

Transience in growing subgraphs via evolving sets

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Abstract. We extend the use of random evolving sets to time-varying conductance models and utilize it to provide tight heat kernel upper bounds. It yields the transience of any uniformly lazy random walk, on \mathbb{Z}^d , $d \geq 3$, equipped with uniformly bounded above and below, independently time-varying edge conductances, of (effectively) non-decreasing in time vertex conductances, thereby affirming part of Conjecture 7.1 (Random walk in changing environment (2015) Preprint).

Résumé. Nous généralisons la méthode basée sur l'évolution aléatoire d'ensembles au cas de modèles de conductances variant avec le temps. Nous l'utilisons pour prouver des bornes supérieures sur le noyau de la chaleur. Ceci montre la transitivité de n'importe quelle marche aléatoire fainéante, dans \mathbb{Z}^d , $d \geq 3$, avec des conductances par arêtes (bornées uniformément supérieurement et inférieurement) variant indépendamment en temps en fonction des conductances par sites. Ceci répond partiellement à la Conjecture 7.1 (Random walk in changing environment (2015) Preprint).

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1. Introduction

There has been much interest in random walks in random environment (see [13]). The challenge often comes from the highly non-reversible nature of the dynamics, which can leave questions as fundamental as recurrence versus transience open. For example, the recurrence of linearly edge reinforced random walk with strong enough reinforcement strength on any graphs is just recently solved [2,22,23]. Many questions in this general area are treated in an ad-hoc manner, and the development of methods in order to fully or partially resolve them is just as interesting as the questions themselves.

The case when the evolution of the environment is independent of the stochastic process is better understood (e.g. [9]), and there are conjectures on the emergence of universality (cf. [1, Conjecture 7.1] and [7, Conjecture 1.2, 1.8, 1.10]). Specifically, [7] conjecture that whenever a graph \mathbb{G}_∞ is recurrent, then any graph sequence $\{\mathbb{G}_t\}_{t \in \mathbb{N}}$ dynamically growing towards \mathbb{G}_∞ is also recurrent, for the discrete time, simple random walk $\{X_t\}_{t \in \mathbb{N}}$ taking steps in $\{\mathbb{G}_t\}_{t \in \mathbb{N}}$; and whenever \mathbb{G}_0 is transient, then any growing sequence $\{\mathbb{G}_t\}$ of uniformly bounded degrees, starting from \mathbb{G}_0 is transient.

Essentially the same phenomenon is conjectured in [1] for the general setting of monotonically time varying conductance models, which are also the focus of the present work. That is, the stochastic process $\{X_t\}_{t \in \mathbb{N}}$ on a locally

finite graph $\mathbb{G} = (V, E)$ constructed as random walk in time varying edge conductances $\{\pi^{(t)}(x, y), t \in \mathbb{N}, (x, y) \in E\}$ which are changed independently of the sample path $t \mapsto X_t$. Specifically, the vertex conductances

$$\pi^{(t)}(x) = \sum_{y \in V} \pi^{(t)}(x, y), \quad x \in V, \tag{1.1}$$

form the time-dependent reversing measure for X_t , and setting $V_t = \{x \in V : \pi^{(t)}(x) > 0\}$, the transition probability of the in-homogeneous Markov chain $X_t \in V_t$ is given by

$$P(t, x; t + 1, y) = \frac{\pi^{(t)}(x, y)}{\pi^{(t)}(x)}, \quad \forall (x, y) \in E, x \in V_t. \tag{1.2}$$

When \mathbb{G} is a tree, [1, Theorem 5.1] proves recurrence of such $\{X_t\}$ provided all edge conductances $\pi^{(t)}(x, y)$ are positive, non-decreasing in t , and bounded above by $\pi^{(\infty)}(x, y)$ of a time-invariant recurrent model, while [1, Theorem 5.2] establishes its transience when all edge conductances are positive, non-increasing in t and bounded below by $\pi^{(\infty)}(x, y)$ of a transient model. Both results apply when $\mathbb{G} = \mathbb{N}$, for which they are complemented by [1, Theorems 4.2 and 4.4] that cover also the non-decreasing transient and non-increasing recurrent cases. We note in passing that all four theorems allow for non-Markovian processes, where edge conductances depend on the past trajectory of the walk, but [1, Section 6] shows that in general (specifically, when $\mathbb{G} = \mathbb{Z}^2$), these results may fail under such dependence. Nevertheless, [1, Conjecture 7.1] proposes that the aforementioned four theorems hold on any locally finite graph \mathbb{G} , provided its time varying edge conductances are independent of the walk’s trajectory (i.e. for the Markovian evolution as in (1.2)).

The present work affirms part of the transient case of this conjecture (and a special case of [7, Conjecture 1.8]), for \mathbb{Z}^d , $d \geq 3$ equipped with uniformly bounded non-decreasing vertex conductances (more generally, extending [1, Theorem 4.2] from $\mathbb{G} = \mathbb{N}$ to all graphs having suitable isoperimetric properties). In contrast, the recurrent direction (i.e. obtaining heat kernel lower bounds), is mostly open.

We prove transience by way of establishing an on-diagonal heat kernel upper bound. The study of heat kernels for diffusions on manifolds and Markov chains on graphs has a long history, dating back at least to the work of De Giorgi, Nash, Moser in the late 1950s and early 60s, and that of Aronson (cf. [3]), investigating properties of solutions of parabolic differential equations. There is a large body of work on Gaussian and sub-Gaussian heat kernel estimates on diverse spaces, their equivalence to functional inequalities, and related stability theory (see [4,6,12,14,24,26,27] and the references therein). In the setting of graphs, some associated *continuous time*, symmetric rate random walks among uniformly elliptic, time dependent conductances have been studied (cf. [5, Section 4], [10, Appendix B] and [11, Theorem 1.1]). In particular, it is by now known that the two-sided Gaussian heat kernel estimates hold for any such random walks on \mathbb{Z}^d , and more generally on any bounded degree graphs satisfying volume doubling plus a uniform Poincaré inequality (cf. [15, Theorem 1.2] and the references therein).

All such continuous time, symmetric rate walks, have *time-independent reversing measure*. Similarly if the discrete time-dependent conductance model of (1.2) satisfies a uniform Sobolev inequality, [4, Section 7] claims some of the Gaussian heat kernel estimates, *provided the reversing measure $\pi^{(t)}(x)$ of (1.1) is held constant in time*, and the walk is uniformly lazy. In contrast, the study of recurrence/transience, and more generally, that of heat kernel estimates, is rather subtle when $t \mapsto \pi^{(t)}(x)$ is not constant. Indeed, some heat kernel estimates are derived in this setting by [25], but as shown in [15, Propositions 1.4 and 1.5], if the time varying vertex conductances are either non-monotone or unbounded, then in general neither the upper/lower Gaussian estimates nor recurrence/transience properties are stable under perturbations (and the same applies for constant speed continuous time random walks).

Random evolving sets have been introduced in [19,20], where they are applied to study the mixing time of possibly non-reversible Markov chains (with the related notion of size-biased evolving sets already inherent in [8]). For static weighted graphs it is known that evolving sets serve well in deducing from an isoperimetric inequality, both the heat kernel upper bound and a Nash inequality. The main tool of this work is the extended notion of random evolving sets in the parabolic (time-varying) context (see Definition 1.12).

Turning to state our main result, we use hereafter A^c for $V \setminus A$ and $\pi^{(t)}(A) = \pi^{(t)}(A, V)$, or more generally $\pi^{(t)}(A, B) = \sum_{x \in A, y \in B} \pi^{(t)}(x, y)$ for any $A \subset V_t, B \subset V$. By analogy to convention, we define the *heat kernel* of $\{X_t\}$ as

$$h(s, x; t, y) := \frac{P(s, x; t, y)}{\pi^{(t)}(y)}, \quad x \in V_s, y \in V_t. \tag{1.3}$$

Definition 1.1. Starting with $\beta(0) = 1$, suppose that

$$\beta(u + 1) := \beta(u) \sup_{x \in V_u} \left\{ \frac{\pi^{(u)}(x)}{\pi^{(u+1)}(x)} \right\}, \quad u \in \mathbb{N}, \tag{1.4}$$

are finite. With $t \mapsto \beta(t)\pi^{(t)}(x)$ non-decreasing, we call vertex conductances $t \mapsto \pi^{(t)}(x)$ *effectively non-decreasing*, if $\eta_\star = \sup_{t > u \geq 0} \{\beta(t)/\beta(u)\} < \infty$ (clearly, $\eta_\star \leq 1$ for non-decreasing $t \mapsto \pi^{(t)}(x)$).

