THE ROUNDING OF THE PHASE TRANSITION FOR DISORDERED PINNING WITH STRETCHED EXPONENTIAL TAILS[1]

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The presence of frozen-in or quenched disorder in a system can often modify the nature of its phase transition. A particular instance of this phenomenon is the so-called rounding effect: it has been shown in many cases that the free energy curve of the disordered system at its critical point is smoother than that of the homogeneous one. In particular some disordered systems do not allow first-order transitions. We study this phenomenon for the pinning of a renewal with stretched-exponential tails on a defect line (the distribution K of the renewal increments satisfies $K(n) \sim c_K \exp(-n^\zeta)$, $\zeta \in (0, 1)$) which has a first order transition when disorder is not present. We show that the critical behavior of the disordered system depends on the value of ζ: when $\zeta > 1/2$ the transition remains of first order, whereas the free energy diagram is smoothed for $\zeta \le 1/2$. Furthermore we show that the rounding effect is getting stronger when ζ diminishes.

1. Introduction. The effect of a quenched disorder on critical phenomena is a central topic in equilibrium statistical mechanics. In many cases, it is expected that the presence of impurities in a system *rounds* or *smoothes* the phase transition in the following sense: the order parameter can be continuous at the phase transition for the disordered system whereas it presents a discontinuity for the pure system (see, e.g., the pioneering work of Imri and Ma [26]). An instance for which this phenomenon is rigorously proved is the magnetization transition of the two-dimensional random field Ising model at low temperature [1].

This phenomenon has been particularly studied for the polymer pinning on a defect line, introduced by Fisher in [15]. Whereas the model can be defined for a renewal with any kind of tail which is heavier than exponential [see (1.2)], the case of power-law tail has focused most of the attention, due to its physical interpretation and its rich mathematical structure. The interested reader can refer to [11, 18, 19] for reviews on the subject. The smoothing of the free energy curve for the pinning model with power-law tails was proved in [23] (with some restriction on the law of the disorder, see [10] for a recent generalization of the result; see also [6, 29] for related models). This confirmed predictions made by theoretical physicists [13]

based on an interpretation of the Harris criterion [25]. Some other consequences of the introduction of disorder such has critical point shift were studied in [3–5, 7, 12, 21, 22, 31].

The present paper aims to study how this phenomenology transposes for renewals which have a much lighter tail: stretched exponential ones. Whereas this issue does not seem to be discussed much in the literature, it is clear from a mathematical point of view that the type of argument used in [23] do not extend to that case (see Section 2.2 for a more detailed discussion). This hints that when renewal tails gets lighter, Harris predictions on disorder relevance might not apply (or at least not in a straight-forward manner). We show that this is indeed the case and provide a necessary and sufficient condition on the return exponent for smoothing of the free energy curve to hold.

Let us finally notice that renewal with stretched exponential tails have recently been the object of a study by Torri [32] with a different perspective: he focuses on the issue of the scaling limit of the process when the environment is heavy tailed.

1.1. *The disordered pinning model.*

Let us shortly introduce the model: set $\tau := (\tau_0, \tau_1, \ldots)$ to be a renewal process of law $\mathbf{P}$, with inter-arrival law $K(\cdot)$, *that is*, $\tau_0 = 0$ and $(\tau_i - \tau_{i-1})_{i \in \mathbb{N}}$ is a sequence of IID positive integer-valued random variables. Set

$$(1.1) \qquad\qquad K(n) := \mathbb{P}[\tau_1 = n].$$

We assume that

$$(1.2) \qquad\qquad \lim_{n \to \infty} n^{-1} \log K(n) = 0.$$

Note that with a slight abuse of notation, τ can also be considered as a subset of $\mathbb{N}$ and we write $\{n \in \tau\}$ for $\{\exists i, \tau_i = n\}$. The random potential $\omega := \{\omega_1, \omega_2, \ldots\}$ is a sequence of IID centered random variables which have unit variance and exponential moments of all order

$$(1.3) \qquad\qquad \lambda(\beta) := \log \mathbb{E}[e^{\beta \omega}] < \infty.$$

Given $\beta > 0$ (the inverse temperature) and $h \in \mathbb{R}$, we define $\mathbf{P}_N^{\beta, h, \omega}$ a measure whose Radon–Nikodym derivative w.r.t. $\mathbf{P}$ is given by

$$(1.4) \qquad \frac{d\mathbf{P}_N^{\beta, h, \omega}}{d\mathbf{P}}(\tau) := \frac{1}{Z_N^{\beta, h, \omega}} \exp\left(\sum_{n=0}^{N} (\beta \omega_n + h)\delta_n\right) \delta_N,$$

where $\delta_n = \mathbf{1}_{\{n \in \tau\}}$ and $Z_N^{\beta, h, \omega}$ is the renormalizing constant which makes $\mathbf{P}_N^{\beta, h, \omega}$ a probability law:

$$(1.5) \qquad\qquad Z_N^{\beta, h, \omega} := \mathbf{E}\left[e^{\sum_{n=1}^{N} (\beta \omega_n + h)\delta_n} \delta_N\right].$$

REMARK 1.1. In the definition (1.4) of $\mathbf{P}_N^{\beta,h,\omega}$, the δ_N corresponds to constraining the end point to be pinned. This conditioning is present for technical reasons and makes some computations easier but is not essential.

By ergodic super-additivity, (see [18], Chapter 4), the limit

$$(1.6) \qquad \mathrm{F}(\beta, h) := \lim_{N \to \infty} \frac{1}{N} \log Z_N^{\beta,h,\omega}$$

exists and is nonrandom. It is nonnegative because of assumption (1.2) and convex in h as a limit of convex functions. The expectation also converges to the same limit

$$(1.7) \qquad \mathrm{F}(\beta, h) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log Z_N^{\beta,h,\omega}.$$

The function F is called the free energy (or sometimes pressure) of the system. Its derivative in h gives the asymptotic contact fraction of the renewal process, that is, the mean number of contact per unit length,

$$(1.8) \qquad \partial_h \mathrm{F}(\beta, h) = \lim_{N \to \infty} \frac{1}{N} \mathbf{E}^{\beta,h,\omega} \left[\sum_{n=1}^{N} \delta_n \right].$$

The above convergence holds by convexity as soon as $\partial_h \mathrm{F}(\beta, h)$ is defined (i.e., everywhere except eventually at a countable number of points). If (1.2) holds, the system undergoes a phase transition from a de-pinned state [$\mathrm{F}(\beta, h) \equiv 0$] to a pinned one [$\mathrm{F}(\beta, h) > 0$ and $\partial_h \mathrm{F}(\beta, h) > 0$] when h varies.

We define $h_c(\beta)$, the critical point at which this transition occurs

$$(1.9) \qquad h_c(\beta) := \min\{h \mid \mathrm{F}(\beta, h) > 0\}.$$

As the renewal process τ we started with is recurrent, we have $h_c(0) = 0$.

REMARK 1.2. From [24], Theorem 2.1, [proved in the context of power-law renewal, but the proof remains valid with assumption (1.2)], the free energy is infinitely differentiable in h in the interval $(h_c(\beta), \infty)$, so that (1.8) holds everywhere except maybe at the critical point.

The phase transition for the pure system, that is, for $\beta = 0$, is very well understood. The pure model is said to be exactly solvable and there is a closed expression for $\mathrm{F}(0, h)$ in terms of the renewal function K (see [15]).

1.2. *Disorder relevance and Harris criterion for power-law renewals.* The disordered system ($\beta > 0$) is much more complicated to analyze and has given rise to a rich literature, most of which devoted to the case where when $n \to \infty$

$$(1.10) \qquad K(n) = c_K n^{-1+\alpha} (1 + o(1))$$

for some $\alpha > 0$. For the pure model, the free energy vanishes like a power of h at the vicinity of $0+$ (see [18], Theorem 2.1).

$$(1.11) \qquad\qquad \mathrm{F}(0, h) = c'_K h^{\max(1, \alpha^{-1})} (1 + o(1)),$$

for $\alpha \neq 1$ (a logarithmic correction is present in the case $\alpha = 1$). The main question for the study of disordered pinning model is how this property of the phase transition is affected by the introduction of disorder. For $\beta > 0$, do we have, at the vicinity of $h_c(\beta)_+$

$$(1.12) \qquad\qquad \mathrm{F}(\beta, h) \approx (h - h_c(\beta))^\nu?$$

In what case does ν equal $\max(1, \alpha^{-1})$, like for the pure system? A first partial answer to that question was given by Giacomin and Toninelli [23] (or in [10] with more generality) where it was shown that

$$(1.13) \qquad\qquad \mathrm{F}(\beta, h) \leq C \left(\frac{h - h_c(\beta)}{\beta} \right)^2,$$

meaning that the quenched critical exponent for the free energy ν, if its exists, satisfies $\nu \geq 2$. In particular, it cannot be equal to the one of the pure system when $\alpha > 1/2$.

