Tube estimates for diffusion processes under a weak Hörmander condition

Paolo Pigato

Université de Lorraine, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54500, France; CNRS, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54500, France; INRIA, Villers-lès-Nancy, F-54600, France. E-mail: paolo.pigato@inria.fr

Abstract. We consider a diffusion process under a local weak Hörmander condition on the coefficients. We find Gaussian estimates for the density in short time and exponential lower and upper bounds for the probability that the diffusion remains in a small tube around a deterministic trajectory (skeleton path). These bounds depend explicitly on the radius of the tube and on the energy of the skeleton path. We use a norm which reflects the non-isotropic structure of the problem, meaning that the diffusion propagates in $\mathbb{R}^2$ with different speeds in the directions σ and $[\sigma, b]$. We establish a connection between this norm and the standard control distance.

Résumé. On considère une diffusion dont les coefficients satisfont une condition d'Hörmander faible locale. On obtient des estimées gaussiennes de la densité en temps court et des bornes inférieures et supérieures exponentielles pour la probabilité que la diffusion reste dans un petit tube autour d'une trajectoire déterministe (« squelette »). Ces bornes dépendent explicitement du rayon du tube et de l'énergie du squelette. On utilise une norme qui prend en compte la structure non isotrope du problème, dans le sens où la diffusion se propage dans $\mathbb{R}^2$ avec des vitesses différentes dans la direction de σ et $[\sigma, b]$. On établit un lien entre cette norme et la distance de contrôle standard.

1. Introduction

In this article we consider the following stochastic differential equation on $[0, T]$:

$$X_t = x_0 + \int_0^t \sigma(X_s) \circ dW_s + \int_0^t b(X_s) \, ds, \tag{1.1}$$

where the diffusion X is two-dimensional and the Brownian Motion W is one-dimensional. $\circ dW_s$ denotes the Stratonovich integral, and we suppose a certain geometric property for the diffusion coefficient (which holds true in particular for the equation associated with the Asian option). Since σ is just a column vector, the ellipticity assumption fails at any point, and the strong Hörmander condition fails as well, so we investigate the regularity of this process assuming a hypoellipticity condition of weak Hörmander type. The prototype of this kind of problems is a two dimensional system where the first component X^1 follows a stochastic dynamic, and the second component X^2 is a deterministic functional of X^1, so the randomness acts indirectly on X^2. Besides the natural application to the Asian option, there are others such as in [23,24]. In these papers the functioning of a neuron is modeled: X^2 is the concentration of some chemicals resulting from a reaction involving the first component X^1. Differently from our setting, though, there are several measurements corresponding to the input X^1, so X^2 is multi-dimensional. The pattern, however, is similar.

We find Gaussian estimates for the density in short time, supposing the process satisfies a weak Hörmander condition. Ben Arous and Léandre investigate the decay of the heat kernel of a hypoelliptic diffusion over the diagonal in their celebrated papers [10,11]. Their framework is different because they work under a strong Hörmander condition and because they are interested in asymptotic results, whereas we provide results holding for finite positive times. In [28] explicit two-sided bounds for the density of diffusion processes are established under strong Hörmander conditions, if the drift is generated by the vector fields of the diffusive part. On the opposite, the problem we consider here is of weak Hörmander type, meaning that the drift has a key role in the propagation of the noise. In this case, the drift gives an additional specific contribution which is usually difficult to handle when trying to estimate the density of the solution. In [7] and [19] bounds are provided for the density of the Asian type SDE and for a chain of SDEs, in a weak Hörmander framework. An analytical approach to a similar density estimate is given by Polidoro, Pascucci and Boscain in [13,33,35].

In this paper, we obtain a more general result than those known in the cited literature, as we allow for a more general coefficient for the Brownian Motion. Indeed we suppose that locally the vector field σ has the same direction of the directional derivative $\partial_\sigma \sigma$, whereas the works mentioned above would apply for $\sigma = (\sigma_1, 0)$ which is a more restrictive condition. Moreover, our coefficients are just locally hypoelliptic. The other novelty is that thanks to our short time non-asymptotic result we are able to find exponential lower and upper bounds for the probability that the diffusion remains in a small tube around a deterministic trajectory. More precisely we consider (1.1) and introduce the associated skeleton path solution of the following ODE:

$$x_t(\phi) = x_0 + \int_0^t \sigma\big(x_s(\phi)\big)\phi_s \, ds + \int_0^t b\big(x_s(\phi)\big) \, ds,$$

for a control $\phi \in L^2[0, T]$. We assume the following weak Hörmander condition: σ, $[\sigma, b]$ span $\mathbb{R}^2$ locally around $x(\phi)$. This is enough to ensure the existence of the density in the case of diffusions (see [32,37]). Similar results are also available for SDEs with coefficients with dependence on time, under very weak regularity assumptions ([17]), SDEs driven by a fractional Brownian Motion ([8]) and for rough differential equations ([16]).

We prove here a tube estimate for (1.1), meaning that we find upper and lower bounds for $\mathbb{P}(\sup_{0 \le t \le T} \|X_t - x_t(\phi)\| \le R)$, explicitly depending on the energy of the skeleton path and on the radius of the tube, that can be time-dependent. Several works have considered this subject, starting from Stroock and Varadhan in [38], where such result is used to prove the support theorem for diffusion processes. In their work $\|\cdot\|$ is the Euclidean norm, but later on different norms have been used to take into account the regularity of the trajectories (about this, see for example [9] and [21]). This problem is interesting for physicists because of the Onsager–Machlup functional (see [14,25]), and is also related to large and moderate deviation theory (see [12,22]).

Since we work under Hörmander-type conditions, in order to give accurate estimates we consider a norm accounting for the non-diffusive time scale of the process. Indeed, thanks to the Hörmander condition, the noise propagates in the whole $\mathbb{R}^2$, but with with speed $t^{1/2}$ in the direction σ and $t^{3/2}$ in the direction $[\sigma, b]$. We also introduce a suitable control metric, adapting the classic control-Carathéodory distance, which is equivalent to this norm.

We apply techniques based on the recent work by Bally and Caramellino ([1–3]) on density estimates for random variables. In Section 3 we recall some of these results and derive an upper and a lower bound for the density in a fairly abstract framework, starting from the Malliavin–Thalmaier representation formula for the density. The importance of these abstract estimates may go beyond our particular problem.

This paper is organized as follows. In Section 2 we introduce notations and state our main results: the short-time density estimate and the tube estimate. In Section 3 we develop the Malliavin calculus techniques that we apply to estimate the density of our diffusion. In Section 4 we apply these techniques, finding the short-time density estimates mentioned above. In Section 5 we use the short-time result and a concatenation procedure to prove the tube estimate.

2. Notations and results

2.1. *Notations*

We start introducing some notations. We write $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, \ldots, n\}^k$ for a multi-index with length $|\alpha| = k$ and $\partial_x^\alpha = \partial_{x_{\alpha_1}} \cdots \partial_{x_{\alpha_k}}$. For $f, g : \mathbb{R}^n \to \mathbb{R}^n$ we recall the definition of the directional derivative of f in the direction g

as

$$\partial_g f(x) = (\nabla f) g(x) = \sum_{i=1}^{n} g^i(x) \partial_{x_i} f(x).$$

The *Lie bracket* $[f, g]$ in x is defined as

$$[f, g](x) = \partial_f g(x) - \partial_g f(x).$$

We denote by M^T the transpose of a 2×2 matrix M. We also use the notation $\lambda_*(M)$ for the smallest singular value of M, and $\lambda^*(M)$ for the largest one. We recall that singular values are the square roots of the eigenvalues of MM^T, and that, when M is symmetric and semi-definite, singular values coincide with the eigenvalues of M. In particular, when M is a covariance matrix, $\lambda_*(M)$ and $\lambda^*(M)$ are the smallest and the largest eigenvalues of M.

If M is invertible we also associate to M the norm on $\mathbb{R}^2$

$$|\xi|_M = \sqrt{\langle (MM^T)^{-1}\xi, \xi \rangle} = |M^{-1}\xi|.$$

For two 2×2 positive semi-definite symmetric matrices B_1, B_2, we write $B_1 \le B_2$ for

$$\xi^T B_1 \xi \le \xi^T B_2 \xi, \quad \text{for all } \xi \in \mathbb{R}^2.$$

As we said, we consider the diffusion

$$X_t = x_0 + \int_0^t \sigma(X_s) \circ dW_s + \int_0^t b(X_s) \, ds, \tag{2.1}$$

where X is in dimension two, W is in dimension one. For $x \in \mathbb{R}^2$, we set

$$A(x) = \big(\sigma(x), [\sigma, b](x)\big) \tag{2.2}$$

and, for any $R > 0$,

$$A_R(x) = \big(R^{1/2}\sigma(x), R^{3/2}[\sigma, b](x)\big). \tag{2.3}$$

2.2. *Density estimate*

In the first part of the paper we prove an estimate for the density of the solution of (2.1). We consider the following assumptions on the coefficients:

A1 The "first order" weak Hörmander condition holds at the initial point of the diffusion:

$$\lambda_*\big(A(x_0)\big) > 0.$$

A2 $\sigma, b \in \mathcal{C}^5(\mathbb{R}^2)$ and there exists a constant $\rho > 0$ such that, $\forall x \in \mathbb{R}^2$:

$$\sum_{1 \le |\alpha| \le 5} \big|\partial_x^\alpha \sigma(x)\big| + \big|\partial_x^\alpha b(x)\big| \le \rho.$$

A3 There exist a neighborhood $V \subset \mathbb{R}^2$ of x_0 and a differentiable scalar function $\kappa_\sigma : V \to \mathbb{R}$ such that for all $x \in V$

$$\partial_\sigma \sigma(x) = \kappa_\sigma(x)\sigma(x). \tag{2.4}$$

We suppose that $\sum_{0 \le |\alpha| \le 1} |\partial_x^\alpha \kappa_\sigma(x_0)| \le \rho$. If $\sigma(x) = (\sigma_1(x), 0)$, the Asian option stochastic differential equation, this property holds true with $\kappa_\sigma = \partial_{x_1}\sigma_1$.

We prove the following Gaussian bound:

Theorem 4.5. *Suppose* **A1**, **A2**, **A3** *hold. Let* $(X_t)_{t\in[0,T]}$ *be the solution of* (2.1), *and for* $t \in [0, T]$, *let* $p_t(x_0, y)$ *be the density of* X_t *at* y. *Then there exist constants* L, C, δ^* *such that, for any* $r > 0$, *if* $0 < \delta \leq \delta^* \exp(-Lr^2)$, *setting* $\hat{x}_0 = x_0 + b(x_0)\delta$, *for* $|y - \hat{x}_0|_{A_\delta(x_0)} \leq r$

$$\frac{1}{C\delta^2} \exp\left(-C|y - \hat{x}_0|^2_{A_\delta(x_0)}\right) \leq p_\delta(x_0, y) \leq \frac{C}{\delta^2} \exp\left(-C^{-1}|y - \hat{x}_0|^2_{A_\delta(x_0)}\right). \tag{2.5}$$

This estimate is local around the point $\hat{x}_0 = x_0 + \delta b(x_0)$. Since we assume the weak Hörmander condition only at x_0, it is not possible to obtain global lower bounds. Indeed the "local" weak Hörmander condition ensures the existence of the density ([26]), but not its positivity. See Example 2.3 for more details on this aspect.

2.3. *Tube estimate*

We suppose $\sigma, b \in \mathcal{C}^5(\mathbb{R}^2)$. For $x \in \mathbb{R}^2$ define

$$n(x) = \sum_{k=0}^{5} \sum_{|\alpha|=k} \left|\partial_x^\alpha b(x)\right| + \left|\partial_x^\alpha \sigma(x)\right|,$$

and set $\lambda(x) = \lambda_*(A(x))$. We take now a control $\phi \in L^2[0, T]$, and the associated *skeleton path* solution of

$$x_t(\phi) = x_0 + \int_0^t \sigma\left(x_s(\phi)\right)\phi_s \, ds + \int_0^t b\left(x_s(\phi)\right) ds. \tag{2.6}$$

We denote by $L(\mu, h)$ the class of non-negative functions which have the property

$$f(t) \leq \mu f(s) \quad \text{for } |t - s| \leq h. \tag{2.7}$$

These functions have been used in [7], in the choice of an "elliptic evolution sequence," and in [6]. They allow us to control the variation of the quantities we are concerned with, along the skeleton path. In Section 5, when considering the tube estimate, we assume that:

H1 There exists a function $\lambda. : [0, T] \to (0, 1]$ such that

$$\lambda(y) \geq \lambda_t, \quad \forall \left|y - x_t(\phi)\right| < 1, \forall t \in [0, T].$$

H2 There exists a function $n. : [0, T] \to [1, \infty)$ such that

$$n(y) \leq n_t, \quad \forall \left|y - x_t(\phi)\right| < 1, \forall t \in [0, T].$$

H3 There exists a differentiable scalar function $\kappa_\sigma : \mathbb{R}^2 \to \mathbb{R}$ s.t.

$$\partial_\sigma \sigma(y) = \kappa_\sigma(y)\sigma(y), \quad \forall \left|y - x_t(\phi)\right| < 1, \forall t \in [0, T].$$

We suppose also that $|\kappa_\sigma(y)| \leq n(y)$, $|\nabla \kappa_\sigma(y)| \leq n(y)$.
H4 We suppose $|\phi.|^2, \lambda., n., R. \in L(\mu, h)$, for some $h > 0, \mu \geq 1$.

Notice that the above hypothesis do not involve global controls of our bounds on $\mathbb{R}^2$: they concern the behavior of the coefficients only along the tube, and may vary with $t \in [0, T]$. We stress that also $R.$, the radius of the tube, may vary with t, but that **H4** implies that $\inf_{t\in[0,T]} R_t > 0$. This means that we cannot "squeeze" the tube to 0 at any time.

For $K, q, K_*, q_* > 0$, for $0 \leq t \leq T$, we denote

$$H_t = K \left(\frac{\mu n_t}{\lambda_t}\right)^q,$$

$$R_t^*(\phi) = \exp\left(-K_*\left(\frac{\mu n_t}{\lambda_t}\right)^{q_*} \mu^{2q_*}\right)\left(h \wedge \inf_{0 \leq \delta \leq h}\left\{\delta \Big/ \int_t^{t+\delta} |\phi_s|^2 \, ds\right\}\right).$$

Theorem 5.1. *Let X_t be given by* (2.1), *$x_t(\phi)$ by* (2.6), *and suppose* **H1**, **H2**, **H3**, **H4**. *There exist positive constants K, q, K_*, q_* such that, for H_t and $R_t^*(\phi)$ as above, if $R_t \leq R_t^*(\phi)$ for $0 \leq t \leq T$,*

$$\exp\left(-\int_0^T H_t\left(\frac{1}{R_t} + |\phi_t|^2\right) dt\right) \leq \mathbb{P}\left(\sup_{t \leq T}|X_t - x_t(\phi)|_{A_{R_t}(x_t(\phi))} \leq 1\right)$$

$$\leq \exp\left(-\int_0^T e^{-H_t}\left(\frac{1}{R_t} + |\phi_t|^2\right) dt\right). \quad (2.8)$$

In general, even if R. does not satisfy $R_t \leq R_t^*(\phi)$ for $0 \leq t \leq T$, the lower bound holds in the form

$$\exp\left(-\int_0^T H_t\left(\frac{1}{h} + \frac{1}{R_t} + |\phi_t|^2 dt\right)\right) \leq \mathbb{P}\left(\sup_{t \leq T}|X_t - x_t(\phi)|_{A_{R_t}(x_t(\phi))} \leq 1\right).$$

Remark 2.1. Notice that estimate (2.8) holds for the controls ϕ which belong to the class $L(\mu, h)$, and μ is involved in the definition of H_t. In this sense, H_t depends on the "growth property" (2.7) of ϕ.

Both these theorems can also be stated in a variant of the Carathéodory distance which looks appropriate to our framework. Here we just briefly give the definition, for more details see Appendix 6.2. For $\phi = (\phi_s^1, \phi_s^2) \in L^2((0, 1), \mathbb{R}^2)$, set

$$\|\phi\|_{1,3}^2 = \int_0^1 |\phi_s^1|^2 \, ds + \left(\int_0^1 |\phi_s^2|^2 \, ds\right)^{\frac{1}{3}}$$

and define the class of controls

$$C_A(x, y) = \left\{\phi \in L^2\big((0, 1), \mathbb{R}^2\big) : dv_s = A(v_s)\phi_s \, ds, x = v_0, y = v_1\right\}$$

(recall $A = (\sigma, [\sigma, b])$). We set $d_c(x, y) = \inf\{\|\phi\|_{1,3} : \phi \in C_A(x, y)\}$. Just remark that $\|\phi\|_{1,3}$ accounts of the different speed in the $[\sigma, b]$ direction. We define also the following quasi-distance on $\Omega = \{x \in \mathbb{R}^2 : \lambda_*(A(x)) > 0\}$. For $x, y \in \Omega$,

$$d(x, y) < \sqrt{R} \quad \Leftrightarrow \quad |x - y|_{A_R(x)} < 1.$$

In Section 6.2 we prove that d and d_c are equivalent quasi-distances, and that Theorem 5.1 also holds in the following form:

Corollary 2.2. *Let X_t be given by* (2.1), *$x_t(\phi)$ by* (2.6), *and suppose* **H1**, **H2**, **H3**, **H4**. *There exist constants $C_T > 0$ and $R_* > 0$ depending on σ, b, μ, h such that, if $R_t \leq R_*$ for every $t \in [0, T]$, it holds*

$$\exp\left(-C_T \int_0^T \left(\frac{1}{R_t} + |\phi_t|^2\right) dt\right) \leq \mathbb{P}\big(d_c\big(X_t, x_t(\phi)\big) \leq \sqrt{R_t}, \forall t \in [0, T]\big)$$

$$\leq \exp\left(-\frac{1}{C_T} \int_0^T \left(\frac{1}{R_t} + |\phi_t|^2\right) dt\right).$$

2.4. *Examples and comments*

Example 2.3. As mentioned before, assuming the weak Hörmander condition only in the initial point x_0 ensures the existence of the density $p_\delta(x_0, y)$, but not its positivity. It does not even ensure that the density is positive locally around x_0. In [19], a multidimensional system under a weak Hörmander condition is studied, and a global lower bound for the density is provided, but the coefficients are hypoelliptic uniformly on the whole space where the diffusion propagates.

The fact that we have lower bounds for the density supposing only **A1** might appear contradictory. In fact, our estimates are local around $\hat{x}_0$, the translated initial condition, and there is no contradiction, as we see in the following classical example (see for instance (3.2.6) in [18]). Take

$$X_t^1 = 1 + W_t, \qquad X_t^2 = \int_0^t b_2(X_s^1)\,ds,$$

where

$$b_2(\xi) = \xi^2 1_{\{|\xi|\leq 1\}} + \bar{b}(\xi) 1_{\{|\xi|>1\}}$$

and $\bar{b}$ is chosen non-negative and such that **A2** is satisfied. Weak Hörmander holds at

$$X_0 = x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

but for any

$$y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$$

with $y^2 < 0$, $p_\delta(x_0, y) = 0$, $\forall \delta > 0$. We have

$$\sigma(x_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad b(x_0) = \begin{pmatrix} 0 \\ (x_0^1)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad [\sigma, b](x_0) = \begin{pmatrix} 0 \\ 2x_0^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

In fact, for any fixed $r > 0$, the set $\{y : |y - \hat{x}_0|_{A_\delta(x_0)} \leq r\}$, on which Theorem 4.5 holds, is included in $\mathbb{R} \times \mathbb{R}^+$, the support of X_δ. Indeed y satisfies

$$|y - \hat{x}_0|_{A_\delta(x_0)} = \sqrt{\delta^{-1}(y^1 - 1)^2 + \frac{1}{4}\delta^{-3}(y^2 - \delta)^2} \leq r.$$

For $y^2 < 0$,

$$|y - \hat{x}_0|_{A_\delta(x_0)} \leq r \quad \Rightarrow \quad \frac{1}{2}\delta^{-1/2} \leq r \quad \Rightarrow \quad \delta \geq \frac{1}{4r^2} \geq \delta^* \exp(-2Lr^2)$$

if $\delta^* \leq \frac{1}{4}$, and this is in contrast with condition $\delta \leq \delta^* \exp(-Lr^2)$ of Theorem 4.5.

Example 2.4. Looking at the geometric condition $\partial_\sigma \sigma(x) = \kappa_\sigma(x)\sigma(x)$ (see **A3** and **H3**) on the coefficients, it is easy to see that it holds if $\sigma = (\sigma_1, 0)$. We give here some other simple examples of diffusion coefficient σ satisfying this condition, but with $\sigma_2 \neq 0$:

- If $\sigma = (\sigma_1, \sigma_2)$, with $\sigma_2 = C\sigma_1$ for some constant C, we have that the condition is satisfied with $\kappa_\sigma = \partial_{x_1}\sigma_1 + \partial_{x_2}\sigma_2$. Remark that with $C = 0$ we recover the Asian option SDE.
- If, for α, β, γ constants,

$$\sigma(x_1, x_2) = \begin{pmatrix} \alpha x_1 + \beta \\ \alpha x_2 + \gamma \end{pmatrix}$$

the condition is satisfied with $\kappa_\sigma = \alpha$.
- If, for α, C constants,

$$\sigma(x_1, x_2) = C \begin{pmatrix} (x_1/x_2)^\alpha \\ (x_1/x_2)^{\alpha-1} \end{pmatrix}$$

the condition is satisfied with $\kappa_\sigma = 0$.

