

DISCRETE BECKNER INEQUALITIES VIA THE BOCHNER–BAKRY–EMERY APPROACH FOR MARKOV CHAINS¹

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Discrete convex Sobolev inequalities and Beckner inequalities are derived for time-continuous Markov chains on finite state spaces. Beckner inequalities interpolate between the modified logarithmic Sobolev inequality and the Poincaré inequality. Their proof is based on the Bakry–Emery approach and on discrete Bochner-type inequalities established by Caputo, Dai Pra and Posta and recently extended by Fathi and Maas for logarithmic entropies. The abstract result for convex entropies is applied to several Markov chains, including birth-death processes, zero-range processes, Bernoulli–Laplace models, and random transposition models, and to a finite-volume discretization of a one-dimensional Fokker–Planck equation, applying results by Mielke.

1. Introduction. Convex Sobolev inequalities such as Poincaré and logarithmic Sobolev inequalities play an important role in the analysis of the convergence to stationarity for Markov processes. Besides implying exponential decay of the entropy, it is known that these functional inequalities give useful concentration bounds [6] and hypercontractivity of the corresponding semigroup [18], and they are a natural tool to estimate mixing times [29]. There exists an extensive literature on the derivation of Poincaré inequalities (or spectral gap estimates) and logarithmic Sobolev (or shorter: log-Sobolev) inequalities in the discrete and continuous setting; see, for example, the reviews [18, 23, 29] and the books [1, 4, 31]. An algorithm for the computation of the spectral gap is presented in [13], while corresponding estimates can be found in [9, 10, 14]. For log-Sobolev inequalities, we refer to [7, 11, 24].

There are much less results on Beckner inequalities for Markov chains, which interpolate between the Poincaré inequality and log-Sobolev inequality [5]. Such inequalities are of interest, for instance, in the large-time analysis of Markov chains using general entropies or in numerical analysis, proving the exponential decay of solutions to discretized partial differential equations [12]. We are only aware of the paper by Bobkov and Tetali [6], where estimates on the constant of the Beckner inequality were derived for Bernoulli–Laplace and random transposition models.

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In this paper, we establish new bounds for discrete convex Sobolev and Beckner inequalities for stochastic processes not studied in [6].

The technique of proof is the Bochner–Bakry–Emery method of Caputo et al. [11], which was recently extended by Fathi and Maas in [19] in the context of Ricci curvature bounds. The idea of the Bakry–Emery approach is to relate the second time derivative of the entropy to its entropy production. This relation is achieved by employing a discrete Bochner-type equation which replaces the Bochner identity in the continuous case.

In order to make these ideas precise, consider a time-homogeneous Markov process $(X_t)_{t \geq 0}$ with values in a finite state space S , having an invariant measure π . We assume that the semigroup $(T_t)_{t \geq 0}$, defined on $L^2(\pi)$ by $T_t f(x) = \mathbb{E}[f(X_t) : X_0 = x]$, is strongly right continuous, so that the infinitesimal generator \mathcal{L} exists, $T_t = e^{t\mathcal{L}}$. Given a probability measure μ on S , we denote by μT_t the distribution of X_t assuming that X_0 is distributed according to μ . The rate of convergence of μT_t to the invariant measure π is a major topic in probability theory. It can be achieved by estimating the time derivative of the relative entropy.

Before explaining the entropy decay, we introduce some notation. The relative entropy $h^\phi(\mu|\pi)$ of μ with respect to π is defined by

$$h^\phi(\mu|\pi) = \pi \left[\phi \left(\frac{d\mu}{d\pi} \right) \right] = \sum_{\eta \in S} \pi(\eta) \phi \left(\frac{d\mu}{d\pi} \right) (\eta),$$

where $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a smooth convex function such that $\phi(1) = 0$ and $1/\phi''$ is concave, $\mathbb{R}_+ = [0, \infty)$, and $h^\phi(\mu|\pi)$ is meant to be infinite whenever $\mu \not\ll \pi$ or $\phi(d\mu/d\pi) \notin L^1(\pi)$. The entropy can be defined on the set of probability densities f such that $\phi(f) \in L^1(\pi)$ by

$$\text{Ent}_\pi^\phi(f) = \pi[\phi(f)],$$

so that $h^\phi(\mu|\pi) = \text{Ent}_\pi^\phi(d\mu/d\pi)$. When $\phi_1(s) = s(\log s - 1) + 1$, we obtain the logarithmic entropy and if $\phi_2(s) = s^2 - 2s + 1$, $\text{Ent}_\pi^\phi(f)$ equals the variance of f , $\text{Var}_\pi(f) = \pi[f^2] - \pi[f]^2$. Another example is $\phi_\alpha(s) = (s^\alpha - s)/(\alpha - 1) - s + 1$ for $1 < \alpha \leq 2$, which interpolates between ϕ_1 and ϕ_2 in the sense that $\phi_\alpha(s) \rightarrow \phi_1(s)$ pointwise as $\alpha \rightarrow 1$ and $\phi_\alpha(s) = \phi_2$ if $\alpha = 2$.

Let $\rho_t = d(\mu T_t)/d\pi$ be the probability density of the Markov chain at time $t \geq 0$. We assume in the following that the Markov chain is reversible, that is, the generator is self-adjoint in $L^2(\pi)$. Then ρ_t solves the Kolmogorov equation $\partial_t \rho_t = \mathcal{L} \rho_t$, $t > 0$. The idea of Bakry and Emery [3] is to differentiate the entropy twice with respect to time. A formal computation gives

$$(1) \quad \begin{aligned} \frac{d}{dt} \text{Ent}_\pi^\phi(\rho_t) &= -\mathcal{E}(\phi'(\rho_t), \rho_t), \\ \frac{d^2}{dt^2} \text{Ent}_\pi^\phi(\rho_t) &= \pi[\mathcal{L}\phi'(\rho_t)\mathcal{L}\rho_t + \phi''(\rho_t)(\mathcal{L}\rho_t)^2], \end{aligned}$$

where $\mathcal{E}(f, g) := -\pi[f\mathcal{L}g]$ is the Dirichlet form of \mathcal{L} . Now suppose that the following inequality holds for some $\lambda > 0$:

$$(2) \quad \pi[\mathcal{L}\phi'(\rho)\mathcal{L}\rho + \phi''(\rho)(\mathcal{L}\rho)^2] \geq \lambda\mathcal{E}(\phi'(\rho), \rho), \quad t > 0.$$

This is equivalent to $\partial_t^2 \text{Ent}_\pi^\phi(\rho) + \lambda \partial_t \text{Ent}_\pi^\phi(\rho) \geq 0$, and by Gronwall’s lemma, we conclude that $\partial_t \text{Ent}_\pi^\phi(\rho_t)$ converges to zero with exponential rate. Furthermore, integration over (t, ∞) leads to

$$(3) \quad \frac{d}{dt} \text{Ent}_\pi^\phi(\rho) + \lambda \text{Ent}_\pi^\phi(\rho) \leq 0, \quad t > 0,$$

if we know that $\text{Ent}_\pi^\phi(\rho_t) \rightarrow 0$ as $t \rightarrow \infty$. On the one hand, this implies exponential convergence of the relative entropy to zero, that is, $\text{Ent}_\pi^\phi(\rho_t) \leq \text{Ent}_\pi^\phi(\rho_0)e^{-\lambda t}$. On the other hand, (3) is equivalent to the *convex Sobolev inequality*:

$$(4) \quad \lambda \text{Ent}_\pi^\phi(f) \leq \mathcal{E}(\phi'(f), f),$$

valid for all probability densities f .

It is well known that if the so-called curvature-dimension condition $CD(\lambda, \infty)$ is satisfied, then the convex Sobolev inequality (4) is valid [4], Section 1.16. For instance, if \mathcal{L} is the generator of the Ornstein–Uhlenbeck process, $CD(\lambda, \infty)$ holds with $\lambda = 1$ under the conditions that ϕ is convex and $1/\phi''$ is concave [2]. In the discrete case, the validity of (4) is not known except in the logarithmic case $\phi = \phi_1$. In this paper, we derive general conditions on ϕ that guarantee the validity of (4).

For the special cases $\phi_1(s) = s(\log s - 1) + 1$ and $\phi_2(s) = s^2 - 2s + 1$, we obtain the *modified log-Sobolev inequality* and *Poincaré inequality*, respectively,

$$(5) \quad \lambda_M \text{Ent}_\pi^{\phi_1}(f) \leq \mathcal{E}(\log f, f), \quad \lambda_P \text{Var}_\pi(f) \leq \mathcal{E}(f, f).$$

Note that if \mathcal{L} is the generator of a reversible diffusion process, we may write $\mathcal{E}(\log f, f) = 4\mathcal{E}(f^{1/2}, f^{1/2})$, so the log-Sobolev inequality $\lambda_L \text{Ent}_\pi^{\phi_1}(f) \leq \mathcal{E}(f^{1/2}, f^{1/2})$ and the first inequality in (5) coincide with $\lambda_M = 4\lambda_L$. This is generally not true for Markov processes with jumps [7], but for reversible processes, the relations $4\lambda_L \leq \lambda_M \leq 2\lambda_P$ hold [6, 18].

The aim of this paper is to determine conditions under which there exists a constant $\lambda > 0$ such that the (discrete) convex Sobolev inequality (4) and the exponential entropy decay

$$(6) \quad \text{Ent}_\pi^\phi(\rho_t) \leq e^{-\lambda t} \text{Ent}_\pi^\phi(\rho_0), \quad t > 0,$$

hold. Furthermore, we derive explicit constants $\lambda_B(\alpha) > 0$ such that the (discrete) *Beckner inequality* holds:

$$(7) \quad \lambda_B(\alpha) \text{Ent}_\pi^{\phi_\alpha}(\rho) \leq \frac{\alpha}{\alpha - 1} \mathcal{E}(\rho^{\alpha-1}, \rho), \quad 1 < \alpha \leq 2.$$

The Beckner inequality is related to the modified log-Sobolev and Poincaré inequalities. Indeed, if $\alpha \rightarrow 1$, (7) becomes the modified log-Sobolev inequality

with $\lim_{\alpha \rightarrow 1} \lambda_B(\alpha) = \lambda_M$ and if $\alpha = 2$, (7) equals the Poincaré inequality with $\lambda_B(2) = 2\lambda_P$. For $1 < \alpha < 2$, applying (7) to functions of the form $1 + \varepsilon f$, performing a Taylor expansion, and letting $\varepsilon \rightarrow 0$ shows that $\lambda_B(\alpha) \leq 2\lambda_P$.

According to the above discussion, inequalities (5)–(7) are achieved by proving (2), and the proof of this inequality is based on a discrete Bochner-type identity. The idea to employ such an identity was first presented in [9], elaborated later in [11, 19], and goes back to [8]. The identity is obtained by identifying the Radon–Nikodym derivative of a measure involving the jump rates of the Markov chain [9], Section 2. This allows one to relate terms with different orders of “discrete derivatives” occurring in \mathcal{L} . For details, we refer to Section 2. Our technique of proving (7) is completely different from the work [6], where an iteration method was used to derive discrete Beckner inequalities.

Fathi and Maas [19] extended the results of Caputo et al. [11]. The key idea of [19] (and, by the way, of [28]) is the use of the logarithmic mean

$$\rho^*(\eta, \xi) = \frac{\rho(\eta) - \rho(\xi)}{\log \rho(\eta) - \log \rho(\xi)}$$

in the analysis. The logarithmic mean allows for the discrete chain rule $\rho^* \nabla \log \rho = \nabla \rho$, where $\nabla \rho(\eta, \xi) = \rho(\eta) - \rho(\xi)$, which naturally holds in the continuous case. This chain rule is needed to treat the logarithmic entropy. In the case of general convex entropies, it is natural to replace the logarithmic mean by

$$(8) \quad \widehat{\rho}(\eta, \xi) = \frac{\rho(\eta) - \rho(\xi)}{\phi'(\rho(\eta)) - \phi'(\rho(\xi))}, \quad \phi \text{ convex},$$

which satisfies the discrete chain rule $\widehat{\rho} \nabla \phi'(\rho) = \nabla \rho$ since $\widehat{\rho}$ “approximates” $1/\phi''(\rho)$. When $\phi = \phi_\alpha$, we obtain the power mean

$$\widehat{\rho}(\eta, \xi) = \frac{\alpha - 1}{\alpha} \frac{\rho(\eta) - \rho(\xi)}{\rho(\eta)^{\alpha-1} - \rho(\xi)^{\alpha-1}}, \quad 1 < \alpha < 2.$$

We remark that the idea to enforce a discrete chain rule is well known in the design of structure-preserving numerical schemes and was used, for example, in the construction of entropy-conservative finite-volume fluxes [20] and in the discrete variational derivative method [21].

The novelty of this paper is the identification of the conditions on ϕ that are needed to apply the technique of [11, 19]. It turns out that, besides convexity of ϕ and the concavity of $1/\phi''$, the concavity of

$$(9) \quad \theta(s, t) = \frac{s - t}{\phi'(s) - \phi'(t)}, \quad s \neq t, \quad \theta(s, s) = \frac{1}{\phi''(s)},$$

is needed. This is not surprising since $\theta(s, t)$ is a discrete approximation of $1/\phi''$, and the concavity of $1/\phi''$ is assumed in the continuous case. Conditions on ϕ that guarantee the concavity of θ are stated in Lemma 15. Both the logarithmic mean and the power mean satisfy these conditions; see Lemma 16. The general theory

can be applied to birth-death processes, thus yielding new discrete convex Sobolev inequalities. For other stochastic processes considered in this paper (zero-range processes, Bernoulli–Laplace models, random transposition models), a homogeneity property of θ is needed, which restricts the class of admissible functions ϕ . It turns out that the logarithmic mean and the power mean satisfy this property; see Lemma 16. For the mentioned processes, new discrete Beckner inequalities are derived.