On the other hand, for small β and $\alpha < 1/2$ it was shown by Alexander [3] (see [28, 31] for alternative proofs) that $h_c(\beta) = -\lambda(\beta)$ [recall (1.3)] and that

$$(1.14) \qquad\qquad \mathrm{F}(\beta, u - \lambda(\beta)) \overset{u \to 0+}{\sim} \mathrm{F}(0, u).$$

In view of (1.11), this implies that ν exists and is equal to $\max(1, \alpha^{-1})$ as for the pure model.

Another aspect of the relevance of disorder is the shift of the *quenched* critical point with respect to the annealed one. The annealed critical point is the one corresponding to the phase transition of the annealed partition function obtained by averaging over the environment

$$(1.15) \qquad h^a_c(\beta) := \inf \left\{ h \,\Big|\, \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}[Z_N^{\beta, h, \omega}] > 0 \right\} = -\lambda(\beta).$$

It follows from Jensen's inequality that

$$(1.16) \qquad\qquad h_c(\beta) \geq h^a_c(\beta).$$

The question of whether the above inequality is strict was investigated in [4, 12, 21, 22] yielding the conclusion that $h_c(\beta) > -\lambda(\beta)$ for every $\beta > 0$ and $\alpha \geq 1/2$.

These results were predicted in the Physics literature [13, 16], based on an interpretation of the Harris criterion [25]: when the specific-heat exponent of the pure system [for the pinning this exponent is equal to $2 - \max(1, \alpha^{-1})$] is positive, then disorder affects the critical properties of the system and is said to be relevant,

whereas when the specific-heat exponent is negative disorder is irrelevant for small values of β.

Relevant disorder affects both the location of the critical point which is shifted with respect to the annealed bound (1.16) [4, 12, 21, 22], and the critical exponent of the free energy [10, 23]. Note that the value of ν (and even its existence) when disorder is relevant is an open question even among physicists, let us mention the recent work [14] where heuristics in favor $\nu = \infty$ (infinitely derivable free energy at the critical point) are given for a toy-model.

Let us also mention that in [20, 27], the issue of disorder relevance was studied for an higher dimensional generalization of pinning model, where the set of contact point is given by a level set of the lattice free field on $\mathbb{Z}^d$, $d \geq 2$. The result obtained were in contrast with what occurs in the one dimensional setup: while the critical exponent for the free energy is always affected by disorder [it is equal to 1 for the pure model, and to 2 ($d \geq 3$) or infinity ($d = 2$) with quenched disorder], the critical point coincides with the annealed bound (1.16) at every temperature.

In this paper, we choose to look at renewal processes whose tails are stretched exponential, that is, we assume that there exists $\zeta \in (0, 1)$ such that

$$(1.17) \qquad K(n) \approx \exp(-n^\zeta),$$

in some sense. As τ is positive recurrent (i.e., its inter-arrival has finite mean), the transition of the pure model is of first order, meaning that $\mathrm{F}(0, h)$ is not derivable at $h_c(0) = 0$ positive recurrent. More precisely from [18], Theorem 2.1, we have

$$(1.18) \qquad \mathrm{F}(0, h) \overset{h \searrow 0}{\sim} \frac{h}{\mathbf{E}[\tau_0]}$$

as for the case $\alpha > 1$ in (1.10). Hence, a standard interpretation of the Harris criterion would tell us that disorder should be relevant for every β. This is partially true in the sense that this conclusion is right if one considers only the question of the critical point shift. The method developed in [12] can be adapted almost in a straight-forward manner to show the following.

PROPOSITION 1.3. *When $K(n)$ has stretched-exponential tails, then for all* $\beta > 0$,

$$(1.19) \qquad h_c(\beta) > -\lambda(\beta).$$

The more challenging question is the one about the order of the phase transition. Indeed the smoothing inequality proved in [23] strongly relies of the fact that $K(\cdot)$ has a power-law tail.

We are in fact able to find a necessary sufficient condition on ζ for a smoothing inequality to hold: we prove that when $\zeta > 1/2$, the transition remains of first order for the disordered system, while for $\zeta \leq 1/2$ the transition is rounded. We also give upper and lower bounds, which do not coincide, on the exponent ν, informally defined in (1.12), when rounding occurs, in particular we show that for any value of $\zeta \in (0, 1)$, the disordered phase transition remains of finite order.

2. Presentation of the results.

2.1. *Results*.   We assume here and in what follows that there exists a constant c_K and $\zeta \in (0, 1)$ which is such that

$$(2.1) \qquad K(n) = c_K(1 + o(1))\exp(-n^{\zeta}).$$

The law $K(n)$ as well as the law of ω are considered to be fixed, and constants that are mentioned throughout the proof can depend on both. Unless it is specified, they will not depend on β and h.

We need to assume for our first result, that the law of our product environment satisfies a concentration inequality. We define the Lipschitz seminorm of F by

$$(2.2) \qquad \|F\|_{\mathrm{lip}} = \sup_{x \neq y \in \mathbb{R}^N} \frac{|F(x) - F(y)|}{|x - y|} < \infty,$$

where $|x - y| = \sqrt{\sum(x_i - y_i)^2}$ is the Euclidean norm.

ASSUMPTION 2.1.   There exists constants C_1 and C_2 such that for any N and for any Lipchitz convex function F on $\mathbb{R}^N$, we have

$$(2.3) \qquad \mathbb{P}\big(|F(\omega_1, \dots, \omega_N) - \mathbb{E}[F(\omega_1, \dots, \omega_N)]| \geq u\big) \leq C_1 e^{-\frac{u^2}{C_2\|F\|_{\mathrm{lip}}^2}}.$$

A crucial point here is that inequality is independent of the dimension N. This is the reason why we use concentration for the Euclidean norm rather than for the L_1 norm.

REMARK 2.2.   The concentration assumption is not very restrictive, it holds for bounded ω (see [30], Chapter 4) or when ω satisfies a log-Sobolev inequality (see [30], Chapter 5, in this case there is no convexity required). This second case includes in particular the case of Gaussian variables and many others classic laws.

Our first result states that transition is of first order for the system for $\zeta > 1/2$ (no smoothing holds). Here and in what follows, $x_+ := \max(x, 0)$ denotes the positive part of $x \in \mathbb{R}$.

THEOREM 2.3.   *Assume that Assumption* 2.1 *holds*:

(i) *For $\zeta > 1/2$, there exists a constant c such that for all $\beta > 0$ and $u > 0$,*

$$(2.4) \qquad \mathrm{F}(\beta, h_c(\beta) + u) \geq c\min(1, \beta^{-2})u.$$

(ii) *For $\zeta \leq 1/2$, there exists a constant c such that for $u > 0$ close to zero*

$$(2.5) \qquad \mathrm{F}(\beta, h_c(\beta) + u) \geq \frac{c}{\beta^2}\left(\frac{u}{|\log u|^5}\right)^{\frac{1-\zeta}{\zeta}}.$$

Our second result show that in fact smoothing holds for $\zeta < 1/2$. For this result, we need to assume that the environment is Gaussian. The assumption could be partially relaxed but the exposition of the Gaussian case is much easier. Let us mention that the recent work [10] gives hopes to extend the proof to general ω.

THEOREM 2.4. *Let us assume that the environment is Gaussian. Then for all* $\zeta < 1/2$ *there exists a constant* c (*which depends on* K) *such that in a neighborhood of* $h_c(\beta)$

$$\text{(2.6)} \qquad \text{F}(\beta, h) \leq c \left(\frac{h - h_c(\beta)}{\beta} \right)_+^{2(1-\zeta)}.$$

Finally, with an extra assumption on $K(\cdot)$ we are able to state that the transition is smooth also when $\zeta = 1/2$. We say that $K(n)$ is log convex if $\log K$ can be extended to a convex function on $\mathbb{R}_+$; or equivalently if

$$\text{(2.7)} \qquad \forall n, l \in \mathbb{N} \qquad (n > l > 1) \quad \Rightarrow \quad K(n+1)K(l-1) \geq K(n)K(l).$$

This assumption is necessary to prove positive correlation, or the FKG inequality (see [17]) for the disordered renewal.

THEOREM 2.5. *Assume that* $\log K(n)$ *is a convex function of* n. *Then for* $\zeta = 1/2$,

$$\text{(2.8)} \qquad \text{F}(\beta, h) = o\big((h - h_c(\beta))_+\big).$$

REMARK 2.6. The log-convex assumption is not that restrictive and is rather natural as assumption (1.2) already implies that the derivative of $\log K$ tends to zero. A particular instance of log-convex K is the case where τ is the set of return times to zero of a one-dimensional nearest-neighbor random walk on $\mathbb{Z}$. This is related to log-convexity of the sequence of Catalan numbers (see [9] for a paper on the subject).

2.2. *The smoothing for polynomial tails.* Let us explain briefly in this section why the proof strategy from [23] fails to give any results in the case of stretched exponential renewals (for more details the reader should refer to the original article). For simplicity, we assume here that the environment is Gaussian and that $\beta = 1$.