These examples show that our estimates are applicable to systems where the regimes of propagation are not completely separated, meaning that the one-dimensional Brownian Motion W can act on both the components of X (improving in this sense the results in [7] and [19]). On the other hand, the condition required on $\partial_\sigma \sigma$ has in some sense the same role of "separating" the different speeds of propagation. Indeed, we need this assumption to deal with a term of order t, which is hard to handle because of its fast speed of propagation, in comparison with the speed $t^{3/2}$ associated to $[\sigma, b]$.

For this reason, a multidimensional extension of these results looks quite hard to obtain, especially if we want to consider systems where W is multi-dimensional. This would produce terms of order t, associated to the brackets $[\sigma^i, \sigma^j]$. To handle these terms we could imagine a generalization of the condition on $\partial_\sigma \sigma$, but we believe that this is not an easy task. On the other hand, similar results on a multidimensional system, but of strong Hörmander type, are the subject of the recent work with Bally and Caramellino ([4,5]), and the techniques used in this paper are also applicable to the system studied in [19] (cf. [34]).

Example 2.5. Consider the geometric Asian option with time horizon T on the Black & Scholes model ([20]). This can be expressed as

$$dX_t^1 = \sigma \circ dW_t + r\,dt = \sigma\,dW_t + r\,dt; \qquad X_0^1 = \xi, \qquad dX_t^2 = \frac{X_t^1}{T}\,dt; \qquad X_0^2 = 0.$$

In this case, for $R > 0$ fixed constant,

$$A_R^{-1}(x) = \begin{pmatrix} \sigma R^{1/2} & 0 \\ 0 & \frac{\sigma}{T} R^{3/2} \end{pmatrix}^{-1} = \frac{1}{\sigma}\begin{pmatrix} \frac{1}{R^{1/2}} & 0 \\ 0 & \frac{T}{R^{3/2}} \end{pmatrix}$$

does not depend on x. We take as control $\phi_t = 0$ so $x_t(\phi) = (\xi + rt, \frac{\xi t + rt^2/2}{T})$. We have

$$\left|X_t - x_t(\phi)\right|_{A_R(x_t(\phi))} = \frac{1}{\sigma}\sqrt{\frac{|X_t^1 - (\xi + rt)|^2}{R} + \frac{T^2|X_t^2 - (\xi t + rt^2/2)/T|^2}{R^3}}$$

$$= \frac{1}{\sigma}\sqrt{\frac{|\sigma W_t|^2}{R} + \frac{|\sigma \int_0^t W_s\,ds|^2}{R^3}},$$

and (2.8) gives

$$e^{-C_1 T/R} \le \mathbb{P}\left(\sup_{t \le T}\left\{\frac{|W_t|^2}{R} + \frac{|\int_0^t W_s\,ds|^2}{R^3}\right\} \le 1\right) \le e^{-C_2 T/R}.$$

Example 2.6. Consider a system given by the Black and Scholes model for the price of an asset, and an (arithmetic average) Asian option on that asset with time horizon T (see for instance [15,20,40]). This is a model of real interest in mathematical finance. The associated SDE is

$$dX_t^1 = X_t^1(\sigma \circ dW_t + r\,dt); \qquad X_0^1 = \xi > 0, \qquad dX_t^2 = \frac{X_t^1}{T}\,dt; \qquad X_0^2 = 0,$$

and $X_t^1 = \xi e^{\sigma W_t + rt}$. The stochastic integral is in Stratonovich form so to recover the classical formulation $r \to r + \sigma^2/2$. In this case, for $R > 0$ fixed constant,

$$A_R^{-1}(x) = \begin{pmatrix} \sigma x^1 R^{1/2} & 0 \\ 0 & \frac{\sigma x^1}{T} R^{3/2} \end{pmatrix}^{-1} = \frac{1}{\sigma x^1}\begin{pmatrix} \frac{1}{R^{1/2}} & 0 \\ 0 & \frac{T}{R^{3/2}} \end{pmatrix}.$$

Remark that this matrix is invertible for $x^1 \ne 0$. Since we are working under local non-degeneracy assumptions, our tube estimates hold for any initial condition $\xi > 0$, provided that $R > 0$ is small enough, since this implies the

positivity of the first component of the skeleton path at any time $t > 0$. On the other hand, results requiring "global" non degeneracy, such as the density estimates in [19], do not hold for this model. We take as control $\phi_t = 0$ so $x_t(\phi) = \xi(e^{rt}, \frac{1}{T} \int_0^t e^{rs}\,ds)$. We have

$$
\begin{aligned}
\left| X_t - x_t(\phi) \right|_{A_R(x_t(\phi))} &= \frac{1}{\sigma \xi e^{rt}} \sqrt{\frac{|X_t^1 - \xi e^{rt}|^2}{R} + \frac{T^2 |X_t^2 - \frac{\xi}{T}\int_0^t e^{rs}\,ds|^2}{R^3}} \\
&= \frac{1}{\sigma \xi e^{rt}} \sqrt{\frac{\xi^2 |e^{rt}(e^{\sigma W_t} - 1)|^2}{R} + \frac{\xi^2 |\int_0^t e^{rs + \sigma W_s}\,ds - \int_0^t e^{rs}\,ds|^2}{R^3}} \\
&= \frac{1}{\sigma e^{rt}} \sqrt{\frac{|e^{rt}(e^{\sigma W_t} - 1)|^2}{R} + \frac{|\int_0^t e^{rs}(e^{\sigma W_s} - 1)\,ds|^2}{R^3}}
\end{aligned}
$$

and (2.8) gives

$$
e^{-C_1 T/R} \le \mathbb{P}\left(\sup_{t \le T}\left\{\frac{|e^{\sigma W_t} - 1|^2}{R\sigma^2} + \frac{|\int_0^t e^{r(s-t)}(e^{\sigma W_s} - 1)\,ds|^2}{R^3 \sigma^2}\right\} \le 1 \right) \le e^{-C_2 T/R}.
$$

3. Malliavin calculus and density estimates

3.1. *Notations*

Our main reference for this section is [32]. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Brownian motion $W = (W_t^1, \ldots, W_t^d)_{t \ge 0}$. We denote by $\mathbb{D}^{k,p}$ the space of the random variables which are k times differentiable in the Malliavin sense in L^p, and $\mathbb{D}^{k,\infty} = \bigcap_{p=1}^\infty \mathbb{D}^{k,p}$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ we denote by $D^\alpha F$ the Malliavin derivative of F corresponding to the multi-index α.

$\mathbb{D}^{k,p}$ is the closure of the space of the simple functionals with respect to the Malliavin Sobolev norm

$$
\|F\|_{k,p} = \left[\mathbb{E}|F|^p + \sum_{j=1}^{k} \mathbb{E}|D^{(j)}F|^p \right]^{\frac{1}{p}},
$$

where

$$
\left| D^{(j)}F \right| = \left(\sum_{|\alpha| = j} \int_{[0,T]^j} \left| D_{s_1,\ldots,s_j}^\alpha F \right|^2 ds_1 \cdots ds_j \right)^{1/2}.
$$

For the special case $j = 1$, we use the standard notation

$$
|DF| = \left| D^{(1)}F \right| = \left(\sum_{m=1}^{d} \int_{[0,T]} \left| D_s^m F \right|^2 ds \right)^{1/2}.
$$

Hereafter, for $j \in \mathbb{N} \setminus \{0\}$, we write $D^{(j)}$ for the "derivative of order j" and D^j for the "derivative with respect to W^j."

As usual, we also denote by L the Ornstein–Uhlenbeck operator, i.e. $L = -\delta \circ D$, where δ is the adjoint operator of D.

For a random vector $F = (F_1, \ldots, F_n)$ in the domain of D, we define its *Malliavin covariance matrix* as follows:

$$
\gamma_F^{i,j} = \langle DF_i, DF_j \rangle_{\mathcal{H}} = \sum_{k=1}^{d} \int_0^T D_s^k F_i \times D_s^k F_j\,ds.
$$

We say that F is *non-degenerate* if its Malliavin covariance matrix is invertible and

$$\mathbb{E}\big(|\det \gamma_F|^{-p}\big) < \infty, \quad \forall p \in \mathbb{N}. \tag{3.1}$$

We denote by $\hat{\gamma}_F$ the inverse of γ_F.

3.2. *Localization*

The following notion of localization is introduced in [2]. Consider a random variable $U \in [0, 1]$ and denote

$$d\mathbb{P}_U = U \, d\mathbb{P}.$$

$\mathbb{P}_U$ is a non-negative measure (not a probability measure, in general). We also set $\mathbb{E}_U$ the expectation (integral) w.r.t. $\mathbb{P}_U$, and denote

$$\|F\|_{p,U}^p = \mathbb{E}_U\big(|F|^p\big) = \mathbb{E}\big(|F|^p U\big), \qquad \|F\|_{k,p,U}^p = \|F\|_{p,U}^p + \sum_{j=1}^k \mathbb{E}_U\big(|D^{(j)} F|^p\big).$$

We assume that $U \in \mathbb{D}^{2,\infty}$ and for every $p \geq 1$

$$m_U(p) := 1 + \big(\mathbb{E}_U |D \ln U|^p\big)^{1/p} + \big(\mathbb{E}_U \big|D^{(2)} \ln U\big|^p\big)^{1/p} < \infty$$

(notice that our definition of m_U is slightly different from the definition in [2]: we are taking p-norms instead of moments, and we also consider $D^{(2)}$, whereas in [2] only the first order derivative D appears in m_U). For $F = (F^1, \ldots, F^n)$ such that $F^1, \ldots, F^n \in \mathbb{D}^{2,\infty}$ and $V \in \mathbb{D}^{1,\infty}$, for any localization function U we introduce the localized Malliavin weights

$$H_{i,U}(F, V) = \sum_{j=1}^n V \hat{\gamma}_F^{i,j} L F^j - \big\langle D\big(V \hat{\gamma}_F^{i,j}\big), D F^j\big\rangle - V \hat{\gamma}_F^{i,j} \big\langle D \ln U, D F^j\big\rangle$$

and the vector

$$H_U(F, V) = \big(H_{i,U}(F, V)\big)_{i=1,\ldots,n}.$$

The following *representation formula for the localized density* has been proved in [1].

Theorem 3.1. *Let U be a localizing r.v. such that under $\mathbb{P}_U$ (3.1) holds, i.e.*

$$\mathbb{E}_U\big[|\det \gamma_F|^{-p}\big] < \infty, \quad \forall p \in \mathbb{N}.$$

Then, under $\mathbb{P}_U$ the law of F is absolutely continuous and has a continuous density $p_{F,U}$ which may be represented as

$$p_{F,U}(x) = \sum_{i=1}^n \mathbb{E}_U\big[\partial_i \mathcal{Q}_n(F - x) H_{i,U}(F, 1)\big], \tag{3.2}$$

where $\mathcal{Q}_n$ denotes the Poisson kernel on $\mathbb{R}^n$, i.e. the fundamental solution of the Laplace operator $\Delta \mathcal{Q}_n = \delta_0$. This is given by

$$\mathcal{Q}_1(x) = \max(x, 0); \qquad \mathcal{Q}_2(x) = \mathcal{A}_2^{-1} \ln |x|; \qquad \mathcal{Q}_n(x) = -\mathcal{A}_n^{-1} |x|^{2-n}, \quad n > 2,$$

where $\mathcal{A}_n$ is the area of the unit sphere in $\mathbb{R}^n$.

This is a localized version of the formula

$$p_F(x) = \sum_{i=1}^{n} \mathbb{E}\big[\partial_i \mathcal{Q}_n (F - x) H_i(F, 1)\big],$$

where the Malliavin weights are given by

$$H(F, G) = G\hat{\gamma}_F \times LF - \langle D(\hat{\gamma}_F G), DF\rangle$$

for which we refer to [29]. We recall the following relation between localized weights, which can be easily checked (a similar formula is proved in [2]). For any U, V localizing r.v.s, $F, G \in \mathbb{D}^{2,\infty}$

$$H_U(F, VG) = V H_{UV}(F, G). \tag{3.3}$$

Example 3.2. The following example of localizing function is taken from [2]. Consider the function depending on a parameter $a > 0$:

$$\psi_a(x) = 1_{|x|\leq a} + \exp\left(1 - \frac{a^2}{a^2 - (x - a)^2}\right) 1_{a<|x|<2a},$$

which is a smooth version of the indicator function $1_{\{|x|\leq a\}}$. For $\Theta_i \in \mathbb{D}^{1,\infty}$, $i = 1, \dots, n$, and $r > 0$, we define the localization r.v.

$$U_r = \prod_{i=1}^{n} \psi_r(\Theta_i). \tag{3.4}$$

For this choice of U_r we have that for any $p \geq 1$,

$$m_{U_r}(p) \leq C_p\left(1 + \frac{\|\Theta\|_{2,p}^2}{r^2}\right) \tag{3.5}$$

and

$$\|1 - U_r\|_{1,p} \leq C\left(1 + \frac{\|\Theta\|_{1,2p}}{r}\right) \sum_{i=1}^{n} \mathbb{P}\big(|\Theta_i| \geq r\big)^{1/2p}. \tag{3.6}$$

The proof of (3.5) follows from inequalities

$$\sup_x \big|(\ln \psi_a)'(x)\big|^p \psi_a(x) \leq \frac{4^p}{a^p} \sup_{t\geq 0}\big(t^{2p} e^{1-t}\big) \leq \frac{C_p}{a^p} < \infty \tag{3.7}$$

and

$$\sup_x \big|(\ln \psi_a)''(x)\big|^p \psi_a(x) \leq \frac{8^p}{a^{2p}} \sup_{t\geq 0}\big(t^{3p} e^{1-t}\big) + \frac{2^p}{a^{2p}} \sup_{t\geq 0}\big(t^{2p} e^{1-t}\big) \leq \frac{C_p}{a^{2p}} < \infty. \tag{3.8}$$

Indeed

$$U_r |D \ln U_r|^p = \prod_{i=1}^{n} \psi_r(\Theta_i) \left|\sum_{i=1}^{n}(\ln \psi_r)'(\Theta_i) D\Theta_i\right|^p \leq \prod_{i=1}^{n} \psi_r(\Theta_i)\left(\sum_{i=1}^{n}\big|(\ln \psi_r)'(\Theta_i)\big|^2\right)^{p/2}\left(\sum_{i=1}^{n}|D\Theta_i|^2\right)^{p/2}$$

$$\leq c_p\left(\sum_{i=1}^{n}\big|(\ln \psi_r)'(\Theta_i)\big|^p \psi_r(\Theta_i)\right)|D\Theta|^p.$$

Here we apply (3.7), and find

$$U_r |D \ln U_r|^p \le C_p \frac{|D\Theta|^p}{r^p}. \tag{3.9}$$

This implies $(\mathbb{E}_{U_r} |D \ln U_r|^p)^{1/p} \le C_p \frac{\|\Theta\|_{1,p}}{r}$. We also have, using (3.7) and (3.8),

$$\begin{aligned}
U_r |D^{(2)} \ln U_r|^p &= \prod_{i=1}^n \psi_r(\Theta_i) \left| D\left(\sum_{i=1}^n (\ln \psi_r)'(\Theta_i) D\Theta_i \right) \right|^p \\
&\le C_p \prod_{i=1}^n \psi_r(\Theta_i) \left[\left| \sum_{i=1}^n (\ln \psi_r)''(\Theta_i)(D\Theta_i)^2 \right|^p + \left| \sum_{i=1}^n (\ln \psi_r)'(\Theta_i) D^{(2)}\Theta_i \right|^p \right] \\
&\le C_p \left(\sum_{i=1}^n |(\ln \psi_r)''(\Theta_i)|^p \psi_r(\Theta_i) \right) |D\Theta|^{2p} + C_p \left(\sum_{i=1}^n |(\ln \psi_r)'(\Theta_i)|^p \psi_r(\Theta_i) \right) |D^{(2)}\Theta|^p \\
&\le C_p \left(\frac{|D\Theta|^{2p}}{r^{2p}} + \frac{|D^{(2)}\Theta|^p}{r^p} \right)
\end{aligned}$$

and so

$$\left(\mathbb{E}_{U_r} |D^{(2)} \ln U_r|^p \right)^{1/p} \le C_p \left(\left(\frac{\|\Theta\|_{1,p}}{r} \right)^2 + \frac{\|\Theta\|_{2,p}}{r} \right).$$

This proves (3.5) Moreover, since $D_s U_r = 0$ on $\bigcap_i \{|\Theta_i| < r\} = (\bigcup_i \{|\Theta_i| \ge r\})^c$,

$$D_s(1 - U_r) = -1_{\{\bigcup_i \{|\Theta_i| \ge r\}\}} D_s U_r$$

and from Hölder inequality

$$\mathbb{E} |D_s(1 - U_r)|^p \le (\mathbb{E} 1_{\{\bigcup_i \{|\Theta_i| \ge r\}\}})^{1/2} (\mathbb{E} |D_s U_r|^{2p})^{1/2}.$$

We control the first factor with the tail estimate

$$(\mathbb{E} 1_{\{\bigcup_i \{|\Theta_i| \ge r\}\}})^{1/2} \le C \sum_{i=1}^n \mathbb{P}(|\Theta_i| \ge r)^{1/2},$$

and we also have

$$|D_s U_r|^{2p} \le U_r |D \ln U_r|^{2p},$$

and from (3.9)

$$\left(\mathbb{E} |D_s(1 - U_r)|^p \right)^{1/p} \le C_p \frac{\|\Theta\|_{1,2p}}{r} \sum_{i=1}^n \mathbb{P}(|\Theta_i| \ge r)^{1/2p}.$$

Moreover

$$\mathbb{E} |1 - U_r|^p \le \mathbb{P}(1 - U_r > 0) \le \mathbb{P}(|\Theta_i| > r, \exists i = 1, \ldots, n) \le \sum_{i=1}^n \mathbb{P}(|\Theta_i| > r),$$

so (3.6) is proved.

3.3. *The distance between two local densities*

We discuss some techniques, based on Malliavin calculus, for estimating the density of a random variable. These ideas are based on the recent work of Bally and Caramellino ([2,3]).

In what follows for a given matrix A we consider its Frobenius norm, given as

$$\|A\|_{\mathrm{Fr}} = \sqrt{\sum_{i,j} |A_{i,j}^2|} = \sqrt{\mathrm{Tr}(A^T A)}.$$

We will employ the fact that the Frobenius norm is sub-multiplicative. Take a square $d \times d$ matrix γ, symmetric and positive definite. Recall that we denote by $\lambda^*(\gamma)$ and $\lambda_*(\gamma)$ the largest and the smallest singular values of γ, which in this case coincide with the largest and smallest eigenvalues. From the equivalence between Frobenius and spectral norm we have

$$\lambda^*(\gamma) \leq \|\gamma\|_{\mathrm{Fr}} \leq \sqrt{d}\lambda^*(\gamma).$$

Denoting $\hat{\gamma} = \gamma^{-1}$, it holds $\lambda^*(\hat{\gamma}) = 1/\lambda_*(\gamma)$. So

$$\frac{1}{\lambda_*(\gamma)} \leq \|\hat{\gamma}\|_{\mathrm{Fr}} \leq \frac{\sqrt{d}}{\lambda_*(\gamma)}.$$

For two time dependent matrices A_s, B_s, we have the following "Cauchy–Schwarz" inequality:

$$\left\| \int A_s B_s \, ds \right\|_{\mathrm{Fr}}^2 \leq \int \|A_s\|_{\mathrm{Fr}}^2 \, ds \int \|B_s\|_{\mathrm{Fr}}^2 \, ds.$$

In particular, if $B_s = v_s$ is a vector,

$$\left| \int A_s v_s \, ds \right|^2 \leq \int \|A_s\|_{\mathrm{Fr}}^2 \, ds \int |v_s|^2 \, ds.$$

We fix some notation. Let W be a Brownian Motion in $\mathbb{R}^d$. For two random variables $F = (F_1, \ldots, F_n)$, $G = (G_1, \ldots, G_n)$ in $\mathbb{D}^{3,\infty}$ and a localizing r.v. U, we denote

$$\Gamma_{F,U}(p) = 1 + \left(\mathbb{E}_U \lambda_*(\gamma_F)^{-p}\right)^{1/p},$$

$$\Gamma_{F,G,U}(p) = 1 + \sup_{0 \leq \varepsilon \leq 1} \left(\mathbb{E}_U \lambda_*(\gamma_{G+\varepsilon(F-G)})^{-p}\right)^{1/p},$$

$$n_{F,G,U}(p) = 1 + \|F\|_{3,p,U} + \|G\|_{3,p,U} + \|LF\|_{1,p,U} + \|LG\|_{1,p,U},$$

$$\Delta_2(F,G) = \left|D(F-G)\right| + \left|D^{(2)}(F-G)\right| + \left|L(F-G)\right|.$$

We also write $n_{F,U}(p)$ for $n_{F,0,U}(p)$. Moreover, in all the above notations, when $U = 1$, i.e. the localization is "trivial," we omit it in the notation. Remark that notations $n_{F,U}$ and $n_{F,G}$, although similar, denote different things. Since we are differentiating with respect to a Brownian Motion, as a direct consequence of Meyer's inequality (see for instance [32]), we have

$$n_{F,G,U}(p) \leq 1 + C\left(\|F\|_{3,p} + \|G\|_{3,p}\right)$$

for every F, G, U.

We now give the main result of this section, comparing the densities of the laws of two random variables under $\mathbb{P}_U$.

Theorem 3.3. *Let U be a localizing r.v. with $m_U(32n) < \infty$. Let $F = (F_1, \ldots, F_n)$, $G = (G_1, \ldots, G_n) \in \mathbb{D}^{3,32n}$. Suppose $\Gamma_{G,U}(p) < \infty$ and $\Gamma_{F,U}(p) < \infty$ for any $p > 1$. Then there exists a constant C_1 such that*

$$p_{G,U}(y) - C_1 \|\Delta_2(F, G)\|_{32n,U} \le p_{F,U}(y) \le p_F(y).$$

If, in addition, $\Gamma_F(32n) < \infty$, there exists a constant C_2 such that

$$p_F(y) \le p_{G,U}(y) + C_2 (\|\Delta_2(F, G)\|_{32n,U} + \|1 - U\|_{1,14n}).$$

Remark 3.4. We can take

$$C_1 = C [m_U(32n) \Gamma_{G,U}(32n) n_{F,G,U}(32n)]^{24n^2},$$

$$C_2 = C [m_U(32n) \Gamma_F(32n) n_{F,G}(32n)]^{24n^2},$$

where C is a constant depending only on the dimension n.

