

## SCALING LIMIT OF THE CORRECTOR IN STOCHASTIC HOMOGENIZATION

BY JEAN-CHRISTOPHE MOURRAT AND JAMES NOLEN<sup>1</sup>

*CNRS and Duke University*

In the homogenization of divergence-form equations with random coefficients, a central role is played by the corrector. We focus on a discrete space setting and on dimension 3 and more. Under a minor smoothness assumption on the law of the random coefficients, we identify the scaling limit of the corrector, which is akin to a Gaussian free field. This completes the argument started in [*Ann. Probab.* **44** (2016) 3207–3233].

**1. Introduction.** We consider a random conductance problem on  $\mathbb{Z}^d$  associated with the discrete divergence-form operator  $\nabla^* A \nabla$ , where the coefficients of  $A$  are independent, identically distributed random variables, bounded away from 0 and infinity. The main object of this paper is the stationary random corrector  $\phi$ , satisfying  $\nabla^* A(\xi + \nabla \phi(\xi)) = 0$  with  $\xi \in \mathbb{R}^d$  being a fixed vector,  $d \geq 3$ . This random function  $\phi$  plays a central role [2, 6, 7, 9] in homogenization theory for the operator  $\nabla^* A \nabla$ , a discrete analogue of the random elliptic operators considered in [8, 13]. Our main result is that for  $d \geq 3$  the appropriately rescaled corrector converges to a Gaussian field that has the homogeneity of a Gaussian free field.

To make this statement precise, we need to introduce some notation. We view  $\mathbb{Z}^d$  as a graph with edges between nearest neighbors, and we denote by  $\mathbb{B} = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d \mid |x - y| = 1\}$  the set of (nonoriented) edges. Let  $(\mathbf{e}_1, \dots, \mathbf{e}_d)$  be the canonical basis of  $\mathbb{Z}^d$ . For every edge  $e \in \mathbb{B}$ , there exists a unique pair  $(\underline{e}, i) \in \mathbb{Z}^d \times \{1, \dots, d\}$  such that  $e$  links  $\underline{e}$  to  $\underline{e} + \mathbf{e}_i$ . Given such a pair, we write  $\bar{e} = \underline{e} + \mathbf{e}_i$ . We call  $\underline{e}$  the *base point* of the edge  $e$ . For  $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ , we let  $\nabla f : \mathbb{B} \rightarrow \mathbb{R}$  be the gradient of  $f$ , defined by

$$\nabla f(e) = f(\bar{e}) - f(\underline{e}).$$

We write  $\nabla^*$  for the formal adjoint of  $\nabla$ , that is, for  $F : \mathbb{B} \rightarrow \mathbb{R}$ ,  $\nabla^* F : \mathbb{Z}^d \rightarrow \mathbb{R}$  is defined via

$$(\nabla^* F)(x) = \sum_{i=1}^d F((x - \mathbf{e}_i, x)) - F((x, x + \mathbf{e}_i)).$$

---

Received March 2015; revised April 2016.

<sup>1</sup>Supported by NSF Grants DMS-10-07572 and DMS-13-51653.

*MSC2010 subject classifications.* 35B27, 35J15, 35R60, 60G60.

*Key words and phrases.* Stochastic homogenization, scaling limit, Gaussian free field.

Given a family  $\{a(e)\}_{e \in \mathbb{B}}$  of positive real numbers and a function  $F : \mathbb{B} \rightarrow \mathbb{R}$ , we let  $AF(e) = a(e)F(e)$ . This provides us with a precise definition of the operator  $\nabla^* A \nabla$ .

In order to facilitate the derivation of the result (and match the assumptions made in [11] for the same reason), we make the simplifying assumption that the coefficients  $\{a(e)\}_{e \in \mathbb{B}}$  are constructed as follows: we give ourselves a family  $\{\zeta_e\}_{e \in \mathbb{B}}$  of independent standard Gaussian random variables, and define  $a(e) := \mathbf{a}(\zeta_e)$ , where  $\mathbf{a} : \mathbb{R} \rightarrow \mathbb{R}$  is a twice differentiable function with bounded first and second derivatives, and taking values in a compact subset of  $(0, \infty)$ . Without loss of generality, we may assume that the underlying probability space is  $\Omega = \mathbb{R}^{\mathbb{B}}$  equipped with the Borel product  $\sigma$ -algebra  $\mathcal{F}$ , and that the underlying probability measure  $\mathbb{P}$  is the product measure on  $(\Omega, \mathcal{F})$  whose marginal distributions are all standard Gaussian,  $N(0, 1)$ . The random variable  $\zeta_e : \Omega \rightarrow \mathbb{R}$  is then defined by the coordinate map;  $\zeta_e$  is simply the  $e$ th component of the element  $\zeta \in \Omega$ . Due to the product structure of  $\mathbb{P}$ , the  $\{\zeta_e\}_{e \in \mathbb{B}}$  are independent. In this way, any  $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$  may be regarded as a Borel-measurable function of the family  $\{\zeta_e\}_{e \in \mathbb{B}}$ . We use  $\langle \cdot \rangle$  to denote expectation with respect to  $\mathbb{P}$ .

Gloria and Otto showed in [7] that for  $d \geq 3$ , for every given  $\xi \in \mathbb{R}^d$ , there exists a random, stationary  $\phi^{(\xi)} : \mathbb{Z}^d \rightarrow \mathbb{R}$ , called the *corrector*, satisfying

$$(1.1) \quad \nabla^* A(\xi + \nabla \phi^{(\xi)}) = 0.$$

Also,  $\langle |\phi^{(\xi)}(0)|^p \rangle < \infty$  for all  $p \geq 1$ . If we interpret the random variables  $\{a(e)\}_{e \in \mathbb{B}}$  as conductances, the function  $x \cdot \xi + \phi^{(\xi)}(x)$  represents a potential with macroscopic gradient  $\xi$ , and the value  $a(e)(\xi + \nabla \phi^{(\xi)})(e)$  represents a current across edge  $e$ . The effective conductivity of the network is the matrix  $A_h$  defined by

$$\xi \cdot A_h \xi = \sum_{e=(0, e_i)} \langle (\xi + \nabla \phi^{(\xi)})(e) a(e) (\xi + \nabla \phi^{(\xi)})(e) \rangle.$$

Let  $\mathcal{G}_h$  be the Green function of the continuum differential operator  $-\nabla \cdot A_h \nabla$ . The main result of [11] is that there exists a deterministic  $d \times d$  matrix  $\mathbf{Q}^{(\xi)}$  and a constant  $C < \infty$  such that letting

$$\mathcal{K}_\xi(x) := \int_{\mathbb{R}^d} \nabla \mathcal{G}_h(y) \cdot \mathbf{Q}^{(\xi)} \nabla \mathcal{G}_h(x - y) dy,$$

we have

$$(1.2) \quad \langle |\phi^{(\xi)}(0) \phi^{(\xi)}(x)| \rangle - \mathcal{K}_\xi(x) \leq C \frac{1 \vee \log^2 |x|}{|x|^{d-1}} \quad (x \in \mathbb{Z}^d).$$

The matrix  $\mathbf{Q}^{(\xi)}$  is positive semidefinite, and  $\mathbf{Q}_{j,k}^{(\xi)}$  is defined in terms of  $\nabla \phi^{(\xi)}$ ,  $\nabla \phi^{(e_j)}$  and  $\nabla \phi^{(e_k)}$ . We refer to [11] for a more precise description of the matrix  $\mathbf{Q}^{(\xi)}$ . The function  $\mathcal{K}_\xi(x)$  has the same homogeneity as the Green function:  $\mathcal{K}_\xi(\lambda x) = \lambda^{2-d} \mathcal{K}_\xi(x)$  for all  $x \in \mathbb{R}^d$ ,  $\lambda > 0$ .