The main idea in [23] is the following. Let $h = h_c(\beta) + u$ be fixed, and N be chosen very large. We look at a system at the critical point $h_c(\beta)$ (for which the free energy is zero): in a typical segment of length N the empirical mean of ω should be of order 0 due to the law of large number; however, with probability of order $\exp(-Nu^2/2)$ the empirical mean is larger than u. In that case, the system does not locally look critical and the partition function corresponding to the segment

should be of order $e^{N\mathrm{F}(\beta,h_c(\beta)+u)}$, if N is chosen sufficiently large to avoid finite size effects.

The distance between these segments of length N which give an unusually "good" contribution to the partition function should be typically huge: of order $\exp(u^2N/2)$, and thus the cost making a huge jump between two consecutive good segments to avoid bad environment should be of order $K(\exp(u^2N/2))$. As the free energy at criticality is zero, the strategy visiting all the "good" segments and avoiding all the bad ones should not give an exponentially large contribution to the partition function, hence the cost of making the large jump should completely compensate for the energy reward one gets when visiting a good segment. For this reason, one must have for sufficiently large N

$$(2.9) \qquad\qquad K\big(\exp\big(u^2N/2\big)\big)e^{N\mathrm{F}(\beta,h_c(\beta)+u)} < 1.$$

In the case where K has a power-law tail, this immediately yields a quadratic bound on the free energy. However, when K has a lighter tail, (2.9) fails to give any interesting information, as $K(\exp(u^2N/2))$ decays super-exponentially.

Some elements of this strategy can somehow be recycled (this is what is done in Section 5) provided we have some information about the behavior of finite volume systems (see Lemma 3.1). However, as will be seen, this fails to give a quadratic bound on the free energy.

2.3. *The case of renewals with exponential and sub-exponential tails.* Another instance of pinning model with absence of smoothing has been exhibited in [2]: disordered pinning of transient renewals with exponential tails [the case $K(n) = O(\exp(-nb))$ for some $b > 0$]. However, let us mention that this case is quite special. When the tail of the renewal is exponential Remark 1.1 is not valid anymore. On the contrary, the behavior of the system crucially depends on whether one pins the renewal at the end:

- The free energy $\mathrm{F}(\beta, h)$ defined by (1.7), which corresponds to a system constrained to be pinned, is negative for small values of h.
- The free energy of the system with no constraints is obtained by considering the best of two strategies: either the walk will avoid the wall completely or it will try to pin the end point. The reward for this is equal to $\max(0, \mathrm{F}(\beta, h))$, which is easily shown to have a first-order transition in h.

Here, the mechanism which triggers a first-order phase transition is completely different: we have to perform an analysis of local fluctuation of the environment to see whether or not the benefit of a good rare region is sufficient to compensate the cost of a large jump coming to it. An upper bound on the fluctuations is obtained via concentration. To obtain a lower-bound we choose to restrict to the Gaussian model for simplicity but similar ideas could in principle be implemented by the use of tilting (like in [10]).

3. Preliminaries.

3.1. *Notation.* The dependence in β and h will frequently be omitted to lighten the notation. When A is an event for τ we set

$$(3.1) \qquad Z_N^\omega(A) := \mathbf{E}\big[e^{\sum_{n=1}^N (\beta \omega_n + h)\delta_n} \delta_N \mathbf{1}_A\big].$$

For $k \in \mathbb{N}$, the shift operator θ^k acting on the sequence ω is defined by

$$(3.2) \qquad \theta^k \omega_n := \omega_{n+k}.$$

For any couple of integers $a \le b$, one sets

$$(3.3) \qquad Z_{[a,b]}^\omega = e^{(\beta \omega_a + h)\mathbf{1}_{a>0}} Z_{b-a}^{\theta^a \omega}$$

to be the partition function associated to the segment $[a, b]$ (with the convention that $Z_0^\omega = 1$). Note that the environment at the starting point of the interval a is taken into account only for $a > 0$ (for technical reasons).

For $\varepsilon > 0$, one defines

$$(3.4) \qquad \begin{aligned} \mathcal{A}^\varepsilon &:= \big\{\tau : \#(\tau \cap (0, N]) \le \varepsilon N, N \in \tau\big\}, \\ \mathcal{B}^\varepsilon &:= \big\{\tau : \#(\tau \cap (0, N]) > \varepsilon N, N \in \tau\big\}, \end{aligned}$$

the set of renewals whose contact fraction is smaller, respectively, larger, than ε.

3.2. *Finite volume bounds for the free energy.* The following result allows us to estimate the free energy only knowing the value of $\frac{1}{N}\mathbb{E}[\log Z_N^\omega]$, for a given N.

LEMMA 3.1. *There exists a constant c such that for every N, β and h,*

$$(3.5) \qquad \begin{aligned} &\frac{1}{N}\mathbb{E}\big[\log Z_N^\omega\big] \le \mathrm{F}(\beta, h), \\ &\frac{1}{N}\mathbb{E}\big[\log Z_N^\omega\big] \ge \mathrm{F}(\beta, h) - \frac{N^{\zeta-1}}{1 - 2^{\zeta-1}} - \frac{2(\lambda(-\beta) + h)_+ + c}{N}. \end{aligned}$$

PROOF. The first inequality is a consequence of the following super-multiplicative property:

$$(3.6) \qquad Z_{N+M}^\omega \ge Z_N^\omega \times Z_M^{\theta^N \omega}$$

(see, e.g., the proof of Proposition 4.2 in [18]). For the second one, the proof is similar to [24], Proposition 2.7, we have

$$(3.7) \qquad Z_{2N}^\omega = \mathbf{E}\big[e^{\sum_{n=1}^{2N}(\beta \omega_n + h)\delta_n}\delta_N \delta_{2N}\big] + \mathbf{E}\big[e^{\sum_{n=1}^{2N}(\beta \omega_n + h)\delta_n}(1 - \delta_N)\delta_{2N}\big].$$

The first term is equal to $Z_N^\omega Z_N^{\theta^N \omega}$. For the second term, by comparing the weight of each τ to the one of $\tau \cup \{N\}$ one obtains

$$\mathbf{E}\big[e^{\sum_{n=1}^{2N}(\beta\omega_n + h)\delta_n}(1 - \delta_N)\delta_{2N}\big]$$

$$(3.8) \qquad \leq Z_N^\omega Z_N^{\theta^N \omega} e^{-\beta\omega_N - h} \max_{0 \leq a < N < b \leq 2N} \frac{K(b-a)}{K(N-a)K(b-N)}$$

$$\leq C e^{-\beta\omega_N - h} Z_N^\omega Z_N^{\theta^N \omega} \exp(N^\zeta),$$

for some constant $C > 1$. The last line of (3.8) is obtained by observing that for any choice of $0 \leq a < N < b \leq 2N$

$$(N-a)^\zeta + (b-N)^\zeta - (b-a)^\zeta \leq N^\zeta.$$

Hence, taking the log and expectation in (3.7) we have

$$\frac{1}{2N}\mathbb{E}\big[\log Z_{2N}^\omega\big]$$

$$\leq \frac{1}{N}\mathbb{E}\big[\log Z_N^\omega\big] + \frac{1}{N}\mathbb{E}\big[\log(1 + Ce^{-\beta\omega_N - h}\exp(N^\zeta))\big]$$

$$(3.9) \qquad \leq \frac{1}{N}\mathbb{E}\big[\log Z_N^\omega\big] + \frac{1}{N}\log(1 + e^{\lambda(-\beta)-h}C\exp(N^\zeta))$$

$$\leq \frac{1}{N}\mathbb{E}\big[\log Z_N^\omega\big] + N^{\zeta-1} + \frac{1}{N}\log(1 + Ce^{\lambda(-\beta)-h})$$

$$\leq \frac{1}{N}\mathbb{E}\big[\log Z_N^\omega\big] + N^{\zeta-1} + \frac{1}{N}\big[\log(2C) + (\lambda(-\beta) - h)_+\big],$$

where the first inequality is simply Jensen's inequality. Then we iterate the inequality and obtain

$$(3.10) \qquad \begin{aligned} \mathrm{F}(\beta, h) &\leq \frac{1}{N}\mathbb{E}\big[\log Z_N^\omega\big] + \frac{N^{\zeta-1}}{1 - 2^{\zeta-1}} \\ &\quad + \frac{2}{N}\big[\log(2C) + (\lambda(-\beta) - h)_+\big]. \qquad \square \end{aligned}$$

3.3. *The FKG inequality for* log-*convex renewals.* For the proof of Theorem 2.5 (and only then), we need to use the fact that the presence of renewal point are positively correlated. This is were the assumption of log convexity of the function K.

In this subsection τ denotes a subset of $\{1, \ldots, N\}$ which contains N, and with some abuse of notation $\mathbf{P}_N^{\beta, h, \omega}$ is considered to be a law on $\mathcal{P}(\{1, \ldots, N\})$ (the set of subsets of $\{1, \ldots, N\}$).

