\infty x^{u-1} e^{-x} \, dx \geq \lambda^{u-1} e^{-\lambda}.$$

Similarly, for $u < 1$, we use (3.3) to see that for any $\lambda > 0$

$$(3.7) \qquad \int_\lambda^\infty x^{u-1} e^{-x} \, dx = \lambda^{u-1} e^{-\lambda} - (1-u) \int_\lambda^\infty x^{u-2} e^{-x} \, dx,$$

and since, for $\lambda > 2(1-u)$

$$(3.8) \qquad \begin{aligned} (1-u) \int_\lambda^\infty x^{u-2} e^{-x} \, dx &\leq \frac{\lambda}{2} \int_\lambda^\infty x^{u-2} e^{-x} \, dx \\ &\leq \frac{1}{2} \int_\lambda^\infty x^{u-1} e^{-x} \, dx, \end{aligned}$$

we get

$$(3.9) \qquad \int_\lambda^\infty x^{u-1} e^{-x} \, dx \geq \frac{2}{3} \lambda^{u-1} e^{-\lambda}.$$

Combining (3.6) and (3.9) we get the lower bound in (3.1). $\quad \square$

LEMMA 3.2. *Let* $\{\xi_{u,v}^{(i)}\}_{i=1}^n$ *be independent. Then for all* $\varepsilon, q > 0$, $n \geq 10$ *and* $(n^\varepsilon/(q\Gamma(u) \log n)) \geq 3/2$,

$$(3.10) \qquad P\left(\max_{1 \leq i \leq n} \xi_{u,v}^{(i)} \geq \frac{(1-\varepsilon) \log n}{v} \right) \geq 1 - e^{-q}.$$

PROOF.   We have

$$(3.11) \qquad \begin{aligned} P\left(\max_{1 \leq i \leq n} \xi_{u,v}^{(i)} > \frac{(1-\varepsilon) \log n}{v} \right) &= 1 - P\left(\max_{1 \leq i \leq n} \xi_{u,v}^{(i)} \leq \frac{(1-\varepsilon) \log n}{v} \right) \\ &= 1 - \prod_{i=1}^n \left(1 - P\left(\xi_{u,v}^{(i)} > \frac{(1-\varepsilon) \log n}{v} \right) \right). \end{aligned}$$

By (3.2), for $n^\varepsilon/(q\Gamma(u) \log n) \geq 3/2$,

$$(3.12) \qquad P\left(\xi_{u,v}^{(i)} > \frac{(1-\varepsilon) \log n}{v} \right) \geq \frac{2 e^{-(1-\varepsilon) \log n}}{3\Gamma(u)(1-\varepsilon) \log n} \geq \frac{q}{n}.$$

Using this and (3.11), we see that

$$P\left(\max_{1 \leq i \leq n} \xi_{u,v}^{(i)} > \frac{(1-\varepsilon) \log n}{v} \right) \geq 1 - \left(1 - \frac{q}{n} \right)^n > 1 - e^{-q}. \qquad \square$$

The next lemma follows immediately from (1.6). It is useful because in applying the permanental inequality sometimes we do not want to consider all the diagonal elements of the nonsingular M-matrix A. We use it in the proof of Lemma 1.1

For a sequence $\{v_i\}_{i=1}^k$, we define $\{v_i^*\}_{i=1}^k$ to be the nondecreasing rearrangement of $\{v_i\}_{i=1}^k$.

LEMMA 3.3. *Let $X = (X_1 \ldots, X_n)$ be an R^n valued random variable defined by (1.4) with an $n \times n$ nonsingular M-matrix A with diagonal elements a_i, $1 \le n$. Then for all $p \ge 1$,*

$$(3.13) \qquad P\Big(\max_{1 \le i \le n} X_i \ge \lambda\Big) \ge P\Big((a_{[n/p]}^*)^{-1} \max_{1 \le i \le [n/p]} \xi_{\alpha,1}^{(i)} \ge \lambda\Big),$$

where $\{\xi_{\alpha,1}^{(i)}, 1 \le i \le [n/p]\}$ are independent.

PROOF. Using (1.6), we see that

$$
\begin{aligned}
P\Big(\max_{1 \le i \le n} X_i \ge \lambda\Big) &\ge P\Big(\max_{1 \le i \le n} a_i^{-1} \xi_{\alpha,1}^{(i)} \ge \lambda\Big) \\
&= P\Big(\max_{1 \le i \le n} (a^*)_i^{-1} \xi_{\alpha,1}^{(i)} \ge \lambda\Big) \\
(3.14) \qquad\qquad &\ge P\Big(\max_{1 \le i \le [n/p]} (a_i^*)^{-1} \xi_{\alpha,1}^{(i)} \ge \lambda\Big) \\
&\ge P\Big((a_{[n/p]}^*)^{-1} \max_{1 \le i \le [n/p]} \xi_{\alpha,1}^{(i)} \ge \lambda\Big). \qquad \square
\end{aligned}
$$

PROOF OF LEMMA 1.1. By Lemma 3.3, for any sequence $t_1, \ldots, t_n \in T$,

$$
\begin{aligned}
P\Big(\sup_{t \in T} X_t \ge \lambda\Big) &\ge P\Big(\max_{1 \le i \le n} X_{t_i} \ge \lambda\Big) \\
(3.15) \qquad\qquad &\ge P\Big((a_{[n/p]}^*(t_1, \ldots, t_n))^{-1} \max_{1 \le i \le [n/p]} \xi_{\alpha,1}^{(i)} \ge \lambda\Big) \\
&\ge P\Big(\max_{1 \le i \le [n/p]} \xi_{\alpha,1}^{(i)} \ge a_{[n/p]}^*(t_1, \ldots, t_n)\lambda\Big).
\end{aligned}
$$

