Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 4, 1606–1656
DOI: 10.1214/16-AIHP765

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Universality for random matrix flows with time-dependent density

László Erdős[1] and Kevin Schnelli[2]

IST Austria, Klosterneuburg, Austria. E-mail: lerdos@ist.ac.at; kevin.schnelli@ist.ac.at

Received 28 September 2015; revised 20 April 2016; accepted 11 May 2016

Abstract. We show that the Dyson Brownian Motion exhibits local universality after a very short time assuming that local rigidity and level repulsion of the eigenvalues hold. These conditions are verified, hence bulk spectral universality is proven, for a large class of Wigner-like matrices, including deformed Wigner ensembles and ensembles with non-stochastic variance matrices whose limiting densities differ from Wigner's semicircle law.

Résumé. Nous démontrons que le mouvement Brownien de Dyson établit l'universalité des statistiques spectrales locales après un temps très court, en supposant la rigidité locale et la répulsion de valeurs propres. Ces conditions sont satisfaites, et donc l'universalité spectrale est démontrée au centre du spectre, pour une large classe des matrices aléatoires du type Wigner, y compris les ensembles de Wigner deformés et des ensembles dont la matrice des variances est non-stochastique, dont les densités asymptotiques diffèrent de la loi du demi-cercle de Wigner.

MSC: 60B20; 60H30

Keywords: Random matrix; Local eigenvalue statistics; Universality; Dyson Brownian motion

Contents

1. Introduction and motivation

In his groundbreaking paper [56], Wigner conjectured that the eigenvalue gap distribution of large random matrices is universal and that it serves as a ubiquitous model for the local spectral statistics of many quantum systems. The

[1] Supported in part by ERC Advanced Grant RANMAT No. 338804.
[2] Supported by ERC Advanced Grant RANMAT No. 338804.

Gaussian case was fully understood in the subsequent works of Dyson, Gaudin and Mehta; see [43] for a summary. This simplest case can be generalized in two directions. For invariant ensembles, the joint density function of the eigenvalues can be explicitly expressed in terms of a Vandermonde determinant; a formula that can also be interpreted as the Gibbs measure of a gas of one-dimensional particles with a logarithmic interaction. For specific values of the inverse temperature $\beta = 1, 2, 4$, the correlation functions may be expressed and analyzed using asymptotics of orthogonal polynomials [31] and universality was proved under various conditions on the potential in [17,18,48,49], with many consecutive works following. This method, however, is not applicable for other values of β even in the Gaussian case, where the correlation functions were described in [53]. Universality for general β-ensembles was first established recently in [12,13] for $\beta \geq 1$, with different proofs given later in [6,51] that also hold for $\beta > 0$ albeit with more restrictions on the potential.

Among the non-invariant ensembles, the most prominent case is the $N \times N$ symmetric or hermitian Wigner matrix characterized by the independence of the entries (up to the constraint imposed by the symmetry class). Beyond the Gaussian case there is no explicit formula for the eigenvalue distribution in general, but in the hermitian case ($\beta = 2$) and for distributions with a Gaussian component, the correlation functions can still be expressed using an algebraic identity (Harish–Chandra–Itzykson–Zuber integral). A rigorous analysis of this approach yielded universality for hermitian Wigner matrices with a substantial Gaussian component, [7,33]. The first proof of hermitian Wigner universality for an arbitrary smooth distribution was given in [22], the smoothness condition was later removed in [23,52]. Lacking the algebraic identity, the symmetric case ($\beta = 1$) required a completely different approach based on the analysis of the Dyson Brownian motion. The basic observation of Dyson [19] was that the eigenvalues of a matrix ensemble, embedded in a simple stochastic flow (*Dyson matrix flow*), evolve autonomously and satisfy a system of N stochastic differential equations, called the *Dyson Brownian Motion* (*DBM*). The universal eigenvalue statistics emerge in the bulk spectrum as a consequence of the invariant measure of the DBM. Local statistics require to understand only the local equilibration mechanism which occurs on a very small time scale that can be bridged by perturbative methods. The rigorous theory of this idea was initiated in [25] and developed in a series of papers [26,30] leading to the complete proof of the Wigner–Dyson–Mehta universality conjecture for Wigner matrices in all symmetry classes; see [27] for a summary. More recently two stronger versions of the bulk universality have been proved. In contrast to the previous results that required a local averaging, the universality of each single gap was shown to be universal in [28], while the universality of correlation functions at each fixed energy was obtained in [14]. These papers heavily relied on a new tool from [28], the concept of Hölder regularity theory for the parabolic equation with random coefficients given by the DBM.

In all these works on the spectral universality for Wigner matrices, the global limiting density was the semicircle law; in particular it did not change in time under the DBM. The same Gaussian measure and its localized versions could be used as equilibrium reference measures for all times. The main idea was to artificially speed up the global convergence by considering the *local relaxation flow* [25] and then to prove that the additional local relaxation terms do not substantially modify the local statistics thanks to a-priori bounds on the location of the particles. These bounds are called *rigidity estimates* and they directly follow from short scale versions of the Wigner semicircle law that are called *local laws*.

The method of local relaxation flow has two main limitations that are related. First, it operates with global measures, in particular, quite precise rigidity information is needed for *all* eigenvalues. This is clearly unnecessary (and in some cases hard to obtain); far away eigenvalues should not influence local statistics too much. Second, if the initial matrix of the Dyson matrix flow does not obey the semicircle law, then the density changes with time following the *semicircular flow*, related to the complex Burgers equation for its Stieltjes transform. The time dependence of the density was originally not incorporated in the method of the local relaxation flow. This second limitation was tackled very recently in [40], where universality for *deformed* Wigner matrices with large diagonal elements was proved. Using ideas from hydrodynamic limits [57], a *global* reference measure was constructed as an invariant β-ensemble with a "time" parameter so that its equilibrium density trails the semicircular flow. This equilibrium measure was then used as a basis to construct the local relaxation flow. Once the fast convergence to the reference measure is established one can infer to the universality of the β-ensemble [12,13], or, alternatively, one can use the uniqueness of the local Gibbs measure established in [28] to conclude universality with a tiny Gaussian component. This result is then easily complemented by a standard Green function comparison method to remove the Gaussian component entirely.

As a technical input for the analysis in [40], the global rigidity for the reference β-ensemble is required, which is not available for the case when the equilibrium density is supported on several intervals. In particular, the result of [40] is limited to the deformed Wigner ensembles with a single interval support that excludes the case when the diagonal has a strongly bimodal distribution.

We remark that bulk universality for special classes of deformed Wigner matrices in the hermitian symmetry class has also been proven with different methods. The local sine kernel statistics for the sum of a GUE and a diagonal matrix with two eigenvalues $\pm a$ of equal multiplicity has been obtained with Riemann–Hilbert method [4,11,16]. In particular, the density in this model is supported on two disjoint intervals if a is sufficiently large. The GUE matrix can be replaced with an arbitrary Hermitian matrix if the first four moments of its single entry distribution matches those of the Gaussian [45]. A much more general class of deformations of the GUE has been tackled in [50] relying on a version of the Harish–Chandra–Itzykson–Zuber integral. Using Green function comparison techniques [29] and the local laws from [35,36,39], one can replace the GUE with any hermitian Wigner matrix under the four moment matching condition.

Random matrices whose limiting densities are supported on several intervals arise in other prominent contexts as well. We call symmetric or hermitian matrix ensembles, $H = (h_{ij})$, *Wigner-like* if their entries are independent (up to the symmetry constraint). If, in addition, the matrix elements are centered, $\mathbb{E}h_{ij} = 0$, and the sum of the variances $S_{ij} = \mathbb{E}|h_{ij}|^2$ in each row is constant, say one, i.e.,

$$\sum_{j=1}^{N} S_{ij} = \mathrm{const} = 1, \quad \forall i, \tag{1.1}$$

then the limiting density is the semicircle law. If either condition is violated, the limiting density is generally not the semicircle law and typically it may be supported on several intervals. The case of $H = W + A$, where W is a standard Wigner matrix with i.i.d. centered entries and A is a deterministic matrix (representing the nonzero expectations $\mathbb{E}h_{ij}$), was considered in [35,36], where local laws and rigidity were established. If condition (1.1) is dropped, then an even richer class of possible limiting densities arise. These were extensively analyzed in [1,2], where all possible density shapes are classified, local laws and rigidity are proven.

In the current paper, we prove bulk universality for all these models. As in the previous papers using DBM, the key part is to show universality for matrices with a tiny Gaussian component.

Beyond these applications, our main result is formulated on a more conceptual level. Dyson argued in [19] that the local equilibrium of the DBM is attained after a very short time *irrespective of the global density.* In fact, the global density equilibrates on a time scale of order one, while the local equilibration time is of order $1/N$. The local equilibration is solely due to the logarithmic interaction in the DBM, while the evolution of the global density is given by the semicircular flow. In this paper we fully decouple the effects of these two processes. In the main Theorem 2.1 we prove bulk local universality for the DBM assuming that it satisfies rigidity and level repulsion, but *only locally*. On the global scale only a very weak version of rigidity is required, in particular the condition is insensitive to outliers or to the behavior at the edges. These assumptions can then easily be verified from local laws in each model.

After completing this manuscript, we learned that similar results were obtained independently in [38].

Notational conventions: We use the symbol $O(\cdot)$ and $o(\cdot)$ for the standard big-O and little-o notation. The notations O, o, $\ll$, $\gg$, refer to the limit $N \to \infty$. Here $a \ll b$ means $|a| \le N^{-\xi}|b|$, for some small $\xi > 0$. We use c and C, C' to denote positive constants that do not depend on N. Sometimes we use subscripts or superscripts to distinguish N-independent constants, e.g., c_0, c_1, c' etc. Their value may change from line to line. Similarly, we will use $\xi > 0$ for a small, respectively $D > 0$ for a large positive exponent, mainly appearing in various rigidity bounds. Their precise values are immaterial; at the end of the proof it may be chosen sufficiently small, respectively sufficiently large, depending on all other exponents along the proof. Finally, we use double brackets to denote index sets, i.e., for $n_1, n_2 \in \mathbb{R}$,

$$[\![n_1, n_2]\!] := [n_1, n_2] \cap \mathbb{Z}, \qquad \mathbb{N}_N := [\![1, N]\!].$$

2. Main results

In this section, we give a detailed description of our model, including all assumptions, and state our main results. We start with introducing basic concepts such as the Stieltjes transform, the semicircular flow and the Dyson Brownian motion (DBM).

2.1. *Stieltjes transform*

Given a probability measure, ν, on $\mathbb{R}$, define its Stieltjes transform, m_ν, by

$$m_\nu(z) := \int_{\mathbb{R}} \frac{\mathrm{d}\nu(v)}{v - z}, \quad z \in \mathbb{C}^+ := \{z \in \mathbb{C}, \operatorname{Im} z > 0\}. \tag{2.1}$$

Note that m_ν is an analytic function in upper half plane. In the following we usually write $z = E + \mathrm{i}\eta$, $E \in \mathbb{R}$, $\eta > 0$, and we refer to E as an "energy" and to z as the spectral parameter. For given $\eta > 0$, we let P_η denote the Poisson kernel defined by

$$P_\eta(E) := \frac{1}{\pi} \frac{\eta}{E^2 + \eta^2}, \quad E \in \mathbb{R}, \tag{2.2}$$

and we note that $\int_{\mathbb{R}} P_\eta(E)\,\mathrm{d}E = 1$ and $P_{\eta_1+\eta_2}(E) = (P_{\eta_1} * P_{\eta_2})(E)$, for all $\eta, \eta_1, \eta_2 > 0$, $E \in \mathbb{R}$, where $*$ denotes the convolution on $\mathbb{R}$. We further remark that

$$\frac{1}{\pi} \operatorname{Im} m_\nu(E + \mathrm{i}\eta) = (P_\eta * \nu)(E). \tag{2.3}$$

Assuming that ν admits a density, which we also denote by ν, we can recover ν from m_ν through the Stieltjes inversion formula

$$\nu(E) = \frac{1}{\pi} \lim_{\eta \searrow 0} \operatorname{Im} m_\nu(E + \mathrm{i}\eta) = \lim_{\eta \searrow 0} (P_\eta * \nu)(E), \quad E \in \mathbb{R}. \tag{2.4}$$

The Hilbert transform, $(\mathrm{T}\nu)$, of ν is defined by as the principal value integral

$$(\mathrm{T}\nu)(E) := \int_{\mathbb{R}} \frac{\mathrm{d}\nu(v)}{v - E}, \quad E \in \mathbb{R}. \tag{2.5}$$

2.2. *Semicircular flow*

We next introduce the semicircular or classical flow. Let $\mathcal{M}(\mathbb{R})$ denote the set of probability measures on $\mathbb{R}$. Then the semicircular flow is the process $\mathbb{R}^+ \times \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$, $(t, \varrho) \mapsto \mathcal{F}_t[\varrho]$ obtained via its Stieltjes transform as follows. For $t = 0$, set $\mathcal{F}_0[\varrho] := \varrho$. For $t > 0$, let $m_t(z)$ satisfy

$$m_t(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\varrho(y)}{\mathrm{e}^{-t/2} y - z - (1 - \mathrm{e}^{-t}) m_t(z)}, \qquad \operatorname{Im} m_t(z) > 0, \quad z \in \mathbb{C}^+. \tag{2.6}$$

It is straightforward to check [47] that (2.6) has indeed a unique solution such that $\liminf_{\eta \searrow 0} \operatorname{Im} m_t(E + \mathrm{i}\eta) < \infty$, for any $E \in \mathbb{R}$, $t > 0$. In fact, for $t > 0$, m_t has a continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$ [8] that we also denote by m_t. Set then

$$\mathcal{F}_t[\varrho](E) := \frac{1}{\pi} \lim_{\eta \searrow 0} \operatorname{Im} m_t(E + \mathrm{i}\eta), \quad t > 0, E \in \mathbb{R}, \tag{2.7}$$

so that $\mathcal{F}_t[\varrho]$ is defined through its density $\mathcal{F}_t[\varrho](E)$, $E \in \mathbb{R}$. In particular, for $t > 0$, $\mathcal{F}_t[\varrho]$ is an absolutely continuous measure. (For simplicity we use the same symbol for absolutely continuous measures and their densities.)

Further, it is easy to check that $m_t(z)$ converges pointwise to

$$m_0(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\varrho_0(y)}{y - z}, \tag{2.8}$$

for all $z \in \mathbb{C}^+$, as $t \searrow 0$. It follows that $\mathcal{F}_t[\varrho]$ converges weakly to ϱ as $t \searrow 0$. Starting from (2.6) and (2.7), one also checks that

$$\mathcal{F}_{t+s}[\varrho] = \mathcal{F}_t \circ \mathcal{F}_s[\varrho] \equiv \mathcal{F}_t\big[\mathcal{F}_s[\varrho]\big], \quad t \geq s, \varrho \in \mathcal{M}(\mathbb{R}).$$

In fact, using the additive free convolution, the flow $t \mapsto \mathcal{F}_t$ can be endowed with a (w^*-continuous) semigroup structure [42,44,54]; see also [32,55] for reviews. Yet, we will not pursue this point of view in the present paper.

In the following, we often write $\varrho_t := \mathcal{F}_t[\varrho]$ with $\varrho_0 = \varrho$ and we call $t \mapsto \varrho_t$ the *semicircular flow* started at ϱ. Recalling (2.1) it is clear that m_t is the Stieltjes transform of ϱ_t and we simply write $m_t \equiv m_{\varrho_t}$. We remark that the standard semicircle law, ϱ_{sc}, is invariant under the semicircular flow, i.e., $\mathcal{F}_t[\varrho_{\mathrm{sc}}] = \varrho_{\mathrm{sc}}$, for all $t \geq 0$, and that $\varrho_t = \mathcal{F}_t[\varrho]$ converges weakly to ϱ_{sc}, as $t \nearrow \infty$, for any $\varrho \in \mathcal{M}(\mathbb{R})$. This follows directly from (2.6) and the fact that the Stieltjes transform, $m_{\varrho_{\mathrm{sc}}} \equiv m_{\mathrm{sc}}$, of ϱ_{sc} satisfies $m_{\mathrm{sc}}(z) = -(m_{\mathrm{sc}}(z) + z)^{-1}$, $z \in \mathbb{C}^+$.

For $N \in \mathbb{N}$ and fixed $t \geq 0$, let $\boldsymbol{\gamma}(t) \equiv (\gamma_k(t))$ denote the set of N-quantiles with respect the density ϱ_t, where $\gamma_k(t)$ is the smallest number satisfying

$$\int_{-\infty}^{\gamma_k(t)} \varrho_t(x)\,\mathrm{d}x = \frac{k}{N}, \quad \varrho_t = \mathcal{F}_t[\varrho], \tag{2.9}$$

for all $t \geq 0$. It is straightforward to check that $\gamma_k(t)$ inside the "bulk," i.e., where ϱ_t is strictly positive, is a continuous function of t. This follows from the (weak) continuity of the flow $t \mapsto \varrho_t$. Moreover, the points $\boldsymbol{\gamma}(t)$ in the bulk approximately satisfy a gradient flow of a classical particle system with a logarithmic two-body interaction potential between the particles (see Lemma 4.3 below). We refer to Appendix A for a more detailed discussion.

2.3. *Dyson Brownian motion*

Fix $N \in \mathbb{N}$ and let $\mathring{F}^{(N)} \subset \mathbb{R}^N$ denote the set

$$\mathring{F}^{(N)} := \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N) \subset \mathbb{R}^N : \lambda_1 < \lambda_2 < \cdots < \lambda_N\}, \tag{2.10}$$

and denote its closure by $F^{(N)}$.

Dyson Brownian motion (DBM) is given by the following stochastic differential equation (SDE)

$$\mathrm{d}\lambda_i(t) = \sqrt{\frac{2}{\beta N}}\,\mathrm{d}B_i(t) - \frac{1}{N}\sum_{j \neq i} \frac{1}{\lambda_j(t) - \lambda_i(t)}\,\mathrm{d}t - \frac{\lambda_i(t)}{2}\,\mathrm{d}t, \quad i \in \mathbb{N}_N, \beta \geq 1, \tag{2.11}$$

with fixed initial condition $\boldsymbol{\lambda}(t = 0) \in F^{(N)}$, where $\beta \geq 1$ is a fixed parameter with the interpretation of inverse temperature, and where $(B_i)_{i=1}^N$ are a collection of independent standard Brownian motions in some probability space $(\Omega, \mathbb{P})$. We denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$.

It is well known, see Section 4.3.1 of [3], that (2.11) with $\beta \geq 1$ has a unique strong solution, $\boldsymbol{\lambda}(t)$, for any initial condition $\boldsymbol{\lambda}(0) \in F^{(N)}$. Further, for any $t > 0$, we have $\boldsymbol{\lambda}(t) \in \mathring{F}^{(N)}$ almost surely.

The equilibrium measure for the DBM is the Gaussian invariant ensemble explicitly given by

$$\mu_{\mathrm{G}}(\boldsymbol{\lambda})\,\mathrm{d}\boldsymbol{\lambda} \equiv \mu_{\beta,\mathrm{G}}^{(N)}(\boldsymbol{\lambda})\,\mathrm{d}\boldsymbol{\lambda} = \frac{1}{Z_{\beta,\mathrm{G}}^{(N)}} \mathrm{e}^{-\beta N \mathcal{H}_{\mathrm{G}}}\,\mathrm{d}\boldsymbol{\lambda}, \qquad \mathcal{H}_{\mathrm{G}} := \sum_{i=1}^N \frac{1}{4}\lambda_i^2 - \frac{1}{N}\sum_{1 \leq i < j \leq N} \log(\lambda_j - \lambda_i), \tag{2.12}$$

where $\mathrm{d}\boldsymbol{\lambda} := \mathbf{1}(\boldsymbol{\lambda} \in F^{(N)})\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2 \cdots \mathrm{d}\lambda_N$ and where $Z_{\beta,\mathrm{G}}^{(N)}$ is a normalization. For fixed β, we denote by $\mathbb{E}^{\mathrm{G}}$ the expectation with respect the measure μ_{G} in (2.12).

Consider next a sequence of vectors $\boldsymbol{\lambda}^{(N)}(0) = (\lambda_1^{(N)}(0), \ldots, \lambda_N^{(N)}(0)) \in F^{(N)}$, $N \in \mathbb{N}$. Let $\boldsymbol{\lambda}^{(N)}(t) = (\lambda_1^{(N)}(t), \ldots, \lambda_N^{(N)}(t)) \in F^{(N)}$ denote the sequence of vectors such that, for each $N \in \mathbb{N}$, $\boldsymbol{\lambda}^{(N)}(t) \in F^{(N)}$ is the solution to (2.11) with initial condition $\boldsymbol{\lambda}^{(N)}(0)$. For simplicity we abbreviate $\boldsymbol{\lambda}(t) \equiv \boldsymbol{\lambda}^{(N)}(t)$, respectively $\lambda_i(t) = \lambda_i^{(N)}(t)$, $i \in \mathbb{N}_N$, in the following.

Assume that there is a probability measure, ϱ_0^∞, on $\mathbb{R}$ such that

$$\varrho_0^{(N)} := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i(0)} \xrightarrow{w} \varrho_0^\infty,$$

as $N \to \infty$, i.e., the empirical distribution of the initial data $\boldsymbol{\lambda}^{(N)}(0)$ converges weakly to ϱ_0^∞. Then, under some mild technical assumptions on $\boldsymbol{\lambda}^{(N)}(0)$, Proposition 4.3.10 of [3] states that

$$\varrho_t^{(N)} := \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(t)} \xrightarrow{w} \varrho_t^\infty, \quad t \geq 0, \tag{2.13}$$

as $N \to \infty$, where $\varrho_t^\infty := \mathcal{F}_t[\varrho_0^\infty]$ denotes the semicircular flow started from ϱ_0^∞, cf., Section 2.2.

2.4. *Main result*

In this subsection, we state our main result. We need one more definition: A labeling ℓ is a random variable $\ell : \mathbb{R} \to \mathbb{Z}$, $x \mapsto \ell(x)$ such that $\ell(x+1) - \ell(x) = 1$ and $\ell(x) = \ell(\lfloor x \rfloor)$.

Theorem 2.1. *Let $\boldsymbol{\lambda}(t)$, $t \geq 0$, be the solution to the DBM in (2.11) with deterministic initial condition $\boldsymbol{\lambda}(0)$. Given any small positive $\epsilon > 0$ and any small $\delta \in [0, 1/20]$, with $\epsilon \geq 2\delta$, consider times $t_1, t_2 \in \mathbb{R}^+$ with $N^{-1+\epsilon} \leq t_2 - t_1 \leq N^{-\epsilon}$. Let ϱ be a probability measure on $\mathbb{R}$. Denote by $\varrho_t \equiv \mathcal{F}_t[\varrho]$ the semicircular flow started from ϱ. Choose $E_* \in \mathbb{R}$ such that $\varrho_{t_1}(E_*) > c$, for some small $c > 0$.*

Assume that $\boldsymbol{\lambda}(t)$ and ϱ are such that the following conditions are satisfied.

(1) *At time t_1, the density $\varrho_{t_1} \equiv \mathcal{F}_{t_1}[\varrho]$ is regular in the following sense. There is a constant $\Sigma > 0$, independent of N, such that the Stieltjes transform $m_{\varrho_{t_1}}$ of ϱ_{t_1}, i.e.,*

$$m_{\varrho_{t_1}}(z) = \int_{\mathbb{R}} \frac{\varrho_{t_1}(y)\,\mathrm{d}y}{y - z}, \quad z \in \mathbb{C}^+, \tag{2.14}$$

extends to a continuous function on $\mathcal{D}_\Sigma := \{z = E + \mathrm{i}\eta \in \mathbb{C} : E \in [E_* - \Sigma, E_* + \Sigma], \eta \geq 0\}$, and satisfies

$$\left|m_{\varrho_{t_1}}(z)\right| \leq C, \qquad \left|\partial_z^n m_{\varrho_{t_1}}(z)\right| \leq C\left(N^\delta\right)^n, \quad n = 1, 2, \tag{2.15}$$

uniformly on $\mathcal{D}_\Sigma$, for some constant C. Moreover, ϱ_{t_1} has finite second moment and satisfies

$$\varrho_{t_1}(E) \geq c, \quad E \in [E_* - \Sigma, E_* + \Sigma], \tag{2.16}$$

for some $c > 0$.

(2) *The process $\boldsymbol{\lambda}(t)$ is* rigid *and is related to ϱ_t in the sense that there is a small $\sigma \equiv \sigma(\Sigma) > 0$, independent of N, such that the following holds.*

  (a) Strong rigidity inside I_σ: *There is a time-independent labeling ℓ such that $\gamma_{\ell(i)}(t_1) \in [E_* - \Sigma/4, E_* + \Sigma/4]$, for all $i \in I_\sigma = [\![L - \sigma N, L + \sigma N]\!]$, where $L \in N_{\mathbb{N}}$ is the largest integer such that $\gamma_{\ell(L)}(t_1) \leq E_*$. Moreover, for any (small) $\xi > 0$ and any (large) $D > 0$ we have*

$$\mathbb{P}\left(\left|\lambda_i(t) - \gamma_{\ell(i)}(t)\right| \leq \frac{N^\xi}{N}, \forall t \in [t_1, t_2]\right) \geq 1 - N^{-D}, \quad \forall i \in I_\sigma, \tag{2.17}$$

  for large enough $N \geq N_0(\xi, D)$, where $(\gamma_i(t))$ are N-quantiles with respect to the measure ϱ_t; see (2.9).

  (b) Weak rigidity outside I_σ: *For any $\xi \in (0, \delta)$ and any (large) $D > 0$, we have*

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{k:|L-k|\geq\sigma N} \frac{1}{\lambda_k(t) - E_*} - \int_{\mathbb{R}\setminus \boldsymbol{I}(t)} \frac{\varrho_t(x)\,\mathrm{d}x}{x - E_*}\right| \leq \frac{N^\xi}{N^\delta}, \forall t \in [t_1, t_2]\right) \geq 1 - N^{-D}, \tag{2.18}$$

  for large enough $N \geq N_0(\xi, D)$, where $\boldsymbol{I}(t) := [\gamma_{\ell(L-\sigma N)}(t), \gamma_{\ell(L+\sigma N)}(t)]$.

(3) Level repulsion inside I_σ: *For any $i \in I_\sigma$ and $t \in [t_1, t_2]$,*

$$\mathbb{P}\left(\left|\lambda_i(t) - \lambda_{i\pm1}(t)\right| \leq \frac{u}{N}, \left|\lambda_i(t) - \gamma_{\ell(i)}(t)\right| \leq \frac{N^\xi}{N}\right) \leq N^\delta u^{\beta+1}, \quad u > 0, \tag{2.19}$$

for large enough N.

(4) Hölder continuity of DBM: *For any* (*small*) $\xi > 0$ *and any* (*large*) $D > 0$, *we have*

$$\mathbb{P}\big(\big|\lambda_k(t) - \lambda_k(s)\big| \le N^{\xi}\sqrt{t-s}, \forall t, s \in [t_1, t_2], t \ge s\big) \ge 1 - N^{-D}, \quad \forall k \in \mathbb{N}_N \setminus I_\sigma, \tag{2.20}$$

for large enough $N \ge N_0(\xi, D)$.

Then, there are small constants $\mathfrak{f}, \chi, \alpha > 0$, *such that the following holds. Fix* $n \in \mathbb{N}$ *and let* $\mathcal{O} : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be smooth and compactly supported. Fix any* $T \in [t_1 + N^{2\delta}/N, t_2]$. *Set* $\varrho_* := \varrho_T(\gamma_L(T))$. *Then*

$$\mathbb{E}\big[\mathcal{O}\big(\big((N\varrho_*)\big(\lambda_{i_0}(T) - \lambda_{i_0+j}(T)\big)\big)_{j=1}^n\big)\big] = \mathbb{E}^{\mathrm{G}}\big[\mathcal{O}\big(\big((N\varrho_\#)(\lambda_{i_0'} - \lambda_{i_0'+j})\big)_{j=1}^n\big)\big] + O\big(N^{-\mathfrak{f}}\big), \tag{2.21}$$

for N sufficiently large, for any $i_0, i_0' \in \mathbb{N}_N$ *satisfying* $|i_0 - L| \le N^{\chi}$, $|i_0' - L'| \le N^{\chi}$ *with any* $L' \in [\alpha N, (1-\alpha)N]$, *and where* $\varrho_\# := \varrho_{\mathrm{sc}}(\gamma_{L',\mathrm{sc}})$ *denotes the density of the semicircle law* ϱ_{sc} *at the location of the* L'*th* N*-quantile of* ϱ_{sc}.

Remark 2.2. The formula (2.21) expresses the *single gap universality*, i.e., that the joint distribution of n consecutive gaps coincides with that of a Gaussian invariant ensemble for any fixed n. Single gap universality clearly implies the weaker *averaged gap universality*, where (2.21) is averaged over N^b consecutive i_0's, for some $0 < b < 1$. It is well known that averaged gap universality implies the *averaged energy universality*, i.e., the universality of the local correlation functions around an energy E, averaged over E near the reference energy E_*; see e.g., Section 7 of [26].

Remark 2.3. The measure ϱ in Theorem 2.1 is assumed to be deterministic, but it may depend on N in contrast to the measure $\varrho_0^{(\infty)}$ of (2.13) which is indeed the limiting object as $N \to \infty$. Consequently, the semicircular flow $\varrho_t = \mathcal{F}_t[\varrho]$ will also be N-dependent. Typically one expects that ϱ_t converges weakly to ϱ_t^{∞}, yet the speed of convergence may be very slow and hence not be compatible with Assumption (2) of Theorem 2.1. In Section A.2, we discuss Assumption (1) in more detail.

Notice that the initial condition $\boldsymbol{\lambda}(0)$ of the DBM and the initial data ϱ of the semicircular flow do not have to be related; this will allow us for an additional freedom in the applications. We only require that $\boldsymbol{\lambda}(t)$ is close to the quantiles of ϱ_t in a short time interval $t \in [t_1, t_2]$ and only locally near the reference energy E_*. We also allow for a possible relabeling ℓ that can be used to accommodate outliers in applications. At first reading the reader may ignore ℓ and consider $\ell(i) = i$ for simplicity.

2.5. *Random matrix flow and universality*

In this subsection, we briefly explain how Theorem 2.1 can be used to prove bulk universality for many random matrix ensembles H. We will follow the three-step strategy initiated in a series of works [24,26,29]; see [27] for a concise summary.

Step 1 is to prove a local law, which includes rigidity for the eigenvalues and bounds on the resolvent matrix elements $G(z) = (H-z)^{-1}$ down almost to the scale of the eigenvalue spacing, i.e., for $\eta = \operatorname{Im} z \gg N^{-1}$. This step is typically model dependent, mainly because the limiting density of the eigenvalues varies from model to model. The key tool is the self-consistent equation for the Stieltjes transform of the density (and its vector version for the individual matrix elements G_{ii}); its solvability and stability properties need to be investigated for each model.

Step 2 is to prove universality for matrices with a small Gaussian component that can conveniently be generated by running a matrix valued Ornstein–Uhlenbeck process. Theorem 2.1 is used in this step and it replaces the previous argument that relied on a global equilibrium measure and its version with relaxation. As advertised in the introduction, Theorem 2.1 requires rigidity information only locally, in particular it also applies to models where the limiting density is supported on several intervals. Step 2 is model independent once the input conditions of Theorem 2.1 are verified.

Finally, *Step 3* is a perturbation argument which is also very general. Using the Green function comparison strategy [29] and the moment matching (introduced first in [52] in the context of random matrices), one can remove the tiny Gaussian component. The main input here is the a priori bound on the resolvent matrix elements obtained in Step 1.

More concretely, consider a random $N \times N$ hermitian or symmetric matrix $H_t = H_t^*$ with matrix elements (h_{ij}). Suppose the matrix elements are time-dependent and they satisfy the Ornstein–Uhlenbeck (OU) process

$$\mathrm{d}h_{ij} = \frac{\mathrm{d}B_{ij}}{\sqrt{N}} - \frac{1}{2}h_{ij}\,\mathrm{d}t, \quad i, j \in \mathbb{N}_N, i \le j, \tag{2.22}$$

where $(B_{ij} : i < j)$ are independent complex Brownian motions with variance t and (B_{ii}) are independent real Brownian with variance t for $\beta = 2$; while for $\beta = 1$, $(B_{ij} : i < j)$ are independent real Brownian motions with variance t and B_{ii} are real Brownian motion with variance $2t$. It is easy to check that the solution to (2.22), H_t, with initial condition H_0, satisfies the distributional equality

$$H_t \sim \mathrm{e}^{-t/2} H_0 + \big(1 - \mathrm{e}^{-t}\big)^{1/2} U, \tag{2.23}$$

where U is Gaussian, i.e., belongs to the GUE ($\beta = 2$), respectively to the GOE ($\beta = 1$), and U is independent of H_0.