Now let us introduce some definition. A function $f : \mathcal{P}(\{1, \ldots, N\}) \to \mathbb{R}$ is said to be increasing if

$$(3.11) \qquad \forall \tau, \tau' \in \mathcal{P}(\{1, \ldots, N\}) \qquad \tau \subset \tau' \quad \Rightarrow \quad f(\tau) \leq f(\tau').$$

Note that the result was proved in [8] for renewal processes in continuous time. Our proof is essentially similar and is based on the use of the celebrated FKG criterion from [17] but we choose to include it for the sake of completeness.

PROPOSITION 3.2. *Assume that the function K is* log-*convex. Then for all β, ω, h and N, the $\mathbf{P}_N^{\beta,h,\omega}$ satisfies the FKG inequality. For all increasing functions f and g,*

$$(3.12) \qquad \mathbf{E}_N^{\beta,h,\omega}[f(\tau)g(\tau)] \geq \mathbf{E}_N^{\beta,h,\omega}[f(\tau)]\mathbf{E}_N^{\beta,h,\omega}[g(\tau)].$$

PROOF. From [17], Proposition 1, it is sufficient to check that for any τ and τ' we have

$$(3.13) \qquad \mathbf{P}_N^{\beta,h,\omega}(\tau \cup \tau')\mathbf{P}_N^{\beta,h,\omega}(\tau \cap \tau') \geq \mathbf{P}_N^{\beta,h,\omega}(\tau)\mathbf{P}_N^{\beta,h,\omega}(\tau').$$

For $\sigma \subset \{0, \ldots, N\}$ whose elements are $\sigma_0 = 0 < \sigma_1 < \cdots < \sigma_m = N$, one sets

$$K(\sigma) = \prod_{i=1}^m K(\sigma_i - \sigma_{i-1}).$$

The reader can check that after simplification (3.13) is equivalent to

$$(3.14) \qquad K(\tau \cup \tau')K(\tau \cap \tau') \geq K(\tau)K(\tau').$$

This inequality is obviously true when $\tau' \subset \tau$. Then we can reason by induction and it is sufficient to check is that if $a \notin \tau \cup \tau'$ and (3.14) holds for τ and τ' then it holds for τ and $\tau' \cup \{a\}$. To this purpose, we only need to check that for any τ, τ' and $a \notin \tau \cup \tau'$ we have

$$(3.15) \qquad \frac{K(\tau \cup \tau' \cup \{a\})}{K(\tau \cup \tau')} \geq \frac{K(\tau' \cup \{a\})}{K(\tau')}.$$

Let us set

$$(3.16) \qquad \begin{aligned} \alpha_1 &:= \sup\{x < a : x \in \tau \cup \tau'\}, & \beta_1 &:= \inf\{x > a : x \in \tau \cup \tau'\}, \\ \alpha_2 &:= \sup\{x < a : x \in \tau'\}, & \beta_2 &:= \inf\{x > a : x \in \tau'\}. \end{aligned}$$

We remark that

$$\alpha_2 \leq \alpha_1 < a < \beta_1 \leq \beta_2.$$

The inequality (3.15) is equivalent to

$$(3.17) \qquad \frac{K(\beta_1 - a)K(a - \alpha_1)}{K(\beta_1 - \alpha_1)} \geq \frac{K(\beta_2 - a)K(a - \alpha_2)}{K(\beta_2 - \alpha_2)}.$$

By convexity of $\log K$, the function

$$(3.18) \qquad (\alpha, \beta) \mapsto \frac{K(\beta_1 - a)K(a - \alpha_1)}{K(\beta_1 - \alpha_1)}$$

is nonincreasing in β and nondecreasing in α on the set $\{(\alpha, \beta) : \alpha < a < \beta\}$. Thus, (3.17) holds. $\square$

4. Proof of Theorem 2.3.

4.1. *Decomposition of the proof.*   The key point consists in proving the following upper bound to $Z_N^\omega(\mathcal{A}^\varepsilon)$ [recall (3.4) and (1.5)].

PROPOSITION 4.1.   *There exists positive constants ε_0 and C such that for all $\varepsilon \leq \varepsilon_0$ we have almost surely, for all N sufficiently large, for all $h \leq 1$ and $\beta > 0$,*

$$(4.1) \quad \frac{1}{N} \log Z_N^\omega(\mathcal{A}^\varepsilon) \leq \frac{1}{2} F(h, \beta) + \sup_{l \geq (3\varepsilon)^{-1}} \left(C\beta \sqrt{\frac{\varepsilon \log l}{l}} - \frac{1}{8} l^{\zeta - 1} (\log l)^{-2} \right).$$

The restriction $h \leq 1$ is chosen for convenience but does not convey any particular significance ($h < c$ for some $c > 0$ would be just as good). The proof of this statement is postponed to Section 4.2.

Now, we observe that if ε is chosen to be larger than the asymptotic contact fraction $\partial_h F(\beta, h)$, the left-hand side of (4.1) converges to the free energy.

LEMMA 4.2.   *For every $h > h_c(\beta)$ when $\varepsilon > \partial_h F(\beta, h)$, we have*

$$(4.2) \qquad\qquad \liminf_{N \to \infty} \mathbf{P}_N^{\beta, h, \omega}[\mathcal{A}^\varepsilon] > 0.$$

As a consequence,

$$(4.3) \qquad\qquad \limsup_{N \to \infty} \frac{1}{N} \log Z_N^\omega(\mathcal{A}^\varepsilon) = F(\beta, h).$$

REMARK 4.3.   Without much more efforts, one can even prove in facts that the limit in (4.2) is equal to one, but this is not necessary for our purpose.

The idea to prove Theorem 2.3 is to use (4.1) where ε is replaced by $2\partial_h F(\beta, h)$ and $\frac{1}{N} \log Z_N^\omega(\mathcal{A}^\varepsilon)$ is replaced by its limit: $F(\beta, h)$. This gives a differential inequality in h which once integrated gives the claimed bounds on the free energy. Details follow at the end of the section.

PROOF OF LEMMA 4.2.   For simplicity (and with no loss in generality), assume that $\varepsilon = (1 + \delta)\partial_h F(\beta, h)$ for some $\delta < 1$ By (1.8) for N sufficiently large,

$$(4.4) \quad \frac{1}{N} \mathbf{E}_N^{\beta, h, \omega}\left[\sum_{n=1}^N \delta_n \right] \leq (1 + \delta)(1 - \delta/2)\partial_h F(\beta, h) = (1 - \delta/2)\varepsilon.$$

As

$$(4.5) \qquad\qquad \frac{1}{N} \mathbf{E}_N^{\beta, h, \omega}\left[\sum_{n=1}^N \delta_N \right] \geq \varepsilon \mathbf{P}_N^{\beta, h, \omega}[\mathcal{B}^\varepsilon],$$

this implies

$$(4.6) \qquad \mathbf{P}_N^{\beta,h,\omega}[\mathcal{B}^\varepsilon] \leq 1 - \delta/2. \qquad \square$$

PROOF OF THEOREM 2.3. Let ε_0 be such that Proposition 4.1 holds (in the case $\zeta > 1/2$, we will also require ε_0 to satisfy another condition). If $h \leq 1$ is such that

$$(4.7) \qquad \varepsilon_h := 2\partial_h \mathrm{F}(\beta, h) \leq \varepsilon_0.$$

Then combining (4.3) with Proposition 4.1, we have

$$(4.8) \qquad \mathrm{F}(\beta, h) \leq \sup_{l \geq (3\varepsilon_h)^{-1}} \left(2C\beta \sqrt{\frac{\varepsilon_h \log l}{l}} - \frac{1}{4} l^{\zeta - 1} (\log l)^{-2} \right).$$

Let us start with the case $\zeta > 1/2$. Let us assume that (recall Remark 1.2),

$$(4.9) \qquad \lim_{h \to h_c(\beta)+} \partial_h \mathrm{F}(\beta, h) < \frac{1}{2} \varepsilon_0 \min(1, \beta^{-2}).$$

From a standard convexity argument (see [18], Proposition 5.1), we have $h_c(\beta) \leq 0$. Thus, we can find $h \in (h_c(\beta), 1]$ such that

$$\varepsilon_h \leq \varepsilon_0 \min(1, \beta^{-2}).$$

Then for this value of h the right-hand side of (4.8) is smaller than

$$\sup_{l \geq (3\varepsilon_0)^{-1}} \left(2C \sqrt{\frac{\varepsilon_0 \log l}{l}} - \frac{1}{4} l^{\zeta - 1} (\log l)^{-2} \right)$$

which is equal to zero if ε_0 has been chosen sufficiently small. Hence, we obtain a contradiction as $\mathrm{F}(\beta, h) > 0$.

Let us move to the case $\zeta \leq 1/2$. We can assume that

$$(4.10) \qquad \lim_{h \to h_c(\beta)+} \partial_h \mathrm{F}(\beta, h) = 0$$

as if not, there is nothing to prove.