Theorem 1.2. Suppose the walk is uniformly lazy, namely $\inf_{t,x} P(t, x; t + 1, x) \geq \gamma$ for some $\gamma \in (0, 1/2]$ and $\beta(u)$ of (1.4) are finite. Fixing $d > 1$, we consider the isoperimetric growth function

$$\psi_{d,\beta}(t) := \sum_{u=0}^{t-1} (\beta(u)^{1/d} \kappa_u)^2, \quad \kappa_u := \inf_{A \subset V_u, 0 < |A| < \infty} \left\{ \frac{\pi^{(u)}(A, A^c)}{\pi^{(u)}(A)^{(d-1)/d}} \right\}, \tag{1.5}$$

with $\psi_d(t)$ in case the factors $\beta(u)^{1/d}$ are omitted. If for fixed $\lambda \in (0, 1/2]$,

$$\exists r \in (s, t), \quad \frac{\psi_{d,\beta}(r) - \psi_{d,\beta}(s)}{\psi_{d,\beta}(t) - \psi_{d,\beta}(s)} \in [\lambda, 1 - \lambda], \tag{1.6}$$

then for some $c_+ = c_+(d, \gamma, \lambda)$ finite, any $t > s \geq 0, x \in V_s$ and $y \in V_t$

$$h(s, x; t, y) \leq c_+ \beta(t) (\psi_{d,\beta}(t) - \psi_{d,\beta}(s))^{-d/2}. \tag{1.7}$$

Let $\eta_0 := \sup_x \pi^{(0)}(x)$ (positive). For $t \mapsto \pi^{(t)}(x)$ effectively non-decreasing and uniformly bounded (i.e. $C := \sup_{t,x} \pi^{(t)}(x) < \infty$), we further have that for some $c_\star = c_\star(d, \gamma, \eta_0, \eta_\star, C)$ finite and all s, x, t, y as above,

$$\pi^{(s)}(x) h(s, x; t, y) \leq c_\star (e + \psi_d(t) - \psi_d(s))^{-d/2}. \tag{1.8}$$

Remark 1.3. If the right-hand side of (1.8) is summable over t , then $\sum_t P(0, x; t, y)$ is finite for any $x \in V_0, y \in V$. Hence, the process $\{X_t\}$ is then *transient in the strong sense* that starting at any non-random $X_0 \in V_0$ yields a finite expected number of visits to any $y \in V$ (and in particular, w.p.1. the sample path $t \mapsto X_t$ visits any $y \in V$ only finitely many times).

Remark 1.4. Assuming κ_u are bounded away from zero, even for polynomially growing $u \mapsto \beta(u)$ the right-hand side of (1.7) yields the optimal $(t - s)^{-d/2}$ bound. For example, this applies when $\sup_x |\pi^{(t)}(x)/\pi(x) - 1| \rightarrow 0$ at rate t^{-1} . In contrast, for exponentially growing $u \mapsto \beta(u)$ the right-hand side of (1.7) is $O(1)$, so carries no information. Indeed, the latter happens for the recurrent random walk among oscillating $[1 - \epsilon, 1 + \epsilon]$ -valued edge conductances on $\mathbb{Z}^2 \times \mathbb{Z}_+$ which is given in [15, Proposition 1.5(i)].

Remark 1.5. For $d > p \geq 1$ the d -dimensional Sobolev ℓ_p -inequality holds on \mathbb{G}_u , if

$$\widehat{\kappa}_u := \inf_{|\text{supp}(f)| < \infty} \left\{ \frac{\|\nabla f\|_{p,u}}{\|f\|_{pd/(d-p),u}} \right\} \tag{1.9}$$

is positive, with the corresponding functional norms for $q \geq 1$,

$$\|f\|_{q,u} := \left(\sum_{x \in V_u} |f(x)|^q \pi^{(u)}(x) \right)^{1/q},$$

$$\|\nabla f\|_{q,u} := \left(\frac{1}{2} \sum_{x,y \in V_u} |f(y) - f(x)|^q \pi^{(u)}(x,y) \right)^{1/q}.$$

Recall that for $d > 1$, the Sobolev ℓ_1 -inequality is equivalent to the isoperimetric inequality of (1.5) with $\widehat{\kappa}_u = \kappa_u$, whereas for $d > 2$, the Sobolev ℓ_2 -inequality is implied by the isoperimetric inequality (see [17, Theorem 3.2.7]).

For uniformly lazy walk and *time-independent conductances*, it is shown in [4] that the Sobolev ℓ_2 -inequality with uniformly positive $\widehat{\kappa}_u = \kappa$ yields the Gaussian heat kernel full upper bound (via the discrete integral maximum principle), and a matching on-diagonal lower bound holds under additional volume condition.

Remark 1.6. In case of *delayed random walk* one specifies only $\{\pi^{(t)}(x, y), x \neq y\}$. Then, assuming that for some $\gamma \in (0, 1/2]$,

$$\sup_{t,x} \pi^{(t)}(x, \{x\}^c) \leq 1 - \gamma,$$

one lets $\pi^{(t)}(x, x) := 1 - \pi^{(t)}(x, \{x\}^c)$. It results with $\pi^{(t)}(x) = 1$ for all t, x and the uniformly lazy transition probabilities $P(t, x; t + 1, y) = \pi^{(t)}(x, y)$ then satisfy the heat-kernel upper bound (1.8).

Here is a direct consequence of Theorem 1.2 (thanks to Remark 1.3).

Corollary 1.7. *Suppose \mathbb{G} of bounded degree satisfies a uniform isoperimetric inequality of order $d > 2$ (e.g. the lattice $\mathbb{G} = \mathbb{Z}^d$), and consider a uniformly lazy walk $\{X_t\}$ on \mathbb{G} equipped with uniformly elliptic and bounded edge conductances (namely, $\pi^{(t)}(x, y) \in [C_1^{-1}, C_1]$ for all t and edges or self-loops (x, y) , with C_1 a universal finite constant).*

If $t \mapsto \pi^{(t)}(x)$ are effectively non-decreasing, then for any law of X_0 the expected number of visits by $\{X_t\}$ to $y \in V$ is finite (so w.p.1. the sample path visits each site finitely many times).

Indeed, in the setting of Corollary 1.7 we have (1.5) holding with κ_u at least some universal positive constant times the edge-isoperimetric constant for \mathbb{G} , hence uniformly bounded away from zero. This yields the linear growth of $\psi_d(\cdot)$ with $P(s, x; t, y) \leq c_\star(t - s)^{-d/2}$, hence the stated strong transience (when $d > 2$), uniformly in X_0 .

We note in passing that having only $\pi^{(t)}(x) \in [C^{-1}, C]$ for all $x \in V$, is not enough (for example the graph \mathbb{Z}^d without all edges connecting finite box \mathbb{B}_r to \mathbb{B}_r^c has uniformly bounded vertex conductances, but $\kappa_u = 0$ in (1.5) and starting at $X_0 = 0$ any random walk on this graph is confined to \mathbb{B}_r , hence recurrent).

The analog of Corollary 1.7 applies also for the continuous time, constant speed random walk, the definition of which we provide next.

Definition 1.8. Suppose $\mathbb{G} = (V, E)$ is locally finite graph equipped with RCLL edge conductances $t \mapsto \pi^{(t)}(x, y)$ such that $\pi^{(t)}(x) > 0$ for all x . The V -valued stochastic process $\{Y_t\}$ of RCLL sample path $t \mapsto Y_t$ is called a constant speed random walk (in short CSRW), if it waits i.i.d. $\exp(1)$ times between successive jumps, and if $Y_{T-} = x$ just prior to the current random jump time T , then the process jumps across each $(x, y) \in E$ with probability $\pi^{(T)}(x, y)/\pi^{(T)}(x)$.

Definition 1.9. We call RCLL vertex conductances $t \mapsto \pi^{(t)}(x)$ effectively non-decreasing, if for Lebesgue a.e. $t_k \uparrow \infty$, the sequence $k \mapsto \pi^{(t_k)}(x)$ is effectively non-decreasing (see Definition 1.1).

Proposition 1.10. *Suppose graph $\mathbb{G} = (V, E)$ of bounded degree that satisfies a uniform isoperimetric inequality of order $d > 2$ (e.g. the lattice $\mathbb{G} = \mathbb{Z}^d$), is equipped with uniformly elliptic and bounded RCLL edge conductances*

(namely, $\pi^{(t)}(x, y) \in [C_1^{-1}, C_1]$ for all $t \geq 0$ and $(x, y) \in E$, with C_1 some universal finite constant). Assuming further that $t \mapsto \pi^{(t)}(x)$ are effectively non-decreasing, w.p.1. the sample path $t \mapsto Y_t$ of the CSRW returns to any $y \in V$ only finitely many times.

In many non-elliptic settings we get fast enough isoperimetric growth for (1.8) to yield the desired a.s. transience. Even when it does not, such result may be obtained by taking advantage of a-priori bounds on the support of the relevant evolving set. We next deal with one such example, which partially resolves the open question raised in [7, Remark 1.12].

Proposition 1.11. *Let \mathbb{D}_0 denote the unique infinite cluster of the correlated percolation model of [21, Theorem 1.2] (which includes as special case the Bernoulli(p) bond percolation at super-critical $p > p_c(\mathbb{Z}^d)$), on \mathbb{Z}^d , $d > 2$, conditioned to contain the origin. Starting with X_0 at the origin, the sample path of any uniformly lazy SRW on growing connected sub-graphs $\{\mathbb{D}_t\}$ of the lattice \mathbb{Z}^d sharing the vertex set $\mathcal{V}(\mathbb{D}_0)$ of \mathbb{D}_0 (with uniformly bounded self-loops, hence vertex, conductances), is strongly transient in the sense of Remark 1.3.*

As mentioned before, our key tool is the evolving set process $\{S_t\}$, where S_t is the following random finite subset of V_t , $t \geq 0$.