The eigenvalues of H_t, here denoted by $\boldsymbol{\lambda}(t)$, satisfy [19] the SDE (2.11), with $\beta = 1$ or $\beta = 2$, where the initial condition $\boldsymbol{\lambda}(0)$ is given by the eigenvalues of the initial matrix H_0. We will run the OU process until time $t_2 = o(1)$. Let ϱ denote the limiting density of H_0. We fix an energy E_* in the bulk spectrum of H_0, i.e., $\varrho(E_*) \geq c > 0$; it is easy to see that E_* stays in the bulk of H_t as well for any $t \leq t_2$. The assumptions of Theorem 2.1 can then, via the identification (2.23), be checked from the matrix flow H_t in the time slice $t \in [t_1, t_2]$. The typical choice is $t_2 = N^{-\epsilon}$ and $t_1 = t_2 - N^{-1+2\delta}$, with some small positive exponents $\epsilon \ll \delta$.

Assumption (2) can be checked from a local law for the random matrix H_t. We need such information not only for the original matrix H_0, but along the whole OU flow. Typically, however, when the local law is proven for some matrix H_0, it also holds for H_t, i.e., for H_0 with a Gaussian convolution. Notice that the strong form of rigidity, an almost optimal bound on $\lambda_i(t) - \gamma_i(t)$ expressed in (2.17), is needed only for eigenvalues near E_* in the bulk. Much weaker information is needed for far away eigenvalues; the condition (2.18) involves controlling the density only on the macroscopic scale. In terms of the Stieltjes transform, $m_t^{(N)}(z) := \frac{1}{N}\operatorname{Tr}(H_t - z)^{-1}$, $z \in \mathbb{C}^+$, of the empirical density, Assumption (2) follows if the bounds

$$\big|m_t^{(N)}(z) - m_t(z)\big| \leq \frac{N^\xi}{N\eta}, \quad \text{for } z = E + i\eta, \eta \in \big[N^{-1+\xi}, c\Sigma\big], |E - E_*| \leq \Sigma, \tag{2.24}$$

$$\big|m_t^{(N)}(E_* + \mathrm{i}\eta) - m_t(E_* + \mathrm{i}\eta)\big| \leq \frac{N^\xi}{N^\delta}, \quad \text{for } \eta \in [c\Sigma, 1], \tag{2.25}$$

hold with high probability, for any $t \in [t_1, t_2]$. Indeed, (2.25) directly implies (2.18). By a simple application of the Helffer–Sjöstrand formula (e.g., following the proof of Lemma 8.1 in [21]), we see from (2.24) that

$$\big|\#\big\{\lambda_j(t) \in [E_1, E_2]\big\} - \#\big\{\gamma_j(t) \in [E_1, E_2]\big\}\big| \leq CN^\xi, \quad \forall E_1, E_2 \in \left[E - \frac{1}{2}\Sigma, E + \frac{1}{2}\Sigma\right]. \tag{2.26}$$

In particular, rigidity between the $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\gamma}(t)$ sequences holds on scale $N^{-1+\xi}$ within $[E - \frac{1}{2}\Sigma, E + \frac{1}{2}\Sigma]$. This implies $|\lambda_i(t) - \gamma_{\ell(i)}(t)| \leq N^{-1+\xi}$, for any $i \in I_\sigma$, up to an overall shift in the labeling that is encoded in the labeling function $\ell(i)$. We only need to show that the labeling $\ell(i)$ is time-independent, i.e., that along the whole time interval $t \in [t_1, t_2]$ it is *the same* element of the $\boldsymbol{\gamma}(t)$ sequence that stays close to a given element of $\boldsymbol{\lambda}(t)$ within the rigidity precision $N^{-1+\xi}$. We call this property the *persistent trailing* of DBM by the flow of the quantiles. Given (2.26), it is sufficient to check this for one element of the sequence; e.g., that if $|\lambda_L(t_1) - \gamma_{\ell(L)}(t_1)| \leq N^{-1+\xi}$ with some shifted index $\ell(L)$, then $|\lambda_L(t) - \gamma_{\ell(L)}(t)| \leq N^{-1+\xi}$, for all $t \in [t_1, t_2]$. Notice that persistent trailing is a nontrivial feature of the DBM since the length of the time interval $t_2 - t_1 = N^{-1+2\delta}$ is much bigger than the rigidity scale $N^{-1+\xi}$. Nevertheless, in Proposition B.1 in Appendix B we show that there is an event Ξ_0 in the probability space of the Brownian motions with $\mathbb{P}(\Xi_0) \geq 1 - N^{-\xi/2}$ such that $\gamma_{\ell(L)}(t)$ persistently trails $\lambda_L(t)$. It is easy to see that the universality in Theorem 2.1 also holds if Assumptions (2)–(3) are valid only on the event Ξ_0.

Level repulsion estimates of the form of Assumption (3) for random matrix ensembles can be obtained using the method of [24]. This approach requires two inputs: strong local rigidity as in (2.17) and smoothness of the distribution of the matrix elements of H. The former is already verified by Assumption (2), the latter needs a slight extension of [24] to "almost-smooth" distributions, where smoothing may be provided by the OU process. Indeed, in Appendix B of [14] it was shown that H_t satisfies level repulsion in the form (2.19), if $t = N^{-c\delta}$ with some small constant $c > 0$ (another merit of the proof in [14] is that it also presents the necessary modifications to cover symmetric matrices as well, while [24] was written for hermitian matrices only). So we will choose $\epsilon = \frac{c}{2}\delta$ in the definition $t_2 = N^{-\epsilon}$ to guarantee that (2.19) holds for any $t \in [t_1, t_2]$. Notice that the only reason to run the DBM up to a relatively large time

$t_2 = N^{-\epsilon}$ is to guarantee that the smoothing effect is substantial to yield level repulsion. If the distribution of H_0 were smooth initially, so level repulsion in its original form [24] applied, we could have chosen $t_1 = 0$, $t_2 = N^{-1+2\delta}$ with some small $\delta > 0$.

Finally, Assumption (4) can easily checked as follows. For any two $N \times N$ matrices $A = A^*$, $B = B^*$, we have $\mathrm{dist}\{\mathrm{Sp}(A), \mathrm{Sp}(B)\} \le \|A - B\|_\infty$, where $\mathrm{Sp}(A)$, $\mathrm{Sp}(B)$ denote the spectra of A, B and where $\|\cdot\|_\infty$ denotes the operator norm. Also recall that the operator norm of U is bounded by a constant with overwhelming probability; see, e.g., Exercise 2.1.30 of [3]. Thus, choosing $A = H_t$, $B = H_0$, we see that Assumption (4) is satisfied provided that $\|H_0\|_\infty \le N^{\xi/2}$ with overwhelming probability. This bound can be easily proven for all matrix models we have in mind.

Having checked the assumptions, the conclusion of Theorem 2.1 is that gap universality holds for any matrix with a substantial Gaussian component of size $t_2 \sim N^{-\epsilon}$. The rest is a standard moment matching and Green function comparison argument that we sketch for completeness.

Given an initial Wigner-like matrix $\widehat{H}$ for which we eventually wish to prove universality, we choose $t_2 = N^{-\epsilon}$ with a sufficiently small $\epsilon > 0$. By moment matching (see, e.g., Lemma 6.5 of [29]), we construct another matrix H_0 such that the solution H_{t_2} at time t_2 of the matrix Ornstein–Uhlenbeck process (2.22) with initial condition H_0 is close to $\widehat{H}$ in the four moment sense. Choosing $T = t_2$ in Theorem 2.1, we obtain gap universality for H_T which also implies universality of local correlation functions at E with a small averaging in the energy parameter E around E_*. The local eigenvalue statistics of $\widehat{H}$ and H_T coincide by the Green function comparison theorem introduced in [29]. More precisely, the method of [29] gives coincidence in the sense of correlation functions while Theorem 1.10 of [34] extends the Green function comparison method to individual eigenvalues, hence to gaps as well. This completes our sketch on how to apply Theorem 2.1 for random matrix models.

2.6. *Strategy of the proof of Theorem* 2.1

Our proof of Theorem 2.1 is divided into five parts.

Part (*i*), carried out in Section 4.1, is to localize the problem: we choose an integer $K \gg 1$ such that

$$N^{\delta} K^{10} \le N, \qquad K \le N^{\delta}. \tag{2.27}$$

We consider the conditional measure on $\mathcal{K} = 2K + 1$ consecutive *internal points* $\mathbf{x} = (\lambda_{L-K}, \dots, \lambda_{L+K})$, labeled by $I := [\![L - K, L + K]\!]$, conditioned on the remaining $N - \mathcal{K}$ *external points* $\mathbf{y} = (\lambda_i : |i - L| > K)$. The index L is chosen so that $\gamma_{\ell(L)}$ is close to E_*, where γ_k is the kth N-quantile of the density at t_1. In the equilibrium setup this corresponds to the local Gibbs measure with boundary conditions given by $\mathbf{y}$ (this idea was first introduced in [13] in the β-ensemble context). In our non-equilibrium setup, we work in the path space and condition on the whole trajectory $\mathbf{Y} = \{\mathbf{y}(t) : t \in [T_1, t_2]\}$, starting at some time $T_1 \ge t_1$ chosen later. The configuration interval $\boldsymbol{J}(t) = [y_{L-K-1}(t), y_{L+K+1}(t)]$ for the conditional measure is time-dependent, but by rigidity it is quite close to the corresponding interval $[\gamma_{\ell(L-K-1)}(t), \gamma_{\ell(L+K+1)}(t)]$ given by the quantiles that remains practically constant owing to the removal of the mean velocity. Still, $\boldsymbol{J}(t)$ may wiggle on the rigidity scale N^{ξ}/N which is much bigger than our target scale, $1/N$, the size of the gap, so that we cannot tolerate this imprecision. Furthermore, similarly to the basic idea of the local relaxation flow [24,26] we want to achieve universality by showing that the measure converges to a (local) reference equilibrium measure. The local Gibbs measures with boundary condition $\mathbf{y}(t)$ change too quickly to serve as useful reference measures.

Part (*ii*) of the proof is to understand the dynamics on a macroscopic scale, i.e., to control the semicircular flow and the induced dynamics on the time-dependent quantiles $\gamma_i(t)$. This analysis is of interest itself and it is deferred to the Appendix A since it requires quite different tools than the main part of the proof. The key information (collected in Section 4.2) is that the quantiles in the bulk move coherently with a local mean velocity that varies in time on the macroscopic scale. Since we concentrate on the vicinity of a fixed energy E_* and on a small time window, by a simple linear shift we can achieve that the mean velocity is negligible near E_*.

Therefore, in *Part* (*iii*), carried out in Section 4.3, we define a *time-independent* local measure, ω_{T_1} with exterior points $\widetilde{\gamma}_k$, $k \in I^c$. These exterior points coincide with $y_k(T_1)$ for k far away from L while they are given by a typical configuration $\mathbf{z}$ of an auxiliary quadratic β-ensemble for k near the boundary of I (with a smooth interpolation in between). The auxiliary ensemble is chosen in such a way that the local density around E_* matches. Using the rigidity bounds for both $\mathbf{y}$ and $\mathbf{z}$, we establish that ω_{T_1} satisfies the logarithmic Sobolev inequality (LSI) and the corresponding dynamics approaches to equilibrium on a time scale of order K/N. Furthermore, we show that the measure ω_{T_1} is rigid by using a general criterion for rigidity of local measures given in Theorem 4.2 of [28] together with the careful choice of the auxiliary ensemble. Moreover, we notice that ω_{T_1} satisfies a level repulsion bound due to Theorem 4.3 of [28]. Finally, Theorem 4.1 of [28] implies that the gap statistics of ω_{T_1} are universal.

Part (*iv*), carried out in Section 4.4, is to consider $\widetilde{x}_i(t)$, $i \in I$, $t \geq T_1$, the solution of the local DBM with exterior points $\widetilde{\gamma}_k$, $k \in I^c$, and with initial condition $\widetilde{x}_i(T_1) = x_i(T_1)$. Writing the distribution of $\widetilde{\mathbf{x}}(t)$ as $g_t\omega_{T_1}$, we derive fast convergence to equilibrium, i.e., for times $t \geq T_1' := T_1 + K(K/N)$ the measure $g_t\omega_{T_1}$ is exponentially close to equilibrium in the relative entropy sense. This information can be used to transfer rigidity and level repulsion from ω_{T_1} to $g_t\omega_{T_1}$, furthermore it shows that the gap statistics of $\widetilde{\mathbf{x}}(t)$ are the same as those of ω_{T_1}, hence are universal.

The next idea, *Part* (*v*), is to *couple* the evolution of $\widetilde{\mathbf{x}}$ to $\mathbf{x}$ by using the same Brownian motions in the DBMs. This basic coupling idea first appeared in [14] in this context; its main advantage is that taking the difference of the original DBM and the DBM for $\widetilde{\mathbf{x}}$, we see that the difference vector $\mathbf{v} := \mathbf{x} - \widetilde{\mathbf{x}}$ satisfies a system of ordinary differential equations (ODEs); the stochastic differentials drop out. Roughly speaking, these ODEs have the form (see (5.7))

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -(\mathcal{B}\mathbf{v})_i + F_i, \qquad (\mathcal{B}\mathbf{v})_i := \sum_{j \in I} B_{ij}(v_i - v_j) + W_i v_i, \tag{2.28}$$

with time-dependent coefficients B_{ij}, W_i and a "forcing term" F_i that all depend on the paths $\mathbf{x}(t), \widetilde{\mathbf{x}}(t)$. These coefficients are crudely given by

$$B_{ij} = \frac{1}{N(x_i - x_j)(\widetilde{x}_i - \widetilde{x}_j)}, \qquad W_i = \frac{1}{N} \sum_{k \notin I} \frac{1}{(x_i - y_k)(\widetilde{x}_i - \widetilde{\gamma}_k)}, \qquad F_i = \frac{1}{N} \sum_{k \notin I} \frac{y_k - \widetilde{\gamma}_k}{(x_i - y_k)(\widetilde{x}_i - \widetilde{\gamma}_k)}.$$

The equation (2.28) is very similar to the basic equation studied in [28] but the forcing term is new. The key result of [28] is a Hölder regularity theory for (2.28) without forcing, under suitable conditions on the coefficients. We extend this statement to include the forcing term; here we rely on the finite speed of propagation, proved also in [28]. Hölder regularity in this context yields that, after some time of order K^c/N, $c \sim 1/100$, the discrete derivative $v_{i+1} - v_i$ is much smaller than its naive size $1/N$. Since $v_{i+1} - v_i = x_{i+1} - x_i - (\widetilde{x}_{i+1} - \widetilde{x}_i)$, we see that the gaps of $\mathbf{x}$ and $\widetilde{\mathbf{x}}$ coincide to leading order. Since the gaps of $\widetilde{\mathbf{x}}$ were shown to be universal in the previous step, we obtain that the gaps of $\mathbf{x}(T)$, $T := T_1' + K^c N^{-1}$, are universal.

There are several technical complications behind this scheme, most importantly we need to regularize the local singularity in the kernel B_{ij} when $x_i \approx x_{i\pm 1}$. In fact, two different regularizations are used; the regularization of the dynamics in Section 5.1 is borrowed from Section 3.1 of [14], while the regularization of the equilibrium measure ω_{T_1} explained at the end of Section 4.5 is similar to the one in Section 9.3 of [28] but with a different choice of regularization scale.

3. Concepts

In this section we recall essential concepts that will be used in the proof of Theorem 2.1.

3.1. *Definition of general β-ensembles*

We first recall the notion of β-ensembles or log-gases. Let $N \in \mathbb{N}$ and recall the definition of the set $\digamma^{(N)} \subset \mathbb{R}^N$ in (2.10). Consider the probability distribution on $\digamma^{(N)}$ given by

$$\mu_{\beta,V}^{(N)}(\mathrm{d}\boldsymbol{\lambda}) \equiv \mu_V(\mathrm{d}\boldsymbol{\lambda}) := \frac{1}{Z_V} \mathrm{e}^{-\beta N \mathcal{H}} \mathrm{d}\boldsymbol{\lambda}, \qquad \mathcal{H} \equiv \mathcal{H}(\boldsymbol{\lambda}) := \sum_{i=1}^N \frac{1}{2} V(\lambda_i) - \frac{1}{N} \sum_{1 \leq i < j \leq N} \log(\lambda_j - \lambda_i), \tag{3.1}$$

where $\beta > 0$, $\mathrm{d}\boldsymbol{\lambda} := \mathbf{1}(\boldsymbol{\lambda} \in \digamma^{(N)}) \, \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \cdots \mathrm{d}\lambda_N$, and $Z_V \equiv Z_{\beta,V}^{(N)}$ is a normalization. Here V is a N-independent potential, i.e., a real-valued, sufficiently regular function on $\mathbb{R}$ to be specified in each case. In the following, we often omit the parameters N and β from the notation. We use $\mathbb{P}^{\mu_V}$ and $\mathbb{E}^{\mu_V}$ to denote the probability and the expectation with respect to μ_V. We view μ_V as a Gibbs measure of N particles on $\mathbb{R}$ with a logarithmic interaction, where the parameter $\beta > 0$ may be interpreted as the inverse temperature. We refer to the variables (λ_i) as particles or points and we call the system a β-log-gas or a β-ensemble. We assume that the potential V is a C^4 function on $\mathbb{R}$ such that its second derivative is bounded below, i.e., we have

$$\inf_{x\in\mathbb{R}} V''(x) \geq -2C_V, \tag{3.2}$$

for some constant $C_V \geq 0$, and we further assume that

$$V(x) > (2+c)\log\big(1+|x|\big), \quad x \in \mathbb{R}, \tag{3.3}$$

for some $c > 0$, for large enough $|x|$. It is also well known, see, e.g., [15], that under these conditions the measure is normalizable, $Z_V < \infty$. Further, the averaged density of the empirical spectral measure, $\varrho_V^{(N)}$, defined as

$$\varrho_V^{(N)} := \mathbb{E}^{\mu_V} \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}, \tag{3.4}$$

converges weakly in the limit $N \to \infty$ to a continuous function, ϱ_V, the equilibrium density, of compact support. It is well known that ϱ_V satisfies

$$V'(x) = -2\int_{\mathbb{R}} \frac{\varrho_V(y)\,\mathrm{d}y}{y-x}, \quad x \in \operatorname{supp}\varrho_V. \tag{3.5}$$

In fact, equality in (3.5) holds if and only if $x \in \operatorname{supp}\varrho_V$.

Viewing the points $\boldsymbol{\lambda} = (\lambda_i)$ as points or particles on $\mathbb{R}$, we define the quantile of the kth particle, γ_k, under the β-ensemble μ_V by

$$\int_{-\infty}^{\gamma_k} \varrho_V(x)\,\mathrm{d}x = \frac{k}{N}. \tag{3.6}$$

For a detailed discussion of general β-ensemble and the proof of the properties mentioned above we refer, e.g., to [3,13].

Assume for the moment that the minimizer ϱ_V is supported on a single interval $[a,b]$, and that V is "regular" in the sense of [37], i.e., the equilibrium density of V is positive on (a,b) and vanishes like a square root at each of the endpoints of $[a,b]$. From [12,13] we then have the following rigidity result.

Proposition 3.1. *Let $V \in \mathbb{C}^4(\mathbb{R})$ be a "regular" potential and assume that ϱ_V is supported on a single interval. Then, for any $\xi > 0$ there are constants $c_0, c_1 > 0$, such that*

$$\mathbb{P}\big(|\lambda_k - \gamma_k| \geq N^{\xi} N^{-\frac{2}{3}} \check{k}^{-\frac{1}{3}}\big) \leq \mathrm{e}^{-c_0 N^{c_1}}, \quad 1 \leq k \leq N, \tag{3.7}$$

where $\check{k} := \min\{k, N-k+1\}$, for N sufficiently large.

Proposition 3.1 will only be used as an auxiliary result (see Section 4.3.2 below), since, for most potentials of interests in the present paper, the equilibrium density ϱ_V is not supported on a single interval. The extension of Proposition 3.1 to that settings has not been established.

Finally, for the Gaussian case, $V(x) = x^2/2$, we write μ_G instead of μ_V, since μ_G is the equilibrium measure for the DBM. More precisely, the Gaussian distribution on $\mathcal{E}^{(N)}$ is given by

$$\mu_G(\mathrm{d}\boldsymbol{\lambda}) = \frac{1}{Z_G}\mathrm{e}^{-\beta N \mathcal{H}_G}\,\mathrm{d}\boldsymbol{\lambda}, \qquad \mathcal{H}_G(\boldsymbol{\lambda}) := \sum_{i=1}^N \frac{1}{4}\lambda_i^2 - \frac{1}{N}\sum_{1\leq i<j\leq N} \log(\lambda_j - \lambda_i), \tag{3.8}$$

where $Z_G \equiv Z_{\beta,G}^{(N)}$ is the normalization.

3.2. *Dyson Brownian motion*

Consider the DBM, $\boldsymbol{\lambda}(t)$, $t \geq 0$, on $\digamma^{(N)}$ of (2.11) with initial condition $\boldsymbol{\lambda}(0) \in \digamma^{(N)}$. Denote by $f_0 \mu_{\mathrm{G}}$, the distribution of $\boldsymbol{\lambda}(0)$[3] and let $f_t \mu_{\mathrm{G}}$ denote the distribution of $\boldsymbol{\lambda}(t)$. Then $f_t \equiv f_{t,N}$ satisfies the forward equation

$$\partial_t f_t = \mathscr{L} f_t, \tag{3.9}$$

where

$$\mathscr{L} = \mathscr{L}_N := \sum_{i=1}^N \frac{1}{\beta N} \partial_i^2 + \sum_{i=1}^N \left(-\frac{1}{2}\lambda_i - \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i} \right) \partial_i, \quad \partial_i = \frac{\partial}{\partial \lambda_i}, \tag{3.10}$$

or in short $\mathscr{L} = \frac{1}{\beta N}\Delta - (\nabla \mathcal{H}_{\mathrm{G}}) \cdot \nabla$, with $\mathcal{H}_{\mathrm{G}}$ as in (3.8).

3.3. *Relative entropy, Bakry–Émery criterion and the logarithmic Sobolev inequality*

A cornerstone in our proof is the analysis of the relaxation of the dynamics (3.9). Such an approach was first introduced in Section 5.1 of [25]. The presentation here follows [26].

Let μ be a probability measure on $\digamma^{(N)}$ be given by a general Hamiltonian $\mathcal{H}$:

$$\mathrm{d}\mu(\mathbf{x}) = \frac{1}{Z} \mathrm{e}^{-\beta N \mathcal{H}(\mathbf{x})}\, \mathrm{d}\mathbf{x}, \tag{3.11}$$

and let L be the generator, symmetric with respect to the measure $\mathrm{d}\mu$, defined by the Dirichlet form

$$D(f) = D_\mu(f) = -\int f L f \,\mathrm{d}\mu := \frac{1}{\beta N} \sum_j \int (\partial_j f)^2 \,\mathrm{d}\mu, \quad \partial_j = \partial_{x_j}. \tag{3.12}$$

The relative entropy of two absolutely continuous probability measures on $\mathbb{R}^N$ is given by

$$S(\widetilde{\nu}|\nu) := \int \frac{\mathrm{d}\widetilde{\nu}}{\mathrm{d}\nu} \log\left(\frac{\mathrm{d}\widetilde{\nu}}{\mathrm{d}\nu}\right) \mathrm{d}\nu.$$

If $\mathrm{d}\widetilde{\nu} = f\,\mathrm{d}\nu$, then we use the notation $S_\nu(f) := S(f\nu|\nu)$. The entropy can be used to control the total variation norm via the well-known inequalities

$$\int |f-1|\,\mathrm{d}\nu \leq \sqrt{2S_\nu(f)}, \qquad \mathbb{P}^{f\nu}(A) \leq \mathbb{P}^{\nu}(A) + \sqrt{2S_\nu(f)}, \tag{3.13}$$

for any ν-measurable event A.

Let f_t be the solution to the evolution equation $\partial_t f_t = L f_t$, $t > 0$, with a given initial condition f_0. Assuming that the Hamiltonian $\mathcal{H}$ satisfies

$$\nabla^2 \mathcal{H}(\boldsymbol{\lambda}) = \operatorname{Hess} \mathcal{H}(\boldsymbol{\lambda}) \geq \vartheta, \quad \boldsymbol{\lambda} \in \digamma^{(N)}, \tag{3.14}$$

the Bakry–Émery criterion [5] yields the logarithmic Sobolev inequality (LSI)

$$S_\mu(f) \leq \frac{2}{\vartheta} D_\mu(\sqrt{f}), \quad f = f_0 \in \mathrm{L}^\infty\big(\mathbb{R}^N, \mathrm{d}\boldsymbol{\lambda}\big), \tag{3.15}$$

and the exponential relaxation of the entropy and Dirichlet form

$$S_\mu(f_t) \leq \mathrm{e}^{-2\vartheta t} S_\mu(f_0), \qquad D_\mu(\sqrt{f_t}) \leq \mathrm{e}^{-2\vartheta t} D_\mu(\sqrt{f_0}), \quad t > 0.$$

[3]Strictly speaking, the distribution of $\boldsymbol{\lambda}(0)$ may not allow a density f_0 with respect to μ_{G}, but for $t > 0$, $\boldsymbol{\lambda}(t)$ admits such a density. Our proofs are not affected by this technicality.

We assume from now on that $\mathcal{H}$ is given by $\mathcal{H}_{\mathrm{G}}$ (see (3.8)), L is given by $\mathscr{L}$ (see (3.10)) and that the equilibrium measure is the Gaussian one, $\mu = \mu_{\mathrm{G}}$. We then have the convexity inequality

$$\langle \mathbf{v}, \nabla^2 \mathcal{H}_{\mathrm{G}}(\mathbf{x})\mathbf{v}\rangle \geq \frac{1}{2}\|\mathbf{v}\|^2 + \frac{1}{N}\sum_{i<j}\frac{(v_i - v_j)^2}{(x_i - x_j)^2} \geq \frac{1}{2}\|\mathbf{v}\|^2, \quad \mathbf{v}\in\mathbb{R}^N. \tag{3.16}$$

This guarantees that μ_{G} satisfies the LSI with $\vartheta = 1/2$ and the relaxation time is of order one.

3.4. *Localized measures*

Following [28], we choose $K \in [\![N^{\varpi}, N^{1/10}]\!]$, for some small $0 < \varpi < 1/10$ and pick $L \in [\![K, N-K]\!]$. We denote by $I = [\![L-K, L+K]\!]$ a set of $\mathcal{K} := 2K+1$ consecutive indices around L. Recall the definition of the set $\digamma^{(N)} \subset \mathbb{R}^N$ in (2.10). For $\boldsymbol{\lambda} \in \digamma^{(N)}$, we rename the points as

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N) = (y_1, \ldots, y_{L-K-1}, x_{L-K}, \ldots, x_{L+K}, y_{L+K+1}, \ldots, y_N), \tag{3.17}$$

and we call $\boldsymbol{\lambda}$ a configuration (of N particles or points on the real line). Note that on the right side of (3.17) the points retain their original indices and are in increasing order,

$$\mathbf{x} = (x_{L-K}, \ldots, x_{K+L}) \in \digamma^{(\mathcal{K})}, \qquad \mathbf{y} = (y_1, \ldots, y_{L-K-1}, y_{L+K+1}, \ldots, y_N) \in \digamma^{(N-\mathcal{K})}. \tag{3.18}$$

We refer to $\mathbf{x}$ as the *internal points or particles* and to $\mathbf{y}$ as the *external points or particles*. In the following, we often fix the external points and consider the conditional measures on the internal points: Let ν be a measure with density on $\digamma^{(N)}$. Then we denote by $\nu^{\mathbf{y}}$ the measure obtained by conditioning on $\mathbf{y}$, i.e., for $\boldsymbol{\lambda}$ of the form (3.17),

$$\nu^{\mathbf{y}}(\mathrm{d}\mathbf{x}) \equiv \nu^{\mathbf{y}}(\mathbf{x})\,\mathrm{d}\mathbf{x} := \frac{\nu(\boldsymbol{\lambda})\,\mathrm{d}\mathbf{x}}{\int \nu(\boldsymbol{\lambda})\,\mathrm{d}\mathbf{x}} = \frac{\nu(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}}{\int \nu(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}},$$

where, with slight abuse of notation, $\nu(\mathbf{x},\mathbf{y})$ stands for $\nu(\boldsymbol{\lambda})$. We refer to the fixed external points $\mathbf{y}$ as boundary conditions of the measure $\nu^{\mathbf{y}}$. For fixed $\mathbf{y} \in \digamma^{(N-\mathcal{K})}$, all (x_i) lie in the configuration interval

$$\boldsymbol{J}_{\mathbf{y}} := [y_{L-K-1}, y_{L+K+1}]. \tag{3.19}$$

Thus $\nu^{\mathbf{y}}$ is supported on $(\boldsymbol{J}_{\mathbf{y}})^{\mathcal{K}} \cap \digamma^{(\mathcal{K})}$, but with a slight abuse of terminology we often say that $\nu^{\mathbf{y}}$ is supported on $\boldsymbol{J}_{\mathbf{y}}$. In case $\nu = f\mu$, we define the conditioned density by $f^{\mathbf{y}}\mu^{\mathbf{y}} = (f\mu)^{\mathbf{y}}$.

For a potential V, we consider the β-ensemble μ_V of (3.1). For K, L and $\mathbf{y}$ fixed, we can write $\mu_V^{\mathbf{y}}$ as

$$\mu_V^{\mathbf{y}}(\mathrm{d}\mathbf{x}) = \frac{1}{Z_V^{\mathbf{y}}}\mathrm{e}^{-\beta N \mathcal{H}^{\mathbf{y}}(\mathbf{x})}\,\mathrm{d}\mathbf{x}, \qquad \mathcal{H}^{\mathbf{y}}(\mathbf{x}) = \sum_{i\in I}\frac{1}{2}V^{\mathbf{y}}(x_i) - \frac{1}{N}\sum_{\substack{i,j\in I\\ i<j}}\log|x_j - x_i|, \tag{3.20}$$

$x_i \in \boldsymbol{J}_{\mathbf{y}}$, with $Z_V^{\mathbf{y}} \equiv Z_{\beta,V}^{\mathbf{y}}$ a normalization and with the *external potential*

$$V^{\mathbf{y}}(x) = V(x) - \frac{2}{N}\sum_{i\notin I}\log|x - y_i|. \tag{3.21}$$

4. Localizing the measures

Having introduced the basic concepts in Section 3, we now start with the actual proof of Theorem 2.1. In this section, we establish *Parts* (*i*)–(*iv*) outlined in Section 2.6. In Section 4.1, we prepare the definition of the local equilibrium measure by introducing the precise definition of internal and external points. We also show that for most external trajectories we have rigidity for the internal points; see Lemma 4.1. In Section 4.2, we discuss regularity properties of the semicircular flow and its quantiles. We further convert the problem into a stationary frame by removing the mean velocity. The regularity properties of the semicircular flow are used in Section 4.3 to define the reference points $\widetilde{\boldsymbol{\gamma}}$ that smoothly interpolate between the quantiles of an appropriately chosen β-ensemble and the external points at time T_1. Using $\widetilde{\boldsymbol{\gamma}}$ as new external points we introduce a reference measure ω_{T_1} on the internal points. This measure will serve as equilibrium measure of the reference local DBM $\widetilde{\mathbf{x}}(t)$ defined in Section 4.4. To complete *Part* (*iv*) we also show the fast relaxation of $\widetilde{\mathbf{x}}(t)$.

As a preparation of *Part* (*v*) in Section 5, we establish in Section 4.5 a few technical properties such as level repulsion and rigidity for $\widetilde{\mathbf{x}}(t)$ and for its equilibrium measure ω_{T_1}. We close Section 4 by stating the universality of the local statistics of the reference measure ω_{T_1}. In fact, for the level repulsion we need to introduce a regularization of the logarithmic potential on a tiny scale.