We denote by  $C_{\text{loc}}^\alpha = C_{\text{loc}}^\alpha(\mathbb{R}^d)$  the local Besov space with exponent of regularity  $\alpha$  and exponents of regularity  $(\infty, \infty)$ . For  $\alpha \in (0, 1)$ , the space  $C_{\text{loc}}^\alpha$  is the space of locally  $\alpha$ -Hölder continuous functions. For  $\alpha < 0$ , if a distribution  $\Phi$  belongs to  $C_{\text{loc}}^\alpha$ , then for every  $\psi \in C_c^\infty$  and every  $x \in \mathbb{R}^d$ , we have

$$(1.3) \quad \sup_{\lambda \in (0, 1]} \lambda^{-\alpha} (\Phi, \lambda^{-d} \psi((\cdot - x)/\lambda)) < \infty,$$

where  $(\cdot, \cdot)$  denotes the duality pairing. This is essentially a characterization of the space  $C_{\text{loc}}^\alpha$ , up to the fact that we prefer to work with a smaller, separable version of this space. We refer to [5] for a more precise discussion.

In the present article, we show that  $\phi^{(\xi)}$  converges to a Gaussian field. The following theorem is our main result.

**THEOREM 1.1** (Scaling limit of the corrector). *Recall that we assume  $d \geq 3$ , and let  $\Phi_\varepsilon^{(\xi)}$  be the (random) distribution defined by*

$$(1.4) \quad \Phi_\varepsilon^{(\xi)}(f) := \varepsilon^{\frac{d}{2}+1} \sum_{x \in \mathbb{Z}^d} f(\varepsilon x) \phi^{(\xi)}(x)$$

for  $f \in C_c^\infty$ . For every  $\alpha < -d$ , the distribution  $\Phi_\varepsilon^{(\xi)}$  converges in law to  $\overline{\Phi}^{(\xi)}$  as  $\varepsilon \rightarrow 0$  in the topology of  $C_{\text{loc}}^\alpha$ , where  $\overline{\Phi}^{(\xi)}$  is the Gaussian random field such that for every smooth, compactly supported function  $f$ ,  $\Phi^{(\xi)}(f)$  is a centred Gaussian with variance

$$(1.5) \quad \sigma^2(f) := \int f(x) \mathcal{K}_\xi(y - x) f(y) dx dy.$$

Furthermore, for any  $\beta < 1 - d/2$ , the law of  $\overline{\Phi}$  is supported on  $C_{\text{loc}}^\beta$ .

In [1] (Conjecture 5), it was conjectured that the appropriate scaling limit for the corrector is a Gaussian free field. While Theorem 1.1 shows that the limit is a Gaussian field, the covariance structure can be different from the Gaussian free field. Indeed, if  $Q^{(\xi)}$  is proportional to  $A_h$ , then an integration by parts shows that  $\mathcal{K}_\xi(x)$  is proportional to the Green function; but not so otherwise. Moreover, a computation in the regime of small ellipticity contrast shows that  $Q^{(\xi)}$  is not necessarily a multiple of  $A_h$ .

From (1.2), one infers that for every smooth and compactly supported function  $f$ , one has

$$\text{Var}[\Phi_\varepsilon^{(\xi)}(f)] \xrightarrow{\varepsilon \rightarrow 0} \sigma^2(f).$$

In order to prove Theorem 1.1, it suffices to prove that (1) the fluctuations of  $\Phi_\varepsilon^{(\xi)}(f)$  are Gaussian, and that (2) the law of  $\Phi_\varepsilon^{(\xi)}$  is tight in  $C_{\text{loc}}^\alpha$ . For the first part, we will in fact be more precise and give a rate of convergence:

PROPOSITION 1.2 (Gaussian fluctuations). *Let  $d_K$  denote the Kantorovich–Wasserstein distance*

$$d_K(X, W) = \sup\{|h(X) - h(W)| \|h'\|_\infty \leq 1\},$$

and let  $Y$  be a standard Gaussian random variable. For every  $f \in C_c^\infty$ , we have

$$(1.6) \quad d_K(\Phi_\varepsilon(f), \sigma(f)Y) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Moreover, if  $\sigma(f) > 0$  and  $\sigma_\varepsilon = \sqrt{\text{Var}(\Phi_\varepsilon(f))}$ , then

$$(1.7) \quad d_K(\sigma_\varepsilon^{-1}\Phi_\varepsilon(f), Y) \lesssim \varepsilon^{d/2} \log |\varepsilon|$$

as  $\varepsilon \rightarrow 0$ .

Here and elsewhere, the relation  $g_1 \lesssim g_2$  is used to mean that  $g_1 \leq Cg_2$  for some constant  $C > 0$  that depends only on  $d \geq 3$  and on the coefficient function  $\mathbf{a}$  [specifically, on  $(\max \mathbf{a})/(\min \mathbf{a})$  and on  $\|\mathbf{a}'\|_\infty$  and  $\|\mathbf{a}''\|_\infty$ ]. Proposition 1.2 is proved in Section 2. We then prove the following tightness result in Section 3.

PROPOSITION 1.3 (Tightness). *For every  $\alpha < -d$ , the sequence of (random) distributions  $(\Phi_\varepsilon^{(\xi)})_{\varepsilon \in (0, 1]}$  is tight in  $C_{\text{loc}}^\alpha$ .*

The last statement in Theorem 1.1 follows from the first estimate in Proposition 3.1 below and from Theorem 1.1 of [5] applied to the family of distributions which has the single element  $\overline{\Phi}$ , since

$$\lambda^{\frac{d}{2}-1} \langle |\overline{\Phi}(f_\lambda)|^p \rangle^{1/p} = \lambda^{\frac{d}{2}-1} \lim_{\varepsilon \rightarrow 0} \langle |\Phi_\varepsilon(f_\lambda)|^p \rangle^{1/p} \leq C(p, f) < \infty$$

holds for all  $\lambda \in (0, 1]$  and  $p \geq 1$ .

REMARK 1.4. Theorem 1.1 can be reformulated as the joint convergence in law of  $(\Phi_\varepsilon^{(\mathbf{e}_1)}, \dots, \Phi_\varepsilon^{(\mathbf{e}_d)})$  to a Gaussian vector field. Indeed, tightness for the product topology follows from the tightness of each of the coordinates. The limit law is then uniquely identified since Theorem 1.1 gives a characterisation of the limit law of every linear combination of  $(\Phi_\varepsilon^{(\mathbf{e}_1)}, \dots, \Phi_\varepsilon^{(\mathbf{e}_d)})$ . The covariance structure of the limiting field can be inferred by a polarization (with respect to  $\xi$ ) of the left-hand side of (1.2).

**2. Proof of Proposition 1.2.** From now on, we drop the dependence on  $\xi$  in the notation for simplicity, writing  $\phi$  and  $\Phi_\varepsilon$  instead of  $\phi^{(\xi)}$  and  $\Phi_\varepsilon^{(\xi)}$ , respectively.

If  $\sigma(f) = 0$ , then the distribution of  $\sigma(f)W$  is  $\delta_0$ . In this case, (1.6) follows immediately from Chebyshev’s inequality and the fact that  $\langle \Phi_\varepsilon(f) \rangle = 0$ : for all

Lipschitz functions  $h$  with  $\|h'\| \leq 1$ ,

$$\begin{aligned}
 (2.1) \quad & \left| \langle h(\Phi_\varepsilon(f)) \rangle - \langle h(\sigma(f)W) \rangle \right| = \left| \langle h(\Phi_\varepsilon(f)) \rangle - h(0) \right| \\
 & \leq \langle |h(\Phi_\varepsilon(f)) - h(0)| \rangle \\
 & \leq \|h'\| \langle |\Phi_\varepsilon(f)| \rangle \leq \sqrt{\text{Var}(\Phi_\varepsilon(f))}.
 \end{aligned}$$

Hence,

$$d_K(\Phi_\varepsilon(f), \sigma(f)W) \leq \sqrt{\text{Var}(\Phi_\varepsilon(f))} \rightarrow 0$$

holds in this case.