The paper is organized as follows. We detail the Bochner–Bakry–Emery method in Section 2. The validity of the discrete Beckner inequality (7) is reduced to the validity of a modification of (2). In Section 3, we apply the general technique to four stochastic processes (as in [19]): birth-death processes, zero-range processes, Bernoulli–Laplace models, and random transposition models. Furthermore, the results for birth-death processes are applied to a finite-volume discretization of a one-dimensional Fokker–Planck equation, yielding exponential decay of the discrete entropy. The proof consists of a combination of the convex Sobolev inequality for birth-death processes and the results of Mielke [28], who proved exponential decay for the logarithmic entropy.

Our main conclusion is that the Bochner–Bakry–Emery approach is sufficiently flexible to be applicable to power functions and, in certain cases, to general convex functions.

2. The Bochner method. Let an irreducible and reversible Markov chain on a finite state space S be given and let π be the invariant measure. We write the generator \mathcal{L} in the form

$$\mathcal{L}f(\eta) = \sum_{\gamma \in G} c(\eta, \gamma) \nabla_{\gamma} f(\eta),$$

where G is the set of allowed moves (represented by functions $\gamma : S \rightarrow S$), the mapping $c : S \times G \rightarrow [0, \infty)$ represents the jump rates, and $\nabla_{\gamma} f(\eta) = f(\gamma\eta) - f(\eta)$. We observe that the generator of every finite Markov chain can be written in this form. We assume the following two properties: For any $\gamma \in G$, there exists $\gamma^{-1} \in G$ satisfying $\gamma^{-1}\gamma\eta = \eta$ for all $\eta \in S$ with $c(\eta, \gamma) > 0$. Furthermore, the reversibility condition

$$\pi \left[\sum_{\gamma \in G} c(\eta, \gamma) F(\eta, \gamma) \right] = \pi \left[\sum_{\gamma \in G} c(\eta, \gamma) F(\gamma\eta, \gamma^{-1}) \right]$$

holds for all $F : S \times G \rightarrow \mathbb{R}$. Under reversibility, the Dirichlet form can be written as

$$(10) \quad \mathcal{E}(f, g) = \frac{1}{2} \pi \left[\sum_{\gamma \in G} c(\eta, \gamma) \nabla_{\gamma} f(\eta) \nabla_{\gamma} g(\eta) \right].$$

For the discrete Bochner-type identity, we suppose as in [11] the following.

ASSUMPTION 1. There exists a function $R : S \times G \times G \rightarrow \mathbb{R}$ such that:

- (i) $R(\eta, \gamma, \delta) = R(\eta, \delta, \gamma)$ for all $\eta \in S, \gamma, \delta \in G$;
- (ii) for all bounded functions $\psi : S \times G \times G \rightarrow \mathbb{R}$,

$$\pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) \psi(\eta, \gamma, \delta) \right] = \pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) \psi(\gamma \eta, \gamma^{-1}, \delta) \right].$$

- (iii) $\gamma \delta \eta = \delta \gamma \eta$ for all $\eta \in S, \gamma, \delta \in G$ with $R(\eta, \gamma, \delta) > 0$.

The following lemma, which extends Lemma 2.3 in [11], was proven in [19], Lemma 3.3. It expresses a discrete Bochner-type identity.

LEMMA 1. Let $\chi, \psi : S \rightarrow \mathbb{R}$ and let $\beta : S \times S \rightarrow \mathbb{R}$ be symmetric. Then

$$\begin{aligned} & \pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) \beta(\eta, \delta \eta) \nabla_{\delta} \chi(\eta) \nabla_{\gamma} \psi(\eta) \right] \\ &= \frac{1}{4} \pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) \nabla_{\gamma} (\beta(\eta, \delta \eta) \nabla_{\delta} \chi(\eta)) \nabla_{\delta} \nabla_{\gamma} \psi(\eta) \right]. \end{aligned}$$

The key estimate is contained in the following proposition that is an extension of Theorem 3.5 in [19] from the logarithmic case to the case of general convex functions.

PROPOSITION 2. Let $\phi \in C^4((0, \infty); (0, \infty))$ be convex such that $\phi(1) = 0$, $1/\phi''$ is concave on $(0, \infty)$, and let θ , defined in (9), be concave. Assume that there exists a function R satisfying Assumption 1 and define $\Gamma(\eta, \gamma, \delta) = c(\eta, \gamma)c(\eta, \delta) - R(\eta, \gamma, \delta)$ for $\eta \in S$ and $\gamma, \delta \in G$. Then, for any positive probability density ρ ,

$$\begin{aligned} & \pi [\mathcal{L}\phi'(\rho)\mathcal{L}\rho + \phi''(\rho)(\mathcal{L}\rho)^2] \\ (11) \quad & \geq \pi \left[\sum_{\gamma, \delta \in G} \Gamma(\eta, \gamma, \delta) (\nabla_{\gamma} \phi'(\rho(\eta)) \nabla_{\delta} \rho(\eta) + \phi''(\rho(\eta)) \nabla_{\gamma} \rho(\eta) \nabla_{\delta} \rho(\eta)) \right]. \end{aligned}$$

REMARK 3. In Lemma 15 (see the Appendix), conditions on ϕ are stated guaranteeing the concavity of θ . We introduce the following notation:

$$(12) \quad \widehat{\rho}(\eta, \delta \eta) = \theta(\rho(\eta), \rho(\delta \eta)) = \frac{\rho(\delta \eta) - \rho(\eta)}{\phi'(\rho(\delta \eta)) - \phi'(\rho(\eta))} = \frac{\nabla_{\delta} \rho(\eta)}{\nabla_{\delta} \phi'(\rho(\eta))},$$

$$(13) \quad \widehat{\rho}_1(\eta, \delta \eta) = \partial_1 \theta(\rho(\eta), \rho(\delta \eta)) = -\frac{1}{\nabla_{\delta} \phi'(\rho(\eta))} + \frac{\nabla_{\delta} \rho(\eta) \phi''(\rho(\eta))}{(\nabla_{\delta} \phi'(\rho(\eta)))^2},$$

$$(14) \quad \widehat{\rho}_2(\eta, \delta \eta) = \partial_2 \theta(\rho(\eta), \rho(\delta \eta)) = \widehat{\rho}_1(\delta \eta, \eta),$$

where $\partial_1 \theta$ and $\partial_2 \theta$ are the partial derivatives of θ with respect to the first and second variable, respectively.

PROOF OF PROPOSITION 2. The first term on the left-hand side of (11) can be written as follows, using the definitions of \mathcal{L} , $\widehat{\rho}$, and Γ :

$$\begin{aligned} \pi[\mathcal{L}\phi'(\rho)\mathcal{L}\rho] &= \pi\left[\sum_{\gamma,\delta\in G} c(\eta,\gamma)c(\eta,\delta)\nabla_\gamma\phi'(\rho)\nabla_\delta\rho(\eta)\right] \\ &= \pi\left[\sum_{\gamma,\delta\in G} c(\eta,\gamma)c(\eta,\delta)\widehat{\rho}(\eta,\delta\eta)\nabla_\gamma\phi'(\rho(\eta))\nabla_\delta\phi'(\rho(\eta))\right] \\ &= \pi\left[\sum_{\gamma,\delta\in G} R(\eta,\gamma,\delta)\widehat{\rho}(\eta,\delta\eta)\nabla_\gamma\phi'(\rho(\eta))\nabla_\delta\phi'(\rho(\eta))\right] \\ &\quad + \pi\left[\sum_{\gamma,\delta\in G} \Gamma(\eta,\gamma,\delta)\widehat{\rho}(\eta,\delta\eta)\nabla_\gamma\phi'(\rho(\eta))\nabla_\delta\phi'(\rho(\eta))\right]. \end{aligned}$$

By Lemma 1 with $\beta(\eta,\delta\eta) = \widehat{\rho}(\eta,\delta\eta)$, the first term on the right-hand side of the previous equation can be rewritten, leading to $\pi[\mathcal{L}\phi'(\rho)\mathcal{L}\rho] = A_1 + A_2$, where

$$\begin{aligned} A_1 &= \frac{1}{4}\pi\left[\sum_{\gamma,\delta\in G} R(\eta,\gamma,\delta)\nabla_\gamma(\widehat{\rho}(\eta,\delta\eta)\nabla_\delta\phi'(\rho(\eta)))\nabla_\delta\nabla_\gamma\phi'(\rho(\eta))\right], \\ A_2 &= \pi\left[\sum_{\gamma,\delta\in G} \Gamma(\eta,\gamma,\delta)\widehat{\rho}(\eta,\delta\eta)\nabla_\gamma\phi'(\rho(\eta))\nabla_\delta\phi'(\rho(\eta))\right]. \end{aligned}$$

Next, we reformulate the second term on the left-hand side of (11), using the definitions of \mathcal{L} , $\widehat{\rho}_1$, and Γ :

$$\begin{aligned} \pi[\phi''(\rho)(\mathcal{L}\rho)^2] &= \pi\left[\sum_{\gamma,\delta\in G} c(\eta,\gamma)c(\eta,\delta)\nabla_\gamma\rho(\eta)\nabla_\delta\rho(\eta)\phi''(\rho(\eta))\right] \\ &= \pi\left[\sum_{\gamma,\delta\in G} c(\eta,\gamma)c(\eta,\delta)\nabla_\gamma\rho(\eta)\widehat{\rho}_1(\eta,\delta\eta)(\nabla_\delta\phi'(\rho(\eta)))^2\right] \\ &\quad + \pi\left[\sum_{\gamma,\delta\in G} c(\eta,\gamma)c(\eta,\delta)\nabla_\gamma\rho(\eta)\nabla_\delta\phi'(\rho(\eta))\right] \\ &= \pi\left[\sum_{\gamma,\delta\in G} R(\eta,\gamma,\delta)\nabla_\gamma\rho(\eta)\widehat{\rho}_1(\eta,\delta\eta)(\nabla_\delta\phi'(\rho(\eta)))^2\right] \\ &\quad + \pi\left[\sum_{\gamma,\delta\in G} \Gamma(\eta,\gamma,\delta)\nabla_\gamma\rho(\eta)\widehat{\rho}_1(\eta,\delta\eta)(\nabla_\delta\phi'(\rho(\eta)))^2\right] \\ &\quad + \pi\left[\sum_{\gamma,\delta\in G} c(\eta,\gamma)c(\eta,\delta)\nabla_\gamma\rho(\eta)\nabla_\delta\phi'(\rho(\eta))\right] \\ &=: B_1 + B_2 + (A_1 + A_2). \end{aligned}$$

Then the left-hand side of (11) is given by

$$\pi[\mathcal{L}\phi'(\rho)\mathcal{L}\rho + \phi''(\rho)(\mathcal{L}\rho)^2] = (B_1 + 2A_1) + (B_2 + 2A_2),$$

and we will estimate $B_1 + 2A_1$ and $B_2 + 2A_2$ separately.

First, we treat $B_2 + 2A_2$. Inserting the definition of $\widehat{\rho}(\eta, \delta\eta)$ and rearranging the terms, we find that

$$\begin{aligned} B_2 + 2A_2 &= \pi \left[\sum_{\gamma, \delta \in G} \Gamma(\eta, \gamma, \delta) \widehat{\rho}_1(\eta, \delta\eta) \nabla_\gamma \rho(\eta) (\nabla_\delta \phi'(\rho(\eta)))^2 \right] \\ &\quad + 2\pi \left[\sum_{\gamma, \delta \in G} \Gamma(\eta, \gamma, \delta) \widehat{\rho}(\eta, \delta\eta) \nabla_\gamma \phi'(\rho(\eta)) \nabla_\delta \phi'(\rho(\eta)) \right] \\ &= \pi \left[\sum_{\gamma, \delta \in G} \Gamma(\eta, \gamma, \delta) \nabla_\gamma \phi'(\rho(\eta)) \nabla_\delta \rho(\eta) \right] \\ &\quad + \pi \left[\sum_{\gamma, \delta \in G} \Gamma(\eta, \gamma, \delta) \nabla_\gamma \rho(\eta) \nabla_\delta \rho(\eta) \phi''(\rho(\eta)) \right], \end{aligned}$$

which is exactly the right-hand side of (11). Thus, it remains to prove that $B_1 + 2A_1 \geq 0$.