4.1. *Localization at time T_1*

Let $K \in \mathbb{N}_N$ satisfy (2.27), with $K \geq \lfloor N^{\varpi} \rfloor$, $0 < \varpi < 1/10$, i.e.,

$$K^{10} N^{\delta} \leq N, \qquad K \geq N^{\varpi}.$$

Recall the constant $\sigma > 0$ from the assumptions of Theorem 2.1. Let $\chi \in (0, \varpi)$ be a small constant, to be chosen later on. Note that $N^{\chi} \ll K$. Then introduce the intervals of integers

$$I := [\![L-K, L+K]\!], \qquad I_0 := [\![L-K^5-N^{\chi}K^4, L+K^5+N^{\chi}K^4]\!], \qquad I_\sigma := [\![L-\sigma N, L+\sigma N]\!], \tag{4.1}$$

and we denote by I^c, I_0^c, I_σ^c the complements of I, I_0, I_σ in $\mathbb{N}_N$. Note that $I \subset I_0 \subset I_\sigma$. For a configuration $\boldsymbol{\lambda} \in F^{(N)}$, we introduce $\mathbf{x}$ and $\mathbf{y}$ as in (3.18).

Fix a small $\xi > 0$. Let

$$\mathcal{G}_s^1 := \bigl\{ \mathbf{y} \in F^{(N-\mathcal{K})} : \bigl| y_k - \gamma_{\ell(k)}(s) \bigr| \leq N^{\xi}/N, \forall k \in I_\sigma \bigr\}, \tag{4.2}$$

respectively,

$$\mathcal{G}_s^2 := \Biggl\{ \mathbf{y} \in F^{(N-\mathcal{K})} : \Biggl| \frac{1}{N} \sum_{k \in I_\sigma^c} \frac{1}{y_k - x} - \int_{y \in [y_{L-\sigma N}, y_{L+\sigma N}]^c} \frac{\varrho_s(y)\,\mathrm{d}y}{y-x} \Biggr| \leq \frac{N^{\xi}}{N^{\delta}}, \\ \forall x \in [y_{L-K^5}, y_{L+K^5}] \Biggr\}, \tag{4.3}$$

where $\mathcal{K} = 2K+1$. Note that for each $s \geq 0$, we choose the labeling in $\mathcal{G}_s^1$ to be the one of Assumption (2) of Theorem 2.1. We then set

$$\mathcal{G}_s := \mathcal{G}_s^1 \cap \mathcal{G}_s^2. \tag{4.4}$$

For any $\mathbf{Y} := \{\mathbf{y}(s) \in F^{(N-\mathcal{K})} : s \in [t_1, t_2]\}$ trajectory, we define the conditional measure $\mathbb{P}^{\mathbf{Y}}$ on the $\mathbf{X} := \{\mathbf{x}(s) \in F^{(N-\mathcal{K})} : s \in [t_1, t_2]\}$ trajectories in the usual way. We use $\mathbb{P}^{\mathbf{Y}}$ to denote the conditional measure on the whole $\mathbf{X}$ trajectories, while for any fixed time s, we use $\mathbb{P}^{\mathbf{y}(s)}$ for the conditional measure (on the $\mathbf{x}(s)$ configurations, for any fixed s). We set

$$\mathcal{G} := \Bigl\{ \mathbf{Y} = \bigl\{ \mathbf{y}(s) : s \in [t_1, t_2] \bigr\} : \\ \mathbf{y}(s) \in \mathcal{G}_s, \forall s : \mathbb{P}^{\mathbf{Y}}\bigl(\bigl| x_i(s) - \gamma_{\ell(i)}(s) \bigr| \leq N^{\xi}/N, \forall i \in I \bigr) \geq 1 - N^{-D}, \\ \text{and} \sup_{t,s \in [t_1,t_2]} \max_{k \in I^c} \bigl| y_k(t) - y_k(s) \bigr| \leq N^{\xi} \sqrt{t-s} \Bigr\}, \tag{4.5}$$

for small $\xi > 0$ and large $D > 0$.

Lemma 4.1. *Let $\boldsymbol{\lambda}(t)$, $t \geq 0$ be the DBM on $\mathcal{F}^{(N)}$ of* (2.11) *with fixed initial condition $\boldsymbol{\lambda}(0)$. Let $x_i(t) \equiv \lambda_i(t)$, $i \in I$, respectively $y_k(t) \equiv \lambda_k(t)$, $k \notin I$, for all $t \geq 0$. Under the assumptions of Theorem* 2.1, *for any* (*small*) $\xi > 0$ *and any* (*large*) $D > 0$, *we have*

$$\mathbb{P}\big(\mathbf{y}(s) \in \mathcal{G}_s, s \in [t_1, t_2]\big) \geq 1 - N^{-D}, \tag{4.6}$$

and

$$\mathbb{P}(\mathcal{G}) \geq 1 - N^{-D}, \tag{4.7}$$

for large enough $N \geq N_0(\xi, D)$, where $\mathbb{P}$ is with respect the Brownian motions (B_i) in (2.11).

Proof. Both estimates (4.6) and (4.7) follow directly from the assumptions (1)–(4) in Theorem 2.1. □

Throughout the rest of this section we will consider the trajectory $\mathbf{Y} \in \mathcal{G}$ as fixed. Nevertheless, all estimates will be uniform on $\mathcal{G}$. In particular, all constants only depend on the constants in (4.5), the constants δ, ϵ and σ of Theorem 2.1 as well as the parameter $\xi > 0$.

4.2. *Regularity of the semicircular flow and removal of mean drift*

Consider the DBM, $\boldsymbol{\lambda}(t)$ of (2.11) with initial condition $\boldsymbol{\lambda}(0)$ and the semicircular flow $\varrho_t = \mathcal{F}_t[\varrho]$. We first study some regularity properties of ϱ_t for $t \in [t_1, t_2]$. The following result is proven in Section A.1.2 of the Appendix A.

Lemma 4.2. *Under the assumptions of Theorem* 2.1, *the semicircular flow ϱ_t satisfies*

$$C^{-1} \leq \varrho_t(E) \leq C, \qquad \big|\partial_E \varrho_t(E)\big| \leq C N^{\delta}, \tag{4.8}$$

for all $E \in [E_* - \Sigma/2, E_* + \Sigma/2]$ and all $t \in [t_1, t_2]$.

Let $L \in \mathbb{N}_N$ be as in Theorem 2.1. In particular, we have $\varrho_{t_1}(\gamma_L(t_1)) \geq c > 0$. Then, we have

$$\big|x_i(t) - \gamma_{\ell(i)}(t)\big| \leq \frac{N^{\xi}}{N}, \quad i \in I, \mathbf{y}(t) \in \mathcal{G}_t, \tag{4.9}$$

with high probability for N sufficiently large, uniformly in $t \in [t_1, t_2]$, for some labeling ℓ that will be fixed throughout the paper. Recall from (2.9) that the quantiles $\boldsymbol{\gamma}$ are determined by

$$\int_{-\infty}^{\gamma_k(t)} \varrho_t(x)\,\mathrm{d}x = \frac{k}{N}. \tag{4.10}$$

The evolution of $\boldsymbol{\gamma}$ is studied in the Appendix A where the following result is proved.

Lemma 4.3. *Under the assumptions of Theorem* 2.1, *the quantiles (γ_k) defined through* (4.10), *satisfy*

$$\dot{\gamma}_{\ell(i)}(t) = -\int_{\mathbb{R}} \frac{\varrho_t(y)\,\mathrm{d}y}{y - \gamma_{\ell(i)}(t)} - \frac{\gamma_{\ell(i)}(t)}{2}, \quad \ell(i) \in I_{\sigma}, \tag{4.11}$$

for all $t > 0$, where $\dot{\gamma}_{\ell(i)}(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\gamma_{\ell(i)}(t)$. In particular, we have $|\dot{\gamma}_{\ell(L)}| \leq C$. Further, we have

$$\dot{\gamma}_{\ell(i)}(t) = -\frac{1}{N}\sum_{k=1}^{N} \frac{1}{\gamma_k(t) - \gamma_{\ell(i)}(t)} - \frac{\gamma_{\ell(i)}(t)}{2} + O\bigg(\frac{N^{\delta}}{N}\bigg), \quad \ell(i) \in I_{\sigma}, \tag{4.12}$$

uniformly in $t \in [T_1, t_2]$. Moreover, we have the estimates

$$\big|\dot{\gamma}_{\ell(i)}(t) - \dot{\gamma}_{\ell(L)}(t)\big| \leq C N^{-1+\delta}|i - L|, \qquad \big|\dot{\gamma}_{\ell(i)}(t) - \dot{\gamma}_{\ell(i)}(T_1)\big| \leq C N^{\delta}(t - T_1), \tag{4.13}$$

uniformly in $t \in [T_1, t_2]$ and $\ell(i) \in I_{\sigma}$.

Equation (4.11) shows that the points $(\gamma_{\ell(i)}(t))$ approximately satisfy a gradient flow evolution of particles with quadratic confinement and interacting via the mean field potential $\frac{1}{N}\log|x-y|$.

Lemma 4.3 is proved in Section A.1.2 of Appendix A. Let us briefly mention how the constant Σ in Assumption (1) and the constant σ in Assumption (2) can be related. For given $\Sigma > 0$, we can choose $\sigma > 0$ such that $\gamma_{\ell(i)}(t_1)$, for any $i \in I_\sigma = [\![L-\sigma N, L+\sigma N]\!]$, all lie inside $[E_* - \Sigma/4, E_* + \Sigma/4]$. Then we know from Lemma 4.3 that $\gamma_{\ell(i)}(t) \in [E_* - \Sigma/2, E_* + \Sigma/2]$ for all $t \in [t_1, t_2]$. By Lemma 4.2, we have control over ϱ_t on $[E_* - \Sigma/2, E_* + \Sigma/2]$.

For simplicity of notation, we henceforth drop the labeling ℓ and simply write, with some abuse of notation, $\gamma_i(t) \equiv \gamma_{\ell(i)}(t)$. From (4.13) and (4.9), we conclude that

$$\left|\lambda_L(t) - \lambda_L(T_1)\right| \le \upsilon_L(t-T_1) + o(t-T_1) + O\left(\frac{N^\xi}{N}\right), \quad t \in [T_1, t_2],$$

with high probability, where we have set $\upsilon_L := \dot{\gamma}_L(T_1)$. We denote by $\overline{\boldsymbol{\lambda}}(t)$, $t \ge T_1$ the process obtained from $\boldsymbol{\lambda}(t)$ by setting

$$\overline{\boldsymbol{\lambda}}(t) := \boldsymbol{\lambda}(t) - \upsilon_L(t-T_1), \quad t \ge T_1. \tag{4.14}$$

Thus $\overline{\boldsymbol{\lambda}}(t)$ satisfies the SDE

$$\mathrm{d}\overline{\boldsymbol{\lambda}}(t) = \sqrt{\frac{2}{\beta N}}\,\mathrm{d}B_i(t) - \upsilon_L\,\mathrm{d}t - \frac{1}{N}\sum_{j\ne i}\frac{1}{\overline{\lambda}_j(t) - \overline{\lambda}_i(t)}\,\mathrm{d}t - \frac{\overline{\lambda}_i(t) + \upsilon_L(t-T_1)}{2}\,\mathrm{d}t, \quad i \in \mathbb{N}_N, \beta \ge 1,$$

for $t \ge T_1$. In the following we write $\overline{x}_i \equiv \overline{\lambda}_i$, $i \in I$, respectively $\overline{y}_k \equiv \overline{\lambda}_k$, $k \notin I$, so that $\overline{\mathbf{Y}} = \{\overline{\mathbf{y}}(t) \in \digamma^{(N-\mathcal{K})} : t \in [T_1, t_2]\}$. Having shifted the original process $(\boldsymbol{\lambda}(t))$ as in (4.14), we also shift the distribution ϱ_t and the quantiles $\boldsymbol{\gamma}$, for $t \ge T_1$, accordingly:

$$\overline{\varrho}_t(x) := \varrho_t\big(x + \upsilon_L(t-T_1)\big), \qquad \overline{\gamma}_i(t) := \gamma_i(t) - \upsilon_L(t-T_1), \quad t \in [T_1, t_2],$$

$x \in \mathbb{R}$, $i \in \mathbb{N}_N$. In a similar way, we introduce the events $\overline{\mathcal{G}}_s^1$, $\overline{\mathcal{G}}_s^2$, $\overline{\mathcal{G}}_s$ and $\overline{\mathcal{G}}$ by replacing the quantities without bars with bars in (4.2), (4.3), (4.4) and (4.5).

4.3. *The reference measure ω_{T_1}*

In this section, we introduce the reference measure ω_{T_1} that will serve as local equilibrium measure.

4.3.1. *The reference points $\widetilde{\gamma}_j$*

Once $\mathbf{Y} \in \mathcal{G}$, thus also $\overline{\mathbf{Y}} \in \overline{\mathcal{G}}$, is fixed, we introduce time-independent "reference points," $\widetilde{\boldsymbol{\gamma}} \equiv (\widetilde{\gamma}_k)$, as follows: For $k \in \mathbb{N}_N$, let

$$\iota_k := \begin{cases} \frac{k-(L-K^5-N^\chi K^4)}{N^\chi K^4} & \text{if } L-K^5-N^\chi K^4 \le k \le L-K^5, \\ \frac{L+K^5+N^\chi K^4-k}{N^\chi K^4} & \text{if } L+K^5 \le k \le L+K^5+N^\chi K^4, \\ 1 & \text{if } L-K^5 \le k \le L+K^5, \\ 0 & \text{else,} \end{cases}$$

with $\chi > 0$ as in (4.1), i.e., ι_k is a linearly mollified cutoff of the indicator function $\mathbf{1}(k \in I_\sigma)$. Set

$$\widetilde{\gamma}_k := \iota_k z_k + (1-\iota_k) y_k(T_1), \quad k \in I^c, \tag{4.15}$$

where the external points $\mathbf{z} \equiv (z_k) \in \digamma^{(N-\mathcal{K})}$ in (4.15) will be chosen in Section 4.3.2 below. Note that $\widetilde{\gamma}_k = z_k$, for$|L-k| \le K^5$;$\widetilde{\gamma}_k = y_k(T_1) = \overline{y}_k(T_1)$, for $|L-k| \ge K^5 + N^\chi K^4$. Thus the sequence $\widetilde{\gamma}$ smoothly interpolates between the external points $\mathbf{y}(T_1)$ from the DBM and $\mathbf{z}$. The external points $\mathbf{z}$ are constructed from an appropriate β-ensemble whose equilibrium density has a single interval support. This will guarantee rigidity; in particular,

$$|z_{k+1} - z_k| \le C \frac{N^{\xi}}{N^{2/3}\check{k}^{1/3}}, \tag{4.16}$$

with $\check{k} = \min\{k, N-k+1\}$, for all $k \in [\![1, N-1]\!]$; see (4.21) below.

Anticipating the precise choice of $\mathbf{z}$, we mention that they are chosen such that

$$z_{L-K-1} = y_{L-K-1}(T_1), \qquad z_{L+K+1} = y_{L+K+1}(T_1). \tag{4.17}$$

In fact, this choice will assure that the configuration interval of the localized measures, both with $\mathbf{y}(T_1)$ and with $\widetilde{\boldsymbol{\gamma}}$ as external points, will have the same (and time-independent) support

$$\boldsymbol{J}_{\mathbf{y}(T_1)} = \boldsymbol{J}_{\mathbf{z}} = [z_-, z_+] = \big[y_{L-K-1}(T_1), y_{L+K+1}(T_1)\big], \tag{4.18}$$

where $z_- \equiv z_{L-K-1}$, $z_+ \equiv z_{L+K+1}$. We next estimate the size of the interval $\boldsymbol{J}_{\mathbf{z}}$.

Lemma 4.4. *Let $\boldsymbol{J}_{\mathbf{z}}$ be as in (4.18) and assume that K satisfies (2.27). Then, we have*

$$|\boldsymbol{J}_{\mathbf{z}}| = \frac{\mathcal{K}}{N\varrho_{T_1}(z_\circ)} + O\bigg(\frac{N^{\xi}}{N}\bigg), \tag{4.19}$$

on $\mathcal{G}^1_{T_1}$, where $z_\circ := (z_{L+K+1} - z_{L-K-1})/2$.

Proof. We mainly follow the proof of Lemma 4.5 in [28]. First, we write, by (4.18),

$$|\boldsymbol{J}_{\mathbf{z}}| = z_+ - z_- = y_{L+K+1}(T_1) - y_{L-K-1}(T_1) = \gamma_{L+K+1}(T_1) - \gamma_{L-K-1}(T_1) + O\big(N^{-1+\xi}\big),$$

where we used that $\mathbf{y}(T_1) \in \mathcal{G}^1_{T_1}$. Next, we note that by Assumption (1) of Theorem 2.1 we have $\varrho_{T_1}(x) = \varrho_{T_1}(\overline{z}) + O(N^{\delta}|x - \overline{z}|)$. Thus from (2.9),

$$\mathcal{K} = N\int_{\gamma_{L-K-1}(T_1)}^{\gamma_{L+K+1}(T_1)} \varrho_{T_1}(x)\,\mathrm{d}x = N\varrho_{T_1}(y)|\boldsymbol{J}_{\mathbf{z}}| + O\big(N^{1+\delta}|\boldsymbol{J}_{\mathbf{z}}|^2\big) + O\big(N^{\xi}\big),$$

where we used that $\varrho_{T_1} \sim 1$. Since $KN^{\delta} \ll N$, by (2.27), we get (4.19). $\square$

4.3.2. *Construction of an auxiliary β-ensemble*

We now turn to the choice of the reference points $\mathbf{z}$ introduced first at the beginning of Section 4.3.1. We construct a global β-ensemble, μ_{AUX}, with potential V_{AUX} and equilibrium density ϱ_{AUX} such that it has a single interval support and such that the density matches with ϱ_{T_1} at $\gamma_L(T_1)$. The main properties of μ_{AUX} are summarized in the next lemma.

Lemma 4.5. *There exists a β-ensemble $\mu_{\mathrm{AUX}} \equiv \mu^{(N)}_{\mathrm{AUX}}$, with quadratic potential V_{AUX} and equilibrium density ϱ_{AUX}, and a set of external configurations $\mathbf{z} \in \digamma^{(N-\mathcal{K})}$, with $z_{L-K-1} = y_{L-K-1}(T_1)$, $z_{L+K+1} = y_{L+K+1}(T_1)$, such that the following holds for N sufficiently large.*

(1) *The limiting equilibrium density ϱ_{AUX} of μ_{AUX} is a shifted semicircle law with finite variance satisfying, for any $\xi > 0$,*

$$\varrho_{\mathrm{AUX}}(y) = \varrho_{T_1}(y) + O\big(N^{\xi}/K\big) + O\big(N^{\delta}|y - z_{L-K-1}|\big), \qquad y \in \mathbb{R}. \tag{4.20}$$

(2) *The external points $\mathbf{z}$ satisfy, for any $\xi > 0$,*

$$|z_k - \gamma_{\mathrm{AUX},k}| \le C\frac{N^{\xi}}{N^{2/3}\check{k}}, \qquad k \in I^c, \tag{4.21}$$

where $(\gamma_{\mathrm{AUX},k})$ *are the quantiles of the equilibrium density* ϱ_{AUX}, *i.e.*, $\int_{-\infty}^{\gamma_{\mathrm{AUX},k}} \mathrm{d}\varrho_{\mathrm{AUX}} = k/N$, *and* $\check{k} = \min\{k, N-k+1\}$. *In particular, since* V_{AUX} *is "regular," the rigidity estimate* (4.16) *holds.*

(3) *The localized measure* $\mu^{\mathbf{z}}_{\mathrm{AUX}}$ *satisfies, for any* $\xi > 0$,

$$\mathbb{P}^{\mu^{\mathbf{z}}_{\mathrm{AUX}}}\big(|x_i - \alpha_i| \geq N^{\xi}/N, \forall i \in I\big) \leq C N^{\xi} \frac{K}{N}, \tag{4.22}$$

where (α_i) *are* $\mathcal{K}$ *equidistant points in* $\boldsymbol{J}_{\mathbf{z}} = [z_{L-K-1}, z_{L+K+1}] = \boldsymbol{J}_{\mathbf{y}(T_1)}$, *i.e.*,

$$\alpha_i = z_\circ + \frac{i-L}{2K+1}|\boldsymbol{J}_{\mathbf{z}}|, \qquad z_\circ = \frac{1}{2}(z_{L-K-1} + z_{L+K+1}). \tag{4.23}$$

Proof. The proof is split into three steps. *Step 1*: We introduce the quadratic potential $V_{\mathrm{G}(\varsigma)}(x) := x^2/2\varsigma^2$, with some $\varsigma > 0$, and consider the β-ensemble, $\mu_{\mathrm{G}(\varsigma)}$, with Hamiltonian

$$\mathcal{H}_{\mathrm{G}(\varsigma)} := \frac{1}{2}\sum_{i=1}^{N} V_{\mathrm{G}(\varsigma)}(\lambda_i) - \frac{1}{N}\sum_{\substack{i,j=1\\ i<j}}^{N} \log|\lambda_j - \lambda_i|.$$

It is easy to check that the limiting equilibrium density, $\varrho_{\mathrm{G}(\varsigma)}$, of $\mu_{\mathrm{G}(\varsigma)}$ satisfies $\varrho_{\mathrm{G}(\varsigma)}(x) = \varrho_{\mathrm{sc}}(x/\varsigma)/\varsigma$, with $\varrho_{\mathrm{sc}}(x) = \frac{2}{\pi}\sqrt{(4-x^2)_+}$ the standard semicircle law. Similarly, the quantiles, $(\gamma_{\mathrm{G}(\varsigma),i})$, of $\varrho_{\mathrm{G}(\varsigma)}$ satisfy $\gamma_{\mathrm{G}(\varsigma),i} = \varsigma\gamma_{\mathrm{sc},i}$, where $\gamma_{\mathrm{sc},i}$ denote the quantiles with respect the standard semicircle law, i.e., $\int_{-\infty}^{\gamma_{\mathrm{sc},i}} \mathrm{d}\varrho_{\mathrm{sc}} = i/N$. Thus $\varrho_{\mathrm{G}(\varsigma)}(\gamma_{\mathrm{G}(\varsigma),i}) = \varrho_{\mathrm{sc}}(\gamma_{\mathrm{sc},i})/\varsigma$. In particular, we can fix ς such that $\varrho_{T_1}(\gamma_L(T_1)) = \varrho_{\mathrm{G}(\varsigma)}(\gamma_{\mathrm{G}(\varsigma),L})$, i.e., we set

$$\varsigma := \frac{\varrho_{\mathrm{sc}}(\gamma_{\mathrm{sc},L})}{\varrho_{T_1}(\gamma_L(T_1))}.$$

We next choose boundary conditions $\tilde{\mathbf{y}}$ with the following properties: (1) For any $\xi > 0$,

$$|\tilde{y}_k - \gamma_{\mathrm{sc},k}| \geq N^{\xi}/N, \quad \forall k \in I^c, \tag{4.24}$$

for $N \geq N_0(\xi)$ (i.e., $\tilde{\mathbf{y}}$ are rigid in the sense of sense of (3.7) with $V = V_{\mathrm{G}(\varsigma)}$); (2) for any $\xi > 0$, there are $c_0', c_1' > 0$ such that

$$\mathbb{P}^{\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma)}}\big(|x_i - \tilde{\alpha}_i| \geq N^{\xi}/N, \forall i \in I\big) \leq \mathrm{e}^{-c_0' N^{c_1'}}, \tag{4.25}$$

where $\tilde{\alpha}_i$ are the $\mathcal{K}$ equidistant points in the configuration interval $\boldsymbol{J}_{\tilde{\mathbf{y}}} = [\tilde{y}_{L-K-1}, \tilde{y}_{L+K+1}]$. The precise choice of $\tilde{\mathbf{y}}$ is unimportant for our argument, as long as $\tilde{\mathbf{y}}$ satisfy (4.24) and (4.25). That we can choose a $\tilde{\mathbf{y}}$ such that (4.24) and (4.25) are satisfied follows from Proposition 3.1 and an application of Markov's inequality.

Step 2: The length of the configuration intervals $\boldsymbol{J}_{\mathbf{y}(T_1)}$ and $\boldsymbol{J}_{\tilde{\mathbf{y}}}$ may differ slightly. Using the scale invariance of the Gaussian measure, we now adjust ς and $\tilde{\mathbf{y}}$ to guarantee that the lengths of the configuration intervals agree: Following the proof of Lemma 4.4 or the proof of Lemma 4.5 in [28], we get from the rigidity estimates for $\mu_{\mathrm{G}(\varsigma)}$ that

$$|\boldsymbol{J}_{\tilde{\mathbf{y}}}| = |\tilde{y}_{L-K-1} - \tilde{y}_{L+K+1}| = \frac{\mathcal{K}}{N\varrho_{\mathrm{G}(\varsigma)}(\gamma_{\mathrm{G}(\varsigma),L})} + O\big(N^{-1}N^{\xi}\big),$$

and from Lemma 4.4 that

$$|\boldsymbol{J}_{\mathbf{y}(T_1)}| = \big|y_{L-K-1}(T_1) - y_{L+K+1}(T_1)\big| = \frac{\mathcal{K}}{N\varrho_{T_1}(\gamma_L)} + O\big(N^{-1}N^{\xi}\big).$$

Using that $\varrho_{T_1}(\gamma_L) = \varrho_{\mathrm{G}(\varsigma)}(\gamma_{\mathrm{G}(\varsigma),L})$, by our choice of ς, we hence conclude that

$$s := \frac{|\boldsymbol{J}_{\tilde{\mathbf{y}}}|}{|\boldsymbol{J}_{\mathbf{y}(T_1)}|} = 1 + O\big(N^{\xi}K^{-1}\big). \tag{4.26}$$

Setting $\tilde{\mathbf{z}} := \tilde{\mathbf{y}}/s$ we have $|\boldsymbol{J}_{\tilde{\mathbf{z}}}| = |\boldsymbol{J}_{\mathbf{y}(T_1)}|$ and

$$\mathbb{P}^{\mu^{\tilde{\mathbf{z}}}_{\mathrm{G}(\varsigma)}}\big(|x_i - \tilde{\alpha}_i/s| \geq N^{\xi}/N, \forall i \in I\big) = \mathbb{P}^{\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma')}}\big(|x_i - \tilde{\alpha}_i| \geq sN^{\xi}/N, \forall i \in I\big), \tag{4.27}$$

where we have set $\varsigma' := s\varsigma$. Using the rigidity of $\widetilde{\mathbf{y}}$ we get, similarly to (4.38), that $\nabla^2_{\mathbf{x}}\mathcal{H}^{\widetilde{\mathbf{y}}}(\mathbf{x}) \geq \frac{cN}{K}$, for all $\mathbf{x} \in (\boldsymbol{J}_{\widetilde{\mathbf{y}}})^{\mathcal{K}} \cap F^{\mathcal{K}}$. Thus the logarithmic Sobolev inequality

$$S\big(\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma')}|\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma)}\big) \leq \frac{CK}{N} D\big(\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma')}|\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma)}\big),$$

with the local Dirichlet form

$$D\big(\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma')}|\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma)}\big) := \frac{1}{\beta N}\sum_{i\in I}\int \bigg(\partial_i\bigg(\frac{\mathrm{d}\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma')}}{\mathrm{d}\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma)}}\bigg)(\mathbf{x})\bigg)^2 \mathrm{d}\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma)}(\mathbf{x})$$

holds. A straightforward calculation together with (4.26) then shows that

$$S\big(\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma')}|\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma)}\big) \leq \frac{CK}{N^2}\sum_{i\in I}\int \big|\partial_i \mathrm{e}^{-\beta N\sum_{j\in I}(V_{\mathrm{G}(\varsigma')}(x_j) - V_{\mathrm{G}(\varsigma)}(x_j))}\big|^2 \mathrm{d}\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma)} \leq C\frac{N^{2\xi}K^2}{N^2}.$$

Thus, using first (4.27) and then the entropy inequality (3.13), we get

$$\begin{aligned}
\mathbb{P}^{\mu^{\tilde{\mathbf{z}}}_{\mathrm{G}(\varsigma)}}\big(|x_i - \tilde{\alpha}_i/s| \geq N^{\xi}/N, \forall i \in I\big) &= \mathbb{P}^{\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma')}}\big(|x_i - \tilde{\alpha}_i| \geq sN^{\xi}/N, \forall i \in I\big) \\
&\leq \mathbb{P}^{\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma)}}\big(|x_i - \tilde{\alpha}_i| \geq sN^{\xi}/N, \forall i \in I\big) + \sqrt{2S\big(\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma')}|\mu^{\tilde{\mathbf{y}}}_{\mathrm{G}(\varsigma)}\big)} \\
&\leq C\mathrm{e}^{-N^{\xi}} + C\frac{N^{\xi}K}{N},
\end{aligned} \tag{4.28}$$

where we used (4.25) (with an additional factor s) to get the last line.

Step 3: Finally, we achieve that $\boldsymbol{J}_{\tilde{\mathbf{z}}} = \boldsymbol{J}_{\mathbf{y}(T_1)}$ by a simple shift in the energy: we replace $V_{\mathrm{G}(\varsigma')}(x)$ by $V_{\mathrm{G}(\varsigma')}(x - b)$, $b := y_{L-K-1}(T_1) - \tilde{y}_{L-K-1}$, $x \in \mathbb{R}$. We now choose μ_{AUX} as the Gaussian measure defined by the potential $V_{\mathrm{G}(\varsigma')}(\cdot - b)$ and we set $z_i := \tilde{z}_i - b$, for $i \in [\![1, N]\!]$. With these choices, (4.28) asserts that

$$\mathbb{P}^{\mu^{\mathbf{z}}_{\mathrm{AUX}}}\big(|x_i - \alpha_i| \geq N^{\xi}/N, \forall i \in I\big) \leq C\frac{N^{\xi}K}{N}, \tag{4.29}$$

where α_i are the $\mathcal{K}$ equidistant points in the interval $\boldsymbol{J}_{\mathbf{z}} = \boldsymbol{J}_{\mathbf{y}(T_1)} = [y_{L-K-1}(T_1), y_{L+K+1}(T_1)]$.

In sum, we have established the following. We consider the β-ensemble μ_{AUX} with quadratic potential, whose equilibrium density ϱ_{AUX} is a semicircle law with radius $\sqrt{2\varsigma'}$ which is centered at b. Taylor expanding the densities ϱ_{T_1} and ϱ_{AUX} around $y_{L-K-1}(T_1)$ and recalling (4.26) as well as (4.8), we obtain (4.20). This proves statement (1) of Lemma 4.5. The points $\mathbf{z} = (z_i)$ are rigid as follows from (4.24) and the choices $z_i = \widetilde{y}_i/s$, $z_i = \widetilde{z}_i - b$, $i \in \mathbb{N}_N$. This immediately implies statement (2) of Lemma 4.5. Finally, the rigidity statement (3) of Lemma 4.5 for the localized measure $\mu^{\mathbf{y}}_{\mathrm{AUX}}$ was obtained in (4.29). This concludes the proof of Lemma 4.5. □

We conclude this subsection with a straightforward technical result that will be used in the next section. Recall the definition of the interval of integers I, I_0 and I_σ in (4.1).

Corollary 4.6. *Let* $\mathbf{z}$ *be as in Lemma* 4.5 *and let* $\widetilde{\boldsymbol{\gamma}}$ *be defined as in* (4.15). *Then, for any* $\xi > 0$,

$$\big|\widetilde{\gamma}_k - \gamma_k(T_1)\big| \leq C\frac{N^{\xi}}{N} + C\cdot\mathbf{1}(k \in I_0)\bigg(\frac{N^{\xi}|\widetilde{\gamma}_k - \widetilde{\gamma}_L|}{K} + N^{\delta}|\widetilde{\gamma}_k - \widetilde{\gamma}_L|^2\bigg), \quad k \in I_\sigma \setminus I, \tag{4.30}$$

for N sufficiently large. Moreover, we have, for any $\xi > 0$,

$$\widetilde{\gamma}_k - \widetilde{\gamma}_{k-1} \le N^{\xi}/N, \quad k \in I_\sigma \setminus I, \tag{4.31}$$

for N sufficiently large.