Definition 1.12. Starting with $S_0 = \{x\}$ for $x \in V_0$, sequentially for $t = 0, 1, 2, \dots$ we let U_{t+1} denote a Uniform(0, 1) random variable which is independent of $\{S_s, X_s, U_s, 0 \leq s \leq t\}$, and form

$$S_{t+1} = \left\{ y \in V_{t+1} : \frac{\pi^{(t)}(S_t, y)}{\pi^{(t+1)}(y)} \geq U_{t+1} \right\}.$$

Assuming $t \rightarrow \pi^{(t)}(x)$ are non-decreasing, it follows that $V_t \subseteq V_{t+1}$ and for every $y \in V_{t+1}$

$$\mathbb{P}(y \in S_{t+1} | S_t) = \frac{\pi^{(t)}(S_t, y)}{\pi^{(t+1)}(y)} \tag{1.10}$$

(the right-hand side of (1.10) is well defined [0, 1]-valued and any y accessible from S_t must be in V_{t+1}).

Remark 1.13. For uniformly lazy random walk having $\pi^{(t)}(x)$ independent of t (so w.l.o.g. $V_t = V$ for all t), one has the analogue of [20, Lemma 8]. That is, if (S_t) is an evolving set process, then the sequence (S_t^c) is also an evolving set process of the same transition probability. The proof in [20, p. 253] can be reproduced using $\pi^{(t+1)}(x) = \pi^{(t)}(x)$ for all t, x , and noting that for Uniform(0, 1) random variable $U \stackrel{(d)}{=} 1 - U$.

We further utilize the concept of conditioned (or size-biased) evolving set, upon adapting it to our *parabolic* time-dependent setting. In particular, it yields the following extension of [18, Theorem 17.23] originally due to [8].

Definition 1.14. We say that $(S_t \subseteq V_t)$ is the conditioned evolving set, starting at $S_0 = \{x\}$, if it has the transition kernel

$$\widehat{K}(t, A; t + 1, B) = \frac{\pi^{(t+1)}(B)}{\pi^{(t)}(A)} K(t, A; t + 1, B), \tag{1.11}$$

where $K(\cdot; \cdot)$ is the transition kernel of the unconditioned evolving set of Definition 1.12.

Proposition 1.15. *Suppose $t \mapsto \pi^{(t)}(x)$ are non-decreasing and (X_t, S_t) starting from $(X_0, S_0) = (x, \{x\})$ follows the time-varying Markov transition kernel P^* on $V \times 2^V$, given for $x \in A \cap V_t$, $\pi^{(t)}(x, y) > 0$, by*

$$\begin{aligned} P^*(t, (x, A); t + 1, (y, B)) &= P(t, x; t + 1, y) \mathbb{P}(S_{t+1} = B | y \in S_{t+1}, S_t = A) \mathbb{1}_{\{y \in B\}} \\ &= \frac{P(t, x; t + 1, y) K(t, A; t + 1, B) \pi^{(t+1)}(y) \mathbb{1}_{\{y \in B\}}}{\pi^{(t)}(A, y)}. \end{aligned}$$

- (a) The marginal process $t \mapsto X_t$ is a time in-homogeneous Markov process having the transition kernel P , and the marginal process $t \mapsto S_t$ is another time in-homogeneous Markov chain whose transition kernel is $\widehat{K}(\cdot, \cdot)$ of (1.11).
- (b) For any $t, x \in V_0$ and $w \in S_t$,

$$\mathbb{P}_{x, \{x\}}^*(X_t = w | S_0, \dots, S_t) = \frac{\pi^{(t)}(w)}{\pi^{(t)}(S_t)}.$$

We next list a few open problems.

Problem 1.16. For time-independent conductances [4] relies, in the setting of Remark 1.5, on using the time-reversed chain.

- (a) Can this idea be extended to monotone and genuinely time varying path of reversing measures $t \mapsto \{\pi^{(t)}(x), x \in V\}$?
- (b) Alternatively, does the bound (1.8) hold for uniformly elliptic, uniformly lazy and bounded edge conductances for which $t \mapsto \pi^{(t)}(x)$ are strictly monotone decreasing in t ?
- (c) Is it possible to establish for monotone increasing reversing measures a Gaussian type off-diagonal upper bound and somewhat comparable lower bounds?

Problem 1.17. Extend Proposition 1.11 to allow adding new vertices as \mathbb{D}_t evolves.

- (a) For example, start with \mathbb{D}_0 the unique infinite cluster of super-critical Bernoulli bond percolation on \mathbb{Z}^d , $d > 2$ and end with the full lattice $\mathbb{D}_\infty = \mathbb{Z}^d$.
- (b) Alternatively, consider finite graphs $\{\mathbb{D}_t\}$ that grow to a transient infinite graph \mathbb{D}_∞ of uniformly bounded degrees. Slow growth can yield recurrence of the walk, with a sharp phase transition from recurrence to transience in terms of the growth rate predicted for $\mathbb{D}_\infty = \mathbb{Z}^d$, $d > 2$ (see [7, Theorem 1.4, Conjecture 1.2]). Extend the scope of evolving sets to resolve this prediction.

Section 2 is devoted to the proof of Theorem 1.2 which partly builds on [20] (and at places also on [17, Ch. 3]), while Propositions 1.10, 1.11 and 1.15 are proved in Section 3.

2. Proof of Theorem 1.2

We start with two key facts about the evolving set process of Definition 1.12, in case $t \mapsto \pi^{(t)}(x)$ are non-decreasing.

Lemma 2.1. *The sequence $\{\pi^{(t)}(S_t)\}$ is a martingale and for any $t \geq 0$, $x \in V_0$ and $y \in V$*

$$P(0, x; t, y) = \frac{\pi^{(t)}(y)}{\pi^{(0)}(x)} \mathbb{P}_{\{x\}}(y \in S_t). \tag{2.1}$$

Proof. Fixing hereafter the starting state $S_0 = \{x\}$ in V_0 , we have from (1.10) that,

$$\begin{aligned} \mathbb{E}(\pi^{(t+1)}(S_{t+1}) | S_t) &= \mathbb{E} \left[\sum_{z \in V_{t+1}} \mathbb{I}_{\{z \in S_{t+1}\}} \pi^{(t+1)}(z) | S_t \right] \\ &= \sum_{z \in V_{t+1}} \mathbb{P}(z \in S_{t+1} | S_t) \pi^{(t+1)}(z) = \sum_{z \in V_{t+1}} \frac{\pi^{(t)}(S_t, z)}{\pi^{(t+1)}(z)} \pi^{(t+1)}(z) = \pi^{(t)}(S_t). \end{aligned}$$

That is, $\{\pi^{(t)}(S_t)\}$ is a martingale.

Turning to confirm the identity (2.1), note first that when $t = 0$, both sides of it equal $\mathbb{I}_{\{y=x\}}$. Next, if this identity holds for t , then using Chapman–Kolmogorov, our induction hypothesis, the formula for $P(t, z; t + 1, y)$ and (1.10),

we find that

$$\begin{aligned}
 P(0, x; t + 1, y) &= \sum_{z \in V_t} P(0, x; t, z) P(t, z; t + 1, y) \\
 &= \sum_{z \in V_t} \frac{\pi^{(t)}(z)}{\pi^{(0)}(x)} \mathbb{P}_{\{x\}}(z \in S_t) P(t, z; t + 1, y) \\
 &= \frac{1}{\pi^{(0)}(x)} \mathbb{E}_{\{x\}} \left[\sum_{z \in S_t} \pi^{(t)}(z) P(t, z; t + 1, y) \right] \\
 &= \frac{1}{\pi^{(0)}(x)} \mathbb{E}_{\{x\}} [\pi^{(t)}(S_t, y)] \\
 &= \frac{1}{\pi^{(0)}(x)} \mathbb{E}_{\{x\}} [\pi^{(t+1)}(y) \mathbb{P}(y \in S_{t+1} | S_t)] = \frac{\pi^{(t+1)}(y)}{\pi^{(0)}(x)} \mathbb{P}_{\{x\}}(y \in S_{t+1}).
 \end{aligned}$$

Thus, by induction (2.1) holds for all t . □

The next result is essential to our proof and the only place where we utilize the assumed isoperimetric inequality (1.5).

Lemma 2.2. For some $\tilde{c} = \tilde{c}(\gamma)$ positive, $\beta := \alpha - 2/d$, any $\alpha \in (0, 1)$, $t \geq 0$ and $x \in V_0$,

$$\mathbb{E}_{\{x\}} [\pi^{(t+1)}(S_{t+1})^\alpha - \pi^{(t)}(S_t)^\alpha | S_t] \leq -\tilde{c}\alpha(1 - \alpha)\kappa_t^2 \pi^{(t)}(S_t)^\beta \mathbb{I}_{\{\pi^{(t)}(S_t) > 0\}}. \tag{2.2}$$

Further, for $\alpha > 1$ we have the converse bound

$$\mathbb{E}_{\{x\}} [\pi^{(t+1)}(S_{t+1})^\alpha - \pi^{(t)}(S_t)^\alpha | S_t] \geq \tilde{c}\alpha(\alpha - 1)\kappa_t^2 \pi^{(t)}(S_t)^\beta \mathbb{I}_{\{\pi^{(t)}(S_t) > 0\}}. \tag{2.3}$$