To this end, we reformulate B_1 , employing Assumption 1(i)–(ii) and identity (14):

$$\begin{aligned} (15) \quad B_1 &= \pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) \widehat{\rho}_1(\eta, \delta\eta) \nabla_\gamma \rho(\eta) (\nabla_\delta \phi'(\rho(\eta)))^2 \right] \\ &= \pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) \widehat{\rho}_1(\delta\eta, \eta) \nabla_\gamma \rho(\delta\eta) (\nabla_{\delta^{-1}} \phi'(\rho(\delta\eta)))^2 \right] \end{aligned}$$

$$(16) \quad = \pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) \widehat{\rho}_2(\eta, \delta\eta) \nabla_\gamma \rho(\delta\eta) (\nabla_\delta \phi'(\rho(\eta)))^2 \right],$$

since $\nabla_{\delta^{-1}} \phi'(\rho(\delta\eta)) = -\nabla_\delta \phi'(\rho(\eta))$. Averaging (15) and (16) gives

$$\begin{aligned} B_1 &= \frac{1}{2} \pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) (\widehat{\rho}_1(\eta, \delta\eta) \nabla_\gamma \rho(\eta) + \widehat{\rho}_2(\eta, \delta\eta) \nabla_\gamma \rho(\delta\eta)) \right. \\ &\quad \left. \times (\nabla_\delta \phi'(\rho(\eta)))^2 \right]. \end{aligned}$$

By (41) from Lemma 15 (see the Appendix) with $u = \rho(\gamma\eta)$, $v = \rho(\gamma\delta\eta)$, $s = \rho(\eta)$, and $t = \rho(\delta\eta)$, it follows that

$$\widehat{\rho}_1(\eta, \delta\eta) \nabla_\gamma \rho(\eta) + \widehat{\rho}_2(\eta, \delta\eta) \nabla_\gamma \rho(\delta\eta) \geq \nabla_\gamma \widehat{\rho}(\eta, \delta\eta),$$

and we infer from the definition of A_1 that

$$(17) \quad B_1 + 2A_1 \geq \frac{1}{2}\pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) \{ \nabla_\gamma \widehat{\rho}(\eta, \delta\eta) (\nabla_\delta \phi'(\rho(\eta)))^2 + \nabla_\gamma (\widehat{\rho}(\eta, \delta\eta) \nabla_\delta \phi'(\rho(\eta))) \nabla_\delta \nabla_\gamma \phi'(\rho(\eta)) \} \right].$$

The following identity has been used in the proof of Theorem 3.5 in [19]:

$$(18) \quad \begin{aligned} & \nabla_\gamma \widehat{\rho}(\eta, \delta\eta) (\nabla_\delta \psi(\eta))^2 + \nabla_\gamma (\widehat{\rho}(\eta, \delta\eta) \nabla_\delta \psi(\eta)) \nabla_\delta \nabla_\gamma \psi(\eta) \\ &= \widehat{\rho}(\gamma\eta, \gamma\delta\eta) (\nabla_\gamma \nabla_\delta \psi(\eta))^2 - \widehat{\rho}(\eta, \delta\eta) \nabla_\delta \psi(\gamma\eta) \nabla_\delta \psi(\eta) \\ & \quad + \widehat{\rho}(\gamma\eta, \delta\gamma\eta) \nabla_\delta \psi(\gamma\eta) \nabla_\delta \psi(\eta). \end{aligned}$$

It can be verified by elementary computations. Taking $\psi(\eta) = \phi'(\rho(\eta))$, the left-hand side of (18) equals the expression in the curly brackets of (17), and we conclude that

$$\begin{aligned} B_1 + 2A_1 &\geq \frac{1}{2}\pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) \widehat{\rho}(\gamma\eta, \gamma\delta\eta) (\nabla_\gamma \nabla_\delta \phi'(\rho(\eta)))^2 \right] \\ & \quad - \frac{1}{2}\pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) \widehat{\rho}(\eta, \delta\eta) \nabla_\delta \phi'(\rho(\gamma\eta)) \nabla_\delta \phi'(\rho(\eta)) \right] \\ & \quad + \frac{1}{2}\pi \left[\sum_{\gamma, \delta \in G} R(\eta, \gamma, \delta) \widehat{\rho}(\gamma\eta, \delta\gamma\eta) \nabla_\delta \phi'(\rho(\gamma\eta)) \nabla_\delta \phi'(\rho(\eta)) \right]. \end{aligned}$$

It follows from Assumption 1(ii)–(iii) that the second and third term on the right-hand side cancel. The first term being nonnegative, we infer that $B_1 + 2A_1 \geq 0$, which completes the proof. \square

The following corollary is a consequence of Proposition 2.

COROLLARY 4. *Let $\phi \in C^4((0, \infty); (0, \infty))$ be convex such that $\phi(1) = 0$, $1/\phi''$ is concave on $(0, \infty)$, and let θ , defined in (9), be concave. Suppose that there exists a constant $\lambda > 0$ such that for all positive probability densities ρ ,*

$$(19) \quad \begin{aligned} & \pi \left[\sum_{\gamma, \delta \in G} \Gamma(\eta, \gamma, \delta) (\nabla_\gamma \phi'(\rho(\eta)) \nabla_\delta \rho(\eta) + \phi''(\rho(\eta)) \nabla_\gamma \rho(\eta) \nabla_\delta \rho(\eta)) \right] \\ & \geq \frac{\lambda}{2}\pi \left[\sum_{\gamma \in G} c(\eta, \gamma) \nabla_\gamma \phi'(\rho(\eta)) \nabla_\gamma \rho(\eta) \right]. \end{aligned}$$

Then the convex Sobolev inequality (4), the decay of the Dirichlet form

$$(20) \quad \mathcal{E}(\phi'(e^{t\mathcal{L}}\rho), e^{t\mathcal{L}}\rho) \leq e^{-\lambda t} \mathcal{E}(\phi'(\rho), \rho), \quad t > 0,$$

and the decay of the entropy (6) hold for all positive probability densities ρ .

PROOF. By Proposition 2 and representation (10) of the Dirichlet form, it follows from (19) that

$$\pi[\mathcal{L}\phi'(\rho)\mathcal{L}\rho] + \pi[(\mathcal{L}\rho)^2\phi''(\rho)] \geq \lambda\mathcal{E}(\phi'(\rho), \rho).$$

Taking into account (1), this inequality is equivalent to

$$(21) \quad \frac{d^2}{dt^2} \text{Ent}_\pi^\phi(\rho_t) \geq -\lambda \frac{d}{dt} \text{Ent}_\pi^\phi(\rho_t).$$

Using Gronwall's lemma, we infer that $0 = \lim_{t \rightarrow \infty} (-\partial_t \text{Ent}_\pi^\phi(\rho_t))$. Furthermore, as π is an invariant measure, $\rho_t \rightarrow 1$ and $\text{Ent}_\pi(\rho_t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore, integrating (21) over $(0, \infty)$, we conclude that

$$-\mathcal{E}(\phi'(\rho_0), \rho_0) = \frac{d}{dt} \text{Ent}_\pi^\phi(\rho_0) \leq -\lambda \text{Ent}_\pi^\phi(\rho_0),$$

and this is exactly the convex Sobolev inequality (4). \square

3. Examples. In this section, we consider some stochastic processes analyzed in [11, 19] but for logarithmic entropies only. For birth-death processes, we are able to allow for general convex entropies, while for the remaining cases (zero-range processes, Bernoulli–Laplace models, Random transposition models), only power entropies with $\phi = \phi_\alpha$ can be considered. The reason is that we need additional features of ϕ that seem to be satisfied only under certain homogeneity properties. These features are summarized in Lemma 16. Our notation follows that of [11].

3.1. *Birth-death processes.* We investigate birth-death processes on $\mathbb{N} = \{0, 1, 2, \dots\}$ with generator

$$\mathcal{L}f(n) = a(n)\nabla_+ f(n) + b(n)\nabla_- f(n), \quad n \in \mathbb{N},$$

where a and b are nonnegative functions on \mathbb{N} satisfying $b(0) = 0$. The function a represents the rate of birth, the function b the rate of death. The set of allowed moves is given by $G = \{+, -\}$, where $+(n) = n + 1$ for $n \in \mathbb{N}$ and $-(n) = n - 1$ for $n \geq 1$, $-(0) = 0$. In particular, $\nabla_\pm f(n) = f(n \pm 1) - f(n)$. According to the notation of Section 2, $c(n, +) = a(n)$ and $c(n, -) = b(n)$.

Since we considered in the previous section finite state spaces, we need to assume that the transition rates $a(n)$ and $b(n)$ vanish for sufficiently large values of n , say n^* , in order to fit into this framework. Another possibility is to consider finitely supported test functions. According to [26], this case may be covered by using the results of Daneri and Savaré [16]. We expect that the result below still holds for countable Markov chains, but we leave the proof for future works; also see [19], Remark 4.2.

We suppose that this Markov chain is irreducible and reversible, that is, there exists a probability measure π on \mathbb{N} satisfying the detailed-balance condition:

$$(22) \quad a(n)\pi(n) = b(n+1)\pi(n+1), \quad n \in \mathbb{N}.$$

The following theorem is a consequence of Corollary 4, applied to birth-death processes.

THEOREM 5. *Let $\lambda > 0$ and let ϕ satisfy the assumptions stated in Proposition 2. Assume that a is nonincreasing and b is nondecreasing on $\{0, \dots, n^*\}$, and*

$$(23) \quad \begin{aligned} & a(n) - a(n + 1) + b(n + 1) - b(n) \\ & + \Theta(a(n) - a(n + 1), b(n + 1) - b(n)) \geq \lambda \end{aligned}$$

for all $n \in \mathbb{N}$, where

$$\Theta(A, B) := \inf_{s, t > 0} \theta(s, t)(A\phi''(s) + B\phi''(t)), \quad A, B \geq 0,$$

and $\theta(s, t) = (s - t)/(\phi'(s) - \phi'(t))$ for $s \neq t$. Then the convex Sobolev inequality (5) and the decay estimates (6) and (20) hold with constant λ .

The mapping Θ generalizes the function in [19], Section 4.1. For the special case $\phi(s) = \phi_\alpha(s) = (s^\alpha - s)/(\alpha - 1) - s + 1$, Lemma 18 in the Appendix shows that $\Theta(A, B) \geq (\alpha - 1)(A + B)$. Moreover, $\Theta(A, B) = A + B$ if $\alpha = 2$. Figure 1 illustrates the “sharpness” of the inequality $\Theta(A, B) \geq (\alpha - 1)(A + B)$ for α close to one.

REMARK 6. Estimates for Poincaré inequalities for Markov chains are given in, for example, [14, 15, 27]. The same criterion as in (23) was obtained in [28], Theorem 5.1 and [19], Theorem 4.1 for the logarithmic entropy ($\alpha \rightarrow 1$). From Lemma 18, we conclude that the Beckner constant can be estimated by $\lambda \geq \alpha(a(n) - a(n + 1) + b(n - 1) - b(n))$. There exist sufficient and necessary conditions on π and $a(n)$ such that an interpolation between the Poincaré and log-Sobolev inequality holds, but without estimates on the constant [31], Theorem 6.2.4.

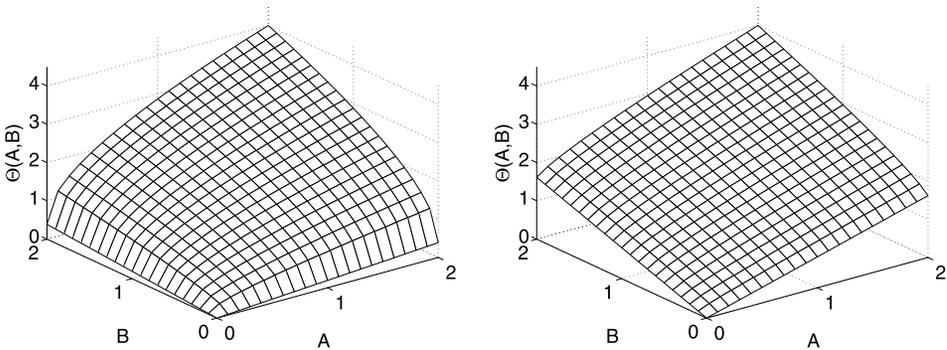


FIG. 1. Illustration of $\Theta(A, B)$, defined in (9), for $\alpha = 1.01$ (left) and $\alpha = 1.8$ (right).

PROOF. We define as in [11], Section 3:

$$R(n, +, +) = a(n)a(n+1), \quad R(n, -, -) = b(n)b(n-1),$$

$$R(n, +, -) = R(n, -, +) = a(n)b(n).$$

This function satisfies Assumption 1. In particular, (ii) follows from the detailed-balance condition (22). As before, we set $\Gamma(n, \gamma, \delta) = c(n, \gamma)c(n, \delta) - R(n, \gamma, \delta)$ for $\gamma, \delta \in G$. According to Corollary 4, we only need to verify (19). The left-hand side equals

$$\begin{aligned} & \pi \left[\sum_{\gamma, \delta \in G} \Gamma(n, \gamma, \delta) (\nabla_\gamma \phi'(\rho(n)) \nabla_\delta \rho(n) + \nabla_\gamma \rho(n) \nabla_\delta \rho(n) \phi''(\rho(n))) \right] \\ &= \pi [a(n)(a(n) - a(n+1)) (\nabla_+ \phi'(\rho(n)) \nabla_+ \rho(n) + (\nabla_+ \rho(n))^2 \phi''(\rho(n)))] \\ & \quad + \pi [b(n)(b(n) - b(n-1)) \\ & \quad \times (\nabla_- \phi'(\rho(n)) \nabla_- \rho(n) + (\nabla_- \rho(n))^2 \phi''(\rho(n)))]], \end{aligned}$$

since the sum over all $\gamma, \delta \in G$ consists of four terms $(+, +)$, $(-, -)$, $(+, -)$, and $(-, +)$, and because of $\Gamma(n, +, -) = \Gamma(n, -, +) = 0$, only two terms do not vanish. Now, we perform the change $n \mapsto n+1$ in the second term and replace $\pi(n+1)b(n+1)$ by $\pi(n)a(n)$, according to the detailed-balance condition (22). Observing that $b(0) = 0$ and $\nabla_- \rho(n+1) = -\nabla_+ \rho(n)$, this leads to

