Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2018, Vol. 54, No. 1, 51–74
https://doi.org/10.1214/16-AIHP794

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Hydrostatics and dynamical large deviations for a reaction-diffusion model

C. Landim[a,b] and K. Tsunoda[c]

[a]*IMPA, Estrada Dona Castorina 110, CEP 22460 Rio de Janeiro, Brasil*
[b]*CNRS UMR 6085, Université de Rouen, Avenue de l'Université, BP 12, Technopôle du Madrillet, F76801 Saint-Étienne-du-Rouvray, France.*
E-mail: landim@impa.br
[c]*Institute of Mathematics for Industry, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan.*
E-mail: k-tsunoda@imi.kyushu-u.ac.jp

Received 14 January 2016; revised 5 September 2016; accepted 12 September 2016

Abstract. We consider the superposition of a symmetric simple exclusion dynamics, speeded-up in time, with a spin-flip dynamics in a one-dimensional interval with periodic boundary conditions. We prove the hydrostatics and the dynamical large deviation principle.

Résumé. On considère la superposition de l'exclusion simple symétrique accélérée en temps avec une dynamique non-conservative sur un intervalle uni-dimensionnel avec des conditions périodiques. On démontre le comportement hydrostatique et un principe de grandes déviations dynamique.

MSC: Primary 82C22; secondary 60F10; 82C35

Keywords: Reaction-diffusion equations; Hydrostatics; Dynamical large deviations

1. Introduction

In recent years, the large deviations of interacting particle systems have attracted much attention as an important step in the foundation of a thermodynamic theory of nonequilibrium stationary states [4,6,8,17]. Notwithstanding the absence of explicit expressions for the stationary states, large deviations principles for the empirical measure under the stationary state have been derived from a dynamical large deviations principle [5,11,20], extending to an infinite-dimensional setting [9,19] for the Freidlin and Wentzell approach [23].

We consider in this article interacting particle systems in which a symmetric simple exclusion dynamics, speeded-up diffusively, is superposed to a non-conservative Glauber dynamics. De Masi, Ferrari and Lebowitz [13] proved that the macroscopic evolution of the empirical measure is described by the solutions of the reaction-diffusion equation

$$\partial_t \rho = (1/2)\Delta\rho + B(\rho) - D(\rho), \tag{1.1}$$

where Δ is the Laplacian and $F = B - D$ is a reaction term determined by the stochastic dynamics. They also proved that the equilibrium fluctuations evolve as generalized Ornstein-Uhlenbeck processes.

A large deviation principle for the empirical measure has been obtained in [24] in the case where the initial distribution is a local equilibrium. The lower bound of the large deviations principle was achieved only for smooth trajectories. More recently, [11] extended the large deviations principle to a one-dimensional dynamics in contact with reservoirs and proved the lower bound for general trajectories in the case where the birth and the death rates, $B(\rho)$ and $D(\rho)$, respectively, are monotone, concave functions.

In this article, we first present a law of large numbers for the empirical measure under the stationary state [18,26]. More precisely, denote by μ_N the stationary state on a one-dimensional torus with N points of the superposition of a Glauber dynamics with a symmetric simple exclusion dynamics speeded-up by N^2. This probability measure is not known explicitly and it exhibits long range correlations [2]. Let V_ϵ denote an ϵ-neighborhood of the set of solutions of the elliptic equation

$$(1/2)\Delta\rho + F(\rho) = 0. \tag{1.2}$$

Theorem 2.2 asserts that for any $\epsilon > 0$, $\mu_N(V_\epsilon^c)$ vanishes as $N \to \infty$. In contrast with previous results, equation (1.2) may not have a unique solution so that equation (1.1) may not have a global attractor, what prevents the use of the techniques developed in [20,28]. This result solves partially a conjecture raised in Section 4.2 of [10].

The main results of this article concern the large deviations of the Glauber-Kawasaki dynamics. We first prove a full large deviations principle for the empirical measure under the sole assumption that B and D are concave functions. These assumptions encompass the case in which the potential $F(\rho) = B(\rho) - D(\rho)$ presents two or more wells, and open the way to the investigation of the metastable behavior of this dynamics. Previous results in this directions include [3,14,15].

We also prove that the large deviations rate function is lower semicontinuous and has compact level sets. These properties play a fundamental role in the proof of the static large deviation principle for the empirical measure under the stationary state μ_N [9,19].

The main difficulty in the proof of the lower bound of the large deviation principle comes from the presence of exponential terms in the rate function, denoted in this introduction by I. In contrast with conservative dynamics, for a trajectory $u(t,x)$, $I(u)$ is not expressed as a weighted H_{-1} norm of $\partial_t u - (1/2)\Delta u - F(u)$. This forces the development of new tools to prove that smooth trajectories are I-dense.

Both the large deviations of the empirical measure under the stationary state and the metastable behavior of the dynamics in the case where the potential admits more than one well are investigated in [21] based on the results presented in this article.

Comments on the proof. The proof of the law of large numbers for the empirical measure under the stationary state μ_N borrows ideas from [20,28]. On the one hand, by [13], the evolution of the empirical measure is described by the solutions of the reaction-diffusion equation (1.1). On the other hand, by [12], for any density profile γ, the solution ρ_t of (1.1) with initial condition γ converges to some solution of the semilinear elliptic equation (1.2). Assembling these two facts, we show in the proof of Theorem 2.2 that the empirical measure eventually reaches a neighborhood of the set of all solutions of the semilinear elliptic equation (1.2).

The proof that the rate function I is lower semicontinuous and has compact level set is divided in two steps. Denote by $Q(\pi)$ the energy of a trajectory π, defined in (2.4). Following [29], we first show in Proposition 4.2 that the energy of a trajectory π is bounded by the sum of its rate function with a constant: $Q(\pi) \le C_0(I(\pi)+1)$. It is not difficult to show that a sequence in the set $\{\pi : Q(\pi) \le a\}$, $a > 0$, which converges weakly also converges in L^1. The lower semicontinuity of the rate function I follows from these two facts. Let π_n be a sequence which converges weakly to π. We may, of course, assume that the sequence $I(\pi_n)$ is bounded. In this case, by the two results presented above, π_n converges to π in L^1. As the rate function $I(\cdot)$, defined in (2.5), is given by $\sup_G J_G(\cdot)$, where the supremum is carried over smooth functions, and since for each such function J_G is continuous for the L^1 topology, $J_G(\pi) = \lim_n J_G(\pi_n) \le \liminf_n I(\pi_n)$. To conclude the proof of the lower semicontinuity of I, it remains to maximize over G. The proof that the level sets are compact is similar.

Note that the previous argument does not require a bound of the H_{-1} norm of $\partial_t\pi$ in terms of $I(\pi)$ and $Q(\pi)$. Actually, such a bound does not hold in the present context. For example, let ρ represent the solution of the hydrodynamic equation (1.1) starting from some initial condition γ. Due to the reaction term, the H_{-1} norm of $\partial_t\rho$ might be infinite, while $I(\rho) = 0$ and $Q(\rho) < \infty$. The fact that a bound on the H_{-1} norm of $\partial_t\pi$ is not used, may simplify the earlier proofs of the regularity of the rate function in the case of conservative dynamics [7,20].

The main difficulty in the proof of the lower bound lies in the I-density of smooth trajectories: each trajectory π with finite rate function should be approachable by a sequence of smooth trajectories π_n such that $I(\pi_n)$ converges to $I(\pi)$. We use in this step the hydrodynamic equation and several convolutions with mollifiers to smooth the paths.

The concavity of B and D are used in this step and only in this one. We emphasize that we can not use Theorem 2.4 in [24] in our setting due to the large deviations which come from initial configurations. Therefore we need to prove the I-density, Theorem 5.2. It is possible that the theory of Orlicz spaces may allow to weaken these assumptions. Similar difficulties appeared in the investigation of the large deviations of a random walk driven by an exclusion process and of the exclusion process with a slow bond [1,22].

This article is organized as follows. In Section 2, we introduce a reaction-diffusion model and state the main results. In Section 3 we prove the law of large numbers for the empirical measure under the stationary state. In Section 4, we present the main properties of the rate function I. In Section 5, we prove that the smooth trajectories are I-dense and we prove Theorem 2.5, the main result of the article. In the Appendix, we recall some results on the solution of the hydrodynamic equation (1.1).

2. Notation and results

Throughout this article, we use the following notation. $\mathbb{N}_0$ stands for the set $\{0, 1, \ldots\}$. For a function $f : X \to \mathbb{R}$, defined on some space X, let $\|f\|_\infty = \sup_{x\in X} |f(x)|$. We will use $C_0 > 0$ and $C > 0$ as a notation for a generic positive constant which may change from line to line.

2.1. *Reaction-diffusion model*

We fix some notation and define the model. Let $\mathbb{T}_N$ be the one-dimensional discrete torus $\mathbb{Z}/N\mathbb{Z} = \{0, 1, \ldots, N-1\}$. The state space of our process is given by $X_N = \{0, 1\}^{\mathbb{T}_N}$. Let η denote a configuration in X_N, x a site in $\mathbb{T}_N$, $\eta(x) = 1$ if there is a particle at site x, otherwise $\eta(x) = 0$.

We consider in the set $\mathbb{T}_N$ the superposition of the symmetric simple exclusion process (Kawasaki) with a spin-flip dynamics (Glauber). This model was introduced by De Masi, Ferrari and Lebowitz in [13] to derive a reaction-diffusion equation from a microscopic dynamics. More precisely, the stochastic dynamics is a Markov process on X_N whose generator $\mathcal{L}_N$ acts on functions $f : X_N \to \mathbb{R}$ as

$$\mathcal{L}_N f = \frac{N^2}{2}\mathcal{L}_K f + \mathcal{L}_G f,$$

where $\mathcal{L}_K$ is the generator of a symmetric simple exclusion process (Kawasaki dynamics),

$$(\mathcal{L}_K f)(\eta) = \sum_{x\in\mathbb{T}_N} \big[f\big(\eta^{x,x+1}\big) - f(\eta)\big],$$

and where $\mathcal{L}_G$ is the generator of a spin flip dynamics (Glauber dynamics),

$$(\mathcal{L}_G f)(\eta) = \sum_{x\in\mathbb{T}_N} c(x,\eta)\big[f\big(\eta^{x}\big) - f(\eta)\big].$$

In these formulas, $\eta^{x,x+1}$ (resp. η^x) represents the configuration obtained from η by exchanging (resp. flipping) the occupation variables $\eta(x)$, $\eta(x+1)$ (resp. $\eta(x)$):

$$\eta^x(z) = \begin{cases} \eta(z) & \text{if } z \neq x, \\ 1-\eta(z) & \text{if } z = x, \end{cases} \qquad \eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases}$$

Moreover, $c(x,\eta) = c(\eta(x-M), \ldots, \eta(x+M))$, for some $M \ge 1$ and some strictly positive cylinder function $c(\eta)$, that is, a function which depends only on a finite number of variables $\eta(y)$. Note that the exclusion dynamics has been speeded-up by a factor N^2, and that the Markov process generated by $\mathcal{L}_N$ is irreducible because $c(\eta)$ is a strictly positive function.

2.2. *Hydrodynamic limit*

We briefly discuss in this subsection the limiting behavior of the empirical measure.

Denote by $\mathbb{T}$ the one-dimensional continuous torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$. Let $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{T})$ be the space of nonnegative measures on $\mathbb{T}$, whose total mass bounded by 1, endowed with the weak topology. For a measure π in $\mathcal{M}_+$ and a continuous function $G : \mathbb{T} \to \mathbb{R}$, denote by $\langle \pi, G\rangle$ the integral of G with respect to π:

$$\langle \pi, G\rangle = \int_{\mathbb{T}} G(u)\pi(du).$$

The space $\mathcal{M}_+$ is metrizable. Indeed, if $f_{2k}(u) = \cos(\pi k u)$ and $f_{2k+1}(u) = \sin(\pi k u)$, $k \in \mathbb{N}_0$, one can define the distance d on $\mathcal{M}_+$ as

$$d(\pi_1, \pi_2) := \sum_{k=0}^{\infty} \frac{1}{2^k} \big| \langle \pi_1, f_k\rangle - \langle \pi_2, f_k\rangle \big|.$$

Denote by $C^m(\mathbb{T})$, m in $\mathbb{N}_0 \cup \{\infty\}$, the set of all real functions on $\mathbb{T}$ which are m times differentiable and whose mth derivative is continuous. Given a function G in $C^2(\mathbb{T})$, we shall denote by ∇G and ΔG the first and second derivative of G, respectively.

Let $\{\eta_t^N : N \geq 1\}$ be the continuous-time Markov process on X_N whose generator is given by $\mathcal{L}_N$. Let $\pi^N : X_N \to \mathcal{M}_+$ be the function which associates to a configuration η the positive measure obtained by assigning mass N^{-1} to each particle of η,

$$\pi^N(\eta) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta(x)\delta_{x/N},$$

where δ_u stands for the Dirac measure which has a point mass at $u \in \mathbb{T}$. Denote by π_t^N the empirical measure process $\pi^N(\eta_t^N)$.