The lower bound for $p_{F,U}$ is a version of Proposition 2.5. in [2], where here we have specified as possible choice for the exponent $p = 32n$. Moreover, we find here that in m_U and $n_{F,G,U}$ we need to consider one more order of derivatives with respect to [2]. Similar estimates can be found also in [3].

Before proceeding with the proof we need some preliminary results. We start with an estimate for the localized Malliavin weights and for the difference of weights:

Lemma 3.5. *Let U be a localizing r.v., $V \in \mathbb{D}^{1,\infty}$, $F = (F_1, \ldots, F_n) \in \mathbb{D}^{3,\infty}$. Suppose $\Gamma_{F,U}(q) < \infty$ for any $q > 1$. For fixed $p \ge 1$, $p_i \ge 1$, $i = 1, \ldots, 4$, with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{2}{p_3} + \frac{3}{p_4}$, there exists a constant C depending only on p and the dimension n such that*

$$\|H_U(F, V)\|_{p,U} \le C \|V\|_{1,p_1} m_U(p_2) \Gamma_{F,U}(p_3)^2 n_{F,U}(p_4)^3. \tag{3.10}$$

Moreover if $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{3}{p_3} + \frac{5}{p_4}$ and $V \in \mathbb{D}^{2,\infty}$,

$$\|H_U(F, V)\|_{1,p,U} \le C \|V\|_{2,p_1} m_U(p_2) \Gamma_{F,U}(p_3)^3 n_{F,U}(p_4)^5. \tag{3.11}$$

Let now $G = (G_1, \ldots, G_n) \in \mathbb{D}^{3,\infty}$. If $\Gamma_{F,G,U}(q) < \infty$ for any $q > 1$, for fixed $p_i \ge 1$, $i = 1, \ldots, 5$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{3}{p_3} + \frac{4}{p_4} + \frac{1}{p_5}$, it also holds

$$\|H_U(F, V) - H_U(G, V)\|_{p,U} \le C \|V\|_{1,p_1} m_U(p_2) \Gamma_{F,G,U}(p_3)^3 n_{F,G,U}(p_4)^4 \|\Delta_2(F, G)\|_{p_5,U}. \tag{3.12}$$

Proof. Consider the weight:

$$H_U(F, V) = V [\hat{\gamma}_F \times LF - \langle D\hat{\gamma}_F, DF \rangle] - \langle \hat{\gamma}_F(DV + V D \ln U), DF \rangle. \tag{3.13}$$

Recall that $D^{(k)}$ means "derivative of order k" and D^k means "derivative with respect to W^k." We first consider $D\gamma_F$ and have the following estimate:

$$\sum_{l=1}^d \int \|D_s^l \gamma_F\|_{\text{Fr}}^2 ds$$

$$= \sum_{l=1}^d \int \left\| \left(\sum_{k=1}^d \int_0^t D_s^l D_u^k F_i \times D_u^k F_j + D_u^k F_i \times D_s^l D_u^k F_j \, du \right)_{i,j} \right\|_{\text{Fr}}^2 ds$$

$$\le 4 |D^{(2)} F|^2 |DF|^2.$$

We now consider $D\hat{\gamma}_F$. From the chain rule and the derivative of the inversion of matrices,

$$D^k\hat{\gamma}_F = -\hat{\gamma}_F\big(D^k\gamma_F\big)\hat{\gamma}_F. \tag{3.14}$$

So, applying also the previous estimate

$$\sum_{k=1}^{d}\int\big\|D_s^k\hat{\gamma}_F\big\|_{\text{Fr}}^2\,ds \le \|\hat{\gamma}_F\|_{\text{Fr}}^4\sum_{k=1}^{d}\int\big\|D_s^k\gamma_F\big\|_{\text{Fr}}^2\,ds \le 4\|\hat{\gamma}_F\|_{\text{Fr}}^4|DF|^2\big|D^{(2)}F\big|^2.$$

From (3.13) we see that

$$\big|H_U(F,V)\big| \le |V|\Bigg(\|\hat{\gamma}_F\|_{\text{Fr}}|LF| + \bigg(\sum_{k=1}^{d}\int\big\|D^k\hat{\gamma}_F\big\|_{\text{Fr}}^2\,ds\bigg)^{1/2}|DF|\Bigg)$$

$$+ \|\hat{\gamma}_F\|_{\text{Fr}}\big(|DV| + |V||D\ln U|\big)|DF|$$

$$\le C\big(|V| + |DV|\big)\big(1 + |D\ln U|\big)\big(|DF| + |LF|\big)\Bigg(\|\hat{\gamma}_F\|_{\text{Fr}} + \bigg(\sum_{k=1}^{d}\int\big\|D^k\hat{\gamma}_F\big\|_{\text{Fr}}^2\,ds\bigg)^{1/2}\Bigg)$$

$$\le C\big(|V| + |DV|\big)\big(1 + |D\ln U|\big)\big(1 + |DF| + \big|D^{(2)}F\big| + |LF|\big)^3\big(1 + \|\hat{\gamma}_F\|_{\text{Fr}}\big)^2.$$

Now

$$\big\|H_U(F,V)\big\|_{p,U} \le C\|V\|_{1,p_1}m_U(p_2)\Gamma_{F,U}(p_3)^2 n_{F,U}(p_4)^3,$$

for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{2}{p_3} + \frac{3}{p_4}$, follows easily applying Hölder and Minkowski inequalities for L_p norms.

The estimate of $\|H_U(F,V)\|_{1,p,U}$ follows using very similar techniques. The part giving the "main" contribution is $D^{(2)}\hat{\gamma}_F$, for which, iterating (3.14), it is not difficult to see

$$\big|D^{(2)}\hat{\gamma}_F\big| \le C\big(|DF| + \cdots + \big|D^{(3)}F\big|\big)^4\|\hat{\gamma}_F\|_{\text{Fr}}^3.$$

This term is also multiplied by $|DF|$, so we have the estimate of the term giving the main contribution. We leave out the similar estimate of the other terms.

When considering the difference $\|H_U(F,V) - H_U(G,V)\|_{p,U}$, we use similar arguments and the following property of norms: $|ab - cd| \le |a-c||b| + |c||b-d|$. As before the main contribution comes from $D(\hat{\gamma}_F - \hat{\gamma}_G)$, so we consider this and leave out the estimates of the other terms. We remark that

$$\hat{\gamma}_F - \hat{\gamma}_G = \hat{\gamma}_F(\gamma_G - \gamma_F)\hat{\gamma}_G$$

and differentiate this product, finding

$$\big|D(\hat{\gamma}_F - \hat{\gamma}_G)\big| \le C\big(1 + \|\hat{\gamma}_F\|_{\text{Fr}} \vee \|\hat{\gamma}_G\|_{\text{Fr}}\big)^3\big(1 + |D\gamma_F| \vee |D\gamma_G|\big)\big(|\gamma_F - \gamma_G| + \big|D(\gamma_F - \gamma_G)\big|\big),$$

where

$$1 + |D\gamma_F| \vee |D\gamma_G| \le C\Bigg(1 + \sum_{i=1}^{2}\big|D^{(i)}F\big| \vee \big|D^{(i)}G\big|\Bigg)^2.$$

We have

$$|\gamma_F - \gamma_G| \le C\big|D(F-G)\big|\big|D(F+G)\big|$$

and

$$\left| D(\gamma_F - \gamma_G) \right| \le C \left(\left| D(F - G) \right| + \left| D^{(2)}(F - G) \right| \right) \left(\left| D(F + G) \right| + \left| D^{(2)}(F + G) \right| \right).$$

Multiplying with $|DF|$, and applying Hölder inequality, we prove the statement. □

Lemma 3.6. *Let U be a localizing r.v., $F = (F_1, \ldots, F_n)$, $G = (G_1, \ldots, G_n) \in \mathbb{D}^{3,\infty}$. If $\Gamma_{F,G,U}(q) < \infty$ for any $q > 1$, there exists a constant C depending only on the dimension n such that*

$$\left| p_{F,U}(y) - p_{G,U}(y) \right| \le C \left[m_U(32n) \Gamma_{F,G,U}(32n) n_{F,G,U}(32n) \right]^{12n^2} \left\| \Delta_2(F,G) \right\|_{32n,U}.$$

Proof. We write the densities using (3.2):

$$
\begin{aligned}
p_{F,U}(y) - p_{G,U}(y) &= E_U \left(\left\langle \nabla \mathcal{Q}_n(F - y), H_U(F, 1) \right\rangle - \left\langle \nabla \mathcal{Q}_n(G - y), H_U(G, 1) \right\rangle \right) \\
&= E_U \left\langle \nabla \mathcal{Q}_n(F - y), H_U(G, 1) - H_U(F, 1) \right\rangle \\
&\quad + E_U \left\langle \nabla \mathcal{Q}_n(G - y) - \nabla \mathcal{Q}_n(F - y), H_U(G, 1) \right\rangle \\
&= I + J.
\end{aligned}
$$

We recall the following inequality proved in [1]. For $p > n$, with $p' = p/(p-1)$,

$$\left(\mathbb{E}_U \left| \nabla \mathcal{Q}_n(F - y) \right|^{p'} \right)^{1/p'} \le C_{p,n} \left(\mathbb{E}_U \left| H_U(F, 1) \right|^p \right)^{p \frac{n-1}{p-n}}.$$

In particular, for $p = 2n$ (fixed from now on), applying (3.10) with $k = 0$, $p_1 = p_2 = p_3 = p_4 = 7p = 14n$,

$$
\begin{aligned}
&\left(\mathbb{E}_U \left| \nabla \mathcal{Q}_n(F - y) \right|^{2n/(2n-1)} \right)^{(2n-1)/(2n)} \\
&\quad \le C \left(\mathbb{E}_U \left| H_U(F, 1) \right|^{2n} \right)^{2(n-1)} \\
&\quad \le C \left[m_U(14n) \Gamma_{F,U}(14n)^2 n_{F,U}(14n)^3 \right]^{4n(n-1)}.
\end{aligned}
\tag{3.15}
$$

We use now Lemma 3.5 to estimate I and J. From Hölder inequality

$$
\begin{aligned}
I &= \mathbb{E}_U \left| \left\langle \nabla \mathcal{Q}_n(F - y), H_U(G, 1) - H_U(F, 1) \right\rangle \right| \\
&\le \left\| \nabla \mathcal{Q}_n(F - y) \right\|_{\frac{2n}{2n-1}, U} \left\| H_U(G, 1) - H_U(F, 1) \right\|_{2n, U}
\end{aligned}
$$

and we have just provided the estimate for the first factor. For the second we apply (3.12) with $p_1 = p_2 = p_3 = p_4 = p_5 = 20n$

$$
\begin{aligned}
&\left\| H_U(F, 1) - H_U(G, 1) \right\|_{2n, U} \\
&\quad \le C m_U(20n) \Gamma_{F,G,U}(20n)^3 n_{F,G,U}(20n)^4 \left\| \Delta_2(F, G) \right\|_{20n, U}.
\end{aligned}
$$

We now study J. For $\lambda \in [0, 1]$ we denote $F_\lambda = G + \lambda(F - G)$. With a Taylor expansion, applying Hölder inequality, integrating again by parts and denoting $V_{j,k} = H_{j,U}(G, 1)(F - G)_k$.

$$
\begin{aligned}
&\mathbb{E}_U \left\langle \nabla \mathcal{Q}_n(F - y) - \nabla \mathcal{Q}_n(G - y), H_U(G, 1) \right\rangle \\
&\quad = \sum_{k,j=1}^d \int_0^1 \mathbb{E}_U \left(\partial_k \partial_j \mathcal{Q}_n(F_\lambda - y) H_{j,U}(G, 1)(F - G)_k \right) d\lambda \\
&\quad = \sum_{k,j=1}^d \int_0^1 \mathbb{E}_U \left(\partial_j \mathcal{Q}_n(F_\lambda - y) H_{k,U} \left(F_\lambda, H_{j,U}(G, 1)(F - G)_k \right) \right) d\lambda
\end{aligned}
$$

$$= \sum_{k,j=1}^{d} \int_{0}^{1} \mathbb{E}_U \big(\partial_j \mathcal{Q}_n(F_\lambda - y) H_{k,U}(F_\lambda, V_{j,k}) \big) \, d\lambda.$$

Now, applying first (3.10) and then (3.11), with some computations in the same fashion as before, it is possible to show

$$\big\| \big(H_{k,U}(F_\lambda, V_{j,k}) \big)_{j=1,\ldots,n} \big\|_{2n,U}$$
$$\leq C m_U (32n)^2 \Gamma_{F,G,U}(32n)^5 n_{F,G,U}(32n)^8 \|F - G\|_{1,32n,U}.$$

From (3.15) and Hölder as before,

$$|J| \leq C \big[m_U(32n) \Gamma_{F,G,U}(32n)^2 n_{F,G,U}(32n)^3 \big]^{4n^2} \|F - G\|_{1,32n,U}.$$

The statement follows. $\qquad\qquad\square$

Lemma 3.7. *Let U be a localizing r.v., $F = (F_1, \ldots, F_n)$, $G = (G_1, \ldots, G_n) \in \mathbb{D}^{3,\infty}$. If $\Gamma_{F,U}(q) < \infty$, $\Gamma_{G,U}(q) < \infty$ for any $q > 1$, there exists a constant C depending only on the dimension n such that*

$$\big| p_{F,U}(y) - p_{G,U}(y) \big|$$
$$\leq C \big[m_U(32n)(\Gamma_{F,U} \vee \Gamma_{G,U})(32n) n_{F,G,U}(32n) \big]^{24n^2} \big\| \Delta_2(F,G) \big\|_{32n,U}.$$

Proof. We denote in this proof $M = \hat{\gamma}_G(\gamma_{F_\lambda} - \gamma_G)$, and define, as in (3.4),

$$V = \prod_{1 \leq i,j \leq n} \psi_{1/(8n^2)}(M_{i,j}). \tag{3.16}$$

We have from Lemma 3.6 that if $\Gamma_{F,G,UV}(q)$ is finite for $q > 0$

$$\big| p_{F,UV}(y) - p_{G,UV}(y) \big|$$
$$\leq C \big[m_{UV}(32n) \Gamma_{F,G,UV}(32n) n_{F,G,UV}(32n) \big]^{12n^2} \big\| \Delta_2(F,G) \big\|_{32n,UV}. \tag{3.17}$$

Remark

$$\hat{\gamma}_G - \hat{\gamma}_{F_\lambda} = \hat{\gamma}_G(\gamma_{F_\lambda} - \gamma_G)\hat{\gamma}_{F_\lambda},$$

so

$$\|\hat{\gamma}_{F_\lambda} - \hat{\gamma}_G\|_{\mathrm{Fr}} \leq \big\| \hat{\gamma}_G(\gamma_{F_\lambda} - \gamma_G) \big\|_{\mathrm{Fr}} \|\hat{\gamma}_{F_\lambda}\|_{\mathrm{Fr}}.$$

On $V \neq 0$ we have $\|\hat{\gamma}_G(\gamma_{F_\lambda} - \gamma_G)\|_{\mathrm{Fr}} \leq 1/2$, because of definition (3.16), so

$$\|\hat{\gamma}_{F_\lambda}\|_{\mathrm{Fr}} \leq 2\|\hat{\gamma}_G\|_{\mathrm{Fr}}$$

and therefore

$$\Gamma_{F,G,UV}(32n) \leq 2\Gamma_{G,UV}(32n) \leq 2\Gamma_{G,U}(32n). \tag{3.18}$$

Now, using (3.3) with $G = 1$,

$$p_{F,U(1-V)}(y) = \mathbb{E}_{U(1-V)} \big[\nabla Q(F - y), H_{U(1-V)}(F, 1) \big]$$
$$= \mathbb{E}_U \big[\nabla Q(F - y), (1 - V) H_{U(1-V)}(F, 1) \big]$$
$$= \mathbb{E}_U \big[\nabla Q(F - y), H_U(F, 1 - V) \big]$$

which implies, using as before (3.10) and (3.15)

$$
\begin{aligned}
p_{F,U(1-V)}(y) &= E_{U(1-V)}\langle \nabla \mathcal{Q}_d(F-y), H_U(F, 1-V)\rangle \\
&\le C\big[m_U(14n)\Gamma_{F,U}(14n)^2 n_{F,U}(14n)^3\big]^{4n(n-1)} \big\|H_U(F, 1-V)\big\|_{2n,U} \\
&\le C\big[m_U(24n)\Gamma_{F,U}(24n)^2 n_{F,U}(24n)^3\big]^{8n(n-1)+1} \|1-V\|_{1,4n,U}
\end{aligned}
$$

and, using (3.6),

$$
\|1-V\|_{1,4n,U} \le C\big\|\hat{\gamma}_G(\gamma_{F_\lambda} - \gamma_G)\big\|_{1,4n,U}.
$$

Now, we first apply Hölder inequality and then

$$
|\gamma_{F_\lambda} - \gamma_G| \le C\big|D(F_\lambda - G)\big|\big|D(F_\lambda + G)\big|
$$

and

$$
\big|D(\gamma_{F_\lambda} - \gamma_G)\big| \le C\big(\big|D(F_\lambda - G)\big| + \big|D^{(2)}(F_\lambda - G)\big|\big)\big(\big|D(F_\lambda + G)\big| + \big|D^{(2)}(F_\lambda + G)\big|\big).
$$

We find

$$
\|1-V\|_{1,4n,U} \le C\Gamma_{G,U}(12n)n_{F,G,U}(12n)\|F-G\|_{2,12n,U}
$$

so

$$
p_{F,U(1-V)}(y) \le C\big[m_U(24n)(\Gamma_{F,U} \vee \Gamma_{G,U})(24n)^2 n_{F,G,U}(24n)^3\big]^{8n^2} \big\|\Delta_2(F,G)\big\|_{32n,U}.
$$

We conclude writing

$$
\begin{aligned}
\big|p_{F,U}(y) - p_{G,U}(y)\big| &= \big|p_{F,UV}(y) + p_{F,U(1-V)}(y) - p_{G,UV}(y) - p_{G,U(1-V)}(y)\big| \\
&\le \big|p_{F,UV}(y) - p_{G,UV}(y)\big| + p_{F,U(1-V)}(y) + p_{G,U(1-V)}(y)
\end{aligned}
$$

and the statement follows easily. $\qquad\square$

Proof of Theorem 3.3. Let V as in the last proof. We can write

$$
\begin{aligned}
p_{F,U}(y) \ge p_{F,UV}(y) &\ge p_{G,UV}(y) - \big|p_{F,UV}(y) - p_{G,UV}(y)\big| \\
&= p_{G,U}(y) - p_{G,U(1-V)}(y) - \big|p_{F,UV}(y) - p_{G,UV}(y)\big|.
\end{aligned}
$$

From (3.10) and (3.15) as before

$$
p_{G,U(1-V)}(y) \le C\big[m_U(14n)\Gamma_{G,U}(14n)^2 n_{F,G,U}(14n)^3\big]^{8n^2} \big\|\Delta_2(F,G)\big\|_{32n,U}.
$$

Using also (3.17) and (3.18) we obtain the desired lower bound for p_F.

For the upper bound we apply Proposition 3.2 localizing on $1-U$. We have

$$
p_{F,1-U}(x) = \mathbb{E}_{(1-U)}\big[\nabla \mathcal{Q}_n(F-x)H_{(1-U)}(F,1)\big] = \mathbb{E}\big[\nabla \mathcal{Q}_n(F-x)H_{(1-U)}(F,1)(1-U)\big].
$$

From (3.3), $H(F, 1-U) = (1-U)H_{(1-U)}(F,1)$, so

$$
p_{F,1-U}(x) = \mathbb{E}\big[\nabla \mathcal{Q}_n(F-x)H(F,1-U)\big].
$$

Now we apply Hölder and find

$$
p_{F,1-U}(x) = \big\|\nabla \mathcal{Q}_n(F-x)\big\|_{\frac{2n}{2n-1}} \big\|H(F,1-U)\big\|_{2n}.
$$

We use (3.15), with $U = 1$, to deal with the gradient of the Poisson kernel:

$$\left(\mathbb{E}\left|\nabla \mathcal{Q}_n(F - y)\right|^{2n/(2n-1)}\right)^{(2n-1)/(2n)} \le C\left(\Gamma_F(14n)^2 n_F(14n)^3\right)^{4n(n-1)}.$$

Now consider the non-localized version of (3.10):

$$\left\|H(F, V)\right\|_p \le C\|V\|_{1,14n}\Gamma_F(14n)^2 n_F(14n)^3$$

and take $V = 1 - U$. We obtain

$$p_{F,1-U} \le C\|1 - U\|_{1,14n}\left[\Gamma_F(14n)^2 n_F(14n)^3\right]^{4n^2}. \tag{3.19}$$

We apply now the lower bound result to $p_{G,U}$, interchanging the roles of F and G, and find

$$p_{F,U}(y) \le p_{G,U}(y) + \left[m_U(32n)\Gamma_{F,U}(32n)n_{F,G}(32n)\right]^{24n^2}\left\|\Delta_2(F, G)\right\|_{32n,U}.$$

Putting together this inequality and (3.19), we have the upper bound. □

3.4. *Density estimates via local inversion*

We recall some results from [4]. We see how to use the local inversion theorem to transfer a known estimate for a Gaussian random variable to its image via a function η such that

$$\eta \in C^3\left(\mathbb{R}^n, \mathbb{R}^n\right), \eta(0) = 0, \quad \lambda^*\left(\nabla \eta(0)\right) \le \frac{1}{2}.$$

Define, for $h > 0$,

$$c_*(\eta, h) = \sup_{|x| \le 2h} \max_{i,j}\left|\partial_i \eta^j(x)\right|$$

and

$$c_2(\eta) = \max_{i,j=1,\dots,n} \sup_{|x| \le 1}\left|\partial_{ij}^2 \eta(x)\right|, \qquad c_3(\eta) = \max_{i,j,k=1,\dots,n} \sup_{|x| \le 1}\left|\partial_{ijk}^3 \eta(x)\right|.$$

Let now Θ be a n-dimensional centered Gaussian variable with covariance matrix Q, non-degenerate. Denote by $\underline{\lambda}$ and $\overline{\lambda}$ the lower and the upper eigenvalues of Q. Suppose to have $r > 0$ such that

$$c_*(\eta, 16r) \le \frac{1}{2n}\sqrt{\frac{\underline{\lambda}}{\overline{\lambda}}}, \qquad r \le h_\eta = \frac{1}{16n^2(c_2(\eta) + \sqrt{c_3(\eta)})}. \tag{3.20}$$

We take a localizing function as in (3.4): $U_r = \prod_{i=1}^n \psi_r(\Theta_i)$. We also define $\Phi(\theta) = \theta + \eta(\theta)$.