Proof. Note that $\pi^{(t)}(S_t) = 0$ iff $S_t = \emptyset$, in which case by Definition 1.12 also $S_{t+1} = \emptyset$ and our claim trivially holds. Assuming hereafter that $\pi^{(t)}(S_t) > 0$, since U_{t+1} is independent of S_t we deduce from (1.10) that for every $y \in V_{t+1}$

$$\begin{aligned}
 p_\star(y, t) &:= \mathbb{P}(y \in S_{t+1} | U_{t+1} \leq 1/2, S_t) \\
 &= \mathbb{P}\left(U_{t+1} \leq \frac{\pi^{(t)}(S_t, y)}{\pi^{(t+1)}(y)} \mid U_{t+1} \leq 1/2, S_t \right) \\
 &= 1 \wedge \frac{2\pi^{(t)}(S_t, y)}{\pi^{(t+1)}(y)}.
 \end{aligned} \tag{2.4}$$

Next, let

$$\begin{aligned}
 \Delta_t &:= \frac{1}{\pi^{(t)}(S_t)} \sum_{y \in V_{t+1}} \pi^{(t+1)}(y) p_\star(y, t) \\
 &= \frac{1}{\pi^{(t)}(S_t)} \sum_{y \in V_{t+1}} [\pi^{(t+1)}(y) \wedge 2\pi^{(t)}(S_t, y)].
 \end{aligned} \tag{2.5}$$

By assumption, our lazy random walk is such that $\pi^{(t)}(y, y) \geq \gamma \pi^{(t)}(y)$ for some $\gamma \in (0, 1/2)$. Consequently, for any $y \in S_t$,

$$\pi^{(t)}(S_t, y) \geq \pi^{(t)}(y, y) \geq \gamma \pi^{(t)}(y) \geq \frac{\gamma}{1 - \gamma} \pi^{(t)}(S_t^c, y). \tag{2.6}$$

Now, since $t \mapsto \pi^{(t)}(y)$ is non-decreasing, it follows from (2.4) and (2.6) that for $y \in S_t$,

$$\begin{aligned} \pi^{(t+1)}(y)p_\star(y, t) &= \pi^{(t+1)}(y) \wedge 2\pi^{(t)}(S_t, y) \\ &\geq \pi^{(t)}(y) \wedge 2\pi^{(t)}(S_t, y) \\ &= \pi^{(t)}(S_t, y) + \pi^{(t)}(S_t^c, y) \wedge \pi^{(t)}(S_t, y) \geq \pi^{(t)}(S_t, y) + \frac{\gamma}{1-\gamma} \pi^{(t)}(S_t^c, y). \end{aligned}$$

Likewise, for $y \in S_t^c$,

$$\pi^{(t+1)}(y)p_\star(y, t) \geq \pi^{(t)}(S_t, y) + \frac{\gamma}{1-\gamma} \pi^{(t)}(S_t, y).$$

Letting

$$R_t := \frac{\pi^{(t)}(S_t, S_t^c)}{\pi^{(t)}(S_t)}, \quad \Gamma_t := \frac{\pi^{(t+1)}(S_{t+1})}{\pi^{(t)}(S_t)}$$

we find upon combining the preceding inequalities with the definition (2.5) of Δ_t , that

$$\Delta_t \geq \frac{1}{\pi^{(t)}(S_t)} \left[\pi^{(t)}(S_t) + \frac{2\gamma}{1-\gamma} \pi^{(t)}(S_t, S_t^c) \right] = 1 + \frac{2\gamma}{1-\gamma} R_t. \tag{2.7}$$

Further, with $\pi^{(t)}(S_t)$ a martingale and U_{t+1} independent of S_t , we have that

$$1 = \mathbb{E}(\Gamma_t | S_t) = \frac{1}{2} \mathbb{E}(\Gamma_t | U_{t+1} \leq 1/2, S_t) + \frac{1}{2} \mathbb{E}(\Gamma_t | U_{t+1} > 1/2, S_t).$$

But, from the definition of Δ_t and of $p_\star(y, t)$ we deduce that

$$\mathbb{E}(\Gamma_t | U_{t+1} \leq 1/2, S_t) = \Delta_t, \quad \mathbb{E}(\Gamma_t | U_{t+1} > 1/2, S_t) = 2 - \Delta_t.$$

Considering first $\alpha \in (0, 1)$, by Jensen's inequality and the preceding identities,

$$\begin{aligned} \mathbb{E}(\Gamma_t^\alpha | S_t) &= \frac{1}{2} \mathbb{E}(\Gamma_t^\alpha | U_{t+1} \leq 1/2, S_t) + \frac{1}{2} \mathbb{E}(\Gamma_t^\alpha | U_{t+1} > 1/2, S_t) \\ &\leq \frac{1}{2} [\mathbb{E}(\Gamma_t | U_{t+1} \leq 1/2, S_t)]^\alpha + \frac{1}{2} [\mathbb{E}(\Gamma_t | U_{t+1} > 1/2, S_t)]^\alpha \\ &= \frac{1}{2} \Delta_t^\alpha + \frac{1}{2} (2 - \Delta_t)^\alpha =: f_\alpha(\Delta_t - 1). \end{aligned} \tag{2.8}$$

Next note that the even function $f_\alpha(\cdot)$ is non-increasing on $[0, 1]$ when $\alpha \in (0, 1)$ and non-decreasing on $[0, 1]$ for any other $\alpha \in \mathbb{R}$. Further, $f_\alpha(0) = 1$ and $f_\alpha''(y) = \alpha(\alpha - 1)f_{\alpha-2}(y)$. Hence, for $y \in [0, 1]$,

$$f_\alpha(y) \leq 1 + \alpha(\alpha - 1) \frac{y^2}{2}, \quad \alpha \in (0, 1), \tag{2.9}$$

$$f_\alpha(y) \geq 1 + \alpha(\alpha - 1) \frac{y^2}{8}, \quad \alpha \geq 1. \tag{2.10}$$

It thus follows from (2.7)–(2.9) that when $\alpha \in (0, 1)$,

$$\mathbb{E}(\Gamma_t^\alpha | S_t) \leq f_\alpha(\Delta_t - 1) \leq f_\alpha\left(\frac{2\gamma}{1-\gamma} R_t\right) \leq 1 - \frac{2\alpha(1-\alpha)\gamma^2}{(1-\gamma)^2} R_t^2. \tag{2.11}$$

Our assumption that \mathbb{G} is locally finite, and the construction of the evolving set $\{S_t\}$ guarantees the finiteness of each S_t . Hence, from (1.5) we have that for any $t \geq 0$,

$$R_t \geq \kappa_t \pi^{(t)}(S_t)^{-1/d}. \tag{2.12}$$

Thus, from (2.11) we conclude that for some positive $\tilde{c} = \tilde{c}(\gamma)$ and all t ,

$$\begin{aligned} \mathbb{E} \left[\frac{\pi^{(t+1)}(S_{t+1})^\alpha}{\pi^{(t)}(S_t)^\alpha} - 1 \middle| S_t \right] &\leq -\frac{2\alpha(1-\alpha)\gamma^2}{(1-\gamma)^2} R_t^2 \\ &\leq -\tilde{c}\alpha(1-\alpha)\kappa_t^2 \pi^{(t)}(S_t)^{-2/d}, \end{aligned} \tag{2.13}$$

and multiplying both sides by $\pi^{(t)}(S_t)^\alpha$ yields the upper bound of (2.2).

Turning to the proof of (2.3), similarly to the derivation of (2.8) and (2.11) we get from (2.7) and (2.10) that when $\alpha > 1$,

$$\mathbb{E}(\Gamma_t^\alpha | S_t) \geq f_\alpha(\Delta_t - 1) \geq f_\alpha \left(\frac{2\gamma}{1-\gamma} R_t \right) \geq 1 + \frac{\alpha(\alpha-1)\gamma^2}{2(1-\gamma)^2} R_t^2.$$

Using (2.12) we find, similarly to the derivation of (2.13), that now,

$$\mathbb{E} \left[\frac{\pi^{(t+1)}(S_{t+1})^\alpha}{\pi^{(t)}(S_t)^\alpha} - 1 \middle| S_t \right] \geq \tilde{c}\alpha(\alpha-1)\kappa_t^2 \pi^{(t)}(S_t)^{-2/d}, \tag{2.14}$$

ending with (2.3). □

Our next lemma embeds $\{\pi^{(t)}(S_t)\}$ as the integer time samples of a continuous martingale (assuming as before that $t \mapsto \pi^{(t)}(x)$ are non-decreasing).