So let us suppose that  $\sigma(f) > 0$ . We wish to prove (1.6) and (1.7). Our proof will be based on the following proposition, which is a version of Theorem 2.2 in [4] and Theorem 3.1 (and Remark 3.6) of [12], stated in a form that is convenient for our purpose.

For a random variable  $F \in L^2(\Omega)$ , we say that  $U = \partial_e F \in L^2(\Omega)$  is the weak derivative with respect to  $\zeta_e$  [recall that  $\zeta \sim N(0, 1)$ ] if the following holds: for any finite subset  $\Lambda \subset \mathbb{B}$  and any smooth, compactly supported function  $\eta : \mathbb{R}^{|\Lambda|} \rightarrow \mathbb{R}$ , we have

$$(2.2) \quad \langle U\eta(\zeta) \rangle = \langle F\zeta_e\eta(\zeta) \rangle - \left\langle F \frac{\partial \eta}{\partial \zeta_e}(\zeta) \right\rangle,$$

where  $\eta(\zeta)$  depends only on  $\{\zeta_{e'}\}_{e' \in \Lambda}$ . We will write  $\partial_e^* = \zeta_e - \partial_e$  for the adjoint of the derivative operator:  $\langle (\partial_e F)\eta \rangle = \langle F\partial_e^* \eta \rangle$ .

**PROPOSITION 2.1.** *Let  $F \in L^2(\Omega)$  be such that  $\langle F \rangle = 0$ ,  $\langle F^2 \rangle = 1$ . Assume also that  $F$  has weak derivatives satisfying  $\sum_e \langle |\partial_e F|^4 \rangle^{1/2} < \infty$  and  $\langle |\partial_{e'} \partial_e F|^4 \rangle < \infty$  for all  $e, e' \in \mathbb{B}$ . Let  $Y \sim N(0, 1)$ . Then*

$$(2.3) \quad \sup_{\|h'\|_\infty \leq 1} \langle h(F) - h(Y) \rangle \leq \sqrt{\frac{5}{\pi}} \sqrt{\sum_{e'} \left( \sum_e \langle |\partial_e F|^4 \rangle^{1/4} \langle |\partial_{e'} \partial_e F|^4 \rangle^{1/4} \right)^2}.$$

A proof of Proposition 2.1 is given later in Section 4. We note that except for the numerical constant, the same result holds if the weak derivatives are replaced by the so-called Glauber derivatives, hence providing a version of the result that applies to functions of independent random variables that are not necessarily Gaussian; see Remark 2.3 of [11].

We will apply this proposition to  $F = \sigma_\varepsilon^{-1} \Phi_\varepsilon(f)$ , where  $\sigma_\varepsilon = \sqrt{\text{Var}(\Phi_\varepsilon(f))} \rightarrow \sigma(f) > 0$ . Without loss of generality, we may assume that  $f(x) = 0$  when  $|x| > 1$ . For any edges  $e, e'$ ,

$$\partial_e \Phi = \varepsilon^{\frac{d}{2}+1} \sum_{x \in \mathbb{Z}^d} f(\varepsilon x) \partial_e \phi(x),$$

and

$$\partial_{e'} \partial_e \Phi = \varepsilon^{\frac{d}{2}+1} \sum_{x \in \mathbb{Z}^d} f(\varepsilon x) \partial_{e'} \partial_e \phi(x).$$

We recall from [7] that the stationary process  $\phi(x)$  is defined by the limit

$$(2.4) \quad \phi(x) = \lim_{\mu \rightarrow 0} \phi_\mu(x), \quad \text{in } L^p(\Omega)$$

for every  $p \geq 1$ , where the stationary process  $\phi_\mu(x)$  satisfies the regularized corrector problem

$$-\mu \phi_\mu + \nabla^* A(\xi + \nabla \phi_\mu) = 0$$

with  $\mu > 0$ . The random variable  $\phi_\mu(x)$  with  $\mu > 0$  is differentiable with respect to each variable  $\{\zeta_e\}$  (Lemma 2.4 of [7]), and  $\partial_e \phi_\mu(x)$  satisfies

$$-\mu \partial_e \phi_\mu + \nabla^* A \nabla \partial_e \phi_\mu = -\nabla^* \cdot (\partial_e a)(\xi + \nabla \phi_\mu), \quad x \in \mathbb{Z}^d$$

and is given by the formula

$$(2.5) \quad \partial_e \phi_\mu(x) = -(\partial_e a(e)) \nabla_e G_\mu(x, e) (\nabla \phi_\mu(e) + \xi_e), \quad x \in \mathbb{Z}^d,$$

where  $G_\mu$  is the Green function associated with operator  $-\mu + \nabla^* A \nabla$ , and  $\nabla_e G(x, e)$  denotes  $G(x, z + e_i) - G(x, z)$  where  $e = [z, z + e_i]$ . Notice that in (2.5),  $\partial_e a(e) = \mathbf{a}'(\zeta_e)$  and  $\nabla \phi_\mu(e) = \phi_\mu(z + e_i) - \phi_\mu(z)$  if  $e = [z, z + e_i]$ .

As mentioned in Remark 4.8 of [11], the weak derivatives  $\partial_e \phi$  and  $\partial_e \partial_{e'} \phi$  may be obtained in the limit as  $\mu \rightarrow 0$ , in the usual way. Specifically, as  $\mu \rightarrow 0$  we have  $\phi_\mu(x) \rightarrow \phi(x)$  and  $G_\mu(x, e) \rightarrow G_0(x, e) = G(x, e)$  in  $L^p(\Omega)$  for any  $p$ , where  $G$  denotes the Green function associated with the operator  $\nabla^* A \nabla$ . In view of (2.5), this implies that the limit

$$(2.6) \quad \lim_{\mu \rightarrow 0} \partial_e \phi_\mu(x) = -(\partial_e a(e)) \nabla_e G(x, e) (\nabla \phi(e) + \xi_e)$$

holds in  $L^p(\Omega)$  for any  $p \geq 1$ . For any test function  $\eta$  [in the sense of (2.2)], we have

$$(2.7) \quad \langle \phi, \partial_e^* \eta(\zeta) \rangle = \lim_{\mu \rightarrow 0} \langle \phi_\mu, \partial_e^* \eta(\zeta) \rangle = \lim_{\mu \rightarrow 0} \langle \partial_e \phi_\mu, \eta(\zeta) \rangle.$$

Combining (2.6) and (2.7), we obtain the Green function identity for  $\partial_e \phi$ :

$$\partial_e \phi(x) = -(\partial_e a(e)) \nabla_e G(x, e) (\nabla \phi(e) + \xi_e).$$

Similarly, if  $e' = [y, y + e_j]$ , then  $\partial_{e'} \partial_e \phi_\mu$  satisfies

$$(2.8) \quad \begin{aligned} & \mu \partial_{e'} \partial_e \phi_\mu - \nabla^* A \nabla (\partial_{e'} \partial_e \phi_\mu) \\ &= \nabla^* \cdot (\partial_{e'} \partial_e a)(\nabla \phi_\mu + \xi) + \nabla^* \cdot (\partial_e a(e)) (\nabla \partial_{e'} \phi_\mu) \\ & \quad + \nabla^* \cdot (\partial_{e'} a(e)) (\nabla \partial_e \phi_\mu). \end{aligned}$$