$$\begin{aligned} & \pi \left[\sum_{\gamma, \delta \in G} \Gamma(n, \gamma, \delta) (\nabla_\gamma \phi'(\rho(n)) \nabla_\delta \rho(n) + \nabla_\gamma \rho(n) \nabla_\delta \rho(n) \phi''(\rho(n))) \right] \\ &= \pi [a(n)(a(n) - a(n+1)) (\nabla_+ \phi'(\rho(n)) \nabla_+ \rho(n) + (\nabla_+ \rho(n))^2 \phi''(\rho(n)))] \\ & \quad + \pi [a(n)(b(n+1) - b(n)) \\ & \quad \times (\nabla_+ \phi'(\rho(n)) \nabla_+ \rho(n) + (\nabla_+ \rho(n))^2 \phi''(\rho(n+1)))] \\ &= \pi [a(n)(a(n) - a(n+1) + b(n+1) - b(n)) \nabla_+ \phi'(\rho(n)) \nabla_+ \rho(n)] \\ & \quad + \pi [a(n)((a(n) - a(n+1)) \phi''(\rho(n)) + (b(n+1) - b(n)) \phi''(\rho(n+1))) \\ & \quad \times \widehat{\rho}(n, n+1) \nabla_+ \phi'(\rho(n)) \nabla_+ \rho(n)] \\ & \geq \lambda \pi [a(n) \nabla_+ \phi'(\rho(n)) \nabla_+ \rho(n)], \end{aligned}$$

where in the last step we employed (23). Using again the detailed-balance condition (22) and the identity $\nabla_- \rho(n) = -\nabla_+ \rho(n-1)$, the right-hand side of (19) becomes

$$\begin{aligned} & \frac{\lambda}{2} \pi \left[\sum_{\gamma \in G} c(n, \gamma) \nabla_\gamma \phi'(\rho(n)) \nabla_\gamma \rho(n) \right] \\ &= \frac{\lambda}{2} \pi [a(n) \nabla_+ \phi'(\rho(n)) \nabla_+ \rho(n)] + \frac{\lambda}{2} \pi [b(n) \nabla_- \phi'(\rho(n)) \nabla_- \rho(n)] \end{aligned}$$

$$\begin{aligned} &= \frac{\lambda}{2}\pi[a(n)\nabla_+\phi'(\rho(n))\nabla_+\rho(n)] + \frac{\lambda}{2}\pi[a(n)\nabla_+\phi'(\rho(n))\nabla_+\rho(n)] \\ &= \lambda\pi[a(n)\nabla_+\phi'(\rho(n))\nabla_+\rho(n)]. \end{aligned}$$

Combining the above computations, inequality (19) follows. \square

3.2. *Zero-range processes.* A zero-range process describes a stochastically interacting particle system that may exhibit phase separation; see, for example, [17]. The system consists of finitely many particles moving in a finite number of sites $\{1, 2, \dots, L\}$. We adopt the notation of [11]. Let $\eta_x \in \mathbb{N}$ denote the number of particles at $x \in \{1, 2, \dots, L\}$. Then the state space is $S = \mathbb{N}^L$. The configuration is changed by moving a particle from an (occupied) site x to another site y . Correspondingly, the set G of allowed moves is given by self-mappings of S which are of the form $\eta \mapsto \eta^{xy}$, where $x, y \in \{1, 2, \dots, L\}$, $x \neq y$, and

$$\eta_z^{xy} = \begin{cases} \eta_z & \text{if } z \notin \{x, y\} \text{ or } \eta_x = 0, \\ \eta_z - 1 & \text{for } z = x \text{ and } \eta_x > 0, \\ \eta_z + 1 & \text{for } z = y \text{ and } \eta_x > 0. \end{cases}$$

We denote by xy the mapping $\eta \mapsto \eta^{xy}$ [such that $xy(\eta) = \eta^{xy}$] and set $\nabla_{xy}f(\eta) = f(\eta^{xy}) - f(\eta)$ for $\eta \in S$.

The jump rates are functions $c_x : \mathbb{N} \rightarrow \mathbb{R}_+$ for $x \in \{1, 2, \dots, L\}$ satisfying $c_x(0) = 0$ and $c_x(n) > 0$ for $n > 0$. They describe the rate at which a particle is moved from site x to site y , with randomly chosen y , with uniform probability on $\{1, 2, \dots, L\}$. Then the rate $c(\eta, xy)$ for moving a particle from x to y is $c_x(\eta_x)/L$, and the generator of the Markov chain becomes

$$\mathcal{L}f(\eta) = \frac{1}{L} \sum_{x,y} c_x(\eta_x) \nabla_{xy}f(\eta),$$

where the sum extends to all $x, y \in \{1, 2, \dots, L\}$. The number of particles $N = \sum_{1 \leq x \leq L} \eta_x$ is conserved. We define the probability measure π_N on configurations with N particles by

$$\pi_N(\eta) = \frac{1}{Z_N} \prod_{x=1}^L \prod_{k=1}^{\eta_x} \frac{1}{c_x(k)},$$

where $Z_N > 0$ the (finite) normalization constant. Since

$$(24) \quad \pi[c_x(\eta_x)g(\eta)] = \pi[c_y(\eta_y)g(\eta^{yx})]$$

holds for all functions $g : S \rightarrow \mathbb{R}$, the Markov chain is reversible with respect to π_N . In the following, we fix the number of particles N and omit the subscript N .

THEOREM 7. Let $\phi(s) = (s^\alpha - s)/(\alpha - 1) - s + 1$ and $1 < \alpha < 2$. Assume that there exist constants $0 \leq \delta < 2^{2-\alpha}c$ such that

$$(25) \quad c \leq c_x(n+1) - c_x(n) \leq c + \delta \quad \text{for } x \in \{1, 2, \dots, G\}, n \geq 0.$$

Then the Beckner inequality (7) and the decay estimates (6) and (20) hold with $\lambda = \alpha c - (3 + 2^{\alpha-2} - \alpha)\delta$.

REMARK 8. In the case of constant rates, the spectral gap is of the order of $L^2/(L^2 + N^2)$ [30]. Note that our bound $\lambda = 2(c - \delta)$ for $\alpha = 2$ does not depend on either L or N . It was shown in [9] that the lower bound in (25) is sufficient to derive the spectral-gap estimate $\lambda \geq c$. In the homogeneous case $\delta = 0$, we have even $\lambda = 2c$. As pointed out in [11], a condition on the growth of the rates is necessary for the modified logarithmic Sobolev inequality. Our bound $\lambda = c - 5\delta/2$ for $\alpha \rightarrow 1$ is the same as in [19], Theorem 4.3.

PROOF. We define as in [11], Section 4, the function

$$R(\eta, xy, uv) = \frac{1}{L^2} \begin{cases} c_x(\eta_x)c_u(\eta_u) & \text{for } x \neq u, \\ c_x(\eta_x)c_u(\eta_u - 1) & \text{for } x = u, \end{cases}$$

which satisfies Assumption 1. It follows that $\Gamma(\eta, xy, uv) = 0$ if $x \neq u$ and

$$\Gamma(\eta, xy, uv) = L^{-2}c_x(\eta_x)(c_x(\eta_x) - c_x(\eta_x - 1)) \quad \text{if } x = u,$$

and the left-hand side of (19) can be written as

$$\begin{aligned} & \pi \left[\sum_{\gamma, \delta \in G} \Gamma(\eta, \gamma, \delta) (\nabla_\gamma \rho^{\alpha-1}(\eta) \nabla_\delta \rho(\eta) + (\alpha - 1) \nabla_\gamma \rho(\eta) \nabla_\delta \rho(\eta) \rho^{\alpha-2}(\eta)) \right] \\ &= \frac{1}{L^2} \pi \left[\sum_{x, y, v} c_x(\eta_x) (c_x(\eta_x) - c_x(\eta_x - 1)) \nabla_{xv} \rho^{\alpha-1}(\eta) \nabla_{xy} \rho(\eta) \right] \\ & \quad + \frac{\alpha - 1}{L^2} \pi \left[\sum_{x, y, v} c_x(\eta_x) (c_x(\eta_x) - c_x(\eta_x - 1)) \nabla_{xy} \rho(\eta) \nabla_{xv} \rho(\eta) \rho^{\alpha-2}(\eta) \right] \\ &= C_1 + C_2. \end{aligned}$$

For future reference, we denote the right-hand side of (19) (without the constant λ) by

$$A = \frac{1}{2L} \pi \left[\sum_{x, y} c_x(\eta_x) \nabla_{xy} \rho^{\alpha-1}(\eta) \nabla_{xy} \rho(\eta) \right].$$

The estimate of the term C_1 is similar to $\tilde{B}_1(\rho, \psi)$ in the proof of Theorem 4.3 in [19] [take $\psi(\eta) = \rho^{\alpha-1}(\eta)$]. First, we interchange y and v and then

use $\nabla_{xv}\rho^{\alpha-1}(\eta) = \nabla_{xy}\rho^{\alpha-1}(\eta) + \nabla_{yv}\rho^{\alpha-1}(\eta^{xy})$ as well as the lower bound $c_x(\eta_x) - c_x(\eta_x - 1) \geq c$:

$$\begin{aligned}
 C_1 &= \frac{1}{L^2}\pi \left[\sum_{x,y,v} c_x(\eta_x)(c_x(\eta_x) - c_x(\eta_x - 1)) \right. \\
 (26) \quad &\times (\nabla_{xy}\rho^{\alpha-1}(\eta) + \nabla_{yv}\rho^{\alpha-1}(\eta^{xy}))\nabla_{xy}\rho(\eta) \left. \right] \\
 &\geq 2cA + \frac{1}{L^2}\pi \left[\sum_{x,y,v} c_x(\eta_x)(c_x(\eta_x) - c_x(\eta_x - 1))\nabla_{yv}\rho^{\alpha-1}(\eta^{xy})\nabla_{xy}\rho(\eta) \right].
 \end{aligned}$$

Note that the term involving $\nabla_{xy}\rho^{\alpha-1}(\eta)$ does not depend on v , so the sum over x, y, v equals L times the sum over x, y . Employing the reversibility condition (24) and exchanging x and y in the second term yields

$$\begin{aligned}
 C_1 &\geq 2cA \\
 (27) \quad &+ \frac{1}{L^2}\pi \left[\sum_{x,y,v} c_y(\eta_y)(c_x(\eta_x^{yx}) - c_x(\eta_x^{yx} - 1))\nabla_{yv}\rho^{\alpha-1}(\eta)\nabla_{xy}\rho(\eta^{yx}) \right] \\
 &= 2cA - \frac{1}{L^2}\pi \left[\sum_{x,y,v} c_x(\eta_x)(c_y(\eta_y + 1) - c_y(\eta_y))\nabla_{xv}\rho^{\alpha-1}(\eta)\nabla_{xy}\rho(\eta) \right].
 \end{aligned}$$

We average (26) and (27) and employ again the identity $\nabla_{xy}\rho^{\alpha-1}(\eta) + \nabla_{yv}\rho^{\alpha-1}(\eta^{xy}) = \nabla_{xv}\rho^{\alpha-1}(\eta)$:

$$\begin{aligned}
 C_1 &\geq cA + \frac{1}{2L^2}\pi \left[\sum_{x,y,v} c_x(\eta_x)((c_x(\eta_x) - c_x(\eta_x - 1)) - (c_y(\eta_y + 1) - c_y(\eta_y))) \right. \\
 &\quad \times \nabla_{xv}\rho^{\alpha-1}(\eta)\nabla_{xy}\rho(\eta) \left. \right].
 \end{aligned}$$

Setting $C_0 := (c_x(\eta_x) - c_x(\eta_x - 1)) - (c_y(\eta_y + 1) - c_y(\eta_y))$, the bounds (25) imply that $|C_0| \leq \delta$. Hence, by Young’s inequality,

$$\begin{aligned}
 &C_0\nabla_{xv}\rho^{\alpha-1}(\eta)\nabla_{xy}\rho(\eta) \\
 &= C_0\widehat{\rho}(\eta, \eta^{xy})\nabla_{xv}\rho^{\alpha-1}(\eta)\nabla_{xy}\rho^{\alpha-1}(\eta) \\
 &\geq -\frac{1}{2}|C_0|\widehat{\rho}(\eta, \eta^{xy})((\nabla_{xy}\rho^{\alpha-1}(\eta))^2 + (\nabla_{xv}\rho^{\alpha-1}(\eta))^2) \\
 &\geq -\frac{\delta}{2}(\nabla_{xy}\rho(\eta)\nabla_{xy}\rho^{\alpha-1}(\eta) + (\nabla_{xv}\rho^{\alpha-1}(\eta))^2\widehat{\rho}(\eta, \eta^{xy})).
 \end{aligned}$$

This yields

$$(28) \quad C_1 \geq \left(c - \frac{\delta}{2}\right)A - \frac{\delta}{4L^2}\pi \left[\sum_{x,y,v} c_x(\eta_x)(\nabla_{xv}\rho^{\alpha-1}(\eta))^2\widehat{\rho}(\eta, \eta^{xy}) \right].$$

Next, we rewrite $B = (C_2 - C_1)/2$. By definition (13) of $\widehat{\rho}_1$ and the reversibility condition (24),

$$\begin{aligned} B &= \frac{1}{2L^2} \pi \left[\sum_{x,y,v} c_x(\eta_x) (c_x(\eta_x) - c_x(\eta_x - 1)) \right. \\ &\quad \left. \times (\nabla_{xy} \rho^{\alpha-1}(\eta))^2 \widehat{\rho}_1(\eta, \eta^{xy}) \nabla_{xv} \rho(\eta) \right] \\ &= \frac{1}{2L^2} \pi \left[\sum_{x,y,v} c_y(\eta_y) (c_x(\eta_x + 1) - c_x(\eta_x)) \right. \\ &\quad \left. \times (\nabla_{xy} \rho^{\alpha-1}(\eta^{yx}))^2 \widehat{\rho}_1(\eta^{yx}, \eta) \nabla_{xv} \rho(\eta^{yx}) \right] \\ &= \frac{1}{2L^2} \pi \left[\sum_{x,y,v} c_x(\eta_x) (c_y(\eta_y + 1) - c_y(\eta_y)) \right. \\ &\quad \left. \times (\nabla_{xy} \rho^{\alpha-1}(\eta))^2 \widehat{\rho}_2(\eta, \eta^{xy}) (\rho(\eta^{xv}) - \rho(\eta^{xy})) \right]. \end{aligned}$$