Therefore, by continuity of the cumulative distribution function
$P(\max_{1 \le i \le [n/p]} \xi_{\alpha,1}^{(i)} \le s)$, we have

$$(3.16) \qquad P\Big(\sup_{t \in T} X_t \ge \lambda\Big) \ge P\Big(\max_{1 \le i \le [n/p]} \xi_{\alpha,1}^{(i)} \ge \inf_{(t_1, \ldots, t_n) \in T^n} a_{[n/p]}^*(t_1, \ldots, t_n)\lambda\Big),$$

which is (1.9). $\square$

PROOF OF THEOREM 1.3. By (1.10), all we need to do is to show that

$$(3.17) \qquad\qquad \lim_{n \to \infty} P\Big(\max_{1 \le i \le [n/p]} \xi_{\alpha,1}^{(i)} \ge \log n\Big) = 1.$$

This follows immediately from Lemma 3.2. $\square$

4. Permanental inequality. We examine the implications of (1.6) and explain why we refer to it as a Sudakov-type inequality.

It follows from (1.6), the permanental inequality that

$$(4.1) \qquad E\Big(\max_{1 \le i \le n} (2X_i)^{1/2} \Big) \ge E\Big(\max_{1 \le i \le n} (2\xi_{\alpha,1}^{(i)}/a_i)^{1/2} \Big).$$

If $K = A^{-1}$ is symmetric and positive definite and $\alpha = 1/2$, then $X = (\eta_1^2/2, \dots, \eta_n^2/2)$, where $(\eta_1, \dots, \eta_n)$ is a Gaussian vector with covariance $\{K_{i,j}\}$. In this case by (4.1),

$$(4.2) \qquad E\Big(\max_{1 \le i \le n} |\eta_i| \ge \lambda \Big) \ge \sqrt{2} E\Big(\max_{1 \le i \le n} (\xi_{\alpha,1}^{(i)}/a_i)^{1/2} \Big).$$

Note that $(\xi_{\alpha,1}^{(i)})^{1/2}$, $1 \le i \le n$, are the absolute values of a sequence of independent normal random variable with variance $1/2$. Therefore, we can rewrite (4.2) as

$$(4.3) \quad E\Big(\max_{1 \le i \le n} |\eta_i| \Big) \ge \sqrt{2} E\Big(\max_{1 \le i \le n} |\zeta_i|/\sqrt{2a_i} \Big) \ge \Big(\max_{1 \le i \le n} \sqrt{a_i} \Big)^{-1} E\Big(\max_{1 \le i \le n} |\zeta_i| \Big),$$

where ζ_i, $1 \le i \le n$, are independent normal random variables with mean zero and variance 1. This is what we get from the permanental inequality for a mean zero normal random vector $(\eta_1, \dots, \eta_n)$ with covariance matrix K.

By Fernique's comparison principle [9], Lemma 5.5.3,

$$(4.4) \qquad E\Big(\max_{1 \le i \le n} \eta_i \Big) \ge E\Big(\max_{1 \le i \le n} \rho_i \Big),$$

where $(\rho_1, \dots, \rho_n)$ is a mean zero Gaussian random variable satisfying

$$(4.5) \qquad E(\rho_i - \rho_j)^2 \le E(\eta_i - \eta_j)^2 = K_{i,i} + K_{j,j} - 2K_{i,j}.$$

This can be achieved when ρ_i, $1 \le i \le n$, are independent normal random variable with variance $(\sigma_n^*)^2/2$ where

$$(4.6) \qquad (\sigma_n^*)^2 = \inf_{1 \le i, j \le n, i \ne j} K_{i,i} + K_{j,j} - 2K_{i,j}.$$

With this choice of ρ_i, $1 \le i \le n$ we get

$$(4.7) \qquad E\Big(\max_{1 \le i \le n} \eta_i \Big) \ge \frac{\sigma_n^*}{\sqrt{2}} E\Big(\max_{1 \le i \le n} \zeta_i \Big).$$

This inequality is essentially Sudakov's inequality.

If we ignore the presence or absence of absolute values, we see that if

$$(4.8) \qquad \max_{1 \le i \le n} a_i \le \frac{2}{(\sigma_n^*)^2},$$

then (4.3), which follows from the permanental inequality, gives a stronger lower bound for $E(\max_{1 \le i \le n} \eta_i)$ than (4.7), which is what we get using the Sudakov inequality. In Lemma 5.2, we show that (4.8) holds when the matrix K is symmetric and constant on the diagonals.