For h sufficiently close to the critical point, we hence have $\varepsilon_h \leq \varepsilon_0$ (and $h \leq 1$), and hence equation (4.8) holds. Computing the maximum in the left-hand side of (4.8), we obtain

$$(4.11) \qquad \mathrm{F}(\beta, h) \leq \begin{cases} C(\beta^2 \varepsilon_h |\log \varepsilon_h|^5)^{\frac{1-\zeta}{1-2\zeta}} |\log \varepsilon_h|^{-2} & \text{for } \zeta < 1/2, \\ \exp(-c(\beta^2 \varepsilon_h)^{-1/5}) & \text{for } \zeta = 1/2. \end{cases}$$

Replacing ε_h by its value (4.7) and ignoring the $|\log \varepsilon_h|^{-2}$ factor in the first line, we obtain for all h such that $h - h_c(\beta) \in (0, u_0(\beta)]$, with $u_0(\beta)$ sufficiently small

$$(4.12) \qquad \mathrm{F}^{\frac{2\zeta - 1}{1-\zeta}} |\log \mathrm{F}|^5 \partial_h \mathrm{F} \geq c\beta^{-2}.$$

Integrating the above inequality between $h_c(\beta)$ and h yields the result. $\quad \square$

4.2. *Proof of Proposition* 4.1. A key tool in the proof is the following concentration inequality. What is crucial in this step is to control the Lipshitz norm in order to obtain better concentration when the contact fraction is smaller.

LEMMA 4.4. *When Assumption* 2.1 *holds then for any event* $A \subset \mathcal{A}^{\varepsilon}$

$$(4.13) \qquad \mathbb{P}\big[\log Z_N^{\omega}(A) - \mathbb{E}\big[\log Z_N^{\omega}(A)\big] \geq t\big] \leq C_1 \exp\left(-\frac{t^2}{C_2 \beta^2 N \varepsilon}\right).$$

PROOF. For any pair of environment ω and ω', the following holds:

$$\left|\log \frac{Z_N^{\omega}(A)}{Z_N^{\omega'}(A)}\right| \leq \beta \max_{\{\tau \subset [0,N]:|\tau \cap [0,N]| \leq \varepsilon N\}} \sum_{x \in \tau} |\omega_x - \omega_x'|$$

$$(4.14)$$

$$\leq \beta \sqrt{\varepsilon N} \sqrt{\sum_{x=1}^{N} \omega_x^2}.$$

Hence, the Lipshitz norm of

$$\omega \mapsto \log Z_N^{\omega}(A)$$

is smaller than $\beta \sqrt{\varepsilon N}$. It is also a convex function, thus the results follows from Assumption 2.1. $\square$

Given $\tau \in \mathcal{A}^{\varepsilon}$, we want to introduce a notion of "typical" size of jumps in τ, we call it $\ell(\tau)$. The idea is to choose $\ell(\tau)$ in a way that the interval $[0, N]$ is mostly spanned by jumps whose size is of order ℓ. Let us set

$$\ell(\tau) := \min\Big\{r \geq (3\varepsilon)^{-1} :$$

$$(4.15)$$

$$\sum_{\{n:\tau_n \leq N\}} (\tau_n - \tau_{n-1})\mathbf{1}_{\{(\tau_n - \tau_{n-1}) \in [r, 2r)\}} \geq \frac{N}{(\log r)^2}\Big\}.$$

Let us check quickly if this definition makes sense.

LEMMA 4.5. *If* $\varepsilon \leq \varepsilon_0$ *is sufficiently small, then for any* $N \geq 0$, *for any* $\tau \in \mathcal{A}^{\varepsilon}$, *we have*

$$(4.16) \quad \Big\{r \geq (3\varepsilon)^{-1} : \sum_{\{n:\tau_n \leq N\}} (\tau_n - \tau_{n-1})\mathbf{1}_{\{(\tau_n - \tau_{n-1}) \in [r, 2r)\}} \geq \frac{N}{(\log r)^2}\Big\} \neq \varnothing.$$

Moreover, we have for all $\tau \in \mathcal{A}^{\varepsilon}$

$$(4.17) \qquad \sum_{\{n:\tau_n \leq N\}} (\tau_n - \tau_{n-1})\mathbf{1}_{\{(\tau_n - \tau_{n-1}) \in [r, 2r)\}} \leq \frac{N}{2}.$$

PROOF. We work by contradiction and assume that the set is empty. Let us set $r_0 := \lceil (3\varepsilon)^{-1} \rceil$ and $r_k = 2^k r_0$ for $k \geq 1$. Then we have, provided ε_0 is chosen in an adequate manner

$$\sum_{n=1}^{\infty} (\tau_n - \tau_{n-1}) \mathbf{1}_{\{(\tau_n - \tau_{n-1}) \geq (3\varepsilon)^{-1}; \tau_n \leq N\}}$$

$$(4.18) \qquad \leq N \sum_{k=0}^{\infty} \frac{1}{(\log r_k)^2}$$

$$\leq N \sum_{k=0}^{\infty} \frac{1}{(k \log 2 + |\log(3\varepsilon)|)^2} \leq \frac{N}{6}.$$

On the other hand, using the definition of $\mathcal{A}^{\varepsilon}$ we have

$$(4.19) \qquad \sum_{n=1}^{\infty} (\tau_n - \tau_{n-1}) \mathbf{1}_{\{(\tau_n - \tau_{n-1}) < (3\varepsilon)^{-1}; \tau_n \leq N\}} < (3\varepsilon)^{-1} \#(\tau \cap (0, N]) \leq \frac{N}{3}.$$

Thus, we obtain the contradiction to the fact that if $N \in \tau$

$$\sum_{n=1}^{\infty} (\tau_n - \tau_{n-1}) \mathbf{1}_{\{\tau_n \leq N\}} = N.$$

Using the same argument as in (4.18), we obtain

$$(4.20) \qquad \sum_{n=1}^{\infty} (\tau_n - \tau_{n-1}) \mathbf{1}_{\{(\tau_n - \tau_{n-1}) \in [(3\varepsilon)^{-1}, \ell(\tau)); \tau_n \leq N\}} \leq \frac{N}{6},$$

which together with (4.19) allows us to conclude that (4.17) holds. $\square$

Define $\mathcal{L}(\tau)$ and $L(\tau)$ to be the set of indices and the number of jumps whose size is larger than $\ell(\tau)$

$$(4.21) \qquad \begin{aligned} \mathcal{L}(\tau) &:= \{n : \tau_n \leq N; (\tau_n - \tau_{n-1}) \geq \ell(\tau)\}, \\ L(\tau) &:= \#\mathcal{L}(\tau). \end{aligned}$$

Note that by definition

$$(4.22) \qquad L(\tau) \in \left[\frac{N}{2\ell(\tau)(\log \ell(\tau))^2}, \frac{N}{\ell(\tau)} \right].$$

For fixed $l, L \in \mathbb{N}$, $l \in [(3\varepsilon)^{-1}, N]$ and $L \in [\frac{N}{2l(\log l)^2}, \frac{N}{l}]$, we set

$$(4.23) \qquad \mathcal{T}(l, L) := \{(\mathbf{t}', \mathbf{t}) \in ([0, N] \cap \mathbb{Z})^{2L} : \forall i \in [1, L], t_i' \geq t_{i-1}, t_i \geq t_i' + l\}$$

which is the set of possible locations for $(\tau_{n-1}, \tau_n)_{n \in \mathcal{L}(\tau)}$. For $(\mathbf{t}', \mathbf{t}) \in \mathcal{T}(l, L)$, we set

$$(4.24) \qquad A_{(\mathbf{t}', \mathbf{t})} := \{\tau \in \mathcal{A}_{\varepsilon} : \ell(\tau) = l; \{(\tau_{n-1}, \tau_n)\}_{n \in \mathcal{L}(\tau)} = \{(t_i', t_i)\}_{i=1}^{L}\}.$$
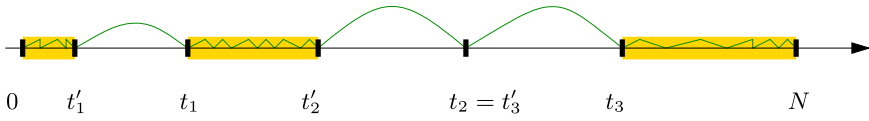
FIG. 1. *A schematic representation of a set* $(\mathbf{t}', \mathbf{t}) \in \mathcal{T}(l, 3)$, *and a renewal in* $\tau \in A_{(\mathbf{t}', \mathbf{t})}$ *(in green). In the yellow region, the length of the jumps must be either smaller than* l. *A consequence that* $\ell(\tau) = l$ *is that the total length of the yellow region is smaller than* $N/2$.