Lemma 2.3. *There exists a martingale $(M_u, u \geq 0)$ of a.s. continuous sample path, such that $M_i = \pi^{(i)}(S_i)$ for $i \in \mathbb{N}$ and $\tau = \inf\{u \geq 0 : M_u \leq 0\}$ is $\mathbb{N} \cup \{\infty\}$ -valued.*

Proof. With $\Phi(\cdot)$ the standard normal CDF and $(B_s, s \geq 0)$ a standard Brownian motion, let $S_0 = \{x\}$ and $U_{i+1} = \Phi(B_{i+1} - B_i)$ the i.i.d. Uniform(0, 1) variables used to construct S_{i+1} from S_i in Definition 1.12. The process $\{S_i\}$ is then adapted to $\mathcal{F}_u := \sigma\{B_s, s \in [0, u]\}$. Considering the \mathcal{F}_u -adapted process

$$M_u := \mathbb{E}[\pi^{(i+1)}(S_{i+1}) | \mathcal{F}_u], \quad \forall u \in [i, i+1), i \in \mathbb{N}, \tag{2.15}$$

we have by the independence of Brownian increments and Lemma 2.1, that for any $i \in \mathbb{N}$,

$$M_i = \mathbb{E}[\pi^{(i+1)}(S_{i+1}) | \mathcal{F}_i] = \mathbb{E}[\pi^{(i+1)}(S_{i+1}) | S_i] = \pi^{(i)}(S_i). \tag{2.16}$$

Clearly, (M_u, \mathcal{F}_u) is a (Doob) martingale within each interval $[i, i+1)$. Upon plugging (2.16) at $i+1$ within (2.15), the martingale property extends to $[i, i+1]$, which by the law of iterated expectations yields that (M_u, \mathcal{F}_u) is a martingale for all $u \geq 0$. Turning to the continuity of $u \mapsto M_u$, for any $i \in \mathbb{N}$, $y \in V_{i+1}$ and $A \subseteq V_i$ let

$$H_i(A, y) := \Phi^{-1} \left(\frac{\pi^{(i)}(A, y)}{\pi^{(i+1)}(y)} \right).$$

By Definition 1.12 and the independence of Brownian increments, we have that for any $s \in [0, 1)$ and $i \in \mathbb{N}$,

$$\begin{aligned} M_{i+s} &= \sum_{y \in V_{i+1}} \pi^{(i+1)}(y) \mathbb{P}(H_i(S_i, y) \geq B_{i+1} - B_i | S_i, B_{i+s} - B_i) \\ &= \sum_{y \in V_{i+1}} \pi^{(i+1)}(y) \Phi \left(\frac{H_i(S_i, y) - B_{i+s} + B_i}{\sqrt{1-s}} \right). \end{aligned} \tag{2.17}$$

With $s \mapsto B_{i+s}$ continuous, each term of the sum on the right-hand side of (2.17) is continuous in $s \in [0, 1)$. Having \mathbb{G} locally finite, only finitely many $y \in V$ for which $H_i(S_i, y) \neq -\infty$ contribute to that sum, hence $u \mapsto M_u$ is continuous on $[i, i + 1)$. Further, a.s. $H_i(S_i, y) \neq B_{i+1} - B_i$ for all $y \in V_{i+1}$, in which case by the continuity of $u \mapsto B_u$ at $i + 1$,

$$\lim_{s \uparrow 1} \Phi \left(\frac{H_i(S_i, y) - B_{i+s} + B_i}{\sqrt{1-s}} \right) = \mathbb{I}\{H_i(S_i, y) \geq B_{i+1} - B_i\}.$$

Upon comparing (2.17) with Definition 1.12, this extends the continuity of $u \mapsto M_u$ to $[i, i + 1]$ and thereby to all $u \geq 0$.

Finally, M_u is non-negative by (2.15), whereas by (2.17) it is strictly positive on $[i, i + 1)$ unless $H_i(S_i, y) = -\infty$ for all y , namely $S_i = \emptyset$ (in which case $M_u = 0$ for all $u \geq i$). \square

Proof of Theorem 1.2. It suffices to prove (1.7) and (1.8) for $s = 0$, as $s \in (0, t)$ then follows by considering the edge conductances $\{\pi^{(s+\cdot)}\}$ starting at $X_s = x \in V_s$ (and consequently, using $\beta(u)/\beta(s)$ and $\psi_{d,\beta}(t) - \psi_{d,\beta}(s)$ instead of $\beta(u)$ and $\psi_{d,\beta}(t)$).

Fixing hereafter $s = 0$, we start with a short derivation of the sub-optimal bound $P(0, x; t, y) \leq C' \psi_d(t)^{-(1-\alpha)d/2}$ for $\alpha \in (0, 1)$, non-decreasing $t \mapsto \pi^{(t)}(y) \leq C$, and some $C' = C'(d, \alpha, \gamma, C)$ finite. Indeed, (2.1) then result with $P(0, x; t, y) \leq C^{1-\alpha} m_t$ for $m_t = \mathbb{E}_{\{x\}}[M_t^\alpha]/M_0$ and $M_t = \pi^{(t)}(S_t)$. Further, with $\beta = \alpha - \delta(1 - \alpha)$, the elementary bound

$$\mathbb{E}[Z^\beta \mathbb{I}_{Z>0}] \geq (\mathbb{E}[Z^\alpha])^{1+\delta}, \quad (2.18)$$

holds for $Z = M_t/M_0 \geq 0$ of mean one and $\delta > 0$. Taking the expectation of (2.2), it thus follows from (2.18) that for $\delta = 2/((1 - \alpha)d)$,

$$m_{t+1} \leq m_t \exp(-\tilde{\alpha}(1 - \alpha)\kappa_t^2 m_t^\delta), \quad (2.19)$$

and consequently $m_t \leq c' \psi_d(t)^{-1/\delta}$ for some $c'(\alpha, d, \gamma)$ finite, as claimed.

However, the sharp bound (1.8) (where $\alpha = 0$), requires the more elaborate argument provided next, where we first derive (1.8) out of (1.7) in case $\pi^{(u)}(x)$ are effectively non-decreasing and uniformly bounded. Indeed, by its definition in (1.5),

$$\kappa_i \leq \inf_{v \in V_i} \pi^{(i)}(\{v\})^{1/d} \leq C^{1/d}, \quad \forall i \geq 0 \quad (2.20)$$

and consequently $\beta(u)^{1/d} \kappa_u \leq (\eta_* C)^{1/d}$. Thus, condition (1.6) holds (for $\lambda = 1/3$) whenever $\psi_{d,\beta}(t) \geq 3(\eta_* C)^{2/d}$. Since $\pi^{(t)}(x) \leq C$, it follows from (1.4) that

$$\beta(t) \geq \sup_x \left\{ \frac{\pi^{(0)}(x)}{\pi^{(t)}(x)} \right\} \geq \frac{\eta_0}{C},$$

hence the condition (1.6) holds whenever

$$\xi(t) := (\eta_*/\beta(t))^{2/d} \psi_{d,\beta}(t) \geq 3(\eta_*^2 C^2/\eta_0)^{2/d}, \quad (2.21)$$

in which case multiplying the inequality (1.7) by $\pi^{(0)}(x)$ yields the bound

$$\pi^{(0)}(x)h(0, x; t, y) \leq c_+ C \eta_* \xi(t)^{-d/2} \leq c_*(e + \xi(t))^{-d/2}, \quad (2.22)$$

for some $c_* = c_*(d, c_+, \eta_0, \eta_*, C)$ finite. Next, recall that by (1.3) and (2.1), for any $t \in \mathbb{N}$,

$$\pi^{(0)}(x)h(0, x; t, y) = \mathbb{P}_{\{x\}}(y \in S_t) \leq \mathbb{P}_{\{x\}}(S_t \neq \emptyset) = \mathbb{P}_{\{x\}}(M_t \neq 0), \quad (2.23)$$

for the continuous, non-negative \mathbb{P} -martingale $\{M_u\}_{u \geq 0}$ of Lemma 2.3. In view of (2.23), the left-hand side of (2.22) is at most one, hence increasing c_* guarantees that (2.22) trivially holds whenever (2.21) fails. Having effectively non-decreasing $t \mapsto \pi^{(t)}(x)$, implies further that $\xi(t) \geq \psi_d(t)$ and thus (1.8) is a consequence of (2.22).

Turning to the proof of (1.7), note that multiplying all edge conductances $\{\pi^{(u)}(x, y)\}$ by a common factor does not effect the transition probabilities of the associated random walk at step u . Hence, re-defining the edge conductances

$$\widehat{\pi}^{(u)}(x, y) = \beta(u)\pi^{(u)}(x, y), \quad u \in \mathbb{N}, (x, y) \in E,$$

results with $h(s, x; t, y) = \beta(t)\widehat{h}(s, x; t, y)$, $\psi_{d,\beta}(\cdot) = \widehat{\psi}_d(\cdot)$ and non-decreasing $u \mapsto \widehat{\pi}^{(u)}(x)$. We consequently proceed to bound the right-hand side of (2.23), for non-decreasing $u \mapsto \pi^{(u)}(x)$ and $\beta(u) \equiv 1$. To this end, we utilize the stopping times

$$\tau_k := \inf\{u \geq 0 : M_u \geq e^k\}, \quad T'_k := \inf\{i \in \mathbb{N} \cap (\tau_k, \infty) : M_i = 0\} \tag{2.24}$$

and note that for $r \in (0, t)$ of (1.6) and any $k \in \mathbb{Z}$,

$$\{M_t \neq 0\} \subseteq \{\tau_k > r\} \cup \{\tau_k \leq r, T'_k > t\}. \tag{2.25}$$

Further, for $\widetilde{M} := \sup_{u \geq 0} \{M_u\}$ and $E_k := \{e^k \leq \widetilde{M} < e^{k+1}\}$, by Doob's inequality

$$\mathbb{P}_{\{x\}}(E_k) \leq \mathbb{P}_{\{x\}}(\widetilde{M} \geq e^k) \leq \pi^{(0)}(x)e^{-k}. \tag{2.26}$$

Thus, fixing $\varepsilon \in (0, 1)$ and setting $k_0 := \lfloor \log \pi^{(0)}(x) \rfloor$, $L := \lceil \log(\varepsilon^2 \psi_d(t)^{d/2}) \rceil$, we get from (2.25) and (2.26) that

$$\begin{aligned} \mathbb{P}_{\{x\}}(M_t \neq 0) &\leq \mathbb{P}_{\{x\}}(\widetilde{M} \geq e^L) + \sum_{k=k_0}^{L-1} \mathbb{P}_{\{x\}}(\{M_t \neq 0\} \cap E_k) \\ &\leq \pi^{(0)}(x) \left[e^{-L} + \sum_{k=k_0}^{L-1} e^{-k} \mathbb{P}_{\{x\}}(\tau_k > r | \widetilde{M} \geq e^k) + \sum_{k=k_0}^{L-1} e^{-k} \mathbb{P}_{\{x\}}(T'_k > t | E_k, \tau_k \leq r) \right]. \end{aligned} \tag{2.27}$$