Proof. Recall that $\widetilde{\gamma}_k = y_k(T_1)$ for $k \notin I_0$. Since $\mathbf{y}(T_1) \in \mathcal{G}^1_{T_1}$ we immediately get

$$\left|\widetilde{\gamma}_k - \gamma_k(T_1)\right| \le C\frac{N^{\xi}}{N}, \tag{4.32}$$

for $k \in I_\sigma \setminus I_0$. Next, assume first that $K+1 \le L-k \le K^5$. Then we have $\widetilde{\gamma}_k = z_k$, and we can write

$$\int_{\widetilde{\gamma}_k}^{y_{K-L-1}(T_1)} \varrho_{\mathrm{AUX}}(y)\,\mathrm{d}y = \frac{|L-K-1-k|}{N} + O\left(\frac{N^{\xi}}{N}\right),$$

where we used $y_{K-L-1}(T_1) = z_{K-L-1}$, the rigidity estimate in (4.21) and the fact that $(\gamma_{\mathrm{AUX},k})$ are the quantiles of ϱ_{AUX}. Using (4.20), we hence can write

$$\begin{aligned}\int_{\widetilde{\gamma}_k}^{y_{K-L-1}(T_1)} \varrho_{T_1}(y)\,\mathrm{d}y &= \frac{|L-K-1-k|}{N} + O\left(\frac{N^{\xi}}{N}\right) \\ &\quad + O\left(\frac{N^{\xi}|\widetilde{\gamma}_k - y_{L-K-1}(T_1)|}{K}\right) + O\left(N^{\delta}\left|\widetilde{\gamma}_k - y_{L-K-1}(T_1)\right|^2\right).\end{aligned}$$

On the other hand, since $\mathbf{y}(T_1) \in \mathcal{G}^1_{T_1}$, i.e., $|y_{L-K-1}(T_1) - \gamma_{L-K-1}(T_1)| \le CN^{\xi}/N$, and using that $\gamma_k(T_1)$ are the quantiles with respect to ϱ_{T_1}, we have

$$\int_{\gamma_k(T_1)}^{y_{K-L-1}(T_1)} \varrho_{T_1}(y)\,\mathrm{d}y = \frac{|L-K-1-k|}{N} + O\left(\frac{N^{\xi}}{N}\right).$$

Comparing these last two equations and using the lower bound on the density ϱ_{T_1}, we conclude that

$$\left|z_k - \gamma_k(T_1)\right| \le C\frac{N^{\xi}}{N} + C\frac{N^{\xi}|\widetilde{\gamma}_k - \widetilde{\gamma}_L|}{K} + CN^{\delta}|\widetilde{\gamma}_k - \widetilde{\gamma}_L|^2, \tag{4.33}$$

for k such that $K+1 \le L-k \le K^5$. Here, we also used that $\widetilde{\gamma}_L - y_{L-K-1}(T_1) \le CK/N$. The same argument applies to the case $K+1 \le k-L \le K^5$.

It remains to consider the transition regime $K^5 \le |L-k| \le K^5 + N^{\chi}K^4$. Using the definition of $\widetilde{\boldsymbol{\gamma}}$ in (4.15), we can estimate

$$\begin{aligned}\left|\widetilde{\gamma}_k - \gamma_k(T_1)\right| &\le \iota_k\left|z_k - \gamma_k(T_1)\right| + (1-\iota_k)\left|y_k(T_1) - \gamma_k(T_1)\right| \\ &\le C\frac{N^{\xi}}{N} + C\frac{N^{\xi}|\widetilde{\gamma}_k - \widetilde{\gamma}_L|}{K} + CN^{\delta}|\widetilde{\gamma}_k - \widetilde{\gamma}_L|^2,\end{aligned}$$

for such k, where we used (4.32), (4.33) and the rigidity of $\mathbf{y}(T_1) \in \mathcal{G}^1_{T_1}$. This proves (4.30).

The estimate (4.31) follows directly from the rigidity of $\mathbf{y}(T_1)$, $\mathbf{z}$ and (4.30). □

4.3.3. *Definition of the reference measure ω_{T_1}*

Given the $\widetilde{\gamma}_j$, we define the local "reference" measure

$$\omega_{T_1}(\mathbf{x})\,\mathrm{d}\mathbf{x} := \frac{1}{Z_{T_1}}\mathrm{e}^{-\beta N\mathcal{H}_{T_1}(\mathbf{x})}\,\mathrm{d}\mathbf{x}, \qquad \mathcal{H}_{T_1}(\mathbf{x}) := \sum_{i\in I} V^{\widetilde{\boldsymbol{\gamma}}}(x_i) - \frac{1}{N}\sum_{\substack{i,j\in I\\ i<j}} \log(x_j - x_i), \tag{4.34}$$

where the external potential is given by

$$V^{\widetilde{\boldsymbol{\gamma}}}(x) := \frac{1}{2}V(x) - \frac{1}{N}\sum_{k\notin I}\log|x-\widetilde{\gamma}_k|, \qquad V(x) = \frac{x^2}{2} + 2\upsilon_L x. \tag{4.35}$$

The subscript T_1 in ω_{T_1} indicates that the external points $\widetilde{\boldsymbol{\gamma}}$ in the construction of this measure were obtained in (4.15) by matching the external points $\mathbf{y}(T_1)$ of the original DBM at time T_1. Note that this measure as well as the measure $\widetilde{\mathbb{P}}^{\overline{\mathbf{Y}}}$ are supported on the fixed configuration interval

$$\boldsymbol{J}_{\mathbf{y}(T_1)} = \big[y_{L-K-1}(T_1), y_{L+K+1}(T_1)\big].$$

4.4. *Local DBM and relaxation*

Having $\overline{\mathbf{Y}} \in \overline{\mathcal{G}}$ fixed, we consider the DBM on the $\overline{\mathbf{x}}$-variables given by the stochastic differential equation (SDE)

$$\begin{aligned} \mathrm{d}\overline{x}_i(t) &= \sqrt{\frac{2}{\beta N}}\,\mathrm{d}B_i(t) - \upsilon_L\,\mathrm{d}t - \frac{1}{N}\sum_{\substack{j\in I\\ j\neq i}}\frac{1}{\overline{x}_j(t)-\overline{x}_i(t)}\,\mathrm{d}t \\ &\quad - \frac{1}{N}\sum_{k\notin I}\frac{1}{\overline{y}_k(t)-\overline{x}_i(t)}\,\mathrm{d}t - \frac{\overline{x}_i(t)+\upsilon_L(t-T_1)}{2}\,\mathrm{d}t, \end{aligned} \tag{4.36}$$

$i\in I$, $t \geq T_1$, with $(B_i)_{i\in I}$ a collection of independent standard Brownian motions. We let $\mathbb{P}^{\overline{\mathbf{Y}}}$ denote the associated path space measure.

For $t \geq T_1$, we define an approximate coupled dynamics, $\widetilde{\mathbf{x}}(t)$ by letting

$$\mathrm{d}\widetilde{x}_i(t) = \sqrt{\frac{2}{\beta N}}\,\mathrm{d}B_i(t) - \upsilon_L\,\mathrm{d}t - \frac{1}{N}\sum_{\substack{j\in I\\ j\neq i}}\frac{1}{\widetilde{x}_j(t)-\widetilde{x}_i(t)}\,\mathrm{d}t - \frac{1}{N}\sum_{k\notin I}\frac{1}{\widetilde{\gamma}_k-\widetilde{x}_i(t)}\,\mathrm{d}t - \frac{\widetilde{x}_i(t)}{2}\,\mathrm{d}t, \tag{4.37}$$

$i \in I$, with initial condition $\widetilde{\mathbf{x}}(T_1) = \overline{\mathbf{x}}(T_1)$. The corresponding path space measure is denoted by $\widetilde{\mathbb{P}}^{\overline{\mathbf{Y}}}$. Going from (4.36) to (4.37) we replaced the time-dependent external points $\mathbf{y}(t)$ by the time-independent reference points $\widetilde{\boldsymbol{\gamma}}$ and we neglected the drift term $\upsilon_L(t-T_1)\,\mathrm{d}t/2$. Note that the Brownian motions in (4.36) and (4.37) are the same. We remark that the measure ω_{T_1} defined in (4.34) is the equilibrium measure of the SDE (4.37).

We write the distribution of $\widetilde{\mathbf{x}}(t)$ as $g_t\omega_{T_1}$ (for $t \geq T_1$). Since they are supported on the same configuration interval, the measures $g_t\omega_{T_1}$ (for $t > T_1$) and ω_{T_1} are both absolutely continuous with respect to the Lebesgue measure, hence also to each other.

We next compare the measures ω_{T_1} and $g_t\omega_{T_1}$ for $t \geq T_1$. We show that the process $(\widetilde{\mathbf{x}}(t))$ equilibrates on a time scale $\sim K/N$, i.e., the local statistics of $g_t\omega_{T_1}$ and ω_{T_1} are very close beyond times $t \geq T_1' := T_1 + K(K/N)$, with $t \leq t_2$.

Since ω_{T_1} is supported on an interval of size $O(K/N)$ (see Lemma 4.4), the Hessian of its Hamiltonian $\mathcal{H}_{T_1}$ from (4.34) satisfies

$$\mathcal{H}_{T_1}''(\mathbf{x}) \geq \min_{i\in I}\frac{1}{N}\sum_{k\notin I}\frac{1}{(x_i-\widetilde{\gamma}_k)^2} \geq \min_{i\in I}\frac{1}{N}\sum_{k:K<|k-L|\leq K^5}\frac{1}{(x_i-z_k)^2} \geq \frac{cN}{K}, \tag{4.38}$$

for all $\mathbf{x} \in (\boldsymbol{J}_{\mathbf{y}(T_1)})^{\mathcal{K}} \cap \digamma^{(\mathcal{K})}$, where we used (4.16). Thus, recalling the discussion in Section 3.3, ω_{T_1} satisfies the logarithmic Sobolev inequality

$$S_{\omega_{T_1}}(f) \leq \frac{CK}{N}D_{\omega_{T_1}}(\sqrt{f}); \tag{4.39}$$

cf., (3.15). Further, for $t \geq T_1 + \frac{1}{2}K(K/N)$ the process $(\widetilde{\mathbf{x}}(t))$ has become absolutely continuous with respect to Lebesgue measure, and one can easily prove that

$$S(g_t\omega_{T_1}|\omega_{T_1}) \leq N^C, \quad t \geq T_1 + \frac{1}{2}K(K/N); \tag{4.40}$$

for some large C; see, e.g., Lemma 4.7 in [20]. Therefore, running the Bakry–Émery argument of Section 3.3 from time $T_1 + \frac{1}{2}K(K/N)$ to time $T_1' = T_1 + K(K/N)$ and using the initial entropy estimate (4.40), we immediately get the following result.

Lemma 4.7. *For any $t \geq T_1' = T_1 + K(K/N)$, we have*

$$D_{\omega_{T_1}}(\sqrt{g_t}) + S_{\omega_{T_1}}(g_t) \leq \mathrm{e}^{-cK}, \quad t \in \big[T_1', t_2\big], \tag{4.41}$$

for some $c > 0$. In particular, the statistics of $\widetilde{\mathbf{x}}(t)$ for any $t \in [T_1', t_2]$ are the same as the statistics of the local equilibrium measure ω_{T_1} as follows from

$$\left|\int O(g_t - 1)\,\mathrm{d}\omega_{T_1}\right| \leq \|O\|_\infty\sqrt{2S_{\omega_{T_1}}(g_t)} \leq C\mathrm{e}^{-cK/2}, \tag{4.42}$$

for any bounded observable O.

4.5. *Three measures and their properties*

Having $\overline{\mathbf{Y}} \in \overline{\mathcal{G}}$ fixed and having constructed the external points $\mathbf{z}$, we have, up to this point, introduced three distinct measures on the internal particles:

(1) ω_{T_1} is given by an explicit formula in (4.34). It is a local β-ensemble on $\boldsymbol{J}_{\mathbf{z}}$ which we refer to as the "reference" measure.
(2) $g_t\omega_{T_1}$ is the distribution of $\widetilde{\mathbf{x}}(t)$ from the dynamics (4.37) on $\boldsymbol{J}_{\mathbf{z}}$.
(3) $\mathbb{P}^{\overline{\mathbf{y}}(t)}$ is the measure of the $\overline{\mathbf{x}}(t)$ dynamics (4.36) at time t, it is also the conditional measure $\mathbb{P}^{\overline{\mathbf{Y}}}$ of the original measure $\mathbb{P}$, conditioned on the $\overline{\mathbf{Y}}$-trajectory at time $t \geq t_1$. This measure is also on $\mathcal{K}$ particles, but now the configuration interval is time-dependent $\boldsymbol{J}_{\overline{\mathbf{y}}(t)} := [\overline{y}_{L-K-1}(t), \overline{y}_{L+K+1}(t)]$.

In the remainder of this subsection, we establish level repulsion and rigidity for the measures ω_{T_1} and $g_t\omega_{T_1}$:

Definition 4.8. We say that the measure ν (on $\mathcal{K}$-point configurations labeled with I, $|I| = \mathcal{K}$, in a fixed interval $\boldsymbol{J}$) is rigid with exponent ξ if

$$\nu\big(|\mathbf{x}_i - \alpha_i| > N^\xi/N, \forall i \in I\big) \leq C\mathrm{e}^{-cN^\xi}, \tag{4.43}$$

where α_i are the $\mathcal{K}$ equidistant points in $\boldsymbol{J}$ and where $c > 0$. The path-space measure Q for times $[T_1, t_2]$ on the same configuration interval $\boldsymbol{J}$ is rigid with exponent ξ if

$$Q\Big(\sup_{s\in[T_1,t_2]}\big|\mathbf{x}_i(s) - \alpha_i\big| > N^\xi/N, \forall i \in I\Big) \leq C\mathrm{e}^{-cN^\xi}. \tag{4.44}$$

Note that if for all t the fixed-time marginals Q_t of a space time measure Q satisfy rigidity, then Q satisfies rigidity (since the trajectories typically have some mild continuity; see Section 9.3 of [28]).

We will establish the following main technical input. Recall that $T_1' = T_1 + K(K/N)$.

Proposition 4.9. *Let $\xi > 0$ be sufficiently small and let K satisfy (2.27). Then, for any $\overline{\mathbf{Y}} \in \overline{\mathcal{G}}$ and any $t \in [T_1, t_2]$ the following holds.*

(1) ω_{T_1} (*i.e., the local "reference" measure*) *is rigid with exponent* ξ *and satisfies*

$$\max_{i\in I}\mathbb{E}^{\omega_{T_1}}\frac{1}{[N|x_i-x_{i\pm 1}|]^p}\leq C_p \tag{4.45}$$

(*with* $x_{i\pm 1}(t)=y_{L\pm(K+1)}(T_1)$ *if* $i=L\pm K$), *for any* $p<2$.

(2) $g_t\omega_{T_1}$ (*i.e., the time marginals of the process* $\widetilde{\boldsymbol{X}}=\{\widetilde{\mathbf{x}}(s):s\in[T_1',t_2]\}$) *is rigid with exponent* ξ, *moreover, the whole process* $\{\widetilde{\mathbf{x}}(s):s\in[T_1',t_2]\}$ *with measure* $\widetilde{\mathbb{P}}^{\widetilde{\mathbf{Y}}}$ *is rigid with exponents* ξ. *Furthermore,*

$$\max_{i\in I}\mathbb{E}^{g_t\omega_{T_1}}\frac{1}{[N|\widetilde{x}_i(t)-\widetilde{x}_{i\pm 1}(t)|]^p}\leq C_p \tag{4.46}$$

(*with* $\tilde{x}_{i\pm 1}(t)=y_{L\pm(K+1)}(T_1)$ *if* $i=L\pm L$), *for any* $p<2$ *and* $t\geq T_1'$.

To simplify the exposition, we split the proof of Proposition 4.9 according to its statements.

4.5.1. *Proof of statement* (1) *of Proposition* 4.9

We start with the rigidity of the reference measure ω_{T_1}. For notational simplicity, we write in the following

$$\gamma_k\equiv\gamma_{\ell(k)}(T_1),\quad k\in I_\sigma,$$

where the labeling ℓ is chosen according to (4.2).

Proof of rigidity of ω_{T_1}. We first recall the following general result of Theorem 4.2 (see also the remark after Lemma 4.5) of [28]. For any local equilibrium measure $\mu^{\mathbf{y}}$ on $\mathcal{K}$ points with potential $V^{\mathbf{y}}$ on an interval $\boldsymbol{J}$ of size $|\boldsymbol{J}|\sim K/N$ rigidity (with exponent $\xi>0$) in the sense of Definition 4.8 holds if the following two conditions are satisfied:

$$\big(V^{\mathbf{y}}\big)'(x)=\varrho(y_\circ)\log\frac{d_+(x)}{d_-(x)}+O\bigg(\frac{N^{c\xi}}{Nd(x)}\bigg), \tag{4.47}$$

and

$$\big|\mathbb{E}^{\mu^{\mathbf{y}}}x_i-\alpha_i\big|\leq N^{c\xi}/N,\quad \forall i\in I, \tag{4.48}$$

where $y_\circ$ is the midpoint of the interval $\boldsymbol{J}$, $d(x)$ is the distance of x to the boundary of $\boldsymbol{J}$,

$$d_-(x):=d(x)+\varrho(y_\circ)N^{\xi}/N,\qquad d_+(x):=\max\big\{|x-y_{L-K-1}|,|x-y_{L+K+1}|\big\}+\varrho(y_\circ)N^{\xi}/N, \tag{4.49}$$

and α_i are the $\mathcal{K}$ equidistant points in $\boldsymbol{J}$.

We now apply this result with the choices $\mathbf{y}=\widetilde{\boldsymbol{\gamma}}$ and $\boldsymbol{J}=\boldsymbol{J}_{\mathbf{z}}=\boldsymbol{J}_{\mathbf{y}(T_1)}$ to the reference measure ω_{T_1}. The condition (4.47) will follow from the global condition (4.52) below and from the fact that the reference points $\widetilde{\boldsymbol{\gamma}}$ are rigid in the sense of Corollary 4.6. The details are as follows.

To check condition (4.47), we introduce the supplemental potential $\widetilde{V}^{\widetilde{\boldsymbol{\gamma}}}$ by setting

$$\widetilde{V}^{\widetilde{\boldsymbol{\gamma}}}(x):=\frac{1}{2}x^2+2\upsilon_L x-\frac{1}{N}\sum_{k:|k-L|\geq K+N^{\xi}}\log|x-\widetilde{\gamma}_k|. \tag{4.50}$$

We then have

$$\big|\big(V^{\widetilde{\boldsymbol{\gamma}}}\big)'(x)-\big(\widetilde{V}^{\widetilde{\boldsymbol{\gamma}}}\big)'(x)\big|\leq\frac{1}{N}\sum_{k:K<|k-L|<K+N^{\xi}}\frac{1}{|\widetilde{\gamma}_k-x|}\leq\frac{N^{\xi}}{Nd(x)},\quad x\in\boldsymbol{J}_{\mathbf{z}}. \tag{4.51}$$

To control $\widetilde{V}^{\widetilde{\gamma}}$, we can follow Appendix A of [28]. First, recall from Lemma 4.3 that υ_L satisfies

$$\upsilon_L = \dot{\gamma}_L = -\int_{\mathbb{R}} \frac{\varrho_{T_1}(y)\,\mathrm{d}y}{y-\gamma_L} - \frac{\gamma_L}{2}, \quad \gamma_L \equiv \gamma_L(T_1). \tag{4.52}$$

Thus,

$$\begin{aligned}\left|\upsilon_L + \frac{x}{2} + \int_{\mathbb{R}} \frac{\varrho_{T_1}(y)\,\mathrm{d}y}{y-x}\right| &= \left|\upsilon_L + \frac{\gamma_L}{2} + \int_{\mathbb{R}} \frac{\varrho_{T_1}(y)\,\mathrm{d}y}{y-\gamma_L}\right| + O\big(\left|\operatorname{Re} m_{T_1}(\gamma_L) - \operatorname{Re} m_{T_1}(x)\right|\big) + O\big(|x-\gamma_L|\big) \\ &\le C\frac{N^{\delta}K}{N} + C\frac{K}{N}, \quad x \in \boldsymbol{J}_{\mathbf{z}},\end{aligned} \tag{4.53}$$

where we used (4.52). To bound the second and third term on the right we used Assumptions (1) of Theorem 2.1 and the estimate on $|\boldsymbol{J}_{\mathbf{z}}|$ in (4.19). We may thus split

$$\big(\widetilde{V}^{\widetilde{\gamma}}\big)' = \Omega_1 + \Omega_2 + \Omega_3 + O\left(N^{\delta}\frac{K}{N}\right),$$

with

$$\begin{aligned}\Omega_1(x) &:= -\int_{\gamma_{L-K-N^{\xi}}}^{\gamma_{L+K+N^{\xi}}} \frac{\varrho_{T_1}(y)\,\mathrm{d}y}{y-x}, \\ \Omega_2(x) &:= -\int_{\widetilde{\gamma}_1}^{\gamma_{L-K-N^{\xi}}} \frac{\varrho_{T_1}(y)\,\mathrm{d}y}{y-x} + \frac{1}{N}\sum_{k=1}^{L-K-N^{\xi}} \frac{1}{\widetilde{\gamma}_k - x}, \\ \Omega_3(x) &:= -\int_{\gamma_{L+K+N^{\xi}}}^{\widetilde{\gamma}_N} \frac{\varrho_{T_1}(y)\,\mathrm{d}y}{y-x} + \frac{1}{N}\sum_{k=L+K+N^{\xi}}^{N} \frac{1}{\widetilde{\gamma}_k - x}.\end{aligned}$$

To estimate Ω_1 we use that $\varrho_{T_1}(y) = \varrho_{T_1}(x) + O(N^{\delta}|y-x|)$ (cf., (4.8) and (4.19)) to get

$$\Omega_1(x) = -\varrho_{T_1}(y_\circ)\log\frac{d_+(x)}{d_-(x)} + O\big(N^{\delta}K/N\big) + O\left(\frac{N^{2\xi}}{N^2 d(x)}\right), \quad x \in \boldsymbol{J}_{\mathbf{z}}. \tag{4.54}$$

To obtain the third line, we used

$$\gamma_{L+K+N^{\xi}} - x = (\gamma_{L+K+N^{\xi}} - \gamma_{L+K+1}) + (\gamma_{L+K+1} - x) = d_+(x) + O\big(N^{\xi}N^{-1}\big),$$

respectively $x - \gamma_{L+K+N^{\xi}} = d_-(x) + O(N^{\xi}N^{-1})$, where we used the definition of $\widetilde{\boldsymbol{\gamma}}$ in (4.15) and the definition of $d_{\pm}$ in (4.49).

We next estimate Ω_2 (Ω_3 is estimated in the very same way): We split

$$\frac{1}{N}\sum_{k=1}^{L-K-N^{\xi}} \frac{1}{\widetilde{\gamma}_k - x} = \frac{1}{N}\sum_{k=1}^{M-1} \frac{1}{y_k(T_1) - x} + \frac{1}{N}\sum_{k=M}^{L-K-N^{\xi}} \frac{1}{\widetilde{\gamma}_k - x}, \tag{4.55}$$

with $M = L - \sigma N$, such that we can estimate on one hand

$$\frac{1}{N}\sum_{k=1}^{M-1} \frac{1}{y_k(T_1) - x} = \int_{\widetilde{\gamma}_1}^{\gamma_{M-1}} \frac{\varrho_{T_1}(y)\,\mathrm{d}y}{y-x} + O\left(\frac{N^{\xi}}{N^{\delta}}\right), \tag{4.56}$$

since $\mathbf{y}(T_1) \in \mathcal{G}_{T_1}$; cf., (4.3). On the other hand, we estimate

$$\frac{1}{N}\sum_{k=M}^{L-K-N^\xi}\frac{1}{\widetilde{\gamma}_k - x} = \sum_{k=M}^{L-K-N^\xi}\int_{\gamma_{k-1}}^{\gamma_k}\frac{\varrho_{T_1}(y)\,\mathrm{d}y}{\widetilde{\gamma}_k - x}$$
$$= \int_{\gamma_{M-1}}^{\gamma_{L-K-N^\xi}}\frac{\varrho_{T_1}(y)\,\mathrm{d}y}{y-x} + O\Bigg(\sum_{k=M}^{L-K-N^\xi}\int_{\gamma_{k-1}}^{\gamma_k}\frac{|\widetilde{\gamma}_k - y|\varrho_{T_1}(y)\,\mathrm{d}y}{(y-x)^2}\Bigg). \tag{4.57}$$

Using Corollary 4.6 and recalling the definition of I from (4.1), we can bound the remainder in the above equation as

$$\sum_{k=M}^{L-K-N^\xi}\int_{\gamma_{k-1}}^{\gamma_k}\frac{|\widetilde{\gamma}_k - y|\varrho_{T_1}(y)\,\mathrm{d}y}{(y-x)^2} \le C\frac{N^\xi}{N}\int_{\gamma_{M-1}}^{\gamma_{L-K-N^\xi}}\frac{\mathrm{d}y}{(y-x)^2}$$
$$+ C\int_{\gamma_{L-K^5-N^\chi K^4-1}}^{\gamma_{L-K-N^\xi}}\frac{(N^\xi/K)(y-x)+N^\delta(y-x)^2}{(y-x)^2}\,\mathrm{d}y$$
$$\le C\frac{N^\xi}{Nd(x)} + C\frac{N^\xi}{K}\log\bigg(\frac{x-\gamma_{L-K^5-N^\chi K^4-1}}{x-\gamma_{L-K-N^\xi}}\bigg) + CN^\delta\frac{K^5}{N}. \tag{4.58}$$

Thus, using that $d(x) \le |\boldsymbol{J}_{\mathbf{y}(T_1)}| \le CK/N$, i.e., $K \ge cNd(x)$, and that K satisfies (2.27), we have

$$\sum_{k=M}^{L-K-N^\xi}\int_{\gamma_{k-1}}^{\gamma_k}\frac{|\widetilde{\gamma}_k - y|\varrho_{T_1}(y)\,\mathrm{d}y}{(y-x)^2} \le C\frac{N^{2\xi}}{Nd(x)},$$

where we bounded the logarithmic term on the right side of (4.58) by N^ξ. Hence, combining this last estimate with (4.56), we find

$$\big|\Omega_2(x)\big| \le C\frac{N^{2\xi}}{Nd(x)} + C\frac{N^\xi}{N^\delta} \le C\frac{N^{2\xi}}{Nd(x)}, \tag{4.59}$$

where we used that $K \ge cNd(x)$ and that K satisfies (2.27). The same bounds holds for Ω_3.

Combining (4.59), (4.54) and (4.51), we get (4.47) for $(V^{\widetilde{\boldsymbol{\gamma}}})'$ (with $c = 2$).

It remains to check (4.48) with the external points $\mathbf{y} = \widetilde{\boldsymbol{\gamma}}$, i.e., $|\mathbb{E}^{\omega_{T_1}}x_i - \alpha_i| \le N^{c\xi}/N$, $i \in I$. First, we note that from (4.29) we have $|\mathbb{E}^{\mu^{\mathbf{z}}_{\mathrm{AUX}}}x_i - \alpha_i| \le CN^\xi/N$. Then, using the logarithmic Sobolev inequality (4.39), we bound the relative entropy

$$S\big(\mu^{\mathbf{z}}_{\mathrm{AUX}}|\omega_{T_1}\big) \le C\frac{K}{N}\frac{1}{N}\sum_{i\in I}\mathbb{E}^{\omega_{T_1}}\big|\partial_i \mathrm{e}^{-\beta N\sum_i[V^{\mathbf{z}}(x_i)-V^{\widetilde{\boldsymbol{\gamma}}}(x_i)]}\big|^2$$
$$\le CK\mathbb{E}^{\mu_{\mathrm{AUX}}}\sum_{i\in I}\Bigg|\frac{1}{2}V'_{\mathrm{AUX}}(x_i) - \frac{1}{2}x_i - \upsilon_L - \frac{1}{N}\sum_{k:|k-L|\ge K^5}\bigg[\frac{1}{\widetilde{\gamma}_k - x_i} - \frac{1}{z_k - x_i}\bigg]\Bigg|^2. \tag{4.60}$$

Note that when k is close to the interval I in the summation above, i.e., when $|k-L| \le K^5$, then the corresponding terms exactly cancel by the choice of $\widetilde{\boldsymbol{\gamma}}$ in (4.15).

To bound the right side of (4.60), we first recall that we have from (3.5) the equilibrium relation

$$V'_{\mathrm{AUX}}(x) = -2\int_{\mathbb{R}}\frac{\varrho_{\mathrm{AUX}}(y)}{y-x}\,\mathrm{d}y, \qquad x \in \operatorname{supp}\varrho_{\mathrm{AUX}}, \tag{4.61}$$

for the auxiliary β-ensemble μ_{AUX}. We denote by $(\gamma_{\mathrm{AUX},i})_{i=1}^N$ the quantiles of the measure ϱ_{AUX} and let $\gamma_{\mathrm{AUX},0} = a_{\mathrm{AUX}}$, $\gamma_{\mathrm{AUX},N} = b_{\mathrm{AUX}}$, where $a_{\mathrm{AUX}}, b_{\mathrm{AUX}}$ are the endpoints of the support of ϱ_{AUX}.

We then bound the summation in (4.60) for all $k \le L - K^5$ as follows (the case $k \ge L + K^5$ is treated in the very same way),

$$\begin{aligned}\left|\sum_{k=1}^{L-K^5}\int_{\gamma_{\mathrm{AUX},k-1}}^{\gamma_{\mathrm{AUX},k}}\left(\frac{\varrho_{\mathrm{AUX}}(y)\,\mathrm{d}y}{z_k-x_i}-\frac{\varrho_{\mathrm{AUX}}(y)\,\mathrm{d}y}{y-x_i}\right)\right| &\le C\sum_{k=1}^{L-K^5}\int_{\gamma_{\mathrm{AUX},k-1}}^{\gamma_{\mathrm{AUX},k}}\frac{|y-z_k|\varrho_{\mathrm{AUX}}(y)\,\mathrm{d}y}{(y-x_i)^2}\\ &\le C\sum_{k=1}^{L-K^5}\frac{N^{\xi}}{N^{2/3}k^{1/3}}\int_{\gamma_{\mathrm{AUX},k-1}}^{\gamma_{\mathrm{AUX},k}}\frac{\varrho_{\mathrm{AUX}}(y)\,\mathrm{d}y}{(y-x_i)^2}\le C\frac{N^{\xi}}{K^5},\end{aligned}\tag{4.62}$$

for all $i \in I$, where we used the rigidity of $\mathbf{z}$ (see (4.21)) and that ϱ_{AUX} vanishes like a square root at the endpoints $a_{\mathrm{AUX}}, b_{\mathrm{AUX}}$ of its support (recall that ϱ_{AUX} is a rescaled and re-centered semicircle).

On the other hand, reasoning exactly as in (4.54), we find that

$$\int_{\gamma_{\mathrm{AUX},L-K^5}}^{\gamma_{\mathrm{AUX},L+K^5}}\frac{\varrho_{\mathrm{AUX}}(y)}{y-x_i}\,\mathrm{d}y=\int_{\gamma_{\mathrm{AUX},L-K^5}}^{\gamma_{\mathrm{AUX},L+K^5}}\frac{\varrho_{\mathrm{AUX}}(x_i)+O(|y-x_i|)}{y-x_i}\,\mathrm{d}y=O\big(K^{-4}\big)+O\big(K^5/N\big),\tag{4.63}$$

for $i \in I$. We therefore get, combining (4.61), (4.62) and (4.63),

$$\left|\frac{1}{2}V'_{\mathrm{AUX}}(x_i)+\frac{1}{N}\sum_{k:|L-k|\ge K^5}\frac{1}{z_k-x_i}\right|\le\frac{C}{K^4},\quad i\in I.\tag{4.64}$$

Second, using the definition $\widetilde{\boldsymbol{\gamma}}$ in (4.15), we obtain similarly to (4.55) and (4.56),

$$\left|\frac{1}{N}\sum_{k=1}^{L-K^5}\frac{1}{\widetilde{\gamma}_k-x_i}-\int_{\widetilde{\gamma}_1}^{\gamma_{L-K^5}}\frac{\varrho_{T_1}(y)\,\mathrm{d}y}{y-x_i}\right|\le\left|\frac{1}{N}\sum_{k=M}^{L-K^5}\frac{1}{\widetilde{\gamma}_k-x_i}-\int_{\gamma_M}^{\gamma_{L-K^5}}\frac{\varrho_{T_1}(y)\,\mathrm{d}y}{y-x_i}\right|+C\frac{N^{\xi}}{N^{\delta}},\tag{4.65}$$

with $M = L - \sigma N$. The first term on the right side of (4.65) can be controlled, similarly to (4.57) and (4.58), as

$$\left|\frac{1}{N}\sum_{k=M}^{L-K^5}\frac{1}{\widetilde{\gamma}_k-x_i}-\int_{\gamma_M}^{\gamma_{L-K^5}}\frac{\varrho_{T_1}(y)\,\mathrm{d}y}{y-x_i}\right|\le C\frac{N^{\xi}}{K^5}+CN^{\delta}\frac{K^5}{N}+C\frac{N^{\xi+\chi}}{K^2},$$

where we used $|\widetilde{\gamma}_{L-K^5}-x_i|\sim K^5/N$ and the assumption on K in (2.27). A similar estimate holds for the summations over $[\![L+K^5, N]\!]$. Further, repeating the arguments of (4.54), we get

$$\int_{\gamma_{L-K^5}}^{\gamma_{L+K^5}}\frac{\varrho_{T_1}(y)}{y-x_i}\,\mathrm{d}y=O\big(K^{-4}\big)+O\bigg(N^{\delta}\frac{K^5}{N}\bigg).$$

Thus, combining the last two estimates and recalling (4.53) as well as (2.27) we find

$$\left|\frac{1}{2}x_i+\upsilon_L+\frac{1}{N}\sum_{k:|k-L|\ge K^5}\frac{1}{\widetilde{\gamma}_k-x_i}\right|\le\frac{C}{K^4}+C\frac{N^{\xi+\chi}}{K^2}.\tag{4.66}$$

Plugging (4.66) and (4.64) into (4.60) we get $S(\mu^{\mathbf{z}}_{\mathrm{AUX}}|\omega_{T_1})\le CN^{2\xi+2\chi}K^{-2}$, which finally leads, in combination with (4.29), to

$$\begin{aligned}\mathbb{P}^{\omega_{T_1}}\big(|x_i-\alpha_i|\ge N^{\xi}N^{-1},\forall i\in I\big)&\le\mathbb{P}^{\mu^{\mathbf{z}}_{\mathrm{AUX}}}\big(|x_i-\alpha_i|\ge N^{\xi}N^{-1},\forall i\in I\big)+\sqrt{2S\big(\mu^{\mathbf{z}}_{\mathrm{AUX}}|\omega_{T_1}\big)}\\ &\le C\frac{N^{\xi}K}{N}+C\frac{N^{\xi+\chi}}{K},\end{aligned}\tag{4.67}$$

where we used (3.13). Together with the a priori bound $|x_i - \alpha_i| \le C(K/N)$, this implies

$$\left|\mathbb{E}^{\omega_{T_1}} x_i - \alpha_i\right| \le C\frac{N^\xi}{N} + C\frac{K}{N}\frac{N^{\xi+\chi}}{K} \le C\frac{N^{\xi+\chi}}{N}, \quad i \in I. \tag{4.68}$$

Thus, choosing, e.g., $\chi = \xi$, we get the bound (4.48) for the measure ω_{T_1}.