REMARK 4.1. The Sudakov inequality is very useful in giving necessary conditions for a Gaussian process to be bounded, but it can be a very weak lower bound for many Gaussian random variables. We point this out because the permanental inequality has the same limitations. Evaluating the right-hand side of (4.7), we get

$$(4.9) \qquad E\left(\max_{1 \le i \le n} \eta_i\right) \ge C\sigma_n^*(\log n)^{1/2}$$

for some constant $C > 0$, for all n sufficiently large. If we take the limit as $n \to \infty$, as we do when considering whether a Gaussian process is bounded, this is only useful when

$$(4.10) \qquad \limsup_{n \to \infty} \sigma_n^*(\log n)^{1/2} > 0.$$

Let $\{B(t), t \in [0, 1]\}$ be Brownian motion and consider $(B(1/n), B(2/n), \dots, B(1))$. Then the Sudakov inequality, (4.9), gives

$$(4.11) \qquad E\left(\max_{1 \le i \le n} B(i/n)\right) \ge C\left(\frac{\log n}{n}\right)^{1/2},$$

whereas

$$(4.12) \qquad E\left(\sup_{t \in [0, 1]} B(t)\right) = \sqrt{2/\pi}.$$

5. Diagonals of nonsingular M-matrices.

We now show that (1.14) holds for a large class of nonsingular M-matrices. In the following, we will make the assumption that $K_{i,i} \ge \max_{j \ne i} K_{j,i}$ and that $A = K^{-1}$ has positive row sums. Considering the nature of the kernel of many important permanental processes this is a reasonable assumption (see e.g., [9], (3.107), (3.109) and Theorem 13.1.2).

LEMMA 5.1. *Let A be an $n \times n$ nonsingular M-matrix with positive row sums and set $K = A^{-1}$. Assume that $K_{i,i} \ge \max_{j \ne i} K_{j,i}$. Then*

$$(5.1) \qquad A_{i,i} \le \frac{1}{K_{i,i} - \max_{j \ne i} K_{j,i}}.$$

PROOF. Using the facts that A is an M-matrix and $\sum_{j=1}^{n} A_{i,j} K_{j,i} = 1$ and $\sum_{j \ne i} |A_{i,j}| \le A_{ii}$, we see that

$$(5.2) \qquad \begin{aligned} K_{i,i} A_{i,i} &= 1 + \sum_{j \ne i} |A_{i,j}| K_{j,i} \\ &\le 1 + \max_{j \ne i} K_{j,i} \sum_{j \ne i} |A_{i,j}| \le 1 + \max_{j \ne i} K_{j,i} A_{i,i}, \end{aligned}$$

which gives (5.1). $\square$

Set

$$(5.3) \qquad \sigma_{i,j}^2 = K_{i,i} + K_{j,j} - (K_{i,j} + K_{j,i}) \quad \text{and} \quad (\sigma_n^*)^2 = \inf_{i,j: i \neq j} \sigma_{i,j}^2.$$

The fact that we can write these as squares follows from our assumption that $K_{i,i} \geq \max_{j \neq i} K_{j,i}$.

LEMMA 5.2. *Under the hypotheses of Lemma 5.1 assume also that K is constant along the diagonal and that*

$$(5.4) \qquad |K_{i,j} - K_{j,i}| \leq C\sigma_{i,j}^2 \qquad \text{for } C < 1.$$

Then

$$(5.5) \qquad A_{i,i} \leq \frac{2}{(1-C)(\sigma_n^*)^2}.$$

PROOF. Consider (5.1) and set $K_{j*,i} = \max_{j \neq i} K_{j,i}$. We have

$$K_{i,i} - K_{j*,i} = \frac{1}{2}\big(K_{i,i} + K_{j*,j*} - (K_{j*,i} + K_{i,j*})\big) - \frac{1}{2}(K_{j*,i} - K_{i,j*})$$

$$(5.6) \qquad \geq \frac{1}{2}\big(\sigma_{i,j*}^2 - |K_{j*,i} - K_{i,j*}|\big)$$

$$\geq \frac{(1-C)\sigma_{i,j*}^2}{2} \geq \frac{(1-C)(\sigma_n^*)^2}{2}.$$

Using this in (5.1), we get (5.5). □

PROOF THEOREM 1.4. This follows from Lemma 5.2 with $t_j = j\delta/n$, $j = 1, \ldots, n$, for some $\delta > 0$ and Corollary 1.1 with $d_{s,t}$ replaced by $\sigma_{s,t}$. □

REMARK 5.1. If $K + K^T$ is positive definite, it is easy to see that $\sigma_{i,j}$ is a metric on $\{1, \ldots, n\}$, because we can define an n-dimensional mean zero Gaussian random variable $\{X_i, i \in \{1, \ldots, n\}\}$ with covariance $(K + K^T)/2$ and

$$(5.7) \qquad \sigma_{i,j} = \big(E(X_i - X_j)^2\big)^{1/2} \quad \text{and} \quad \sigma_n^* = \inf_{i,j: i \neq j} \big(E(X_i - X_j)^2\big)^{1/2}.$$

We can remove the assumption that the kernel is constant on the diagonal.

LEMMA 5.3. *Let A be an $n \times n$ nonsingular M-matrix with positive row sums and set $K = A^{-1}$. Assume that $K_{i,i} > \max_{j \neq i} K_{j,i}$. Choose $r_i = K_{i,i}/\widehat{K}$ for some constant $\widehat{K}$. Set*

$$(5.8) \qquad \widehat{\sigma}_{i,j}^2 = 2\widehat{K} - \frac{K_{i,j}}{r_j} - \frac{K_{j,i}}{r_i}$$

and assume that

$$(5.9) \qquad \left| \frac{K_{i,j}}{r_j} - \frac{K_{j,i}}{r_i} \right| \le C \widehat{\sigma}_{i,j}^2, \qquad C < 1,$$

for all i, j. Set

$$(5.10) \qquad (\widehat{\sigma}_n^*)^2 = \inf_{i,j : i \ne j} \widehat{\sigma}_{i,j}^2.$$

Then

$$(5.11) \qquad r_i A_{i,i} \le \frac{2}{(1-C)(\widehat{\sigma}_n^*)^2}.$$

PROOF. Let $X = (X_1, \ldots, X_n)$ be the α-permanental vector with kernel K. Then $Y = (Y_1/r_1, \ldots, Y_n/r_n)$ is the α-permanental vector with kernel $K_Y =: K R^{-1}$, where $R = \mathrm{diag}(r_1, \ldots, r_n)$. It follows from the assumption that $K_{i,i} > \max_{j \ne i} K_{j,i}$ that this also holds for K_Y. Let $A_Y = K_Y^{-1} = RA$. We see that A_Y is a nonsingular M-matrix with positive row sums. Consequently, (5.11) follows from Lemma 5.2. $\square$

We have the following generalization of Theorem 1.4.