Lemma 3.8. *The density p_{G,U_r} of*

$$G := \Phi(\Theta) = \Theta + \eta(\Theta)$$

under $\mathbb{P}_{U_r}$ has the following bounds on $B(0, r)$:

$$\frac{1}{C \det Q^{1/2}}\exp\left(-\frac{C}{\underline{\lambda}}|z|^2\right) \le p_{G,U_r}(z) \le \frac{C}{\det Q^{1/2}}\exp\left(-\frac{1}{C\overline{\lambda}}|z|^2\right).$$

This result is proved in [4] under a slightly stronger constraint on r, but going trough the proof it is easy to see that what we suppose here is enough. For details see [34]. The proof is quite standard and follows from the local inversion theorem (see [36] for a standard version of this theorem).

4. Density estimates of the diffusion process

In this section we prove lower and upper bounds for the density of X_δ.

4.1. *Development*

In this section, in order to lighten the notation, we do not mention the dependence of the parameters on the initial condition (so, for example, we write A instead of $A(x_0)$, and so on). We need to introduce some notation. Consider a small time $\delta \in (0, 1]$. We define

- The translated initial condition

$$\hat{x}_0 = x_0 + b(x_0)\delta.$$

- The matrices $\bar{A}$ and $\bar{A}_\delta$ as

$$\bar{A} = \big(\sigma + \delta\partial_b\sigma, [\sigma, b]\big)$$

and

$$\bar{A}_\delta = \big(\delta^{1/2}(\sigma + \delta\partial_b\sigma), \delta^{3/2}[\sigma, b]\big).$$

Recall (2.2), (2.3), and remark that **A1** implies that these matrices are always invertible if δ is small enough.
- The Gaussian r.v.

$$\Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} \delta^{-1/2}W_\delta \\ \delta^{-3/2}\int_0^\delta (\delta - s)\,dW_s \end{pmatrix}.$$

- The polynomial of degree 3 and direction $\sigma(x_0)$ (recall κ_σ defined in (2.4)):

$$\eta(u) = \left(\frac{\kappa_\sigma(x_0)}{2}u^2 + \frac{(\partial_\sigma\kappa_\sigma + \kappa_\sigma^2)(x_0)}{6}u^3 \right)\sigma(x_0). \tag{4.1}$$

- The principal term

$$G = \Theta + \tilde{\eta}_\delta(\Theta), \tag{4.2}$$

where $\tilde{\eta}_\delta(\Theta) = \bar{A}_\delta^{-1}\eta(\delta^{1/2}\Theta_1)$.
- The remainder R_δ:

$$\begin{aligned} R_\delta = &\int_0^\delta \int_0^s \big(\partial_b\sigma(X_u) - \partial_b\sigma(x_0)\big)\,du \circ dW_s \\ &+ \int_0^\delta \int_0^s \big(\partial_\sigma b(X_u) - \partial_\sigma b(x_0)\big) \circ dW_u\,ds \\ &+ \int_0^\delta \int_0^s \partial_b b(X_u)\,du\,ds \\ &+ \int_0^\delta \int_0^s \int_0^u \big(\partial_\sigma\partial_\sigma\sigma(X_v) - \partial_\sigma\partial_\sigma\sigma(x_0)\big) \circ dW_v \circ dW_u \circ dW_s \\ &+ \int_0^\delta \int_0^s \int_0^u \partial_b\partial_\sigma\sigma(X_v) \circ dv \circ dW_u \circ dW_s. \end{aligned} \tag{4.3}$$

Notice that $R_\delta \sim \mathcal{O}(\delta^2)$. We also denote $\tilde{R}_\delta := \bar{A}_\delta^{-1}R_\delta$.

We now prove that the following decomposition holds:

$$X_\delta = \hat{x}_0 + \bar{A}_\delta(G + \tilde{R}_\delta). \tag{4.4}$$

This is a main tool in our approach. Indeed, we find Gaussian bounds for the density of the variable $F := \bar{A}_\delta^{-1}(X_\delta - \hat{x}_0) = G + \tilde{R}_\delta$ in the Euclidean metric of $\mathbb{R}^2$. The fact that in Theorem 4.5 the bounds for the diffusion are in the $A_\delta(x_0)$-norm follows from the change of variable suggested by (4.4).

Let us prove (4.4). We write the stochastic Taylor development of X_t with a remainder of order t^2:

$$X_t = x_0 + b(x_0)t + U_t + R_t,$$

where

$$U_t = \sigma(x_0)W_t + \partial_\sigma \sigma(x_0) \int_0^t W_s \circ dW_s$$

$$+ \partial_\sigma \partial_\sigma \sigma(x_0) \int_0^t \int_0^s W_u \circ dW_u \circ dW_s$$

$$+ \partial_b \sigma(x_0) \int_0^t s \, dW_s + \partial_\sigma b(x_0) \int_0^t W_s \, ds.$$

Now we write

$$\int_0^t W_s \, ds = \int_0^t (t - s) \, dW_s,$$

$$\int_0^t s \, dW_s = -\int_0^t (t - s) \, dW_s + t W_t.$$

Therefore

$$U_t = \big(\sigma(x_0) + t\partial_b\sigma(x_0)\big)W_t + \big(\partial_\sigma b(x_0) - \partial_b\sigma(x_0)\big)\int_0^t (t-s)\,dW_s$$

$$+ \partial_\sigma\sigma(x_0)\frac{W_t^2}{2} + \partial_\sigma\partial_\sigma\sigma(x_0)\frac{W_t^3}{6}.$$

So we have the following decomposition of X_t:

$$X_t = x_0 + b(x_0)t + \big(\sigma(x_0) + t\partial_b\sigma(x_0)\big)W_t + [\sigma, b](x_0)\int_0^t (t-s)\,dW_s + \eta(W_t) + R_t, \tag{4.5}$$

where x_0 is the initial condition. Remark that **A3** implies that both the coefficients of η have the same direction as $\sigma(x_0)$:

$$\eta(u) = \frac{\partial_\sigma\sigma(x_0)}{2}u^2 + \frac{\partial_\sigma\partial_\sigma\sigma(x_0)}{6}u^3 = \left(\frac{\kappa_\sigma(x_0)}{2}u^2 + \frac{(\partial_\sigma\kappa_\sigma + \kappa_\sigma^2)(x_0)}{6}u^3\right)\sigma(x_0).$$

4.2. *Preliminary estimates*

We introduce the following class of constants:

$$\mathcal{C} = \left\{ C > 0 : C = K\left(\frac{\rho}{\lambda_*(A(x_0))}\right)^q, \exists K, q \geq 1 \right\}. \tag{4.6}$$

We stress that the constants defined above depend on the parameters of the diffusion through the ratio $\rho/\lambda_*(A(x_0))$ (cf. **A1**, **A2**), but K, q do not depend on σ, b. We will also denote by $1/\mathcal{C} = \{\delta > 0 : 1/\delta \in \mathcal{C}\}$.

We keep using the notations of the previous development.

Lemma 4.1. *There exist L_1, L_2, K_1, K_2 positive constants not depending on the parameters, $\delta^* \in 1/\mathcal{C}$ such that: for any fixed $r > 0$ and δ such that $\delta \le \delta^* \exp(-2L_1 r^2)$, let $G = \Theta + \tilde{\eta}_\delta(\Theta)$ be the r.v. defined in (4.2); let U_r be the localizing r.v. defined in (3.4), and p_{G,U_r} the local density of G; then the following estimate holds for $|z| \le r$:*

$$K_1 \exp\left(-L_1|z|^2\right) \le p_{G,U_r}(z) \le K_2 \exp\left(-L_2|z|^2\right). \tag{4.7}$$

Proof. In what follows, $C \in \mathcal{C}$, and may vary from line to line (meaning that K, q may vary in (4.6)). We start by computing the derivatives of η:

$$\eta(y) = \left(\frac{\kappa_\sigma}{2}y^2 + \frac{\partial_\sigma \kappa_\sigma + \kappa_\sigma^2}{6}y^3\right)\sigma,$$

$$\eta'(y) = \left(\kappa_\sigma y + \frac{\partial_\sigma \kappa_\sigma + \kappa_\sigma^2}{2}y^2\right)\sigma,$$

$$\eta''(y) = \left(\kappa_\sigma + \left(\partial_\sigma \kappa_\sigma + \kappa_\sigma^2\right)y\right)\sigma,$$

$$\eta'''(y) = \left(\partial_\sigma \kappa_\sigma + \kappa_\sigma^2\right)\sigma.$$

By the definition of $\bar{A}_\delta^{-1}$,

$$\bar{A}_\delta^{-1}\delta^{1/2}(\sigma + \delta\partial_b\sigma) = (1, 0)^T.$$

Therefore

$$\bar{A}_\delta^{-1}\sigma = \delta^{-1/2}(1, 0)^T - \bar{A}_\delta^{-1}\delta\partial_b\sigma.$$

By (6.2) and (6.4) (see Section 6.1) we have $|\bar{A}_\delta^{-1}\delta\partial_b\sigma| \le C\delta^{-1/2}$, so that $|\bar{A}_\delta^{-1}\sigma| \le C\delta^{-1/2}$. We stress that this upper bound is $\delta^{-1/2}$ in contrast with $\delta^{-3/2}$ in (6.2), because $\bar{A}_\delta$ works in the specific direction σ. Now we can estimate the norms of $\tilde{\eta}_\delta$ and its derivatives. Since they are collinear with σ, we have

$$\left|\tilde{\eta}_\delta(u)\right| = \left|\bar{A}_\delta^{-1}\eta\left(\delta^{1/2}u_1\right)\right| \le C\left(|u_1|^2\delta^{1/2} + |u_1|^3\delta\right),$$

$$\left|\partial_{u_1}\tilde{\eta}_\delta(u)\right| = \left|\bar{A}_\delta^{-1}\delta^{1/2}\eta'\left(\delta^{1/2}u_1\right)\right| \le C\left(|u_1|\delta^{1/2} + |u_1|^2\delta\right),$$

$$\left|\partial_{u_1}^2\tilde{\eta}_\delta(u)\right| = \left|\bar{A}_\delta^{-1}\delta\eta''\left(\delta^{1/2}u_1\right)\right| \le C\left(\delta^{1/2} + |u_1|\delta\right),$$

$$\left|\partial_{u_1}^3\tilde{\eta}_\delta(u)\right| = \left|\bar{A}_\delta^{-1}\delta^{3/2}\eta'''\left(\delta^{1/2}u_1\right)\right| \le C\delta,$$

$$\left|\partial_{u_2}\tilde{\eta}_\delta(u)\right| = 0.$$

So, referring to the notation of Section 3.4, we have

$$c_*(\tilde{\eta}_\delta, h) = \sup_{|u| \le 2h} \max_{i,j}\left|\partial_i \tilde{\eta}_\delta^j(u)\right| \le Ch\delta^{1/2},$$

$$c_2(\tilde{\eta}_\delta) = \max_{i,j} \sup_{|u| \le 1}\left|\partial_{i,j}^2 \tilde{\eta}_\delta(u)\right| \le C\delta^{1/2}, \tag{4.8}$$

$$c_3(\tilde{\eta}_\delta) = \max_{i,j,k} \sup_{|u| \le 1}\left|\partial_{i,j,k}^3 \tilde{\eta}_\delta(u)\right| \le C\delta.$$

We first want to apply Lemma 3.8 to $G = \Theta + \tilde{\eta}_\delta(\Theta)$. Here $n = 2$, and the covariance matrix of Θ is

$$\gamma_\Theta = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}.$$

It has 2 positive eigenvalues, $0 < \lambda_1 < \lambda_2$, and $\det(\gamma_\Theta) = 1/12$. We are supposing here $\delta \leq \delta^* \exp(-2L_1 r^2) \leq \delta^*/r^2$. Since

$$h_{\tilde{\eta}_\delta} = \frac{1}{64(c_2(\tilde{\eta}_\delta) + \sqrt{c_3(\tilde{\eta}_\delta)})} \geq \frac{1}{C_1\sqrt{\delta}} \geq \frac{r}{C_1\sqrt{\delta^*}}$$

and

$$c_*(\tilde{\eta}_\delta, 16r) \leq C_2 r\sqrt{\delta} \leq C_2\sqrt{\delta^*},$$

choosing $\delta^* \leq \frac{1}{16}\frac{\lambda_1}{\lambda_2}\frac{1}{C_1^2 C_2^2}$ the conditions (3.20) are satisfied:

$$c_*(\tilde{\eta}_\delta, 16r) \leq \frac{1}{4}\sqrt{\frac{\lambda_1}{\lambda_2}}, \quad r \leq h_{\tilde{\eta}_\delta}. \tag{4.9}$$

So there exist L_1, L_2, K_1, K_2 universal constants, such that for $|z| \leq r$,

$$K_1 \exp(-L_1|z|^2) \leq p_{G,U_r}(z) \leq K_2 \exp(-L_2|z|^2). \qquad \square$$

The following lemma is a slight modification of Lemma 2.3.1 in [32].

Lemma 4.2. *Let γ be a symmetric non-negative definite $n \times n$ matrix. We assume that, for fixed $p \geq 2$, $\mathbb{E}[\|\gamma\|_{\mathrm{Fr}}^{p+1}] < \infty$, and that $\exists \varepsilon_0 > 0$ s.t. for $\varepsilon \leq \varepsilon_0$,*

$$\sup_{|\xi|=1} \mathbb{P}[\langle \gamma\xi, \xi \rangle < \varepsilon] \leq \varepsilon^{p+2n}.$$

Then there exist a constant C depending only on the dimension n such that

$$\mathbb{E}[\lambda_*(\gamma)^{-p}] \leq C\mathbb{E}[\|\gamma\|_{\mathrm{Fr}}^{p+1}]\varepsilon_0^{-p}.$$

We consider now

$$F = \bar{A}_\delta^{-1}(X_\delta - \hat{x}_0). \tag{4.10}$$

We will use the general estimates of Section 3. We denote by D the Malliavin derivative with respect to W, the Brownian motion driving (2.1). We first prove that the moments of $\lambda^*(\gamma_F^{-1}) = \lambda_*(\gamma_F)^{-1}$ are bounded, and these bounds do not depend on δ. This result looks interesting by itself, since it means that we are able to account precisely of the scaling of the diffusion in the two main directions σ and $[\sigma, b]$. In this particular case this is a refinement of the classical result on the bounds of the Malliavin covariance under the (weak) Hörmander condition (cf. [27,31,32]).

Lemma 4.3. *Let $F = \bar{A}_\delta^{-1}(X_\delta - \hat{x}_0)$. For any $p > 1$, there exists $C \in \mathcal{C}$ such that for any $\delta \leq 1$, $\Gamma_F(p) \leq e^C$.*

Proof. Following [32] we define the tangent flow of X as the derivative with respect to the initial condition of X, $Y_t := \partial_x X_t$. We also denote its inverse $Z_t = Y_t^{-1}$. They satisfy the following stochastic differential equations

$$Y_t = \mathrm{Id} + \int_0^t \nabla\sigma(X_s)Y_s \circ dW_s + \int_0^t \nabla b(X_s)Y_s\, ds,$$

$$Z_t = \mathrm{Id} - \int_0^t Z_s \nabla\sigma(X_s) \circ dW_s - \int_0^t Z_s \nabla b(X_s)\, ds.$$

The Malliavin derivative of X is

$$D_s X_t = Y_t Z_s \sigma(X_s),$$

so

$$D_s F = D_s \bar{A}_\delta^{-1}(X_\delta - \hat{x}_0) = \bar{A}_\delta^{-1} Y_\delta Z_s \sigma(X_s).$$

We define

$$\bar{\gamma}_\delta = \int_0^\delta A_\delta^{-1} Z_s \sigma(X_s)\sigma(X_s)^T Z_s^T A_\delta^{-1,T} \, ds.$$

Then

$$\gamma_F = \langle DF, DF \rangle = \bar{A}_\delta^{-1} Y_\delta A_\delta \bar{\gamma}_\delta A_\delta^T Y_\delta^T \bar{A}_\delta^{-1,T}.$$

Remark that

$$\gamma_F^{-1} = \bar{A}_\delta^T Z_\delta^T A_\delta^{-1,T} \bar{\gamma}_\delta^{-1} A_\delta^{-1} Z_\delta \bar{A}_\delta,$$

and that in this representation we have both A_δ and its "perturbed" version $\bar{A}_\delta$. We have to check the integrability of $\lambda_*(\gamma_F)^{-1} = \lambda^*(\gamma_F^{-1})$. Recall that $\lambda^*(\cdot)$ is a norm on the set of matrices, and that for two 2×2 matrices M_1, M_2, $\lambda^*(M_1 M_2) \le 2\lambda^*(M_1)\lambda^*(M_2)$. We have

$$\lambda_*(\gamma_F)^{-1} \le 4\lambda^*\big(\bar{\gamma}_\delta^{-1}\big)\lambda^*\big(A_\delta^{-1} Z_\delta \bar{A}_\delta\big)^2.$$

We need to bound $A_\delta^{-1} Z_\delta \bar{A}_\delta$, which we expect to be close to the identity matrix for small δ, and $\bar{\gamma}_\delta^{-1}$.

We take care first of the moments of $\lambda^*(\bar{\gamma}_\delta^{-1})$. We use the following representation, holding for general ϕ, which follows applying Ito's formula (details in [32])

$$Z_t\phi(X_t) = \phi(x_0) + \int_0^t Z_s[\sigma, \phi](X_s) \, dW_s^k + \int_0^t Z_s\left\{[b, \phi] + \frac{1}{2}[\sigma, [\sigma, \phi]]\right\}(X_s) \, ds. \tag{4.11}$$

Taking $\phi = \sigma$ the representation above reduces to

$$Z_t\sigma(X_t) = \sigma(x_0) + \int_0^t Z_s[b, \sigma](X_s) \, ds$$

$$= \sigma(x_0) + t[b, \sigma](x_0) + L_t, \tag{4.12}$$

with $L_t = \int_0^t Z_s[b, \sigma](X_s) - Z_0[b, \sigma](x_0) \, ds$. Notice that $L_t \sim \mathcal{O}(t^{3/2})$. Using **A2** one gets

$$\mathbb{E}\left[\lambda^*\left(\int_0^{\delta\varepsilon} L_s L_s^T \, ds\right)^q\right] \le \mathbb{E}\left[\left\|\int_0^{\delta\varepsilon} L_s L_s^T \, ds\right\|_{\mathrm{Fr}}^q\right] \le e^{C'}(\delta\varepsilon)^{4q}, \quad \forall q > 0, \exists C' \in \mathcal{C}$$

($e^{C'}$ comes from Gronwall inequality). We have

$$A_\delta^{-1} Z_s \sigma(X_s) = A_\delta^{-1}\big(\sigma(x_0) + s[b, \sigma](x_0) + L_s\big)$$

$$= \frac{1}{\delta^{1/2}}\begin{pmatrix} 1 \\ -s/\delta \end{pmatrix} + A_\delta^{-1} L_s.$$

For constant c and fixed ε, we introduce the stopping time

$$S_\varepsilon = \inf\left\{s \ge 0 : \lambda^*\left(\int_0^s L_u L_u^T \, du\right) \ge c(\delta\varepsilon)^3\right\} \wedge \delta.$$

We have

$$\lambda^*\left(A_\delta^{-1}\int_0^{S_\varepsilon} L_u L_u^T \, du A_\delta^{-1,T}\right) \le 4\lambda^*\big(A_\delta^{-1}\big)^2 \lambda^*\left(\int_0^{S_\varepsilon} L_u L_u^T \, du\right) \le \frac{C''}{\delta^3} c(\delta\varepsilon)^3,$$

where $C'' \in \mathcal{C}$. We fix $c = \frac{1}{64C''}$, so

$$\lambda^* \left(A_\delta^{-1} \int_0^{S_\varepsilon} L_u L_u^T \, du \, A_\delta^{-1,T} \right) \le \frac{\varepsilon^3}{64}. \tag{4.13}$$