In the last step, we interchanged x and y and used the identity $\widehat{\rho}_1(\eta^{xy}, \eta) = \widehat{\rho}_2(\eta, \eta^{xy})$. Averaging the expressions for B involving $\widehat{\rho}_1$ and $\widehat{\rho}_2$ gives

$$\begin{aligned} B &= \frac{1}{4L^2} \pi \left[\sum_{x,y,v} c_x(\eta_x) (\nabla_{xy} \rho^{\alpha-1}(\eta))^2 \rho(\eta^{xv}) \right. \\ &\quad \left. \times ((c_x(\eta_x) - c_x(\eta_x - 1)) \widehat{\rho}_1(\eta, \eta^{xy}) + (c_y(\eta_y + 1) - c_y(\eta_y)) \widehat{\rho}_2(\eta, \eta^{xy})) \right] \\ &\quad - \frac{1}{4L^2} \pi \left[\sum_{x,y,v} c_x(\eta_x) (\nabla_{xy} \rho^{\alpha-1}(\eta))^2 \right. \\ &\quad \left. \times ((c_x(\eta_x) - c_x(\eta_x - 1)) \widehat{\rho}_1(\eta, \eta^{xy}) \rho(\eta) \right. \\ &\quad \left. + (c_y(\eta_y + 1) - c_y(\eta_y)) \widehat{\rho}_2(\eta, \eta^{xy}) \rho(\eta^{xy})) \right] \\ &= B_1 + B_2. \end{aligned}$$

The term B_1 is estimated by using condition (25) (note that $\widehat{\rho}_1, \widehat{\rho}_2 \geq 0$ since θ is nondecreasing in both variables) and then employing the assumption $c \geq 2^{\alpha-2\delta}$ and interchanging y and v :

$$\begin{aligned} B_1 &\geq \frac{c}{4L^2} \pi \left[\sum_{x,y,v} c_x(\eta_x) (\nabla_{xy} \rho^{\alpha-1}(\eta))^2 \rho(\eta^{xv}) (\widehat{\rho}_1(\eta, \eta^{xy}) + \widehat{\rho}_2(\eta, \eta^{xy})) \right] \\ &\geq \frac{2^{\alpha-2\delta}}{4L^2} \pi \left[\sum_{x,y,v} c_x(\eta_x) (\nabla_{xv} \rho^{\alpha-1}(\eta))^2 \rho(\eta^{xy}) (\widehat{\rho}_1(\eta, \eta^{xv}) + \widehat{\rho}_2(\eta, \eta^{xv})) \right] \\ &= B_3. \end{aligned}$$

We employ condition (25) once more and Lemma 17(i) (see the Appendix) to estimate B_2 :

$$\begin{aligned} B_2 &\geq -\frac{c+\delta}{4L^2}\pi\left[\sum_{x,y,v}c_x(\eta_x)(\nabla_{xy}\rho^{\alpha-1}(\eta))^2(\widehat{\rho}_1(\eta,\eta^{xy})\rho(\eta)+\widehat{\rho}_2(\eta,\eta^{xy})\rho(\eta^{xy}))\right] \\ &= -\frac{c+\delta}{4L^2}(2-\alpha)\pi\left[\sum_{x,y,v}c_x(\eta_x)(\nabla_{xy}\rho^{\alpha-1}(\eta))^2\widehat{\rho}(\eta,\eta^{xy})\right] \\ &= -\frac{1}{2}(2-\alpha)(c+\delta)A. \end{aligned}$$

Consequently,

$$(29) \quad B \geq -\frac{1}{2}(2-\alpha)(c+\delta)A + B_3.$$

We add (28) and (29):

$$(30) \quad \begin{aligned} C_1 + B &\geq \left(c - \frac{\delta}{2} - \frac{1}{2}(2-\alpha)(c+\delta)\right)A + B_4, \\ \text{where } B_4 &= B_3 - \frac{\delta}{4L^2}\pi\left[\sum_{x,y,v}c_x(\eta_x)(\nabla_{xv}\rho^{\alpha-1}(\eta))^2\widehat{\rho}(\eta,\eta^{xy})\right]. \end{aligned}$$

We wish to estimate B_4 from below by a multiple of A . To this end, we employ the reversibility and interchange x and v in the second term in B_4 :

$$\begin{aligned} &\pi\left[\sum_{x,y,v}c_x(\eta_x)(\nabla_{xv}\rho^{\alpha-1}(\eta))^2\widehat{\rho}(\eta,\eta^{xy})\right] \\ &= \pi\left[\sum_{x,y,v}c_v(\eta_v)(\nabla_{xv}\rho^{\alpha-1}(\eta^{vx}))^2\widehat{\rho}(\eta^{vx},\eta^{vy})\right] \\ &= \pi\left[\sum_{x,y,v}c_x(\eta_x)(\nabla_{xv}\rho^{\alpha-1}(\eta))^2\widehat{\rho}(\eta^{xv},\eta^{xy})\right]. \end{aligned}$$

Then, averaging those two expressions for B_4 that involve $\widehat{\rho}(\eta,\eta^{xy})$ and $\widehat{\rho}(\eta^{xv},\eta^{xy})$,

$$\begin{aligned} B_4 &= \frac{\delta}{8L^2}\pi\left[\sum_{x,y,v}c_x(\eta_x)(\nabla_{xv}\rho^{\alpha-1}(\eta))^2(2^{\alpha-1}\rho(\eta^{xy}))(\widehat{\rho}_1(\eta,\eta^{xv})+\widehat{\rho}_2(\eta,\eta^{xv}))\right] \\ &\quad - \frac{\delta}{8L^2}\pi\left[\sum_{x,y,v}c_x(\eta_x)(\nabla_{xv}\rho^{\alpha-1}(\eta))^2(\widehat{\rho}(\eta,\eta^{xy})+\widehat{\rho}(\eta^{xv},\eta^{xy}))\right]. \end{aligned}$$

We employ Lemma 17(ii) in the form

$$\begin{aligned} &2^{\alpha-1}\rho(\eta^{xy})(\widehat{\rho}_1(\eta,\eta^{xv})+\widehat{\rho}_2(\eta,\eta^{xv})) - (\widehat{\rho}(\eta,\eta^{xy})+\widehat{\rho}(\eta^{xv},\eta^{xy})) \\ &\geq -2^{\alpha-1}\widehat{\rho}(\eta,\eta^{xv}), \end{aligned}$$

which leads to

$$B_4 \geq -\frac{2^{\alpha-1}\delta}{8L^2}\pi\left[\sum_{x,y,v}c_x(\eta_x)(\nabla_{xv}\rho^{\alpha-1}(\eta))^2\widehat{\rho}(\eta,\eta^{xv})\right] = -\frac{2^{\alpha-1}\delta}{4}A.$$

Hence, we infer from (30) that

$$C_1 + B \geq \left(c - \frac{\delta}{2} - \frac{1}{2}(2 - \alpha)(c + \delta) - \frac{\delta}{4}2^{\alpha-1}\right)A.$$

Finally, by definition of B ,

$$C_1 + C_2 = 2(C_1 + B) \geq (2c - \delta - (2 - \alpha)(c + \delta) - 2^{\alpha-2}\delta)A = \lambda A.$$

This shows (19), and an application of Corollary 4 completes the proof. \square

3.3. Bernoulli–Laplace models. We consider again a system of particles moving in a finite set of sites $\{1, 2, \dots, L\}$ but in contrast to the previous subsection, we assume that at most one particle per site is allowed, that is, $S = \{0, 1\}^L$. The set of allowed moves is $G = \{xy : x, y \in \{1, 2, \dots, L\}, x \neq y\}$, and the moves are of the form $xy : \eta \mapsto \eta^{xy}$ for $\eta \in S$, where $\eta^{xy} = \eta$ if $\eta_x(1 - \eta_y) = 0$ and otherwise,

$$\eta_z^{xy} = \begin{cases} \eta_z & \text{if } z \notin \{x, y\}, \\ 0 & \text{for } z = x, \\ 1 & \text{for } z = y. \end{cases}$$

We associate to each site x a Poisson clock of constant intensity $\lambda_x > 0$. When the clock of site x rings, we choose randomly a site y . If $\eta_x = 1$ and $\eta_y = 0$ [i.e., if $\eta_x(1 - \eta_y) = 1$], the particle at x moves to y ; otherwise [i.e., if $\eta_x(1 - \eta_y) = 0$], nothing happens. Therefore, the transition rates are given by $c(\eta, xy) = (\lambda_x/L)\eta_x(1 - \eta_y)$, and the generator reads as

$$\mathcal{L}f(\eta) = \frac{1}{L} \sum_{xy \in G} \lambda_x \eta_x (1 - \eta_y) \nabla_{xy} f(\eta),$$

where, as in the previous subsection, $\nabla_{xy} f(\eta) = f(\eta^{xy}) - f(\eta)$.

Let $N \leq L$ be the number of particles in the system. There exists a unique stationary distribution π_N , which is given by [11], Section 5

$$\pi_N(\eta) = \frac{1}{Z_{L,N}} \prod_{x=1}^L \left(\frac{1}{1 + \lambda_x}\right)^{\eta_x} \left(\frac{\lambda_x}{1 + \lambda_x}\right)^{1 - \eta_x},$$

where $Z_{L,N} > 0$ is a normalization constant. In the following, we write π instead of π_N , as the number of particles is fixed. Reversibility holds for π , and it reads as

$$(31) \quad \pi \left[\sum_{xy \in G} c(\eta, xy) F(\eta, xy) \right] = \pi \left[\sum_{xy \in G} c(\eta, xy) F(\eta^{xy}, yx) \right]$$

for arbitrary functions $F : S \times G \rightarrow \mathbb{R}$.

THEOREM 9. *Let $\phi(s) = (s^\alpha - s)/(\alpha - 1) - s + 1$ and $1 < \alpha < 2$. Assume that there exist constants $0 \leq \delta \leq 2^{2-\alpha}c$ such that*

$$(32) \quad c \leq \lambda_x \leq c + \delta \quad \text{for } x \in \{1, 2, \dots, L\}.$$

Then the Beckner inequality (7) and the decay estimates (6) and (20) hold with $\lambda = \alpha c - (\frac{5}{2} + 2^{\alpha-3} - \alpha)\delta$.

REMARK 10. For the modified log-Sobolev inequality, the bound in [11] reads as $\lambda = c - \delta$, and the bound in [19] equals $\lambda = c - 7\delta/4$ (for $\delta < 4c/7$). Our result coincides with that in [19] for $\alpha \rightarrow 1$. In [22], the bound $1 \leq \lambda \leq 2$ was proved in case $c = 1, \delta = 0$. Further bounds, depending on L and N , were collected in [6], Examples 3.11.

Concerning the Beckner inequality, Bobkov and Tetali [6], Section 4, derived for the homogeneous case $c = L/(N(L - N))$ and $\delta = 0$ the constant $\lambda \geq \alpha(L + 2)/(2N(L - N))$. Our constant $\lambda = (\alpha L - 2\alpha + 4)/(N(L - N))$ (see the proof below) is larger for $L > 2$ and all $1 < \alpha \leq 2$.