Applying Theorem 4.2 of [28] as mentioned at the beginning of the proof, we see that the measure ω_{T_1} satisfies rigidity with exponent ξ. □

The level repulsion estimate (4.45) in statement (2) of Proposition 4.9 is proved using the explicit Vandermonde structure of ω_{T_1}. The proof is essentially identical to the proof of Theorem 4.3 in [28] given Section 7.2 of [28]. We therefore leave the details aside.

4.5.2. *Proof of statement* (2) *of Proposition* 4.9

The rigidity for $g_t\omega_{T_1}$, with fixed $t \ge T_1'$ in the sense of Definition 4.8 immediately follows from the rigidity for ω_{T_1} and the entropy estimate (4.41). Using the stochastic continuity of $(\widetilde{\mathbf{x}}(t))$ and the rigidity of $g_t\omega_{T_1}$ a sufficiently large set of discrete times, we can conclude that $\widetilde{\mathbb{P}}^{\mathbf{Y}}$ itself is rigid; see Section 9.3 of [28] for details.

It remains to prove the level repulsion for $g_t\omega$ given in (4.46).

Proof of (4.46). The level repulsion bound (4.46) follows from (4.45) and the entropy bound (4.41). More precisely, we have to introduce $\omega_{T_1}^{\varepsilon_*}$, an ε_*-regularization in the ω_{T_1} measure in the same way as in Section 9.3 of [28]. The parameter $\varepsilon_* = \mathrm{e}^{-K^c}$ will be chosen tiny with a small $c > 0$. This regularization modifies the interaction terms in (4.37) and in the Hamiltonian $\mathcal{H}_{T_1}$. In the latter the log becomes $\log_{\varepsilon_*}$ defined as

$$\log_{\varepsilon_*}(x) := \mathbf{1}(x \ge \varepsilon_*)\log(x) + \mathbf{1}(x \le \varepsilon_*)\left\{\log\varepsilon_* + \frac{x-\varepsilon_*}{\varepsilon_*} - \frac{1}{2\varepsilon_*^2}(x-\varepsilon_*)^2\right\}. \tag{4.69}$$

This has the property that $\partial_x^2\log_{\varepsilon_*}(x)$ is the same, $-x^{-2}$, as before if $x > \varepsilon_*$, but it remains bounded by ε_*^{-2} for all x. The Hamiltonian is still convex. The support of the measure $\omega_{T_1}^{\varepsilon_*}$ is not $\boldsymbol{J}_{\mathbf{z}}$ but the whole $\mathbb{R}$, but it is still overwhelmingly supported on $\boldsymbol{J}_{\mathbf{z}}$. In particular, ω_{T_1} and $\omega_{T_1}^{\varepsilon_*}$ are close in entropy sense, see (9.57) from [28],

$$S\big(\omega_{T_1}|\omega_{T_1}^{\varepsilon_*}\big) \le CK^C\varepsilon_*^2. \tag{4.70}$$

As a consequence, by the entropy inequality (3.13) we may transfer the rigidity bounds from the measure ω_{T_1} to the measure $\omega_{T_1}^{\varepsilon_*}$, i.e., we have

$$\mathbb{P}^{\omega_{T_1}^{\varepsilon_*}}\big(|x_i - \alpha_i| \ge N^\xi/N\big) \le \mathrm{e}^{-K^c}. \tag{4.71}$$

Similar modifications occur in the SDE (4.37); the $(\widetilde{x}_i - \widetilde{x}_j)^{-1}$ and also the $(\widetilde{x}_i - \widetilde{\gamma}_k)^{-1}$ terms get regularized to

$$(\widetilde{x}_i - \widetilde{x}_j)_{\varepsilon_*}^{-1} := \partial_x\log_{\varepsilon_*}(\widetilde{x}_i - \widetilde{x}_j),$$

and they will be uniformly bounded by ε_*^{-1}. Now we can prove (4.46) with the regularization, since we can use the entropy inequality (3.13) to get

$$\mathbb{E}^{g_t\omega_{T_1}^{\varepsilon_*}}\frac{1}{[N|x_i - x_{i+1}|_{\varepsilon_*}]^p} \le \mathbb{E}^{\omega_{T_1}^{\varepsilon_*}}\frac{1}{[N|x_i - x_{i+1}|_{\varepsilon_*}]^p} + \varepsilon_*^{-p}\sqrt{2S_{\omega_{T_1}^{\varepsilon_*}}(g_t)} \le C_pK^\xi, \tag{4.72}$$

Here in estimating the first term we used that the level repulsion bounds hold for the regularized measure

$$\mathbb{P}^{\omega_{T_1}^{\varepsilon_*}}(x_{i+1} - x_i \le s/N) \le CK^\xi s^2, \quad s \ge K^\xi\varepsilon_*,$$

see (9.58) of [28], i.e., we have

$$\mathbb{E}^{\omega_{T_1}^{\varepsilon_*}} \frac{1}{[N|x_i - x_{i+1}|_{\varepsilon_*}]^p} \le C_p K^{\xi}. \tag{4.73}$$

The exponential smallness of the entropy $S_{\omega_{T_1}^{\varepsilon_*}}(g_t)$ is proven exactly the same way as the proof of (4.41), since the Bakry–Émery type convexity argument remains valid for the equilibrium measure $\omega_{T_1}^{\varepsilon_*}$ as well. This exponential smallness wins over ε_*^{-p} if the constant c in the definition of $\varepsilon_* = \exp(-K^c)$ is small.

Since the only purpose of this regularization is to prove (4.46), we will not carry the ε_* superscript throughout the proof, i.e., we continue to write ω_{T_1} everywhere, although we really mean $\omega_{T_1}^{\varepsilon_*}$. As we have seen, the key input information on ω_{T_1} for our whole analysis, the rigidity (4.71), holds for the regularized measure. The other input, the level repulsion in the form (4.45) holds with an additional factor K^{ξ}, see (4.73), that plays no role in the applications of this estimate. □

4.6. *Local statistics of ω_{T_1}*

In this subsection, we show that the gap statistics of the localized reference measure ω_{T_1} are universal, i.e., are given by the statistics of the Gaussian invariant ensemble up to negligible errors for large N. The precise universality statement for ω_{T_1} is as follows.

Theorem 4.10. *There is a small universal constants $\mathfrak{e}, \chi, \alpha > 0$, such that for any $\mathbf{y} \in \mathcal{G}_{T_1}$ (see (4.4)), for any fixed j and for any smooth compactly supported function $\mathcal{O}$ of n variables, we have*

$$\begin{aligned}\mathbb{E}^{\omega_{T_1}} \mathcal{O}\big(\big((N\varrho_{T_1}(\gamma_L))(x_{i_0} - x_{i_0+j})\big)_{j=1}^n\big) &= \mathbb{E}^{\mathrm{G}} \mathcal{O}\big(\big((N\varrho_{\#})(x_{i_0'} - x_{i_0'+j})\big)_{j=1}^n\big) \\ &\quad + O\big(\|O'\|_\infty N^{-\mathfrak{e}}\big),\end{aligned} \tag{4.74}$$

for N sufficiently large, for any $i_0, i_0' \in \mathbb{N}_N$ satisfying $|i_0 - L| \le N^{\chi}$, $|i_0' - L'| \le N^{\chi}$ with any $L' \in [\alpha N, (1-\alpha)N]$, and where $\varrho_\# := \varrho_{\mathrm{sc}}(\gamma_{L',\mathrm{sc}})$ denotes the density of the semicircle law ϱ_{sc} at the location of the L'th N-quantile of ϱ_{sc}.

In short, Theorem 4.10 assures that the gap statistics of the localized measure ω_{T_1} in the bulk is determined by the Gaussian invariant ensemble in the limit of large N. This result follows from Theorem 4.1 of [28] and the properties of ω_{T_1} established in this section so far.

Proof of Theorem 4.10. Theorem 4.1 of [28], as stated, directly compares two local measures, but together with Proposition 5.3 in [28] it can also be stated as a direct universality result: if the conditions of Theorem 4.1 of [28] hold for a local measure, then it has universal local gap statistics.

Theorem 4.1 (see also remark after Lemma 4.5) in [28] has two types of conditions.

(1) Regularity of the external potential in the sense of Definition 4.4 of [28]. For the case at hand, the external potential $V^{\widetilde{\boldsymbol{\gamma}}}$ defined in (4.35) is regular if

$$\big(V^{\widetilde{\boldsymbol{\gamma}}}\big)'(x) = \varrho_{T_1}(z_\circ)\log\frac{d_+(x)}{d_-(x)} + O\bigg(\frac{N^{c\xi}}{Nd(x)}\bigg), \tag{4.75}$$

$$\big(V^{\widetilde{\boldsymbol{\gamma}}}\big)''(x) \ge \frac{c}{d(x)}, \quad x \in \boldsymbol{J}_{\mathbf{z}} = [z_-, z_+], \tag{4.76}$$

hold, with some fixed constant c, where $z_\circ = (z_+ + z_-)/2$, $d(x) = \min\{|x - z_+|, |x - z_-|\}$ and $d_\pm(x)$ as in (4.49). Here we used the notation $z_- = z_{L-K-1}$, $z_+ = z_{L+K+1}$.

In proving (4.47) with external points $\widetilde{\boldsymbol{\gamma}}$, we already established (4.75). The convexity estimate (4.76) follows from the rigidity of $\widetilde{\boldsymbol{\gamma}}$: there is a constant $c > 0$ such that

$$\big(V^{\widetilde{\boldsymbol{\gamma}}}\big)''(x) = V''(x) + \frac{2}{N}\sum_{j \notin I}\frac{1}{(\widetilde{\gamma}_j - x)^2} \ge \frac{1}{2} + \frac{c}{d(x)}.$$

(2) The second input for Theorem 4.1 of [28] is

$$\left|\mathbb{E}^{\omega_{T_1}} x_i - \alpha_i\right| \le C\frac{N^{c\xi}}{N}, \quad i \in I, \tag{4.77}$$

where (α_i) denote the $\mathcal{K}$ equidistant points in $\boldsymbol{J}_{\mathbf{z}}$. We have already established (4.77) in (4.68).

Based upon these two inputs, Theorem 4.1 of [28] implies (4.74). □

5. Universal gap statistics for small times

In Section 4, we showed that the equilibrium measure ω_{T_1} of the dynamics (4.37) has universal gap statistics, i.e., the distribution of the rescaled eigenvalue gaps in the bulk of the spectrum under ω_{T_1} coincide with those of the corresponding Gaussian invariant ensemble up to negligible errors for sufficiently large N. In the present section, we compare the gaps of the two dynamics (4.37) and (4.36). We proceed in three steps that are outlined in the Sections 5.1, 5.2 and 5.3. In Section 5.4, we then complete the proof of Theorem 2.1.

As in Section 4, we will fix a $\mathbf{Y} \in \mathcal{G}$, or equivalently $\overline{\mathbf{Y}} \in \overline{\mathcal{G}}$, but do not always indicate this choice in the notation. All estimates obtained will be uniform on $\mathcal{G}$, so that we can integrate out $\mathbf{Y}$ at the very end of Section 5.4.

5.1. *Step* 1: *Small scale regularization*

First we introduce a small regularization in (4.36) starting from the time T_1. This regularization is only needed for the critical case $\beta = 1$, where the level repulsion, cf., Assumptions (3) of Theorem 2.1, is weakest. Level repulsion and the regularization introduced below will allow use to bound the kernel B_{ij} defined below in (5.8) as $\mathbb{E}|B_{ij}| \lesssim N$; see (5.14). For $\beta > 1$, a similar bound may be obtained without any regularization. In the following we carry the regularization along since the case $\beta = 1$ is the hardest.

This regularization procedure is the same as in Section 3.1 of [14], but it is different from the regularization in the ω_{T_1} measure and in the $\widetilde{\mathbf{x}}$ dynamics explained in part (2) of the proof of Proposition 4.9 (where the regularization parameter was called ε_*). Choose

$$\varepsilon_{jk} := \begin{cases} \varepsilon \cdot \mathbf{1}(j,k \in I_\sigma) & \text{if } j \ge k, \\ -\varepsilon \cdot \mathbf{1}(j,k \in I_\sigma) & \text{if } j < k, \end{cases} \quad \text{with } \varepsilon := N^{-10C_1}, \tag{5.1}$$

for a large $C_1 > 1$. (Note that by the above choice $\varepsilon_* \ll \varepsilon$.)

Define the regularized version of (4.36) as

$$\begin{aligned} \mathrm{d}\widehat{x}_i(t) &= \sqrt{\frac{2}{\beta N}}\,\mathrm{d}B_i(t) - \upsilon_L\,\mathrm{d}t + \frac{1}{N}\sum_{j\in I}\frac{1}{\overline{x}_i(t) - \overline{x}_j(t) + \varepsilon_{ij}}\,\mathrm{d}t \\ &\quad + \frac{1}{N}\sum_{k\notin I}\frac{1}{\overline{x}_i(t) - \overline{y}_k(t) + \varepsilon_{ik}} - \frac{\widehat{x}_i(t) + \upsilon_L t}{2}\,\mathrm{d}t, \quad i \in I, t \in [T_1, t_2], \end{aligned} \tag{5.2}$$

with initial condition $\widehat{\mathbf{x}}(T_1) = \overline{\mathbf{x}}(T_1)$, where the Brownian motions (B_i) are the same as in (4.36) and (4.37). Note that $\widehat{\mathbf{x}}$ may not preserve the ordering of the particles, but we will not need this property.

Lemma 5.1. *Define the event*

$$\Xi^1 := \bigcap_{t\in[T_1,t_2]}\left\{\max_{i\in I}\left|\overline{x}_i(t) - \widehat{x}_i(t)\right| \le N^{-5C_1}\right\}. \tag{5.3}$$

Under the conditions of Theorem 2.1, especially the level repulsion assumption (2.19), there is a set $\overline{\mathcal{G}}^* \subset \overline{\mathcal{G}}$ with $\mathbb{P}(\overline{\mathcal{G}}^*) \ge 1 - N^{-C_1}$ such that $\mathbb{P}^{\overline{\mathbf{Y}}}(\Xi^1) \ge 1 - CN^{-C_1}$ holds for any $\overline{\mathbf{Y}} \in \overline{\mathcal{G}}^*$. In particular, the local statistics of $\overline{\mathbf{x}}(t)$ and $\widehat{\mathbf{x}}(t)$ are asymptotically the same for any $t \in [T_1, t_2]$.

Proof. Let $\mathcal{R}$ be the rigidity set

$$\mathcal{R} := \{ |\overline{x}_i(t) - \overline{\gamma}_i(t)| \le N^{\xi}/N : t \in [T_1, t_2], i \in I \}.$$

First we claim that $\mathbb{P}(\mathcal{R} \cap \Xi^1) \ge 1 - N^{-2C_1}$. This estimate can be proved following the argument in Section 3.1 of [14] for $\upsilon_L = 0$. Mutatis mutandis the same proof applies for $\upsilon_L \ne 0$. As an input, we need a level repulsion estimate of the form

$$\mathbb{E}\mathbf{1}(\mathcal{R}) \frac{1}{[N|\overline{x}_{i+1}(t) - \overline{x}_i(t) + \varepsilon|]^2} \le N^{\delta+\xi} |\log \varepsilon|, \quad \forall t \in [T_1, t_2], i \in I, \tag{5.4}$$

that follows from (2.19). Therefore, by conditioning, there is an event $\overline{\mathcal{G}}^*$ such that $\mathbb{P}(\overline{\mathcal{G}}^*) \ge 1 - N^{-C_1}$ and

$$\mathbb{P}^{\overline{\mathbf{Y}}}(\mathcal{R} \cap \Xi^1) \ge 1 - N^{-C_1}, \quad \forall \overline{\mathbf{Y}} \in \overline{\mathcal{G}}^*.$$

Since $\mathbb{P}(\overline{\mathcal{G}}) \ge 1 - N^{-D}$ for any $D > 0$, see (4.7), without loss of generality we can assume that $\overline{\mathcal{G}}^* \subset \overline{\mathcal{G}}$. Note that for $\overline{\mathbf{Y}} \in \overline{\mathcal{G}}$ we have $\mathbb{P}^{\overline{\mathbf{Y}}}(\mathcal{R}) \ge 1 - N^{-D}$, for any large $D > 0$; cf., (4.5). Choosing D larger than C_1, completes the proof. □

5.2. *Step* 2: *Hölder regularity*

To compare the gaps of $\widehat{\mathbf{x}}$ and $\widetilde{\mathbf{x}}$, we introduce

$$\mathbf{v} \equiv \mathbf{v}(t) := \mathrm{e}^{(t-T_1')/2}\big(\widehat{\mathbf{x}}(t) - \widetilde{\mathbf{x}}(t)\big), \quad t \ge T_1'. \tag{5.5}$$

Subtracting (5.2) from (4.37) and dropping the t argument for brevity, we have

$$\begin{aligned} \frac{\mathrm{d}v_i}{\mathrm{d}t} = &-\frac{1}{N} \sum_{\substack{j \in I \\ j \ne i}} \frac{v_i - v_j}{(\overline{x}_i - \overline{x}_j + \varepsilon_{ij})(\widetilde{x}_i - \widetilde{x}_j)} - v_i \frac{1}{N} \sum_{k \notin I} \frac{1}{(\overline{x}_i - \overline{y}_k + \varepsilon_{ik})(\widetilde{x}_i - \widetilde{\gamma}_k)} - \frac{1}{2}\mathrm{e}^{(t-T_1')/2} \upsilon_L(t - T_1) \\ &+ \frac{1}{N} \sum_{\substack{j \in I \\ j \ne i}} \frac{(\widehat{x}_i - \overline{x}_i) - (\widehat{x}_j - \overline{x}_j) + \varepsilon_{ij}}{(\overline{x}_i - \overline{x}_j + \varepsilon_{ij})(\widetilde{x}_i - \widetilde{x}_j)} + \frac{1}{N} \sum_{k \notin I} \frac{(\overline{y}_k - \widetilde{\gamma}_k) + (\widehat{x}_i - \overline{x}_i) + \varepsilon_{ik}}{(\overline{x}_i - \overline{y}_k + \varepsilon_{ik})(\widetilde{x}_i - \widetilde{\gamma}_k)}, \quad t \ge T_1', \end{aligned} \tag{5.6}$$

$i \in I$. We rewrite (5.6) in the form

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -(\mathscr{B}\mathbf{v})_i + F_i^{(1)} + F_i^{(2)}, \qquad (\mathscr{B}\mathbf{v})_i := \sum_{j \in I} B_{ij}(v_i - v_j) + W_i v_i, \tag{5.7}$$

with time-dependent (symmetric) coefficients,[4] $i, j \in I$,

$$B_{ij} := \frac{1}{N(\overline{x}_i - \overline{x}_j + \varepsilon_{ij})(\widetilde{x}_i - \widetilde{x}_j)}, \qquad W_i := \frac{1}{N} \sum_{k \notin I} \frac{1}{(\overline{x}_i - \overline{y}_k + \varepsilon_{ik})(\widetilde{x}_i - \widetilde{\gamma}_k)}, \tag{5.8}$$

and with the "forcing terms"

$$F_i^{(1)} := \frac{1}{N} \sum_{\substack{j \in I \\ j \ne i}} \frac{(\widehat{x}_i - \overline{x}_i) - (\widehat{x}_j - \overline{x}_j) + \varepsilon_{ij}}{(\overline{x}_i - \overline{x}_j + \varepsilon_{ij})(\widetilde{x}_i - \widetilde{x}_j)} - \frac{1}{2}\mathrm{e}^{(t-T_1')/2} \upsilon_L(t - T_1), \tag{5.9}$$

$$F_i^{(2)} := \frac{1}{N} \sum_{k \notin I} \frac{\widehat{x}_i - \overline{x}_i + \varepsilon_{ik}}{(\overline{x}_i - \overline{y}_k + \varepsilon_{ik})(\widetilde{x}_i - \widetilde{\gamma}_k)} + \frac{1}{N} \sum_{k \notin I} \frac{\overline{y}_k - \widetilde{\gamma}_k}{(\overline{x}_i - \overline{y}_k + \varepsilon_{ik})(\widetilde{x}_i - \widetilde{\gamma}_k)}. \tag{5.10}$$

[4]Sometimes we write $B_{i,j}$ instead of B_{ij} to clarify the notation.

(Since $\overline{x}_i, \widetilde{x}_i, \widehat{x}_i$ and $\overline{y}_k$ depend on time, we have $B_{ij} \equiv B_{ij}(t)$, $W_i \equiv W_i(t)$, etc.)

We first study in Section 5.2.1 the "free dynamics" generated by $\mathscr{B}$, show that it is Hölder continuous, and then deal with the forcing terms $(F_i^{(1)})$, $(F_i^{(2)})$ via a perturbative argument in Section 5.3 to establish Hölder continuity for the full dynamics.

5.2.1. *Hölder regularity of the free dynamics*

Let $\widetilde{\mathbf{v}}$ solve (5.7) without the forcing terms, i.e.,

$$\frac{\mathrm{d}\widetilde{v}_i}{\mathrm{d}t} = -(\mathscr{B}\widetilde{\mathbf{v}})_i = -\sum_j B_{ij}(\widetilde{v}_i - \widetilde{v}_j) - W_i \widetilde{v}_i, \quad t \geq T_1', \tag{5.11}$$

with initial condition $\widetilde{\mathbf{v}}(T_1') = \mathbf{v}(T_1')$. We will view equation (5.11) as a discrete heat equation with a long-range hopping described by (B_{ij}) instead of the customary discrete Laplacian. It turns out that the celebrated De Giorgi–Nash–Moser regularity theory extends to this setup [28]. Hölder continuity in this setting means that $\widetilde{v}_{i+1} - \widetilde{v}_i$ is small. Adding back the forcing terms this will imply that

$$(v_{i+1} - v_i)(t) = \mathrm{e}^{(t-T_1')/2}\big[(\widehat{x}_{i+1} - \widehat{x}_i) - (\widetilde{x}_{i+1} - \widetilde{x}_i)\big](t)$$

is also small, for times slightly beyond T_1'. The original De Giorgi–Nash–Moser regularity theory requires uniformly bounded coefficients, however $B_{i+1,i}$ defined in (5.8) is unbounded if $\widetilde{x}_{i+1}$ is close to $\widetilde{x}_i$. Nevertheless level repulsion guarantees that in a certain space–time average sense $B_{i+1,i}$ is bounded with high probability. This motivates the following definitions.

Let $\mathcal{T} := [T_1', T_1'']$, where T_1'' is defined in Lemma 5.2 below. Mimicking Definition 9.7 in [28], we say that the equation (5.11) is *regular* at a space–time point $(Z, \theta) \in I \times \mathcal{T}$ with exponent $\rho > 0$ if

$$\sup_{t\in\mathcal{T}} \sup_{1\leq M\leq \mathcal{K}} \frac{1}{N^{-1} + |t-\theta|} \int_t^\theta \frac{1}{M} \sum_{i\in I:|i-Z|\leq M} \sum_{j\in I:|j-Z|\leq M} \big|B_{ij}(s)\big|\,\mathrm{d}s \leq N^{1+\rho}. \tag{5.12}$$

Furthermore, we say that the equation is *strongly regular* at a space–time point $(Z, \theta) \in I \times \mathcal{T}$ with exponent $\rho > 0$ if it is regular at all points $\{Z\} \times \{\theta + \Omega\}$, where the set Ω is defined as

$$\Omega := \left\{ -\frac{K}{N} \cdot 2^{-m}\big(1 + 2^{-k}\big) : 0 \leq m, k \leq C \log K \right\}.$$

From Theorem 10.1 of [28] and Lemma 5.1 we obtain the following result.

Lemma 5.2. *Let $c_1 \sim 1/100$ and choose $T_1'' = T_1' + K^{c_1}/N$. Then, there is an event Ξ^2 and constants C and $\rho \equiv \rho(\xi) > 0$ such that on the event Ξ^2 the equation (5.11) is strongly regular at (L, T_1''), and*

$$\mathbf{1}\big(\Xi^1 \cap \Xi^2\big)\big|\widetilde{v}_{i+1}\big(T_1''\big) - \widetilde{v}_i\big(T_1''\big)\big| \leq C N^{-1+\xi} K^{-\mathfrak{q}/4}, \quad |i - L| \leq C, \tag{5.13}$$

where $\mathfrak{q} > 0$ is a universal constant. Moreover, we have the estimate $\mathbb{P}^{\overline{\mathbf{Y}}}(\Xi^1 \cap \Xi^2) \geq 1 - N^{-\rho/8}$, for N sufficiently large.

Proof. We apply the Hölder regularity result, Theorem 10.1 of [28], to the evolution equation (5.11). Thanks to the regularization introduced in Step 1, cf., Section 5.1, we have, for any $i, j \in I$ and $t \in [T_1', t_2]$, that

$$\begin{aligned}
\mathbb{E}B_{ij} &\leq N\left(\max_{i\in I} \mathbb{E}\frac{1}{[N|\overline{x}_i - \overline{x}_{i-1} + \varepsilon|]^p}\right)^{1/p}\left(\max_{i\in I} \mathbb{E}\frac{1}{[N|\widetilde{x}_i - \widetilde{x}_{i-1}|]^q}\right)^{1/q} \\
&\leq N\left(\max_{i\in I} \frac{1}{(N\varepsilon)^\phi} \mathbb{E}\frac{1}{[N|\overline{x}_i - \overline{x}_{i-1} + \varepsilon|]^2}\right)^{1/p}\left(\max_{i\in I} \mathbb{E}\frac{1}{[N|\widetilde{x}_i - \widetilde{x}_{i-1}|]^q}\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\le N\big(\varepsilon^{-\phi}N^{\delta+\xi+\phi}|\log\varepsilon|\big)^{1/p}C_{q(\phi)}\\
&\le CN^{1+\delta+2\xi},
\end{aligned}
\tag{5.14}
$$

where we first applied Hölder's inequality with conjugate exponents p, q, with $p = 2+\phi$, $\phi > 0$, then used (5.4) and (4.46), and finally chose ϕ sufficiently small depending on C_1 in (5.1).

Notice that to guarantee regularity in the sense of (5.12) (modulo a constant factor), instead of taking suprema over all $s \in \mathcal{T}$, $M \in [\![1, \mathcal{K}]\!]$, it suffices to take suprema over a dyadic sequence of times $s_k = \theta \pm 2^{-k}$ and parameters $M_l = 2^l$, $k, l \in [\![1, C\log N]\!]$, since space–time averages on comparable scales are comparable. Using (5.14), setting $\rho := \delta + 3\xi$ and applying Markov inequality, for any fixed values of s and M, the space–time average in (5.12) is bounded by $N^{1+\rho}$ with probability at least $1 - N^{-\rho/2}$. Taking the union bound for not more than $C(\log N)^2$ times, we can guarantee regularity at any space–time point with probability at least $1 - N^{-\rho/3}$. Since the definition of strong regularity requires regularity at not more than $C(\log N)^2$ space–time points, an additional union bound guarantees strong regularity at any fixed space–time point with probability at least $1 - N^{-\rho/4}$. Defining

$$
\widetilde{\Xi}^2 := \big\{\text{equation (5.11) is strongly regular at } \big(L, T_1''\big)\big\},
$$

this proves that $\mathbb{P}^{\overline{\mathbf{Y}}}(\widetilde{\Xi}^2) \ge 1 - N^{-\rho/4}$ and verifies condition $(C1)_\rho$ in Theorem 10.1 of [28] on $\widetilde{\Xi}^2$.

The other condition $(C2)_\xi$ in Theorem 10.1 of [28] concerns large distance estimates of B_{ij}. More precisely, condition $(C2)_\xi$ requires that

$$
B_{ij}(t) \ge \frac{N^{1-\xi}}{|i-j|^2}, \quad t \in \big[T_1', T_1''\big],
\tag{5.15}
$$

for any i, j with $|L-i| \le K/C$, $|L-j| \le K/C$, and that

$$
\frac{N\mathbf{1}(\min\{|L-i|, |L-j|\} \ge K/C)}{C|i-j|^2} \le B_{ij}(t) \le \frac{CN}{|i-j|^2}, \quad t \in \big[T_1', T_1''\big],
\tag{5.16}
$$

for any $|i-j| \ge C'N^\xi$, with some constants $C, C' > 10$. Further, W_i is required to satisfy

$$
\frac{CN^{1-\xi}}{\Delta_i} \le W_i(s) \le \frac{CN^{1+\xi}}{\Delta_i}, \quad t \in \big[T_1', T_1''\big],
\tag{5.17}
$$

where $\Delta_i := \min\{L+K+1-i, L-K-1-i\}$. Using the rigidity estimates for $\mathbf{x}, \widetilde{\mathbf{x}}$ of Lemma 4.9, it is easy to check that, for any $\xi > 0$, there is an event $\widehat{\Xi}^2$ and constants c, with $\mathbb{P}^{\overline{\mathbf{Y}}}(\widehat{\Xi}^2) \ge 1 - \mathrm{e}^{-cN^\xi}$, such that (5.17), (5.16) and (5.15) hold. Set $\Xi^2 = \widetilde{\Xi}^2 \cap \widehat{\Xi}^2$. Then for all sufficiently small $\xi > 0$, we have $\mathbb{P}(\Xi^2) \ge 1 - CN^{-\rho/4}$.

Let $\|A\|_\infty := \sup_{i\in I}|A_i|$, $A \in \mathbb{C}^N$. Then the conclusion of Theorem 10.1 of [28] for the equation (5.11) is that

$$
\mathbf{1}\big(\Xi^2\big)\big|\widetilde{v}_{i+1}\big(T_1''\big) - \widetilde{v}_i\big(T_1''\big)\big| \le CK^{-\mathfrak{q}/4}\big\|\tilde{\mathbf{v}}\big(T_1'\big)\big\|_\infty, \quad |i-L| \le C,
\tag{5.18}
$$

where $\mathfrak{q} > 0$ is a universal exponent and where $T_1'' = T_1' + K^{c_1}/N$. More precisely, (5.18) follows from (10.6) of [28] after rescaling space and time by setting the constant α equal to $1/4$ (this α is different from the α used in the present paper).