Fix arbitrarily $T > 0$. For a topological space X and an interval $I = [0, T]$ or $[0, \infty)$, denote by $C(I, X)$ the set of all continuous trajectories from I to X endowed with the uniform topology. Let $D(I, X)$ be the space of all right-continuous trajectories from I to X with left-limits, endowed with the Skorokhod topology. For a probability measure ν in X_N, denote by $\mathbb{P}_\nu^N$ the measure on $D([0, T], X_N)$ induced by the process η_t^N starting from ν.

Let $\nu_\rho = \nu_\rho^N$, $0 \leq \rho \leq 1$, be the Bernoulli product measure with the density ρ. Define the continuous functions $B, D : [0, 1] \to \mathbb{R}$ by

$$B(\rho) = \int \big[1 - \eta(0)\big] c(\eta)\, d\nu_\rho, \qquad D(\rho) = \int \eta(0) c(\eta)\, d\nu_\rho.$$

Since $B(1) = 0$, $D(0) = 0$ and B, D are polynomials in ρ,

$$B(\rho) = (1 - \rho)\tilde{B}(\rho), \qquad D(\rho) = \rho \tilde{D}(\rho), \tag{2.1}$$

where $\tilde{B}(\rho)$, $\tilde{D}(\rho)$ are polynomials.

The next result was proved by De Masi, Ferrari and Lebowitz in [13] for the first time. We refer to [13,24,25] for its proof.

Theorem 2.1. *Fix $T > 0$ and a measurable function $\gamma : \mathbb{T} \to [0, 1]$. Let $\nu = \nu_N$ be a sequence of probability measures on X_N associated to γ, in the sense that*

$$\lim_{N\to\infty} \nu_N \left(\left| \langle \pi^N, G\rangle - \int_{\mathbb{T}} G(u)\gamma(u)\, du \right| > \delta \right) = 0,$$

for every $\delta > 0$ and every continuous function $G : \mathbb{T} \to \mathbb{R}$. Then, for every $t \geq 0$, every $\delta > 0$ and every continuous function $G : \mathbb{T} \to \mathbb{R}$, we have

$$\lim_{N\to\infty} \mathbb{P}_\nu^N \left(\left| \langle \pi_t^N, G \rangle - \int_{\mathbb{T}} G(u)\rho(t,u)\, du \right| > \delta \right) = 0,$$

where $\rho : [0,\infty) \times \mathbb{T} \to [0,1]$ is the unique weak solution of the Cauchy problem

$$\begin{cases} \partial_t \rho = (1/2)\Delta\rho + F(\rho) & \text{on } \mathbb{T}, \\ \rho(0,\cdot) = \gamma(\cdot), \end{cases} \tag{2.2}$$

where $F(\rho) = B(\rho) - D(\rho)$.

The definition, existence and uniqueness of weak solutions of the Cauchy problem (2.2) are discussed in the Appendix.

2.3. *Hydrostatic limit*

We examine in this subsection the asymptotic behavior of the empirical measure under the stationary state. Fix $N \geq 1$ large enough. Since the Markov process η_t^N is irreducible and the cardinality of the state space X_N is finite, there exists a unique invariant probability measure for the process η_t^N, denoted by μ_N. Let $\mathcal{P}_N$ be the probability measure on $\mathcal{M}_+$ defined by $\mathcal{P}_N = \mu_N \circ (\pi^N)^{-1}$.

For each $p \geq 1$, let $L^p(\mathbb{T})$ be the space of all real pth integrable functions $G : \mathbb{T} \to \mathbb{R}$ with respect to the Lebesgue measure: $\int_{\mathbb{T}} |G(u)|^p\, du < \infty$. The corresponding norm is denoted by $\|\cdot\|_p$:

$$\|G\|_p^p := \int_{\mathbb{T}} \left|G(u)\right|^p du.$$

In particular, $L^2(\mathbb{T})$ is a Hilbert space equipped with the inner product

$$\langle G, H \rangle = \int_{\mathbb{T}} G(u)H(u)\, du.$$

For a function G in $L^2(\mathbb{T})$, we also denote by $\langle G \rangle$ the integral of G with respect to the Lebesgue measure: $\langle G \rangle := \int_{\mathbb{T}} G(u)\, du$.

Let $\mathcal{E}$ be the set of all classical solutions of the semilinear elliptic equation:

$$(1/2)\Delta\rho + F(\rho) = 0 \quad \text{on } \mathbb{T}. \tag{2.3}$$

Classical solution means a function $\rho : \mathbb{T} \to [0,1]$ in $C^2(\mathbb{T})$ which satisfies the equation (2.3) for any $u \in \mathbb{T}$. We sometimes identify $\mathcal{E}$ with the set of all absolutely continuous measures whose density are a classical solution of (2.3):

$$\left\{\pi \in \mathcal{M}_+ : \pi(du) = \rho(u)\, du, \rho \text{ is a classical solution of the equation (2.3)}\right\}.$$

Theorem 2.2. *The measure $\mathcal{P}_N$ asymptotically concentrates on the set $\mathcal{E}$. Namely, for any $\delta > 0$, we have*

$$\lim_{N\to\infty} \mathcal{P}_N \left(\pi \in \mathcal{M}_+ : \inf_{\bar{\pi}\in\mathcal{E}} d(\pi, \bar{\pi}) \geq \delta \right) = 0.$$

If the set $\mathcal{E}$ is a singleton, it follows from Theorem 2.2 that the sequence $\{\mathcal{P}_N : N \geq 1\}$ converges:

Corollary 2.3. *Assume that there exists a unique classical solution $\overline{\rho} : \mathbb{T} \to [0,1]$ of the semilinear elliptic equation (2.3). Then $\mathcal{P}_N$ converges to the Dirac measure concentrated on $\overline{\rho}(u)\, du$ as $N \to \infty$.*

Remark 2.4. In [14,15], De Masi et al. examined the dynamics introduced above in the case of the double well potential $F(\rho) = -V'(\rho) = a(2\rho - 1) - b(2\rho - 1)^3$, $a, b > 0$, which is symmetric around the density $1/2$. They proved that, starting from a product measure with mean $1/2$, the unstable equilibrium of the ODE $\dot{x}(t) = -V'(x(t))$, the empirical density remains in a neighborhood of $1/2$ in a time scale of order $\log N$. Bodineau and Lagouge in Subsection of [10] conjectured that Theorem 2.2 remains true if we replace $\mathcal{E}$ by the set of all stable equilibrium solutions of the equation (2.3). This conjecture is proved in [21] and follows from the large deviation principle for the sequence $\{\mathcal{P}_N : N \geq 1\}$.

2.4. *Dynamical large deviations*

Denote by $\mathcal{M}_{+,1}$ the closed subset of $\mathcal{M}_+$ of all absolutely continuous measures with density bounded by 1:

$$\mathcal{M}_{+,1} = \big\{\pi \in \mathcal{M}_+(\mathbb{T}) : \pi(du) = \rho(u)\,du, 0 \leq \rho(u) \leq 1 \text{ a.e. } u \in \mathbb{T}\big\}.$$

Fix $T > 0$, and denote by $C^{m,n}([0,T] \times \mathbb{T})$, m, n in $\mathbb{N}_0 \cup \{\infty\}$, the set of all real functions defined on $[0,T] \times \mathbb{T}$ which are m times differentiable in the first variable and n times on the second one, and whose derivatives are continuous. Let $Q_\eta = Q_\eta^N$, $\eta \in X_N$, be the probability measure on $D([0,T], \mathcal{M}_+)$ induced by the measure-valued process π_t^N starting from $\pi^N(\eta)$.

Fix a measurable function $\gamma : \mathbb{T} \to [0,1]$. For each path $\pi(t, du) = \rho(t,u)\,du$ in $D([0,T], \mathcal{M}_{+,1})$, define the energy $\mathcal{Q} : D([0,T], \mathcal{M}_{+,1}) \to [0,\infty]$ as

$$\mathcal{Q}(\pi) = \sup_{G \in C^{0,1}([0,T]\times\mathbb{T})} \left\{ 2\int_0^T dt\,\langle \rho_t, \nabla G_t\rangle - \int_0^T dt \int_{\mathbb{T}} du\, G^2(t,u) \right\}. \tag{2.4}$$

It is known that the energy $\mathcal{Q}(\pi)$ is finite if and only if ρ has a generalized derivative and this generalized derivative is square integrable on $[0,T] \times \mathbb{T}$:

$$\int_0^T dt \int_{\mathbb{T}} du \big|\nabla \rho(t,u)\big|^2 < \infty.$$

Moreover, it is easy to see that the energy $\mathcal{Q}$ is convex and lower semicontinuous.

For each function G in $C^{1,2}([0,T] \times \mathbb{T})$, define the functional $\bar{J}_G : D([0,T], \mathcal{M}_{+,1}) \to \mathbb{R}$ by

$$\begin{aligned} \bar{J}_G(\pi) = {} & \langle \pi_T, G_T\rangle - \langle \gamma, G_0\rangle - \int_0^T dt \Big\langle \pi_t, \partial_t G_t + \frac{1}{2}\Delta G_t \Big\rangle \\ & - \frac{1}{2}\int_0^T dt \big\langle \chi(\rho_t), (\nabla G_t)^2 \big\rangle - \int_0^T dt \big\{ \big\langle B(\rho_t), e^{G_t} - 1 \big\rangle + \big\langle D(\rho_t), e^{-G_t} - 1 \big\rangle \big\}, \end{aligned}$$

where $\chi(r) = r(1-r)$ is the mobility. Let $J_G : D([0,T], \mathcal{M}_+) \to [0,\infty]$ be the functional defined by

$$J_G(\pi) = \begin{cases} \bar{J}_G(\pi) & \text{if } \pi \in D([0,T], \mathcal{M}_{+,1}), \\ \infty & \text{otherwise.} \end{cases}$$

We define the large deviation rate function $I_T(\cdot|\gamma) : D([0,T], \mathcal{M}_+) \to [0,\infty]$ as

$$I_T(\pi|\gamma) = \begin{cases} \sup J_G(\pi) & \text{if } \mathcal{Q}(\pi) < \infty, \\ \infty & \text{otherwise,} \end{cases} \tag{2.5}$$

where the supremum is taken over all functions G in $C^{1,2}([0,T] \times \mathbb{T})$.

We review here an explicit formula for the functional I_T at smooth trajectories obtained in Lemma 2.1 of [24]. Let ρ be a function in $C^{2,3}([0,T] \times \mathbb{T})$ with $c \leq \rho \leq 1 - c$, for some $0 < c < 1/2$, and set $\pi(t,du) = \rho(t,u)\,du$. Then there exists a unique solution $H \in C^{1,2}([0,T] \times \mathbb{T})$ of the partial differential equation

$$\partial_t \rho = (1/2)\Delta\rho - \nabla\big(\chi(\rho)\nabla H\big) + B(\rho)e^H - D(\rho)e^{-H},$$

with some initial profile γ. In the case, $I_T(\pi|\gamma)$ can be expressed as

$$I_T(\pi|\gamma) = \frac{1}{2}\int_0^T dt\langle\chi(\rho_t), (\nabla H_t)^2\rangle + \int_0^T dt\langle B(\rho_t), f(H_t)\rangle + \int_0^T dt\langle D(\rho_t), f(-H_t)\rangle,$$

where $f(a) = 1 - e^a + ae^a$.

The following theorem is one of main results of this paper.

Theorem 2.5. *Assume that the functions B and D are concave on $[0,1]$. Fix $T > 0$ and a measurable function $\gamma : \mathbb{T} \to [0,1]$. Assume that a sequence η^N of initial configurations in X_N is associated to γ, in the sense that*

$$\lim_{N\to\infty}\langle\pi^N(\eta^N), G\rangle = \int_{\mathbb{T}} G(u)\gamma(u)\,du$$

for every continuous function $G : \mathbb{T} \to \mathbb{R}$. Then, the measure Q_{η^N} on $D([0,T], \mathcal{M}_+)$ satisfies a large deviation principle with the rate function $I_T(\cdot|\gamma)$. That is, for each closed subset $\mathcal{C} \subset D([0,T], \mathcal{M}_+)$,

$$\overline{\lim_{N\to\infty}} \frac{1}{N}\log Q_{\eta^N}(\mathcal{C}) \le -\inf_{\pi\in\mathcal{C}} I_T(\pi|\gamma),$$

and for each open subset $\mathcal{O} \subset D([0,T], \mathcal{M}_+)$,

$$\underline{\lim_{N\to\infty}} \frac{1}{N}\log Q_{\eta^N}(\mathcal{O}) \ge -\inf_{\pi\in\mathcal{O}} I_T(\pi|\gamma).$$

Moreover, the rate function $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.

Remark 2.6. Jona-Lasinio, Landim and Vares [24] proved the dynamical large deviations principle stated above, but the lower bound was obtained only for smooth trajectories. Bodineau and Lagouge [11] proved the lower bound for one-dimensional reaction-diffusion models in contact with reservoirs in the case where B and D are concave, monotone functions.