Passing to the limit  $\mu \rightarrow 0$  in the same way as above, we obtain the analogous representation for  $\partial_{e'}\partial_e\phi$ . For  $e \neq e'$ ,  $\partial_{e'}\partial_e a = 0$ , and we have

$$\begin{aligned} \partial_{e'}\partial_e\phi(x) &= -(\partial_e a(e))\nabla_e G(x, e)(\nabla_e\partial_{e'}\phi(z)) - (\partial_{e'} a(e'))\nabla_{e'} G(x, e')(\nabla_{e'}\partial_e\phi(y)) \\ (2.9) \quad &= (\partial_e a(e))(\partial_{e'} a(e'))\nabla_e G(x, e)(\nabla_e\nabla_{e'} G(e, e')(\nabla\phi(e') + \xi_{e'})) \\ &\quad + (\partial_e a(e))(\partial_{e'} a(e'))\nabla_{e'} G(x, e')(\nabla_{e'}\nabla_e G(e', e)(\nabla\phi(e) + \xi_e)), \end{aligned}$$

while for  $e' = e$ , we have

$$\begin{aligned} \partial_e^2\phi(x) &= 2(\partial_e a(e))^2\nabla_e G(x, e)(\nabla_e\nabla_e G(e, e)(\nabla_e\phi(e) + \xi_e)) \\ (2.10) \quad &\quad - (\partial_e^2 a(e))\nabla_e G(x, e)(\nabla_e\phi(e) + \xi_e). \end{aligned}$$

Therefore, applying the generalized Hölder inequality and the fact that  $|\partial_e a| = |\mathbf{a}'(\zeta_e)| \lesssim 1$  and  $|\partial_e^2 a| = |\mathbf{a}''(\zeta_e)| \lesssim 1$

$$\begin{aligned} &\langle |\partial_e\partial_{e'}\phi(x)|^p \rangle^{1/p} \\ (2.11) \quad &\lesssim \langle |\nabla_e G(x, e)|^{3p} \rangle^{\frac{1}{3p}} \langle |\nabla_e\nabla_{e'} G(e, e')|^{3p} \rangle^{\frac{1}{3p}} \langle |\nabla\phi(e') + \xi|^{3p} \rangle^{\frac{1}{3p}} \\ &\quad + \langle |\nabla_e G(x, e')|^{3p} \rangle^{\frac{1}{3p}} \langle |\nabla_{e'}\nabla_e G(e', e)|^{3p} \rangle^{\frac{1}{3p}} \langle |\nabla\phi(e) + \xi|^{3p} \rangle^{\frac{1}{3p}}. \end{aligned}$$

Also,

$$(2.12) \quad \langle |\partial_e\phi(x)|^p \rangle^{1/p} \lesssim \langle |\nabla_e G(x, e)|^{2p} \rangle^{\frac{1}{2p}} \langle |\nabla\phi(x) + \xi|^{2p} \rangle^{\frac{1}{2p}}.$$

Gloria and Otto [7] proved that for all  $p \geq 1$ ,

$$\langle |\nabla\phi(e) + \xi|^p \rangle^{1/p} \lesssim 1.$$

Marahrens and Otto [10] proved that the Green function satisfies

$$\langle |\nabla_e G(0, e)|^p \rangle^{1/p} \lesssim \frac{1}{(1 + |e|)^{d-1}},$$

and

$$\langle |\nabla_{e'}\nabla_e G(e', e)|^p \rangle^{1/p} \lesssim \frac{1}{(1 + |e - e'|)^d}$$

for all  $p \geq 2$ . By combining these crucial estimates with (2.11) and (2.12), we obtain

$$\begin{aligned} &\langle |\partial_e\partial_{e'}\phi(x)|^p \rangle^{1/p} \\ (2.13) \quad &\lesssim \left( \frac{1}{(1 + |x - e|)^{d-1}} + \frac{1}{(1 + |x - e'|)^{d-1}} \right) \cdot \frac{1}{(1 + |e - e'|)^d}, \end{aligned}$$

and

$$(2.14) \quad \langle |\partial_e\phi(x)|^p \rangle^{1/p} \lesssim \frac{1}{(1 + |x - e|)^{d-1}}.$$

Applying (2.14) and (2.13) to  $\partial_e \Phi$  and  $\partial_{e'} \partial_e \Phi$ , and recalling that  $f(x) = 0$  when  $|x| > 1$ , we have

$$\begin{aligned}
 \varepsilon^{-\frac{d}{2}-1} \langle |\partial_e \Phi|^p \rangle^{1/p} &\lesssim \sum_{|x| \leq \varepsilon^{-1}} \langle |\partial_e \phi(x)|^p \rangle^{1/p} \\
 (2.15) \qquad \qquad \qquad &\lesssim \sum_{|x| \leq \varepsilon^{-1}} \frac{1}{(1 + |x - e|)^{d-1}},
 \end{aligned}$$

and

$$\begin{aligned}
 \varepsilon^{-\frac{d}{2}-1} \langle |\partial_e \partial_{e'} \Phi|^p \rangle^{1/p} \\
 (2.16) \qquad \lesssim \sum_{|x| \leq \varepsilon^{-1}} \langle |\partial_e \partial_{e'} \phi(x)|^p \rangle^{1/p} \\
 \lesssim \frac{1}{(1 + |e - e'|)^d} \sum_{|x| \leq \varepsilon^{-1}} \left( \frac{1}{(1 + |x - e|)^{d-1}} + \frac{1}{(1 + |x - e'|)^{d-1}} \right).
 \end{aligned}$$

Now, we are prepared to apply Proposition 2.1. The fact that  $\sum_e \langle |\partial_e \Phi|^4 \rangle^{1/2} < \infty$  and  $\langle |\partial_{e'} \partial_e \Phi|^4 \rangle < \infty$  follows from (2.15), (2.16) and the following lemma, which is proved later.

LEMMA 2.2. *For all  $e \in \mathbb{Z}^d$  and  $\varepsilon \in (0, 1]$ ,*

$$(2.17) \qquad \sum_{|x| \leq \varepsilon^{-1}} \frac{1}{(1 + |x - e|)^{d-1}} \lesssim \frac{\varepsilon^{-1}}{(1 + |\varepsilon e|)^{d-1}}.$$

By the estimates above, we have

$$\begin{aligned}
 \varepsilon^{-d-2} \sum_e \langle |\partial_e \partial_{e'} \Phi|^4 \rangle^{1/4} \langle |\partial_e \Phi|^4 \rangle^{1/4} \\
 \leq \sum_e \frac{1}{(1 + |e - e'|)^d} \left( \sum_{|x| \leq \varepsilon^{-1}} \left( \frac{1}{(1 + |x - e|)^{d-1}} + \frac{1}{(1 + |x - e'|)^{d-1}} \right) \right) \\
 \times \left( \sum_{|x| \leq \varepsilon^{-1}} \frac{1}{(1 + |x - e|)^{d-1}} \right).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \varepsilon^{-d-2} \sum_e \langle |\partial_e \partial_{e'} \Phi|^4 \rangle^{1/4} \langle |\partial_e \Phi|^4 \rangle^{1/4} \\
 (2.18) \qquad \leq \varepsilon^{-2} \sum_e \frac{1}{(1 + |e - e'|)^d} \left( \frac{1}{(1 + |\varepsilon e|)^{d-1}} + \frac{1}{(1 + |\varepsilon e'|)^{d-1}} \right) \\
 \times \frac{1}{(1 + |\varepsilon e|)^{d-1}}.
 \end{aligned}$$



To bound this expression, we use the following estimate, proved later.