Noting that e^{-L} is of $O(\psi_d(t)^{-d/2})$ size, the remainder of the proof consists of three steps. First, by the continuity of our non-negative martingale, and the lower bound of (2.3) on its quadratic variation, we show in *Step I* that conditioning on $\{\widetilde{M} \geq e^k\}$ transforms the law of $\{S_0, \dots, S_r\}$ to that of Definition 1.14. Then, *Step II* shows that the probability of $\max_{i \leq r} \{\pi^{(i)}(S_i)\}$ not exceeding e^k for such size-biased evolving sets, is at most $O(\exp(-c\psi_d(r)e^{-2k/d}))$ and as a result the left sum in (2.27) is at most $O(\psi_d(r)^{-d/2})$ (see (2.35)). Noting that under $\{\tau_k \leq r\}$ the probability of $E_k = \{\tau_{k+1} = \infty\}$ is bounded away from zero, *Step III* controls the right sum over k in (2.27), as $\{E_k, \tau_k \leq r\}$ dictates a downward path $e^{a_{i+1}}$ driving $u \mapsto \pi^{(u)}(S_u)$, $u = \lceil \tau_k \rceil + i$, to zero at $u = t$, or else the super-martingale $Q_{i \wedge \sigma} \geq 0$ with $Q_0 \leq c_5 e^{k/2} \psi_d(t)^{-d}$, must exceed $O(e^{-3k/2})$, an event whose probability is $O(e^{2k} \psi_d(t)^{-d})$.

Step I. The \mathbb{P} -martingale (M_u, \mathcal{F}_u) is non-negative, continuous, hence converges \mathbb{P} -almost surely to a finite limit M_∞ . Further, $M_u = M_0 + W_{\langle M \rangle_u}$ for a standard Brownian motion $(W_s, s \geq 0)$, time changed by the quadratic variation $\langle M \rangle_u$ (e.g. [16, Theorem 3.4.6, Problem 3.4.7]). In particular, having a.s. finite M_∞ implies the same for $\langle M \rangle_\infty$. In view of Lemma 2.3, for any $i \in \mathbb{N}$,

$$\langle M \rangle_i \geq \sum_{j=1}^i \mathbb{E}[M_j^2 - M_{j-1}^2 | \mathcal{F}_{j-1}] \geq 2\widetilde{c} \sum_{j=0}^{i-1} \kappa_j^2 M_j^{2-2/d},$$

with the right inequality due to Lemma 2.2 (for $\alpha = 2 > 2/d$). Since $\psi_d(\infty) = \infty$, it then follows that

$$\liminf_{i \rightarrow \infty} \frac{\langle M \rangle_i}{\psi_d(i)} \geq 2\widetilde{c} \liminf_{i \rightarrow \infty} \frac{1}{\psi_d(i)} \sum_{j=0}^{i-1} \kappa_j^2 M_j^{2-2/d} = 2\widetilde{c} M_\infty^{2-2/d}.$$

We thus see that with probability one, if $M_\infty > 0$ then $\langle M \rangle_\infty = \infty$, out of which we deduce that necessarily $M_\infty = 0$. The a.s. convergence to zero of M_t allows us in turn to deduce that for any $u \geq 0$ and $z > 0$,

$$\mathbb{P}\left(\sup_{t \geq u} \{M_t\} \geq z | \mathcal{F}_u\right) = \frac{M_u}{z} \wedge 1. \tag{2.28}$$

Indeed, in case $M_u = 0$ the martingale condition implies that a.s. $M_t \equiv 0$ for all $t \geq u$, whereas for $M_u \in (0, z)$ we get (2.28) by applying for example [16, Problem 1.3.28(i)].

Turning to bound the left-sum in (2.27), note that subject to $\mathbb{I}\{\tau_k > r\}$, the probability of $\{\tilde{M} \geq e^k\}$ given \mathcal{F}_r is precisely the left-hand side of (2.28) for $z = e^k$ and $u = r$. With the unconditional probability given by (2.28) with $u = 0$, it thus follows that

$$\mathbb{P}_{\{x\}}(\tau_k > r | \tilde{M} \geq e^k) = \mathbb{E}_{\{x\}} \left[\frac{M_r}{M_0} \mathbb{I}(\tau_k > r) \right] \leq \mathbb{E}_{\{x\}} \left[\frac{M_r}{M_0} \mathbb{I}(T_k > r) \right], \tag{2.29}$$

where $T_k = \inf\{i \in \mathbb{N} : \pi^{(i)}(S_i) \geq e^k\}$ is the discrete-time analog of τ_k of (2.24) (hence necessarily $T_k \geq \tau_k$). Next note that the right-hand side of (2.29) equals $\widehat{\mathbb{P}}(T_k > r)$ for the martingale change of measure

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}(S_0, \dots, S_r) = \frac{\pi^{(r)}(S_r)}{\pi^{(0)}(x)}.$$

The measure $\widehat{\mathbb{P}}$ is thus given by the time-in-homogeneous Doob h -transform of the evolving sets process, for $h(t, A) = \pi^{(t)}(A)$, namely the measure corresponding to the transition kernel $\widehat{K}(\cdot, \cdot)$ of (1.11). That is, $\widehat{\mathbb{P}}$ is the law of the conditioned (size-biased) evolving set of Definition 1.14.

Step II. Under $\widehat{\mathbb{P}}$ with probability one S_i are non-empty and $Y_i := \pi^{(i)}(S_i)^{-1/2} \mathbb{I}_{\{T_k > i\}}$ finite, whereby from Markov's inequality and (2.29) we deduce that for any k ,

$$\begin{aligned} \mathbb{P}_{\{x\}}(\tau_k > r | \tilde{M} \geq e^k) &\leq \widehat{\mathbb{P}}_{\{x\}}(T_k > r) \\ &= \widehat{\mathbb{P}}_{\{x\}}(Y_r > e^{-k/2}) \leq e^{k/2} \widehat{\mathbb{E}}_{\{x\}}(Y_r). \end{aligned} \tag{2.30}$$

Further, by Lemma 2.2 with $\alpha = 1/2$ and $c = \tilde{c}/8 > 0$, we have that $\widehat{\mathbb{P}}$ -a.e. if $Y_i > 0$, namely $T_k > i$, then

$$\begin{aligned} \widehat{\mathbb{E}}_{\{x\}}(Y_{i+1} | Y_i) &= \mathbb{E}_{\{x\}} \left(\frac{\pi^{(i+1)}(S_{i+1})^{1/2} \mathbb{I}_{\{T_k > i+1\}}}{\pi^{(i)}(S_i)} \middle| Y_i \right) \\ &\leq Y_i^2 \mathbb{E}_{\{x\}}(\pi^{(i+1)}(S_{i+1})^{1/2} | Y_i) \leq Y_i (1 - 2c\kappa_i^2 Y_i^{4/d}). \end{aligned}$$

Note that either $Y_i = 0$, that is $\{T_k \leq i\}$, in which case necessarily $Y_{i+1} = 0$ and the preceding inequality holds, or else by definition $Y_i > e^{-k/2}$. Thus, $\widehat{\mathbb{P}}$ -a.e. for all i and Y_i ,

$$\widehat{\mathbb{E}}_{\{x\}}(Y_{i+1} | Y_i) \leq Y_i [1 - 2c\kappa_i^2 (Y_i^{4/d} \vee e^{-2k/d})]. \tag{2.31}$$

Recall [20, Lemma 12] that $E[2Zf(2Z)] \geq (EZ)f(EZ)$ for any $Z \geq 0$ and non-decreasing $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$. In particular, with $l_i := \widehat{\mathbb{E}}_{\{x\}}(Y_i)$ and $f(y) = (y/2)^{4/d} \vee e^{-2k/d}$, we deduce upon taking the expectation of (2.31) that

$$l_{i+1} \leq l_i - c\kappa_i^2 l_i f(l_i) \leq l_i e^{-c\kappa_i^2 f(l_i)}. \tag{2.32}$$

With $f(l_i)$ strictly positive it thus follows that either $l_i = 0$, or else

$$\int_{l_{i+1}}^{l_i} \frac{dz}{zf(z)} \geq \frac{1}{f(l_i)} \int_{l_{i+1}}^{l_i} \frac{dz}{z} = \frac{1}{f(l_i)} \log \frac{l_i}{l_{i+1}} \geq c\kappa_i^2. \tag{2.33}$$

Hence, if $l_r > 0$ then by (2.32), $l_i > 0$ for $i < r$ and summing (2.33) over $0 \leq i < r$, yields

$$c\psi_d(r) \leq \int_{l_r}^{\infty} (2^{4/d} z^{-1-4/d}) \wedge (e^{2k/d} z^{-1}) dz \tag{2.34}$$