Remark 2.7. Proposition 4.2 asserts that there exists a finite constant C_0 such that if π is a trajectory with finite energy, $Q(\pi) < \infty$, then $Q(\pi) \le C_0(I_T(\pi|\gamma) + 1)$. In the case where B and D are concave functions, we can use Theorem 5.2, which asserts that the smooth trajectories are $I_T(|\gamma)$-dense, to prove the same bound without the assumption that the trajectory π has finite energy. In particular, in this case we can define the rate function $I_T(|\gamma)$ simply as

$$I_T(\pi|\gamma) = \sup_G J_G(\pi).$$

Remark 2.8. In the proof that the rate function $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets we do not use a bound on the H_{-1} norm of $\partial_t\rho$ in terms of its rate function $I_T(\pi|\gamma)$. Actually, as mentioned in the introduction, such a bound does not hold for reaction-diffusion models. Therefore, the arguments presented here permit to simplify the proof of the regularity of the rate function in other models, such as the weakly asymmetric simple exclusion process [7,20].

3. Proof of Theorem 2.2

We prove in this section Theorem 2.2. Our approach is a generalization of the one developed in [20,28], but it does not require the existence of a global attractor for the underlying dynamical system. The method can be applied to any dynamics which fulfills two conditions: the macroscopic evolution of the empirical measure is described by a hydrodynamic equation, and for any initial condition the solution of this equation converges to a stationary profile as time goes to infinity. For instance, the boundary driven reaction-diffusion models examined in [11].

Recall from Section 2.3 the definition of the measure μ_N on X_N, the map π^N from X_N to $\mathcal{M}_+$ and the measure $\mathcal{P}_N = \mu_N \circ (\pi^N)^{-1}$ on $\mathcal{M}_+$. Denote by $\mathbf{Q}^N$ the probability measure on the Skorokhod space $D([0,\infty),\mathcal{M}_+)$ induced by the measure-valued process π_t^N under the initial distribution $\mathcal{P}_N$. Since the measure μ_N is stationary under the dynamics, $\mathcal{P}_N(\mathcal{B}) = \mathbf{Q}^N(\pi : \pi_T \in \mathcal{B})$, for each $T > 0$ and Borel set $\mathcal{B} \subset \mathcal{M}_+$.

Lemma 3.1. *The sequence $\{\mathbf{Q}^N : N \ge 1\}$ is tight and all its limit points $\mathbf{Q}^*$ are concentrated on absolutely continuous paths $\pi(t,du) = \rho(t,u)\,du$ whose density ρ is nonnegative and bounded above by* 1:

$$\mathbf{Q}^*\big\{\pi : \pi(t,du) = \rho(t,u)\,du,\ \textit{for } t \in [0,\infty)\big\} = 1,$$
$$\mathbf{Q}^*\big\{\pi : 0 \le \rho(t,u) \le 1,\ \textit{for } (t,u) \in [0,\infty) \times \mathbb{T}\big\} = 1.$$

The proof of this lemma is similar to the one of Proposition 3.1 in [27].

Let $\mathcal{A}$ be the set of all trajectories $\pi(t,du) = \rho(t,u)\,du$ in $D([0,\infty),\mathcal{M}_{+,1})$ whose density ρ is a weak solution to the Cauchy problem (2.2) for some initial profile $\rho_0 : \mathbb{T} \to [0,1]$.

Lemma 3.2. *All limit points $\mathbf{Q}^*$ of the sequence $\{\mathbf{Q}^N : N \ge 1\}$ are concentrated on paths $\pi(t,du) = \rho(t,u)\,du$ in $\mathcal{A}$*:

$$\mathbf{Q}^*(\mathcal{A}) = 1.$$

The proof of this lemma is similar to the one of Lemma A.1.1 in [25].

Proof of Theorem 2.2. Fix a positive $\delta > 0$. Let $\mathcal{E}_\delta$ be the δ-neighborhood of $\mathcal{E}$ in $\mathcal{M}_+$:

$$\mathcal{E}_\delta := \Big\{\pi \in \mathcal{M}_+ : \inf_{\bar\pi \in \mathcal{E}} d(\pi,\bar\pi) < \delta\Big\}.$$

Denote by $\mathcal{E}_\delta^c$ the complement of the set $\mathcal{E}_\delta$. The assertion of Theorem 2.2 can be rephrased as

$$\lim_{N\to\infty} \mathcal{P}_N\big(\mathcal{E}_\delta^c\big) = 0.$$

Therefore, to conclude the theorem it is enough to show that any limit point of the sequence $\mathcal{P}_N(\mathcal{E}_\delta^c)$ is equal to zero.

Fix $T > 0$. Since the measure μ_N is invariant under the dynamics,

$$\mathcal{P}_N\big(\mathcal{E}_\delta^c\big) = \mathbf{Q}^N\big(\pi : \pi_T \in \mathcal{E}_\delta^c\big). \tag{3.1}$$

Let $\mathbf{Q}^*$ be a limit point of $\{\mathbf{Q}^N : N \ge 1\}$ and take a subsequence N_k so that the sequence $\{\mathbf{Q}^{N_k} : k \ge 1\}$ converges to $\mathbf{Q}^*$ as $k \to \infty$. Note that the set $\{\pi : \pi_T \in \mathcal{E}_\delta^c\}$ is not closed in $D([0,\infty),\mathcal{M}_+)$. However, we claim that

$$\varlimsup_{k\to\infty} \mathbf{Q}^{N_k}\big(\pi : \pi_T \in \mathcal{E}_\delta^c\big) \le \mathbf{Q}^*\big(\big\{\pi : \pi_T \in \mathcal{E}_\delta^c\big\} \cap \mathcal{A}\big), \tag{3.2}$$

where $\mathcal{A}$ is the set introduced just before Lemma 3.2. Indeed, denote by $\overline{\{\pi : \pi_T \in \mathcal{E}_\delta^c\}}$ the closure of the set $\{\pi : \pi_T \in \mathcal{E}_\delta^c\}$ under the Skorokhod topology. By definition of the weak topology and by Lemma 3.2,

$$\varlimsup_{k\to\infty} \mathbf{Q}^{N_k}\big(\pi : \pi_T \in \mathcal{E}_\delta^c\big) \le \mathbf{Q}^*\big(\overline{\big\{\pi : \pi_T \in \mathcal{E}_\delta^c\big\}}\big) = \mathbf{Q}^*\big(\overline{\big\{\pi : \pi_T \in \mathcal{E}_\delta^c\big\}} \cap \mathcal{A}\big).$$

It remains to prove that

$$\overline{\big\{\pi : \pi_T \in \mathcal{E}_\delta^c\big\}} \cap \mathcal{A} = \big\{\pi : \pi_T \in \mathcal{E}_\delta^c\big\} \cap \mathcal{A}.$$

Let π be a path in $\overline{\{\pi : \pi_T \in \mathcal{E}_\delta^c\}} \cap \mathcal{A}$. Then there exists a sequence $\{\pi^n : n \geq 1\}$ such that π^n converges to π in $D([0,\infty), \mathcal{M}_+)$ as $n \to \infty$ and π_T^n belongs to $\mathcal{E}_\delta^c$ for any $n \geq 1$. Since $\mathcal{A}$ is contained in $C([0,\infty), \mathcal{M}_{+,1})$, the sequence $\{\pi^n : n \geq 1\}$ converges to π under the uniform topology. Hence π_T^n converges to π_T. Since $\mathcal{E}_\delta^c$ is closed in $\mathcal{M}_+$, π_T also belongs to $\mathcal{E}_\delta^c$, which proves (3.2).

Fix a path $\pi(t,du) = \rho(t,u)\,du$ in $\mathcal{A}$. By Proposition A.6, there exists a density profile ρ_∞ in $\mathcal{E}$ such that ρ_t converges to ρ_∞ in $C^2(\mathbb{T})$. Hence,

$$\mathcal{A} \subset \bigcup_{j\geq 1} \bigcap_{k \geq j} \{\pi_k \in \mathcal{E}_\delta\}. \tag{3.3}$$

By (3.1) and (3.2),

$$\overline{\lim_{N\to\infty}}\, \mathcal{P}_N\big(\mathcal{E}_\delta^c\big) \leq \mathbf{Q}^*\big(\big\{\pi : \pi_k \in \mathcal{E}_\delta^c\big\} \cap \mathcal{A}\big) \quad \text{for all } k \geq 1.$$

Since this bound holds for any $k \geq 1$,

$$\overline{\lim_{N\to\infty}}\, \mathcal{P}_N\big(\mathcal{E}_\delta^c\big) \leq \underline{\lim_{k\to\infty}}\, \mathbf{Q}^*\big(\big\{\pi_k \in \mathcal{E}_\delta^c\big\} \cap \mathcal{A}\big) \leq \mathbf{Q}^*\bigg(\bigcap_{j\geq 1} \bigcup_{k\geq j} \big\{\pi_k \in \mathcal{E}_\delta^c\big\} \cap \mathcal{A}\bigg).$$

This latter set is empty in view of (3.3), which completes the proof of the theorem. □

4. The rate function $I_T(\cdot|\gamma)$

We prove in this section that the large deviations rate function is lower semicontinuous and has compact level sets. These properties play a fundamental role in the proof of the static large deviation principle, cf. [9,19]. One of the main steps in the proof of these properties is Proposition 4.2. It asserts that there exists a finite constant C_0 such that for all trajectory $\pi(t,du) = \rho(t,u)$ whose density ρ has finite energy, we have $Q(\pi) \leq C_0(I_T(\pi|\gamma) + 1)$. Such bound was first proved in [29].

Proposition 4.1. *Let π be a path in $D([0,T], \mathcal{M}_+)$ such that $I_T(\pi|\gamma)$ is finite. Then $\pi(0,du) = \gamma(u)\,du$ and π belongs to $C([0,T], \mathcal{M}_{+,1})$.*

Proof. The proof of this proposition is similar to the one of Lemma 3.5 in [5]. Actually, the computation performed in the proof of Lemma 3.5 in [5] gives that, for any g in $C^2(\mathbb{T})$ and any $0 \leq s < t \leq T$,

$$\big|\langle \pi_t, g\rangle - \langle \pi_s, g\rangle\big| \leq C\alpha_{s,r}\big\{I_T(\pi|\gamma) + 1\big\}, \tag{4.1}$$

for some positive constant $C = C(g)$, which depends only on g. In the inequality (4.1), the constant $\alpha_{s,r}$ is given by $(\log(r-s)^{-1})^{-1}$. Equation (4.1) implies the desired continuity. □

The next proposition plays an important role in the proof of Theorem 4.7.

Proposition 4.2. *There exists a constant $C_0 > 0$ such that, for any path $\pi(t,du) = \rho(t,u)\,du$ in $D([0,T], \mathcal{M}_{+,1})$ with finite energy, we have*

$$\int_0^T dt \int_{\mathbb{T}} du \frac{|\nabla \rho(t,u)|^2}{\chi(\rho(t,u))} \leq C_0\big\{I_T(\pi|\gamma) + 1\big\}.$$

We fix some notation before proving Proposition 4.2.

Let $H^1(\mathbb{T})$ be the Sobolev space of functions G with generalized derivatives ∇G in $L^2(\mathbb{T})$. $H^1(\mathbb{T})$ endowed with the scalar product $\langle \cdot, \cdot \rangle_{1,2}$, defined by

$$\langle G, H\rangle_{1,2} = \langle G, H\rangle + \langle \nabla G, \nabla H\rangle,$$

is a Hilbert space. The corresponding norm is denoted by $\|\cdot\|_{1,2}$:

$$\|G\|_{1,2}^2 := \int_{\mathbb{T}} \big|G(u)\big|^2 \, du + \int_{\mathbb{T}} \big|\nabla G(u)\big|^2 \, du.$$

For a Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ and $T > 0$, we denote by $L^2([0,T], \mathbb{B})$ the Banach space of measurable functions $U : [0,T] \to \mathbb{B}$ for which

$$\|U\|_{L^2([0,T],\mathbb{B})}^2 = \int_0^T \|U_t\|_{\mathbb{B}}^2 \, dt < \infty$$

holds. For each $p \geq 1$ and $T > 0$, let $L^p([0,T] \times \mathbb{T})$ be the space of all real pth integrable functions $U : [0,T] \times \mathbb{T} \to \mathbb{R}$ with respect to the Lebesgue measure: $\int_0^T dt \int_{\mathbb{T}} |U(t,u)|^p \, du < \infty$.