PROOF. We need to verify the condition in Corollary 4. As in [11], we choose

$$R(\eta, xy, uv) = L^{-2}\lambda_x\lambda_u\eta_x(1 - \eta_y)\eta_u(1 - \eta_v) \quad \text{for } |\{x, y, u, v\}| = 4$$

and $R(\eta, xy, uv) = 0$ otherwise. The notation $|\{x, y, u, v\}| = 4$ means that the four variables are pairwise different. Then $\Gamma(\eta, xy, uv) = 0$ if $|\{x, y, u, v\}| = 4$ and

$$\Gamma(\eta, xy, uv) = L^{-2}\lambda_x\lambda_u\eta_x(1 - \eta_y)\eta_u(1 - \eta_v)$$

otherwise. The sum of $\Gamma(\eta, \gamma, \delta)$ over $\gamma, \delta \in G$ in the left-hand side of (19) vanishes if (x, y, u, v) are pairwise different. Therefore, the sum consists of three terms: $(\gamma, \delta) = (xy, xy)$, $(\gamma, \delta) = (xy, uy)$, and $(\gamma, \delta) = (xy, xv)$, and it follows that

$$\begin{aligned} & \pi \left[\sum_{\gamma, \delta \in G} \Gamma(\eta, \gamma, \delta) (\nabla_\gamma \rho^{\alpha-1}(\eta) \nabla_\delta \rho(\eta) + (\alpha - 1) \nabla_\gamma \rho(\eta) \nabla_\delta \rho(\eta) \rho^{\alpha-2}(\eta)) \right] \\ &= \frac{1}{L^2} \pi \left[\sum_{x,y} \lambda_x^2 \nabla_{xy} \rho^{\alpha-1}(\eta) \nabla_{xy} \rho(\eta) + \sum_{|\{x,y,u\}|=3} \lambda_x \lambda_u \nabla_{xy} \rho^{\alpha-1}(\eta) \nabla_{uy} \rho(\eta) \right. \\ & \quad \left. + \sum_{|\{x,y,v\}|=3} \lambda_x^2 \nabla_{xy} \rho^{\alpha-1}(\eta) \nabla_{xv} \rho(\eta) \right] \\ & \quad + \frac{\alpha - 1}{L^2} \pi \left[\sum_{x,y} \lambda_x^2 \nabla_{xy} \rho(\eta) \nabla_{xy} \rho(\eta) \rho^{\alpha-2}(\eta) \right. \\ & \quad \left. + \sum_{|\{x,y,u\}|=3} \lambda_x \lambda_u \nabla_{xy} \rho(\eta) \nabla_{uy} \rho(\eta) \rho^{\alpha-2}(\eta) \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{|\{x,y,v\}|=3} \lambda_x^2 \nabla_{xy} \rho(\eta) \nabla_{xv} \rho(\eta) \rho^{\alpha-2}(\eta) \Big] \\
& = C_1 + C_2.
\end{aligned}$$

Observe that the right-hand side of (19) (without the constant λ) reads as

$$\begin{aligned}
(33) \quad A & = \frac{1}{2} \pi \left[\sum_{\gamma \in G} c(\eta, \gamma) \nabla_{\gamma} \rho^{\alpha-1}(\eta) \nabla_{\gamma} \rho(\eta) \right] \\
& = \frac{1}{2L} \pi \left[\sum_{xy \in G} \lambda_x \nabla_{xy} \rho^{\alpha-1}(\eta) \nabla_{xy} \rho(\eta) \right],
\end{aligned}$$

since $\nabla_{xy} \rho(\eta) = 0$ whenever $\eta_x(1 - \eta_y) = 0$, so the factor $\eta_x(1 - \eta_y)$ can be omitted.

As in the previous subsection, we estimate $B = (C_2 - C_1)/2$, recalling definition (13) of $\widehat{\rho}_1$:

$$\begin{aligned}
(34) \quad B & = \frac{1}{2L^2} \pi \left[\sum_{x,y} \lambda_x^2 (\nabla_{xy} \rho^{\alpha-1}(\eta))^2 \widehat{\rho}_1(\eta, \eta^{xy}) \nabla_{xy} \rho(\eta) \right] \\
& + \frac{1}{2L^2} \pi \left[\sum_{|\{x,y,u\}|=3} \lambda_x \lambda_u (\nabla_{xy} \rho^{\alpha-1}(\eta))^2 \widehat{\rho}_1(\eta, \eta^{xy}) \nabla_{uy} \rho(\eta) \right] \\
& + \frac{1}{2L^2} \pi \left[\sum_{|\{x,y,v\}|=3} \lambda_x^2 (\nabla_{xy} \rho^{\alpha-1}(\eta))^2 \widehat{\rho}_1(\eta, \eta^{xy}) \nabla_{xv} \rho(\eta) \right] \\
& = B_1 + B_2 + B_3.
\end{aligned}$$

The estimations of B_1 , B_2 , and B_3 are the same as in the proof of Theorem 4.6 in [19] after taking $\psi(\eta) = \rho^{\alpha-1}(\eta)$ in $\widetilde{\mathcal{B}}_2(\rho, \psi)$. The key point is the use of Lemma 17(iii). In contrast to [19], the factor $2 - \alpha$ appears. Therefore, following [19] and taking into account (33), we conclude that

$$\begin{aligned}
(35) \quad B_1 & \geq -\frac{\delta}{2L} (2 - \alpha) A, \\
B_2 & \geq -\frac{1}{2L} (N - 1)(c + \delta)(2 - \alpha) A, \\
B_3 & \geq \frac{c}{4L^2} \pi \left[\sum_{|\{x,y,v\}|=3} \lambda_x \eta_x (1 - \eta_y)(1 - \eta_v) (\nabla_{xy} \rho^{\alpha-1}(\eta))^2 \rho(\eta^{xv}) \right. \\
& \quad \left. \times (\widehat{\rho}_1(\eta, \eta^{xy}) + \widehat{\rho}_2(\eta, \eta^{xy})) \right] - \frac{1}{2L} (L - N - 1)(c + \delta)(2 - \alpha) A.
\end{aligned}$$

Since we assumed that $\delta \leq 2^{2-\alpha} c$, we can estimate the factor in the first term of B_3 by $c/(4L^2) \geq 2^{\alpha-4} \delta/L^2$.

Next, we estimate C_1 . This expression consists of three terms. We interchange x and u in the second term and y and v in the third term. Then $C_1 = B_4 + B_5 + B_6$, where

$$\begin{aligned}
 B_4 &= \frac{1}{L^2} \pi \left[\sum_{x,y} \lambda_x^2 \nabla_{xy} \rho(\eta) \nabla_{xy} \rho^{\alpha-1}(\eta) \right], \\
 B_5 &= \frac{1}{L^2} \pi \left[\sum_{|\{x,y,u\}|=3} \lambda_x \lambda_u \nabla_{xy} \rho(\eta) \nabla_{uy} \rho^{\alpha-1}(\eta) \right], \\
 B_6 &= \frac{1}{L^2} \pi \left[\sum_{|\{x,y,v\}|=3} \lambda_x^2 \nabla_{xy} \rho(\eta) \nabla_{xv} \rho^{\alpha-1}(\eta) \right].
 \end{aligned}$$

By condition (32), $B_4 \geq (2c/L)A$. The term B_6 is estimated by employing the reversibility (31), averaging, and using (32), similar to the estimate of J_6 in the proof of Theorem 4.6 in [19]. The result is

$$\begin{aligned}
 (36) \quad B_6 &\geq \frac{1}{2L} (L - N - 1)(2c - \delta)A - B_7, \\
 \text{where } B_7 &= \frac{\delta}{4L^2} \pi \left[\sum_{|\{x,y,v\}|=3} \lambda_x \eta_x (1 - \eta_y)(1 - \eta_v) (\nabla_{xv} \rho^{\alpha-1}(\eta))^2 \widehat{\rho}(\eta, \eta^{xy}) \right].
 \end{aligned}$$

Similarly, replacing $\psi(\eta)$ by $\rho^{\alpha-1}(\eta)$ in J_5 in the proof of Theorem 4.6 in [19], we have $B_5 \geq (c/L)(N - 1)A$.

It remains to rewrite B_7 . For this, we employ the reversibility, average the original and the resulting expressions, and interchange y and v . This yields (see the computation of J_7 in [19])

$$\begin{aligned}
 B_7 &= \frac{\delta}{8L^2} \pi \left[\sum_{|\{x,y,v\}|=3} \lambda_x \eta_x (1 - \eta_y)(1 - \eta_v) \right. \\
 &\quad \left. \times (\nabla_{xy} \rho^{\alpha-1}(\eta))^2 (\widehat{\rho}(\eta^{xv}, \eta^{xy}) + \widehat{\rho}(\eta, \eta^{xv})) \right].
 \end{aligned}$$

Combining estimate (35) for B_3 and (36), together with the above estimate for B_7 and applying Lemma 17(ii), we infer that

$$\begin{aligned}
 &B_3 + B_6 \\
 &\geq \frac{1}{2L} (L - N - 1)(\alpha c - (3 - \alpha)\delta)A \\
 &\quad + \frac{\delta}{8L^2} \pi \left[\sum_{|\{x,y,v\}|=3} \lambda_x \eta_x (1 - \eta_y)(1 - \eta_v) (\nabla_{xy} \rho^{\alpha-1}(\eta))^2 \right. \\
 &\quad \left. \times (2^{\alpha-1} \rho(\eta^{xv})(\widehat{\rho}_1(\eta, \eta^{xy}) + \widehat{\rho}_2(\eta, \eta^{xy})) - (\widehat{\rho}(\eta^{xv}, \eta^{xy}) + \widehat{\rho}(\eta^{xv}, \eta))) \right] \\
 &\geq \frac{1}{4L} (L - N - 1)(2\alpha c - 2(3 - \alpha)\delta - 2^{\alpha-1}\delta)A.
 \end{aligned}$$

It remains to summarize the estimates:

$$\begin{aligned} C_1 + C_2 &= 2B + 2C_1 = 2(B_1 + B_2) + 2(B_4 + B_5) + 2(B_3 + B_6) \\ &\geq -\frac{(2-\alpha)}{L}(\delta + (N-1)(c+\delta))A + \frac{2}{L}(2c + (N-1)c)A \\ &\quad + \frac{1}{2L}(L-N-1)(2\alpha c - 2(3-\alpha)\delta - 2^{\alpha-1}\delta)A \\ &= \frac{1}{L}((\alpha L + 4 - 2\alpha)c \\ &\quad + ((\alpha - 2^{\alpha-2} - 3)L + (1 + 2^{\alpha-2})N + (3 + 2^{\alpha-2} - \alpha))\delta)A. \end{aligned}$$

Arguing as in [19], we may suppose that $N \geq L/2$. Because of $4 - 2\alpha \geq 0$, $(1 + 2^{\alpha-2})N/L \geq (1 + 2^{\alpha-2})/2$, and $3 + 2^{\alpha-2} - \alpha \geq 0$, we infer that

$$\begin{aligned} C_1 + C_2 &\geq \left(\frac{1}{L}(\alpha L + 4 - 2\alpha)c + \left(\alpha - \frac{5}{2} - 2^{\alpha-3}\right)\delta\right)A \\ &\geq \left(\alpha c - \left(\frac{5}{2} + 2^{\alpha-3} - \alpha\right)\delta\right)A \end{aligned}$$

which completes the proof. \square

3.4. Random transposition model. The random transposition model is a random walk on the group of permutations. Let S_n be the set of permutations on $\{1, 2, \dots, n\}$ and T_n the set of all transpositions in S_n . Given $1 \leq i, j \leq n$, we denote by $\tau_{ij} \in T_n$ the transposition that interchanges i and j , that is, $\tau_{ij}(i) = j$, $\tau_{ij}(j) = i$, and $\tau_{ij}(k) = k$ for $k \neq i, j$. The composition of two permutations $\sigma_1, \sigma_2 \in S_n$ is denoted by $\sigma_1\sigma_2$.

We define a graph structure on the group S_n by saying that two permutations are neighbors if they differ by precisely one transposition. Thus every vertex $\sigma \in S_n$ has $\binom{n}{2} = n(n-1)/2$ neighbors given by $\{\tau_{ij}\sigma\}_{1 \leq i, j \leq n}$, and the set of edges is $E_n = \{\{\sigma, \tau_{ij}\sigma\} : 1 \leq i, j \leq n, \sigma \in S_n\}$. We write $\sigma \leftrightarrow \tau\sigma$ if $\{\sigma, \tau\sigma\} \in E_n$. The random walk on (S_n, E_n) is then defined by the transition rates $c(\sigma, \tau) = 2/(n(n-1))$ if $\sigma \leftrightarrow \tau\sigma$ and $c(\sigma, \tau) = 0$ otherwise. The generator of the Markov chain reads as

$$\mathcal{L}f(\sigma) = \frac{2}{n(n-1)} \sum_{\tau \in T_n} \nabla_{\tau} f(\sigma),$$

where $\nabla_{\tau} f(\sigma) = f(\tau \circ \sigma) - f(\sigma)$. The uniform measure $\pi(\sigma) = 1/n!$ for all $\sigma \in S_n$ is reversible for the above transition rates $c(\sigma, \tau)$. To simplify the notation, we write $\nabla_{ij} = \nabla_{\tau}$ if $\tau = \tau_{ij}$, $\sigma_{ij} = \tau_{ij} \circ \sigma$ and $\sigma_{ijk} = \tau_{ij} \circ \tau_{jk} \circ \sigma$.

THEOREM 11. *Let $\phi(s) = (s^{\alpha} - s)/(\alpha - 1) - s + 1$ and $1 < \alpha < 2$. For $n \geq 2$, the Beckner inequality (7) and the decay estimates (6) and (20) hold with constant $\lambda = 8/(n(n-1))$.*

REMARK 12. Diaconis and Saloff-Coste [18], Section 4.3, report that the logarithmic Sobolev constant satisfies the bounds $1/(3n \log n) \leq \lambda \leq 1/(n - 1)$; also see [22], Theorem 1. Our bound is worse by a factor of $1/n$. The bound $\lambda \geq \alpha(n + 2)/(n(n - 1))$ was derived in [6], Section 4. It is usually better than our bound $\lambda = 8/(n(n - 1))$; for very small numbers of n [namely $n < (8/\alpha) - 2$], our result is superior.