LEMMA 2.3. *Let  $p > 0$ . For all  $e' \in \mathbb{Z}^d$  and  $\varepsilon \in (0, 1/2]$ ,*

$$(2.19) \quad \sum_e \frac{1}{(1 + |e - e'|)^d} \frac{1}{(1 + |\varepsilon e|)^p} \lesssim \frac{|\log \varepsilon|}{(1 + |\varepsilon e'|)^d} + \frac{|\log \varepsilon|}{(1 + |\varepsilon e'|)^p}.$$

Using Lemma 2.3, we conclude that

$$\begin{aligned} \sum_e \frac{1}{(1 + |e - e'|)^d} \frac{1}{(1 + |\varepsilon e|)^{2(d-1)}} &\lesssim \frac{|\log \varepsilon|}{(1 + |\varepsilon e'|)^d} + \frac{|\log \varepsilon|}{(1 + |\varepsilon e'|)^{2(d-1)}} \\ &\lesssim \frac{|\log \varepsilon|}{(1 + |\varepsilon e'|)^d} \end{aligned}$$

and

$$\begin{aligned} \sum_e \frac{1}{(1 + |e - e'|)^d} \frac{1}{(1 + |\varepsilon e|)^{d-1}} &\lesssim \frac{|\log \varepsilon|}{(1 + |\varepsilon e'|)^d} + \frac{|\log \varepsilon|}{(1 + |\varepsilon e'|)^{d-1}} \\ &\lesssim \frac{|\log \varepsilon|}{(1 + |\varepsilon e'|)^{d-1}}. \end{aligned}$$

These bounds combined with (2.18) imply that

$$\begin{aligned} &\varepsilon^{-2d-4} \sum_{e'} \left( \sum_e \langle |\partial_e \partial_{e'} \Phi|^4 \rangle^{1/4} \langle |\partial_e \Phi|^4 \rangle^{1/4} \right)^2 \\ (2.20) \quad &\lesssim \varepsilon^{-4} \sum_{e'} \left( \frac{|\log \varepsilon|}{(1 + |\varepsilon e'|)^d} \right)^2 + \varepsilon^{-4} \sum_{e'} \frac{1}{(1 + |\varepsilon e'|)^{2(d-1)}} \left( \frac{|\log \varepsilon|}{(1 + |\varepsilon e'|)^{d-1}} \right)^2 \\ &\lesssim \varepsilon^{-4-d} (\log \varepsilon)^2. \end{aligned}$$

In view of this estimate, we see that the random variable  $F_\varepsilon = \sigma_\varepsilon^{-1} \Phi_\varepsilon$  satisfies

$$\sum_{e'} \left( \sum_e \langle |\partial_e \partial_{e'} F_\varepsilon|^4 \rangle^{1/4} \langle |\partial_e F_\varepsilon|^4 \rangle^{1/4} \right)^2 \lesssim \sigma_\varepsilon^{-4} \varepsilon^d (\log \varepsilon)^2.$$

By Proposition 2.1, we conclude that

$$\sup_{\|h'\|_\infty \leq 1} \langle h(F_\varepsilon) - h(Y) \rangle \lesssim \sigma_\varepsilon^{-2} \varepsilon^{d/2} \log \varepsilon \lesssim \varepsilon^{d/2} \log \varepsilon$$

as  $\varepsilon \rightarrow 0$ , since  $\lim_{\varepsilon \rightarrow 0} \sigma_\varepsilon = \sigma > 0$  in this case. This proves (1.7). Finally,

$$\begin{aligned} \langle h(\Phi_\varepsilon) - h(\sigma Y) \rangle &= \langle h(\Phi_\varepsilon) - h(\sigma_\varepsilon Y) \rangle + \langle h(\sigma_\varepsilon Y) - h(\sigma Y) \rangle \\ (2.21) \quad &= \sigma_\varepsilon \langle \hat{h}_\varepsilon(\sigma_\varepsilon^{-1} \Phi_\varepsilon) - \hat{h}_\varepsilon(Y) \rangle + \langle h(\sigma_\varepsilon Y) - h(\sigma Y) \rangle \\ &= \sigma_\varepsilon \langle \hat{h}_\varepsilon(F_\varepsilon) - \hat{h}_\varepsilon(Y) \rangle + \langle h(\sigma_\varepsilon Y) - h(\sigma Y) \rangle, \end{aligned}$$

where  $\hat{h}_\varepsilon(\cdot) = \sigma_\varepsilon^{-1}h(\sigma_\varepsilon\cdot)$  also satisfies  $\|\hat{h}'_\varepsilon\|_\infty \leq 1$ . Hence,

$$\sup_{\|h'\|_\infty \leq 1} (h(\Phi_\varepsilon) - h(\sigma Y)) \lesssim \varepsilon^{d/2} \log \varepsilon + |\sigma_\varepsilon - \sigma|,$$

which implies that (1.6) holds as well.

**PROOF OF LEMMA 2.2.** If  $|e| > 2\varepsilon^{-1}$  and  $|x| \leq \varepsilon^{-1}$ , then  $|x - e| > |e|/2 \geq 1$ . So, clearly

$$\sum_{|x| \leq \varepsilon^{-1}} \frac{1}{(1 + |x - e|)^{d-1}} \lesssim \frac{\varepsilon^{-d}}{|e|^{d-1}} = \frac{\varepsilon^{-1}}{|\varepsilon e|^{d-1}}$$

in this case. For  $|e| < 2\varepsilon^{-1}$ ,

$$\sum_{|x| \leq \varepsilon^{-1}} \frac{1}{(1 + |x - e|)^{d-1}} \leq \int_{|y| < 3\varepsilon^{-1}} \frac{1}{(1 + |y|)^{d-1}} dy \lesssim \varepsilon^{-1}.$$

So, (2.17) holds in this case as well.  $\square$

**PROOF OF LEMMA 2.3.** First, consider the sum over edges satisfying  $|e| < \varepsilon^{-1}$ :

$$\begin{aligned} \sum_{|e| < \varepsilon^{-1}} \frac{1}{(1 + |e - e'|)^d} \frac{1}{(1 + |\varepsilon e|)^p} &\lesssim \sum_{|e| < \varepsilon^{-1}} \frac{1}{(1 + |e - e'|)^d} \\ (2.22) \qquad \qquad \qquad &\lesssim \begin{cases} |\log \varepsilon|, & \forall e', \\ \varepsilon^{-d} & \forall |e'| > \varepsilon^{-1} \\ |e'|^d, & \end{cases} \\ &\lesssim \frac{|\log \varepsilon|}{(1 + |\varepsilon e'|)^d}. \end{aligned}$$

Next, consider the sum over edges satisfying  $|e| > \varepsilon^{-1}$ :

$$\begin{aligned} \sum_{|e| \geq \varepsilon^{-1}} \frac{1}{(1 + |e - e'|)^d} \frac{1}{(1 + |\varepsilon e|)^p} \\ (2.23) \qquad \qquad \qquad &\lesssim \sum_{|e| \geq \varepsilon^{-1}} \frac{1}{(1 + |e - e'|)^d} \frac{1}{|\varepsilon e|^p} \\ &\lesssim \varepsilon^{-p} \int_{|x| > \varepsilon^{-1}} \frac{1}{(1 + |x - e'|)^d} \frac{1}{|x|^p} dx. \end{aligned}$$

Let  $x = \varepsilon^{-1}z$ ,  $e' = \varepsilon^{-1}w$ . Then

$$(2.24) \quad \sum_{|e| \geq \varepsilon^{-1}} \frac{1}{(1 + |e - e'|)^d} \frac{1}{(1 + |\varepsilon e|)^p} \lesssim \varepsilon^{-d} \int_{|z| > 1} \frac{1}{(1 + \varepsilon^{-d}|z - w|^d)} \frac{1}{|z|^p} dz.$$