Fix a path $\pi(t, du) = \rho(t,u) \, du$ in $D([0,T], \mathcal{M}_{+,1})$ with finite energy. For a smooth function $G : [0,T] \times \mathbb{T} \to \mathbb{R}$ and for a bounded function H in $L^2([0,T], H^1(\mathbb{T}))$, define the functionals

$$L_G(\pi) = \langle \pi_T, G_T \rangle - \langle \pi_0, G_0 \rangle - \int_0^T dt \langle \pi_t, \partial_t G_t \rangle,$$

$$B_H^1(\pi) = \frac{1}{2} \int_0^T dt \langle \nabla \rho_t, \nabla H_t \rangle - \frac{1}{2} \int_0^T dt \big\langle \chi(\rho_t), (\nabla H_t)^2 \big\rangle,$$

$$B_H^2(\pi) = \int_0^T dt \big\{ \big\langle B(\rho_t), e^{H_t} - 1 \big\rangle + \big\langle D(\rho_t), e^{-H_t} - 1 \big\rangle \big\}.$$

Note that, for paths $\pi(t, du)$ such that $\pi(0, du) = \gamma(u) \, du$,

$$\sup_{H \in C^{1,2}([0,T] \times \mathbb{T})} \big\{ L_H(\pi) + B_H^1(\pi) - B_H^2(\pi) \big\} = I_T(\pi | \gamma). \tag{4.2}$$

Consider the function $\phi : \mathbb{R} \to [0, \infty)$ defined by

$$\phi(r) := \begin{cases} \frac{1}{Z} \exp\{-\frac{1}{(1-r^2)}\} & \text{if } |r| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the constant Z is chosen so that $\int_{\mathbb{R}} \phi(r) \, dr = 1$. For each $\delta > 0$, let

$$\phi^\delta(r) := \frac{1}{\delta} \phi\left(\frac{r}{\delta}\right).$$

Since the support of the function ϕ^δ is contained in $[-\delta, \delta]$, the function ϕ^δ can be regarded as a function on $\mathbb{T}$. To distinguish convolution in time from convolution in space, we denote by $\psi^\delta : \mathbb{T} \to [0, \infty)$ the function ϕ^ε defined on $\mathbb{T}$ with $\varepsilon = \delta$.

Denote by $f * g$ the space or time convolution of two functions f, g:

$$(f * g)(a) = \int f(a-b) g(b) \, db,$$

where the integral runs over $\mathbb{R}$ in the case where f, g are functions of time and over $\mathbb{T}$ in the case where f and g are functions of space.

Throughout this section, we adopt the following notation: For a bounded measurable function $\rho : [0,T]\times\mathbb{T}\to\mathbb{R}$, define the smooth approximation in space, time and space-time by

$$\rho^{\varepsilon}(t,u) := \big[\rho(t,\cdot)*\psi^{\varepsilon}\big](u) = \int_{\mathbb{T}} \rho(t,u+v)\psi^{\varepsilon}(v)\,dv,$$

$$\rho^{\delta}(t,u) := \big[\rho(\cdot,u)*\phi^{\delta}\big](t) = \int_{-\delta}^{\delta} \rho(t+r,u)\phi^{\delta}(r)\,dr,$$

$$\rho^{\varepsilon,\delta}(t,u) := \int_{-\delta}^{\delta} dr \int_{\mathbb{T}} dv\,\rho(t+r,u+v)\psi^{\varepsilon}(v)\phi^{\delta}(r).$$

In the above formulas, we extend the definition of ρ to $[-1,T+1]$ by setting $\rho_t=\rho_0$ for $-1\le t\le 0$ and $\rho_t=\rho_T$ for $T\le t\le T+1$. Remark that we use similar notation, ρ^{ε} and ρ^{δ}, for different objects. However, ρ^{ε} and ρ^{δ} always represent a smooth approximation of ρ in space and time, respectively. For each $\pi(t,du)=\rho(t,u)\,du$, we also define paths $\pi^{\varepsilon}(t,du)=\rho^{\varepsilon}(t,u)\,du$, $\pi^{\delta}(t,du)=\rho^{\delta}(t,u)\,du$ and $\pi^{\varepsilon,\delta}(t,du)=\rho^{\varepsilon,\delta}(t,u)\,du$.

We summarize some properties of ρ^{ε} in the next proposition. The proof is elementary and is thus omitted.

Proposition 4.3. *Let $\rho:[0,T]\times\mathbb{T}\to\mathbb{R}$ be a function in $L^2([0,T],H^1(\mathbb{T}))$. Then, for each $\varepsilon>0$, ρ^{ε} and $\nabla\rho^{\varepsilon}$ converges to ρ and $\nabla\rho$ in $L^2([0,T]\times\mathbb{T})$, respectively. Moreover, if ρ is bounded in $[0,T]\times\mathbb{T}$ and the application $\langle\rho_t,g\rangle$ is continuous on the time interval $[0,T]$ for any function g in $C^{\infty}(\mathbb{T})$, then, for each $\varepsilon>0$, ρ^{ε} is uniformly continuous on $[0,T]\times\mathbb{T}$.*

For each $a>0$, define the functions $h=h_a$ and χ_a on [0, 1] by

$$h(\rho) := \frac{1}{2(1+2a)}\big\{(\rho+a)\log(\rho+a)+(1-\rho+a)\log(1-\rho+a)\big\},$$

$$\chi_a(\rho) := (\rho+a)(1-\rho+a).$$

Note that $h''=(2\chi_a)^{-1}$.

Until the end of this section, $0<C_0<\infty$ represents a constant independent of ε, δ and a and which may change from line to line.

Lemma 4.4. *Let $R^{\varepsilon,\delta}$ be the difference between $L_H(\pi^{\varepsilon,\delta})$ and $L_{H^{\varepsilon,\delta}}(\pi)$:*

$$R^{\varepsilon,\delta} = L_H\big(\pi^{\varepsilon,\delta}\big) - L_{H^{\varepsilon,\delta}}(\pi),$$

where $H=h_a'(\rho^{\varepsilon,\delta})$. Then, for any fixed $\varepsilon>0$, $R^{\varepsilon,\delta}$ converges to 0 as $\delta\downarrow 0$.

Proof. Keep in mind that $H=h_a'(\rho^{\varepsilon,\delta})$ depends on ε and δ, although this does not appears in the notation, and recall that C_0 represents a constant independent of ε, δ and a which may change from line to line. A change of variables shows that

$$\begin{aligned} L_H\big(\pi^{\varepsilon,\delta}\big) &= \big\langle\rho_T^{\delta},H_T^{\varepsilon}\big\rangle - \big\langle\rho_0^{\delta},H_0^{\varepsilon}\big\rangle - \int_0^T dt\big\langle\rho_t^{\delta},\partial_t H_t^{\varepsilon}\big\rangle \\ &= \big\langle\rho_T,H_T^{\varepsilon,\delta}\big\rangle - \big\langle\rho_0,H_0^{\varepsilon,\delta}\big\rangle - \int_0^T dt\big\langle\rho_t^{\delta},\partial_t H_t^{\varepsilon}\big\rangle + R_1^{\varepsilon,\delta}, \end{aligned}$$

where

$$R_1^{\varepsilon,\delta} := R^{\varepsilon,\delta,T} - R_0^{\varepsilon,\delta,0} \quad\text{and}\quad R^{\varepsilon,\delta,t} := \big\langle\rho_t^{\delta}-\rho_t,H_t^{\varepsilon}\big\rangle + \big\langle\rho_t,H_t^{\varepsilon}-H_t^{\varepsilon,\delta}\big\rangle$$

for $0\le t\le T$.

From a simple computation it is easy to see that

$$\int_0^T dt\langle\rho_t^\delta,\partial_t H_t^\varepsilon\rangle=\int_0^T dt\langle\rho_t,\partial_t H_t^{\varepsilon,\delta}\rangle+R_2^{\varepsilon,\delta},$$

where $|R_2^{\varepsilon,\delta}|\le C_0\delta\|\partial_t H^\varepsilon\|_\infty$. To conclude the proof, it is enough to show that, for each fixed $\varepsilon>0$, $R_1^{\varepsilon,\delta}$ and $\delta\|\partial_t H^\varepsilon\|_\infty$ converge to zero as $\delta\downarrow 0$.

Fix $\varepsilon>0$. We first prove that

$$\lim_{\delta\downarrow 0} R^{\varepsilon,\delta,t}=0\quad\text{for } t=0 \text{ and } t=T. \tag{4.3}$$

We prove this assertion for $t=T$, the argument being similar for $t=0$. A change of variables shows that

$$R^{\varepsilon,\delta,T}=\langle\rho_T^{\varepsilon,\delta}-\rho_T^\varepsilon,H_T\rangle+\langle\rho_T^\varepsilon,H_T-H_T^\delta\rangle.$$

By Proposition 4.3, $\rho^\varepsilon(\cdot,u)$ is continuous for any $u\in\mathbb{T}$. Therefore, for any $(t,u)\in[0,T]\times\mathbb{T}$,

$$\begin{aligned}&\lim_{\delta\downarrow 0}\rho^{\varepsilon,\delta}(t,u)=\rho^\varepsilon(t,u),\\&\lim_{\delta\downarrow 0}H^\delta(T,u)=h_a'\big(\rho^\varepsilon(T,u)\big)=\lim_{\delta\downarrow 0}H(T,u).\end{aligned}\tag{4.4}$$

Since h' is bounded and continuous on [0, 1], (4.3) is proved by letting $\delta\downarrow 0$ and by the bounded convergence theorem.

It remains to show that $\delta\|\partial_t H^\varepsilon\|_\infty$ converges to 0 as $\delta\downarrow 0$. An elementary computation gives that, for any $(t,u)\in[0,T]\times\mathbb{T}$,

$$\partial_t H^\varepsilon(t,u)=\int_{\mathbb{T}}dv\,h''\big(\rho^{\varepsilon,\delta}(t,u+v)\big)\psi^\varepsilon(v)\int_{-\delta}^{\delta}dr\,\rho^\varepsilon(t+r,u+v)\big(\phi^\delta\big)'(r).$$

Since ϕ^δ is a symmetric function, a change of variables shows that

$$\int_{-\delta}^{\delta}dr\,\rho^\varepsilon(t+r,u+v)\big(\phi^\delta\big)'(r)=\int_{-\delta}^{0}dr\big\{\rho^\varepsilon(t+r,u+v)-\rho^\varepsilon(t-r,u+v)\big\}\big(\phi^\delta\big)'(r).$$

By Proposition 4.3, ρ^ε is uniformly continuous on $[-1,T+1]\times\mathbb{T}$. On the other hand, $\delta\int_{-\delta}^0(\phi^\delta)'(r)\,dr=\phi(0)$. Therefore, the last expression multiplied by δ converges to 0 as $\delta\downarrow 0$ uniformly in $(t,u)\in[0,T]\times\mathbb{T}$. Since h'' and ψ^ε are uniformly bounded, $\delta\|\partial_t H^\varepsilon\|_\infty$ converges to 0 as $\delta\downarrow 0$. □

Lemma 4.5. *For any path $\pi(t,du)=\rho(t,u)\,du$ such that $\mathcal{Q}(\pi)<\infty$ and for $i=1,2$,*

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}B^i_{H^{\varepsilon,\delta}}(\pi)=B^i_{h'(\rho)}(\pi).$$

Moreover, there exists a positive constant $C_0<\infty$, independent of $a>0$, such that

$$\int_0^T dt\int_{\mathbb{T}}du\frac{(\nabla\rho(t,u))^2}{\chi_a(\rho(t,u))}\le C_0B^1_{h'(\rho)}(\pi),\qquad \big|B^2_{h'(\rho)}(\pi)\big|\le C_0. \tag{4.5}$$

Proof. Throughout this proof, $C(a)$ expresses a constant depending only on $a>0$ which may change from line to line.