THEOREM 5.1. *Let u be the potential density of a transient Markov process in R^1. Set*

$$(5.12) \qquad \widehat{\sigma}_{s,t}^2 = 2 - \frac{u(s,t)}{u(t,t)} - \frac{u(t,s)}{u(s,s)}$$

and assume that

$$(5.13) \qquad \left| \frac{u(s,t)}{u(t,t)} - \frac{u(t,s)}{u(s,s)} \right| \le C \widehat{\sigma}_{s,t}^2, \qquad C < 1,$$

for all $|s|, |t|$ sufficiently small. Then

$$(5.14) \qquad \limsup_{n \to \infty} \left(\sup_{(t_1, \ldots, t_n)} (\widehat{\sigma}_n^*)^2 (t_1, \ldots, t_n) \right) \log n = \infty$$

implies that

$$(5.15) \qquad \sup_t \frac{Y_t}{u(t,t)} = \infty \qquad a.s.,$$

where Y_t is the α-permanental process with kernel u.

PROOF. The proof is the same as the proof of Theorem 1.4. $\square$

In Lemma 5.1, we assume that $K_{i,i} > \max_{j \ne i} K_{j,i}$. The following example shows that we can still get an inequality like (5.5) when this condition does not hold.

EXAMPLE 5.1. Consider the covariance matrix $\mathcal{B}$ of $(B(1), \ldots, B(n))$, where $\{B(t), t \in R_+\}$ is standard Brownian motion. Obviously, $\mathcal{B}_{i,i} - \max_{j \neq i} \mathcal{B}_{i,j} = 0$. However, $\mathcal{B}^{-1}$ is a tri-diagonal matrix with all diagonal elements equal to 2, except that $(\mathcal{B}^{-1})_{n,n} = 1$ and all off diagonal elements that are not zero equal to -1. In this case,

$$(5.16) \qquad (\mathcal{B}^{-1})_{i,i} \leq 2 = \frac{2}{(\sigma_n^*)^2}.$$

[Here, $(\sigma_n^*)^2 = \min_{i \neq j} E(B(i) - B(j))^2 = 1$.]

We can use this to create another interesting example. Let D be a diagonal matrix with entries $1, \ldots, n$. Let $\widetilde{\mathcal{B}} = D^{-1}\mathcal{B}$. This matrix has entries 1 on and above the diagonal and $\widetilde{\mathcal{B}}_{i,j} = j/i$ for $1 < j < i$. The diagonal entries of $\widetilde{\mathcal{A}} = (\widetilde{\mathcal{B}})^{-1}$ are $\widetilde{\mathcal{A}}_{i,i} = 2B_{i,i} = 2i$, $1 \leq i \leq n-1$ and $\widetilde{\mathcal{A}}_{n,n} = B_{n,n} = n$. Set

$$(5.17) \qquad \phi_{i,j}^2 = \widetilde{\mathcal{B}}_{i,i} + \widetilde{\mathcal{B}}_{j,j} - \widetilde{\mathcal{B}}_{i,j} - \widetilde{\mathcal{B}}_{j,i}$$

and

$$(5.18) \qquad (\phi_n^*)^2 = \min_{\substack{1 \leq i,j \leq n \\ i \neq j}} \phi_{i,j}^2.$$

The minimum is achieved at $\phi_{n,n-1}^2 = 1/n$. Therefore, we have

$$(5.19) \qquad (\mathcal{A})_{i,i} \leq \frac{2}{(\phi_n^*)^2} = 2n, \qquad 1 \leq i \leq n.$$

The maximum on the left-hand side of (5.19) is $(\mathcal{A})_{n-1,n-1} = 2(n-1)$, since $(\mathcal{A})_{n,n} = n$.

LEMMA 5.4. *When u is the potential density of a transient Markov process in R^1, (5.13) always holds with $C = 1$.*

PROOF. We need to show that

$$(5.20) \qquad \left| \frac{u(s,t)}{u(t,t)} - \frac{u(t,s)}{u(s,s)} \right| \leq 2 - \frac{u(s,t)}{u(t,t)} - \frac{u(t,s)}{u(s,s)}.$$

Without loss of generality, we assume that $u(s,t)/u(t,t) \geq u(t,s)/u(s,s)$. Then (5.20) is equivalent to

$$(5.21) \qquad \frac{u(s,t)}{u(t,t)} \leq 1.$$

It follows from [9], Lemma 3.4.3, that when u is the potential density of a transient Markov process, in R^1, this always holds. $\quad\square$

6. Permanental processes with a kernel that is the potential density of a Lévy process.

PROOF OF THEOREM 1.5. It follows from Lemma 1.2 that (1.27) is the same as (1.18). Therefore, (1.28) follows from (1.19) with $t_1, \ldots, t_n$ replaced by δ/n, $2\delta/n, \ldots, \delta$. □

The next lemma is used in the proof of Theorem 1.6.