Now we suppose to be on the event $\{ \frac{S_\varepsilon}{\delta} \ge \varepsilon \}$. Applying first inequality

$$\langle (v + R)(v + R)^T \xi, \xi \rangle \ge \frac{1}{2} \langle v v^T \xi, \xi \rangle - \langle R R^T \xi, \xi \rangle,$$

which holds for any vectors v, R, ξ, and then (4.12), we obtain

$$\begin{aligned}
\bar{\gamma}_\delta &= \int_0^\delta A_\delta^{-1} Z_s \sigma(X_s) \sigma(X_s)^T Z_s^T A_\delta^{-1,T} \, ds \\
&\ge \int_0^{S_\varepsilon} A_\delta^{-1} Z_s \sigma(X_s) \sigma(X_s)^T Z_s^T A_\delta^{-1,T} \, ds \\
&\ge \frac{1}{2} \int_0^{S_\varepsilon} \frac{1}{\delta} \begin{pmatrix} 1 & -s/\delta \\ -s/\delta & (s/\delta)^2 \end{pmatrix} ds - A_\delta^{-1} \int_0^{S_\varepsilon} L_s L_s^T \, ds \, A_\delta^{-1,T}.
\end{aligned}$$

We have

$$\int_0^{S_\varepsilon} \frac{1}{\delta} \begin{pmatrix} 1 & -s/\delta \\ -s/\delta & (s/\delta)^2 \end{pmatrix} ds \ge \int_0^{\delta\varepsilon} \frac{1}{\delta} \begin{pmatrix} 1 & -s/\delta \\ -s/\delta & (s/\delta)^2 \end{pmatrix} ds \ge \begin{pmatrix} \varepsilon & -\frac{\varepsilon^2}{2} \\ -\frac{\varepsilon^2}{2} & \frac{\varepsilon^3}{3} \end{pmatrix} \ge \mathrm{Id}_2 \, \frac{\varepsilon^3}{16},$$

so, from (4.13),

$$\langle \bar{\gamma}_\delta \xi, \xi \rangle \ge \frac{1}{2} \frac{\varepsilon^3}{16} |\xi|^2 - \frac{\varepsilon^3}{64} |\xi|^2 = \frac{\varepsilon^3}{64} |\xi|^2, \quad \forall |\xi| = 1.$$

Now, remark that $t \to \lambda^* (\int_0^t L_s L_s^T \, ds)$ is increasing. For any $q > 0$

$$\begin{aligned}
\mathbb{P}(S_\varepsilon < \delta\varepsilon) &\le \mathbb{P} \left(\lambda^* \left(\int_0^{\delta\varepsilon} L_s L_s^T \, ds \right)^q \ge c^q (\delta\varepsilon)^{3q} \right) \\
&\le \frac{\mathbb{E}[\lambda^* (\int_0^{\delta\varepsilon} L_s L_s^T \, ds)^q]}{c^q (\delta\varepsilon)^{3q}} \\
&\le \frac{e^{C'} (\delta\varepsilon)^{4q}}{c^q (\delta\varepsilon)^{3q}} \le \frac{e^{C'}}{c^q} (\delta\varepsilon)^q \le \varepsilon^{q/2}
\end{aligned}$$

for $\delta \le 1$, for $\varepsilon \le \varepsilon_0 = e^{-C'''}$ with $C''' \in \mathcal{C}$. Therefore, for any q, for any $\varepsilon \le \varepsilon_0$, $\delta \le 1$,

$$\mathbb{P}\big(\langle \bar{\gamma}_\delta \xi, \xi \rangle < \varepsilon^3/64 \big) \le \mathbb{P}[S_\varepsilon < \delta\varepsilon] \le \varepsilon^{q/2}.$$

Now we apply Lemma 4.2. We obtain

$$\mathbb{E}\lambda^* \big(\bar{\gamma}_\delta^{-1} \big)^q = \mathbb{E}\lambda_* (\bar{\gamma}_\delta)^{-q} \le e^C$$

for $\delta \le 1$, $C \in \mathcal{C}$.

We consider now $A_\delta^{-1} Z_\delta \bar{A}_\delta$. Applying (4.11) and **A3**, one can prove that

$$Z_t \sigma(x_0) = \big(1 - \kappa_\sigma(x_0) W_t \big) \sigma(x_0) + J_t,$$

with $J_t \sim \mathcal{O}(t)$. So

$$Z_\delta \bar{A}_\delta = \left(\sqrt{\delta}\left(1 - \kappa_\sigma(x_0)W_\delta\right)\sigma(x_0), \, 0\right) + M_\delta,$$

where M_δ is a 2×2 matrix with $\mathbb{E}\lambda^*(M_\delta)^q \leq e^C \delta^{3q/2}$, $C \in \mathcal{C}$. This estimate follows again from **A2**. Since $A_\delta = (\delta^{1/2}\sigma(x_0), \delta^{3/2}[\sigma, b](x_0))$

$$A_\delta^{-1}\left(\sqrt{\delta}\left(1 - \kappa_\sigma(x_0)W_\delta\right)\sigma(x_0), \, 0\right) = \begin{pmatrix} 1 - \kappa_\sigma(x_0)W_\delta & 0 \\ 0 & 0 \end{pmatrix}$$

and $\mathbb{E}|1 - \kappa_\sigma(x_0)W_\delta|^q \leq C \in \mathcal{C}$. Clearly $\mathbb{E}\lambda^*(A_\delta^{-1}M_\delta)^q \leq e^C$, $C \in \mathcal{C}$, so

$$\mathbb{E}\lambda^*\left(A_\delta^{-1}Z_\delta\bar{A}_\delta\right)^q \leq e^C, \quad C \in \mathcal{C}. \qquad \square$$

4.3. *Two-sided bound for the density of X_δ*

In this section we prove the short time density estimate (2.5). We start with the following lemma, which is a density estimate for the "renormalized" random variable F (see (4.10)). We use Theorem 3.3 to recover estimates for p_F from (4.7). We will need the preliminary estimates of Section 4.2.

Lemma 4.4. *Recall* (4.6), *the definition of $\mathcal{C}$, and that, for fixed $\delta > 0$, we set $F = \bar{A}_\delta^{-1}(X_\delta - \hat{x}_0)$ and p_F is its density.*

(1) *There exist $C, C^*, L \in \mathcal{C}$ such that the following holds. We set $\delta^* = e^{-C^*}$. For any fixed $r > 0$, if $\delta \leq \delta^* \exp(-Lr^2)$, for $|z| \leq r$ we have*

$$\frac{1}{C}\exp\left(-C|z|^2\right) \leq p_F(z).$$

(2) *There exists $\delta^* \in 1/\mathcal{C}$; $C, L \in \mathcal{C}$ such that: for any fixed $r > 0$, if $\delta \leq \delta^* \exp(-Lr^2)$, for $|z| \leq r$, we have*

$$p_F(z) \leq e^C \exp\left(-C^{-1}|z|^2\right).$$

Proof. We apply Theorem 3.3. Here $n = 2$, so $32n = 64$.
   (1) (Lower bound). Let L_1 be the constant in Lemma 4.1. We first prove the lower bound for $r \geq \frac{1}{\sqrt{L_1}} =: \tilde{r}$.

We start checking that C_1 in Remark 3.4 is in $\mathcal{C}$. From (3.5) and $r \geq \frac{1}{\sqrt{L_1}}$,

$$m_{U_r}(64) \leq C\left(1 + \frac{\|\Theta\|_{2,64}^2}{r^2}\right) \leq C \in \mathcal{C}.$$

Recall that $G = \Theta + \tilde{\eta}_\delta(\Theta)$, where Θ is a Gaussian with covariance (and also Malliavin covariance matrix) given by

$$\gamma_\Theta = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}.$$

This matrix has 2 positive eigenvalues, $0 < \lambda_1 < \lambda_2$. Recall also that the Malliavin derivative D is taken with respect to the Brownian motion W driving (2.1). We consider $\Gamma_{G,U_r} = 1 + (\mathbb{E}_{U_r}\lambda_*(\gamma_G)^{-p})^{1/p}$.

$$\begin{aligned}
\langle \gamma_G \xi, \xi \rangle &= \int_0^\delta \langle D_s G, \xi \rangle^2 \\
&\geq \int_0^\delta \frac{1}{2}\langle D_s\Theta, \xi \rangle^2 - \left(D_s\tilde{\eta}_\delta(\Theta), \xi\right)^2 ds \\
&= S_1 + S_2.
\end{aligned}$$

We have

$$S_2 = \int_0^\delta \langle \nabla \tilde{\eta}_\delta(\Theta) D_s \Theta, \xi \rangle^2 \, ds = \int_0^\delta \langle D_s \Theta, \nabla \tilde{\eta}_\delta(\Theta)^T \xi \rangle^2 \, ds \le \lambda_2 \|\nabla \tilde{\eta}_\delta(\Theta)\|_{\text{Fr}}^2 |\xi|^2$$

and $S_1 \ge \lambda_1/2$, so

$$\lambda_*(\gamma_G) \ge \lambda_1 \left(\frac{1}{2} - \frac{\lambda_2}{\lambda_1} \|\nabla \tilde{\eta}_\delta(\Theta)\|_{\text{Fr}}^2 \right).$$

Recall $c_*(\tilde{\eta}_\delta, h) = \sup_{|x| \le 2h} \max_{i,j} |\partial_i \tilde{\eta}_\delta^j(x)|$, so on the event $\{U_r \ne 0\}$ we have $|\Theta| \le 4r$ and $\|\nabla \tilde{\eta}_\delta(\Theta)\|_{\text{Fr}} \le 2c_*(\tilde{\eta}_\delta, 16r)$. We proved in (4.9) that $c_*(\tilde{\eta}_\delta, 16r) \le \frac{1}{4}\sqrt{\frac{\lambda_1}{\lambda_2}}$, so it follows

$$\|\nabla \tilde{\eta}_\delta(\Theta)\|_{\text{Fr}} \le \frac{1}{2}\sqrt{\frac{\lambda_1}{\lambda_2}},$$

and therefore $\lambda_*(\gamma_G) \ge \lambda_1/4$, which implies $\Gamma_{G,U_r}(64) \le C$. Recall (4.4) and (4.10). Standard computations using **A2** and Gronwall lemma give $n_{F,G,U_r}(p) \le e^C$, $C \in \mathcal{C}$, so from Theorem 3.3 we have that $\exists C \in \mathcal{C}$ such that for $|z| \le r$

$$p_F(z) \ge p_{G,U_r}(z) - e^C \|\tilde{R}_\delta\|_{64,U_r} \ge K_1 \exp(-L_1|z|^2) - e^C \|\tilde{R}_\delta\|_{64,U_r}.$$

Recall (4.3). By using **A2**, one can show that $\|R_\delta\|_{2,p} \le e^C \delta^2$, with $C \in \mathcal{C}$. So, from (6.1) with $\bar{A}_\delta$ instead of A_δ,

$$\|\tilde{R}_\delta\|_{64,U_r} = \|\bar{A}_\delta^{-1} R_\delta\|_{64,U_r} \le e^C \delta^2/\delta^{3/2} = e^C \sqrt{\delta},$$

so there exists $\bar{C} \in \mathcal{C}$ such that $p_F(z) \ge K_1 \exp(-L_1|z|^2) - e^{\bar{C}}\sqrt{\delta}$. We have that, for $r \ge \tilde{r}$, if

$$\delta \le \left(\frac{K_1 \exp(-\bar{C}) \exp(-L_1 r^2)}{2} \right)^2 = \left(\frac{K_1 \exp(-\bar{C})}{2} \right)^2 \exp(-2L_1 r^2) \tag{4.14}$$

the following lower bound holds for $|z| \le r$:

$$p_F(z) \ge \frac{K_1}{2} \exp(-L_1|z|^2)$$

and this implies Lemma 4.4(1) for $r \ge \tilde{r}$. We take now $0 < r \le \tilde{r}$. Remark that $\exp(-2) = \exp(-2L_1\tilde{r}^2)$. We can suppose $\delta^* \le (\frac{K_1 \exp(-\bar{C}-1)}{2})^2$, so

$$\delta \le \left(\frac{K_1 \exp(-\bar{C}-1)}{2} \right)^2 = \left(\frac{K_1 \exp(-\bar{C})}{2} \right)^2 \exp(-2L_1\tilde{r}^2).$$

If $|z| \le r$, then $|z| \le \tilde{r}$, and we apply what we have just proved for $r \ge \tilde{r}$, taking $\tilde{r}$ as radius. The following holds:

$$p_F(z) \ge \frac{K_1}{2} \exp(-L_1|z|^2).$$

(2) (Upper bound). The proof of the upper bound follows again from Theorem 3.3. We deal with C_2 exactly as for the lower bound, with the difference that we need a bound for $\Gamma_F(64)$ instead of $\Gamma_{G,U_r}(64)$. This is proved in Lemma 4.3. As before, we first suppose $r \ge \frac{1}{\sqrt{L_2}}$, where L_2 is the constant in Lemma 4.1. We obtain

$$p_F(z) \le K_2 \exp(-L_2|z|^2) + e^{\bar{C}}(\sqrt{\delta} + \|1 - U_r\|_{1,28})$$

$\bar{C} \in \mathcal{C}$. We fix $L \in \mathcal{C}$ and take $\delta \leq \exp(-Lr^2)$, and we also need to prove that $\|1 - U_r\|_{1,28}$ decays as $C \exp(-C^{-1}|z|^2)$ for $|z| \leq r$. This follows from (3.6): $\exists C \in \mathcal{C}$ such that

$$\|1 - U_r\|_{1,28} \leq \sum_{i=1,2} \mathbb{P}(|\Theta_i| > r)^{\frac{1}{56}} C(1 + 1/r) \leq C e^{-C^{-1}r^2}.$$

We have the desired result for $r \geq \frac{1}{\sqrt{L_2}}$. Now, we take $r \leq \frac{1}{\sqrt{L_2}}$. If $|z| \leq r$, then $|z| \leq \frac{1}{\sqrt{L_2}}$, and we can apply the result already proved for $r \geq \frac{1}{\sqrt{L_2}}$, taking $\frac{1}{\sqrt{L_2}}$ as radius. Then, we prove as in (1) that the result can be extended to all $r > 0$. $\qquad\square$

We are now ready to prove the main theorem in short time.

Theorem 4.5. *Suppose* **A1**, **A2**, **A3** *hold. Let* $(X_t)_{t \in [0,T]}$ *be the solution of* (2.1)*, and for* $t \in [0, T]$*, let* $p_t(x_0, y)$ *be the density of* X_t *at* y.

(1) *There exist* $C, C^*, L \in \mathcal{C}$ *such that the following holds. We set* $\delta^* = e^{-C^*}$*. For any fixed* $r > 0$*, if* $0 < \delta \leq \delta^* \exp(-Lr^2)$*, setting* $\hat{x}_0 = x_0 + b(x_0)\delta$*, for* $|y - \hat{x}_0|_{A_\delta(x_0)} \leq r$ *we have*

$$\frac{1}{C\delta^2} \exp\left(-C|y - \hat{x}_0|_{A_\delta(x_0)}^2\right) \leq p_\delta(x_0, y).$$

(2) *There exists* $\delta^* \in 1/\mathcal{C}$*,* $L, C \in \mathcal{C}$ *such that: for any fixed* $r > 0$*, if* $0 < \delta \leq \delta^* \exp(-Lr^2)$*, setting* $\hat{x}_0 = x_0 + b(x_0)\delta$*, for* $|y - \hat{x}_0|_{A_\delta(x_0)} \leq r$*, we have*

$$p_\delta(x_0, y) \leq \frac{e^C}{\delta^2} \exp\left(-C^{-1}|y - \hat{x}_0|_{A_\delta(x_0)}^2\right).$$

Proof. We write the expectation of $f(X_\delta)$ for a function f with support included in $B(0, r)$. We get

$$\mathbb{E}\left[f(X_\delta)\right] = \mathbb{E}\left[f(\hat{x}_0 + \bar{A}_\delta F)\right] = \int f(\hat{x}_0 + \bar{A}_\delta z) p_F(z) \, dz.$$

With δ, r satisfying the hypothesis of Lemma 4.4, we can apply the previous density estimates to p_F. Then the change of variable $y = \hat{x}_0 + \bar{A}_\delta z$ gives that, for $|y - \hat{x}_0|_{\bar{A}_\delta(x_0)} \leq r$, we obtain respectively

(1) $\frac{1}{C|\det \bar{A}_\delta(x_0)|} \exp(-C|y - \hat{x}_0|_{\bar{A}_\delta(x_0)}^2) \leq p_\delta(x_0, y)$,

(2) $p_\delta(x_0, y) \leq \frac{e^C}{|\det \bar{A}_\delta(x_0)|} \exp(-C^{-1}|y - \hat{x}_0|_{\bar{A}_\delta(x_0)}^2)$,

where $p_\delta(x_0, y)$ is the density of X_δ in y. These estimates and the equivalence between $|\cdot|_{A_\delta}$ and $|\cdot|_{\bar{A}_\delta}$ (see (6.4) in Section 6.1) imply the thesis. $\qquad\square$

Remark 4.6. In the proof of Lemma 4.4 we have used **A2**, the assumption of uniformly bounded derivatives, to say that $n_{F,G,U_r}(p) \leq e^C$ and $\|R_\delta\|_{2,p} \leq e^C \delta^2$, $C \in \mathcal{C}$. If we also ask that

$$|\sigma(x)| + |b(x)| \leq \rho, \quad \forall x \in \mathbb{R}^2 \tag{4.15}$$

we have that $n_{F,G,U_r} \leq \tilde{C}$ and $\|R_\delta\|_{2,p} \leq \tilde{C}\delta^2$, $\tilde{C} \in \mathcal{C}$. This holds because, supposing the boundedness of the coefficients, we do not need anymore to use the Gronwall lemma to estimate the moments, but a direct computation is enough. These are standard estimates. In particular, in (4.14) we have $1/\bar{C}$ instead of $\exp(-\bar{C})$. As a consequence, if we also suppose (4.15), the lower bound in Lemma 4.4 and Theorem 4.5 holds for $\delta^* \in 1/\mathcal{C}$. In particular, taking $r^* = (L \vee C)^{-1/2}$ in Theorem 4.5(1) we can state that: $\exists r^*, \delta^* \in 1/\mathcal{C}, C \in \mathcal{C}$ such that for $\delta \leq \delta^*$, for $|y - \hat{x}_0|_{A_\delta(x_0)} \leq r^*$

$$\frac{1}{C\delta^2} \leq p_\delta(x_0, y).$$

On the other hand, in the upper bound we cannot get rid of the exponential dependence in the constant. Indeed, the estimate on $\Gamma_F(64)$ of Lemma 4.3 is involved (the estimate on the "non-degeneracy" of the rescaled diffusion F). This has an exponential dependence on the parameters, even supposing (4.15), because it involves the moments of Z_t, the inverse of the flow of X, and in this estimate we always need to use Gronwall lemma. Anyways, taking $r^* = (L)^{-1/2}$ in Theorem 4.5(2) we find that: $\exists r^*, \delta^* \in 1/\mathcal{C}$, $C \in \mathcal{C}$ such that for $\delta \leq \delta^*$, for $|y - \hat{x}_0|_{A_\delta(x_0)} \leq r^*$

$$p_\delta(x_0, y) \leq \frac{e^C}{\delta^2}.$$

We put together those two inequalities in the following two-sided bound, which is the formulation that will be used to prove the tube estimate: $\exists r^*, \delta^* \in 1/\mathcal{C}$, $C \in \mathcal{C}$ such that for $\delta \leq \delta^*$, for $|y - \hat{x}_0|_{A_\delta(x_0)} \leq r^*$

$$\frac{1}{C\delta^2} \leq p_\delta(x_0, y) \leq \frac{e^C}{\delta^2}. \tag{4.16}$$

5. Tube estimates of the diffusion process

As an application of Theorem 4.5 we prove the tube estimate. We suppose in this section $\sigma, b \in C^5(\mathbb{R}^2)$ and set, for $x \in \mathbb{R}^2$,

$$n(x) = \sum_{k=0}^{5} \sum_{|\alpha|=k} \left|\partial_x^\alpha b(x)\right| + \left|\partial_x^\alpha \sigma(x)\right|, \qquad \lambda(x) = \lambda_*\big(A(x)\big).$$

We consider the diffusion (2.1) on $[0, T]$, and the skeleton path (2.6): for $\phi \in L^2[0, T]$, let

$$x_t(\phi) = x_0 + \int_0^t \sigma\big(x_s(\phi)\big)\phi_s \, ds + \int_0^t b\big(x_s(\phi)\big) \, ds, \quad \text{for } t \in [0, T].$$

Recall **H1**, **H2**, **H3**, **H4**:

$$\lambda(y) \geq \lambda_t, \qquad n(y) \leq n_t, \qquad \partial_\sigma \sigma(y) = \kappa_\sigma(y)\sigma(y), \quad \forall \left|y - x_t(\phi)\right| < 1, \forall t \in [0, T].$$

Moreover, defining $(R_t)_{t \in [0,T]}$ the time-dependent radius of the tube, we suppose that

$$n_\cdot : [0, T] \to [1, \infty), \qquad R_\cdot : [0, T] \to (0, 1],$$

$$\lambda_\cdot : [0, T] \to (0, 1], \qquad |\phi_\cdot|^2 : [0, T] \to (0, \infty)$$

are in $\in L(\mu, h)$, for some $h > 0$, $\mu \geq 1$, where $L(\mu, h)$ is the class of non-negative functions which have the property

$$f(t) \leq \mu f(s) \quad \text{for } |t - s| \leq h.$$

Denote, for $0 \leq t \leq T$, for K_*, q_* positive universal constants,

$$R_t^*(\phi) = \exp\left(-K_*\left(\frac{\mu n_t}{\lambda_t}\right)^{q_*}\mu^{2q_*}\right)\left(h \wedge \inf_{0 \leq \delta \leq h}\left\{\delta / \int_t^{t+\delta} |\phi_s|^2 \, ds\right\}\right). \tag{5.1}$$

Theorem 5.1. *Let $(X_t)_{t\in[0,T]}$ be a process verifying* (2.1), *and $x_t(\phi)$ the skeleton path defined above. If* **H1**, **H2**, **H3**, **H4** *are satisfied, there exist positive universal constants $\bar{K}, \bar{q}$ such that*

$$\exp\left(-\int_0^T \bar{K}\left(\frac{\mu n_t}{\lambda_t}\right)^{\bar{q}}\left(\frac{1}{h} + \frac{1}{R_t} + |\phi_t|^2 \, dt\right)\right) \leq \mathbb{P}\left(\sup_{t\leq T}|X_t - x_t(\phi)|_{A_{R_t}(x_t(\phi))} \leq 1\right).$$

Moreover, there exist positive universal constants $\bar{K}, \bar{q}, K_*, q_*$ such that if $R_. \leq R_.^*(\phi)$

$$\mathbb{P}\Big(\sup_{t \leq T}|X_t - x_t(\phi)|_{A_{R_t}(x_t(\phi))} \leq 1\Big)$$

$$\leq \exp\bigg(-\int_0^T \bar{K}\Big(\frac{\mu n_t}{\lambda_t}\Big)^{\bar{q}}\bigg(\frac{\exp(-K_*(\frac{\mu n_t}{\lambda_t})^{q_*})}{R_t} + |\phi_t|^2\bigg) dt\bigg).$$

Remark 5.2. Remark that for $R_t \leq R_t^*(\phi) \leq h \exp(-K_*(\frac{\mu n_t}{\lambda_t})^{q_*})$ the statement in (2.8) is implied by this one.