Let $\pi(t,du)=\rho(t,u)\,du$ be a path in $D([0,T],\mathcal{M}_{+,1})$ such that $Q(\pi)<\infty$. We first show that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}B^1_{H^{\varepsilon,\delta}}(\pi)=B^1_{h'(\rho)}(\pi). \tag{4.6}$$

Since $\nabla\rho^\varepsilon = \rho * \nabla\psi^\varepsilon$, by Proposition 4.3, $\nabla\rho^\varepsilon$ is uniformly continuous in $[0, T] \times \mathbb{T}$. Therefore, for any $(t, u) \in [0, T] \times \mathbb{T}$, we have

$$\lim_{\delta\downarrow 0} \nabla\rho^{\varepsilon,\delta}(t, u) = \nabla\rho^\varepsilon(t, u),$$

$$\lim_{\delta\downarrow 0} \nabla H^{\varepsilon,\delta}(t, u) = \int_{\mathbb{T}} dv\, \psi^\varepsilon(v) h_a''\big(\rho^\varepsilon(t, u+v)\big)\nabla\rho^\varepsilon(t, u+v).$$

Hence, by the bounded convergence theorem and a change of variables,

$$\lim_{\delta\downarrow 0} B^1_{H^{\varepsilon,\delta}}(\pi) = \frac{1}{2}\int_0^T dt\big\{\big\langle\nabla\rho_t^\varepsilon, h_a''\big(\rho_t^\varepsilon\big)\nabla\rho_t^\varepsilon\big\rangle - \big\langle\chi(\rho_t), \big(\big[h_a''\big(\rho_t^\varepsilon\big)\nabla\rho_t^\varepsilon\big]^\varepsilon\big)^2\big\rangle\big\}. \tag{4.7}$$

On the one hand, since for any fixed $a > 0$ h_a'' is bounded, and since by Proposition 4.3, $\nabla\rho^\varepsilon$ converges to $\nabla\rho$ in $L^2([0, T] \times \mathbb{T})$,

$$\lim_{\varepsilon\downarrow 0}\int_0^T dt\big\langle h_a''\big(\rho_t^\varepsilon\big)\big[\nabla\rho_t^\varepsilon - \nabla\rho_t\big]^2\big\rangle = 0.$$

As ρ has finite energy and h_a'' is bounded, the family $\{h_a''(\rho^\varepsilon)[\nabla\rho]^2; \varepsilon > 0\}$ is uniformly integrable. Moreover, since h_a'' is Lipschitz continuous, by Proposition 4.3, $h_a''(\rho^\varepsilon)$ converges to $h_a''(\rho)$ as $\varepsilon \downarrow 0$ in measure, that is, for any $b > 0$, the Lebesgue measure of the set $\{(t, u) \in [0, T] \times \mathbb{T}; |h_a''(\rho^\varepsilon(t, u)) - h_a''(\rho(t, u))| \geq b\}$ converges to 0 as $\varepsilon \downarrow 0$. Therefore

$$\lim_{\varepsilon\downarrow 0}\int_0^T dt\big\langle h_a''\big(\rho_t^\varepsilon\big)[\nabla\rho_t]^2\big\rangle = \int_0^T dt\big\langle h_a''(\rho_t)[\nabla\rho_t]^2\big\rangle. \tag{4.8}$$

On the other hand, by Schwarz inequality,

$$\limsup_{\varepsilon\downarrow 0}\int_0^T dt\big\langle\chi(\rho_t)\big\{\big[h_a''\big(\rho_t^\varepsilon\big)\nabla\rho_t^\varepsilon - h_a''(\rho_t)\nabla\rho_t\big]^\varepsilon\big\}^2\big\rangle$$

$$\leq \limsup_{\varepsilon\downarrow 0}\int_0^T dt\big\langle\chi(\rho_t)\big\{h_a''\big(\rho_t^\varepsilon\big)\nabla\rho_t^\varepsilon - h_a''(\rho_t)\nabla\rho_t\big\}^2\big\rangle.$$

We may now repeat the arguments presented to estimate the first term on the right hand side of (4.7) to show that the last expression vanishes.

Since χ is a bounded function, to complete the proof of (4.6), it remains to show that

$$\limsup_{\varepsilon\downarrow 0}\int_0^T dt\big\langle\big\{\big[h_a''(\rho_t)\nabla\rho_t\big]^\varepsilon - h_a''(\rho_t)\nabla\rho_t\big\}^2\big\rangle = 0.$$

We estimate the previous integral by the sum of two terms, the first one being

$$\int_0^T dt\big\langle\big\{\big[h_a''(\rho_t)\nabla\rho_t\big]^\varepsilon - \big[h_a''(\rho_t)\big]^\varepsilon\nabla\rho_t\big\}^2\big\rangle$$

$$\leq C(a)\int_0^T dt\int_{\mathbb{T}} dv\,\psi^\varepsilon(v)\big\langle\big\{\nabla\rho_t(u+v) - \nabla\rho_t(u)\big\}^2\big\rangle,$$

where we used Schwarz inequality and the fact that h_a'' is uniformly bounded. This expression vanishes as $\varepsilon \to 0$ because $\nabla\rho$ belongs to $L^2([0, T] \times \mathbb{T})$. The second term in the decomposition is

$$\int_0^T dt\big\langle[\nabla\rho_t]^2\big\{\big[h_a''(\rho_t)\big]^\varepsilon - h_a''(\rho_t)\big\}^2\big\rangle. \tag{4.9}$$

By the argument leading to (4.8), the expression (4.9) converges to 0 as $\varepsilon \downarrow 0$.

We turn to the proof that

$$\lim_{\varepsilon\downarrow 0}\overline{\lim_{\delta\downarrow 0}}\big|B^2_{H^{\varepsilon,\delta}}(\pi) - B^2_{h'(\rho)}(\pi)\big| = 0. \tag{4.10}$$

Since B, D and h' are bounded functions, the difference appearing in the previous formula is less than or equal to

$$C(a)\bigg\{\int_0^T \big\|e^{H_t^{\varepsilon,\delta}} - e^{h'(\rho_t)}\big\|_1\,dt + \int_0^T \big\|e^{-H_t^{\varepsilon,\delta}} - e^{-h'(\rho_t)}\big\|_1\,dt\bigg\}$$
$$\le C(a)\int_0^T \big\|H_t^{\varepsilon,\delta} - h'(\rho_t)\big\|_1\,dt.$$

By Proposition 4.3, ρ^ε is uniformly continuous in $[0,T]\times\mathbb{T}$. Therefore letting $\delta\to 0$, the previous expression converges to

$$C(a)\int_0^T dt\big\|\big[h'\big(\rho_t^\varepsilon\big)\big]^\varepsilon - h'(\rho_t)\big\|_1\,dt$$
$$\le C(a)\bigg\{\int_0^T \big\|\big[h'\big(\rho_t^\varepsilon\big)\big]^\varepsilon - h'\big(\rho_t^\varepsilon\big)\big\|_1\,dt + \int_0^T \big\|h'\big(\rho_t^\varepsilon\big) - h'(\rho_t)\big\|_1\,dt\bigg\}.$$

Since h' is Lipschitz continuous and ρ^ε converges to ρ in $L^2([0,T]\times\mathbb{T})$, the second integral vanishes in the limit as $\varepsilon\downarrow 0$. On the other hand, the first integral is bounded above by

$$C(a)\int_0^T dt\int_{\mathbb{T}} dv\,\psi^\varepsilon(v)\int_{\mathbb{T}} du\big|\rho_t^\varepsilon(u+v) - \rho_t^\varepsilon(u)\big|$$
$$\le C(a)\int_0^T dt\int_{\mathbb{T}} dv\,\psi^\varepsilon(v)\int_{\mathbb{T}} du\big|\rho_t(u+v) - \rho_t(u)\big|.$$

This last integral vanishes in the limit as $\varepsilon\downarrow 0$ because ρ belongs to $L^2([0,T]\times\mathbb{T})$.

To proof of the first bound in (4.5) is elementary and left to the reader. To prove the second one, recall from (2.1) that there exist polynomials $\tilde B$, $\tilde D$ such that $B(\rho) = (1-\rho)\tilde B(\rho)$ and $D(\rho) = \rho\tilde D(\rho)$. From this fact, it is easy to see that the second bound in (4.5) holds for some finite constant C_0, independent of $a>0$. □

Proof of Proposition 4.2. We may assume, without loss of generality, that $I_T(\pi|\gamma)$ is finite. From the variational formula (4.2) and Lemma 4.4,

$$L_H\big(\pi^{\varepsilon,\delta}\big) + B^1_{H^{\varepsilon,\delta}}(\pi) - B^2_{H^{\varepsilon,\delta}}(\pi) - R^{\varepsilon,\delta} \le I_T(\pi|\gamma), \tag{4.11}$$

where H stands for the function $h'(\rho^{\varepsilon,\delta})$.

Since $\rho^{\varepsilon,\delta}$ is smooth, an integration by parts yields the identity

$$L_H\big(\pi^{\varepsilon,\delta}\big) = \big\langle h\big(\rho_T^{\varepsilon,\delta}\big)\big\rangle - \big\langle h\big(\rho_0^{\varepsilon,\delta}\big)\big\rangle.$$

There exists, therefore, a constant C_0, independent of ε, δ and a, such that

$$\big|L_H\big(\pi^{\varepsilon,\delta}\big)\big| \le C_0.$$

In (4.11), let $\delta\downarrow 0$ and then $\varepsilon\downarrow 0$. It follows from the previous bound, and from Lemmas 4.4 and 4.5 that

$$\int_0^T dt\int_{\mathbb{T}} du\,\frac{|\nabla\rho(t,u)|^2}{\chi_a(\rho(t,u))} \le C_0\big\{I_T(\pi|\gamma) + 1\big\}.$$

It remains to let $a\downarrow 0$ and to use Fatou's lemma. □

Corollary 4.6. *The density ρ of a path $\pi(t,du)=\rho(t,u)\,du$ in $D([0,T],\mathcal{M}_{+,1})$ is the weak solution of the Cauchy problem* (2.2) *with initial profile γ if and only if the rate function $I_T(\pi|\gamma)$ is equal to* 0. *Moreover, in that case*

$$\int_0^T dt \int_{\mathbb{T}} du \frac{|\nabla\rho(t,u)|^2}{\chi(\rho(t,u))} < \infty. \tag{4.12}$$

Proof. If the density ρ of a path $\pi(t,du)=\rho(t,u)\,du$ in $D([0,T],\mathcal{M}_{+,1})$ is the weak solution of the Cauchy problem (2.2), then for any G in $C^{1,2}([0,T]\times\mathbb{T})$ we have

$$\begin{aligned} J_G(\pi) = &-\frac{1}{2}\int_0^T dt\big\langle\chi(\rho_t),(\nabla G_t)^2\big\rangle \\ &-\int_0^T dt\big\{\big\langle B(\rho_t),e^{G_t}-G_t-1\big\rangle+\big\langle D(\rho_t),e^{-G_t}+G_t-1\big\rangle\big\}. \end{aligned}$$

Since $e^x-x-1\ge 0$ for any x in $\mathbb{R}$, $I_T(\pi|\gamma)=0$. In addition, the bound (4.12) follows from Proposition 4.2.

On the other hand, if $I_T(\pi|\gamma)$ is equal to 0, then, for any G in $C^{1,2}([0,T]\times\mathbb{T})$ and ε in $\mathbb{R}$, we have $J_{\varepsilon G}(\pi)\le 0$. Note that $J_0(\pi)$ is equal to 0. Hence the derivative of $J_{\varepsilon G}(\pi)$ in ε at $\varepsilon=0$ is equal to 0. This implies that the density ρ is a weak solution of the Cauchy problem (2.2). □

Theorem 4.7. *The function $I_T(\cdot|\gamma): D([0,T],\mathcal{M}_+)\to[0,\infty]$ is lower semicontinuous and has compact level sets.*

Proof. For each $q\ge 0$, let E_q be the level set of the rate function $I_T(\cdot|\gamma)$:

$$E_q := \big\{\pi\in D\big([0,T],\mathcal{M}_+\big)\big|I_T(\pi|\gamma)\le q\big\}.$$

Let $\{\pi^n : n\ge 1\}$ be a sequence in $D([0,T],\mathcal{M}_+)$ such that π^n converges to some element π in $D([0,T],\mathcal{M}_+)$. We show that $I_T(\pi|\gamma)\le\liminf_{n\to\infty}I_T(\pi^n|\gamma)$. If $\liminf I_T(\pi^n|\gamma)$ is equal to ∞, the conclusion is clear. Therefore, we may assume that the set $\{I_T(\pi^n|\gamma): n\ge 1\}$ is contained in E_q for some $q>0$. From the lower semicontinuity of the energy $\mathcal{Q}$ and Proposition 4.2, we have

$$\mathcal{Q}(\pi)\le\varliminf_{n\to\infty}\mathcal{Q}\big(\pi^n\big)\le C(q+1)<\infty.$$

Since π^n belongs to $D([0,T],\mathcal{M}_{+,1})$, so does π.

Let ρ and ρ^n be the density of π and π^n respectively. We now claim that the sequence $\{\rho^n : n\ge 1\}$ converges to ρ in $L^1([0,T]\times\mathbb{T})$. Indeed, by the triangle inequality,

$$\begin{aligned} &\int_0^T \big\|\rho_t-\rho_t^n\big\|_1\,dt \\ &\quad\le\int_0^T\big\|\rho_t-\rho_t^\varepsilon\big\|_1\,dt+\int_0^T\big\|\rho_t^\varepsilon-\rho_t^{n,\varepsilon}\big\|_1\,dt+\int_0^T\big\|\rho_t^{n,\varepsilon}-\rho_t^n\big\|_1\,dt, \end{aligned} \tag{4.13}$$

where $\rho_t^{n,\varepsilon}=\rho_t^n*\psi^\varepsilon$. The first term on the right hand side in (4.13) can be computed as

$$\begin{aligned} \int_0^T\big\|\rho_t-\rho_t^\varepsilon\big\|_1\,dt &\le\int_0^T dt\int_{\mathbb{T}}du\int_{\mathbb{T}}dv\,\psi^\varepsilon(v)\big|\rho(t,u+v)-\rho(t,u)\big| \\ &\le\int_0^T dt\int_{\mathbb{T}}du\int_{\mathbb{T}}dv\,\psi^\varepsilon(v)\int_u^{u+v}dw\big|\nabla\rho(t,w)\big|. \end{aligned}$$

Note that $\operatorname{supp}\psi_\varepsilon\subset[-\varepsilon,\varepsilon]$. From the fundamental inequality $2ab\le A^{-1}a^2+Ab^2$, for any $A>0$, the above expression can be bounded above by

$$\frac{\mathcal{Q}(\pi)}{2A}+\frac{AT\varepsilon}{2}.$$

Similarly, the last term on the right hand side in (4.13) can be bounded above by

$$\int_0^T \left\| \rho_t^{\varepsilon,n} - \rho_t^n \right\|_1 dt \le \frac{\mathcal{Q}(\pi^n)}{2A} + \frac{AT\varepsilon}{2}.$$

Since, for fixed $\varepsilon > 0$, $\rho_t^{\varepsilon,n}$ converges to ρ_t^{ε} weakly as $n \to \infty$ for a.e. $t \in [0,T]$, letting $n \to \infty$ in (4.13) gives that

$$\overline{\lim_{n\to\infty}} \int_0^T \left\| \rho_t - \rho_t^n \right\|_1 dt \le C(q,T)\left\{ \frac{1}{A} + A\varepsilon \right\},$$

for some constant $C(q,T) > 0$ which depends on q and T. Optimizing in A and letting $\varepsilon \downarrow 0$, we complete the proof of the claim made above (4.13).