LEMMA 6.1. *Suppose that ℓ and h are positive, quasi-monotonic slowly varying functions* (*see* [3], *Section* 2.7) *at infinity. Let $\mathcal{R}(\lambda)$ and $\mathcal{I}(\lambda)$ be functions on $[0, \infty]$ such that*

$$(6.1) \qquad \mathcal{R}(\lambda) = \frac{h(\lambda)}{\lambda} \quad and \quad |\mathcal{I}(\lambda)| = \frac{\ell(\lambda)}{\lambda}.$$

If $\mathcal{R} \in L^1$ and

$$(6.2) \qquad \int_{1/z}^{\infty} \mathcal{R}(\lambda) \, d\lambda \geq B\ell(1/|z|)$$

as $|z| \to 0$ with $B > \frac{\pi}{2}$, then

$$(6.3) \qquad \left| \int_0^{\infty} \sin(\lambda z) \mathcal{I}(\lambda) \, d\lambda \right| \leq C \int_0^{\infty} (1 - \cos(\lambda z)) \mathcal{R}(\lambda) \, d\lambda$$

for some $C < 1$, and all $|z|$ sufficiently small. Furthermore,

$$(6.4) \qquad \int_0^{\infty} (1 - \cos(\lambda z)) \mathcal{R}(\lambda) \, d\lambda \sim \int_{1/|z|}^{\infty} \mathcal{R}(\lambda) \, d\lambda$$

as $|z| \to 0$.

PROOF. It suffices to show (6.3) for $z > 0$. By [11], (1.43),

$$(6.5) \qquad \int_0^{\infty} \frac{1_{\{\lambda z \leq 1\}} - e^{i\lambda z}}{\lambda} \ell(\lambda) \, d\lambda \sim \ell(1/z) \int_0^{\infty} \frac{1_{\{\lambda z \leq 1\}} - e^{i\lambda z}}{\lambda} \, d\lambda$$

as $z \to 0$. Taking the imaginary part of (6.5), we see that

$$(6.6) \qquad \int_0^{\infty} \sin(\lambda z) \mathcal{I}(\lambda) \, d\lambda = \int_0^{\infty} \frac{\sin(\lambda z)}{\lambda} \ell(\lambda) \, d\lambda \sim \ell(1/z) \int_0^{\infty} \frac{\sin(s)}{s} \, ds$$

as $z \to 0$. Therefore, by [6], 3.721,

$$(6.7) \qquad \int_0^{\infty} \sin(\lambda z) \mathcal{I}(\lambda) \, d\lambda \sim \frac{\pi}{2} \ell(1/z)$$

as $z \to 0$.

To use below, we note that by a change of variables

$$\int_0^\infty \frac{1_{\{\lambda z \le 1\}} - \cos(\lambda z)}{\lambda}\, d\lambda = \int_0^\infty \frac{1_{\{s \le 1\}} - \cos(s)}{s}\, ds$$

(6.8)

$$= 2\int_0^1 \frac{\sin^2(s/2)}{s}\, ds - \int_1^\infty \frac{\cos(s)}{s}\, ds.$$

Therefore, the first integral in (6.8) is a constant which we denote by c_0. It is easy to see that $c_0 < \infty$. The first of the last two integrals in (6.8) is bounded by $1/4$, and by integration by parts, that the second of these last two integrals is bounded by 2.

By (6.5), (6.8) and [9], Theorem 14.7.2,

$$\int_0^\infty \big(1 - \cos(\lambda z)\big)\mathcal{R}(\lambda)\, d\lambda$$

(6.9)

$$= \int_0^\infty \big(1_{\{\lambda z \le 1\}} - \cos(\lambda z)\big)\mathcal{R}(\lambda)\, d\lambda + \int_{1/z}^\infty \mathcal{R}(\lambda)\, d\lambda$$

$$\sim h(1/z)\int_0^\infty \frac{1_{\{\lambda z \le 1\}} - \cos(\lambda z)}{\lambda}\, d\lambda + \int_{1/z}^\infty \mathcal{R}(\lambda)\, d\lambda$$

$$= c_o h(1/z) + \int_{1/z}^\infty \mathcal{R}(\lambda)\, d\lambda \sim \int_{1/z}^\infty \mathcal{R}(\lambda)\, d\lambda$$

as $z \to 0$. Thus, we obtain (6.3) and also (6.4); see (1.26). □

PROOF OF THEOREM 1.6. The characteristic exponent of this process

$$\psi(\lambda) = -\int_\infty^\infty \big(e^{i\lambda x} - 1 - i\lambda x 1_{\{|x|<1\}}\big)\nu(dx)$$

(6.10)

$$\sim \frac{\pi}{2}|\lambda|g(\lambda) + i(p-q)\lambda \int_{1/\lambda}^1 \frac{g(1/x)}{x}\, dx$$

$$\sim \frac{\pi}{2}|\lambda|g(\lambda) + i(p-q)\lambda \int_1^\lambda \frac{g(s)}{s}\, ds$$

as $\lambda \to \infty$. Note that the $\operatorname{Re}\psi(\lambda) = o(\operatorname{Im}\psi(\lambda))$ as $\lambda \to \infty$. We show how to obtain (6.10) in Section A.2.