Proof of Theorem 5.1. A main point in this proof is the choice a sequence of short time intervals in a way that allows us to apply the short time density estimate. This issue is related to the choice of a an "elliptic evolution sequence" in [6,7]. We fix ϕ from the beginning and write x_t for $x_t(\phi)$ to have a more readable notation.

We introduce also the time-dependent version of (4.6). For $t \in [0, T]$

$$\mathcal{C}_t = \big\{C_t > 0 : C_t = \exp\big(K(n_t/\lambda_t)^q\big), \exists K, q \geq 1\big\}. \tag{5.2}$$

The constants defined above depend on σ, b through the ratio n_t/λ_t locally along the skeleton path. We stress that K, q do not depend on σ, b and do not depend on $t \in [0, T]$. We will also denote by $1/\mathcal{C}_t = \{\delta_t > 0 : 1/\delta_t \in \mathcal{C}_t\}$. We start proving the lower bound.

Step 1 (Time grid and notations): We set, for large q_1, K_1 to be fixed in the sequel,

$$f_R(t) = K_1\Big(\frac{\mu n_t}{\lambda_t}\Big)^{q_1}\bigg(\frac{1}{h} + \frac{1}{R_t} + |\phi_t|^2\bigg).$$

We use this function to split the time interval $[0, T]$ is short-enough sub-intervals (our time grid). Recall **H4**: $|\phi_.|^2, n_., \lambda_., R_. \in L(\mu, h), \exists \mu \geq 1, 0 < h \leq 1$. This implies $f_R \in L(\mu^{2q_1+1}, h)$. We also define

$$\delta(t) = \inf_{\delta > 0}\bigg\{\int_t^{t+\delta} f_R(s)\, ds \geq \frac{1}{\mu^{2q_1+1}}\bigg\}. \tag{5.3}$$

Since

$$\frac{\delta(t)}{h} = \int_t^{t+\delta(t)} \frac{1}{h}\, ds \leq \int_t^{t+\delta(t)} f_R(s)\, ds = \frac{1}{\mu^{2q_1+1}},$$

for any $t \in [0, T]$, $\delta(t) \leq h/\mu^{2q_1+1} \leq h$. Therefore we can use on the intervals $[t, t + \delta(t)]$ the fact that our bounds are in $L(\mu, h)$. If $0 < t - t' \leq h$,

$$\mu^{2q_1+1} f_R(t)\delta(t) \geq \int_t^{t+\delta(t)} f_R(s)\, ds = \frac{1}{\mu^{2q_1+1}} = \int_{t'}^{t'+\delta(t')} f_R(s)\, ds \geq \mu^{-(2q_1+1)} f_R(t)\delta(t'),$$

so $\delta(t')/\delta(t) \leq \mu^{4q_1+2}$. Also the converse holds, and $\delta(\cdot) \in L(\mu^{4q_1+2}, h)$. We set

$$\varepsilon(t) = \bigg(\int_t^{t+\delta(t)} |\phi_s|^2\, ds\bigg)^{1/2}.$$

We have

$$\frac{1}{\mu^{2q_1+1}} = \int_t^{t+\delta(t)} f_R(s)\, ds \geq \int_t^{t+\delta(t)} \frac{f_R(t)}{\mu^{2q_1+1}}\, ds \geq \delta(t)\frac{f_R(t)}{\mu^{2q_1+1}},$$

so

$$\delta(t) \leq \frac{1}{f_R(t)} \leq \frac{R_t}{K_1}\Big(\frac{\lambda_t}{\mu n_t}\Big)^{q_1}. \tag{5.4}$$

Similarly,

$$\frac{1}{\mu^{2q_1+1}} \geq \int_t^{t+\delta(t)} K_1\left(\frac{\mu n_s}{\lambda_s}\right)^{q_1} |\phi_s|^2 \, ds \geq \frac{1}{\mu^{2q_1}} K_1\left(\frac{\mu n_t}{\lambda_t}\right)^{q_1} \varepsilon(t)^2,$$

and we can write both

$$\delta(t) \leq \frac{1}{K_1}\left(\frac{\lambda_t}{\mu n_t}\right)^{q_1} \quad \text{and} \quad \varepsilon(t)^2 \leq \frac{1}{K_1}\left(\frac{\lambda_t}{\mu n_t}\right)^{q_1}. \tag{5.5}$$

We set our time grid as

$$t_0 = 0; \quad t_k = t_{k-1} + \delta(t_{k-1}),$$

and introduce the following notation on the grid:

$$\delta_k = \delta(t_k); \qquad \varepsilon_k = \varepsilon(t_k); \qquad n_k = n_{t_k}; \qquad \lambda_k = \lambda_{t_k}; \qquad X_k = X_{t_k}; \qquad x_k = x_{t_k}; \qquad R_k = R_{t_k}.$$

We also define

$$\hat{X}_k = X_k + b(X_k)\delta_k; \qquad \hat{x}_k = x_k + b(x_k)\delta_k,$$

and for $t_k \leq t \leq t_{k+1}$,

$$\hat{X}_k(t) = X_k + b(X_k)(t - t_k); \qquad \hat{x}_k(t) = x_k + b(x_k)(t - t_k).$$

Moreover we denote

$$|\xi|_k = |\xi|_{A_{\delta_k}(x_k)}; \qquad \mathcal{C}_k = \mathcal{C}_{t_k},$$

and $r_k^* \in \mathcal{C}_k$ the radius r^* associated to (4.16), when taking as initial condition $x_0 = x_k$.

Remark 5.3. Consider $D_k = \{\sup_{t_k \leq t \leq t_{k+1}} |X_t - x_t|_{A_{R_t}(x_t)} \leq 1\}$, and $\Gamma_k = \{|X_k - x_k|_k \leq r_k\}$, where r_k is radius smaller than 1 that will be defined in the sequel. We denote $\mathbb{P}_k$ the conditional probability

$$\mathbb{P}_k(\cdot) = \mathbb{P}(\cdot | W_t, t \leq t_k; \Gamma_k).$$

We will lower bound $\mathbb{P}(\sup_{t \leq T} |X_t - x_t(\phi)|_{A_{R_t}(x_t(\phi))} \leq 1)$ computing the product of the probabilities $\mathbb{P}_k(D_k \cap \Gamma_{k+1})$, and this computation relies on the application of the density estimate in short time. Remark that **A1**, **A3** are local assumption, therefore it is enough to ask for **H1**, **H3** to apply Theorem 4.5. What about **A2** (global) and **H2** (local)? Suppose that we have a process X which, for some external reasons, verifies (2.1) for $t_k \leq t \leq t_{k+1}$, and such that $\sup_{t_k \leq t \leq t_{k+1}} |X_t - x_t|_{A_{R_t}(x_t)} \leq 1$. From **H2**

$$n(y) \leq n_k \quad \text{for } \{y \in \mathbb{R}^2 : |y - x_k| \leq 1\}.$$

A classical theorem (see [39]) tells us that we can define $\bar{\sigma}, \bar{b}$ which coincide with σ, b on $\{y \in \mathbb{R}^2 : |y - x_k| \leq 1\}$, which are differentiable as many times as σ, b but on the whole $\mathbb{R}^2$, and for which

$$n(y) \leq \alpha n_k \quad \text{for all } y \in \mathbb{R}^2, \text{ with } \alpha \text{ constant.}$$

Let $\bar{X}$ be the strong solution to

$$\bar{X}_t = X_k + \int_{t_k}^t \bar{\sigma}(\bar{X}_s) \circ dW_s + \int_{t_k}^t \bar{b}(\bar{X}_s) \, ds, \quad t \in [t_k, t_{k+1}].$$

It is clear that

$$\mathbb{P}(D_k \cap \Gamma_{k+1}) = \mathbb{P}\Big(\Big\{\sup_{t_k \leq t \leq t_{k+1}} |\bar{X}_t - x_t|_{A_{R_t}(x_t)} \leq 1\Big\} \cap \Big\{|\bar{X}_{t_{k+1}} - x_{k+1}|_{k+1} \leq r_{k+1}\Big\}\Big),$$

and therefore we can equivalently prove our estimates supposing that $n(y)$ is globally, and not just locally, bounded by n_k. From now on we assume that $n(y) \leq n_k$ for $y \in \mathbb{R}^2$.

Step 2 (Application of the density estimate): Lemmas 6.3, 6.5, 6.6, 6.7 hold for δ_k and ε_k small enough, and in particular Lemma 6.7 says that

$$\frac{1}{C_k^1}|\xi|_{A_\delta(x_k)} \leq |\xi|_{A_\delta(x_{k+1})} \leq C_k^1 |\xi|_{A_\delta(x_k)}, \tag{5.6}$$

for some $C_k^1 \in \mathcal{C}_k$, for any $\delta \leq \delta_k$. Recall (5.5), and

$$R_k/\mu \leq R_t \leq \mu R_k, \quad \text{for } t_k \leq t \leq t_{k+1},$$

so that $R_t \geq \delta_k$ for $t_k \leq t \leq t_{k+1}$. Moreover we have $|x_{k+1} - \hat{x}_k|_k \leq C_k(\varepsilon_k \vee \sqrt{\delta_k})$, and for all $t_k \leq t \leq t_{k+1}$, applying also (6.1), $|x_t - \hat{x}_k(t)|_{A_{R_t}(x_t)} \leq C_k(\varepsilon_k \vee \sqrt{\delta_k})$ for $t_k \leq t \leq t_{k+1}$. Recall again (5.5), and we fix q_3, K_3 such that, for $q_1 \geq q_3, K_1 \geq K_3$, the Lemmas 6.3, 6.5, 6.6, 6.7 hold and

$$|x_{k+1} - \hat{x}_k|_k \leq r_k^*/8, \tag{5.7}$$

$$\left|\hat{x}_k(t) - x_t\right|_{A_{R_t}(x_t)} \leq \frac{1}{4} \quad \text{for all } t_k \leq t \leq t_{k+1}, \tag{5.8}$$

and moreover $\delta_k \leq \delta_k^*$ associated to (4.16) with initial condition x_k.

Now, $\delta(\cdot) \in L(\mu^{4q_1+2}, h)$ implies $\delta_k/\delta_{k+1} \leq \mu^{4q_1+2}$ and $\delta_{k+1}/\delta_k \leq \mu^{4q_1+2}$. This, (5.6) and (6.1) give

$$\frac{1}{C_k^1 \mu^{2q_1+1}} |\xi|_k \leq |\xi|_{k+1} \leq \mu^{2q_1+1} C_k^1 |\xi|_k, \tag{5.9}$$

where C_k^1 is in $\mathcal{C}_k$, depending on K_3, q_3. We now set, for K_2, q_2 to be fixed in the sequel,

$$r_k = \frac{1}{K_2 \mu^{2q_1+2q_2+1}} \Big(\frac{\lambda_k}{n_k}\Big)^{q_2}, \tag{5.10}$$

and define as we said before

$$\Gamma_k = \big\{|X_k - x_k|_k \leq r_k\big\}, \qquad D_k = \Big\{\sup_{t_k \leq t \leq t_{k+1}} |X_t - x_t|_{A_{R_t}(x_t)} \leq 1\Big\},$$

and $\mathbb{P}_k$ as the conditional probability

$$\mathbb{P}_k(\cdot) = \mathbb{P}(\cdot|W_t, t \leq t_k; \Gamma_k).$$

We find a lower bound for $\mathbb{P}_k(\Gamma_{k+1} \cap D_k)$ using our density estimate in short time. We denote $p_k(X_k, y) = p_{\delta_k}(X_k, y)$ the density of X_{k+1} in y with respect to $\mathbb{P}_k$. We prove that on $\{y : |y - x_{k+1}|_{k+1} \leq r_{k+1}\}$ we can apply (4.16) to $p_k(X_k, \cdot)$ and so there exists $\underline{C}_k \in \mathcal{C}_k$ such that

$$\frac{1}{\underline{C}_k \delta_k^2} \leq p_k(X_k, y). \tag{5.11}$$

We estimate

$$|y - \hat{X}_k|_k \leq |y - x_{k+1}|_k + |x_{k+1} - \hat{x}_k|_k + |\hat{x}_k - \hat{X}_k|_k. \tag{5.12}$$

We already have (5.7). Since we are on $|y - x_{k+1}|_{k+1} \leq r_{k+1}$, from (5.9) and the fact that $r_{k+1}/r_k \leq \mu^{2q_2}$

$$|y - x_{k+1}|_k \leq C_k^1 \mu^{2q_1+1} |y - x_{k+1}|_{k+1} \leq C_k^1 \mu^{2q_1+1} r_{k+1} \leq C_k^1 \mu^{2q_1+2q_2+1} r_k \leq \frac{C_k^1}{K_2} \left(\frac{\lambda_k}{n_k} \right)^{q_2}.$$

It also holds $|\hat{x}_k - \hat{X}_k|_k \leq C_k |x_k - X_k|_k \leq C_k r_k$, for some $C_k \in \mathcal{C}_k$. Similarly, since $R_t \geq \delta_k$, from (6.1) $|\hat{x}_k(t) - \hat{X}_k(t)|_{A_{R_t}(x_t)} \leq C_k r_k$, for all $t_k \leq t \leq t_{k+1}$. Recalling (5.10), we can fix K_2, q_2 such that $|y - x_{k+1}|_k \leq r_k^*/16$, $|\hat{x}_k - \hat{X}_k|_k \leq r_k^*/16$, and

$$\left| \hat{X}_k(t) - \hat{x}_k(t) \right|_{A_{R_t}(x_t)} \leq 1/4, \qquad \text{for all } t_k \leq t \leq t_{k+1}. \tag{5.13}$$

From (5.12), (5.7) this implies $|y - \hat{X}_k|_k \leq r_k^*/4$. We also have $|x_k - X_k|_k \leq r_k$, so we can also fix K_2, q_2 such that $r_k \leq \alpha$ in Lemma 6.5. Therefore

$$\frac{1}{4} |\xi|_k \leq |\xi|_{A_{\delta_k}(X_k)} \leq 4|\xi|_k.$$

So $|y - \hat{X}_k|_{A_{\delta_k}(X_k)} \leq r_k^*$ and (4.16) holds (which means that (5.11) holds). Now, from Lemma 6.5 and (5.9)

$$\left\{ | \cdot - x_{k+1}|_{A_{\delta_k}(X_k)} \leq r_{k+1}/\left(4C_k^1 \mu^{2q_1+1} \right) \right\} \subset \left\{ | \cdot - x_{k+1}|_k \leq r_{k+1}/\left(C_k^1 \mu^{2q_1+1} \right) \right\}$$
$$\subset \left\{ | \cdot - x_{k+1}|_{k+1} \leq r_{k+1} \right\},$$

and $r_{k+1}/(4C_k^1 \mu^{2q_1+1}) \geq r_k/(4C_k^1 \mu^{2q_1+2q_2+1}) = \frac{1}{4C_k^1 K_2 \mu^{4q_1+4q_2+2}} (\frac{\lambda_k}{n_k})^{q_2}$. So

$$\text{Leb}\left(| \cdot - x_{k+1}|_{k+1} \leq r_{k+1} \right) \geq \delta_k^2 \det A(X_k) \left(\frac{1}{4C_k^1 K_2 \mu^{4q_1+4q_2+2}} \left(\frac{\lambda_k}{n_k} \right)^{q_2} \right)^2.$$

Now, from **H1**, $\det A(X_k) \geq \lambda_k$. So, from (5.11),

$$\mathbb{P}_k(\Gamma_{k+1}) \geq \frac{1}{\underline{C}_k} \left(\frac{1}{4C_k^1 K_2 \mu^{4q_1+4q_2+2}} \left(\frac{\lambda_k}{n_k} \right)^{q_2} \right)^2 \lambda_k,$$

where $\underline{C}_k \in \mathcal{C}_k$ is the constant in (4.16). This implies

$$2\mu^{-8q_1} \exp\left(-K_4(\log \mu + \log n_k - \log \lambda_k) \right) \leq \mathbb{P}_k(\Gamma_{k+1})$$

for some constant K_4 (depending on K_2, K_3, q_2, q_3; on the contrary, we keep explicit the dependence in q_1, which is not fixed yet).

Step 3 (Concatenation): Consider now $t_k \leq t \leq t_{k+1}$. Recall the definition

$$D_k = \left\{ \sup_{t_k \leq t \leq t_{k+1}} |X_t - x_t|_{A_{R_t}(x_t)} \leq 1 \right\},$$

and introduce

$$E_k = \left\{ \sup_{t_k \leq t \leq t_{k+1}} \left| X_t - \hat{X}_k(t) \right|_{A_{R_t}(x_t)} \leq \frac{1}{2} \right\}.$$

We decompose

$$|X_t - x_t|_{A_{R_t}(x_t)} \leq \left| X_t - \hat{X}_k(t) \right|_{A_{R_t}(x_t)} + \left| \hat{X}_k(t) - \hat{x}_k(t) \right|_{A_{R_t}(x_t)} + \left| \hat{x}_k(t) - x_t \right|_{A_{R_t}(x_t)},$$

and, from the previous part of the proof, (5.8) gives $|\hat{x}_k(t) - x_t|_{A_{R_t}(x_t)} \leq 1/4$, and (5.13) gives $|\hat{X}_k(t) - \hat{x}_k(t)|_{A_{R_t}(x_t)} \leq 1/4$. So $|X_t - x_t|_{A_{R_t}(x_t)} \leq |X_t - \hat{X}_k(t)|_{A_{R_t}(x_t)} + 1/2$, and therefore $E_k \subset D_k$.

Now we have to estimate E_k. A development of $X_t - \hat{X}_k(t)$ similar to (4.4) gives that the diffusion moves with speed $\delta_k^{1/2}$ in the direction of $\sigma(x_k)$, $\delta_k^{3/2}$ otherwise. Taking the $|\cdot|_{A_{R_t}(x_t)}$ norm we account precisely of this fact. Applying the exponential martingale inequality we find that

$$\mathbb{P}_k\big(E_k^c\big) \leq \exp\left(-\frac{1}{K_5}\left(\frac{\lambda_k}{\mu n_k}\right)^{q_5} \frac{R_k}{\delta_k}\right)$$

for some constants K_5, q_5. From (5.4), $R_k/\delta_k \geq K_1(\mu n_k/\lambda_k)^{q_1}$. We recall that $\lambda_k \leq 1$ and $n_k \geq 1$, so choosing and fixing now q_1, K_1 large enough we conclude

$$\mathbb{P}_k\big(E_k^c\big) \leq \mu^{-8q_1} \exp\big(-K_4(\log\mu + \log n_k - \log\lambda_k)\big) \leq \frac{1}{2}\mathbb{P}_k(\Gamma_{k+1}),$$

so

$$\begin{aligned}
\mathbb{P}_k(\Gamma_{k+1} \cap D_k) \geq \mathbb{P}_k(\Gamma_{k+1} \cap E_k) &\geq \mathbb{P}_k(\Gamma_{k+1}) - \mathbb{P}_k\big(E_k^c\big) \geq \frac{1}{2}\mathbb{P}_k(\Gamma_{k+1}) \\
&\geq \exp\big(-K_6(\log\mu + \log n_k - \log\lambda_k)\big),
\end{aligned} \tag{5.14}$$

for some constant K_6. Let now $N(T) = \max\{k : t_k \leq T\}$. From definition (5.3)

$$\int_0^T f_R(t)\,dt \geq \sum_{k=1}^{N(T)} \int_{t_{k-1}}^{t_k} f_R(t)\,dt \geq \frac{N(T)}{\mu^{2q_1+1}}.$$

From (5.14),

$$\begin{aligned}
\mathbb{P}\Big(\sup_{t\leq T}\big|X_t - x_t(\phi)\big|_{A_{R_t}(x_t(\phi))} \leq 1\Big) &\geq \mathbb{P}\left(\bigcap_{k=1}^{N(T)} \Gamma_{k+1} \cap D_k\right) \\
&\geq \prod_{k=1}^{N(T)} \exp\big(-K_6(\log\mu + \log n_k - \log\lambda_k)\big) \\
&= \exp\left(-K_6 \sum_{k=1}^{N(T)} \log\mu + \log n_k - \log\lambda_k\right).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{k=1}^{N(T)} (\log\mu + \log n_k - \log\lambda_k) &= \mu^{2q_1+1} \sum_{k=1}^{N(T)} \int_{t_k}^{t_k+1} f_R(s)\,ds\,(\log\mu + \log n_k - \log\lambda_k) \\
&\leq \int_0^T \mu^{2q_1+1} f_R(t) \log\left(\frac{\mu^3 n_t}{\lambda_t}\right) dt,
\end{aligned}$$

the lower bound follows.