It follows from this claim that for any function G in $C^{1,2}([0,T] \times \mathbb{T})$,

$$\lim_{n\to\infty} J_G\big(\pi^n\big) = J_G(\pi).$$

This limit implies that $I_T(\pi|\gamma) \le \liminf_{n\to\infty} I_T(\pi^n|\gamma)$, proving that $I_T(\cdot|\gamma)$ is lower-semicontinuous.

The same argument shows that E_q is closed in $D([0,T], \mathcal{M}_+)$. Since it is shown in [24] that E_q is relatively compact in $D([0,T], \mathcal{M}_+)$, E_q is compact in $D([0,T], \mathcal{M}_+)$, and the proof is completed. □

5. $I_T(\cdot|\gamma)$-Density

The lower bound of the large deviations principle stated in Theorem 2.5 has been established in [24] for smooth trajectories. To remove this restriction, we have to show that any trajectory π_t, $0 \le t \le T$, with finite rate function can be approximated by a sequence of smooth trajectories $\{\pi^n : n \ge 1\}$ such that

$$\pi^n \longrightarrow \pi \quad \text{and} \quad I_T\big(\pi^n|\gamma\big) \longrightarrow I_T(\pi|\gamma).$$

This is the content of this section. We first introduce some terminology.

Definition 5.1. Let A be a subset of $D([0,T], \mathcal{M}_+)$. A is said to be $I_T(\cdot|\gamma)$-dense if for any π in $D([0,T], \mathcal{M}_+)$ such that $I_T(\pi|\gamma) < \infty$, there exists a sequence $\{\pi^n : n \ge 1\}$ in A such that π^n converges to π in $D([0,T], \mathcal{M}_+)$ and $I_T(\pi^n|\gamma)$ converges to $I_T(\pi|\gamma)$.

Let Π be the set of all trajectories $\pi(t,du) = \rho(t,u)\,du$ in $D([0,T], \mathcal{M}_{+,1})$ whose density ρ is a weak solution of the Cauchy problem

$$\begin{cases} \partial_t \rho = \frac{1}{2}\Delta\rho - \nabla(\chi(\rho)\nabla H) + B(\rho)e^H - D(\rho)e^{-H} & \text{on } \mathbb{T}, \\ \rho(0,\cdot) = \gamma(\cdot), \end{cases} \tag{5.1}$$

for some function H in $C^{1,2}([0,T] \times \mathbb{T})$.

Theorem 5.2. *Assume that the functions B and D are concave. Then, the set Π is $I_T(\cdot|\gamma)$-dense.*

The proof of Theorem 5.2 is divided into several steps. Throughout this section, denote by $\lambda : [0,T] \times \mathbb{T} \to [0,1]$ the unique weak solution of the Cauchy problem (2.2) with initial profile γ, and assume that the functions B and D are concave.

Let Π_1 be the set of all paths $\pi(t,du) = \rho(t,u)\,du$ in $D([0,T], \mathcal{M}_{+,1})$ whose density ρ is a weak solution of the Cauchy problem (2.2) in some time interval $[0,\delta]$, $\delta > 0$.

Lemma 5.3. *The set Π_1 is $I_T(\cdot|\gamma)$-dense.*

Proof. Fix $\pi(t,du)=\rho(t,u)\,du$ in $D([0,T],\mathcal{M}_{+,1})$ such that $I_T(\pi|\gamma)<\infty$. For each $\delta>0$, set the path $\pi^\delta(t,du)=\rho^\delta(t,u)\,du$ where

$$\rho^\delta(t,u)=\begin{cases}\lambda(t,u) & \text{if } t\in[0,\delta],\\ \lambda(2\delta-t,u) & \text{if } t\in[\delta,2\delta],\\ \rho(t-2\delta,u) & \text{if } t\in[2\delta,T].\end{cases}$$

It is clear that π^δ converges to π in $D([0,T],\mathcal{M}_+)$ as $\delta\downarrow 0$ and that π^δ belongs to Π_1. To conclude the proof it is enough to show that $I_T(\pi^\delta|\gamma)$ converges to $I_T(\pi|\gamma)$ as $\delta\downarrow 0$.

Since the rate function is lower semicontinuous, $I_T(\pi|\gamma)\le\liminf_{\delta\to 0}I_T(\pi^\delta|\gamma)$. Note that $\mathcal{Q}(\pi^\delta)\le 2\mathcal{Q}(\lambda)+\mathcal{Q}(\pi)$. From Corollary 4.6, we have $\mathcal{Q}(\pi^\delta)<\infty$. To prove the upper bound $\limsup_{\delta\to 0}I_T(\pi^\delta|\gamma)\le I_T(\pi|\gamma)$, we now decompose the rate function $I_T(\pi^\delta|\gamma)$ into the sum of the contributions on each time interval $[0,\delta]$, $[\delta,2\delta]$ and $[2\delta,T]$. The first contribution is equal to 0 since the density ρ^δ is a weak solution of the equation (2.2) on this interval. The third contribution is bounded above by $I_T(\pi|\gamma)$ since π^δ on this interval is a time translation of the path π.

On the time interval $[\delta,2\delta]$, the density ρ^δ solves the backward reaction-diffusion equation: $\partial_t\rho^\delta=-(1/2)\Delta\rho^\delta-F(\rho^\delta)$. Therefore, the second contribution can be written as

$$\sup_{G\in C^{1,2}([0,T]\times\mathbb{T})}\left\{\int_0^\delta dt\left\{\langle\nabla\lambda_t,\nabla G_t\rangle-\frac{1}{2}\big\langle\chi(\lambda_t),(\nabla G_t)^2\big\rangle\right\}\right.$$
$$\left.+\int_0^\delta dt\big\{\big\langle B(\lambda_t),1-e^{G_t}-G_t\big\rangle+\big\langle D(\lambda_t),1-e^{-G_t}+G_t\big\rangle\big\}\right\}.$$

By Schwarz inequality, the first integral inside the supremum is bounded above by

$$\frac{1}{2}\int_0^\delta dt\int_{\mathbb{T}}du\frac{|\nabla\lambda(t,u)|^2}{\chi(\lambda(t,u))}. \tag{5.2}$$

On the other hand, taking advantage of the relation (2.1) and of the fact that B and D are bounded functions, a simple computation shows that the second integral inside the supremum in the penultimate displayed equation is bounded above by

$$C\int_0^\delta dt\int_{\mathbb{T}}du\log\frac{1}{\chi(\lambda(t,u))}+C\delta,$$

for some finite constant C independent of δ. By Corollary 4.6, the expression (5.2) converges to 0 as $\delta\downarrow 0$. Hence, to conclude the proof it suffices to show that

$$\lim_{\delta\downarrow 0}\int_0^\delta dt\int_{\mathbb{T}}du\log\chi\big(\lambda(t,u)\big)=0. \tag{5.3}$$

Let $\lambda_t^j:[0,T]\to\mathbb{R}$, $j=0,1$, be the unique solution of the ordinary differential equation

$$\frac{d}{dt}\lambda_t^j=F\big(\lambda_t^j\big), \tag{5.4}$$

with initial condition $\lambda_0^j=j$ and set $\lambda^j(t,u)\equiv\lambda_t^j$ for $(t,u)\in[0,T]\times\mathbb{T}$. Since λ^j is constant in spatial variable, $\Delta\lambda^j=0$. Therefore it follows from (5.4) that λ^j is a unique weak solution of the Cauchy problem (2.2) with initial profile $\lambda_0^j(u)\equiv j$. By Proposition A.5,

$$\lambda_t^0\le\lambda(t,u)\quad\text{and}\quad 1-\lambda_t^1\le 1-\lambda(t,u), \tag{5.5}$$

for any $(t,u) \in [0,T] \times \mathbb{T}$. Since $F(1) < 0 < F(0)$, an elementary computation shows that

$$\lim_{\delta \downarrow 0} \int_0^\delta dt \log \lambda_t^0 = 0 \quad \text{and} \quad \lim_{\delta \downarrow 0} \int_0^\delta dt \log\left(1 - \lambda_t^1\right) = 0. \tag{5.6}$$

By definition of χ and by (5.5),

$$\log \chi\left(\lambda(t,u)\right) = \log \lambda(t,u) + \log\left(1 - \lambda(t,u)\right) \geq \log \lambda_t^0 + \log\left(1 - \lambda_t^1\right).$$

To conclude the proof of (5.3), it remains to recall (5.6). □

Let Π_2 be the set of all paths $\pi(t,du) = \rho(t,u)\,du$ in Π_1 with the property that for every $\delta > 0$ there exists $\varepsilon > 0$ such that $\varepsilon \leq \rho(t,u) \leq 1 - \varepsilon$ for all $(t,u) \in [\delta, T] \times \mathbb{T}$.

Lemma 5.4. *The set Π_2 is $I_T(\cdot|\gamma)$-dense.*

Proof. Fix $\pi(t,du) = \rho(t,u)\,du$ in Π_1 such that $I_T(\pi|\gamma) < \infty$. For each $\varepsilon > 0$, set the path $\pi^\varepsilon(t,du) = \rho^\varepsilon(t,u)\,du$ with $\rho^\varepsilon = (1-\varepsilon)\rho + \varepsilon\lambda$. It is clear that π^ε converges to π in $D([0,T], \mathcal{M}_+)$ as $\varepsilon \downarrow 0$. Let $\lambda^j(t,u) \equiv \lambda_t^j$, $j = 0, 1$, be the weak solution of the equation (2.2) with initial profile $\lambda_0^j(u) \equiv j$. By Proposition A.5, $\varepsilon\lambda^0 \leq \rho^\varepsilon \leq (1-\varepsilon) + \varepsilon\lambda^1$. Moreover it is easy to see that λ^j, $j = 1, 2$, belongs to the set Π_2 since λ^j solves the ordinary differential equation

$$\frac{d}{dt}\lambda_t^j = F\left(\lambda_t^j\right),$$

and $F(1) < 0 < F(0)$. Therefore π^ε belongs to Π_2. To conclude the proof it is enough to show that $I_T(\pi^\varepsilon|\gamma)$ converges to $I_T(\pi|\gamma)$ as $\varepsilon \downarrow 0$.

Since the rate function is lower semicontinuous, $I_T(\pi|\gamma) \leq \liminf_{\varepsilon\downarrow 0} I_T(\pi^\varepsilon|\gamma)$. By the convexity of the energy, $\mathcal{Q}(\pi^\varepsilon) \leq \varepsilon\mathcal{Q}(\lambda) + (1-\varepsilon)\mathcal{Q}(\pi)$, hence $\mathcal{Q}(\pi^\varepsilon) < \infty$. Let G be a function in $C^{1,2}([0,T] \times \mathbb{T})$. Since B, D and χ are concave and Lipschitz continuous,

$$J_G\left(\pi^\varepsilon\right) \leq (1-\varepsilon)J_G(\pi) + \varepsilon J_G(\lambda) + C_0\left\{\varepsilon + \int_0^T \left\|\rho_t^\varepsilon - \rho_t\right\|_1 dt\right\}$$

for some finite constant C_0, which may change from line to line. Therefore,

$$I_T\left(\pi^\varepsilon|\gamma\right) \leq (1-\varepsilon)I_T(\pi|\gamma) + \varepsilon I_T(\lambda|\gamma) + C_0 T\varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives $\limsup_{\varepsilon\downarrow 0} I_T(\pi^\varepsilon|\gamma) \leq I_T(\pi|\gamma)$, which completes the proof. □

Let Π_3 be the set of all paths $\pi(t,du) = \rho(t,u)\,du$ in Π_2 whose density $\rho(t,\cdot)$ belongs to the space $C^\infty(\mathbb{T})$ for any $t \in (0,T]$.