It is the subset of $\mathcal{A}^{\varepsilon}$ for which the jumps of τ whose length is larger than $\ell(\tau)$ (the jumps of "typical" size) exactly span the segments $(t'_i, t_i)_{i=1}^{L}$ (see also Figure 1). We have

$$(4.25) \qquad Z_N^{\omega}(\mathcal{A}^{\varepsilon}) = \sum_{l \in [(3\varepsilon)^{-1}, N] \cap \mathbb{N}} \sum_{L \in [\frac{N}{2l(\log l)^2}, \frac{N}{l}] \cap \mathbb{N}} \sum_{(\mathbf{t}', \mathbf{t}) \in \mathcal{T}(l, L)} Z_N^{\omega}(A_{(\mathbf{t}', \mathbf{t})}).$$

In particular,

$$\log Z_N^{\omega}(\mathcal{A}^{\varepsilon})$$

$$(4.26) \qquad \begin{aligned} &\leq 2 \log N + \max_{\substack{l \in [(3\varepsilon)^{-1}, N] \\ L \in [\frac{N}{2l(\log l)^2}, \frac{N}{l}]}} \Big[\log\big(\#\mathcal{T}(l, L)\big) \\ &\quad + \max_{(\mathbf{t}', \mathbf{t}) \in \mathcal{T}(l, L)} \log Z_N^{\omega}(A_{(\mathbf{t}', \mathbf{t})}) \Big]. \end{aligned}$$

The idea is then to use Lemma 4.4 to find a good bound on the right-hand side. A first easy step is to get an estimate on the cardinal of $\mathcal{T}(l, L)$.

LEMMA 4.6. *There exist constants* C *and* ε_0 *such that for all* $\varepsilon \leq \varepsilon_0$, *and all choices of* N, $l \in [(3\varepsilon)^{-1}, N]$, $L \in [\frac{N}{2l(\log l)^2}, \frac{N}{l}]$ *we have*

$$(4.27) \qquad \#\mathcal{T}(l, L) \leq C \exp(3L \log l).$$

PROOF. The set $\{t_i\}_{i=1}^{L} \cup \{t'_i + 1\}_{i=1}^{L}$ is a subset of $\{1, \dots, N\}$ with $2L$ elements. Hence,

$$(4.28) \qquad \#\mathcal{T}(l, L) \leq \binom{N}{2L} \leq \frac{N^{2L}}{(2L)!} \leq C \left(\frac{2N}{L}\right)^{2L},$$

where we used in the last line that $n! \geq C^{-1}(n/4)^n$. We conclude by observing that for ε small enough we have

$$(4.29) \qquad \log\left(\frac{2N}{L}\right) \leq \log[4l(\log l)^2] \leq \frac{3}{2}\log l. \qquad \square$$

To use Lemma 4.4 efficiently, we must also know about the expected value of $\log Z_N^{\omega}(A_{(\mathbf{t}', \mathbf{t})})$.

LEMMA 4.7. *For any* $(\mathbf{t}', \mathbf{t}) \in \mathcal{T}(l, L)$, *for ε sufficiently small (depending only on K)*

$$(4.30) \qquad \frac{1}{N} \mathbb{E}[\log Z_N^{\omega}(A_{(\mathbf{t}', \mathbf{t})})] \leq \frac{1}{2} \mathrm{F}(\beta, h) + l^{-1} - \frac{1}{4} l^{\zeta - 1} (\log l)^{-2}.$$

PROOF. We have [recall (3.3)]

$$(4.31) \qquad Z_N^{\omega}(A_{(\mathbf{t}', \mathbf{t})}) \leq \left[\prod_{i=1}^{L} Z_{[t_i', t_{i-1}]}^{\omega} K(t_i - t_i') \right] Z_{[t_L, N]}^{\omega}.$$

Hence,

$$\mathbb{E}[\log Z_N^{\omega}(A_{(\mathbf{t}', \mathbf{t})})]$$

$$(4.32) \qquad \leq \sum_{i=1}^{L} \mathbb{E}[\log Z_{[t_i', t_{i-1}]}^{\omega}] + \mathbb{E}[\log Z_{[t_L, N]}^{\omega}] + \sum_{i=1}^{L} \log K(t_i - t_i').$$

From Lemma 3.1, we have

$$\sum_{i=1}^{L} \mathbb{E}[\log Z_{[t_i', t_{i-1}]}^{\omega}] + \mathbb{E}[\log Z_{[t_L, N]}^{\omega}]$$

$$(4.33)$$

$$\leq \left(\sum_{i=1}^{L} (t_i' - t_{i-1}) + (N - t_L') \right) \mathrm{F}(\beta, h) + Lh \leq N \mathrm{F}(\beta, h)/2 + L,$$

where in the last inequality we used the fact that $h \leq 1$, and (4.17). Concerning the last term in (4.32), we have [recall (4.22)] for ε sufficiently small

$$(4.34) \qquad - \sum_{i=1}^{L} \log K(t_i - t_i') \geq \frac{1}{2} \sum_{i=1}^{L} (t_i - t_i')^{\zeta} \geq \frac{1}{2} L l^{\zeta} \geq \frac{1}{4} N l^{\zeta - 1} (\log l)^{-2},$$

which completes the proof. $\square$

LEMMA 4.8. *There exists a constant C such that for N sufficiently large, for all $l \in [(3\varepsilon)^{-1}, N]$ and $L \in [\frac{N}{2l(\log l)^2}, \frac{N}{l}]$*

$$(4.35) \qquad \mathbb{P}\left(M(l, L) \geq C\beta N \sqrt{\frac{\varepsilon \log l}{l}} \right) \leq \frac{1}{N^4},$$

where

$$(4.36) \qquad M(l, L) = \max_{(\mathbf{t}', \mathbf{t}) \in \mathcal{T}(l, L)} \left(\log Z_N^{\omega}(A_{(\mathbf{t}', \mathbf{t})}) - \mathbb{E}[\log Z_N^{\omega}(A_{(\mathbf{t}', \mathbf{t})})] \right).$$

PROOF. From Lemma 4.4 and a standard union bounds, we have for any $u > 0$

$$\mathbb{P}\Big(\max_{(\mathbf{t}',\mathbf{t}) \in \mathcal{T}(l,L)} \big(\log Z_N^\omega(A_{(\mathbf{t}',\mathbf{t})}) - \mathbb{E}[\log Z_N^\omega(A_{(\mathbf{t}',\mathbf{t})})]\big) \geq u\Big)$$

(4.37)

$$\leq (\#\mathcal{T}(l,L))C_1 \exp\Big(-\frac{u^2}{C_2\beta^2 N\varepsilon}\Big).$$

Using Lemma 4.6, and setting $u := C\beta N\sqrt{\frac{\varepsilon \log l}{l}}$, one can conclude provided that C is chosen sufficiently large. $\square$

PROOF OF PROPOSITION 4.1. Using Lemma 4.8 and Lemma 4.7, and Borel–Cantelli's lemma we have, almost surely for all large N, for all $l \in [(3\varepsilon)^{-1}, N]$ and $L \in [\frac{N}{2l(\log l)^2}, \frac{N}{l}]$

$$\frac{1}{N}\max_{(\mathbf{t}',\mathbf{t}) \in \mathcal{T}(l,L)} \log Z_N^\omega(A_{(\mathbf{t}',\mathbf{t})})$$

(4.38)

$$\leq \frac{1}{2}\mathrm{F}(\beta,h) + l^{-1} + C\beta\sqrt{\frac{\varepsilon \log l}{l}} - \frac{1}{4}l^{\zeta-1}(\log l)^{-2}.$$

Combining this with (4.26), we obtain

$$\frac{1}{N}\log Z_N^\omega(\mathcal{A}^\varepsilon) \leq \frac{2\log N}{N} + \frac{1}{2}\mathrm{F}(\beta,h)$$

(4.39)

$$+ \max_{l \geq (3\varepsilon)^{-1}}\Big(\frac{\log\#\mathcal{T}(l,L)}{N} + l^{-1}$$

$$+ C\beta\sqrt{\frac{\varepsilon \log l}{l}} - \frac{1}{4}l^{\zeta-1}(\log l)^{-2}\Big).$$

The terms $\frac{2\log N}{N}$ and $\frac{\log\#\mathcal{T}(l,L)}{N}$ can be neglected if l is sufficiently large (i.e., ε is sufficiently small) and $l^{\zeta-1}(\log l)^{-2}/4$ is replaced by $l^{\zeta-1}(\log l)^{-2}/8$. $\square$

5. Proof of Theorem 2.4: Rounding for $\zeta < 1/2$. The idea to find an upper bound on the free energy is somehow inspired by what is done in [23]. The main difference is that here, we must combine the argument with the finite volume criterion given by Lemma 3.1 to get a result. We use that ω is Gaussian in the following way.

LEMMA 5.1. *For any N if ω are IID Gaussian variables, then the sequence*

$$\Big(\omega_x - \frac{1}{N}\sum_{n=1}^N \omega_n\Big)_{x=1}^N$$

is independent of $\sum_{n=1}^N \omega_n$.

With this observation, we see that changing the value of h by an amount δ is in fact equivalent to changing the empirical mean of the ω by an amount $\delta\beta^{-1}$.

In a first step, we try to control the expectation of the free energy for a typical value of $\sum_{n=1}^{N}\omega_n$.