We first consider the case when $p \ne q$. It follows from (6.10) that

(6.11)
$$\big|\mathcal{I}_\beta(\lambda)\big| \sim \frac{1}{|p-q|\lambda \int_1^\lambda \frac{g(s)}{s}\, ds} := \frac{\ell(\lambda)}{\lambda}$$

and

(6.12)
$$\mathcal{R}_\beta(\lambda) \sim \frac{(\pi/2)g(\lambda)}{|p-q|^2\lambda(\int_1^\lambda \frac{g(s)}{s}\, ds)^2}$$

as $\lambda \to \infty$. Note that

$$(6.13) \qquad \frac{(\pi/2)g(\lambda)}{|p-q|^2\lambda(\int_1^\lambda \frac{g(s)}{s}\,ds)^2} = -\frac{(\pi/2)}{|p-q|}\frac{d}{d\lambda}\ell(\lambda)$$

which implies that

$$(6.14) \qquad \int_{1/z}^\infty \mathcal{R}_\beta(\lambda)\,d\lambda \sim \frac{(\pi/2)}{|p-q|}\ell(1/z).$$

Comparing this with (6.11), we see that (6.2) holds for all $B < (\pi/2)/|p-q|$. Obviously, we can take $B > (\pi/2)$ as long as $p \neq q$. Also, by (1.26) and (6.4),

$$(6.15) \qquad \sigma^2(z) \sim \frac{1}{|p-q|^2}\left(\int_1^{1/z} \frac{g(s)}{s}\,ds\right)^{-1}.$$

Therefore, (1.31) follows from Theorem 1.5.

Note that we require that $\mathcal{R}_\beta \in L^1(R_+)$. That is why we impose the condition in (1.30).

When $p = q$, $\psi(\lambda)$ is real and symmetric and

$$(6.16) \qquad \mathcal{R}_\beta(\lambda) \sim \frac{2}{\pi|\lambda|g(\lambda)}.$$

Condition (6.2) in Lemma 6.1 is trivially satisfied and, by by (1.26) and (6.4)

$$(6.17) \qquad \sigma^2(z) \sim \frac{4}{\pi^2}\int_{1/|z|}^\infty \frac{1}{|\lambda|g(\lambda)}\,d\lambda$$

as $|z| \to 0$. Therefore, (1.33) follows from Theorem 1.5. $\quad\square$

DETAILS FOR (1.34). This is simple for symmetric processes, so we only need to check (1.34) for $p \neq q$. By (1.26), for all Lévy processes,

$$(6.18) \qquad R_\beta(z) = u^\beta(0) - \frac{\sigma^2(z)}{2}.$$

For the processes with Lévy measure given by (1.29), we see by (6.7) that as $z \to 0$

$$(6.19) \qquad H_\beta(z) \sim \frac{\ell(1/z)}{2}.$$

In addition by (1.26), (6.9) and (6.14),

$$(6.20) \qquad \sigma^2(z) \sim \frac{2}{\pi}\int_{1/z}^\infty R(\lambda)\,d\lambda \sim \frac{1}{|p-q|}\ell(1/z).$$

Therefore,

$$(6.21) \qquad H_\beta(z) \sim \frac{|p-q|}{2}\sigma^2(z) \quad \text{and} \quad H_\beta(-z) \sim -\frac{|p-q|}{2}\sigma^2(z).$$

Adding (6.18) and (6.21), we get (1.34).

PROOF OF COROLLARY 1.2.   To show that (1.27) is satisfied, it suffices to show that for all $z > 0$ sufficiently small

$$(6.22) \qquad \left| \int_0^\infty \sin(\lambda z) \mathcal{I}_\beta(\lambda) \, d\lambda \right| \le C \int_0^\infty (1 - \cos(\lambda z)) \mathcal{R}_\beta(\lambda) \, d\lambda$$

for some $C < 1$. To simplify the proof, we assume that $\mathcal{I}_\beta(\lambda) \ge 0$ and take $\mathcal{I}_\beta(\lambda)$ and $\mathcal{R}_\beta(\lambda)$ to be nonincreasing functions. It is easy to extend the proof to the case in which $|\mathcal{I}_\beta(\lambda)|$ and $\mathcal{R}_\beta(\lambda)$ are asymptotic to nonincreasing functions as $\lambda \to \infty$. We have

$$(6.23) \qquad \int_0^\infty \sin(\lambda z) \mathcal{I}_\beta(\lambda) \, d\lambda \le \int_0^{\pi/z} \lambda z \mathcal{I}_\beta(\lambda) \, d\lambda$$

because, since $\mathcal{I}_\beta(\lambda)$ is decreasing, for all $k \ge 1$

$$(6.24) \qquad -\int_{(2k-1)\pi/z}^{(2k)\pi/z} \sin(\lambda z) \mathcal{I}_\beta(\lambda) \, d\lambda \ge \int_{(2k)\pi/z}^{(2k+1)\pi/z} \sin(\lambda z) \mathcal{I}_\beta(\lambda) \, d\lambda.$$

Also

$$
\begin{aligned}
\frac{1}{2} \int_0^\infty (1 - \cos(\lambda z)) \mathcal{R}_\beta(\lambda) \, d\lambda &= \int_0^\infty \sin^2\left(\frac{\lambda z}{2}\right) \mathcal{R}_\beta(\lambda) \, d\lambda \\
&\ge \int_{\pi/(2z)}^\infty \sin^2\left(\frac{\lambda z}{2}\right) \mathcal{R}_\beta(\lambda) \, d\lambda \\
&= \sum_{k=0}^\infty \int_{\pi(1+4k)/(2z)}^{\pi(1+4(k+1))/(2z)} \sin^2\left(\frac{\lambda z}{2}\right) \mathcal{R}_\beta(\lambda) \, d\lambda \\
&\ge \sum_{k=0}^\infty \int_{\pi(1+4k)/(2z)}^{\pi(3+4k)/(2z)} \sin^2\left(\frac{\lambda z}{2}\right) \mathcal{R}_\beta(\lambda) \, d\lambda.
\end{aligned}
$$