Lemma 5.5. *The set Π_3 is $I_T(\cdot|\gamma)$-dense.*

Proof. Fix $\pi(t,du) = \rho(t,u)\,du$ in Π_2 such that $I_T(\pi|\gamma) < \infty$. Since π belongs to the set Π_1, we may assume that the density solves the equation (2.2) in some time interval $[0, 2\delta]$, $\delta > 0$. Take a smooth nondecreasing function $\alpha : [0,T] \to [0,1]$ with the following properties:

$$\begin{cases} \alpha(t) = 0 & \text{if } t \in [0,\delta], \\ 0 < \alpha(t) < 1 & \text{if } t \in (\delta, 2\delta), \\ \alpha(t) = 1 & \text{if } t \in [2\delta, T]. \end{cases}$$

Let $\psi(t,u):(0,\infty)\times\mathbb{T}\to(0,\infty)$ be the transition probability density of the Brownian motion on $\mathbb{T}$ at time t starting from 0. For each $n\in\mathbb{N}$, denote by ψ^n the function

$$\psi^n(t,u):=\psi\left(\frac{1}{n}\alpha(t),u\right)$$

and define the path $\pi^n(t,du)=\rho^n(t,u)\,du$ where

$$\rho^n(t,u)=\begin{cases}\rho(t,u) & \text{if } t\in[0,\delta],\\ (\rho_t*\psi_t^n)(u)=\int_{\mathbb{T}}dv\,\rho(t,v)\psi^n(t,u-v) & \text{if } t\in(\delta,T].\end{cases}$$

It is clear that π^n converges to π in $D([0,T],\mathcal{M}_+)$ as $n\to\infty$. Since the density ρ^n is a weak solution to the Cauchy problem (2.2) in time interval $[0,\delta]$, by Proposition A.4, $\rho^n(t,\cdot)$ belongs to the space $C^\infty(\mathbb{T})$ for $t\in(0,\delta]$. On the other hand, by the definition of ρ^n, it is clear that $\rho^n(t,\cdot)$ belongs to the space $C^\infty(\mathbb{T})$ for $t\in(\delta,T]$. Therefore π^n belongs to Π_3. To conclude the proof it is enough to show that $I_T(\pi^n|\gamma)$ converges to $I_T(\pi|\gamma)$ as $n\to\infty$.

Since the rate function is lower semicontinuous, $I_T(\pi|\gamma)\le\liminf_{n\to\infty}I_T(\pi^\varepsilon|\gamma)$. Note that the generalized derivative of ρ^n is given by

$$\nabla\rho^n(t,u)=\begin{cases}\nabla\rho(t,u) & \text{if } t\in[0,\delta],\\ (\nabla\rho_t*\psi_t^n)(u) & \text{if } t\in(\delta,T].\end{cases}$$

Therefore, by Schwarz inequality, $\mathcal{Q}(\pi^n)\le\mathcal{Q}(\pi)<\infty$.

The strategy of the proof of the upper bound is similar to the one of Lemma 5.3. We decompose the rate function $I_T(\pi^n|\gamma)$ into the sum of the contributions on each time interval $[0,\delta]$, $[\delta,2\delta]$ and $[2\delta,T]$. The first contribution is equal to 0 since the density ρ^n is a weak solution of the Cauchy problem (2.2) on this interval. Since π^n is defined as a spatial average of π, and since the functions B and D are concave, similar arguments to the ones presented in the proof of Lemma 5.4 yield that the third contribution is bounded above by $I_T(\pi|\gamma)+o_n(1)$. Hence it suffices to show that the second contribution converges to 0 as $n\to\infty$.

Since $\partial_t\psi=(1/2)\Delta\psi$, an integration by parts yields that in the time interval $(\delta,2\delta)$,

$$\partial_t\rho^n=\partial_t\rho*\psi^n+\frac{\alpha'(t)}{2n}\Delta\rho*\psi^n.$$

Thus, since ρ is a weak solution of the hydrodynamic equation (2.2) in the time interval $[\delta,2\delta]$, for any function G in $C^{1,2}([0,T]\times\mathbb{T})$,

$$\begin{aligned}&\langle\rho_{2\delta}^n,G_{2\delta}\rangle-\langle\rho_\delta^n,G_\delta\rangle-\int_\delta^{2\delta}dt\langle\rho_t^n,\partial_tG_t\rangle\\&\quad=\int_\delta^{2\delta}dt\left\{\left\langle\rho_t^n,\frac{1}{2}\Delta G_t\right\rangle-\frac{\alpha'(t)}{2n}\langle\nabla\rho_t^n,\nabla G_t\rangle+\langle F_t^n,G_t\rangle\right\},\end{aligned}$$

where $F_t^n=F(\rho_t)*\psi_t^n$. Therefore, the contribution to $I_T(\pi|\gamma)$ of the piece of the trajectory in the time interval $[\delta,2\delta]$ can be written as

$$\begin{aligned}\sup_{G\in C^{1,2}([0,T]\times\mathbb{T})}&\left\{\int_\delta^{2\delta}dt\left(-\frac{\alpha'(t)}{2n}\langle\nabla\rho_t^n,\nabla G_t\rangle-\frac{1}{2}\langle\chi(\rho_t^n),(\nabla G_t)^2\rangle\right)\right.\\&\left.+\int_\delta^{2\delta}dt\langle F_t^nG_t-B(\rho_t^n)(e^{G_t}-1)-D(\rho_t^n)(e^{-G_t}-1)\rangle\right\}.\end{aligned}\tag{5.7}$$

By Schwarz inequality, the first integral inside the supremum is bounded above by

$$\frac{\|\alpha'\|_\infty^2}{8n^2}\int_\delta^{2\delta}dt\int_{\mathbb{T}}du\frac{|\nabla\rho^n(t,u)|^2}{\chi(\rho^n(t,u))}.$$

Since π belongs to Π_2, there exists a positive constant $C(\delta)$, depending only on δ, such that $C(\delta) \le \rho^n \le 1 - C(\delta)$ on time interval $[\delta, 2\delta]$. This bounds together with the fact that $\mathcal{Q}(\pi^n) \le \mathcal{Q}(\pi)$ permit to prove that the previous expression converges to 0 as $n \to \infty$. On the other hand, the second integral inside the supremum (5.7) is bounded above by

$$\int_\delta^{2\delta} dt \big\langle F_t^n m_t^n - B(\rho_t^n)(e^{m_t^n} - 1) - D(\rho_t^n)(e^{-m_t^n} - 1)\big\rangle, \tag{5.8}$$

where

$$m_t^n = \log \frac{F_t^n + \sqrt{(F_t^n)^2 + 4B(\rho_t^n)D(\rho_t^n)}}{2B(\rho_t^n)}.$$

Note that m_t^n is well-defined and that the integrand in (5.8) is uniformly bounded in n because in the time interval $[\delta, 2\delta]$ ρ_t is bounded below by a strictly positive constant and bounded above by a constant strictly smaller than 1. Since $m^n(t,u)$ converges to 0 as $n \to \infty$ for any $(t,u) \in [\delta, 2\delta] \times \mathbb{T}$, the expression in (5.8) converges to 0 as $n \to \infty$. □

Let Π_4 be the set of all paths $\pi(t, du) = \rho(t,u)\, du$ in Π_3 whose density ρ belongs to $C^{\infty,\infty}((0,T] \times \mathbb{T})$.

Lemma 5.6. *The set Π_4 is $I_T(\cdot|\gamma)$-dense.*

Proof. Fix $\pi(t,du) = \rho(t,u)\,du$ in Π_3 such that $I_T(\pi|\gamma) < \infty$. Since π belongs to the set Π_1, we may assume that the density ρ solves the equation (2.2) in the time interval $[0, 3\delta]$ for some $\delta > 0$. Take a smooth nonnegative function $\phi : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\operatorname{supp} \phi \subset [0,1] \quad \text{and} \quad \int_0^1 \phi(s)\, ds = 1.$$

Let α be the function introduced in the previous lemma. For each $\varepsilon > 0$ and $n \in \mathbb{N}$, let

$$\Phi(\varepsilon, s) := \frac{1}{\varepsilon}\phi\left(\frac{s}{\varepsilon}\right), \qquad \alpha_n(t) := \frac{1}{n}\alpha(t),$$

and let $\pi^n(t,du) = \rho^n(t,u)\,du$ where

$$\rho^n(t,u) = \int_0^1 \rho\big(t + \alpha_n(t)s, u\big)\phi(s)\, ds = \int_{\mathbb{R}} \rho(t+s,u)\Phi\big(\alpha_n(t), s\big)\, ds.$$

In the above formula, we extend the definition of ρ to $[0, T+1]$ by setting $\rho_t = \tilde{\lambda}_{t-T}$ for $T \le t \le T+1$, where $\tilde{\lambda} : [0,1] \times \mathbb{T} \to [0,1]$ stands for the unique weak solution of the equation (2.2) with initial profile ρ_T. Note that it follows from the construction of ρ^n that $\rho_t = \rho_t^n$ for any $t \in [0, \delta]$, therefore, ρ^n is a weak solution of the equation (2.2) in time interval $[0, \delta]$.

It is clear that π^n converges to π in $D([0,T], \mathcal{M}_+)$. Since on the time interval $(0, 3\delta)$, the function ρ is smooth in time, for n large enough the function ρ^n is smooth in time on $(0,T] \times \mathbb{T}$. Hence, π^n belongs to Π_4 and $\mathcal{Q}(\pi^n)$ is finite.

The remaining part of the proof is similar to the one of the previous lemma. We only present the arguments leading to the bound $\limsup_{n\to\infty} I_T(\pi^n|\gamma) \le I_T(\pi|\gamma)$. The rate function can be decomposed in three pieces, two of which can be estimated as in Lemma 5.5. We consider the contribution to $I_T(\pi^n|\gamma)$ of the piece of the trajectory corresponding to the time interval $[\delta, 2\delta]$.

The derivative of ρ^n in time on $(\delta, 2\delta)$ is computed as

$$\partial_t \rho^n(t,u) = \int_{\mathbb{R}} \partial_t \rho(t+s,u)\Phi\big(\alpha_n(t), s\big)\, ds + \int_{\mathbb{R}} \rho(t+s,u)\partial_t\big[\Phi\big(\alpha_n(t), s\big)\big]\, ds.$$

It follows from this equation and from the fact that the density ρ solves the hydrodynamic equation (2.2) on the time interval $[\delta, 3\delta]$, that for any function G in $C^{1,2}([0,T]\times\mathbb{T})$,

$$\langle \rho^n_{2\delta}, G_{2\delta}\rangle - \langle \rho^n_{\delta}, G_{\delta}\rangle - \int_\delta^{2\delta} dt \langle \rho^n_t, \partial_t G_t\rangle = \int_\delta^{2\delta} dt \Big\{ \Big\langle \rho^n_t, \frac{1}{2}\Delta G_t \Big\rangle + \langle F^n_t + r^n_t, G_t\rangle \Big\},$$

where

$$F^n(t,u) := \int_{\mathbb{R}} F\big(\rho(t+s,u)\big)\Phi\big(\alpha_n(t),s\big)\,ds,$$
$$r^n(t,u) := \int_{\mathbb{R}} \rho(t+s,u)\partial_t\big[\Phi\big(\alpha_n(t),s\big)\big]\,ds.$$

Therefore, the second contribution can be bounded above by

$$\sup_{G\in C^{1,2}([0,T]\times\mathbb{T})} \Big\{ \int_\delta^{2\delta} dt \big\langle \big(F^n_t + r^n_t\big)G_t - B\big(\rho^n_t\big)\big(e^{G_t}-1\big) - D\big(\rho^n_t\big)\big(e^{-G_t}-1\big)\big\rangle \Big\}. \tag{5.9}$$

We now show that $r^n(t,u)$ converges to 0 as $n\to\infty$ uniformly in $(t,u)\in(\delta,2\delta)\times\mathbb{T}$. Let (t,u) in $(\delta,2\delta)\times\mathbb{T}$. Since $\int_{\mathbb{R}} \partial_t[\Phi(\alpha_n(t),s)]\,ds = \partial_t[\int_{\mathbb{R}}\Phi(\alpha_n(t),s)\,ds] = 0$, $r^n(t,u)$ can be written as

$$\int_{\mathbb{R}} \big\{\rho(t+s,u) - \rho(t,u)\big\}\partial_t\big[\Phi\big(\alpha_n(t),s\big)\big]\,ds.$$

Since ρ is Lipschitz continuous on $[\delta,3\delta]\times\mathbb{T}$, there exists a positive constant $C(\delta)>0$, depending only on δ, such that

$$\big|\rho(t+s,u) - \rho(t,u)\big| \le C(\delta)s,$$

for any $(t,u)\in[\delta,2\delta]\times\mathbb{T}$ and $s\in[0,\delta]$. Therefore $r^n(t,u)$ is bounded above by

$$C(\delta)\int_{\mathbb{R}} s\big|\partial_t\big[\Phi\big(\alpha_n(t),s\big)\big]\big|\,ds.$$

It follows from a simple computation and from the change of variables $\alpha_n(t)s = \bar{s}$ that

$$\int_{\mathbb{R}} s\big|\partial_t\big[\Phi\big(\alpha_n(t),s\big)\big]\big|\,ds \le \frac{\|\alpha'(t)\|_\infty}{n}\int_0^1 \big\{s\phi(s) + s^2\big|\phi'(s)\big|\big\}\,ds.$$

Therefore $r^n(t,u)$ converges to 0 as $n\to\infty$ uniformly in $(t,u)\in(\delta,2\delta)\times\mathbb{T}$.