Step 4 (Upper bound): We define, with the same K_1, q_1 as in Step 1,

$$g_R(t) = K_1\left(\frac{\mu n_t}{\lambda_t}\right)^{q_1}\left(\frac{\exp(-K_*(\frac{\mu n_t}{\lambda_t})^{q_*}\mu^{2q_*})}{R_t} + |\phi_t|^2\right).$$

Because of (5.1), for all $t \in [0, T]$,

$$\frac{\exp(-K_*(\frac{\mu n_t}{\lambda_t})^{q_*}\mu^{2q_*})}{R_t} \geq \frac{1}{h}. \tag{5.15}$$

We define now a new $\delta(t)$

$$\delta(t) = \inf_{\delta > 0}\left\{\int_t^{t+\delta} g_R(s)\,ds \geq \frac{1}{\mu^{2q_1+1}}\right\}$$

and, as before,

$$\varepsilon(t) = \left(\int_t^{t+\delta(t)} |\phi_s|^2\,ds\right)^{1/2}.$$

As in Step 1, using also (5.15), we can check that (5.5) holds also for this choice of δ:

$$\delta(t) \leq \frac{h}{K_1}\left(\frac{\lambda_t}{\mu n_t}\right)^{q_1} \leq \frac{1}{K_1}\left(\frac{\lambda_t}{\mu n_t}\right)^{q_1} \quad \text{and} \quad \varepsilon(t)^2 \leq \frac{1}{K_1}\left(\frac{\lambda_t}{\mu n_t}\right)^{q_1}.$$

In particular, $\delta(t) \leq h$. With these definitions we set a time grid $\{t_k : k = 0, \ldots, N(T)\}$ and all the associated quantities as in Step 1. As we did for the lower bound, since we estimate the probability of remaining in the tube for any $t \in [t_k, t_{k+1}]$, we can suppose that the bound $n(y) \leq n_k$ holds $\forall y \in \mathbb{R}^2$. The short time density estimate (4.16) holds again. Recall now that $R_. \in L(\mu, h)$, and this gives the analogous to (5.9):

$$\frac{1}{C_k^1\sqrt{\mu}}|\xi|_{A_{R_k}(x_k)} \leq |\xi|_{A_{R_{k+1}}(x_{k+1})} \leq C_k^1\sqrt{\mu}|\xi|_{A_{R_k}(x_k)}.$$

We define

$$\Delta_k = \left\{|X_k - x_k|_{A_{R_k}(x_k)} \leq 1\right\},$$

$\tilde{\mathbb{P}}_k$ as the conditional probability $\tilde{\mathbb{P}}_k(\cdot) = \mathbb{P}(\cdot|W_t, t \leq t_k; \Delta_k)$. Now, since $\delta(t) \leq h$, we can apply the fact that $R, \lambda, n \in L(\mu, h)$ and

$$\int_t^{t+\delta(t)} K_1\left(\frac{\mu n_s}{\lambda_s}\right)^{q_1}|\phi|_s^2\,ds \leq \mu^{2q_1}K_1\left(\frac{\mu n_t}{\lambda_t}\right)^{q_1}\int_t^{t+\delta(t)}|\phi|_s^2\,ds,$$

$$\int_t^{t+\delta(t)} K_1\left(\frac{\mu n_s}{\lambda_s}\right)^{q_1}\frac{\exp(-K_*(\frac{\mu n_s}{\lambda_s})^{q_*}\mu^{2q_*})}{R_s}\,ds \leq \mu^{2q_1+1}K_1\left(\frac{\mu n_t}{\lambda_t}\right)^{q_1}\exp\left(-K_*\left(\frac{\mu n_t}{\lambda_t}\right)^{q_*}\right)\frac{\delta(t)}{R_t}.$$

Recall now (5.1)

$$R_t \leq R_t^*(\phi) = \exp\left(-K_*\left(\frac{\mu n_t}{\lambda_t}\right)^{q_*}\mu^{2q_*}\right)\left(\inf_{0\leq\delta\leq h}\left\{\delta\Big/\int_t^{t+\delta}|\phi_s|^2\,ds\right\}\right),$$

which implies

$$\int_t^{t+\delta(t)}|\phi_s|^2\,ds \leq \exp\left(-K_*\left(\frac{\mu n_t}{\lambda_t}\right)^{q_*}\right)\frac{\delta(t)}{R_t}.$$

We obtain

$$1 = \mu^{2q_1+1}\int_t^{t+\delta(t)} g_R(s)\,ds \leq 2\mu^{4q_1+2}K_1\left(\frac{\mu n_t}{\lambda_t}\right)^{q_1}\exp\left(-K_*\left(\frac{\mu n_t}{\lambda_t}\right)^{q_*}\right)\frac{\delta(t)}{R_t}.$$

so

$$\frac{R_t}{\delta(t)} \le 2\mu^{4q_1+2} K_1 \left(\frac{\mu n_t}{\lambda_t}\right)^{q_1} \exp\left(-K_* \left(\frac{\mu n_t}{\lambda_t}\right)^{q_*}\right). \tag{5.16}$$

As we did in Step 1, if q_*, K_* are large enough, R_k is small enough and the upper bound for the density holds on Δ_{k+1}. Because of (5.6),

$$\mathrm{Leb}\big(|\cdot - x_k|_{A_{R_k}(x_{k+1})} \le 1\big) \le \mathrm{Leb}\big(|\cdot - x_k|_{A_{R_k}(x_k)} \le 1\big)(C_k^1)^2 = (C_k^1)^2 \det\big(A(x_k)\big) R_k^2.$$

Now, using the density estimate,

$$\tilde{\mathbb{P}}_k(\Delta_{k+1}) \le (C_k^1)^2 \det\big(A(x_k)\big) e^{\overline{C}_k} \left(\frac{R_k}{\delta_k}\right)^2,$$

where $\overline{C}_k$ is the constant in the upper bound of (4.16). Recall (5.16), for $t = t_k$

$$\frac{R_k}{\delta_k} \le 2\mu^{4q_1+2} K_1 \left(\frac{\mu n_k}{\lambda_k}\right)^{q_1} \exp\left(-K_* \left(\frac{\mu n_k}{\lambda_k}\right)^{q_*}\right)$$

so we chose now K_*, q_* large enough to have

$$\tilde{\mathbb{P}}_k(\Delta_{k+1}) \le \exp(-K_7)$$

for a constant $K_7 > 0$. From the definition of $N(T)$

$$\int_0^T g_R(t)\,dt = \sum_{k=1}^{N(T)} \int_{t_{k-1}}^{t_k} g_R(t)\,dt = \frac{N(T)}{\mu^{2q_1+1}} \le N(T).$$

As before

$$\mathbb{P}\Big(\sup_{t \le T}\big|X_t - x_t(\phi)\big|_{A_{R_t}(x_t(\phi))} \le 1\Big) \le \prod_{k=1}^{N(T)} \tilde{\mathbb{P}}_k(\Delta_{k+1})$$

$$\le \prod_{k=1}^{N(T)} \exp(-K_7) = \exp\big(-K_7\, N(T)\big) \le \exp\Big(-K_7 \int_0^T g_R(t)\Big),$$

and we have the upper bound. $\qquad\square$

6. Matrix norm and control metric

6.1. *Matrix norms*

In this work we use a number of properties of norms associated to the matrix A and A_R. Recall that in general we can associate a norm to a matrix M with full row rank via

$$|y|_M = \sqrt{\big\langle (MM^T)^{-1} y, y\big\rangle}.$$

Recall that, for $R > 0$,

$$A = \big(\sigma, [\sigma, b]\big), \qquad A_R = \big(R^{1/2}\sigma, R^{3/2}[\sigma, b]\big).$$

Lemma 6.1. *For every $y \in \mathbb{R}^2$ and $0 < R \leq R' \leq 1$,*

$$\left(R/R'\right)^{1/2}|y|_{A_R} \geq |y|_{A_{R'}} \geq \left(R/R'\right)^{3/2}|y|_{A_R}, \tag{6.1}$$

$$\frac{1}{R^{1/2}\lambda^*(A)}|y| \leq |y|_{A_R} \leq \frac{1}{R^{3/2}\lambda_*(A)}|y|. \tag{6.2}$$

Proof. Writing explicitly the inequalities (6.1), we easily see that they are verified if $0 < R \leq R' \leq 1$. Taking $R' = 1$, we have

$$R^{1/2}|y|_{A_R} \geq |y|_A \geq R^{3/2}|y|_{A_R}$$

and so

$$\frac{1}{R^{1/2}\lambda^*(A)}|y| \leq |y|_{A_R} \leq \frac{1}{R^{3/2}\lambda_*(A)}|y|. \qquad \square$$

Remark 6.2. Recall the following properties of matrices:

$$\forall \xi, \quad C\,|\xi|_B^2 \geq |\xi|_A^2 \quad \Leftrightarrow \quad C\left(BB^T\right)^{-1} \geq \left(AA^T\right)^{-1} \quad \Leftrightarrow \quad BB^T \leq C\,AA^T$$

and, denoting with M_i the columns of M,

$$\left\langle MM^T\xi, \xi\right\rangle = \sum_i \langle M_i, \xi\rangle^2,$$

so that

$$\lambda_*(M)^2 = \inf_{|\xi|=1} \sum_i \langle M_i, \xi\rangle^2 \quad \text{and} \quad \lambda^*(M)^2 = \sup_{|\xi|=1} \sum_i \langle M_i, \xi\rangle^2.$$

Taking $M = A(x) = (\sigma(x), [\sigma, b](x))$ we have in particular that

$$\lambda_*\left(A(x)\right)^2|\xi|^2 \leq \left\langle\sigma(x), \xi\right\rangle^2 + \left\langle[\sigma, b](x), \xi\right\rangle^2 \leq \lambda^*\left(A(x)\right)^2|\xi|^2 \quad \forall \xi \in \mathbb{R}^2. \tag{6.3}$$

We prove now some equivalences between norms that will be needed especially in the concatenation along the tube. We state them for $t_k = t_0 = 0$ to lighten the notation. Recall that x_0 is the initial condition of (2.1), and that in the concatenation (Section 5) we have

H1 $\lambda_*(A(x)) \geq \lambda_0, \forall|x - x_0| < 1$.
H2 $n(x) \leq n_0, \forall x \in \mathbb{R}^2$ (this is justified in Step 1 of the proof).
H3 $\partial_\sigma \sigma(x) = \kappa_\sigma(x)\sigma(x), \forall|x - x_0| < 1, |\kappa_\sigma| \leq n_0, |\nabla\kappa_\sigma| \leq n_0$.

Moreover, we recall that $\lambda_0 \leq 1$ and $n_0 \geq 1$. In (5.2) we define a class of constants that in the case $t = 0$ is

$$\mathcal{C}_0 = \left\{C > 0 : C = \left(K(n_0/\lambda_0)^q\right), \exists K, q \geq 1\right\}.$$

Lemma 6.3. *There exists $C \in \mathcal{C}_0, \delta^* \in 1/\mathcal{C}_0$ such that for $\delta \leq \delta^*$, with $\hat{x}_0 = x_0 + b(x_0)\delta$, for any $\xi \in \mathbb{R}^2$*

$$\frac{1}{C}|\xi|_{A_\delta(x_0)} \leq |\xi|_{\bar{A}_\delta(x_0)} \leq C|\xi|_{A_\delta(x_0)}, \tag{6.4}$$

$$\frac{1}{C}|\xi|_{A_\delta(x_0)} \leq |\xi|_{A_\delta(\hat{x}_0)} \leq C|\xi|_{A_\delta(x_0)}. \tag{6.5}$$

Remark 6.4. This lemma is used also in Section 4, when $\mathcal{C}_t$ has not yet been defined. It is clear that in that case the constants must be taken in $\mathcal{C}$ defined in (4.6).

Proof of Lemma 6.3. We take $M = A_\delta(x_0)$ and $M = \bar{A}_\delta(x_0)$ in Remark 6.2. Recall that $\lambda_0 \leq 1$ and $n_0 \geq 1$ and notice that

$$\left|\partial_b \sigma(x_0)\right| \leq 4n_0^2 \leq \frac{4n_0^2}{\lambda_*(A(x_0))} \lambda_*\big(A(x_0)\big) \leq C\lambda_*\big(A(x_0)\big), \quad \text{with } C \in \mathcal{C}_0$$

so, from (6.3)

$$\delta^3 \langle \partial_b \sigma(x_0), \xi \rangle^2 \leq \delta^3 C \lambda_*^2\big(A(x_0)\big)|\xi|^2 \leq C\big(\delta\langle\sigma(x_0), \xi\rangle^2 + \delta^3\langle[\sigma, b](x_0), \xi\rangle^2\big).$$

We have

$$\delta\langle\sigma(x_0) + \delta\partial_b\sigma(x_0), \xi\rangle^2 + \delta^3\langle[\sigma, b](x_0), \xi\rangle^2$$
$$\leq 2\delta\langle\sigma(x_0), \xi\rangle^2 + 2\delta^3\langle\partial_b\sigma(x_0), \xi\rangle^2 + \delta^3\langle[\sigma, b](x_0), \xi\rangle^2$$
$$\leq C\big(\delta\langle\sigma(x_0), \xi\rangle^2 + \delta^3\langle[\sigma, b](x_0), \xi\rangle^2\big),$$

so $|\xi|^2_{A_\delta(x_0)} \leq C|\xi|^2_{\bar{A}_\delta(x_0)}$. Analogously,

$$\delta\langle\sigma(x_0), \xi\rangle^2 + \delta^3\langle[\sigma, b](x_0), \xi\rangle^2 \leq C\big(\langle\delta\sigma(x_0) + \delta\partial_b\sigma(x_0), \xi\rangle^2 + \delta^3\langle[\sigma, b](x_0), \xi\rangle^2\big),$$

so $|\xi|^2_{\bar{A}_\delta(x_0)} \leq C|\xi|^2_{A_\delta(x_0)}$. From

$$\left|\sigma(\hat{x}_0) - \sigma(x_0)\right| = \left|\sigma\big(x_0 + b(x_0)\delta\big) - \sigma(x_0)\right| \leq \int_0^\delta \left|\nabla\sigma\big(x_0 + b(x_0)t\big)b(x_0)\right| dt \leq C\delta,$$

applying again Remark 6.2 as in the previous point, also (6.5) follows. $\qquad \square$

The following lemma establish the equivalence of matrix norms of this kind when the matrix is taken in two points that are close in such matrix norms.

Lemma 6.5. *Consider $x_0, x, y \in \mathbb{R}^2$, with $|x - x_0| < 1$. There exist $\alpha \in 1/\mathcal{C}_0$ such that if and $|x - y|_{A_\delta(x)} \leq \alpha$,*

$$\frac{1}{4}|\xi|_{A_\delta(x)} \leq |\xi|_{A_\delta(y)} \leq 4|\xi|_{A_\delta(x)}, \quad \forall\xi \in \mathbb{R}^2.$$

Proof. Remark that (6.2) implies

$$|x - y| \leq \delta^{1/2}C_1|x - y|_{A_\delta(x)} \leq \alpha C_1 \delta^{1/2} \leq \delta^{1/2}$$

for $\alpha \leq 1/C_1$. A Taylor development gives

$$\sigma(x) - \sigma(y) = \nabla\sigma(x)(x - y) + \mathcal{O}\big(|x - y|^2\big),$$

so

$$\langle\sigma(x), \xi\rangle^2 \leq 4\langle\sigma(y), \xi\rangle^2 + 4\langle\nabla\sigma(x)(x - y), \xi\rangle^2 + C_2|x - y|^4|\xi|^2.$$

Since $A_\delta(x)$ is invertible,

$$\nabla\sigma(x)(x - y) = \nabla\sigma(x)A_\delta(x)A_\delta^{-1}(x)(x - y).$$

From Cauchy–Schwarz inequality and $|A_\delta^{-1}(x)(x-y)| \le \alpha$,

$$\left|\langle \nabla\sigma(x)(x-y), \xi\rangle\right| = \left|\langle A_\delta^{-1}(x)(x-y), \left(\nabla\sigma(x)A_\delta(x)\right)^T\xi\rangle\right|$$
$$\le \alpha\left|\left(\nabla\sigma(x)A_\delta(x)\right)^T\xi\right|.$$

We are supposing **H3**, so $\partial_\sigma\sigma = \kappa_\sigma\sigma$ holds in x, and

$$\nabla\sigma(x)A_\delta(x) = \nabla\sigma(x)\left(\delta^{1/2}\sigma(x), \delta^{3/2}[\sigma, b](x)\right)$$
$$= \left(\delta^{1/2}\kappa_\sigma(x)\sigma(x), \delta^{3/2}\partial_{[\sigma,b]}\sigma(x)\right),$$

so

$$\left|\left(\nabla\sigma(x)A_\delta(x)\right)^T\xi\right|^2 = \delta\kappa_\sigma^2(x)\langle\sigma(x), \xi\rangle^2 + \delta^3\langle\partial_{[\sigma,b]}\sigma(x), \xi\rangle^2$$

and therefore

$$\langle\nabla\sigma(x)(x-y), \xi\rangle^2 \le \alpha^2\left(\delta\kappa_\sigma^2(x)\langle\sigma(x), \xi\rangle^2 + \delta^3\langle\partial_{[\sigma,b]}\sigma(x), \xi\rangle^2\right)$$
$$\le C_3\alpha^2\delta\langle\sigma(x), \xi\rangle^2 + C_3\alpha^2\delta^3|\xi|^2.$$

Now,

$$C_2|x-y|^4|\xi|^2 \le C_2C_1^4\alpha^4\delta^2|\xi|^2.$$

So

$$\langle\sigma(x), \xi\rangle^2 \le 4\langle\sigma(y), \xi\rangle^2 + 4C_3\alpha^2\delta\langle\sigma(x), \xi\rangle^2 + 4C_3\alpha^2\delta^3|\xi|^2 + C_2C_1^4\alpha^4\delta^2|\xi|^2.$$

Taking $\alpha \le \frac{1}{8C_3C_2C_1^2}$, this implies

$$\langle\sigma(x), \xi\rangle^2 \le 8\langle\sigma(y), \xi\rangle^2 + \alpha\delta^2|\xi|^2.$$

In the direction $[\sigma, b]$ we have $[\sigma, b](x) - [\sigma, b](y) = \mathcal{O}(|x-y|)$

$$\langle[\sigma, b](x), \xi\rangle^2 \le 2\langle[\sigma, b](y), \xi\rangle^2 + C_4|x-y|^2|\xi|^2 \le 2\langle[\sigma, b](y), \xi\rangle^2 + C_4C_1^2\alpha^2\delta|\xi|^2.$$

We take now $\alpha \le 1/(C_4C_1^2)$, and we conclude that

$$\delta\langle\sigma(x), \xi\rangle^2 + \delta^3\langle[\sigma, b](x), \xi\rangle^2 \le 8\delta\langle\sigma(y), \xi\rangle^2 + 2\delta^3\langle[\sigma, b](y), \xi\rangle^2 + 2\alpha\delta^3|\xi|^2.$$

Using now (6.3) and **H1**,

$$|\xi|^2 \le C_5\left(\langle\sigma(y), \xi\rangle^2 + \langle[\sigma, b](y), \xi\rangle^2\right).$$

So taking $\alpha \le 4/C_5$ we have

$$\delta\langle\sigma(x), \xi\rangle^2 + \delta^3\langle[\sigma, b](x), \xi\rangle^2 \le 16\delta\langle\sigma(y), \xi\rangle^2 + 16\delta^3\langle[\sigma, b](y), \xi\rangle^2.$$

From Remark 6.2 we have $|\xi|_{A_\delta(x)} \le 4|\xi|_{A_\delta(y)}$. The converse inequality follows from an analogous reasoning. Remark that all the conditions we need on α are satisfied taking $\alpha \in 1/\mathcal{C}_0$ small enough, since $|x-x_0| < 1$ and **H1**, **H2**, **H3**. $\square$

We prove now that moving along a control $\phi \in L^2[0, T]$ for a small time, the trajectory remains close to the initial point in the A_δ-norm. Define, for fixed δ,

$$\varepsilon = \left(\int_0^\delta |\phi_s|^2 \, ds \right)^{1/2}.$$

Recall that we have

$$x_t(\phi) = x_0 + \int_0^t \sigma\big(x_s(\phi)\big)\phi_s \, ds + \int_0^t b\big(x_s(\phi)\big) \, ds.$$

Lemma 6.6. *There exist $\delta_*, \varepsilon_* \in 1/\mathcal{C}_0$, $C \in \mathcal{C}_0$ such that if $\delta \le \delta_*$, $\varepsilon \le \varepsilon_*$*

$$\big| x_\delta(\phi) - \big(x_0 + b(x_0)\delta\big) \big|_{\bar{A}_\delta(x_0)} \le C\big(\varepsilon \vee \delta^{1/2}\big).$$

Proof. Via computations analogous to decomposition (4.4) it is possible to write

$$x_\delta(\phi) - \big(x_0 + b(x_0)\delta\big) = \bar{A}_\delta(x_0)(G_\phi + \tilde{R}_{\phi,\delta}),$$

where

$$G_\phi = \Theta_\phi + \tilde{\eta}_\delta(\Theta_\phi), \qquad \Theta_\phi = \begin{pmatrix} \delta^{-1/2} \int_0^\delta \phi_s \, ds \\ \delta^{-3/2} \int_0^\delta (\delta - s)\phi_s \, ds \end{pmatrix}$$

and

$$|\tilde{R}_{\phi,\delta}| \le C\big(\varepsilon \vee \delta^{1/2}\big).$$

Remark that, by Hölder inequality,

$$\left| \delta^{-1/2} \int_0^\delta \phi_s \, ds \right| \le \varepsilon, \qquad \left| \delta^{-3/2} \int_0^\delta (\delta - s)\phi_s \, ds \right| \le \varepsilon$$

so $|\Theta_\phi| \le 2\varepsilon$ and by (4.2) $|\tilde{\eta}_\delta(\Theta_\phi)| \le 4\varepsilon^2$. Therefore $|G_\phi| \le 4\varepsilon$ and

$$\left| \bar{A}_\delta(x_0)^{-1}\big(x_\delta(\phi) - \big(x_0 + b(x_0)\delta\big)\big) \right| = |G_\phi + \tilde{R}_{\phi,\delta}| \le C\big(\varepsilon \vee \delta^{1/2}\big). \qquad \square$$

Lemma 6.7. *There exist $\delta_*, \varepsilon_* \in 1/\mathcal{C}_0$, $C \in \mathcal{C}_0$ such that for $\delta \le \delta_*$, $\varepsilon \le \varepsilon_*$*

$$\frac{1}{C} |\xi|_{A_\delta(x_0)} \le |\xi|_{A_\delta(x_\delta)} \le C |\xi|_{A_\delta(x_0)}.$$

Proof. Recall $\hat{x}_0 = x_0 + \delta b(x_0)$. Applying in this order (6.5), (6.4), Lemma 6.6 we obtain

$$|x_\delta - \hat{x}_0|_{A_\delta(\hat{x}_0)} \le C\big|x_\delta - \big(x_0 + b(x_0)\delta\big)\big|_{A_\delta(x_0)} \le C\big|x_\delta - \big(x_0 + b(x_0)\delta\big)\big|_{\bar{A}_\delta(x_0)} \le C\big(\varepsilon \vee \delta^{1/2}\big).$$

Now, choosing δ_*, ε_* small enough, we can apply Lemma 6.5 to the points $x_\delta, \hat{x}_0$, and

$$\frac{1}{4} |\xi|_{A_\delta(\hat{x}_0)} \le |\xi|_{A_\delta(x_\delta)} \le 4 |\xi|_{A_\delta(\hat{x}_0)}.$$

Now again (6.5) concludes the proof. $\qquad \square$

6.2. *The control metric*

Recall (2.1), (2.2), (2.3). In the spirit of [28], we want to express our results is some control norm. Let

$$\Omega = \big\{ x \in \mathbb{R}^2 : \lambda(x) = \lambda_*\big(A(x)\big) > 0 \big\}.$$

A natural way to associate a quasi-distance to the matrix norm $|\cdot|_{A_R(\cdot)}$ used in this paper is to define

$$d(x, y) < \sqrt{R} \quad \Leftrightarrow \quad |x - y|_{A_R(x)} < 1$$

(we take $\sqrt{R}$ because it is the "diffusive" regime). With this definition, d is a quasi-distance on Ω, verifying the following properties (see [30]):

  (i) for every $x \in \Omega$, for every $r > 0$, the set $\{y \in \Omega : d(x, y) < r\}$ is open,
 (ii) $d(x, y) = 0$ if and only if $x = y$,
(iii) for every compact set $K \Subset \Omega$ there exists $C > 0$ such that $d(x, y) \le C(d(x, z) + d(z, y))$ holds for every $x, y, z \in K$.