PROOF. The right-hand side of (19) (except the factor λ) can be written as

$$\begin{aligned}
 (37) \quad A &= \frac{1}{n(n - 1)} \pi \left[\sum_{\tau \in T_n} \nabla_{\tau} \rho^{\alpha-1}(\sigma) \nabla_{\tau} \rho(\sigma) \right] \\
 &= \frac{1}{2n(n - 1)} \pi \left[\sum_{i \neq j} \nabla_{ij} \rho^{\alpha-1}(\sigma) \nabla_{ij} \rho(\sigma) \right],
 \end{aligned}$$

where the factor $1/2$ takes into account that every transposition (i, j) is counted twice. As in [19], Section 4.4, we define $R(\sigma, (i, j), (k, \ell)) = 4/(n^2(n - 1)^2)$ if $|\{i, j, k, \ell\}| = 4$ and $R(\sigma, (i, j), (k, \ell)) = 0$, otherwise. Then $\Gamma(\sigma, (i, j), (k, \ell)) = 0$ if $|\{i, j, k, \ell\}| = 4$ and

$$\Gamma(\sigma, (i, j), (k, \ell)) = \frac{4}{n^2(n - 1)^2}$$

otherwise. The left-hand side of (19) then becomes

$$\begin{aligned}
 &\pi \left[\sum_{\gamma, \delta} \Gamma(\sigma, \gamma, \delta) (\nabla_{\gamma} \rho^{\alpha-1}(\sigma) \nabla_{\delta} \rho(\sigma) + (\alpha - 1) \nabla_{\gamma} \rho(\sigma) \nabla_{\delta} \rho(\sigma) \rho^{\alpha-2}(\sigma)) \right] \\
 &= \frac{2}{n^2(n - 1)^2} \pi \left[\sum_{i \neq j} \nabla_{ij} \rho^{\alpha-1}(\sigma) \nabla_{ij} \rho(\sigma) \right. \\
 &\quad \left. + 2 \sum_{|\{i, j, k\}|=3} \nabla_{ij} \rho(\sigma) \nabla_{ik} \rho^{\alpha-1}(\sigma) \right] \\
 &\quad + \frac{2(\alpha - 1)}{n^2(n - 1)^2} \pi \left[\sum_{i \neq j} \nabla_{ij} \rho(\sigma) \nabla_{ij} \rho(\sigma) \rho^{\alpha-2}(\sigma) \right. \\
 &\quad \left. + 2 \sum_{|\{i, j, k\}|=3} \nabla_{ij} \rho(\sigma) \nabla_{ik} \rho(\sigma) \rho^{\alpha-2}(\sigma) \right] \\
 &= C_1 + C_2.
 \end{aligned}$$

The expression C_1 can be estimated exactly as in the proof of Theorem 4.8 in [19] using the reversibility and averaging [see the estimate for $\tilde{\mathcal{B}}_1(\rho, \psi)$ for $\psi = \rho^{\alpha-1}$]:

$$C_1 \geq \frac{2}{n - 1} A - \frac{1}{n^2(n - 1)^2} \pi \left[\sum_{|\{i, j, k\}|=3} (\rho^{\alpha-1}(\sigma_{ij}) - \rho^{\alpha-1}(\sigma))^2 \hat{\rho}(\sigma_{ik}, \sigma_{ijk}) \right].$$

We estimate now $B = (C_2 - C_1)/2$:

$$B = \frac{1}{n^2(n-1)^2} \pi \left[\sum_{i \neq j} (\nabla_{ij} \rho^{\alpha-1}(\sigma))^2 \nabla_{ij} \rho(\sigma) \widehat{\rho}_1(\sigma, \sigma_{ij}) \right. \\ \left. + 2 \sum_{|\{i,j,k\}|=3} (\nabla_{ik} \rho^{\alpha-1}(\sigma))^2 \nabla_{ij} \rho(\sigma) \widehat{\rho}_1(\sigma, \sigma_{ik}) \right].$$

Arguing as for $\widetilde{B}_2(\rho, \psi)$ with $\psi = \rho^{\alpha-1}$ in the proof of Theorem 4.8 in [19], it follows that

$$B = \frac{1}{n^2(n-1)^2} \pi \\ \times \left[\sum_{|\{i,j,k\}|=3} (\nabla_{ij} \rho^{\alpha-1}(\sigma))^2 (\rho(\sigma_{ik}) \widehat{\rho}_1(\sigma, \sigma_{ij}) + \rho(\sigma_{ijk}) \widehat{\rho}_2(\sigma, \sigma_{ij})) \right] \\ - \frac{1}{n^2(n-1)^2} \pi \\ \times \left[\sum_{|\{i,j,k\}|=3} (\nabla_{ij} \rho^{\alpha-1}(\sigma))^2 (\rho(\sigma) \widehat{\rho}_1(\sigma, \sigma_{ij}) + \rho(\sigma_{ij}) \widehat{\rho}_2(\sigma, \sigma_{ij})) \right] \\ + \frac{1}{2n^2(n-1)^2} \pi \left[\sum_{i \neq j} (\nabla_{ij} \rho^{\alpha-1}(\sigma))^2 \nabla_{ij} \rho(\sigma) (\widehat{\rho}_1(\sigma, \sigma_{ij}) - \widehat{\rho}_2(\sigma, \sigma_{ij})) \right] \\ = B_1 + B_2 + B_3.$$

Property (iii) of Lemma 17 (applied with $\lambda_1 = \lambda_2 = 1$) implies that $B_3 \geq 0$. Combining B_1 and B_2 , we can apply Lemma 15 with $s = \rho(\sigma)$, $t = \rho(\sigma_{ij})$, $u = \rho(\sigma_{ik})$, and $v = \rho(\sigma_{ijk})$, leading to

$$B \geq B_1 + B_2 \geq \frac{1}{n^2(n-1)^2} \pi \left[\sum_{|\{i,j,k\}|=3} (\nabla_{ij} \rho^{\alpha-1}(\sigma))^2 (\widehat{\rho}(\sigma_{ik}, \sigma_{ijk}) - \widehat{\rho}(\sigma, \sigma_{ij})) \right] \\ = \frac{1}{n^2(n-1)^2} \pi \left[\sum_{|\{i,j,k\}|=3} (\nabla_{ij} \rho^{\alpha-1}(\sigma))^2 \widehat{\rho}(\sigma_{ik}, \sigma_{ijk}) \right] - \frac{2(n-2)}{n(n-1)} A.$$

Adding the estimations for C_1 and B , one term cancels and we end up with

$$C_1 + C_2 = 2(C_1 + B) \geq 2 \left(\frac{2}{n-1} - \frac{2(n-2)}{n(n-1)} \right) A = \frac{8}{n(n-1)} A.$$

This completes the proof. \square

4. Application: Finite-volume discretization of a Fokker–Planck equation.

The Bakry–Emery method has been originally applied to Markov diffusion operators or associated Fokker–Planck equations, and the exponential decay for the

probability densities with an explicit decay rate was shown. In numerical analysis, the aim is to prove this equilibration property for numerical discretizations of Fokker–Planck equations. As these discretizations can, at least in some cases, be interpreted as a Markov chain, one may apply Markov chain theory to achieve this goal. This was done by Mielke [28], Section 5.3, to prove exponential decay of the logarithmic entropy for a finite-volume approximation of a Fokker–Planck equation. The proof is based on diagonal dominance properties of the matrices appearing in (2). Our aim is to extend the exponential decay to power-type entropies by combining Mielkes results and the estimate for birth-death processes from Theorem 5. As a by-product, this provides an alternative proof for the case $\alpha \rightarrow 1$ without using matrix algebra.

More specifically, we consider a finite-volume approximation of the one-dimensional Fokker–Planck equation:

$$(38) \quad \partial_t u = \partial_x (\partial_x u + u \partial_x V), \quad t > 0, u(\cdot, 0) = u_0 \text{ in } \mathbb{R},$$

where $u(x, t)$ describes some probability density and $V(x)$ is a given potential satisfying $e^{-V} \in L^1(\mathbb{R})$. It holds that $\int_{\mathbb{R}} u(x, t) dx = \int_{\mathbb{R}} u_0(x) dx$ for $t > 0$, that is, mass is conserved. We introduce the uniform grid $x_n = n/N$, $n \in \mathbb{Z}$, where $N \in \mathbb{N}$. The quantity $h = 1/N$ is the grid size. The Fokker–Planck equation has the unique steady state $\pi(x) = Ze^{-V(x)}$, where $Z > 0$ is a normalization constant. The symmetric form of (38),

$$\partial_t u = \partial_x \left(\pi \partial_x \left(\frac{u}{\pi} \right) \right),$$

motivates the following numerical scheme. We integrate this equation over $[x_{n-1}, x_n]$:

$$\frac{d}{dt} \frac{1}{h} \int_{x_{n-1}}^{x_n} u(x, t) dx = \frac{1}{h} \left[\pi \partial_x \left(\frac{u(\cdot, t)}{\pi} \right) \right]_{x_{n-1}}^{x_n}.$$

We choose u_n to approximate $\int_{x_{n-1}}^{x_n} u(\cdot, x) dx/h$, $\pi_n = \int_{x_{n-1}}^{x_n} \pi(x) dx/h$, and the numerical flux q_n to approximate $h^{-1}[\pi \partial_x(u/\pi)](x_n)$. We choose as in [28]

$$q_n = \frac{\kappa_n}{h^2} \left(\frac{u_{n+1}}{\pi_{n+1}} - \frac{u_n}{\pi_n} \right), \quad \kappa_n = (\pi_n \pi_{n+1})^{1/2}.$$

Setting $\rho_n = u_n/\pi_n$, the numerical scheme reads as

$$\begin{aligned} \partial_t \rho_n &= \frac{1}{\pi_n} (q_n - q_{n-1}) = \frac{\kappa_n}{h^2 \pi_n} (\rho_{n+1} - \rho_n) + \frac{\kappa_{n-1}}{h^2 \pi_n} (\rho_{n-1} - \rho_n) \\ &= a(n) \nabla_+ \rho_n + b(n) \nabla_- \rho_n, \end{aligned}$$

where we employed the notation of Section 3.1 and $a(n) = \kappa_n/(h^2 \pi_n)$, $b(n) = \kappa_{n-1}/(h^2 \pi_n)$. The right-hand side can be interpreted as the generator of a birth-death process on \mathbb{Z} . The initial datum is given by $\rho_n(0) = u_n(0)/\pi_n$, where

$u_n(0) = \int_{x_{n-1}}^{x_n} u(x, 0) dx/h$. According to [11], Section 3.5, the results of Theorem 5 still hold in that case, and the assumption $b(0) = 0$ is clearly not needed. The entropy is given by

$$\begin{aligned} \text{Ent}_{\pi}^{\phi_{\alpha}}(\rho) &= \sum_{n \in \mathbb{Z}} \pi_n \left(\frac{\rho_n^{\alpha} - \rho_n}{\alpha - 1} - \rho_n + 1 \right) \\ &= \frac{1}{\alpha - 1} \sum_{n \in \mathbb{Z}} \pi_n (\rho_n^{\alpha} - 1), \end{aligned}$$

where $\rho = (\rho_n)_{n \in \mathbb{Z}}$, $1 < \alpha \leq 2$, since conservation of mass implies that $\sum_n \pi_n \rho_n = \sum_n \pi_n$.

THEOREM 13. *Let $V \in C^2([0, 1])$ and $V''(x) \geq \lambda > 0$ for $x \in [0, 1]$. Then*

$$\text{Ent}_{\pi}^{\phi_{\alpha}}(\rho(t)) \leq \text{Ent}_{\pi}^{\phi_{\alpha}}(\rho(0)) e^{-2\alpha\lambda ht}, \quad n \in \mathbb{N},$$

where $\lambda_h = 2h^{-2}\Phi(h^2\lambda/8)$ and

$$\Phi(s^2) = \frac{3\text{erf}(s) - \text{erf}(3s)}{2\text{erf}(s)} \quad \text{with } \text{erf}(s) = \frac{2}{\sqrt{p}} \int_0^s e^{-t^2} dt$$

and $p = 3.14159\dots$ is the number pi (to avoid confusion with the invariant measure π). Moreover, the following discrete Beckner inequality holds:

$$\begin{aligned} &2\lambda_h \sum_{n \in \mathbb{Z}} \pi_n \left(\frac{\rho_n^{\alpha} - \rho_n}{\alpha - 1} - \rho_n + 1 \right) \\ &\leq \frac{1}{\alpha - 1} \sum_{n \in \mathbb{Z}} \frac{\sqrt{\pi_{n+1}\pi_n}}{h^2} (\rho_{n+1}^{\alpha-1} - \rho_n^{\alpha-1})(\rho_{n+1} - \rho_n). \end{aligned}$$

REMARK 14. We remark that $\lambda_h \nearrow \lambda$ as $h \rightarrow 0$ [28], Corollary 5.5. Thus, the decay rate is asymptotically sharp. A modified log-Sobolev inequality with constant λ for a finite-difference approximation was proved in [25] for λ -log-concave potentials by translating the Bakry–Emery condition to the discrete case.

PROOF. Note that $a(n)$ and $b(n)$ satisfy the detailed-balance condition (22). The proof is a consequence of Theorem 5 and the results of Mielke [28], Section 5. In particular, he has shown that $(1 - \lambda_h)\pi_n \geq \sqrt{\pi_{n-1}\pi_{n+1}}$. Consequently,

$$\begin{aligned} a(n) - a(n+1) &= \sqrt{\frac{\pi_{n+1}}{\pi_n}} - \sqrt{\frac{\pi_{n+2}}{\pi_{n+1}}} \geq \lambda_h \sqrt{\frac{\pi_{n+1}}{\pi_n}}, \\ b(n+1) - b(n) &= \sqrt{\frac{\pi_n}{\pi_{n+1}}} - \sqrt{\frac{\pi_{n-1}}{\pi_n}} \geq \lambda_h \sqrt{\frac{\pi_n}{\pi_{n+1}}}. \end{aligned}$$

Using Lemma 18 and the relation between the arithmetic and geometric mean, it follows that

$$\begin{aligned} & a(n) - a(n + 1) + b(n + 1) - b(n) + \Theta(a(n) - a(n + 1), b(n + 1) - b(n)) \\ & \geq \alpha(a(n) - a(n + 1) + b(n + 1) - b(n)) \\ & \geq 2\alpha\sqrt{(a(n) - a(n + 1))(b(n + 1) - b(n))} \geq 2\alpha\lambda_h. \end{aligned}$$

Applying Theorem 5 completes the proof. \square

APPENDIX: PROPERTIES OF THE MEAN FUNCTION

We show some properties for

$$(39) \quad \theta(s, t) = \frac{s - t}{\phi'(s) - \phi'(t)}, \quad 0 < s, t < \infty, s \neq t,$$

with $\theta(s, s) = 1/\phi''(s)$. This function is symmetric and, if ϕ is convex, positive. For the following lemma, we introduce for $0 < s, t < \infty$,

$$Y(s, t) = (\phi')^{-1}((1 - m)\phi'(s) + m\phi'(t)), \quad 0 \leq m \leq 1.$$

We set $Y_1 = \partial Y/\partial s$, $Y_2 = \partial Y/\partial t$, $Y_{11} = \partial^2 Y/\partial s^2$, etc.