Restricting the integral in (2.24) to the set  $|z - w| \leq \varepsilon$ , we have

$$\varepsilon^{-d} \int_{\substack{|z|>1 \\ |z-w|\leq\varepsilon}} \frac{1}{(1 + \varepsilon^{-d}|z - w|^d)} \frac{1}{|z|^p} dz \lesssim \varepsilon^{-d} \int_{\substack{|z|>1 \\ |z-w|\leq\varepsilon}} \frac{1}{|z|^p} dz \lesssim \min(1, |w|^{-p}).$$

Restricting the integral in (2.24) to the set  $|z - w| \geq \varepsilon$ , we have

$$\varepsilon^{-d} \int_{\substack{|z|>1 \\ |z-w|\geq\varepsilon}} \frac{1}{(1 + \varepsilon^{-d}|z - w|^d)} \frac{1}{|z|^p} dz \lesssim \int_{\substack{|z|>1 \\ |z-w|\geq\varepsilon}} \frac{1}{|z - w|^d} \frac{1}{|z|^p} dz.$$

If  $|z| > 3|w|$ , then  $|w - z| > |z|/2$ , so

$$\int_{\substack{|z|>1 \\ |z-w|\geq\varepsilon \\ |z|>3|w|}} \frac{1}{|z - w|^d} \frac{1}{|z|^p} dz \leq \int_{\substack{|z|>1 \\ |z|>3|w|}} \frac{1}{|z|^{p+d}} dz \lesssim \min(1, |w|^{-p}).$$

On the other hand, restricting to  $|z| < 3|w|$ ,

$$\begin{aligned} & \int_{\substack{|z|>1 \\ |z-w|\geq\varepsilon \\ |z|<3|w|}} \frac{1}{|z - w|^d} \frac{1}{|z|^p} dz \\ (2.25) \quad &= \int_{\substack{|z|>1 \\ \varepsilon \leq |z-w| \leq 1 \\ |z|<3|w|}} \frac{1}{|z - w|^d} \frac{1}{|z|^p} dz + \int_{\substack{|z|>1 \\ |z-w|\geq 1 \\ |z|<3|w|}} \frac{1}{|z - w|^d} \frac{1}{|z|^p} dz \\ &\lesssim |\log \varepsilon| \min(1, |w|^{-p}) + \log(1 + |w|) \min(1, |w|^{-p}). \end{aligned}$$

The bound (2.19) now follows by combining these estimates with  $w = \varepsilon e'$ .  $\square$

**3. Proof of Proposition 1.3.** We now proceed to prove Proposition 1.3. The main result of [5] states informally that in order to verify the tightness of a family of distributions  $(f_n)$  in  $C_{loc}^\alpha$ , it suffices to check that the estimate (1.3) holds uniformly over  $f_n$ . More precisely, and by the invariance of the corrector under translations, it suffices to show the following proposition.

**PROPOSITION 3.1.** *For  $f \in C_c^0(\mathbb{R}^d; \mathbb{R})$ , let  $f_\lambda(x) = \lambda^{-d} f(x/\lambda)$ . For all  $p \geq 1$ , there exists a constant  $C = C(p, f)$  such that*

$$\langle |\Phi_\varepsilon(f_\lambda)|^p \rangle^{1/p} \leq C \cdot \begin{cases} \lambda^{1-\frac{d}{2}}, & 0 < \varepsilon \leq \lambda \leq 1, \\ \varepsilon^{\frac{d}{2}+1} \lambda^{-d}, & 0 < \lambda \leq \varepsilon \leq 1. \end{cases}$$

**PROOF.** In the regime  $0 < \lambda \leq \varepsilon \leq 1$  the stated bound is immediate from the fact that the corrector  $\phi$  is in  $L^p(\Omega)$ , since in this case

$$\Phi_\varepsilon(f_\lambda) = \varepsilon^{\frac{d}{2}+1} \sum_{x \in \mathbb{Z}^d} f_\lambda(\varepsilon x) \phi(x)$$

has only a finite number of nonzero terms (independent of  $\lambda$  and  $\varepsilon$ ) and  $\|f\lambda\|_{L^\infty(\mathbb{R}^d)} \leq C\lambda^{-d}$ . In the regime  $0 < \varepsilon \leq \lambda \leq 1$ , observe that for any  $\varepsilon, \lambda \in (0, 1]$ ,

$$\Phi_\varepsilon(f_\lambda) = \lambda^{1-\frac{d}{2}} \Phi_r(f)$$

with  $r = \varepsilon/\lambda$ . Therefore, to prove Proposition 3.1, it suffices to show that

$$(3.1) \quad \langle |\Phi_\varepsilon(f)|^p \rangle^{1/p} \leq C$$

holds for all  $\varepsilon \in (0, 1]$ .

Since  $\phi$  is stationary and  $\langle \phi(0) \rangle = 0$ , we know that  $\langle \Phi_\varepsilon(f) \rangle = 0$ . In particular,

$$(3.2) \quad \langle \Phi_\varepsilon(f)^p \rangle \lesssim 1 \quad \text{uniformly over } \varepsilon > 0$$

holds for  $p = 1$ . Arguing inductively, let us suppose that (3.2) holds for some positive integer  $p = n$ . We claim that (3.2) must also hold for  $p = 2n$ . To prove this claim, we use the identity

$$\langle \Phi_\varepsilon(f)^{2n} \rangle = \langle \Phi_\varepsilon(f)^n \rangle^2 + \text{Var}(\Phi_\varepsilon(f)^n),$$

which, by the induction hypothesis, gives us

$$(3.3) \quad \langle \Phi_\varepsilon(f)^{2n} \rangle \lesssim 1 + \text{Var}(\Phi_\varepsilon(f)^n).$$

We will show that  $\text{Var}(\Phi_\varepsilon(f)^n) \lesssim \langle \Phi_\varepsilon(f)^{2n} \rangle^{1-1/n}$ , hence

$$(3.4) \quad \langle \Phi_\varepsilon(f)^{2n} \rangle \lesssim 1 + \langle \Phi_\varepsilon(f)^{2n} \rangle^{1-1/n}.$$

This bound and Young’s inequality establish the claim that (3.2) also holds for  $p = 2n$ .

To prove that  $\text{Var}(\Phi_\varepsilon(f)^n) \lesssim \langle \Phi_\varepsilon(f)^{2n} \rangle^{1-1/n}$ , we first apply the Gaussian spectral gap inequality [which in this infinite dimensional setting follows from the Helffer–Sjöstrand representation stated in Section 4, i.e., from (4.3) with  $F = G$ ]:

$$(3.5) \quad \text{Var}(\Phi_\varepsilon(f)^n) \leq \sum_e \langle |\partial_e(\Phi_\varepsilon(f)^n)|^2 \rangle.$$

Then we apply Hölder’s inequality to obtain

$$(3.6) \quad \begin{aligned} \text{Var}(\Phi_\varepsilon(f)^n) &\leq \sum_e \langle |n\Phi_\varepsilon(f)^{n-1} \partial_e \Phi_\varepsilon(f)|^2 \rangle \\ &\leq \sum_e \langle \Phi_\varepsilon(f)^{2n} \rangle^{1-1/n} \langle |\partial_e \Phi_\varepsilon(f)|^{2n} \rangle^{1/n} \\ &= \langle \Phi_\varepsilon(f)^{2n} \rangle^{1-1/n} \sum_e \langle |\partial_e \Phi_\varepsilon(f)|^{2n} \rangle^{1/n}. \end{aligned}$$