To complete the proof, it remains to take a supremum in $G\in C^{1,2}([0,T]\times\mathbb{T})$ in formula (5.9) and to let $n\to\infty$. □

Proof of Theorem 5.2. From the previous lemma, all we need is to prove that Π_4 is contained in Π. Let $\pi(t,du) = \rho(t,u)\,du$ be a path in Π_4. There exists some $\delta>0$ such that the density ρ solves the equation (2.2) on time interval $[0,2\delta]$. In particular, the density ρ also solves the equation (5.1) with $H=0$ on time interval $[0,2\delta]$. On the one hand, since the density ρ is smooth on $[\delta,T]$ and there exists $\varepsilon>0$ such that $\varepsilon\le\rho(t,u)\le 1-\varepsilon$ for any $(t,u)\in[\delta,T]\times\mathbb{T}$, from Lemma 2.1 in [24], there exits a unique function H in $C^{1,2}([\delta,T]\times\mathbb{T})$ satisfying the equation (5.1) with ρ on $[\delta,T]$, and it is proved that π belongs to Π. □

Proof of Theorem 2.5. We have already proved in Section 4 that the rate function is lower semicontinuous and that it has compact level sets.

Recall from the beginning of this section the definition of the set Π. It has been proven in [24] that for each closed subset $\mathcal{C}$ of $D([0,T],\mathcal{M}_+)$,

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_{\eta^N}(\mathcal{C}) \le -\inf_{\pi\in\mathcal{C}} I_T(\pi|\gamma),$$

and that for each open subset $\mathcal{O}$ of $D([0,T],\mathcal{M}_+)$,

$$\underline{\lim}_{N\to\infty} \frac{1}{N} \log Q_{\eta^N}(\mathcal{O}) \ge -\inf_{\pi\in\mathcal{O}\cap\Pi} I_T(\pi|\gamma).$$

Since $\mathcal{O}$ is open in $D([0,T],\mathcal{M}_+)$, by Theorem 5.2,

$$\inf_{\pi\in\mathcal{O}\cap\Pi} I_T(\pi|\gamma) = \inf_{\pi\in\mathcal{O}} I_T(\pi|\gamma),$$

which completes the proof. □

Appendix

In sake of completeness, we present in this section results on the Cauchy problem (2.2).

Definition A.1. A measurable function $\rho : [0,T]\times\mathbb{T}\to[0,1]$ is said to be a weak solution of the Cauchy problem (2.2) in the layer $[0,T]\times\mathbb{T}$ if, for every function G in $C^{1,2}([0,T]\times\mathbb{T})$,

$$\begin{aligned}\langle\rho_T,G_T\rangle - \langle\gamma,G_0\rangle - \int_0^T dt\,\langle\rho_t,\partial_t G_t\rangle \\ = \frac{1}{2}\int_0^T dt\,\langle\rho_t,\Delta G_t\rangle + \int_0^T dt\big\langle F(\rho_t),G_t\big\rangle.\end{aligned} \tag{A.1}$$

For each $t\ge 0$, let P_t be the semigroup on $L^2(\mathbb{T})$ generated by $(1/2)\Delta$.

Definition A.2. A measurable function $\rho : [0,T]\times\mathbb{T}\to[0,1]$ is said to be a mild solution of the Cauchy problem (2.2) in the layer $[0,T]\times\mathbb{T}$ if, for any t in $[0,T]$, it holds that

$$\rho_t = P_t\gamma + \int_0^t P_{t-s}F(\rho_s)\,ds. \tag{A.2}$$

The first proposition asserts existence and uniqueness of weak and mild solutions, a well known result in the theory of partial differential equations. We give a brief proof because uniqueness of the Cauchy problem (2.2) plays an important role in the proof of Theorem 2.1.

Proposition A.3. *Definitions A.1 and A.2 are equivalent. Moreover, there exists a unique weak solution of the Cauchy problem (2.2).*

Proof. Since F is Lipschitz continuous, by the method of successive approximation, there exists a unique mild solution of the Cauchy problem (2.2). Therefore to conclude the proposition it is enough to show that the above two notions of solutions are equivalent.

Assume that $\rho : [0,T]\times\mathbb{T}\to[0,1]$ is a weak solution of the Cauchy problem (2.2). Fix a function g in $C^2(\mathbb{T})$ and $0\le t\le T$. For each $\delta>0$, define the function G^δ as

$$G^\delta(s,u) = \begin{cases} (P_{t-s}g)(u) & \text{if } 0\le s\le t, \\ \delta^{-1}(t+\delta-s)g(u) & \text{if } t\le s\le t+\delta, \\ 0 & \text{if } t+\delta\le s\le T. \end{cases}$$

One can approximate G^δ by functions in $C^{1,2}([0,T]\times\mathbb{T})$. Therefore, by letting $\delta\downarrow 0$ in (A.1) with G replaced by G^δ and by a summation by parts,

$$\langle\rho_t, g\rangle = \langle P_t\gamma, g\rangle + \int_0^t \big\langle P_{t-s}F(\rho_s), g\big\rangle ds. \tag{A.3}$$

Since (A.3) holds for any function g in $C^2(\mathbb{T})$, ρ is a mild solution of the Cauchy problem (2.2).

Conversely, assume that $\rho:[0,T]\times\mathbb{T}\to[0,1]$ is a weak solution of the Cauchy problem (2.2). In this case, (A.3) is true for any function g in $C^2(\mathbb{T})$ and any $0\le t\le T$. Differentiating (A.3) in t gives that

$$\frac{d}{dt}\langle\rho_t, g\rangle = \frac{1}{2}\langle\rho_t, \Delta g\rangle + \big\langle F(\rho_t), g\big\rangle.$$

Therefore (A.1) holds for any function $G(t,u)=g(u)$ in $C^2(\mathbb{T})$. It is not difficult to extend this to any function G in $C^{1,2}([0,T]\times\mathbb{T})$. Hence ρ is a weak solution of the Cauchy problem (2.2). □

The following two propositions assert the smoothness and the monotonicity of weak solutions of the Cauchy problem (2.2).

Proposition A.4. *Let ρ be the unique weak solution of the Cauchy problem* (2.2). *Then ρ is infinitely differentiable over* $(0,\infty)\times\mathbb{T}$.

Proposition A.5. *Let ρ_0^1 and ρ_0^2 be two initial profiles. Let ρ^j, $j=1,2$, be the weak solutions of the Cauchy problem* (2.2) *with initial condition ρ_0^j. Assume that*

$$m\big\{u\in\mathbb{T}:\rho_0^1(u)\le\rho_0^2(u)\big\}=1,$$

where m is the Lebesgue measure on $\mathbb{T}$. Then, for any $t\ge 0$, it holds that

$$m\big\{u\in\mathbb{T}:\rho^1(t,u)\le\rho^2(t,u)\big\}=1.$$

The proofs of Propositions A.4 and A.5 can be found in the ones of Proposition 2.1 of [16].

The last proposition asserts that, for any initial density profile γ, the weak solution ρ_t of the Cauchy problem (2.2) converges to some solution of the semilinear elliptic equation (2.3). Recall, from Section 2.3, the definition of the set $\mathcal{E}$.

Proposition A.6. *Let $\rho:[0,\infty)\times\mathbb{T}\to[0,1]$ be the unique weak solution of the Cauchy problem* (2.2). *Then there exists a density profile ρ_∞ in $\mathcal{E}$ such that ρ_t converges to ρ_∞ as $t\to\infty$ in $C^2(\mathbb{T})$.*

The proof of this proposition can be found in the one of Theorem D of [12].

Acknowledgements

The authors wish to thank J. Farfán for the careful reading of the first version of the manuscript and his fruitful comments and an anonymous referee for his or her helpful comments. This research work has been initiated during the second author's stay in Instituto Nacional de Matemática Pura e Aplicada, he acknowledges gratefully hospitality and support. He would like to thank Professor Funaki for his constant encouragement.

References

[1] L. Avena, M. Jara and F. Voellering. Explicit LDP for a slowed RW driven by a symmetric exclusion process. Preprint, Available at arXiv:1409.3013.

[2] G. Basile and G. Jona-Lasinio. Equilibrium states with macroscopic correlations. *Internat. J. Modern Phys. B* **18** (2004) 479–485. MR2069473
[3] O. Benois, C. Landim and M. Mourragui. Hitting times of rare events in Markov chains. *J. Stat. Phys.* **153** (2013) 967–990. MR3131505
[4] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio and C. Landim. Macroscopic fluctuation theory for stationary non equilibrium state. *J. Stat. Phys.* **110** (2002) 635–675. MR1898852
[5] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio and C. Landim. Large deviations for the boundary driven symmetric simple exclusion process. *Math. Phys. Anal. Geom.* **6** (2003) 231–267. MR1997915
[6] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio and C. Landim. Macroscopic fluctuation theory. *Rev. Modern Phys.* **87** (2015) 593–636. MR3403268
[7] L. Bertini, C. Landim and M. Mourragui. Dynamical large deviations for the boundary driven weakly asymmetric simple exclusion process. *Ann. Probab.* **37** (2009) 2357–2403. MR2573561
[8] T. Bodineau and B. Derrida. Current large deviations for asymmetric exclusion processes with open boundaries. *J. Stat. Phys.* **123** (2006) 277–300. MR2227085
[9] T. Bodineau and G. Giacomin. From dynamic to static large deviations in boundary driven exclusion particle systems. *Stochastic Process. Appl.* **110** (2004) 67–81. MR2052137
[10] T. Bodineau and M. Lagouge. Current large deviations in a driven dissipative model. *J. Stat. Phys.* **139** (2010) 201–218. MR2602960
[11] T. Bodineau and M. Lagouge. Large deviations of the empirical currents for a boundary-driven reaction diffusion model. *Ann. Appl. Probab.* **22** (2012) 2282–2319. MR3024969
[12] X.-Y. Chen and H. Matano. Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations. *J. Differential Equations* **78** (1989) 160–190. MR0986159
[13] A. De Masi, P. Ferrari and J. Lebowitz. Reaction diffusion equations for interacting particle systems. *J. Stat. Phys.* **44** (1986) 589–644. MR0857069
[14] A. De Masi, A. Pellegrinotti, E. Presutti and M. E. Vares. Spatial patterns when phases separate in an interacting particle system. *Ann. Appl. Probab.* **22** (1994) 334–371. MR1258880
[15] A. De Masi, E. Presutti and M. E. Vares. Escape from the unstable equilibrium in a random process with infinitely many interacting particles. *J. Stat. Phys.* **44** (1986) 645–696. MR0857070
[16] P. De Mottoni and M. Schatzman. Geometrical evolution of developed interfaces. *Trans. Amer. Math. Soc.* **347** (1995) 1533–1589. MR1672406
[17] B. Derrida, J. Lebowitz and E. Speer. Large deviation of the density profile in the steady state of the open symmetric simple exclusion process. *J. Stat. Phys.* **107** (2002) 599–634. MR1898851
[18] G. Eyink, J. L. Lebowitz and H. Spohn. Hydrodynamics of stationary nonequilibrium states for some lattice gas models. *Comm. Math. Phys.* **132** (1990) 252–283. MR1069212
[19] J. Farfan. Static large deviations of boundary driven exclusion processes. Preprint, Available at arXiv:0908.1798.
[20] J. Farfan, C. Landim and M. Mourragui. Hydrostatics and dynamical large deviations of boundary driven gradient symmetric exclusion processes. *Stochastic Process. Appl.* **121** (2011) 725–758. MR2770905
[21] J. Farfan, C. Landim and K. Tsunoda. Static large deviations for a reaction-diffusion model. Preprint, Available at arXiv:1606.07227.
[22] T. Franco and A. Neumann. Large deviations for the exclusion process with a slow bond. Preprint, Available at arXiv:1501.00225.
[23] M. I. Freidlin and A. D. Wentzell. *Random Perturbations of Dynamical Systems*. Springer, New York, 1998. MR1652127
[24] G. Jona-Lasinio, C. Landim and M. E. Vares. Large deviations for a reaction diffusion model. *Probab. Theory Related Fields* **97** (1993) 339–361. MR1245249
[25] C. Kipnis and C. Landim. *Scaling Limits of Interacting Particle Systems*. Springer, Berlin, 1999. MR1707314
[26] C. Kipnis, C. Landim and S. Olla. Macroscopic properties of a stationary nonequilibrium distribution for a non-gradient interacting particle system. *Ann. Inst. Henri Poincaré* **31** (1995) 191–221. MR1340037
[27] C. Landim, M. Mourragui and S. Sellami. Hydrodynamic limit for a nongradient interacting particle system with stochastic reservoirs. *Theory Probab. Appl.* **45** (2001) 604–623. MR1968722
[28] M. Mourragui and M. Orlandi. Boundary driven Kawasaki process with long-range interaction: Dynamical large deviations and steady states. *Nonlinearity* **26** (2013) 141–175. MR3001766
[29] J. Quastel, F. Rezakhanlou and S. R. S. Varadhan. Large deviations for the symmetric simple exclusion process in dimensions $d \geq 3$. *Probab. Theory Related Fields* **113** (1999) 1–84. MR1670733