Next, recalling from (5.11) that $\tilde{v}_i(T_1') = v_i(T_1')$ and that $v_i(T_1') = \widehat{x}_i(T_1') - \widetilde{x}_i(T_1')$, we get

$$
\big\|\mathbf{v}\big(T_1'\big)\big\|_\infty \le \big\|\widehat{\mathbf{x}}\big(T_1'\big) - \overline{\mathbf{x}}\big(T_1'\big)\big\|_\infty + \big\|\overline{\mathbf{x}}\big(T_1'\big) - \widetilde{\mathbf{x}}\big(T_1'\big)\big\|_\infty \le CN^{-5C_0} + CN^{-1+\xi},
\tag{5.19}
$$

on $\Xi^1 \cap \Xi^2$, where we used Lemma 5.1 and that the processes $(\widetilde{\mathbf{x}}(t))$, $(\overline{\mathbf{x}}(t))$, are both rigid in the sense of Definition 4.8 for $t \in [T_1, t_2]$. Thus, combining (5.18) with (5.19) we get

$$
\mathbf{1}\big(\Xi^1 \cap \Xi^2\big)\big|\widetilde{v}_{i+1}\big(T_1''\big) - \widetilde{v}_i\big(T_1''\big)\big| \le CN^{-1+\xi}K^{-\mathfrak{q}/4}, \quad |i-L| \le C,
\tag{5.20}
$$

where the event $\Xi^1 \cap \Xi^2$ satisfies $\mathbb{P}^{\overline{\mathbf{Y}}}(\Xi^1 \cap \Xi^2) \ge 1 - N^{-\rho/8}$, $\rho \equiv \rho(\xi) > 0$, for sufficiently small $\xi > 0$. $\square$

5.3. *Step* 3: *Removing the forcing terms*

Having established the Hölder regularity of the free dynamics of (5.11), we now deal with the "full dynamics" (5.6). The main result is as follows.

Proposition 5.3. *Let $c_1 \sim 1/100$ and choose $T_1'' = T_1' + K^{c_1}/N$. Then there is an event Ξ and constants C and $c_2, c_3 > 0$ such that, for any $\overline{\mathbf{y}} \in \overline{\mathcal{G}}$,*

$$\mathbf{1}(\Xi) \max_{i \in \frac{1}{4}I} \left| v_i\left(T_1''\right) - \widetilde{v}_i\left(T_1''\right)\right| \le \frac{N^{-c_2}}{N}, \tag{5.21}$$

for N sufficiently large, where $\frac{1}{4}I := [\![L - K/4, L + K/4]\!]$. Moreover the event Ξ is such that $\Xi \subset \Xi^1$ and satisfies $\mathbb{P}(\Xi \cap \mathcal{G}) \ge 1 - N^{-c_3}$, for N sufficiently large.

The proof of Proposition 5.3 is given in the following subsections.

5.3.1. *Removing the forcing terms* $F_i^{(1)}$

Subtracting (5.11) from (5.7) we have

$$\frac{\mathrm{d}(v_i - \widetilde{v}_i)}{\mathrm{d}t} = -\left[\mathcal{B}(\mathbf{v} - \widetilde{\mathbf{v}})\right]_i + F_i, \quad i \in I, t \in \left[T_1', T_1''\right].$$

Eventually, we will choose $i \in \frac{1}{4}I := [\![L - K/4, L + K/4]\!]$, yet here we can take $i \in I$.

Let $\mathcal{U}_{\mathcal{B}}(t,s)$ denote the time-dependent propagator of the equation (5.11) from time s to t, with $s \le t$. From the Duhamel formula we have

$$v_i(t) - \widetilde{v}_i(t) = \int_{T_1'}^{t} \left(\mathcal{U}_{\mathcal{B}}(t,s)F(s)\right)_i \mathrm{d}s, \quad i \in I, t \ge T_1'.$$

Note that $\mathcal{U}_{\mathcal{B}}$ is a contraction in the sup norm by the maximum principle (recall that $W_i \ge 0$). Thus

$$\left|v_i(t) - \widetilde{v}_i(t)\right| \le \int_{T_1'}^{t} \max_{i \in I} \left|F_i^{(1)}(s)\right| \mathrm{d}s + \int_{T_1'}^{t} \left|\left(\mathcal{U}_{\mathcal{B}}(t,s)F^{(2)}(s)\right)_i\right| \mathrm{d}s, \quad t \ge T_1'. \tag{5.22}$$

Fixing $t = T_1''$, using Lemma 5.1 and the choice of ε in (5.1), we estimate

$$\mathbf{1}\left(\Xi^1 \cap \Xi^2\right) \int_{T_1'}^{T_1''} \max_{i \in I} \left|F_i^{(1)}(s)\right| \mathrm{d}s \le \mathbf{1}\left(\Xi^2\right) C N^{-5C_1} \int_{T_1'}^{T_1''} \sum_{i \in I} \sum_{j \in I} \left|B_{ij}(s)\right| \mathrm{d}s + C\upsilon_L\left(T_1'' - T_1\right)^2, \tag{5.23}$$

where we estimated the maximum by the sum. Thus recalling (5.12) and using (5.23), (5.22) we get

$$\begin{aligned}\mathbf{1}\left(\Xi^1 \cap \Xi^2\right) \int_{T_1'}^{T_1''} \max_{i \in I} \left|F_i^{(1)}(s)\right| \mathrm{d}s &\le C N^{-5C_1} K\left(T_1'' - T_1'\right) N^{1+\rho} + C\upsilon_L\left(T_1'' - T_1\right)^2 \\ &\le C K^{1+c_1} N^{-5C_1+\rho} + C\frac{K^{1+c_1}}{N^2}.\end{aligned}$$

Since $C_1 > 1$, we conclude that the effect due to $F^{(1)}$ is below the precision we are interested in, i.e., there is $c > 0$ such that, for any $i \in I$,

$$\mathbf{1}\left(\Xi^1 \cap \Xi^2\right)\left|v_i\left(T_1''\right) - \widetilde{v}_i\left(T_1''\right)\right| \le C N^{-1-c} + \mathbf{1}\left(\Xi^1 \cap \Xi^2\right) \int_{T_1'}^{T_1''} \left|\left(\mathcal{U}_{\mathcal{B}}\left(T_1'', s\right)F^{(2)}(s)\right)_i\right| \mathrm{d}s. \tag{5.24}$$

5.3.2. *Removing the forcing terms $F_i^{(2,\mathrm{in})}$*
To estimate the influences of the forcing terms $(F_i^{(2)})$, we write

$$F_i^{(2)} = F_i^{(2,\mathrm{in})} + F_i^{(2,\mathrm{out})},$$

with

$$F_i^{(2,\mathrm{in})} := F_i^{(2)}\mathbf{1}\left(i \in \frac{1}{2}I\right), \qquad F_i^{(2,\mathrm{out})} := F_i^{(2)}\mathbf{1}\left(i \in I, i \notin \frac{1}{2}I\right),$$

where we introduced $\frac{1}{2}I := [\![L-K/2, L+K/2]\!]$.

To control the inside part $F_i^{(2,\mathrm{in})}$, we use the following lemma. Recall the definition of the event $\mathcal{G}$ in (4.5) and the definitions of the intervals of consecutive integers I_0 and I_σ in (4.1).

Lemma 5.4. *Let K satisfy (2.27) and fix $\overline{\mathbf{Y}} \in \overline{\mathcal{G}}$. Then we have the following estimates.*

(1) *For all $k \in I_\sigma \setminus I$, we have*

$$\left|\overline{y}_k(t) - \overline{y}_k(T_1)\right| \le C\frac{N^\xi}{N} + CN^\delta(t-T_1)\frac{|L-k|}{N} + CN^\delta(t-T_1)^2, \quad t \in [T_1, t_2]. \tag{5.25}$$

(2) *For all $k \in I_0 \setminus I$, we have*

$$\left|\overline{y}_k(t) - \widetilde{\gamma}_k\right| \le C\frac{N^\xi}{N} + C\frac{N^\xi}{K}\frac{|L-k|}{N}, \quad t \in [T_1, t_2]. \tag{5.26}$$

We complement Lemma 5.4 with the estimate

$$\left|\overline{y}_k(t) - \widetilde{\gamma}_k\right| \le CN^\xi\sqrt{t}, \quad t \ge T_1, k \in I_\sigma^c, \tag{5.27}$$

on $\mathcal{G}$, as follows immediately from the definition of $\mathcal{G}$ in (4.5); cf., Assumption (4) of Theorem 2.1.

Proof of Lemma 5.4. To prove (5.25), we estimate

$$\begin{aligned}\left|\overline{y}_k(t) - \overline{y}_k(T_1)\right| &\le \left|\overline{y}_k(t) - \overline{\gamma}_k(t)\right| + \left|\overline{\gamma}_k(t) - \overline{\gamma}_k(T_1)\right| + \left|\overline{y}_k(T_1) - \overline{\gamma}_k(T_1)\right| \\ &\le C\frac{N^\xi}{N} + \left|\overline{\gamma}_k(t) - \gamma_k(T_1)\right|,\end{aligned} \tag{5.28}$$

on the event $\overline{\mathcal{G}}$, where we used the rigidity bound in (4.2) for $k \in I_\sigma$. Next, we write

$$\overline{\gamma}_k(t) - \gamma_k(T_1) = \gamma_k(t) - \upsilon_L(t-T_1) - \gamma_k(T_1) = \int_{T_1}^t \dot{\gamma}_k(s)\,\mathrm{d}s - \upsilon_L(t-T_1).$$

Then by Lemma 4.3 we have

$$\begin{aligned}\int_{T_1}^t \dot{\gamma}_k(s)\,\mathrm{d}s &= \int_{T_1}^t \dot{\gamma}_L(s)\,\mathrm{d}s + O\big(N^{-1+\delta}(t-T_1)|k-L|\big) \\ &= \dot{\gamma}_L(T_1)(t-T_1) + O\big(N^{-1+\delta}(t-T_1)^2\big) + O\big(N^{-1+\delta}(t-T_1)|k-L|\big).\end{aligned}$$

Thus recalling that $\upsilon_L = \dot{\gamma}_L(T_1)$ by definition, we conclude that

$$\left|\overline{\gamma}_k(t) - \gamma_k(T_1)\right| \le CN^{-1+\delta}(t-T_1)^2 + CN^{-1+\delta}(t-T_1)|k-L|. \tag{5.29}$$

Together with (5.28) this implies (5.25).

To bound the left side of (5.26), we split

$$\big|\overline{y}_k(t)-\widetilde{\gamma}_k\big| \le \big|\overline{y}_k(t)-\overline{\gamma}_k(t)\big| + \big|\overline{\gamma}_k(t)-\gamma_k(T_1)\big| + \big|\gamma_k(T_1)-\widetilde{\gamma}_k\big|. \tag{5.30}$$

Then, using the rigidity from the definition of $\mathcal{G}_t^1$ in (4.2), the first term on the right side can be bounded by CN^{ξ}/N. The second term on the right side is controlled by (5.29). To bound the third term on the right side we apply Corollary 4.6 to find

$$\big|\gamma_k(T_1)-\widetilde{\gamma}_k\big| \le C\frac{N^{\xi}}{N} + C\frac{N^{\xi}|\widetilde{\gamma}_k-\widetilde{\gamma}_L|}{K} + CN^{\delta}|\widetilde{\gamma}_k-\widetilde{\gamma}_L|^2.$$

Recalling that $|\widetilde{\gamma}_k-\widetilde{\gamma}_L|\le CK^5/N$, for $k\in I_0\setminus I$, and using that K satisfies (2.27), we get (5.26). □

We now bound the term $F_i^{(2,\mathrm{in})}$. Abbreviate

$$\widetilde{B}_{ik} := \frac{1}{N(\overline{x}_i-\overline{y}_k+\varepsilon_{ik})(\widetilde{x}_i-\widetilde{\gamma}_k)}, \qquad i\in I, k\in I^c. \tag{5.31}$$

Recall the bound on $\mathbf{1}(\Xi^1)|\overline{x}_i-\widehat{x}_i|$ from Lemma 5.1 and the definition of (ϵ_{ik}) in (5.1). Using Lemma 5.4 and recalling that $t-T_1\le CK^2/N$, we obtain

$$\begin{aligned}\mathbf{1}(\Xi^1)\big|F_i^{(2,\mathrm{in})}(s)\big| &\le C\sum_{k\in I_0\setminus I}\left(\frac{1}{N^{5C_1}}+\frac{N^{\xi}}{N}+\frac{N^{\xi}}{K}\frac{|k-L|}{N}\right)\big|\widetilde{B}_{ik}(s)\big| \\ &\quad + C\sum_{k\in I_\sigma\setminus I_0}\left(\frac{N^{\xi}}{N}+N^{\delta}\frac{K^2}{N}\frac{|L-k|}{N}\right)\big|\widetilde{B}_{ik}(s)\big| \\ &\quad + C\sum_{k\notin I_\sigma}\left(N^{\xi}\frac{K}{\sqrt{N}}\right)\big|\widetilde{B}_{ik}(s)\big|, \quad i\in\frac{1}{2}I, s\in\big[T_1',T_1''\big],\end{aligned} \tag{5.32}$$

where we also used (5.27) together with $T_1''-T_1\le CK^2/N$ to get the last term on the right of (5.32).

To perform the sums over k in the first two terms on the right, we recall (5.31) and we note that there are two constants $c,c'>0$, such that

$$|\overline{y}_k-\overline{x}_i|\ge c|L-k|/N, \qquad |\widetilde{\gamma}_k-\widetilde{x}_i|\ge c'|L-k|/N, \quad k\in I_\sigma\setminus I, \tag{5.33}$$

where we used the rigidity estimate for $\mathbf{y}$ (embodied in $\mathcal{G}$; see (4.5)), the rigidity estimate for $\widetilde{\boldsymbol{\gamma}}$ obtained in Corollary 4.6 and the rigidity estimate for $\mathbf{x}$, $\widetilde{\mathbf{x}}$ obtained in Proposition 4.9, as well as the choice $i\in\frac{1}{2}I=[\![L-K/2,L+K/2]\!]$. The summation over $k\notin I_\sigma$ in the third term is estimated using that $|\overline{y}_k-\overline{x}_i||\widetilde{\gamma}_k-\widetilde{x}_i|\ge c''(\sigma)>0$, $k\notin I_\sigma$. Hence, after summing up the right side of (5.32), we get

$$\begin{aligned}\mathbf{1}(\Xi^1)\big|F_i^{(2,\mathrm{in})}(s)\big| &\le \frac{1}{N^{5C_1-1}K}+\frac{N^{\xi}}{K}+\frac{N^{2\xi}}{K}+CN^{\delta}\frac{N^{\xi}}{NK^3}+C\frac{N^{\xi}K}{\sqrt{N}} \\ &\le C\frac{N^{2\xi}}{K}, \quad i\in\frac{1}{2}I, s\in\big[T_1',t_2\big],\end{aligned} \tag{5.34}$$

where we used $5C_1-1>1$ and that K satisfies (2.27). Thus by (5.34) we have

$$\mathbf{1}(\Xi^1)\int_{T_1'}^{T_1''}\big|F_i^{(2,\mathrm{in})}(s)\big|\,\mathrm{d}s \le \big(T_1''-T_1'\big)\frac{N^{\xi}}{K} = \frac{K^{c_1}}{N}\frac{N^{2\xi}}{K},$$

for all $i \in \frac{1}{2}I$, so this error is below the precision we are interested in: For some $c > 0$ we have that

$$\mathbf{1}(\Xi^1)\big|v_i(T_1'') - \widetilde{v}_i(T_1'')\big| \le C\frac{N^{-c}}{N} + \mathbf{1}(\Xi^1)\int_{T_1'}^{T_1''}\big|\big(\mathcal{U}_{\mathcal{B}}(T_1'', s)F^{(2,\mathrm{out})}(s)\big)_i\big|\,\mathrm{d}s, \quad i \in \frac{1}{2}I. \tag{5.35}$$

The outside part $F^{(2,\mathrm{out})}$ is treated with a finite speed of propagation estimate.

5.3.3. *Removing the forcing term* $F^{(2,\mathrm{out})}$

We first recall the finite of propagation estimate for the propagator $\mathcal{U}_{\mathcal{B}}$. Abbreviate for simplicity $\mathcal{U}(t,s) \equiv \mathcal{U}_{\mathcal{B}}(t,s)$ and denote its kernel by $\mathcal{U}_{ij}(t,s)$, $i, j \in I$. By Lemma 9.6 of [28] there is C such that

$$\big|\mathcal{U}_{ij}(t,s)\big| \le C\frac{K^{1/2}\sqrt{N(t-s)+1}}{|i-j|}, \quad i, j \in I, t \ge s \ge T_1. \tag{5.36}$$

on Ξ^2. We refer to (5.36) as a finite speed of propagation estimate.

Next recall that we want to control

$$\max_{i\in\frac{1}{4}I}\sum_{j\in I}\mathcal{U}_{ij}(t,s)F_j^{(2,\mathrm{out})} = \max_{i\in\frac{1}{4}I}\frac{1}{N}\sum_{j\in I\setminus\frac{1}{2}I}\sum_{k\in I^c}\mathcal{U}_{ij}(t,s)F_{jk}^{(2)}(s),$$

where $T_1' \le s \le t \le T_1''$, and where we have introduced

$$F_{jk}^{(2)}(s) := \left(\frac{\widehat{x}_j - \overline{x}_j + \varepsilon_{jk}}{(\overline{x}_j - \overline{y}_k + \varepsilon_{jk})(\widetilde{x}_j - \widetilde{\gamma}_k)} + \frac{\overline{y}_k - \widetilde{\gamma}_k}{(\overline{x}_j - \overline{y}_k + \varepsilon_{jk})(\widetilde{x}_j - \widetilde{\gamma}_k)}\right)(s).$$

With some large C, we next split the summations over k and j as

$$\begin{aligned}\sum_{j\in I}\mathcal{U}_{ij}(t,s)F_j^{(2,\mathrm{out})} &= \frac{1}{N}\sum_{j\in I\setminus\frac{1}{2}I}\sum_{k\in I^c}\mathbf{1}\big(|j-k| \ge CN^{\xi}\big)\mathcal{U}_{ij}(t,s)F_{jk}^{(2)}(s)\\ &\quad + \frac{1}{N}\sum_{j\in I\setminus\frac{1}{2}I}\sum_{k\in I^c}\mathbf{1}\big(|j-k| < CN^{\xi}\big)\mathcal{U}_{ij}(t,s)F_{jk}^{(2)}(s), \quad i \in \frac{1}{4}I.\end{aligned} \tag{5.37}$$

We start with bounding the first term on the right side of (5.37). On the event Ξ^1, we can bound

$$\begin{aligned}\frac{1}{N}\sum_{k\in I^c}\big|\mathbf{1}\big(|j-k| \ge CN^{\xi}\big)F_{jk}^{(2)}(s)\big| &\le C\sum_{k\in I_0\setminus I}\mathbf{1}\big(|j-k| \ge CN^{\xi}\big)\left(\frac{1}{N^{5C_1}} + \frac{N^{\xi}}{N} + \frac{N^{\xi}}{K}\frac{|k-L|}{N}\right)\big|\widetilde{B}_{jk}(s)\big|\\ &\quad + C\sum_{k\in I_\sigma\setminus I_0}\mathbf{1}\big(|j-k| \ge CN^{\xi}\big)\left(\frac{N^{\xi}}{N} + N^{\delta}\frac{K^2}{N}\frac{|k-L|}{N}\right)\big|\widetilde{B}_{jk}(s)\big|\\ &\quad + C\sum_{k\notin I_\sigma}\mathbf{1}\big(|j-k| \ge CN^{\xi}\big)\frac{N^{\xi}K}{\sqrt{N}}\big|\widetilde{B}_{jk}(s)\big|,\end{aligned} \tag{5.38}$$

here we used (5.36) and the Lemmas 5.1 and 5.4. We further used that $s \le t$, $t - T_1 \le CK^2/N$ by assumption. Thus, summing over k, we get

$$\mathbf{1}(\Xi^1)\frac{1}{N}\sum_{k\in I^c}\big|\mathbf{1}\big(|j-k| \ge CN^{\xi}\big)F_{jk}^{(2)}(s)\big| \le C\frac{N^{2\xi}}{|j-L+K+N^{\xi}|}, \quad j \in I \setminus \frac{1}{2}I,$$

where we used the estimates in (5.33). See (5.32) and (5.34) for a similar estimate. Returning to (5.37) we see that the first term on the right side is bounded as

$$\begin{aligned}\frac{1}{N}\sum_{j\in I\setminus\frac{1}{2}I}\sum_{k\notin I}\mathbf{1}\big(|j-k|\geq CN^{\xi}\big)\big|\mathcal{U}_{ij}(t,s)F^{(2)}_{jk}(s)\big| &\leq C\sum_{j:K/2\leq|j-L|\leq K}\frac{N^{2\xi}K^{1/2}\sqrt{N(t-s)+1}}{|j-L+K/4||j-L+K+N^{\xi}|}\\ &\leq CN^{2\xi}K^{-1/2+c_1/2},\end{aligned} \tag{5.39}$$

on $\Xi^1\cap\Xi^2$, where we used that $t-s\leq K^{c_1}/N$.

It remains to control the second term in (5.37). Similar to (5.38), we have on Ξ^1, for $j\in I\setminus\frac{1}{2}I$, that

$$\begin{aligned}\frac{1}{N}\sum_{k\in I^c}\big|\mathbf{1}\big(|j-k|<CN^{\xi}\big)F^{(2)}_{jk}(s)\big| &\leq C\sum_{k\in I_0\setminus I}\mathbf{1}\big(|j-k|<CN^{\xi}\big)\bigg(\frac{1}{N^{5C_1}}+\frac{N^{\xi}}{N}+\frac{N^{2\xi}}{K}\frac{1}{N}\bigg)\big|\widetilde{B}_{jk}(s)\big|\\ &\leq \frac{CN^{\xi}}{N}\sum_{k\in I_0\setminus I}\mathbf{1}\big(|j-k|<CN^{\xi}\big)\big|\widetilde{B}_{jk}(s)\big|.\end{aligned}$$

We thus have, for $i\in\frac{1}{4}I$,

$$\begin{aligned}&\frac{1}{N}\sum_{k\in I^c}\bigg|\int_{T_1'}^{T_1''}\mathrm{d}s\mathbf{1}\big(|j-k|<CN^{\xi}\big)\mathcal{U}_{ij}\big(T_1'',s\big)F^{(2)}_{jk}(s)\bigg|\\ &\quad\leq C\frac{N^{\xi}}{N}\int_{T_1'}^{T_1''}\mathrm{d}s\sum_{k\in I^c}\sum_{j\in I}\mathbf{1}\big(|j-k|<CN^{\xi}\big)\frac{K^{1/2}\sqrt{N(T_1''-s)+1}}{|i-j|}\big|\widetilde{B}_{jk}(s)\big|\\ &\quad\leq C\frac{N^{2\xi}K^{1/2}}{\sqrt{N}K}\sqrt{T_1''-T_1'}\int_{T_1'}^{T_1''}\mathrm{d}s\sum_{j=L+K-\lfloor CN^{\xi}\rfloor}^{L+K}\big|B_{j,j+1}(s)\big|\\ &\qquad+C\frac{N^{2\xi}K^{1/2}}{\sqrt{N}K}\sqrt{T_1''-T_1'}\int_{T_1'}^{T_1''}\mathrm{d}s\sum_{j=L-K}^{L-K+\lceil CN^{\xi}\rceil}\big|B_{j,j-1}(s)\big|,\end{aligned} \tag{5.40}$$

where we used that $|\widetilde{B}_{jk}|\leq|B_{j,j+1}|$, $k>j$, respectively $|\widetilde{B}_{jk}|\leq|B_{j,j-1}|$, $k<j$, for all $k\in I^c$, $j\in I$. (Here and below we use the convention that, for $j\in I$, $B_{j,L\pm(K+1)}=\widetilde{B}_{j,L\pm(K+1)}$, respectively $B_{j,k}=0$ if $|k|>L+K+1$.) To bound the two terms on the right side of (5.40), we use that the evolution equation (5.11) is "regular" at the space–time points $(L+K,T_1'')$ and $(L-K,T_1'')$.

Lemma 5.5. *There is an event Ξ^3 such that evolution equation (5.11) is regular at the space–time points $(L+K,T_1'')$ and $(L-K,T_1'')$ in the sense that*

$$\mathbf{1}\big(\Xi^3\big)\sup_{s\in\mathcal{T}}\sup_{1\leq M\leq\mathcal{K}}\frac{1}{N^{-1}+|s-T_1''|}\int_s^{T_1''}\frac{1}{M}\sum_{i\in I:|i-L\pm K|\leq M}\big|B_{i,i\pm1}(s)\big|\,\mathrm{d}s\leq N^{1+\rho}. \tag{5.41}$$

Moreover, we have the estimate $\mathbb{P}^{\overline{\mathbf{Y}}}(\Xi^1\cap\Xi^2\cap\Xi^3)\geq 1-N^{-\rho/10}$.

Proof. We can follow almost verbatim the first part of the proof of Lemma 5.2. Using (5.4) and (4.46), we can bound $\mathbb{E}|B_{i,i\pm1}|$ as in (5.14). Then dyadic decompositions around the space–time points $(L\pm K,T_1'')$ combined with applications of Markov inequality yield the claim. $\square$

Next, returning to the estimate in (5.40), we conclude from Lemma 5.5 that the first term on the right can be bounded as

$$\mathbf{1}\big(\Xi^1 \cap \Xi^3\big)\frac{N^{2\xi}K^{1/2}}{\sqrt{N}K}\sqrt{T_1''-T_1'}\int_{T_1'}^{T_1''}\mathrm{d}s\sum_{j=L+K-\lfloor CN^{\xi}\rfloor}^{L+K}\big|B_{j,j+1}(s)\big| \le C\frac{N^{3\xi}}{\sqrt{KN}}\big(T_1''-T_1'\big)^{3/2}N^{1+\rho}$$
$$\le CN^{3\xi}K^{3c_1/2}N^{-1+\rho}.$$

Using the same argument to bound the second term on the right side of (5.40), we conclude that

$$\mathbf{1}\big(\Xi^1 \cap \Xi^3\big)\frac{1}{N}\sum_{k\in I^c}\bigg|\int_{T_1'}^{T_1''}\mathrm{d}s\,\mathbf{1}\big(|j-k|<CN^{\xi}\big)\mathcal{U}_{ij}\big(T_1'',s\big)F_{jk}^{(2)}(s)\bigg| \le C\frac{N^{3\xi+\rho}K^{3c_1/2}}{N}, \quad i\in\frac{1}{4}I. \tag{5.42}$$

Summarizing the estimates above, we can now state the proof of Proposition 5.3.

Proof of Proposition 5.3. Let $\Xi := \Xi^1\cap\Xi^2\cap\Xi^3$. Note that $\mathbb{P}(\Xi)\ge 1-N^{-c_2}$, for any $0<c_2\le\rho$ (with $\rho=\delta+3\xi$). Adding up the estimates (5.42), (5.35) and (5.24), and recalling that K satisfies (2.27) and that $c_1\sim 1/100$, we conclude that there is $c_3\equiv c_3(\xi)>0$ such that (5.21) holds, for $\xi>0$ sufficiently small and N large enough. ☐

5.4. *Conclusion of the proof of Theorem 2.1*

Recall from (5.5) the definition of $(v_i(t))$. Combining (5.21) and (5.13) we obtain

$$\mathbf{1}(\Xi)\big|v_{i+1}(t_2)-v_i(t_2)\big| \le N^{-1-c_4}, \quad |i-L|\le C, \tag{5.43}$$

with some small $c_4>0$. Moreover, we have $\mathbb{P}(\Xi\cap\mathcal{G})\ge 1-N^{-c_5}$, for some $c_5>0$. This exactly proves the following result.

Lemma 5.6. *The gap statistics of $\widehat{\mathbf{x}}(T_1'')$ and $\widetilde{\mathbf{x}}(T_1'')$ for indices near L coincide in the limit of large N.*

Combining this with Lemma 5.1, we need only to understand the local statistics of $\widetilde{\mathbf{x}}$. But by Lemma 4.7, this is the same as the gap statistics of the local equilibrium measure ω_{t_0}. The latter one is universal as we showed in Section 4.6. To conclude the proof of Theorem 2.1, we note that we can integrate over $\mathcal{G}$, as follows from Lemma 4.1 and the assumption that the observable $\mathcal{O}$ is compactly supported. Finally, choosing $T_1\ge t_1$ such that $T_1''=T$, we obtain (2.21). This completes the proof of Theorem 2.1.

Appendix A: Semicircular flow

In this appendix we study the semicircular flow in more detail. In Section A.1 we prove Lemma 4.2 and Lemma 4.3. In Section A.2 we discuss the Assumption (1) of Theorem 2.1 in more detail by arguing that it is satisfied for a large number of random matrix models.

A.1. *Classical flow of the density*

Recall from (2.6) that m_t satisfies

$$m_t(z)=\int_{\mathbb{R}}\frac{\varrho(y)\,\mathrm{d}y}{\mathrm{e}^{-t/2}y-(1-\mathrm{e}^{-t})m_t(z)-z}, \qquad \operatorname{Im} m_t(z)>0, \quad \operatorname{Im} z>0, \tag{A.1}$$

for all $t\ge 0$, and that m_t determines a density ϱ_t via the Stieltjes inversion formula, i.e., $\varrho_t(x)=\frac{1}{\pi}\lim_{\eta\searrow 0}\operatorname{Im} m_t(x+\mathrm{i}\eta)$, $x\in\mathbb{R}$. We call the map $t\mapsto\varrho_t$ the semicircular flow started from ϱ.

It was shown in [8] that the density ϱ_t is a real analytic function inside support for fixed $t > 0$. Yet, without any further assumptions on ϱ, estimates on the derivatives of ϱ_t deteriorate for small t, since the equation (A.1) may lose its stability properties (i.e., the denominator on the right side can become singular). This can, for example, be remedied by imposing the conditions in Assumption (1) of Theorem 2.1. Consider for $\Sigma > 0$ the domain

$$\mathcal{D}_\Sigma := \{z = x + \mathrm{i}\eta \in \mathbb{C} : x \in [E - \Sigma, E + \Sigma], \eta \geq 0\}.$$

Denote by m_0 the Stieltjes transform of ϱ. In accordance with the Assumption (1) of Theorem 2.1, we assume here that m_0 extends to a continuous function on $\mathcal{D}_\Sigma$ and that there is a small $\delta \geq 0$ and a constant C such that

$$\sup_{z\in\mathcal{D}} |m_0(z)| \leq C, \qquad \sup_{z\in\mathcal{D}} |\partial_z^n m_0(z)| \leq C(N^\delta)^n, \quad n = 1, 2. \tag{A.2}$$

Lemma A.1. *Consider the semicircular flow ϱ_t started from ϱ. Let ϱ satisfy Assumption (A.2) with exponent $\delta > 0$ and $\Sigma > 0$. Then there is $C' > 0$, such that $m_t(z)$ is uniformly bounded on $\mathcal{D}_\Sigma$, for all $0 \leq t \leq CN^{-2\delta}$. Moreover, there are constants C, C', depending only on ϱ, such that*

$$\sup_{z\in\mathcal{D}_{\Sigma/2}} |m_t(z) - m_0(z)| \leq CtN^\delta, \tag{A.3}$$

for all $0 \leq t \leq C'N^{-2\delta}$. Further, there are constants C, C', depending only on ϱ, such that we have the bounds

$$\sup_{z\in\mathcal{D}_{\Sigma/2}} |\partial_z^n m_t(z)| \leq C(N^\delta)^n, \quad n = 1, 2, 0 \leq t \leq C'N^{-2\delta}. \tag{A.4}$$

Proof. Set $\sigma_t := 1 - \mathrm{e}^{-t}$. Starting from (A.1) we obtain, for $t > 0$,

$$|m_t(z)|^2 \leq \left(\int_{\mathbb{R}} \frac{\varrho(y)\,\mathrm{d}y}{|\mathrm{e}^{-t/2}y - z - \sigma_t m_t(z)|^2}\right) = \frac{\operatorname{Im} m_t(z)}{\eta + \sigma_t \operatorname{Im} m_t(z)}, \quad z \in \mathbb{C}^+,$$

where we first used Schwarz inequality for the probability measure ϱ to get the second line. Then we used once more (A.1). We thus obtain the rough a priori bound

$$|m_t(z)| \leq \sigma_t^{-1/2}, \quad t > 0, \tag{A.5}$$

for $z \in \mathbb{C}^+ \cup \mathbb{R}$. Next, we introduce

$$\widetilde{m}_t(z) := \int_{\mathbb{R}} \frac{\varrho(\mathrm{e}^{t/2}v)\mathrm{e}^{t/2}\,\mathrm{d}v}{v - z}. \tag{A.6}$$

Note that $\widetilde{m}_t$ is uniformly bounded on, say, $\mathcal{D}_{\Sigma/2}$ for, say, $t \leq 1$. This may be seen by writing $\widetilde{m}_t(z) = \mathrm{e}^{t/2}m_0(\mathrm{e}^{t/2}z)$. Then we can write

$$m_t(z) = \widetilde{m}_t(z + \sigma_t m_t(z)).$$

Thus, using the estimates on m_0 in (A.2), we have

$$|m_t(z) - \widetilde{m}_t(z)| = |\widetilde{m}_t(z + \sigma_t m_t(z)) - \widetilde{m}_t(z))| \leq CN^\delta \sigma_t |m_t(z)| \leq CN^\delta \sigma_t^{1/2}, \tag{A.7}$$

$0 < t \leq 1$, on $\mathcal{D}_{\Sigma/2}$, where we used the a priori bound (A.5). It follows that

$$|m_t(z)| \leq |\widetilde{m}_t(z)| + CN^\delta \sigma_t^{1/2} \leq C, \quad 0 < t \leq CN^{-2\delta}.$$

But then reasoning once more as in (A.7), we must have

$$|m_t(z) - \widetilde{m}_t(z)| \leq |\widetilde{m}_t(z + \sigma_t m_t(z)) - \widetilde{m}_t(z))| \leq C\sigma_t N^\delta |m_t(z)| \leq C\sigma_t N^\delta, \tag{A.8}$$

$0 < t \le CN^{-2\delta}$, for all $z \in \mathcal{D}_{\Sigma/2}$. We hence obtain that $|m_t(z)| \le C$ on $\mathcal{D}_\Sigma$ and $0 < t \le CN^{-2\delta}$. Next, we observe that

$$\left|\widetilde{m}_t(z) - m_0(z)\right| = \left|\mathrm{e}^{t/2} m_0\big(\mathrm{e}^{-t/2} z\big) - m_0(z)\right| \le C\big(\mathrm{e}^{t/2} - 1\big) + CN^\delta\big(\mathrm{e}^{t/2} - 1\big). \tag{A.9}$$

Combining (A.8) and (A.9), (A.3) follows since $t \le C' N^{-2\delta}$ by assumption.