Now we show that the last sum is bounded by a constant. Without loss of generality, assume that  $f(x) = 0$  for  $|x| > 1$ . Recall that  $\langle |\partial_e \phi(x)|^p \rangle^{1/p} \lesssim (1 + |x - e|)^{1-d}$  [see (2.14)]. So,

$$\begin{aligned} \langle (\partial_e \Phi(f))^{2n} \rangle^{1/(2n)} &= \varepsilon^{\frac{d}{2}+1} \left\langle \left( \sum_x f(\varepsilon x) \partial_e \phi(x) \right)^{2n} \right\rangle^{1/(2n)} \\ &\lesssim \varepsilon^{\frac{d}{2}+1} \sum_{|x| \leq \varepsilon^{-1}} \langle (\partial_e \phi(x))^{2n} \rangle^{1/(2n)} \\ &\lesssim \varepsilon^{\frac{d}{2}+1} \sum_{|x| \leq \varepsilon^{-1}} \frac{1}{(1 + |x - e|)^{d-1}}. \end{aligned}$$

The last sum is controlled by Lemma 2.2, which leads to

$$\langle (\partial_e \Phi(f))^{2n} \rangle^{1/(2n)} \leq \varepsilon^{\frac{d}{2}+1} \frac{\varepsilon^{-1}}{(1 + |\varepsilon e|)^{d-1}}.$$

Therefore,

$$(3.7) \quad \sum_e \langle |\partial_e \Phi_\varepsilon(f)|^{2n} \rangle^{1/n} \lesssim \varepsilon^{d+2} \sum_e \frac{\varepsilon^{-2}}{(1 + |\varepsilon e|)^{2d-2}} \lesssim 1$$

since  $d \geq 3$ . In view (3.6), we have now established that  $\text{Var}(\Phi_\varepsilon(f)^n) \lesssim \langle \Phi_\varepsilon(f)^{2n} \rangle^{1-1/n}$ .

By induction on  $n$ , we have proved that

$$\langle \Phi_\varepsilon(f)^p \rangle^2 \lesssim 1$$

holds for all  $p \in \{2^n | n = 0, 1, 2, 3, \dots\}$ . Now (3.1) follows by Jensen’s inequality. □

**4. Stein’s method.** In this section, we prove Proposition 2.1, which is a version of Theorem 2.2 in [4] and Theorem 3.1 (and Remark 3.6) in [12], stated in a form that is convenient for our purpose. The basis of the estimate is the following lemma.

LEMMA 4.1 (See [3], Lemma 4.2). *Suppose  $h : \mathbb{R} \rightarrow \mathbb{R}$  is absolutely continuous with bounded derivative, and  $Y \sim N(0, 1)$ . There exists a solution to*

$$(4.1) \quad \varphi'(x) - x\varphi(x) = h(x) - \langle h(Y) \rangle, \quad x \in \mathbb{R}$$

which satisfies  $\|\varphi'\|_\infty \leq \sqrt{\frac{2}{\pi}} \|h'\|_\infty$  and  $\|\varphi''\|_\infty \leq 2\|h'\|_\infty$ .

PROOF OF PROPOSITION 2.1. Let  $F \in L^2(\Omega)$  be such that  $\langle F \rangle = 0, \langle F^2 \rangle = 1$ . By (4.1), we have

$$(4.2) \quad \begin{aligned} \langle h(F) - h(Y) \rangle &= \langle \varphi'(F) - \varphi(F)F \rangle \\ &= \text{Cov}(\langle \varphi'(F) \rangle F - \varphi(F), F). \end{aligned}$$

Since  $\partial_e F \in L^2(\Omega)$  for all  $e \in \mathbb{B}$ , we can estimate this covariance by applying the Helffer–Sjöstrand correlation representation (see [11], Proposition 3.1) to  $\text{Cov}(G, F)$  with  $G = \langle \varphi'(F) \rangle F - \varphi(F)$ . Let  $\partial_e^* = -\partial_e + \zeta_e$  be the adjoint of the derivative operator  $\partial_e$ . Let  $\mathcal{L} = \partial^* \partial$ , where  $\partial F = (\partial_e F)_{e \in \mathbb{B}}$  and for  $K = (K_e)_{e \in \mathbb{B}}$ ,  $\partial^* K = \sum_e \partial_e^* K_e$ . The correlation representation is

$$(4.3) \quad \text{Cov}(G, F) = \sum_{e \in \mathbb{B}} \langle \partial_e G (\mathcal{L} + 1)^{-1} \partial_e F \rangle.$$

From this and (4.2), we obtain

$$(4.4) \quad \begin{aligned} \langle h(F) - h(Y) \rangle &= \sum_e \langle \partial_e G (\mathcal{L} + 1)^{-1} \partial_e F \rangle \\ &= \sum_e \langle (\langle \varphi'(F) \rangle - \varphi'(F)) \partial_e F (\mathcal{L} + 1)^{-1} \partial_e F \rangle. \end{aligned}$$

Observe that  $\langle \varphi'(F) \rangle - \varphi'(F) \in L^\infty$  has zero mean. Moreover, the series  $\sum_e \partial_e F (\mathcal{L} + 1)^{-1} \partial_e F$  converges in  $L^1(\Omega)$  and has mean  $1 = \text{Cov}(F, F)$ . Hence,

$$(4.5) \quad \begin{aligned} \langle h(F) - h(Y) \rangle &= \left\langle (\langle \varphi'(F) \rangle - \varphi'(F)) \left( \sum_e \partial_e F (\mathcal{L} + 1)^{-1} \partial_e F \right) \right\rangle \\ &= \left\langle \varphi'(F) \left( 1 - \sum_e \partial_e F (\mathcal{L} + 1)^{-1} \partial_e F \right) \right\rangle \\ &\leq \|\varphi'\|_\infty \left\langle \left| 1 - \sum_e \partial_e F (\mathcal{L} + 1)^{-1} \partial_e F \right| \right\rangle \\ &\leq \sqrt{\frac{2}{\pi}} \|h'\|_\infty \left\langle \left| 1 - \sum_e \partial_e F (\mathcal{L} + 1)^{-1} \partial_e F \right| \right\rangle. \end{aligned}$$

Assuming that  $\sum_e \langle |\partial_e F|^4 \rangle^{1/2} < \infty$ , we claim that the sum

$$S = \sum_e \partial_e F (\mathcal{L} + 1)^{-1} \partial_e F$$

is in  $L^2(\Omega)$ , and we will estimate the last term in (4.5) by the variance

$$(4.6) \quad \left\langle \left| 1 - \sum_e \partial_e F (\mathcal{L} + 1)^{-1} \partial_e F \right| \right\rangle \leq \sqrt{\text{Var} \left( \sum_e \partial_e F (\mathcal{L} + 1)^{-1} \partial_e F \right)}.$$

Indeed,

$$\|\partial_e F (\mathcal{L} + 1)^{-1} \partial_e F\|_2 \leq \|\partial_e F\|_4 \|(\mathcal{L} + 1)^{-1} \partial_e F\|_4 \leq \|\partial_e F\|_4^2,$$

since  $(\mathcal{L} + 1)^{-1}$  is a contraction on  $L^p(\Omega)$  for any  $p \geq 2$  (see Proposition 3.2 of [11]). Thus,  $S \in L^2(\Omega)$  if  $\sum_e \|\partial_e F\|_4^2 < \infty$ .