(6.25)

Note that if $\pi(1 + 4k)/(2z) \le \lambda \le \pi(3 + 4k)/(2z)$ then

$$(6.26) \qquad \pi/4 + k\pi \le \lambda z/2 \le 3\pi/4 + k\pi,$$

and consequently $\sin^2(\lambda z/2) \ge 1/2$. Therefore,

$$(6.27) \qquad \int_{\pi(1+4k)/(2z)}^{\pi(3+4k)/(2z)} \sin^2\left(\frac{\lambda z}{2}\right) \mathcal{R}_\beta(\lambda) \, d\lambda \ge \frac{1}{2} \int_{\pi(1+4k)/(2z)}^{\pi(3+4k)/(2z)} \mathcal{R}_\beta(\lambda) \, d\lambda.$$

Furthermore, since $\mathcal{R}_\beta(\lambda)$ is decreasing, for all $z > 0$ sufficiently small

$$
\begin{aligned}
\frac{1}{2} &\int_{\pi(1+4k)/(2z)}^{\pi(3+4k)/(2z)} \mathcal{R}_\beta(\lambda) \, d\lambda \\
&\ge \frac{1}{4} \int_{\pi(1+4k)/(2z)}^{\pi(3+4k)/(2z)} \mathcal{R}_\beta(\lambda) \, d\lambda + \frac{1}{4} \int_{\pi(3+4k)/(2z)}^{\pi(5+4k)/(2z)} \mathcal{R}_\beta(\lambda) \, d\lambda \\
&= \frac{1}{4} \int_{\pi(1+4k)/(2z)}^{\pi(1+4(k+1))/(2z)} \mathcal{R}_\beta(\lambda) \, d\lambda.
\end{aligned}
$$

(6.28)

Putting all this together, we see that for all z sufficiently small

$$(6.29) \qquad \int_0^\infty \big(1 - \cos(\lambda z)\big) \mathcal{R}_\beta(\lambda)\, d\lambda \geq \frac{1}{2} \int_{\pi/(2z)}^\infty \mathcal{R}_\beta(\lambda)\, d\lambda.$$

Combining (6.23) and (6.29) and using the hypothesis (1.38), we get (6.22) for some $C < 1$.

We see from (1.16) and (6.29) that

$$(6.30) \qquad \sigma^2(z) \geq C \int_{\pi/(2z)}^\infty \mathcal{R}_\beta(\lambda)\, d\lambda$$

which implies by Theorem 1.5 that if

$$(6.31) \qquad \limsup_{n \to \infty} \left(\int_{\pi n/(2\delta)}^\infty \mathcal{R}_\beta(\lambda)\, d\lambda \right) \log n = \infty,$$

then the α-permanental process with kernel u^β is unbounded almost surely. It is easy to see that, by interpolation, this is equivalent to (1.39). $\quad \square$

REMARK 6.1. We consider (1.38) for $\mathcal{I}_\beta(\lambda)$ and $\mathcal{R}_\beta(\lambda)$ asymptotic to $\mathcal{I}(\lambda)$ and $\mathcal{R}(\lambda)$ as $\lambda \to \infty$ [see (6.11) and (6.12)]. In this case, Corollary 1.2 it is not much cruder than the estimates given in Theorem 1.6. Since in this case $\lambda \mathcal{I}_\beta(\lambda)$ is slowly varying at infinity we see that the left-hand side of (1.38) is asymptotic to $\pi \ell(1/|z|)$ as $|z| \to 0$. By (6.2) and the fact that ℓ is slowly varying, the right-hand side of (1.38) is asymptotic to $C\pi \ell(1/|z|)/(2|p - q|)$ as $|z| \to 0$. Therefore, (1.38) holds for $C > 2|p - q|$. Therefore, Corollary 1.2 gives the results obtained in Example 1.1 when $p \neq q$ and $|p - q| < 1/2$.

APPENDIX

A.1. A property of the potential density of a transient Markov process.

LEMMA A.1. *If $\{u(s,t), s, t \in T\}$ is the potential density of a transient Markov process X_t with state space T, then for all distinct $t_1, \ldots, t_n$ in T, the matrix K with elements $\{u(t_i, t_j)\}_{i,j=1}^n$ is invertible and its inverse is a nonsingular M-matrix.*

PROOF. This proof is a portion of the proof of [9], Theorem 13.1.2. We define the following stopping time:

$$(A.1) \qquad \sigma = \inf\{t \geq 0 | X_t \in \{t_1, \ldots, t_n\} \cap \{X_0\}^c\}$$

(note that σ may be infinite). Let $\{L_t^x; (x, t) \in S \times R^1\}$ be the local times of X. Since $u(t_i, t_j)$ is the 0-potential density of X, we can normalize the local time so that

$$(A.2) \qquad u(t_i, t_j) = E^{t_i}\big(L_\infty^{t_j}\big).$$

Using (A.2) and the strong Markov property, we see that

$$u(t_i, t_j) = E^{t_i}(L_\sigma^{t_j}) + E^{t_i}(E^{X_\sigma}(L_\infty^{t_j}); \sigma < \infty)$$

$$\text{(A.3)} \qquad = d_{i,j} + \sum_{k=1}^n h_{i,k} u(t_k, t_j),$$

where $d_{i,j} = E^{t_i}(L_\sigma^{t_j})$ and $h_{i,k} = P^{t_i}(X_\sigma = t_k)$.