To deal with the derivatives of $m_t(z)$, we first note that we for $z \in \mathcal{D}_{\Sigma/2}$, we have $\mathrm{e}^{t/2}(z + \sigma_t m_t(z)) \in \mathcal{D}_\Sigma$, for $t \le 1$. Thus we can bound, for $z \in \mathcal{D}_{\Sigma/2}$ and $t \le N^{-2\delta}$,

$$\left|\sigma_t \int_{\mathbb{R}} \frac{\varrho(v)\,\mathrm{d}v}{(\mathrm{e}^{-t/2} v - z - \sigma_t m_t(z))^2}\right| = \sigma_t \left|(\partial_z \widetilde{m})\big(z + \sigma_t m_t(z)\big)\right| \le CN^\delta \sigma_t \le CN^{-\delta}, \tag{A.10}$$

where we used the definition of $\widetilde{m}(z)$ in (A.6) and the assumptions in (A.2).

Next, differentiating (A.1) with respect to z, we obtain

$$\big(\partial_z m_t(z)\big)\left(1 - \sigma_t \int_{\mathbb{R}} \frac{\varrho(v)\,\mathrm{d}v}{(\mathrm{e}^{-t/2} v - z - \sigma_t m_t(z))^2}\right) = \int_{\mathbb{R}} \frac{\varrho(v)\,\mathrm{d}v}{(\mathrm{e}^{-t/2} v - z - \sigma_t m_t(z))^2}.$$

Hence, using twice (A.10), we get $|\partial_z m_t(z)| \le CN^\delta$, for $z \in \mathcal{D}_{\Sigma/2}$ and $0 \le t \le CN^{-2\delta}$. Repeating the argument, we see that there is another constant C such that $|\partial_z^2 m_t(z)| \le CN^{2\delta}$, $0 \le t \le N^{-2\delta}$, for all $z \in \mathcal{D}_{\Sigma/2}$. This proves (A.4). □

A.1.1. *Quantiles*

From (A.4), we see that the derivatives of $m_t(z)$ are bounded inside $\mathcal{D}_{\Sigma/2}$ for $t \ll 1$. Without further assumptions on ϱ we have little control on $m_t(z)$ outside $\mathcal{D}_{\Sigma/2}$. Yet, we can circumvent this problem, by introducing a regularization of $m_t(z)$, respectively the measure ϱ_t, as follows. Throughout the rest of this appendix, let $\eta_* > 0$ satisfy

$$\eta_* N^\delta \ll \frac{1}{N}. \tag{A.11}$$

Recall the definition of the Poisson kernel $P_\cdot$ in (2.2). We then set

$$\varrho_t^{\eta_*}(x) := (P_{\eta_*} * \varrho_t)(x) = \frac{1}{\pi} \operatorname{Im} m_t(x + \mathrm{i}\eta_*). \tag{A.12}$$

We claim that

$$\int_{\mathbb{R}} \frac{\varrho_t^{\eta_*}(y)\,\mathrm{d}y}{y - z} = m_t(z + \mathrm{i}\eta_*), \quad z \in \mathbb{C}^+. \tag{A.13}$$

Indeed, using (2.3) and (A.12), we have

$$\frac{1}{\pi} \operatorname{Im} \int_{\mathbb{R}} \frac{\varrho_t^{\eta_*}(y)\,\mathrm{d}y}{y - E - \mathrm{i}\eta} = \big(P_\eta * \varrho_t^{\eta_*}\big)(E) = (P_{\eta+\eta_*} * \varrho_t)(E) = \frac{1}{\pi} \operatorname{Im} m_t(E + \mathrm{i}\eta + \mathrm{i}\eta_*),$$

where we used $P_\eta * P_{\eta_*} = P_{\eta+\eta_*}$. Since $z = E + \mathrm{i}\eta$ and since the Stieltjes transform is analytic in the upper half plane, we get (A.13). In the following we write $m_t^{\eta_*}(z) := m_t(z + \mathrm{i}\eta_*)$. Note that $\varrho_t^{\eta_*}$ is a probability measure. It follows from basic properties of the Poisson kernel that $\varrho_t^{\eta_*}$ converges uniformly on compact sets to ϱ_t as $\eta_* \searrow 0$. Using (A.2) it is then easy to check that $|\varrho_0^{\eta_*}(x) - \varrho_0(x)| \le CN^\delta \eta_* \ll N^{-1}$, for all $x \in [E_* - \Sigma, E_* + \Sigma]$. Since the semicircular flow preserve regularity (for short times see Lemma A.1), we also get $|\varrho_t^{\eta_*}(x) - \varrho_t(x)| \le CN^\delta \eta_* \ll N^{-1}$, for all $x \in [E_* - \Sigma/2, E_* + \Sigma/2]$, $0 \le t \le N^{-2\delta}$.

As a consequence of the regularization in (A.12), $\varrho_t^{\eta_*}$ is smooth with bounded derivatives (in terms of inverse powers of η_*) that all lie in $\mathrm{L}^p(\mathbb{R})$, $1 \le p \le \infty$. Consequently, the following basic properties of the Hilbert transform can be justified easily (see, e.g., [46]): For $n \in \mathbb{N}$,

$$\partial^n \big(\mathrm{T}\varrho_t^{\eta_*}\big) = (-1)^n \big(\mathrm{T}\big(\partial^n \varrho_t^{\eta_*}\big)\big) \tag{A.14}$$

(here ∂^n denotes the nth spatial derivative). Further, we have $\mathrm{T}(\mathrm{T}\varrho_t^{\eta_*}) = -\varrho_t^{\eta_*}$.

Next, we define the continuous quantile, $\gamma_w^{\eta_*}(t)$, $w \in [0, N]$, of the measure $\varrho_t^{\eta_*}$ by

$$\int_{-\infty}^{\gamma_w^{\eta_*}(t)} \varrho_t^{\eta_*}(y)\,\mathrm{d}y = \frac{w}{N}, \quad \varrho_0^{\eta_*} = \varrho^{\eta_*}, \tag{A.15}$$

cf., (2.9). Note that $\gamma_w^{\eta_*}(t)$ is defined for any w by (A.15), since the measure $\varrho_t^{\eta_*}$ is supported on the whole real axis. The measure ϱ_t (without regularization) may be supported on several disjoint intervals. This leads to some ambiguities in the definition of the quantiles, cf., (2.9) for one way of resolving the ambiguity. Using the regularized density $\varrho_t^{\eta_*}$ is another way of avoiding this ambiguity. Nonetheless, we emphasize that the η_*-regularization is simply a technical tool: for every practical purpose we have $\eta_* = 0$ and the reader may forget about it in the subsequent arguments.

Corollary A.2. *Under the assumption of Lemma A.1, the following holds. For $0 \le t \le C'N^{-2\delta}$, with C' sufficiently small, we have the estimates*

$$\left|\varrho_t^{\eta_*}(E) - \varrho^{\eta_*}(E)\right| \le CN^\delta t, \qquad \left|\left(\mathrm{T}\varrho_t^{\eta_*}\right)(E) - \left(\mathrm{T}\varrho^{\eta_*}\right)(E)\right| \le CN^\delta t, \tag{A.16}$$

for all $E \in \mathbb{R}$. Further, for w such that $\gamma_w^{\eta_*}(0) \in [E_* - \Sigma, E_* + \Sigma]$, we have the estimate

$$\left|\gamma_w^{\eta_*}(t) - \gamma_w^{\eta_*}(0)\right| \le CN^{-\delta/2}, \quad 0 \le t \le N^{-2\delta}, \tag{A.17}$$

for some C. In particular, if $\gamma_w^{\eta_*}(0) \in [E_* - \Sigma/4, E_* + \Sigma/4]$, then $\gamma_w^{\eta_*}(t) \in [E_* - \Sigma/2, E_* + \Sigma/2]$, for all $0 \le t \le C'N^{-2\delta}$.

Proof. The estimates in (A.16) follow from (A.3) by noticing that $\varrho_t^{\eta_*}(E) = \pi^{-1}\operatorname{Im} m_t(E + \mathrm{i}\eta_*)$, respectively $(\mathrm{T}\varrho_t^{\eta_*})(E) = \operatorname{Re} m_t(E + \mathrm{i}\eta_*)$. To establish (A.17), we note that, by definition,

$$\int_{-\infty}^{\gamma_w^{\eta_*}(t)} \varrho_t^{\eta_*}(y)\,\mathrm{d}y = \int_{-\infty}^{\gamma_w^{\eta_*}(0)} \varrho^{\eta_*}(y)\,\mathrm{d}y = \frac{w}{N}.$$

Thus, using (A.16), we get

$$\int_{-\infty}^{\gamma_w^{\eta_*}(t)} \varrho^{\eta_*}(y)\,\mathrm{d}y = \int_{-\infty}^{\gamma_w^{\eta_*}(0)} \varrho^{\eta_*}(y)\,\mathrm{d}y + O\left(\sqrt{N^\delta t}\right), \tag{A.18}$$

where we also used that ϱ has finite second moment. By our assumption on w we must have $\varrho^{\eta_*}(\gamma_w^{\eta_*}(0)) \ge c$, for some $c > 0$. We thus get from (A.18) and (A.2) that

$$\varrho^{\eta_*}\left(\gamma_w^{\eta_*}(0)\right)\left|\gamma_w^{\eta_*}(t) - \gamma_w^{\eta_*}(0)\right| \le C\left|\int_{-\infty}^{\gamma_w^{\eta_*}(t)} \varrho^{\eta_*}(y)\,\mathrm{d}y - \int_{-\infty}^{\gamma_w^{\eta_*}(0)} \varrho^{\eta_*}(y)\,\mathrm{d}y\right| \le C\left(N^\delta t\right)^{1/2}.$$

Thus, for $0 \le t \le CN^{-2\delta}$, we get $|\gamma_w^{\eta_*}(t) - \gamma_w^{\eta_*}(0)| \le CN^{-\delta/2}$. □

The estimate (A.17) will serve as a priori bound below. To get precise estimates, we derive next the equation of motion of $\gamma_w^{\eta_*}(t)$ under the semicircular flow $t \to \varrho_t^{\eta_*}$.

Lemma A.3. *For $t > 0$, we have for all $w \in [0, N]$,*

$$\frac{\mathrm{d}\gamma_w^{\eta_*}(t)}{\mathrm{d}t} = -\left(\mathrm{T}\varrho_t^{\eta_*}\right)\left(\gamma_w^{\eta_*}(t)\right) - \frac{\gamma_w^{\eta_*}(t)}{2} - \frac{\eta_*}{2}\frac{(\mathrm{T}\varrho_t^{\eta_*})(\gamma_w^{\eta_*}(t))}{\varrho_t^{\eta_*}(\gamma_w^{\eta_*}(t))}, \tag{A.19}$$

and

$$\frac{\mathrm{d}\gamma_w^{\eta_*}(t)}{\mathrm{d}w} = \frac{1}{N\varrho_t^{\eta_*}(\gamma_w^{\eta_*}(t))}. \tag{A.20}$$

In particular, if $\varrho_t^{\eta_*}(\gamma_w^{\eta_*}(t)) \geq c$*, for some fixed* $c > 0$*, we have the uniform estimates*

$$\frac{\mathrm{d}\gamma_w^{\eta_*}(t)}{\mathrm{d}t} = -\big(\mathrm{T}\varrho_t^{\eta_*}\big)\big(\gamma_w^{\eta_*}(t)\big) - \frac{\gamma_w^{\eta_*}(t)}{2} + O(\eta_*), \qquad \frac{\mathrm{d}\gamma_w^{\eta_*}(t)}{\mathrm{d}w} = O\big(N^{-1}\big). \tag{A.21}$$

Remark A.4. Lemma A.3 directly controls $\gamma_w^{\eta_*}(t)$ in the bulk. With some more effort the third term on the right of (A.19) can by controlled at the edges. For example, assuming that ϱ_t vanishes as a square root at, say, its lowest endpoint, it can also be shown that $\varrho_t^{\eta_*}(\gamma_w^{\eta_*}(t)) \gtrsim \sqrt{\eta_*}$, for small w. Thus the "error" term in (A.19) is of order $\sqrt{\eta_*}$, as $\eta_* \searrow 0$, at the lowest edge of the density $\varrho_t^{\eta_*}$.

Proof. We recall that $m_t(z)$, $z \in \mathbb{C}^+$, defined in (A.1), satisfies the following complex Burgers' equation [47] (see also [54])

$$\frac{\mathrm{d}m_t(z)}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\big(m_t(z)\big(m_t(z) + z\big)\big), \quad z \in \mathbb{C}^+, t \geq 0. \tag{A.22}$$

Indeed differentiating (A.1) with respect to time we obtain (A.22) after a series of elementary manipulations. We use (A.22) with $m_t^{\eta_*}(z) = m_t(z + \mathrm{i}\eta_*)$ replacing $m_t(z)$ in the following.

To deal with the right side of (A.22), we note that

$$m_t^{\eta_*}(z)^2 = \int_{\mathbb{R}^2} \frac{\varrho_t^{\eta_*}(x)\,\mathrm{d}x}{x - z}\frac{\varrho_t^{\eta_*}(y)\,\mathrm{d}y}{y - z} = 2\int_{\mathbb{R}^2} \frac{\varrho_t^{\eta_*}(x)\,\mathrm{d}x}{x - z}\frac{\varrho_t^{\eta_*}(y)\,\mathrm{d}y}{y - x} = 2\int_{\mathbb{R}} \frac{(\mathrm{T}\varrho_t^{\eta_*})(x)\varrho_t^{\eta_*}(x)\,\mathrm{d}x}{x - z}, \tag{A.23}$$

$z \in \mathbb{C}^+$. Plugging (A.23) into (A.22), we get

$$\frac{\mathrm{d}m_t^{\eta_*}(z)}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\bigg(\int_{\mathbb{R}} \frac{2(\mathrm{T}\varrho_t^{\eta_*})(x)\varrho_t^{\eta_*}(x)\,\mathrm{d}x}{x - z} + \int_{\mathbb{R}} \frac{x\varrho_t^{\eta_*}(x)\,\mathrm{d}x}{x - z} + \int_{\mathbb{R}} \frac{\mathrm{i}\eta_*\varrho_t^{\eta_*}(x)}{x - z}\bigg), \quad z \in \mathbb{C}^+,$$

where we used that $\int_{\mathbb{R}} \varrho_t^{\eta_*}(x)\,\mathrm{d}x = 1$. Differentiating the right side with respect to z, we get

$$\frac{\mathrm{d}m_t^{\eta_*}(z)}{\mathrm{d}t} = -\frac{1}{2}\int_{\mathbb{R}} \frac{(2(\mathrm{T}\varrho_t^{\eta_*})(x) + x + \mathrm{i}\eta_*)\varrho_t^{\eta_*}(x)\,\mathrm{d}x}{(x - z)^2}, \quad z \in \mathbb{C}^+.$$

Integrating by parts in x, taking the imaginary part and the limit $\eta \searrow 0$ (with $\eta_* > 0$), we obtain

$$\dot{\varrho}_t^{\eta_*}(E) = \frac{1}{2}\big[\big(2\big(\mathrm{T}\varrho_t^{\eta_*}\big)(E) + E\big)\varrho_t^{\eta_*}(E)\big]' + \frac{\eta_*}{2}\big[\big(\mathrm{T}\varrho_t^{\eta_*}\big)(E)\big]', \tag{A.24}$$

where we use the notation $\dot{a} \equiv \partial_t a$ and $a' \equiv \partial_E a$, for any function $a \equiv a(t, E)$. On the other hand, differentiating the defining equation (A.15) of $\gamma_w^{\eta_*}(t)$ with respect to t, we get

$$\dot{\gamma}_w^{\eta_*}(t) = -\frac{1}{\varrho_t^{\eta_*}(\gamma_w^{\eta_*}(t))}\int_{-\infty}^{\gamma_w^{\eta_*}(t)} \dot{\varrho}_t^{\eta_*}(y)\,\mathrm{d}y. \tag{A.25}$$

Hence, combining (A.25) and (A.24) we get (A.19).

Finally, to establish the first estimate in (A.21), we note that $(\mathrm{T}\varrho_t^{\eta_*})$ is uniformly bounded by Lemma A.1. Together with the assumption $\varrho_t^{\eta_*}(\gamma_w^{\eta_*}(t)) \geq c > 0$, (A.21) follows. To prove (A.20) and the second estimate in (A.21) it suffices to differentiate (A.15) with respect to w. □

Remark A.5. For $\varrho \in \mathcal{M}(\mathbb{R})$, let

$$\mathrm{Ent}[\varrho] := \int_{\mathbb{R}} \frac{1}{2}x^2\varrho(x)\,\mathrm{d}x - \int_{\mathbb{R}} \log|x - y|\,\mathrm{d}\varrho(x)\,\mathrm{d}\varrho(y), \tag{A.26}$$

Voiculescu's free entropy. Then the limiting equation of (A.24), i.e., as $\eta_* \searrow 0$, is the gradient flow of $\mathrm{Ent}[\varrho_t]$ on the Wasserstein space $\mathcal{P}_2(\mathbb{R})$; see [9,10,41].

To conclude this subsection, we give an estimate on the second derivative $\ddot{\gamma}_w^{\eta_*}(t)$ inside the bulk.

Lemma A.6. *Under the assumptions of Lemma A.1 the following holds. For $w \in [0, N]$, such that $\gamma_w^{\eta_*}(0) \in [E_* - \Sigma/4, E_* + \Sigma/4]$ we have*

$$\left|\ddot{\gamma}_w^{\eta_*}(t)\right| \le CN^\delta\big(1 + \left|\dot{\gamma}_w^{\eta_*}(t)\right|\big), \quad 0 < t \le C'N^{-2\delta}. \tag{A.27}$$

Proof. Let $0 < t \le C'N^{-2\delta}$. For notational simplicity, we abbreviate here $\gamma_{w,t} \equiv \gamma_w^{\eta_*}(t)$ and $\varrho_t \equiv \varrho_t^{\eta_*}$. We first compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mathrm{T}\varrho_t)(\gamma_{w,t})\big] &= (\mathrm{T}\dot{\varrho}_t)(\gamma_{w,t}) + \big[(\mathrm{T}\varrho_t)\big]'(\gamma_{w,t})\dot{\gamma}_{w,t} \\ &= -\frac{1}{2}\big[\mathrm{T}\big(\big(2(\mathrm{T}\varrho_t)(\cdot) + \cdot\big)\varrho_t(\cdot)\big)\big]'(\gamma_{w,t}) - \frac{\eta_*}{2}\big[\mathrm{T}(\mathrm{T}\varrho_t)\big]'(\gamma_{w,t}) + [\mathrm{T}\varrho_t]'(\gamma_{w,t})\dot{\gamma}_{w,t},\end{aligned}$$

where we used (A.24) and (A.14). (Here $\mathrm{T}((2(\mathrm{T}\varrho_t)(\cdot) + \cdot)\varrho_t(\cdot))(x)$ denotes the Hilbert transform of the function $y \to ((\mathrm{T}\varrho_t)(y) + y)\varrho_t(y)$ evaluated at x.) Next, we note the identities

$$\frac{1}{2}(\mathrm{T}\varrho_t)^2 - \frac{1}{2}\varrho_t^2 = \mathrm{T}\big((\mathrm{T}\varrho_t)\varrho_t\big),$$

which follows from (A.23) and $(\mathrm{T}(\varrho_t(\cdot)\cdot))(x) = 1 + x(\mathrm{T}\varrho_t)(x)$, $x \in \mathbb{R}$, which can be checked by hand. We hence obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{T}\varrho_t)(\gamma_{w,t}) = &-\frac{1}{2}\big[(\mathrm{T}\varrho_t)^2\big]'(\gamma_{w,t}) + \frac{1}{2}\big[\varrho_t^2\big]'(\gamma_{w,t}) - \frac{1}{2}\big[\cdot(\mathrm{T}\varrho_t)(\cdot)\big]'(\gamma_{w,t}) \\ &+ [\mathrm{T}\varrho_t]'(\gamma_{w,t})\dot{\gamma}_{w,t} + \frac{\eta_*}{2}[\varrho_t]'(\gamma_{w,t}),\end{aligned}$$

where we also used that $(\mathrm{T}(\mathrm{T}\varrho_t)) = -\varrho_t$. Simplifying further and using (A.19), we eventually obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mathrm{T}\varrho_t)(\gamma_{w,t})\big] = &-\frac{1}{2}\big[\varrho_t^2\big]'(\gamma_{w,t}) + \frac{1}{2}\big[(\mathrm{T}\varrho_t)\big](\gamma_{w,t}) + 2\big[(\mathrm{T}\varrho_t)\big]'(\gamma_{w,t})\dot{\gamma}_{w,t} \\ &+ \frac{\eta_*}{2}[\varrho_t]'(\gamma_{w,t}) - \frac{\eta_*}{2}\left[\frac{(\mathrm{T}\varrho_t)'(\mathrm{T}\varrho_t)}{\varrho_t}\right](\gamma_{w,t}).\end{aligned}$$

We further compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left[\frac{(\mathrm{T}\varrho_t)}{\varrho_t}\right](\gamma_{w,t})\right] = &\left[\frac{\partial_t(\mathrm{T}\varrho_t)}{\varrho_t}\right](\gamma_{w,t}) - \left[\frac{(\mathrm{T}\varrho_t)\dot{\varrho}_t}{\varrho_t^2}\right](\gamma_{w,t}) \\ &+ \left[\frac{[(\mathrm{T}\varrho_t)]'}{\varrho_t}\right](\gamma_{w,t})\dot{\gamma}_{w,t} - \left[\frac{(\mathrm{T}\varrho_t)\varrho_t'}{\varrho_t^2}\right](\gamma_{w,t})\dot{\gamma}_{w,t}.\end{aligned} \tag{A.28}$$

We next recall that we have the bounds $|\varrho_t(\gamma_{w,t})| + |(\mathrm{T}\varrho_t)(\gamma_{w,t})| \le C$, as follows from the boundedness of m_t (see Lemma A.1), and

$$\left|\partial_x \operatorname{Im} m_t(x + \mathrm{i}\eta_*)\right| + \left|\partial_x \operatorname{Re} m_t(x + \mathrm{i}\eta_*)\right| \le CN^\delta, \tag{A.29}$$

for any $x \in [E_* - \Sigma/2, E_* + \Sigma/2]$, see (A.4). Thus, recalling that $(\mathrm{T}\varrho_t)^{\eta_*}(E) = \operatorname{Re} m_t(E + \mathrm{i}\eta_*)$, $\pi\varrho_t^{\eta_*}(E) = \operatorname{Im} m_t(E + \mathrm{i}\eta_*)$, we find

$$\left|(\mathrm{T}\varrho_t)'(\gamma_{w,t})\right| + \left|\varrho_t'(\gamma_{w,t})\right| \le CN^\delta, \qquad \left|\dot{\varrho}_t(\gamma_{w,t})\right| \le CN^\delta.$$

Hence, differentiating (A.19) with respect to t and using (A.28), (A.29), we can bound

$$|\ddot{\gamma}_{w,t}| \le \frac{1}{2}|\dot{\gamma}_{w,t}| + C\bigg(1+\eta_* + \frac{\eta_*}{\varrho_t(\gamma_{w,t})} + \frac{\eta_*}{(\varrho_t(\gamma_{w,t}))^2}\bigg)\big(1 + CN^\delta + N^\delta|\dot{\gamma}_{w,t}|\big).$$

Finally, using that $\varrho_t(\gamma_{w,0}) \ge c/2$, for $0 \le t \le CN^{-2\delta}$, as follows from the estimates (A.16) and (A.17), and our assumption $\varrho(\gamma_{w,0}) \ge c > 0$, we immediately (A.27). □

Corollary A.7. *Under the assumptions of Lemma A.6 the following holds true. Let $w, w_0 \in [0, N]$ such that $\gamma_w(0), \gamma_{w_0}(0) \in [E_* - \Sigma/4, E_* + \Sigma/4]$. Then we have the estimates*

$$\big|\dot{\gamma}_{w_0}^{\eta_*}(t)\big| \le C, \qquad \big|\ddot{\gamma}_{w_0}^{\eta_*}(t)\big| \le CN^\delta, \tag{A.30}$$

and

$$\big|\dot{\gamma}_{w}^{\eta_*}(t) - \dot{\gamma}_{w_0}^{\eta_*}(t)\big| \le CN^\delta\frac{|w-w_0|}{N} + C\eta_*, \qquad \big|\dot{\gamma}_{w_0}^{\eta_*}(t) - \dot{\gamma}_{w_0}^{\eta_*}(0)\big| \le CN^\delta t, \tag{A.31}$$

uniformly in $0 \le t \le CN^{-2\delta}$, with constants depending only on δ, ϱ and E and Σ.

Proof. Since $\gamma_w(0), \gamma_{w_0}(0) \in [E_* - \Sigma/4, E_* + \Sigma/4]$, we have by Corollary A.2 that $\gamma_w(t), \gamma_{w_0}(t) \in\in [E_* - \Sigma/2, E_* + \Sigma/2]$, for $t \le CN^{-2\delta}$, in particular we have $\varrho_t(\gamma_w(t)), \varrho_t(\gamma_{w_0}(t)) \ge c > 0$ for such t.

Recalling the identity $(\mathrm{T}\varrho_{t_0}^{\eta_*})(E) = \operatorname{Re} m_{t_0}(E + \mathrm{i}\eta_*)$ (as follows from (A.13) and (2.5)), and the estimates on m_t, $\partial_z m_t$ derived in Lemma A.1, we conclude from that from (A.19) and (A.20) that

$$\big|\dot{\gamma}_{w_0}(t)\big| \le C, \qquad \big|\dot{\gamma}_{w}^{\eta_*}(t) - \dot{\gamma}_{w_0}^{\eta_*}(t)\big| \le CN^{-1+\delta}|w-w_0| + C\eta_*,$$

for all $0 \le t \le CN^{-2\delta}$. We further get from (A.27) that

$$\big|\ddot{\gamma}_{w_0}^{\eta_*}(t)\big| \le CN^\delta, \qquad \big|\dot{\gamma}_{w_0}^{\eta_*}(t) - \dot{\gamma}_{w_0}^{\eta_*}(0)\big| \le CN^\delta t.$$

This proves (A.31). □

A.1.2. *Proofs of Lemma 4.2 and of Lemma 4.3*

Proof of Lemma 4.2. Without lost of generality, we can assume that $t_1 = 0$. Fix $N \in \mathbb{N}$. From Lemma A.1, we directly get $|\varrho_t(x) - \varrho(t)| \ll CN^\delta t$. Thus for $t \le C'N^{-2\delta}$, we get the first estimate in (4.8). Next, we note that first and second derivative of $m_t(z)$, $z = E + \mathrm{i}\eta$, are bounded on $\mathcal{D}_{\Sigma/2}$ by (A.4). Thus the first derivative converges uniformly as $\eta \searrow 0$, $E \in [E_* - \Sigma/2, E_* + \Sigma/2]$, and we have $\pi\partial_E\varrho_t(E) = \lim_{\eta\searrow 0} \operatorname{Im} m_t(E + \mathrm{i}\eta)$. In particular, we obtain from (A.4) the second estimate in (4.8). □

Proof of Lemma 4.3. Without lost of generality, we can assume that $t_1 = 0$ here. Let $\eta_* > 0$ as in (A.11). We then recall that we have $|\varrho_t^{\eta_*}(x) - \varrho_t| \le CN^\delta\eta_*$, for all $x \in [E_* - \Sigma, E_* + \Sigma]$ and $0 \le t \le C'N^{-2\delta}$. This follows directly from the definition of the Poisson kernel in (2.2) and Lemma A.1. Thus, using the definition of $\gamma_w^{\eta_*}(t)$ in (A.15), we must have $|\gamma_i^{\eta_*}(t) - \gamma_i(t)| \le CN^\delta\eta_*$, $i \in I_\sigma$. By Assumption (1) of Theorem 2.1, $m_0(z)$ extends to a continuous function on $\mathcal{D}_\Sigma$. Thus reasoning as in the proof of Lemma A.1, we conclude that $m_t(z)$ extends to a continuous function on $\mathcal{D}_{\Sigma/2}$ for $0 \le t \le C'N^{-2\delta}$. Hence, considering now η_* as a free parameter (not depending on N) and taking $\eta_* \searrow 0$, we conclude from (A.19) that

$$\lim_{\eta_*\searrow 0}\frac{\mathrm{d}\gamma_i^{\eta_*}(t)}{\mathrm{d}t} = -\int_{\mathbb{R}}\frac{\varrho_t(y)\,\mathrm{d}y}{y-\gamma_i(t)} - \frac{\gamma_i(t)}{2}, \quad i \in I_\sigma, \tag{A.32}$$

for all $i \in I_\sigma$ and $0 \le t \le C'N^{-2\delta}$. Here, we also used that $\varrho_t^{\eta_*}(\gamma_i^{\eta_*}(t)) > 0$, $\varrho_t(\gamma_i(t)) > 0$. Further, since $\dot{\gamma}_i^{\eta_*}(t)$ converges uniformly to $\dot{\gamma}_i(t)$ for all $0 \le t \le C'N^{-2\delta}$, we can also exchange derivative and limit on the left side of

(A.32) and we obtain (4.11). In particular, we have $\lim_{\eta_*\searrow 0}\dot{\gamma}_i^{\eta_*}(t)=\dot{\gamma}_i(t)$, $i\in I_\sigma$, $0\le t\le C'N^{-2\delta}$. Thus (4.13) follows from (A.31).

Now we show (4.12). For any fixed t we define the "continuous" quantiles

$$\int_{-\infty}^{\gamma_u(t)}\varrho_t(x)\,\mathrm{d}x=\frac{u}{N},\quad u\in[0,N],\tag{A.33}$$

and also the "half-quantiles" $\widehat{\gamma}_i(t):=\gamma_{i-1/2}(t)$ for any integer i. Since t is fixed throughout the proof, we drop the t argument. From (A.33) we get the regularity of the continuous quantiles:

$$\frac{\mathrm{d}\gamma_u}{\mathrm{d}u}=\frac{1}{N\varrho(\gamma_u)}=O\bigl(N^{-1}\bigr),\qquad \frac{\mathrm{d}^2\gamma_u}{\mathrm{d}u^2}=-\frac{\varrho'(\gamma_u)}{N^2\varrho(\gamma_u)^3}=O\bigl(N^{-2+\delta}\bigr),\tag{A.34}$$

in the bulk regime, where we used (4.8).