Under the additional assumption  $\langle |\partial_{e'} \partial_e F|^4 \rangle < \infty$  for all  $e, e' \in \mathbb{B}$ , we also have  $\partial_{e'} S \in L^2(\Omega)$  for all  $e' \in \mathbb{B}$ , so that  $\text{Var}(S)$  can be estimated by the Gaussian spectral gap inequality. We compute

$$\partial_{e'}(\mathcal{L} + 1)^{-1} \partial_e F = (\mathcal{L} + 2)^{-1} \partial_{e'} \partial_e F,$$

so that

$$\partial_{e'}(\partial_e F(\mathcal{L} + 1)^{-1} \partial_e F) = (\partial_{e'} \partial_e F)(\mathcal{L} + 1)^{-1} \partial_e F + \partial_e F(\mathcal{L} + 2)^{-1} \partial_{e'} \partial_e F.$$

The operator  $(\mathcal{L} + 2)^{-1}$  is also a contraction, satisfying  $\|(\mathcal{L} + 2)^{-1} u\|_p \leq (1/2)\|u\|_p$  for all  $u \in L^p(\Omega)$ ,  $p \geq 2$  (this follows from minor modification to the proof of Proposition 3.2 of [11]). Therefore,

$$\begin{aligned} & \|\partial_{e'}(\partial_e F(\mathcal{L} + 1)^{-1} \partial_e F)\|_2 \\ (4.7) \quad & \leq \|(\partial_{e'} \partial_e F)(\mathcal{L} + 1)^{-1} \partial_e F\|_2 + \|\partial_e F(\mathcal{L} + 2)^{-1} \partial_{e'} \partial_e F\|_2 \\ & \leq \|\partial_{e'} \partial_e F\|_4 \|(\mathcal{L} + 1)^{-1} \partial_e F\|_4 + \|\partial_e F\|_4 \|(\mathcal{L} + 2)^{-1} \partial_{e'} \partial_e F\|_4 \\ & \leq \left(1 + \frac{1}{2}\right) \|\partial_{e'} \partial_e F\|_4 \|\partial_e F\|_4 < \infty. \end{aligned}$$

Hence,  $\partial_{e'} S \in L^2(\Omega)$  for all  $e' \in \mathbb{B}$ .

Now we apply the Gaussian spectral gap inequality:

$$\begin{aligned} (4.8) \quad \text{Var}\left(\sum_e \partial_e F(\mathcal{L} + 1)^{-1} \partial_e F\right) & \leq \sum_{e'} \left\langle \left| \partial_{e'} \sum_e \partial_e F(\mathcal{L} + 1)^{-1} \partial_e F \right|^2 \right\rangle \\ & \leq 2 \sum_{e'} \left\langle \left| \sum_e (\partial_{e'} \partial_e F)(\mathcal{L} + 1)^{-1} \partial_e F \right|^2 \right\rangle \\ & \quad + 2 \sum_{e'} \left\langle \left| \sum_e \partial_e F(\mathcal{L} + 2)^{-1} \partial_{e'} \partial_e F \right|^2 \right\rangle. \end{aligned}$$

To bound the last two sums, we apply Minkowski’s inequality:

$$\begin{aligned} \left\langle \left| \sum_e (\partial_{e'} \partial_e F)(\mathcal{L} + 1)^{-1} \partial_e F \right|^2 \right\rangle^{1/2} & \leq \sum_e \langle |\partial_{e'} \partial_e F(\mathcal{L} + 1)^{-1} \partial_e F|^2 \rangle^{1/2}, \\ \left\langle \left| \sum_e \partial_e F(\mathcal{L} + 2)^{-1} \partial_{e'} \partial_e F \right|^2 \right\rangle^{1/2} & \leq \sum_e \langle |\partial_e F(\mathcal{L} + 2)^{-1} \partial_{e'} \partial_e F|^2 \rangle^{1/2}. \end{aligned}$$

As was seen in (4.7), we have

$$\begin{aligned} (4.9) \quad \langle |\partial_{e'} \partial_e F(\mathcal{L} + 1)^{-1} \partial_e F|^2 \rangle^{1/2} & \leq \langle |\partial_{e'} \partial_e F|^4 \rangle^{1/4} \langle |\partial_e F|^4 \rangle^{1/4}, \\ \langle |\partial_e F(\mathcal{L} + 2)^{-1} \partial_{e'} \partial_e F|^2 \rangle^{1/2} & \leq \frac{1}{2} \langle |\partial_e F|^4 \rangle^{1/4} \langle |\partial_{e'} \partial_e F|^4 \rangle^{1/4}. \end{aligned}$$

Therefore, returning to (4.8), we obtain that

$$\text{Var}\left(\sum_e \partial_e F (\mathcal{L} + 1)^{-1} \partial_e F\right) \leq \frac{5}{2} \sum_{e'} \left( \sum_e \langle |\partial_e F|^4 \rangle^{1/4} \langle |\partial_{e'} \partial_e F|^4 \rangle^{1/4} \right)^2,$$

and thus from (4.5) and (4.6) we conclude

$$(4.10) \quad \langle h(F) - h(Y) \rangle \leq \sqrt{\frac{5}{\pi}} \|h'\|_\infty \sqrt{\sum_{e'} \left( \sum_e \langle |\partial_e F|^4 \rangle^{1/4} \langle |\partial_{e'} \partial_e F|^4 \rangle^{1/4} \right)^2}.$$

[The assumptions on  $F$  in Proposition 2.1 do not guarantee that the right-hand side of (4.10) is finite. In the case that the right-hand side is infinite, the conclusion of the theorem holds trivially.]  $\square$

## REFERENCES

- [1] BERGER, N. and BISKUP, M. (2007). Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields* **137** 83–120. [MR2278453](#)
- [2] BISKUP, M. (2011). Recent progress on the random conductance model. *Probab. Surv.* **8** 294–373. [MR2861133](#)
- [3] CHATTERJEE, S. (2008). A new method of normal approximation. *Ann. Probab.* **36** 1584–1610. [MR2435859](#)
- [4] CHATTERJEE, S. (2009). Fluctuations of eigenvalues and second order Poincaré inequalities. *Probab. Theory Related Fields* **143** 1–40. [MR2449121](#)
- [5] FURLAN, M. and MOURRAT, J.-C. (2015). A tightness criterion for random fields, with application to the Ising model. Preprint. Available at [arXiv:1502.07335](#).
- [6] GLORIA, A., NEUKAMM, S. and OTTO, F. (2015). Quantification of ergodicity in stochastic homogenization: Optimal bounds via spectral gap on Glauber dynamics. *Invent. Math.* **199** 455–515. [MR3302119](#)
- [7] GLORIA, A. and OTTO, F. (2011). An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.* **39** 779–856. [MR2789576](#)
- [8] KOZLOV, S. M. (1979). The averaging of random operators. *Mat. Sb. (N.S.)* **109(151)** 188–202, 327. [MR0542557](#)
- [9] KÜNNEMANN, R. (1983). The diffusion limit for reversible jump processes on  $\mathbf{Z}^d$  with ergodic random bond conductivities. *Comm. Math. Phys.* **90** 27–68. [MR0714611](#)
- [10] MARAHDRENS, D. and OTTO, F. (2015). Annealed estimates on the Green function. *Probab. Theory Related Fields* **163** 527–573. [MR3418749](#)
- [11] MOURRAT, J.-C. and OTTO, F. (2016). Correlation structure of the corrector in stochastic homogenization. *Ann. Probab.* **44** 3207–3233. [MR3551195](#)
- [12] NOURDIN, I. and PECCATI, G. (2009). Stein’s method on Wiener chaos. *Probab. Theory Related Fields* **145** 75–118. [MR2520122](#)
- [13] PAPANICOLAOU, G. C. and VARADHAN, S. R. S. (1981). Boundary value problems with rapidly oscillating random coefficients. In *Random Fields, Vol. I, II (Esztergom, 1979)*. *Colloquia Mathematica Societatis János Bolyai* **27** 835–873. North-Holland, Amsterdam. [MR0712714](#)

ECOLE NORMALE SUPÉRIEURE DE LYON  
CNRS  
LYON  
FRANCE  
E-MAIL: [jean-christophe.mourrat@ens-lyon.fr](mailto:jean-christophe.mourrat@ens-lyon.fr)

DEPARTMENT OF MATHEMATICS  
DUKE UNIVERSITY  
DURHAM, NORTH CAROLINA  
USA  
E-MAIL: [nolen@math.duke.edu](mailto:nolen@math.duke.edu)