Let $D = \{d_{i,j}\}_{1 \le i,j \le n}$ and $H = \{h_{i,j}\}_{1 \le i,j \le n}$. We can write (A.3) as

$$\text{(A.4)} \qquad K = D + HK$$

so that $(I - H)K = D$. Moreover, D is a diagonal matrix with all its diagonal elements strictly positive. This follows because, starting from $X_0 = t_i$, $\sigma > 0$, which implies that each $b_{i,i} > 0$. On the other hand, the process is killed the first time it hits any $t_j \ne t_i$. Thus, starting from t_i, $L_\sigma^{t_j} = 0$, $j \ne i$.

Since D is invertible, both $(I - H)$ and K are invertible and

$$\text{(A.5)} \qquad K^{-1} = D^{-1}(I - H).$$

It is clear that $H \ge 0$. It follows from this that K^{-1} has negative off diagonal elements. Moreover, since $h_{i,i} = 0$ it follows that K^{-1} has positive diagonal elements. Therefore, K is a nonsingular M-matrix. Furthermore,

$$\text{(A.6)} \qquad \sum_{j=1}^n h_{i,j} = P^{t_i}(\sigma < \infty) \le 1 \qquad \forall i = 1, \ldots, n,$$

from which it follows that K^{-1} has positive row sums. $\quad\square$

A.2. Derivation of (6.10). We have

$$\text{(A.7)} \qquad \operatorname{Im} \psi(\lambda) = -(p - q) \int_0^\infty (\sin \lambda x - \lambda x 1_{\{|x| < 1\}}) \nu(dx).$$

Let $\nu_1(dx) := (p - q)\nu(x)$. Then $\operatorname{Im} \psi(\lambda)$ is equal to

$$\text{(A.8)} \qquad -\int_0^{1/\lambda} (\sin \lambda x - \lambda x) \nu_1(dx) + \lambda \int_{1/\lambda}^1 x \nu_1(dx) - \int_{1/\lambda}^\infty \sin \lambda x \nu_1(dx).$$

Using $|\sin \lambda x - \lambda x| \le |\lambda x|^3$ in the first of these integral and $|\sin \lambda| \le 1$ in the the last of these integral we see that their absolute values are both $O(g(\lambda)/\lambda)$ as $\lambda \to \infty$. Consequently,

$$\text{(A.9)} \qquad \operatorname{Im} \psi(\lambda) \sim (p - q)\lambda \int_{1/\lambda}^1 \frac{g(1/x)}{x} \, dx = (p - q)\lambda \int_1^\lambda \frac{g(s)}{s} \, ds$$

as $\lambda \to \infty$.

The asymptotic behavior of $\operatorname{Re} \psi(\lambda)$ as $\lambda \to \infty$ follows from the next lemma.

LEMMA A.2. *Let $g(\cdot)$ be a slowly varying function at infinity. Then*

$$(A.10) \qquad \int_0^\infty (1 - \cos \lambda x) \frac{g(1/x)}{x^2} \, dx \sim \frac{\pi}{2} \lambda g(\lambda)$$

as $\lambda \to \infty$.

PROOF. We write the left-hand side of (A.10) as

$$(A.11) \qquad \lambda g(\lambda) \int_0^\infty \frac{(1 - \cos s)}{s^2} \frac{g(\lambda/s)}{g(\lambda)} \, ds.$$

Consider

$$(A.12) \qquad \int_0^M \frac{(1 - \cos s)}{s^2} \frac{g(\lambda/s)}{g(\lambda)} \, ds + \frac{1}{g(\lambda)} \int_M^\infty \frac{(1 - \cos s)}{s^2} g(\lambda/s) \, ds.$$

Note that by [3], Theorem 1.5.6, for $s \in (0, M]$, $g(\lambda/s)/g(\lambda) \le C(s^{-\varepsilon} \vee 1)$, for any $\varepsilon > 0$, and some constant depending on M and ε. Therefore, by the dominated convergence theorem we see that the limit, as $\lambda \to \infty$ of the first integral in (A.12) is

$$(A.13) \qquad \int_0^M \frac{(1 - \cos s)}{s^2} \, ds.$$

The second integral in (A.12) is bounded by

$$(A.14) \qquad \frac{2}{\lambda g(\lambda)} \int_{M/\lambda}^\infty \frac{g(1/s)}{s^2} \, ds = \frac{2}{\lambda g(\lambda)} \int_0^{\lambda/M} g(v) \, dv.$$

We need a condition on g near 0. This is given implicitly by the statement that ν is a Lévy measure, which requires that

$$(A.15) \qquad \int_1^\infty \frac{g(1/|x|)}{x^2} \, dx = \int_0^1 g(v) \, dv \le C' < \infty.$$

Therefore, since g is slowly varying at infinity, the second integral in (A.14) is bounded by $3g(\lambda/M)/(Mg(\lambda))$ which goes to $3/M$ as $\lambda \to \infty$. Therefore, taking $\lambda \to \infty$, we see that (A.12) is equal to

$$(A.16) \qquad \int_0^M \frac{(1 - \cos s)}{s^2} \, ds + O(1/M)$$

for all M. This gives us (A.10) because, by integration by parts,

$$(A.17) \qquad \int_0^\infty \frac{(1 - \cos s)}{s^2} \, ds = \int_0^\infty \frac{\sin s}{s} \, ds = \frac{\pi}{2}$$

by [6], 3.721. □

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