We say that two quasi-distances $d_1 : \Omega \times \Omega \to \mathbb{R}^+$ and $d_2 : \Omega \times \Omega \to \mathbb{R}^+$ are equivalent if for every compact set $K \Subset \Omega$ there exists a constant C such that for every $x, y \in K$

$$\frac{1}{C} d_1(x, y) \le d_2(x, y) \le C d_1(x, y). \tag{6.6}$$

In particular if d_1 is a distance and d_2 is equivalent with d_1 then d_2 is a quasi-distance.

On the other hand, the distance usually considered in the framework of hypoelliptic stochastic differential equations is the *control distance* defined as follows: denote, for $x, y \in \Omega$,

$$C(x, y) = \big\{ \phi \in L^2(0, 1) : dv_s = \sigma(v_s)\phi_s \, ds, \ x = v_0, \ y = v_1 \big\}. \tag{6.7}$$

The control distance d_c between x and y is

$$d_c(x, y) = \inf \bigg\{ \bigg(\int_0^1 |\phi_s|^2 \, ds \bigg)^{1/2} : \phi \in C(x, y) \bigg\}.$$

Geometrically speaking, this corresponds to take the geodesic (i.e. the length-minimizing curve) joining x and y on the sub-Riemannian manifold associated with the diffusion coefficient σ. In our case this notion looks inadequate: we are supposing just a weak Hörmander condition, and this means that we have to use the drift coefficient b to generate the whole space $\mathbb{R}^2$. Therefore any reasonable associated distance should incorporate b as well. Moreover it should account of the different speed associated to the vector field $[\sigma, b]$. This is the reason for the following definition.

Definition 6.8. We first introduce a function which accounts of the different scale of propagation in the direction $[\sigma, b]$. For $\phi = (\phi_s^1, \phi_s^2) \in L^2((0, 1), \mathbb{R}^2)$,

$$\|\phi\|_{1,3}^2 = \int_0^1 |\phi_s^1|^2 \, ds + \bigg(\int_0^1 |\phi_s^2|^2 \, ds \bigg)^{\frac{1}{3}}.$$

We generalize (6.7) to

$$C_A(x, y) = \big\{ \phi \in L^2\big((0, 1), \mathbb{R}^2\big) : dv_s = A(v_s)\phi_s \, ds, \ x = v_0, \ y = v_1 \big\}.$$

A classic result by Carathéodory says that for any $x, y \in \Omega$ there exist a piecewise smooth $\phi \in C_A(x, y)$. We set

$$d_c(x, y) = \inf \big\{ \|\phi\|_{1,3} : \phi \in C_A(x, y) \big\}.$$

We are interested in establishing an equivalence between d, the quasi-distance defined via the matrix-norm, and d_c, the quasi-distance in terms of the control.

Lemma 6.9. *Let $\xi \in \Omega$. Suppose that there exists a neighborhood U_ξ of ξ such that for all $x \in U_\xi$:*

A1′ $\lambda_*(A(x)) > \underline{\lambda_\xi} > 0$,
A2′ $\sum_{0 \le |\alpha| \le 5} |\partial_x^\alpha \sigma(x)| + |\partial_x^\alpha b(x)| \le \rho_\xi$,
A3′ $\partial_\sigma \sigma(x) = \kappa_\sigma(x)\sigma(x)$, *where κ_σ is a differentiable scalar function, $|\kappa_\sigma(x)| \le \rho_\xi$ and $|\nabla \kappa_\sigma(x)| \le \rho_\xi$.*

Then there exist a neighborhood V_ξ of ξ and a constant C_ξ such that, for any $x, y \in V_\xi$

$$\frac{1}{C_\xi} d(x, y) \le d_c(x, y) \le C_\xi d(x, y). \tag{6.8}$$

Remark 6.10. This implies, using the fact that every open cover of a compact has a finite subcover, Corollary 2.2. Moreover, again via a standard compactness argument, we have that *if* **A1′**, **A2′**, **A3′** *hold for any $\xi \in \Omega$, then d and d_c are equivalent quasi-distances* on Ω.

Proof of Lemma 6.9. We use in this proof some notions on similar metrics and pseudo-metrics for which we refer to [30]. For any $\phi \in L^\infty((0, T), \mathbb{R}^2)$ we set

$$\|\phi\|_{1,3,\infty} = \sup_{0 \le s \le 1} |\phi_s^1| + \sup_{0 \le s \le 1} |\phi_s^2|^{\frac{1}{3}}$$

and define

$$\rho(x, y) = \inf\{\|\phi\|_{1,3,\infty} : \phi \in C_A(x, y)\}.$$

It is also possible to allow only constant linear combinations of the vector fields:

$$\bar{C}_A(x, y) = \{\theta \in \mathbb{R}^2 : dv_s = A(v_s)\theta \, ds, \ x = v_0, \ y = v_1\}. \tag{6.9}$$

Analogously, we define

$$\rho_2(x, y) = \inf\{|\theta^1| + |\theta^2|^{1/3} : \theta \in \bar{C}_A(x, y)\}.$$

In [30] the quasi-distances ρ and ρ_2 are defined in a slightly different way, but clearly equivalent to ours. It is also proved that ρ and ρ_2 are locally equivalent. We use here only the trivial inequality $\rho \le \rho_2$. Remark that the difference between ρ and d_c is that we take $\|\phi\|_{1,3,\infty}$ instead of $\|\phi\|_{1,3}$, so $d_c \le \rho$ follows easily from the fact that the $L^2(0, 1)$ norm is dominated by the $L^\infty(0, 1)$ norm.

We prove that, for fixed ξ, there exist V_ξ and C_ξ such that

$$d(x, y) < \sqrt{R} \quad \Rightarrow \quad \rho_2(x, y) < C_\xi \sqrt{R},$$

for $x, y \in V_\xi$. Since $x, y \in V_\xi$, we can suppose $|x - y| < \gamma_\xi$ small. By definition, $d(x, y) < \sqrt{R}$ means $|x - y|_{A_R(x)} < 1$. We prove that this implies the existence of $\theta \in \bar{C}_A(x, y)$ with $|\theta^1| < C_\xi R^{1/2}$, $|\theta^2| < C_\xi R^{3/2}$. Indeed, for fixed x, consider the function

$$\theta \to \Phi(\theta) = \int_0^1 A(v_s)\theta \, ds,$$

with v satisfying $dv_s = A(v_s)\theta \, ds$, $v_0 = x$. Remark that $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(0) = 0$ and $\nabla \Phi(0) = A(x)$, which is non-degenerate because of **A1′**. Therefore it is locally invertible: there exist two neighborhoods of 0 such that Φ is a diffeomorphism from one to the other, and therefore for $y - x$ in the neighborhood in the image we can find θ such that $\Phi(\theta) = y - x$. Moreover, from the fact that **A1′** and **A2′** are uniform around ξ, the size of the neighborhoods can

be taken uniformly in x. Therefore we can find a neighborhood of ξ such that for given x, y in this neighborhood, there exist θ for which $\Phi(\theta) = y - x$. Again from **A1$'$** and **A2$'$**, we can also suppose that $|\theta| \le C_1^\xi |\Phi(\theta)|$. So, there exists V_ξ neighborhood of ξ such that, for $x, y \in V_\xi$, there exists $\theta \in \bar{C}_A(x, y)$, and moreover

$$|\theta| \le C_1^\xi |\Phi(\theta)| \le C_1^\xi |x - y| < C_1^\xi \gamma_\xi.$$

We now show

$$|\theta^1| < C_\xi R^{1/2}, \qquad |\theta^2| < C_\xi R^{3/2}.$$

It is clear from (6.2) that $|\theta^1| \le |\theta| \le C_1^\xi |x - y| < C_2^\xi R^{1/2}$. Now, with a development similar to (4.5), we can write

$$\int_0^1 A(v_s)\theta \, ds = [\sigma, b](x)\theta^2 + \sigma(x)\theta^1 + \eta(\theta^1) + L(\theta),$$

with $|L(\theta)| \le C_3^\xi |\theta| (|\theta^2| + |\theta^1|^3)$ for $|\theta| < C_1^\xi \gamma_\xi$ and η defined as in (4.1):

$$\eta(u) = \left(\frac{\kappa_\sigma(x)}{2} u^2 + \frac{(\partial_\sigma \kappa_\sigma + \kappa_\sigma^2)(x)}{6} u^3 \right) \sigma(x) = \left(\alpha(x)u^2 + \beta(x)u^3 \right) \sigma(x)$$

(we have used again **A3$'$**). So

$$A(x)^{-1} \int_0^1 A(v_s)\theta \, ds = \begin{pmatrix} \theta^1 + \alpha(x)(\theta^1)^2 + \beta(x)(\theta^1)^3 \\ \theta^2 \end{pmatrix} + A(x)^{-1} L(\theta).$$

Since $|\theta| < C_1^\xi \gamma_\xi$ and $|\theta^1| \le C_2^\xi R^{1/2}$,

$$\left| A(x)^{-1} L(\theta) \right| \le C_4^\xi |\theta| \left(|\theta^2| + |\theta^1|^3 \right) \le C_4^\xi C_1^\xi \left(C_2^\xi \right)^3 \gamma_\xi \left(|\theta^2| + R^{3/2} \right) \le \frac{|\theta^2| + R^{3/2}}{2},$$

choosing $\gamma_\xi \le (2C_4^\xi C_1^\xi (C_2^\xi)^3)^{-1}$. In particular, the second component of $A(x)^{-1} L(\theta)$ is in absolute value smaller than $(|\theta^2| + R^{3/2})/2$. Then the second component of $A(x)^{-1} \int_0^1 A(v_s)\theta \, ds$ is in absolute value larger than $|\theta^2| - (|\theta^2| + R^{3/2})/2 = (|\theta^2| - R^{3/2})/2$. As a consequence, the second component of $A_R(x)^{-1} \int_0^1 A(v_s)\theta \, ds$ is in absolute value larger than $R^{-3/2}(|\theta^2| - R^{3/2})/2$. Since $|\int_0^1 A(v_s)\theta \, ds|_{A_R(x)} = |x - y|_{A_R(x)} \le 1$, we have $R^{-3/2}(|\theta^2| - R^{3/2})/2 \le 1$ and so we conclude $|\theta^2| \le 3R^{3/2}$.

We now prove

$$d_c(x, y) < \frac{\sqrt{R}}{C_6^\xi} \quad \Rightarrow \quad d(x, y) < \sqrt{R}.$$

We suppose $\phi \in C_A(x, y)$ with $\|\phi\|_{1,3} \le \frac{\sqrt{R}}{C_6^\xi}$, which implies

$$\int_0^1 |\phi_s^1|^2 \, ds \le \frac{R}{(C_6^\xi)^2} \quad \text{and} \quad \int_0^1 |\phi_s^2|^2 \, ds \le \frac{R^3}{(C_6^\xi)^6}.$$

Developing as before and applying **A3$'$**,

$$|x - y|_{A_R(x)} = \left| A_R(x)^{-1} \int_0^1 A(v_s)\phi_s \, ds \right| \le C_7^\xi \sqrt{\frac{\int_0^1 |\phi_s^1|^2 \, ds}{R} + \frac{\int_0^1 |\phi_s^2|^2 \, ds + (\int_0^1 |\phi_s^1|^2 \, ds)^3}{R^3}}.$$

Therefore

$$|x - y|_{A_R(x)} \le C_7^\xi \frac{\sqrt{3}}{C_6^\xi} < 1$$

if C_6^ξ is a large enough constant. □

Acknowledgements

References

[1] V. Bally and L. Caramellino. Riesz transform and integration by parts formulas for random variables. *Stochastic Process. Appl.* **121** (6) (2011) 1332–1355. MR2794979

[2] V. Bally and L. Caramellino. Positivity and lower bounds for the density of Wiener functionals. *Potential Anal.* **39** (2) (2013) 141–168. MR3078335

[3] V. Bally and L. Caramellino. On the distances between probability density functions. *Electron. J. Probab.* **19** (2014) 33. MR3296526

[4] V. Bally, L. Caramellino and P. Pigato. Diffusions under a local strong Hörmander condition. Part I: Density estimates. Preprint, 2016. Available at arXiv:1607.04542.

[5] V. Bally, L. Caramellino and P. Pigato. Diffusions under a local strong Hörmander condition. Part II: Tube estimates. Preprint, 2016. Available at arXiv:1607.04544.

[6] V. Bally, B. Fernandez and A. Meda. Estimates for the probability that a Itô process remains near a path. In *Stochastic Processes and Their Applications* 2087–2113, **121**, 2011. MR2819243

[7] V. Bally and A. Kohatsu-Higa. Lower bounds for densities of Asian type stochastic differential equations. *J. Funct. Anal.* **258** (9) (2010) 3134–3164. MR2595738

[8] F. Baudoin and M. Hairer. A version of Hörmander's theorem for the fractional Brownian motion. *Probab. Theory Related Fields* **139** (3–4) (2007) 373–395. MR2322701

[9] G. Ben Arous, M. Gradinaru and M. Ledoux. Hölder norms and the support theorem for diffusions. *Ann. Inst. Henri Poincaré Probab. Stat.* **30** (1994) 415–436. MR1288358

[10] G. Ben Arous and R. Léandre. Décroissance exponentielle du noyau de la chaleur sur la diagonale. I. *Probab. Theory Related Fields* **90** (2) (1991) 175–202. MR1128069

[11] G. Ben Arous and R. Léandre. Décroissance exponentielle du noyau de la chaleur sur la diagonale. II. *Probab. Theory Related Fields* **90** (3) (1991) 377–402. MR1133372

[12] J. M. Bismut. *Large Deviations and the Malliavin Calculus. Progress in Mathematics*. Birkhäuser, Boston, 1984. MR0755001

[13] U. Boscain and S. Polidoro. Gaussian estimates for hypoelliptic operators via optimal control. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **18** (4) (2007) 333–342. MR2349990

[14] M. Capitaine. On the Onsager–Machlup functional for elliptic diffusion processes. In *Séminaire de Probabilités* **XXXIV** 313–328. *Lecture Notes in Mathematics* **1729**. Springer, Berlin, 2000. MR1768071

[15] P. Carr and M. Schröder. Bessel processes, the integral of geometric Brownian motion, and Asian options. *Teor. Veroyatn. Primen.* **48** (3) (2003) 503–533. MR2141348

[16] T. Cass and P. Friz. Densities for rough differential equations under Hörmander's condition. *Ann. of Math. (2)* **171** (3) (2010) 2115–2141. MR2680405

[17] P. Cattiaux and L. Mesnager. Hypoelliptic non-homogeneous diffusions. *Probab. Theory Related Fields* **123** (2002) 453–483. MR1921010

[18] J. M. Coron. *Control and Nonlinearity. Mathematical Surveys and Monographs* **136**. American Mathematical Society, Providence, RI, 2007. MR2302744

[19] F. Delarue and S. Menozzi. Density estimates for a random noise propagating through a chain of differential equations. *J. Funct. Anal.* **259** (6) (2010) 1577–1630. MR2659772

[20] P. Foschi, S. Pagliarani and A. Pascucci. Approximations for Asian options in local volatility models. *J. Comput. Appl. Math.* **237** (2013) 442–459. MR2966919

[21] P. Friz, T. Lyons and D. Stroock. Lévy's area under conditioning. *Ann. Inst. Henri Poincaré Probab. Stat.* **42** (2006) 89–101. MR2196973

[22] A. Guillin. Averaging principle of SDE with small diffusion: Moderate deviations. *Ann. Probab.* **31** (1) (2003) 413–443. MR1959798

[23] R. Höpfner, E. Löcherbach and M. Thieullen. Strongly degenerate time inhomogeneous SDEs: Densities and support properties. Application to a Hodgkin–Huxley system with periodic input. Preprint, 2014. Available at arXiv:1410.0341.

[24] R. Höpfner, E. Löcherbach and M. Thieullen. Ergodicity for a stochastic Hodgkin–Huxley model driven by Ornstein–Uhlenbeck type input. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** (1) (2016) 483–501. MR3449311

[25] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*, 2nd edition. *North-Holland Mathematical Library* **24**. North-Holland Publishing Co., Amsterdam, 1989. MR1011252

[26] S. Kusuoka and D. Stroock. Applications of the Malliavin Calculus. I. In *Stochastic Analysis. Proceedings of the Taniguchi International Symposium on Stochastic Analysis* 271–306. *North-Holland Mathematical Library* **32**. North-Holland, Amsterdam, 1984. MR0780762

[27] S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. II. *J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math.* **32** (1) (1985) 1–76. MR0783181

[28] S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. III. *J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math.* **34** (2) (1987) 391–442. MR0914028

[29] P. Malliavin and A. Thalmaier. *Stochastic Calculus of Variations in Mathematical Finance*. Springer-Verlag, Berlin, 2006. MR2189710

[30] A. Nagel, E. M. Stein and S. Wainger. Balls and metrics defined by vector fields. I. Basic properties. *Acta Math.* **155** (1–2) (1985) 103–147. MR0793239

[31] J. Norris. Simplified Malliavin calculus. In *Seminaire de Probabilités* **XX** 101–130. *Lecture Notes in Mathematics* **1204**. Springer, Berlin, 1986. MR0942019

[32] D. Nualart. *Malliavin Calculus and Related Topics*. Springer, Berlin, 2006. MR2200233

[33] A. Pascucci and S. Polidoro. Harnack inequalities and Gaussian estimates for a class of hypoelliptic operators. *Trans. Amer. Math. Soc.* **358** (11) (2006) 4873–4893. MR2231876

[34] P. Pigato. Tube estimates for hypoelliptic diffusions and scaling properties of stochastic volatility models. Ph.D. thesis, Université Paris-Est, Universitá degli Studi di Padova, 2015.

[35] S. Polidoro. A global lower bound for the fundamental solution of Kolmogorov–Fokker–Planck equations. *Arch. Ration. Mech. Anal.* **137** (4) (1997) 321–340. MR1463798

[36] W. Rudin. *Principles of Mathematical Analysis*, 3rd edition. *International Series in Pure and Applied Mathematics*. McGraw-Hill Book Co., New York, 1976. MR0385023

[37] I. Shigekawa. *Stochastic Analysis*. *Translations of Mathematical Monographs* **224**. American Mathematical Society, Providence, RI, 2004. MR2060917

[38] D. Stroock and S. Varadhan. On the support of diffusion processes with applications to the strong maximum principle. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probabability: Probability Theory* **III** 333–359. Univ. California Press, Berkeley, 1972. MR0400425

[39] H. Whitney. On the extension of differentiable functions. *Bull. Amer. Math. Soc.* **50** (2) (1944) 76–81. MR0009785

[40] M. Yor. On some exponential functionals of Brownian motion. *Adv. in Appl. Probab.* **24** (3) (1992) 509–531. MR1174378