PROPOSITION 5.2.   *There exists a constant C such that for all N sufficiently large and all u,*

$$(5.1) \qquad \mathbb{E}\left[\log Z_N^{\beta,h_c(\beta),\omega}\ \Big|\ \sum_{n=1}^{N}\omega_n \ge u\sqrt{N}\right] \le CN^{\zeta}(1+|u|^{\zeta})e^{\zeta u^2/2}+\beta^2.$$

This will be done using the finite volume criterion of Lemma 3.1: if (5.1) does not hold, one can find a strategy which gives a positive-free energy for $h = h_c(\beta)$, and hence yields a contradiction. Then the idea is to integrate this bound over all values of u to obtain a bound for $\mathbb{E}[\log Z_N^{\beta,h,\omega}]$. Of course, the bound obtained in this manner is a good one only if N is wisely chosen. We can finally conclude using the finite volume criterion Lemma 3.1.

REMARK 5.3.   In our proof, we actually use variant version of the inequality (5.1) where the conditioning is $\sum_{n=1}^{N}\omega_n = u\sqrt{N}$. The inequality (5.1) remains of course valid in that case, by stochastic domination of one conditioned law by the other, and the fact that $\log Z_N^{\beta,h_c(\beta),\omega}$ is an increasing function of ω.

PROOF OF THEOREM 2.4.   Now for $h = h_c(\beta) + v$ one sets $N := \beta^2 v^{-2}$ (assuming that we have chosen v such that N is an integer). We have

$$\mathbb{E}[\log Z_N^{\beta,h,\omega}]$$

$$= \int \frac{1}{\sqrt{2\pi}}\exp(-u^2/2)\frac{1}{N}\mathbb{E}\left[\log Z_N^{\beta,h,\omega}\ \Big|\ \sum_{n=1}^{N}\omega_x = u\sqrt{N}\right]du$$

$$(5.2)$$

$$= \int \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(u-\beta^{-1}v\sqrt{N})^2}{2}\right)$$

$$\times \frac{1}{N}\mathbb{E}\left[\log Z_N^{\beta,h_c(\beta),\omega}\ \Big|\ \sum_{n=1}^{N}\omega_x = u\sqrt{N}\right]du.$$

Using Proposition 5.2, we have the following inequality provided that v is sufficiently small (in which case the β^2 can be neglected):

$$\mathbb{E}[\log Z_N^{\beta,h,\omega}]$$
$$(5.3)$$
$$\le CN^{\zeta}\int\frac{1}{\sqrt{2\pi}}(1+|u|^{\zeta})\exp\left(\frac{\zeta u^2-(u-1)^2}{2}\right)du \le C'N^{\zeta}.$$

Hence, using Lemma 3.1 (the lower-bound part), we obtain

$$(5.4) \qquad \textsc{f}(\beta, h) \leq C'' N^{\zeta - 1} = C'' (v\beta - 1)^{2(\zeta - 1)}. \qquad \square$$

PROOF OF PROPOSITION 5.2. One can assume $u \geq 1$ without loss of generality. Set

$$M := u \exp(u^2/2).$$

Let X_0 be the smallest integer such that

$$(5.5) \qquad \sum_{n = X_0 N + 1}^{(X_0 + 1)N} \omega_n \geq u\sqrt{N}.$$

Then we obtain a lower bound on Z_{NM} by deciding to visit the stretch $[X_0 N, (X_0 + 1)N]$ if $X_0 \leq M - 2$ and to do only a long excursion in the other case [recall (3.3)] (see Figure 2):

$$(5.6) \qquad Z_{MN}^{\omega} \geq K(X_0 N) Z_{[X_0 N, (X_0 + 1)N]}^{\omega} K\big((M - X_0 + 1)N\big) e^{\beta \omega_{NM} + h_c(\beta)}$$

if $X_0 \leq M - 2$, and

$$(5.7) \qquad Z_{MN}^{\omega} \geq K(MN) e^{\beta \omega_{NM} + h_c(\beta)},$$

when $X_0 \geq M - 1$. Taking the expectation one obtains, by translation invariance

$$(5.8) \qquad \begin{aligned} &\mathbb{E}\big[\log\big(K(X_0 N) Z_{[X_0 N, (X_0 + 1)N]}^{\omega} K\big((M - X_0 + 1)N\big)\big) \\ &\quad \mid X_0 \leq (M - 2)\big] \\ &\geq -2(MN)^{\zeta} + h_c(\beta) + \mathbb{E}\left[\log Z_N^{\beta, h_c(\beta), \omega} \,\middle|\, \sum_{n=1}^{N} \omega_x \geq u\sqrt{N}\right]. \end{aligned}$$
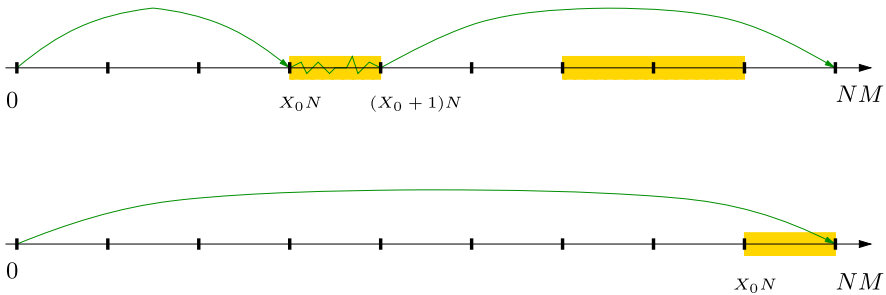

FIG. 2. *Here, we present our strategy to obtain a lower bound on the partition function Z_{NM}. The yellow segments are those which are such that the empirical mean of ω of $uN^{-1/2}$ and $[X_0 N, (X_0 + 1)N]$ is the first of them. We allow pinning only on the segment $[X_0 N, (X_0 + 1)N]$ and only if $X_0 \leq M - 2$.*

We also have (as ω_{MN} is independent of the event $\{X_0 \leq M - 2\}$ its conditional mean is zero)

(5.9)
$$\mathbb{E}\big[\log\big[K(MN)e^{\beta\omega_{NM}+h_c(\beta)}\big] \mid X_0 \geq M - 2\big]$$
$$= \log K(MN) + h_c(\beta).$$

And hence [recall that $h_c(\beta) \geq -\beta^2/2$ for Gaussian environments],

$$\mathbb{E}\big[\log Z_{MN}^\omega\big]$$

(5.10)
$$\geq \mathbb{E}\left[\log Z_N^{\beta,h_c(\beta),\omega} \,\bigg|\, \sum_{n=1}^N \omega_x \geq u\sqrt{N}\right]\mathbb{P}[X_0 \leq M - 2]$$
$$- 2(MN)^\zeta - \beta^2.$$

By standard estimates on Gaussian tails, there exists a constant $c > 0$ such that

$$\forall u > 1 \qquad \mathbb{P}\left[\sum_{n=1}^N \omega_x \geq u\sqrt{N}\right] \geq \frac{c}{u}e^{-u^2/2},$$

and hence, using the definition of M we have

$$\mathbb{P}[X_0 \leq M - 2] > c',$$

for some positive constant c'. This implies [recall Lemma 3.1 and that $\mathrm{F}(\beta, h_c(\beta)) = 0$] that there exists $c'' > 0$ such that

$$0 \geq \mathbb{E}\big[\log Z_{MN}^\omega\big]$$

(5.11)
$$\geq c'\left(\mathbb{E}\left[\log Z_N^{\beta,h_c(\beta),\omega} \,\bigg|\, \sum_{n=1}^N \omega_x \geq u\sqrt{N}\right]\right) - c''u^\zeta e^{\zeta u^2/2}N^\zeta - \beta^2.$$

The above inequality is in fact only valid if one assumes that

$$\mathbb{E}\left[\log Z_N^{\beta,h_c(\beta),\omega} \,\bigg|\, \sum_{n=1}^N \omega_x \geq u\sqrt{N}\right] \geq 0,$$

but if this is not the case there is nothing to prove. $\quad\square$

6. Proof of Theorem 2.5: Rounding for $\zeta = 1/2$.

The case for $\zeta = 1/2$ is a bit more complicated. Assume that

(6.1)
$$\lim_{h \to h_c(\beta)+} \partial_h \mathrm{F}(\beta, h) = c_0 > 0,$$

and let us derive a contradiction. First, we prove that the contact fraction at the critical point, if well defined, cannot be equal to c_0 as there is always a positive probability for the polymer to have a very small contact fraction.