LEMMA 15 (Concavity of θ). *Let $\phi \in C^4((0, \infty); (0, \infty))$ be convex such that $\phi(1) = 0$, and $1/\phi''$ is concave on $(0, \infty)$. If $\phi^{(3)}(s) \leq 0$ for $s > 0$, the function θ , defined in (39), is nondecreasing in s and in t . Furthermore, if additionally*

$$(40) \quad Y_{11} \leq 0, \quad Y_{22} \leq 0, \quad Y_{11}Y_{22} \geq Y_{12}^2 \quad \text{in } (0, \infty)^2, m \in (0, 1),$$

then θ is concave. In this situation, it holds that for all $u, v, s, t > 0$,

$$(41) \quad \theta(u, v) - \theta(s, t) \leq \partial_1\theta(s, t)(u - s) + \partial_2\theta(s, t)(v - t).$$

PROOF. The function θ is nondecreasing in s if and only if $\partial_1\theta(s, t) \geq 0$. Since

$$\partial_1\theta(s, t) = \frac{\phi'(s) - \phi'(t) - (s - t)\phi''(s)}{(\phi'(s) - \phi'(t))^2},$$

it is sufficient to prove the nonnegativity of $G(s, t) = \phi'(s) - \phi'(t) - (s - t)\phi''(s)$. By assumption, the derivative $\partial_1 G(s, t) = -(s - t)\phi^{(3)}(s)$ is nonpositive for $s \in (0, t)$ and nonnegative otherwise. Then $G(s, t) \geq G(t, t) = 0$, and the conclusion follows. The monotonicity in the second variable is shown analogously.

For the proof of the concavity of θ , we observe that

$$\theta(s, t) = \int_0^1 ((\phi')^{-1})'((1 - m)\phi'(s) + m\phi'(t)) dm.$$

Thus, the concavity of θ is equivalent to that one of

$$F(s, t) = ((\phi')^{-1})'((1-m)\phi'(s) + m\phi'(t)) = \frac{1}{\phi''(Y(s, t))}$$

for any $m \in (0, 1)$. Let $0 < s, t < \infty$ and $0 < m < 1$. We claim that if $\phi^{(3)} \leq 0$ and (40) holds, then F is concave. For this, it is sufficient to prove that $F_{11} = \partial^2 F / \partial s^2 \leq 0$, $F_{22} = \partial^2 F / \partial t^2 \leq 0$, and the determinant of the Hessian of F is nonnegative. Because of (40) and $\phi''(Y) \geq 0$, $\phi^{(3)}(Y) \leq 0$, and $(1/\phi'')''(Y) \leq 0$, we obtain

$$\begin{aligned} F_{11} &= -\frac{\phi^{(4)}(Y)}{\phi''(Y)^2} Y_1^2 + 2\frac{\phi^{(3)}(Y)^2}{\phi''(Y)^3} Y_1^2 - \frac{\phi^{(3)}(Y)}{\phi''(Y)^2} Y_{11} \\ &= \left(\frac{1}{\phi''}\right)''(Y) Y_1^2 - \frac{\phi^{(3)}(Y)}{\phi''(Y)^2} Y_{11} \leq 0, \\ F_{22} &= \left(\frac{1}{\phi''}\right)''(Y) Y_2^2 - \frac{\phi^{(3)}(Y)}{\phi''(Y)^2} Y_{22} \leq 0. \end{aligned}$$

Then, using the assumptions and

$$\begin{aligned} F_{12} = F_{21} &= -\frac{\phi^{(4)}(Y)}{\phi''(Y)^2} Y_1 Y_2 + 2\frac{\phi^{(3)}(Y)^2}{\phi''(Y)^3} Y_1 Y_2 - \frac{\phi^{(3)}(Y)}{\phi''(Y)^2} Y_{12} \\ &= \left(\frac{1}{\phi''}\right)''(Y) Y_1 Y_2 - \frac{\phi^{(3)}(Y)}{\phi''(Y)^2} Y_{12}, \\ Y_1 &= (1-m) \frac{\phi''(s)}{\phi''(Y)} \geq 0, \quad Y_2 = m \frac{\phi''(t)}{\phi''(Y)} \geq 0, \\ Y_{12} &= -m(1-m) \frac{\phi''(s)\phi''(t)\phi^{(3)}(Y)}{\phi''(Y)^3} \geq 0, \end{aligned}$$

it follows that

$$\begin{aligned} F_{11} F_{22} - F_{12}^2 &= \left(\frac{\phi^{(3)}(Y)}{\phi''(Y)^2}\right)^2 (Y_{11} Y_{22} - Y_{12}^2) \\ &\quad + \frac{\phi^{(3)}(Y)}{\phi''(Y)^2} \left(\frac{1}{\phi''}\right)''(Y) (2Y_1 Y_2 Y_{12} - Y_1^2 Y_{22} - Y_2^2 Y_{11}) \geq 0. \end{aligned}$$

Finally, inequality (41) follows after Taylor expansion and taking into account the concavity of θ . \square

We claim that the assumptions of Lemma 15 are satisfied for the power mean

$$\theta_\alpha(s, t) = \frac{\alpha - 1}{\alpha} \frac{s - t}{s^{\alpha-1} - t^{\alpha-1}}, \quad 1 < \alpha < 2.$$

LEMMA 16. *Let $1 < \alpha < 2$. The function θ_α is C^∞ , symmetric, positive, increasing and concave on $(0, \infty)^2$. Furthermore, θ_α and its first partial derivatives are positive homogenous, that is, $\theta_\alpha(\lambda s, \lambda t) = \lambda^{2-\alpha}\theta_\alpha(s, t)$, $\partial_1\theta_\alpha(\lambda s, \lambda t) = \lambda^{1-\alpha}\partial_1\theta_\alpha(s, t)$, and $\partial_2\theta_\alpha(\lambda s, \lambda t) = \lambda^{1-\alpha}\partial_2\theta_\alpha(s, t)$ for all $s, t > 0$ and $\lambda > 0$.*

PROOF. The regularity, symmetry, and positivity of θ_α follow from elementary computations. The monotonicity follows from $\phi_\alpha^{(3)}(s) = \alpha(\alpha - 2)s^{\alpha-3} < 0$ for $s > 0$. To show that θ_α is concave, we verify the conditions of Lemma 15. We compute

$$\begin{aligned} Y(s, t) &= ((1 - m)s^{\alpha-1} + mt^{\alpha-1})^{1/(\alpha-1)}, \\ Y_{11}(s, t) &= -m(1 - m)(2 - \alpha)(st)^{\alpha-3}Y(s, t)^{3-2\alpha}t^2, \\ Y_{22}(s, t) &= -m(1 - m)(2 - \alpha)(st)^{\alpha-3}Y(s, t)^{3-2\alpha}s^2, \\ Y_{12}(s, t) &= m(1 - m)(2 - \alpha)(st)^{\alpha-3}Y(s, t)^{3-2\alpha}st, \end{aligned}$$

and it follows that $Y_{11} \leq 0$, $Y_{22} \leq 0$, and $Y_{11}Y_{22} - Y_{12}^2 = 0$. \square

We prove more properties of θ_α , needed in Sections 3.2–3.4.

LEMMA 17 (Properties of θ_α). *Let $1 < \alpha < 2$. The function θ_α satisfies for all $r, s, t > 0$ and $\lambda_1, \lambda_2 > 0$:*

- (i) $s\partial_1\theta_\alpha(s, t) + t\partial_2\theta_\alpha(s, t) = (2 - \alpha)\theta_\alpha(s, t)$;
- (ii) $2^{\alpha-1}r(\partial_1\theta_\alpha(s, t) + \partial_2\theta_\alpha(s, t)) - (\theta_\alpha(r, s) + \theta_\alpha(r, t)) \geq -2^{\alpha-1}\theta_\alpha(s, t)$;
- (iii) $\lambda_1\partial_1\theta_\alpha(s, t)(s - t) - \lambda_2\partial_2\theta_\alpha(s, t)(s - t) \leq (2 - \alpha)|\lambda_1 - \lambda_2|\theta_\alpha(s, t)$.

PROOF. Identity (i) can be obtained by an elementary computation. The proof of (ii) is similar to the proof of Lemma A.2 in [19]. Indeed, setting $u = s/r$ and $v = t/r$ and using the homogeneity properties of θ_α and its first partial derivatives, inequality (ii) is equivalent to

$$2^{\alpha-1}(\partial_1\theta_\alpha(u, v) + \partial_2\theta_\alpha(u, v)) - (\theta_\alpha(1, u) + \theta_\alpha(1, v)) \geq -2^{\alpha-1}\theta_\alpha(u, v).$$

This inequality follows from the concavity and the $(2 - \alpha)$ -homogeneity property of θ_α and from (i):

$$\begin{aligned} \theta_\alpha(1, u) + \theta_\alpha(1, v) &\leq 2\theta_\alpha\left(\frac{u + 1}{2}, \frac{v + 1}{2}\right) \\ &= 2^{\alpha-1}\theta_\alpha(u + 1, v + 1) \\ &\leq 2^{\alpha-1}(\theta_\alpha(u, v) + \partial_1\theta_\alpha(u, v) + \partial_2\theta_\alpha(u, v)). \end{aligned}$$

Finally, by property (i),

$$\begin{aligned} & \lambda_1 \partial_1 \theta_\alpha(s, t)(s - t) - \lambda_2 \partial_2 \theta_\alpha(s, t)(s - t) \\ & \leq \max\{\lambda_1, \lambda_2\}(s \partial_1 \theta_\alpha(s, t) + t \partial_2 \theta_\alpha(s, t)) \\ & \quad - \min\{\lambda_1, \lambda_2\}(t \partial_1 \theta_\alpha(s, t) + s \partial_2 \theta_\alpha(s, t)) \\ & = \max\{\lambda_1, \lambda_2\}(2 - \alpha)\theta_\alpha(s, t) - \min\{\lambda_1, \lambda_2\}(t \partial_1 \theta_\alpha(s, t) + s \partial_2 \theta_\alpha(s, t)). \end{aligned}$$

Choosing $u = t$ and $v = s$ in (41) gives $\partial_1 \theta_\alpha(s, t)(s - t) + \partial_2 \theta_\alpha(s, t)(t - s) \leq 0$, and combining this inequality with property (i) yields

$$\begin{aligned} & -(t \partial_1 \theta_\alpha(s, t) + s \partial_2 \theta_\alpha(s, t)) \\ & = \partial_1 \theta_\alpha(s, t)(s - t) + \partial_2 \theta_\alpha(s, t)(t - s) - (s \partial_1 \theta_\alpha(s, t) + t \partial_2 \theta_\alpha(s, t)) \\ & \leq -(2 - \alpha)\theta_\alpha(s, t), \end{aligned}$$

such that

$$\begin{aligned} & \lambda_1 \partial_1 \theta_\alpha(s, t)(s - t) - \lambda_2 \partial_2 \theta_\alpha(s, t)(s - t) \\ & \leq (\max\{\lambda_1, \lambda_2\} - \min\{\lambda_1, \lambda_2\})(2 - \alpha)\theta_\alpha(s, t) \\ & = |\lambda_1 - \lambda_2|(2 - \alpha)\theta_\alpha(s, t). \end{aligned}$$

This completes the proof. \square

LEMMA 18. Let $\phi_\alpha(s) = (s^\alpha - s)/(\alpha - 1) - s + 1$ and $1 < \alpha < 2$. It holds for all $A, B \geq 0$,

$$\Theta(A, B) := \inf_{s, t > 0} \theta_\alpha(s, t)(A\phi_\alpha''(s) + B\phi_\alpha''(t)) \geq (\alpha - 1)(A + B).$$

PROOF. Since

$$\theta_\alpha(s, t) = \frac{\alpha - 1}{\alpha} \frac{s - t}{s^{\alpha-1} - t^{\alpha-1}} = \frac{1}{\alpha} \int_0^1 ((1 - m)s^{\alpha-1} + mt^{\alpha-1})^{(2-\alpha)/(\alpha-1)} dm,$$

it follows that

$$\begin{aligned} & \theta_\alpha(s, t)(A\phi_\alpha''(s) + B\phi_\alpha''(t)) \\ & = A \int_0^1 \left((1 - m) + m \left(\frac{t}{s} \right)^{\alpha-1} \right)^{(2-\alpha)/(\alpha-1)} dm \\ & \quad + B \int_0^1 \left((1 - m) \left(\frac{s}{t} \right)^{\alpha-1} + m \right)^{(2-\alpha)/(\alpha-1)} dm \\ & \geq A \int_0^1 (1 - m)^{(2-\alpha)/(\alpha-1)} dm + B \int_0^1 m^{(2-\alpha)/(\alpha-1)} dm \\ & = (\alpha - 1)(A + B), \end{aligned}$$

which completes the proof. \square

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