(which trivially holds also when $l_r = 0$). We proceed to rule out having $l_r > 2e^{-k/2}$. Indeed, in that case we get from (2.34) that

$$c\psi_d(r) \leq \int_{l_r}^{\infty} 2^{4/d} z^{-1-4/d} dz = 2^{4/d} (d/4) l_r^{-4/d},$$

whereby $l_r \leq c' \psi_d(r)^{-d/4}$ for $c' = 2(4d/c)^{d/4}$. As $k < L$, this yields in view of (1.6) and our choice of L that

$$\varepsilon^{-1} \psi_d(t)^{-d/4} \leq e^{-(L-1)/2} < 2e^{-k/2} < l_r \leq c' \psi_d(r)^{-d/4} \leq c' 3^{d/4} \psi_d(t)^{-d/4},$$

yielding a contradiction when $\varepsilon = (1/c')3^{-d/4}$. Taking hereafter such ε we thus have that $l_r \leq 2e^{-k/2}$ in which case (2.34) yields

$$c \psi_d(r) \leq e^{2k/d} \int_{l_r}^{2e^{-k/2}} \frac{dz}{z} + 2^{4/d} \int_{2e^{-k/2}}^{\infty} \frac{dz}{z^{1+4/d}} \leq e^{2k/d} (\log(2e^{-k/2}/l_r) + c_0),$$

for some finite $c_0 = c_0(d)$. That is, for $c_1 = 2e^{c_0}$ finite,

$$l_r \leq c_1 e^{-k/2} \exp\{-c \psi_d(r) e^{-2k/d}\}.$$

Plugging this bound in the right-hand side of (2.30), we bound the left sum in (2.27) after change of variable $s = e^{-2k/d} \psi_d(r)$, by

$$\begin{aligned} \sum_{k=k_0}^{L-1} e^{-k} \mathbb{P}_{\{x\}}(\tau_k > r | \tilde{M} \geq e^k) &\leq c_1 \sum_{k=k_0}^{L-1} e^{-k} \exp\{-c \psi_d(r) e^{-2k/d}\} \\ &\leq c_2 \int_0^\infty e^{-cs} (s/\psi_d(r))^{d/2} s^{-1} ds \leq c_3 \psi_d(r)^{-d/2}, \end{aligned} \tag{2.35}$$

for some finite constants $c_j = c_j(d, \gamma)$, $j = 2, 3$.

Step III. Moving next to bound the right sum in (2.27), conditioning on $\{\mathcal{F}_{\lceil \tau_k \rceil}, \tau_k \leq r\}$ we have by the strong Markov property at $\lceil \tau_k \rceil$ that,

$$\tilde{S}_i := S_{\lceil \tau_k \rceil + i}, \quad i \geq 0,$$

is an evolving set process for conductances $\tilde{\pi}^{(i)}(\cdot) := \pi^{(\lceil \tau_k \rceil + i)}(\cdot)$, with which we also associate

$$\tilde{\kappa}_i := \kappa_{\lceil \tau_k \rceil + i}, \quad \tilde{\psi}_d(i) := \psi_d(\lceil \tau_k \rceil + i) - \psi_d(\lceil \tau_k \rceil).$$

Note that if $k \geq k_0$ then $\tau_k > 0$ and hence $M_{\tau_k} = e^k$ whenever $\tau_k < \infty$. Thus, from (2.28) at the stopping time $u = \tau_k \leq r$, we deduce that

$$\mathbb{P}_{\{x\}}(E_k | \tau_k \leq r) = \mathbb{P}_{\{x\}}(\tau_{k+1} = \infty | \tau_k \leq r) = 1 - e^{-1}.$$

Consequently, for $c_4 = 1/(1 - e^{-1})$, $k \geq k_0$ and any $\mathcal{F}_{\lceil \tau_k \rceil + i}$ -stopping time

$$\sigma := \inf\{i \geq 0 : \tilde{\pi}^{(i)}(\tilde{S}_i) > e^{a_{i+1}}\},$$

with $a_{t - \lceil \tau_k \rceil} = -\infty$, one has that

$$\begin{aligned} \mathbb{P}_{\{x\}}(T'_k > t | E_k, \tau_k \leq r) &= c_4 \mathbb{P}_{\{x\}}(T'_k > t, \tau_{k+1} = \infty | \tau_k \leq r) \\ &\leq c_4 \mathbb{P}_{\{x\}}(\sigma < t - \lceil \tau_k \rceil, \tau_{k+1} = \infty | \tau_k \leq r). \end{aligned} \tag{2.36}$$

In particular, we shall employ (2.36) for the non-increasing a_i such that

$$\frac{\tilde{c}}{4} \tilde{\kappa}_i^2 = 2 \int_{a_{i+1}}^{a_i} e^{2z/d} dz, \quad 0 \leq i < t - \lceil \tau_k \rceil. \tag{2.37}$$

To this end, we first show that $(Q_{i \wedge \sigma}, \mathcal{F}_{\lceil \tau_k \rceil + i})$, $i < t - \lceil \tau_k \rceil$ is a super-martingale, for

$$Q_i := e^{-2a_i} \tilde{Y}_i, \quad \tilde{Y}_i := \tilde{\pi}^{(i)}(\tilde{S}_i)^{1/2} \mathbb{I}(\tau_{k+1} > \lceil \tau_k \rceil + i) \mathbb{I}(\tau_k \leq r).$$

Indeed, applying Lemma 2.2 (for $\alpha = 1/2$), to the evolving process $\{\tilde{S}_i\}$, if $\tau_k \leq r$ and $\tau_{k+1} > \lceil \tau_k \rceil + i$ then

$$\mathbb{E}_{\{x\}}[\tilde{Y}_{i+1} | \mathcal{F}_{\lceil \tau_k \rceil + i}] \leq \tilde{Y}_i \left(1 - \frac{\tilde{c}}{4} \tilde{\kappa}_i^2 \tilde{Y}_i^{-4/d} \mathbb{1}_{\{\tilde{Y}_i > 0\}} \right).$$

This inequality trivially holds if either $\{\tau_{k+1} \leq \lceil \tau_k \rceil + i\}$ or $\{\tau_k > r\}$ (whereby both sides are zero), yielding that for $i < t - 1 - \lceil \tau_k \rceil$

$$\mathbb{E}_{\{x\}}[Q_{i+1} | \mathcal{F}_{\lceil \tau_k \rceil + i}] \leq Q_i \exp \left\{ 2(a_i - a_{i+1}) - \frac{\tilde{c}}{4} \tilde{\kappa}_i^2 (\tilde{Y}_i)^{-4/d} \mathbb{1}_{\{\tilde{Y}_i > 0\}} \right\}. \quad (2.38)$$

Recall that our choice of a_i in (2.37), implies that

$$\frac{\tilde{c}}{4} \tilde{\kappa}_i^2 \geq 2(a_i - a_{i+1}) e^{2a_{i+1}/d} \geq 2(a_i - a_{i+1}) (\tilde{Y}_i)^{4/d}, \quad (2.39)$$

when $\tilde{Y}_i \leq e^{a_{i+1}/2}$. Thus, the exponent on the right-hand side of (2.38) is non-positive when both $i < \sigma$ and $\tilde{Y}_i > 0$, in which it follows from (2.38) that

$$\mathbb{E}_{\{x\}}(Q_{(i+1) \wedge \sigma} | \mathcal{F}_{\lceil \tau_k \rceil + i}) \leq Q_{i \wedge \sigma}.$$

As this inequality trivially holds with equality when $i \geq \sigma$, as well as when $\tilde{Y}_i = 0$ (for then also $\tilde{Y}_{i+1} = 0$), we have the claimed super-martingale property.

Now, since $\tilde{Y}_i < e^{(k+1)/2}$, if $\tau_k \leq r$ then by (2.37),

$$Q_0 \leq e^{-2a_0} e^{(k+1)/2} \leq c_5 e^{k/2} (\psi_d(t) - \psi_d(r))^{-d},$$

for some $c_5 = c_5(d, \gamma)$ finite. Further, by the definition of σ , if $i = \sigma < t - \lceil \tau_k \rceil$ then \tilde{S}_i is non-empty, hence $\tilde{\pi}^{(i)}(\tilde{S}_i)^{1/d} \geq \tilde{\kappa}_i$ (see (2.20)). It then follows from (2.37) that

$$e^{2a_i/d} = e^{2a_{i+1}/d} + \frac{\tilde{c}}{4d} \tilde{\kappa}_i^2 \leq e^{2a_{i+1}/d} + \frac{\tilde{c}}{4d} \tilde{\pi}^{(i)}(\tilde{S}_i)^{2/d}$$

which by definition of σ implies that also

$$\tilde{\pi}^{(i)}(\tilde{S}_i) \geq c_6 e^{a_i} \quad (2.40)$$

for $c_6 := (1 + \tilde{c}/(4d))^{-d/2}$ positive. In case $\tau_{k+1} = \infty$ it further suffices to consider only those $i \geq 0$ for which the right-hand side of (2.40) is at most $e^{(k+1)}$, implying in turn that (when also $\tau_k \leq r$),

$$Q_i = e^{-2a_i} \tilde{\pi}^{(i)}(\tilde{S}_i)^{1/2} \geq c_6^2 (c_6 e^{a_i})^{-3/2} \geq c_6^2 e^{-3(k+1)/2}.$$

In conclusion, when $\tau_k \leq r$,

$$\{\sigma < t - \lceil \tau_k \rceil, \tau_{k+1} = \infty\} \subseteq \{\sigma < t - \lceil \tau_k \rceil, Q_\sigma \geq c_6^2 e^{-3(k+1)/2}\}.$$