LEMMA 6.1.   *The following three statements hold*:

(i) *For all $\varepsilon > 0$,*

$$(6.2) \qquad \limsup_{N \to \infty} \mathbb{E}[\mathbf{P}_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^\varepsilon)] < 1.$$

(ii) *For any $u > c_0$,*

$$(6.3) \qquad \lim_{N \to \infty} \mathbb{E}[\mathbf{P}_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^u)] = 0.$$

(iii) *We have*

$$(6.4) \qquad \limsup_{N \to \infty} \mathbb{E}\left[\mathbf{E}_N^{\beta, h_c(\beta), \omega}\left(\sum_{n=1}^N \delta_N\right)\right] < c_0.$$

PROOF.   Point (iii) is a simple consequence of the two first point as

$$(6.5) \qquad \mathbb{E}\left[\mathbf{E}_N^{\beta, h_c(\beta), \omega}\left(\sum_{n=1}^N \delta_N\right)\right] = \int_0^1 \mathbb{E}[\mathbf{P}_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^u)] \, du.$$

Point (ii) is rather easy to prove: Assume that for $u > c_0$ and for some $\delta > 0$ one has

$$(6.6) \qquad \mathbb{P}[\mathbf{P}_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^u) > \delta] > \delta,$$

for infinitely many N. We note that if

$$\mathbf{P}_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^u) > \delta,$$

then

$$(6.7) \qquad \begin{aligned} Z_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^u) &\geq \delta Z_N^{\beta, h_c(\beta), \omega} \\ &\geq \delta K(N) e^{\beta \omega_N + h_c(\beta)}, \end{aligned}$$

where the last inequality is just obtained by considering renewal trajectories with only one contact. Hence, for every $h > h_c(\beta)$ we have

$$(6.8) \qquad Z_N^{\beta, h, \omega} \geq Z_N^{\beta, h, \omega}(\mathcal{B}^u) \geq \delta e^{Nu(h - h_c)} K(N) e^{\beta \omega_N + h_c(\beta)}.$$

This implies (as we know that the limit exists and is nonrandom) that for every $h > h_c(\beta)$

$$(6.9) \qquad \lim_{N \to \infty} \frac{1}{N} \log Z_N^{\beta, h, \omega} \geq u(h - h_c(\beta))$$

which contradicts assumption (6.1) for small h.

To prove (i), let us assume that

$$(6.10) \qquad \lim_{N \to \infty} \mathbb{E}[\mathbf{P}_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^\varepsilon)] = 1,$$

(or that it occurs along a subsequence) and derive a contradiction from it. Set

$$(6.11) \qquad f_N(u) := \mathbb{E}\left[\mathbf{P}_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^\varepsilon) \,\Big|\, \sum_{x=1}^{N-1} \omega_x = u\sqrt{N-1}\right].$$

We have

$$(6.12) \qquad \mathbb{E}[\mathbf{P}_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^\varepsilon)] = \int \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) f_N(u) \, du.$$

As $f_N(u)$ is an increasing function of u this implies that for all $u \in \mathbb{R}$

$$(6.13) \qquad \lim_{N \to \infty} f_N(u) = 1.$$

Fix $u = -10\varepsilon^{-1}$ and let N be sufficiently large so that $f_N(u) \geq 3/4$. Then necessarily

$$(6.14) \qquad \mathbb{P}\left(\mathbf{P}_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^\varepsilon) \geq 1/2 \,\Big|\, \sum_{x=1}^{N-1} \omega_x = u\sqrt{N-1}\right) \geq 1/2.$$

Note that $\mathbf{P}_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^\varepsilon) \geq 1/2$ implies in particular that

$$Z_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^\varepsilon) \geq Z_N^{\beta, h_c(\beta), \omega}((\mathcal{B}^\varepsilon)^c) \geq K(N)e^{\beta\omega_N + h_c(\beta)},$$

and hence (6.14)

$$(6.15) \qquad \mathbb{P}\left(Z_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^\varepsilon) \geq K(N)e^{\beta\omega_N + h_c(\beta)} \,\Big|\, \sum_{x=1}^{N-1} \omega_x = u\sqrt{N-1}\right) \geq 1/2.$$

From Lemma 5.1, replacing u by v in the conditioning is equivalent to replacing ω_n by $\omega_n + (v - u)(N-1)^{-1/2}$ for $n \in \{1, \ldots, N-1\}$. Hence, for $v \geq u$ we have

$$(6.16) \qquad \mathbb{P}\left(Z_N^{\beta, h_c(\beta), \omega}(\mathcal{B}^\varepsilon) \geq e^{\varepsilon(v-u)\sqrt{N-1}}\zeta_N \,\Big|\, \sum_{x=1}^{N-1} \omega_x = v\sqrt{N-1}\right) \geq 1/2,$$

where

$$\zeta_N := K(N)e^{\beta\omega_N + h_c(\beta)}.$$

This implies that for any v (this is obvious for $v \leq u$)

$$(6.17) \qquad \mathbb{P}\left(Z_N^{\beta, h_c(\beta), \omega} \geq e^{\varepsilon(v-u)\sqrt{N-1}}\zeta_N \,\Big|\, \sum_{x=1}^{N-1} \omega_x = v\sqrt{N-1}\right) \geq 1/2.$$

Hence, using the obvious bound $Z_N^{\beta,h_c(\beta),\omega} \geq K(N)e^{\beta\omega_N + h_c(\beta)}$ one obtains

$$
\text{(6.18)} \quad
\begin{aligned}
&\mathbb{E}\left[\log Z_N^{\beta,h_c(\beta),\omega} \;\Big|\; \sum_{x=1}^{N-1} \omega_x = v\sqrt{N-1}\right] \\
&\qquad \geq \log K(N) + h_c(\beta) + \frac{1}{2}\varepsilon(v-u)\sqrt{N-1}.
\end{aligned}
$$

Hence, integrating over v one obtains (recall the value we have chosen for u)

$$
\text{(6.19)} \quad
\begin{aligned}
&\mathbb{E}\big[\log Z_N^{\beta,h_c(\beta),\omega}\big] \\
&\qquad \geq \log K(n) + h_c(\beta) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int \varepsilon(v-u)\sqrt{N-1}\,e^{-\frac{v^2}{2}}\,dv \\
&\qquad = \log K(n) + h_c(\beta) - \frac{\varepsilon u}{2}\sqrt{N-1} \\
&\qquad = \log K(N) + h_c(\beta) + 5\sqrt{N-1} > 0.
\end{aligned}
$$

This contradicts the fact that the free energy is zero. $\square$

Then we can conclude by exhibiting a finite volume bound similar to those of Lemma 3.1 for the free energy derivative.

LEMMA 6.2. *For K log-convex, for any N and h*

$$
\text{(6.20)} \quad \frac{1}{N}\mathbb{E}\left[\mathbf{E}_N^{\beta,h,\omega}\left(\sum_{n=1}^{N}\delta_N\right)\right] \geq \partial_h \mathrm{F}(\beta,h).
$$

PROOF. This is a simple consequence of the FKG inequality, as the number of contact is an increasing function. For $M \geq 1$, on has

$$
\text{(6.21)} \quad
\begin{aligned}
\mathbf{E}_{MN}^{\beta,h,\omega}\left[\sum_{n=1}^{NM}\delta_n\right] &\geq \mathbf{E}_{MN}^{\beta,h,\omega}\left[\sum_{n=1}^{NM}\delta_n \;\Big|\; \delta_{iN} = 1, \forall i \in \{1,\ldots,M-1\}\right] \\
&= \sum_{i=0}^{M}\mathbf{E}_N^{\beta,h,\theta^{iN}\omega}\left[\sum_{n=1}^{N}\delta_n\right]
\end{aligned}
$$

and hence taking the average

$$
\text{(6.22)} \quad \frac{1}{NM}\mathbb{E}\left[\mathbf{E}_{MN}^{\beta,h,\omega}\left[\sum_{n=1}^{NM}\delta_n\right]\right] \leq \frac{1}{N}\mathbb{E}\left[\mathbf{E}_N^{\beta,h,\omega}\left[\sum_{n=1}^{N}\delta_n\right]\right].
$$

The result follows by taking M to infinity. $\square$

PROOF OF THEOREM 2.5. For a fixed N,

$$h \mapsto \frac{1}{N} \mathbb{E}\left[\mathbf{E}_N^{\beta,h,\omega}\left[\sum_{n=1}^N \delta_n \right] \right]$$

is a continuous function. Hence from (6.1), one can find N sufficiently large and $h > h_c$ such that

$$(6.23) \qquad \frac{1}{N} \mathbb{E}\left[\mathbf{E}_N^{\beta,h,\omega}\left[\sum_{n=1}^N \delta_n \right] \right] < c_0.$$

By Lemma 6.2, this implies that $\partial_h F(\beta, h) < c_0$ which yields a contradiction. Hence, one must have a smooth transition. $\square$

REMARK 6.3. In fact, the proof in this section yields a nontrivial result for $\zeta < 1/2$: when K is log-convex we have

$$(6.24) \qquad \lim_{N \to \infty} \frac{1}{N} \mathbb{E}\left[\mathbf{E}_N^{\beta,h_c(\beta),\omega}\left[\sum_{n=1}^N \delta_n \right] \right] = \lim_{h \to h_c(\beta)+} \partial_h F(\beta, h).$$

In other words, the contact fraction at the critical point is equal to the right-derivative of the free energy.

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