Setting $j=\ell(i)$ for brevity, we can write

$$\int_{\mathbb{R}}\frac{\varrho_t(y)\,\mathrm{d}y}{y-\gamma_{\ell(i)}}=\biggl[\int_{-\infty}^{\widehat{\gamma}_{j-\sigma N}}+\int_{\widehat{\gamma}_{j-\sigma N}}^{\widehat{\gamma}_{j+\sigma N+1}}+\int_{\widehat{\gamma}_{j+\sigma N+1}}^{\infty}\biggr]\frac{\varrho(y)\,\mathrm{d}y}{y-\gamma_j}.\tag{A.35}$$

The first integral can be written as (with $\widehat{\gamma}_0=-\infty$ and using (A.33))

$$\sum_{k=0}^{j-\sigma N-1}\int_{\widehat{\gamma}_k}^{\widehat{\gamma}_{k+1}}\frac{\varrho(y)\,\mathrm{d}y}{y-\gamma_j}=\frac{1}{N}\sum_{k=1}^{j-\sigma N-1}\frac{1}{\gamma_k-\gamma_j}+\sum_{k=1}^{j-\sigma N-1}\int_{\widehat{\gamma}_k}^{\widehat{\gamma}_{k+1}}\frac{(\gamma_k-y)\varrho(y)\,\mathrm{d}y}{(y-\gamma_j)(\gamma_k-\gamma_j)}+\int_{-\infty}^{\widehat{\gamma}_1}\frac{\varrho(y)\,\mathrm{d}y}{y-\gamma_j}.\tag{A.36}$$

The last term is $O(N^{-1})$. The error term in the middle is bounded by

$$\Biggl|\sum_{k=1}^{j-\sigma N-1}\int_{\widehat{\gamma}_k}^{\widehat{\gamma}_{k+1}}\frac{(\gamma_k-y)\varrho(y)\,\mathrm{d}y}{(y-\gamma_j)(\gamma_k-\gamma_j)}\Biggr|\le\frac{C}{N}\sum_{k=1}^{j-\sigma N-1}\int_{\widehat{\gamma}_k}^{\widehat{\gamma}_{k+1}}\frac{\varrho(y)\,\mathrm{d}y}{(\gamma_k-\gamma_j)^2}\le C\sum_{k=1}^{j-\sigma N-1}\frac{1}{(k-j)^2}\le CN^{-1}.$$

The third integral in (A.35) is estimated similarly. Finally, for the second integral we write

$$\int_{\widehat{\gamma}_{j-\sigma N}}^{\widehat{\gamma}_{j+\sigma N+1}}\frac{\varrho(y)\,\mathrm{d}y}{y-\gamma_j}=\sum_{k:|k-j|=1}^{\sigma N}\biggl(\frac{1}{N(\gamma_k-\gamma_j)}+\int_{\widehat{\gamma}_k}^{\widehat{\gamma}_{k+1}}\frac{(\gamma_k-y)\varrho(y)\,\mathrm{d}y}{(y-\gamma_j)(\gamma_k-\gamma_j)}\biggr)+\int_{\widehat{\gamma}_j}^{\widehat{\gamma}_{j+1}}\frac{\varrho(y)\,\mathrm{d}y}{y-\gamma_j}.\tag{A.37}$$

In the last integral we Taylor expand $\varrho(y)=\varrho(\gamma_j)+O(N^\delta|\gamma_j-y|)$ by (4.8) to get

$$\int_{\widehat{\gamma}_j}^{\widehat{\gamma}_{j+1}}\frac{\varrho(y)\,\mathrm{d}y}{y-\gamma_j}=\varrho(\gamma_j)\int_{\widehat{\gamma}_j}^{\widehat{\gamma}_{j+1}}\frac{\mathrm{d}y}{y-\gamma_j}+O\bigl(N^{-1}\bigr).$$

Computing the integral explicitly and using $\widehat{\gamma}_j=\gamma_{j-1/2}$, $\widehat{\gamma}_{j+1}=\gamma_{j+1/2}$ we have

$$\int_{\widehat{\gamma}_j}^{\widehat{\gamma}_{j+1}}\frac{\mathrm{d}y}{y-\gamma_j}=\log\biggl|1+\frac{\gamma_{j-1/2}-2\gamma_j+\gamma_{j+1/2}}{\gamma_{j-1/2}-\gamma_j}\biggr|=O\bigl(N^{-1+\delta}\bigr).$$

Here we used $\gamma_{j-1/2}-\gamma_j\ge c/N$ and (A.34) to estimate the second order discrete derivative.

Finally, we estimate the integral in the middle term in (A.37) and we will show that

$$\sum_{k:|k-j|=1}^{\sigma N}\int_{\widehat{\gamma}_k}^{\widehat{\gamma}_{k+1}}\frac{(\gamma_k-y)\varrho(y)\,\mathrm{d}y}{(y-\gamma_j)(\gamma_k-\gamma_j)}=O\bigl(N^{-1+\delta}\log N\bigr).\tag{A.38}$$

Clearly, (A.38) together with (A.35), (A.36), (A.37) and (A.39) imply (4.12).

In the rest of the proof we show (A.38). We write

$$\begin{aligned}
&\sum_{k:|k-j|=1}^{\sigma N}\int_{\widehat{\gamma}_k}^{\widehat{\gamma}_{k+1}}\frac{(\gamma_k-y)\varrho(y)\,\mathrm{d}y}{(y-\gamma_j)(\gamma_k-\gamma_j)}\\
&\quad=\sum_{m=1}^{\sigma N}\biggl(\int_{\widehat{\gamma}_{j-m}}^{\widehat{\gamma}_{j-m+1}}\frac{(\gamma_{j-m}-y)\varrho(y)\,\mathrm{d}y}{(y-\gamma_j)(\gamma_{j-m}-\gamma_j)}+\int_{\widehat{\gamma}_{j+m}}^{\widehat{\gamma}_{j+m+1}}\frac{(\gamma_{j+m}-y)\varrho(y)\,\mathrm{d}y}{(y-\gamma_j)(\gamma_{j+m}-\gamma_j)}\biggr).
\end{aligned}\tag{A.39}$$

In the first integral, we replace $\varrho(y)$ with $\varrho(\gamma_j)$. From Taylor expansion, $|\varrho(y)-\varrho(\gamma_j)|\le CmN^{-1+\delta}$, the error in this replacement is bounded by

$$\frac{CmN^\delta}{N}\int_{\widehat{\gamma}_{j-m}}^{\widehat{\gamma}_{j-m+1}}\frac{|\gamma_{j-m}-y|\,\mathrm{d}y}{|y-\gamma_j||\gamma_{j-m}-\gamma_j|}\le\frac{CmN^\delta}{N^2}\int_{\widehat{\gamma}_{j-m}}^{\widehat{\gamma}_{j-m+1}}\frac{\mathrm{d}y}{|y-\gamma_j||\gamma_{j-m}-\gamma_j|}\le\frac{CN^\delta}{Nm},$$

since $|\gamma_{j-m}-\gamma_j|\sim m/N$. We get a similar error when replacing $\varrho(y)$ with $\varrho(\gamma_j)$ in the second integral in (A.39). These errors, even after summation over m, are still of order $O(N^{-1+\delta}\log N)$, hence negligible. Thus we get that (A.39) equals

$$\sum_{m=1}^{\sigma N}\biggl(\frac{\varrho(\gamma_j)}{\gamma_{j-m}-\gamma_j}\int_{\widehat{\gamma}_{j-m}}^{\widehat{\gamma}_{j-m+1}}\frac{\gamma_{j-m}-y}{y-\gamma_j}\,\mathrm{d}y+\frac{\varrho(\gamma_j)}{\gamma_{j+m}-\gamma_j}\int_{\widehat{\gamma}_{j+m}}^{\widehat{\gamma}_{j+m+1}}\frac{\gamma_{j+m}-y}{y-\gamma_j}\,\mathrm{d}y\biggr)+O\bigl(N^{-1+\delta}\log N\bigr).$$

Next, using (A.34), we have

$$\frac{1}{\gamma_{j-m}-\gamma_j}=-\frac{\varrho(\gamma_j)N}{m}+O\bigl(N^\delta\bigr),\qquad \gamma_{j+m}-\gamma_j=\frac{m}{\varrho(\gamma_j)N}+O\bigl(N^{-2+\delta}m^2\bigr),$$

so we get, after a change of variables, that (A.39) equals

$$\begin{aligned}
&\sum_{m=1}^{\sigma N}\frac{N}{m}\biggl(-\int_{\widehat{\gamma}_{j-m}}^{\widehat{\gamma}_{j-m+1}}\frac{\gamma_{j-m}-y}{y-\gamma_j}\,\mathrm{d}y+\int_{\widehat{\gamma}_{j+m}}^{\widehat{\gamma}_{j+m+1}}\frac{\gamma_{j+m}-y}{y-\gamma_j}\,\mathrm{d}y\biggr)+O\bigl(N^{-1+\delta}\log N\bigr)\\
&\quad=-\sum_{m=1}^{\sigma N}\frac{N}{m}\biggl(\int_{\widehat{\gamma}_{j-m}-\gamma_{j-m}}^{\widehat{\gamma}_{j-m+1}-\gamma_{j-m}}\frac{u\,\mathrm{d}u}{\gamma_j-\gamma_{j-m}-u}+\int_{\widehat{\gamma}_{j+m}-\gamma_{j+m}}^{\widehat{\gamma}_{j+m+1}-\gamma_{j+m}}\frac{u\,\mathrm{d}u}{\gamma_{j+m}-\gamma_j+u}\biggr)+O\bigl(N^{-1+\delta}\log N\bigr).
\end{aligned}$$

The limits of integrations can be approximated as follows:

$$\begin{aligned}
&\widehat{\gamma}_{j-m}-\gamma_{j-m}=-\frac{1}{2N\varrho(\gamma_j)}+O\bigl(mN^{-2+\delta}\bigr),\qquad \widehat{\gamma}_{j+m}-\gamma_{j+m}=-\frac{1}{2N\varrho(\gamma_j)}+O\bigl(mN^{-2+\delta}\bigr),\\
&\widehat{\gamma}_{j-m+1}-\gamma_{j-m}=\frac{1}{2N\varrho(\gamma_j)}+O\bigl(mN^{-2+\delta}\bigr),\qquad \widehat{\gamma}_{j+m+1}-\gamma_{j+m}=\frac{1}{2N\varrho(\gamma_j)}+O\bigl(mN^{-2+\delta}\bigr).
\end{aligned}$$

Replacing these limits with their common values yields negligible errors, for example:

$$\sum_{m=1}^{\sigma N}\frac{N}{m}\int_{\frac{1}{2N\varrho(\gamma_j)}}^{\widehat{\gamma}_{j-m+1}-\gamma_{j-m}}\frac{u\,\mathrm{d}u}{|\gamma_j-\gamma_{j-m}-u|}\le C\sum_{m=1}^{\sigma N}\frac{N}{m}\cdot\frac{mN^\delta}{N^2}\cdot\frac{1}{N}\frac{1}{m/N}=CN^{-1+\delta}\log N.$$

Thus, with the notation $d:=1/2N\varrho(\gamma_j)$, we get

$$(A.39)=-\sum_{m=1}^{\sigma N}\frac{N}{m}\biggl(\int_{-d}^{d}\frac{u\,\mathrm{d}u}{\gamma_j-\gamma_{j-m}-u}+\int_{-d}^{d}\frac{u\,\mathrm{d}u}{\gamma_{j+m}-\gamma_j+u}\biggr)+O\bigl(N^{-1+\delta}\log N\bigr).\tag{A.40}$$

Using that $\gamma_j - \gamma_{j-m} = \gamma_{j+m} - \gamma_j + O(m^2 N^{-2+\delta})$ from (A.34), we can write

$$\frac{1}{\gamma_{j+m} - \gamma_j + u} = \frac{1}{\gamma_j - \gamma_{j-m} + u} + O\big(N^{\delta}\big),$$

for any u in the second integration regime, since here $|u| \le \frac{1}{2N\varrho(\gamma_j)}$ and $\gamma_{j+m} - \gamma_j \ge \frac{m}{N\varrho(\gamma_j)} + O(m^2 N^{-2+\delta})$. Thus, replacing $\gamma_{j+m} - \gamma_j$ with $\gamma_j - \gamma_{j-m}$ in the denominator of the second integral in (A.40) yields an error of order $\sum_m (N/m) N^{-2+\delta} = CN^{1+\delta} \log N$. After this replacement the two integrals in (A.40) cancel out:

$$\int_{-d}^{d} \frac{u \,\mathrm{d}u}{\gamma_j - \gamma_{j-m} - u} + \int_{-d}^{d} \frac{u \,\mathrm{d}u}{\gamma_j - \gamma_{j-m} + u} = \int_{-d}^{d} \frac{u \,\mathrm{d}u}{(\gamma_j - \gamma_{j-m})^2 - u^2} = 0.$$

We have shown that (A.39) is of order $O(N^{-1+\delta} \log N)$, which finishes the proof of (A.38). □

A.2. Remarks on Assumption (1) *of Theorem* 2.1

We conclude this Appendix with some remarks on Assumption (1) of Theorem 2.1. We consider the semicircular flow $\varrho_t = \mathcal{F}_t(\varrho)$ started from ϱ, for $t \ge 0$. As remarked earlier, the semicircle law ϱ_{sc} is invariant under the flow. It is then easy to check that m_{sc}, the Stieltjes transform of ϱ_{sc} satisfies the bound in (2.15) for all t with $\delta = 1$.

For many matrix models the distribution ϱ, and hence also ϱ_t, are not explicit and checking Assumption (1) directly may be not an easy task. In many situations, one can however use the smoothing effect of the semicircle flow to establish these estimates. The following example may be of some interest.

Denote by $C^{0,\alpha}(\mathbb{R})$, $C^{0,\alpha}(\mathbb{C}^+)$ the spaces of uniformly α-Hölder continuous functions on $\mathbb{R}$, $\mathbb{C}^+$. Assume that $\varrho \in \mathbb{C}^{0,\alpha}(\mathbb{R})$, for some $\alpha > 0$. Then the Stieltjes transform, m_0, of ϱ is in $\mathbb{C}^{0,\alpha}(\mathbb{C}^+)$. Adapting the proof of Lemma A.1 one can establish the following result. Abbreviate $\sigma_t = 1 - \mathrm{e}^{-t}$.

Lemma A.8. *Assume that $\varrho \in \mathbb{C}^{0,\alpha}(\mathbb{R})$. Then, $m_t(z)$, the Stieltjes transform of $\varrho_t = \mathcal{F}_t(\varrho)$, is uniformly bounded for all $z \in \mathbb{C}^+ \cup \mathbb{R}$ and $t \ge 0$. Moreover, there is a constant C, depending only on ϱ, such that*

$$\big|m_t(z) - m_0(z)\big| \le C\sigma_t^{\alpha}, \quad 0 \le t \le 1, \tag{A.41}$$

and all $z \in \mathbb{C}^+ \cup \mathbb{R}$. Further, for all $n \in \mathbb{N}$, there is C_n such that we have the bounds

$$\big|\partial_z^n m_t(z)\big| \le C_n\big(\sigma_t \operatorname{Im} m_t(z)\big)^{\alpha - n}, \quad t > 0, \tag{A.42}$$

for all $z \in \mathbb{C}^+ \cup \mathbb{R}$.

Thus, running the semicircular flow from time $t = 0$ to time $t_1 = N^{-\tau_1}$, $\tau_1 > 0$, we see that Lemma A.8 implies the Assumption (1) of Theorem 2.1 for energies inside the "bulk" for the choice $\delta \ge (1 - \alpha)\tau_1$. For the Wigner-like matrices of [1,2] typical choices for α are 1/3 or 1/2.

Appendix B: Persistent trailing of the DBM

In this section, we prove that the time-dependent quantiles $\gamma_k(t)$ persistently trail the DBM up to a time-independent shift in the indices. More precisely, we have the following

Proposition B.1. *Consider a time interval $[t_1, t_2]$ of length $t_2 - t_1 = O(N^{-\epsilon})$ with some small $\epsilon > 2\delta$, where δ, given in (2.15), is the regularity exponent of the initial data of the quantiles. Let $\boldsymbol{\lambda}(t)$ be the solution of (2.11) and let $\boldsymbol{\gamma}(t)$ be given by (2.9). Suppose that*

$$\mathbb{P}\left\{\big|\lambda_i(t) - \lambda_j(t)\big| \le \frac{N^{\xi}|i-j|}{N}, i, j \in I_{\sigma}\right\} \ge 1 - N^{-D}, \tag{B.1}$$

for any D. Fix an index L in the bulk and let $\ell(L)$ such that

$$\big|\lambda_L(t_1)-\gamma_{\ell(L)}(t_1)\big| \le CN^{-1+\xi}. \tag{B.2}$$

Then in the probability space of the Brownian motions $\{B_i(t): i\in\mathbb{N}_N, t\in[t_1,t_2]\}$ we have

$$\mathbb{P}\Big(\sup_{t\in[t_1,t_2]}\big|\lambda_L(t)-\gamma_{\ell(L)}(t)\big| \le CN^{-1+2\xi}\Big) \ge 1-N^{-\xi}. \tag{B.3}$$

Notice that $\gamma_{\ell(L)}(t)$ is a deterministic trajectory. This result therefore also shows that the typical fluctuation of the solution of the DBM is much smaller than the white noise term in (2.11) naively indicates. Indeed, the variance of the integral of this term is

$$\mathbb{E}\bigg|\int_{t_1}^{t_2}\sqrt{\frac{2}{\beta N}}\,\mathrm{d}B_L\bigg|^2 \simeq \frac{t_2-t_1}{N},$$

which would indicate a behavior $|\lambda_L(t_2)-\lambda_L(t_1)| \gtrsim (t_2-t_1)^{1/2}N^{-1/2}$. This is much larger than the actual value $|\lambda_L(t_2)-\lambda_L(t_1)| \le CN^{-1+\xi}$.

Proof of Proposition B.1. Let

$$v_i(t) := \lambda_i(t)-\gamma_{\ell(i)}(t).$$

Subtracting the DBM (2.11) from (4.12) and localizing it for the indices $i\in I$, we get

$$\mathrm{d}v_i(t) = \sqrt{\frac{2}{\beta N}}\,\mathrm{d}B_i(t) - \sum_{j\in I}\mathcal{B}_{ij}(v_i-v_j)\,\mathrm{d}t - \mathcal{W}_i v_i\,\mathrm{d}t + \kappa_i(t)\,\mathrm{d}t, \quad i\in I,$$

with (time-dependent) coefficients

$$\mathcal{B}_{ij} = \frac{1}{N}\frac{1}{(\lambda_j-\lambda_i)(\gamma_j-\gamma_i)}, \quad i,j\in I, \qquad \mathcal{W}_i = \frac{1}{2}+\sum_{k\notin I}\frac{1}{N}\frac{1}{(\lambda_k-\lambda_i)(\gamma_k-\gamma_i)}, \quad i\in I,$$

and a deterministic error term $|\kappa_i(t)|\le N^{-1+\delta}$.

By (B.1) and the spacing of the quantiles in the bulk, we know that

$$\mathcal{B}_{ij}(s) \ge \frac{b}{|i-j|}, \qquad \mathcal{W}_i(s) \ge \frac{b}{||j-L|-K|+1}, \tag{B.4}$$

with $b := N^{1-\xi}$ uniformly in time $s\in[t_1,t_2]$ with very high probability.

Let $\mathcal{U}(s,t)$ be the propagator of the parabolic equation

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\sum_{j\in I}\mathcal{B}_{ij}(u_i-u_j) - \mathcal{W}_i u_i, \quad i\in I, \tag{B.5}$$

then

$$v_i(t) = v_i(t_1) + \int_{t_1}^{t}\sum_j\big[\mathcal{U}(s,t)\big]_{ij}\bigg[\sqrt{\frac{2}{\beta N}}\,\mathrm{d}B_j(s)+\kappa_j(s)\,\mathrm{d}s\bigg]. \tag{B.6}$$

Since the propagator is a contraction in the supremum norm, the $\kappa(s)$ error term, after integration, gives a negligible error at most $C(t_2-t_1)N^{-1+\delta}\le CN^{-1}$. To estimate the main term, notice that the propagator depends on the sigma algebra $\Sigma_s := \{\{B_i(u)\}_{i\in I} : u \in (s,t]\}$ and is independent of the white noise at time s. Therefore

$$\mathbb{E}\big|v_i(t) - v_i(t_1)\big|^2 = \frac{2}{\beta N}\mathbb{E}\int_{t_1}^{t}\sum_j\big|\big[\mathcal{U}(s,t)\big]_{ij}\big|^2\,\mathrm{d}s + O\big(N^{-2}\big). \tag{B.7}$$

Fix $i \in I$, s and t and define $w_j := [\mathcal{U}(s,t)]_{ij}$, which is the same as $[\mathcal{U}(s,t)]_{ji}$ by symmetry. Then for any $\nu > 0$, we have

$$\bigg|\sum_j\big|\big[\mathcal{U}(s,t)\big]_{ij}\big|^2\bigg| \le \big\|\mathcal{U}(s,t)\mathbf{w}\big\|_\infty \le \frac{CN^\xi}{[N(t-s)]^{\frac{1}{1+\nu}}}\|\mathbf{w}\|_{1+\nu},$$

by the heat kernel estimate on the equation (B.5); see Proposition 9.4 in [28] (the conditions of this proposition are guaranteed by (B.4)). By the L^p-contraction of the semigroup for any $p \ge 1$, we have

$$\|\mathbf{w}\|_{1+\nu} = \big\|\mathcal{U}(s,t)\delta_i\big\|_{1+\nu} \le \|\delta_i\|_{1+\nu} = 1.$$

Thus we get

$$\mathbb{E}\big|v_i(t) - v_i(t_1)\big|^2 \le \frac{CN^\xi}{N^{1+\frac{1}{1+\nu}}}\int_{t_1}^{t}\frac{1}{(t-s)^{\frac{1}{1+\nu}}}\,\mathrm{d}s + O\big(N^{-2}\big) \le C_\nu\frac{N^{\xi+2\nu}}{N^2} \le C\frac{N^{2\xi}}{N^2}, \tag{B.8}$$

after choosing $\nu = \xi/2$. Using Doob martingale inequality, one also has

$$\mathbb{E}\sup_{t\le t_2}\big|v_i(t) - v_i(t_1)\big|^2 \le C\mathbb{E}\big|v_i(t_2) - v_i(t_1)\big|^2 \le C\frac{N^{2\xi}}{N^2}.$$

Setting $i = L$ and using Markov inequality and combining it with assumption (B.2), we get (B.3). □

Acknowledgements

The authors thank O. Ajanki and T. Krüger for many valuable discussions at the early stage of the project.

References

[1] O. Ajanki, L. Erdős and T. Krüger Quadratic vector equation on complex upper half plane. Preprint. Available at arXiv:1506.05095.

[2] O. Ajanki, L. Erdős and T. Krüger Universality for general Wigner-type matrices. Preprint. Available at arXiv:1506.05098.

[3] G. Anderson, A. Guionnet and O. Zeitouni. *An Introduction to Random Matrices*. *Cambridge Stud. Adv. Math.* **118**. Cambridge Univ. Press, Cambridge, 2010. MR2760897

[4] A. Aptekarev, P. Bleher and A. Kuijlaars. Large n limit of Gaussian random matrices with external source, part II. *Comm. Math. Phys.* **259** (2005) 367–389. MR2172687

[5] D. Bakry and M. Émery. *Diffusions Hypercontractives, Séminaire de probabilités, XIX, 1983/84*. *Lecture Notes in Math.* **1123**. Springer, Berlin, 1985. MR0889476

[6] F. Bekerman, A. Figalli and A. Guionnet. Transport maps for β-matrix models and universality. *Comm. Math. Phys.* **338** (2) (2015) 589–619. Preprint. Available at arXiv:1311.2315. MR3351052

[7] G. Ben Arous and S. Péché. Universality of local eigenvalue statistics for some sample covariance matrices. *Comm. Pure Appl. Math.* **LVIII** (2005) 1–42. MR2162782

[8] P. Biane. On the free convolution with a semi-circular distribution. *Indiana Univ. Math. J.* **46** (1997) 705–718. MR1488333

[9] P. Biane. Logarithmic Sobolev inequalities, matrix models and free entropy. *Acta Math. Sin. (Engl. Ser.)* **19** (3) (2003) 497–506. MR2014030

[10] P. Biane and R. Speicher. Free diffusions, free energy and free Fisher information. *Ann. Inst. Henri Poincaré Probab. Stat.* **37** (2001) 581–606. MR1851716

[11] P. Bleher and A. Kuijlaars. Large n limit of Gaussian random matrices with external source, part I. *Comm. Math. Phys.* **252** (2004) 43–76. MR2103904

[12] P. Bourgade and H.-T. Erdős Yau. Bulk universality of general β-ensembles with non-convex potential. *J. Math. Phys.* **53** (2012) 095221. Special issue in honor of E. Lieb's 80th birthday. MR2905803

[13] P. Bourgade and H.-T. Erdős Yau. Universality of general β-ensembles. *Duke Math. J.* **163** (6) (2014) 1127–1190. MR3192527

[14] P. Bourgade, H.-T. Erdős Yau and J. Yin Fixed energy universality for generalized Wigner matrices. Preprint. Available at arXiv:1407.5606.

[15] A. Boutet de Monvel, L. Pastur and M. Shcherbina. On the statistical mechanics approach in the random matrix theory: Integrated density of states. *J. Stat. Phys.* **79** (1995) 585–611. MR1327898

[16] T. Claeys and D. Wang. Random matrices with equispaced external source. *Comm. Math. Phys.* **328** (2014) 1023–1077. MR3201219

[17] P. Deift. *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*. *Courant Lecture Notes in Mathematics* **3**. American Mathematical Society, Providence, RI, 1999. MR1677884

[18] P. Deift and D. Gioev. *Random Matrix Theory: Invariant Ensembles and Universality*. *Courant Lecture Notes in Mathematics* **18**. American Mathematical Society, Providence, RI, 2009. MR2514781

[19] F. J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.* **3** (1962) 1191–1198. MR0148397

[20] L. Erdős, A. Knowles, H.-T. Yau and J. Yin. Spectral statistics of Erdős–Rényi graphs II: Eigenvalue spacing and the extreme eigenvalues. *Comm. Math. Phys.* **314** (3) (2012) 587–640. MR2964770

[21] L. Erdős, A. Knowles, H.-T. Yau and J. Yin. Spectral statistics of Erdős–Rényi graphs I: Local semicircle law. *Ann. Probab.* **41** (3B) (2013) 2279–2375. MR3098073

[22] L. Erdős, G. Péché, J. Ramírez, B. Schlein and H.-T. Yau. Bulk universality for Wigner matrices. *Comm. Pure Appl. Math.* **63** (7) (2010) 895–925. MR2662426

[23] L. Erdős, J. Ramirez, B. Schlein, T. Tao, V. Vu and H.-T. Yau. Bulk universality for Wigner Hermitian matrices with subexponential decay. *Math. Res. Lett.* **17** (4) (2010) 667–674. MR2661171

[24] L. Erdős, B. Schlein and H.-T. Yau. Wegner estimate and level repulsion for Wigner random matrices. *Int. Math. Res. Not. IMRN* **3** (2010) 436–479. MR2587574

[25] L. Erdős, B. Schlein and H.-T. Yau. Universality of random matrices and local relaxation flow. *Invent. Math.* **185** (1) (2011) 75–119. MR2810797

[26] L. Erdős, B. Schlein, H.-T. Yau and J. Yin. The local relaxation flow approach to universality of the local statistics for random matrices. *Ann. Inst. Henri Poincaré B, Probab. Stat.* **48** (1) (2012) 1–46. MR2919197

[27] L. Erdős and H.-T. Yau. Universality of local spectral statistics of random matrices. *Bull. Amer. Math. Soc.* **49** (3) (2012) 377–414. MR2917064

[28] L. Erdős and H.-T. Yau. Gap universality of generalized Wigner and β-ensembles. *J. Eur. Math. Soc. (JEMS)* **17** (2015) 1927–2036. MR3372074

[29] L. Erdős, H.-T. Yau and J. Yin. Bulk universality for generalized Wigner matrices. *Probab. Theory Related Fields* **154** (1–2) (2012) 341–407. MR2981427

[30] L. Erdős, H.-T. Yau and J. Yin. Rigidity of eigenvalues of generalized Wigner matrices. *Adv. Math.* **229** (3) (2012) 1435–1515. MR2871147

[31] A. S. Fokas, A. R. Its and A. V. Kitaev. The isomonodromy approach to matrix models in 2D quantum gravity. *Comm. Math. Phys.* **147** (1992) 395–430. MR1174420

[32] F. Hiai and D. Petz. *The Semicircle Law, Free Random Variables and Entropy*, **77**. American Mathematical Society, Providence, RI, 2000. MR1746976

[33] K. Johansson. Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices. *Comm. Math. Phys.* **215** (3) (2001) 683–705. MR1810949

[34] A. Knowles and J. Yin. Eigenvalue distribution of Wigner matrices. *Probab. Theory Related Fields* **154** (3–4) (2013) 543–582. MR3034787

[35] A. Knowles and J. Yin. The isotropic semicircle law and deformation of Wigner matrices. *Comm. Pure Appl. Math.* **66** (11) (2013) 1663–1749. MR3103909

[36] A. Knowles and J. Yin Anisotropic local laws for random matrices. Preprint. Available at arXiv:1410.3516.

[37] A. B. J. Kuijlaars and K. T.-R. McLaughlin. Generic behavior of the density of states in random matrix theory and equilibrium problems in the presence of real analytic external fields. *Comm. Pure Appl. Math.* **53** (2000) 736–785. MR1744002

[38] B. Landon and H.-T. Yau Convergence of local statistics of Dyson Brownian motion. Preprint. Available at arXiv:1504.03605.

[39] J. O. Lee and K. Schnelli. Local deformed semicircle law and complete delocalization for Wigner matrices with random potential. *J. Math. Phys.* **54** (2013) 103504. MR3134604

[40] J. O. Lee, K. Schnelli, B. Stetler and H.-T. Yau Bulk universality for deformed Wigner matrices. Preprint. Available at arXiv:1405.6634.

[41] S. Li, X. D. Li and Y. X. Xie On the law of large numbers for the empirical measure process of generalized Dyson Brownian motion. Preprint. Available at arXiv:1407.7234.

[42] H. Maassen. Addition of freely independent random variables. *J. Funct. Anal.* **106** (1992) 409–438. MR1165862

[43] M. L. Mehta. *Random Matrices*, 3rd edition. Academic Press, New York, 1991. MR1083764

[44] A. Nica and R. Speicher. On the multiplication of free N-tuples of noncommutative random variables. *Amer. J. Math.* **118** (1996) 799–837. MR1400060

[45] S. O'Rourke and V. Vu. Universality of local eigenvalue statistics in random matrices with external source. *Random Matrices Theory Appl.* **03** (2014) 1450005. MR3208886

[46] J. N. Pandey. *The Hilbert Transform of Schwartz Distributions and Applications*. Wiley-Interscience, New York, 1996. MR1363489

[47] L. Pastur. On the spectrum of random matrices. *Theoret. and Math. Phys.* **10** (1972) 67–74. MR0475502

[48] L. Pastur and M. Shcherbina. Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles. *J. Stat. Phys.* **86** (1997) 109–147. MR1435193

[49] L. Pastur and M. Shcherbina. Bulk universality and related properties of Hermitian matrix models. *J. Stat. Phys.* **130** (2) (2008) 205–250. MR2375744

[50] T. Shcherbina. On universality of bulk local regime of the deformed Gaussian unitary ensemble. *Math. Phys. Anal. Geom.* **5** (2009) 396–433. MR2590774

[51] M. Shcherbina. Change of variables as a method to study general β-models: Bulk universality. *J. Math. Phys.* **55** (2014) 043504. MR3390602
[52] T. Tao and V. Vu. Random matrices: Universality of the local eigenvalue statistics. *Acta Math.* **206** (2011) 127–204. MR2784665
[53] B. Valkó and B. Virág. Continuum limits of random matrices and the Brownian carousel. *Invent. Math.* **177** (3) (2009) 463–508. MR2534097
[54] D. Voiculescu. Addition of certain non-commuting random variables. *J. Funct. Anal.* **66** (3) (1986) 323–346. MR0839105
[55] D. Voiculescu, K. J. Dykema and A. Nica. *Free Random Variables: A Noncommutative Probability Approach to Free Products with Applications to Random Matrices, Operator Algebras and Harmonic Analysis on Free Groups*. American Mathematical Society, Providence, RI, 1992. MR1217253
[56] E. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Ann. of Math.* **62** (1955) 548–564. MR0077805
[57] H.-T. Yau. Relative entropy and the hydrodynamics of Ginzburg–Landau models. *Lett. Math. Phys.* **22** (1991) 63–80. MR1121850