Applying Doob's optional stopping to the non-negative super-martingale $\{Q_{i \wedge \sigma}\}$ we further bound the right-hand side of (2.36) by

$$\begin{aligned} c_4 \mathbb{P}_{\{x\}}(\sigma < t - \lceil \tau_k \rceil, Q_\sigma \geq c_6^2 e^{-3(k+1)/2} | \tau_k \leq r) &\leq c_7 e^{3k/2} \mathbb{E}_{\{x\}}(Q_0 | \tau_k \leq r) \\ &\leq c_8 e^{2k} (\psi_d(t) - \psi_d(r))^{-d}, \end{aligned}$$

for some finite $c_j(d, \gamma)$, $j = 7, 8$. In view of (2.36) the right sum of (2.27) is thus bounded by

$$\sum_{k=k_0}^{L-1} e^{-k} \mathbb{P}_{\{x\}}(T'_k > t | E_k, \tau_k \leq r) \leq c_9 e^L (\psi_d(t) - \psi_d(r))^{-d}. \quad (2.41)$$

For our choice of L , the bound (1.7) follows from (2.27), (2.35) and (2.41). \square

3. Proofs of Propositions 1.10, 1.11 and 1.15

Proof of Proposition 1.10. Let $\{\tau_j\}$ be a collection of i.i.d $\exp(2)$ random variables. We simulate the CSRW using $T_k := \sum_{j=1}^k \tau_j$ as our successive Poisson clocks and independently designate that each time T_k the clock rings, with probability $(1/2)$ the walk Y_t stays put, and with probability $(1/2)$ it makes a jump according to the given edge conductances at time T_k . By the thinning property of the Poisson process, the simulated process $t \mapsto Y_t$ is the CSRW of Definition 1.8. On the other hand, the sampled process $X_k = Y_{T_k}$ has the law of $(1/2)$ -lazy discrete time random walk on \mathbb{G} , with time-varying edge conductances $\pi^{(T_k)}(x, y)$ that are in $[C_1^{-1}, C_1]$ for every realization ω of $\{T_k\}$. Consequently, denoting $N_t = \max\{k \in \mathbb{N} : T_k \leq t\}$, a Poisson process of rate 2, we have as in Corollary 1.7 that for some $C_2 = C_2(d, C_1) > 0$, any $t \geq s \geq 0$ and all ω ,

$$\psi_d^\omega(t) - \psi_d^\omega(s) \geq C_2(N_t - N_s).$$

From Definition 1.9 of the effectively non-decreasing RCLL conductances $t \mapsto \pi^{(t)}(x)$ and (1.8), for a.e. $\omega = \{T_k\}$ there exists $c_*^\omega = c_*^\omega(d, C_1)$ finite, such that the quenched heat-kernel bound

$$P^\omega(s, x; t, y) \leq c_*^\omega [e + C_2(N_t - N_s)]^{-d/2} =: c_*^\omega \phi(N_t - N_s), \tag{3.1}$$

applies for the transition probabilities $P^\omega(s, x; t, y)$ of the CSRW $\{Y_t\}$. With $\phi(\cdot)$ positive and decreasing on \mathbb{R}_+ , we have that

$$\phi(N_t - N_s) \leq \phi(0) \mathbb{I}_{\{N_t - N_s \leq t - s\}} + \phi(t - s),$$

and consequently

$$\int_0^\infty \mathbb{E}[(c_*^\omega)^{-1} P^\omega(0, x; t, y)] dt \leq \phi(0) \int_0^\infty \mathbb{P}(N_t \leq t) dt + \int_0^\infty \phi(t) dt \tag{3.2}$$

is finite. Thus, by Fubini’s theorem $\int_0^\infty (c_*^\omega)^{-1} P^\omega(0, x; t, y) dt$ is finite for a.e. ω , which together with the finiteness of c_*^ω implies that $\int_0^\infty P^\omega(0, x; t, y) dt$ is finite. That is, starting at any non-random $x \in V$ we have a finite total local time for the CSRW at any $y \in V$. Hence, for a.e. ω the sampled process at jump times $\{Y_{T_k}\}$, visits every $y \in V$ only finitely often. \square

Proof of Proposition 1.11. In view of Theorem 1.2 with $V = \mathcal{V}(\mathbb{D}_0)$ and Remark 1.3, with $d/2 > 1$, we have the stated claim upon showing that for any $\theta = \theta_{\text{iso}} > 0$, there exists some $T = T(\omega, \theta) < \infty$ and constant $c'(\theta, d) > 0$ such that the isoperimetric growth function satisfies

$$\psi_d(t) \geq c' t^{1-\theta(1-1/d)}, \quad \forall t \geq T, P^\omega\text{-a.s.} \tag{3.3}$$

(as then $\psi_d(t)^{-d/2}$ would be summable upon taking θ sufficiently small).

To this end, let \mathbb{D}_u^ℓ denote the vertices of $\mathbb{D}_u \cap [-\ell, \ell]^d$ and recall that starting at $x = 0$ we have that the evolving set $S_u \subseteq \mathbb{D}_u^\ell$ (because the SRW has at most linear growth in each direction). Here $\pi^{(u)}(x, y) \in \{0, 1\}$ so we are just counting edges. Further, with all degrees of vertices of \mathbb{D}_u within $[1, 2d]$, we replace $\pi^{(u)}(A)$ by the size $|A|^{(u)}$ of $A \cap \mathbb{D}_u$ with $|\partial A|^{(u)} = \pi^{(u)}(A, A^c)$. By [21, Theorem 1.2] (together with Borel–Cantelli lemma), the unique infinite percolation cluster \mathbb{D}_0 of [21] satisfies the following isoperimetric inequality for some $c = c(\theta) > 0$ and all $l \geq l_0(\omega, \theta)$ large enough

$$\inf_{A \subseteq \mathbb{D}_0^l, |A| \leq |\mathbb{D}_0^l|/2} \left\{ \frac{|\partial_{\mathbb{D}_0^l} A|}{|A|^{(d-1)/d}} \right\} \geq cl^{-\theta(1-1/d)}, \quad P^\omega\text{-a.s.} \tag{3.4}$$

Moreover, since to \mathbb{D}_u we only add edges and no new vertices, clearly $|\partial A|^{(u)} \geq |\partial A|^{(0)}$, and $|A|^{(0)} = |A|^{(u)}$, with the inequality (3.4) holding uniformly for all $\{\mathbb{D}_u\}$. Applying (3.4) to sets $S_u \subseteq \mathbb{D}_u^{2u}$, we have that

$$\kappa_u \geq c' u^{-\theta(1-1/d)},$$

yielding all $t \geq 2T$, the claimed growth of (3.3),

$$\psi_d(t) \geq \sum_{u=T}^{t-1} \kappa_u^2 \geq c'^2(t-T)t^{-\theta(1-1/d)} \geq \frac{c'^2}{2}t^{1-\theta(1-1/d)}.$$

□

Proof of Proposition 1.15. With (a) and (b) trivially holding at $t = 0$, we proceed by induction on t . Specifically, we assume that both (a) and (b) hold for some $t \geq 0$. Then, with $\mathbf{S}_t = (S_0, \dots, S_t)$, by the definition of $P(\cdot; \cdot)$ and $P^*(\cdot; \cdot)$, our hypothesis of (b) holding for t implies that for any $v \in B$ such that $\pi^{(t)}(S_t, v) > 0$,

$$\begin{aligned} & \mathbb{P}_{x, \{x\}}^*(X_{t+1} = v, S_{t+1} = B | \mathbf{S}_t) \\ &= \sum_{w \in S_t} \mathbb{P}^*(X_{t+1} = v, S_{t+1} = B | X_t = w, \mathbf{S}_t) \mathbb{P}_{x, \{x\}}^*(X_t = w | \mathbf{S}_t) \\ &= \sum_{w \in S_t} \frac{P(t, w; t+1, v) K(t, S_t; t+1, B) \pi^{(t+1)}(v)}{\pi^{(t)}(S_t, v)} \frac{\pi^{(t)}(w)}{\pi^{(t)}(S_t)} \\ &= \frac{\pi^{(t+1)}(v)}{\pi^{(t)}(S_t)} \frac{\sum_{w \in S_t} \pi^{(t)}(w) P(t, w; t+1, v)}{\pi^{(t)}(S_t, v)} K(t, S_t; t+1, B) \\ &= \frac{\pi^{(t+1)}(v)}{\pi^{(t)}(S_t)} K(t, S_t; t+1, B). \end{aligned} \tag{3.5}$$

By Definition 1.14, the conditioned evolving set is such that $X_{t+1} \in S_{t+1}$ so the left-hand side of (3.5) is zero when $\pi^{(t)}(S_t, v) = 0$. Consequently, summing in (3.5) over $v \in B$ we find that

$$\mathbb{P}_{x, \{x\}}^*(S_{t+1} = B | \mathbf{S}_t) = \frac{\pi^{(t+1)}(B)}{\pi^{(t)}(S_t)} K(t, S_t; t+1, B) = \widehat{K}(S_t, B), \tag{3.6}$$

and thereby verify that our claim (a) extends up to $t + 1$. Further, the ratio of (3.5) and (3.6) results with

$$\mathbb{P}_{x, \{x\}}^*(X_{t+1} = v | S_{t+1} = B, \mathbf{S}_t) = \frac{\pi^{(t+1)}(v)}{\pi^{t+1}(B)},$$

which amounts to the claimed property (b) at $t + 1$. □

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