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EQUIVALENCE OF ENSEMBLES FOR LARGE VEHICLE-SHARING MODELS

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For a class of large closed Jackson networks submitted to capacity constraints, asymptotic independence of the nodes in normal traffic phase is proved at stationarity under mild assumptions, using a local limit theorem. The limiting distributions of the queues are explicit. In the Statistical Mechanics terminology, the equivalence of ensembles—canonical and grand canonical—is proved for specific marginals. The framework includes the case of networks with two types of nodes: single server/finite capacity nodes and infinite servers/infinite capacity nodes, that can be taken as basic models for bike-sharing systems. The effect of local saturation is modeled by generalized blocking and rerouting procedures, under which the stationary state is proved to have product-form. The grand canonical approximation can then be used for adjusting the total number of bikes and the capacities of the stations to the expected demand.

1. Introduction. Many cities are now equipped with bike-sharing systems. Those have met a great success among the public, but they turn out to face some difficulty for adapting to the demand of users. One major inherent problem is due to the spatial heterogeneity of the demand. It results in the common observation that, during the day, some parking stations stay most of the time empty, while others keep many vehicles unused. Some sites may indeed be more popular for picking a vehicle than for returning it. This induces two possible drawbacks for users, namely, finding no vehicle for rent at their starting point, or no parking room at the end of their ride. Indeed, since parked vehicles are locked, each station has only a given number of docking places. This capacity limitation may be most problematic, making customers lose time in an additional run (and then possibly a walk back to their former destination) just for parking their vehicle. On the contrary, finding no vehicle at some station is more simply resolved—using alternative means of transport. A major concern, when designing such a network, is thus to find the best acceptable tradeoff, relating to the total number of offered vehicles, as function of the different capacities of the stations. A low supply of vehicles produces empty stations, while a large supply causes saturation.

The purpose of the present paper is to analyze large, inhomogeneous networks with finite capacity stations. The performance of such models is estimated through

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explicit asymptotics of their stationary state. Performance indicators such as the total rate of failure can then be optimized by adapting the flexible parameters (the total number of vehicles and, possibly, the sizes of the stations). Note, yet, that this may not be sufficient to reach a good quality of service. Ultimately, different devices should be implemented in order to limit service failure, or disparity between stations: For example, setting up a vehicle reservation service, or supplying empty stations with vehicles (moved from saturated stations), or else displaying online updates of the current state of the network (so that one knows where parking room or vehicles are available). In this regard, besides quantifying the weaknesses of the simplest primitive system, we introduce state-dependent routing procedures that may open the way to such improvements.

Time inhomogeneity is another concern: The behavior of users should vary in time, more or less obeying a 24 hours cycle. This issue will not be addressed in this paper, where all processes will be assumed to have constant traffic parameters and to have reached their stationary regime. In this respect, a major issue would be to estimate the relaxation time—or time to reach equilibrium—of such processes, and compare it with the duration of the constant parameter phase of interest.

It has been commonly observed that vehicle-sharing systems can be modeled by closed Jackson networks. In this description, the vehicles play the role of the customers—which number is fixed—while the users act as successive servers at the parking stations. The associated network has two types of nodes: one server nodes, that describe the stations, and infinite server nodes, representing the different routes between the stations. In [6], a detailed analysis of large, infinite capacity, closed Jackson networks at equilibrium is proposed. Application to vehicle sharing systems—at their early experimental stage in the 1990s—is briefly considered. The asymptotics of a model mentioned in [6] is examined in [10] as the number of nodes (stations and routes) is fixed, while the number of customers (vehicles) grows to infinity. Both papers crucially rely on the explicit product-form of the stationary distribution, which is standard in the infinite capacity case.

In [8], assuming complete homogeneity of the traffic and equal finite capacities of the stations, a mean-field asymptotic behavior is obtained for the dynamics of the distribution of vehicles over the different sites. Using the same method, [9] extends the study to systems where inhomogeneity is modeled by clusters. In addition, different alternatives are investigated, as for example, avoiding empty (resp., full) stations when taking (resp., returning) a bike, or returning bikes at the station having the largest available space, between two randomly chosen stations.

The present paper analyzes the impact of finite capacity on large inhomogeneous networks. The models considered thus involve real-world blocking and rerouting mechanisms that we implement on the infinite capacity model of [6] and [10]. The crucial product-forms of the stationary states are proved for a class of state-dependent routings that extends the classical setting of [5]. Asymptotics are obtained as both the number of vehicles and the size of the network increase.

Namely, it is proved that the finite dimensional distributions of the stations occupation process at stationarity converge to products of truncated geometric distributions. If the total number of bikes does not exceed some order of magnitude, stations and routes are moreover asymptotically independent, the latter being approximately distributed as Poisson variables. The approach, at stationarity, is similar to [6] and [10]. As compared to the dynamical point of view of [8] and [9], our results complete the picture of the equilibrium state there obtained for locally homogeneous systems.

Our asymptotics fit the frame of the so-called principle of equivalence of canonical and grand canonical ensembles, in the Statistical Mechanics terminology. The networks of interest are here in some "partially subcritical" phase, in the sense that part of the network—namely, the parking stations—is in normal traffic (or no-condensation) regime. This makes that a local limit theorem can be used. This way of deriving the grand canonical approximation is standard and can be traced back to Khinchin [13]. Dobrushin and Tirozzi [4] have used it for analyzing a family of Gibbs measures on $\mathbb{Z}^2$. It has, since then, proved useful in many different contexts, notably for exclusion and zero-range processes under thermodynamic limit ([14], Appendix 2). Note that zero-range processes are identical to (homogeneous) closed Jackson networks, but due to homogeneity, condensation can only occur in specific attractive cases, which are not considered in [14]. Such supercritical phases are described in [12] and [1]. The generalized Jackson networks with blocking considered in Section 4 contain as particular cases the generalized exclusion processes analyzed in [14].

Contrary to the classical setting of the equivalence of ensembles—or its alternative formulation as Gibbs conditioning principle [3]—the equilibrium states of general closed Jackson networks are nonsymmetrical with respect to the nodes. Malyshev and Yakovlev [15] have used analytical methods to address the problem for classical (one-server nodes) closed Jackson networks, under existence of a limiting density of customers—as in [14]—and of a limiting profile, capturing inhomogeneity, for the traffic parameters. The critical phenomenon of condensation is highlighted and both phases are studied. These results are extended in [6], using local limit theorems, which is more flexible and does not require specification of a limiting density nor parameter profile. Note yet that this method does not lead to a complete and unified treatment of supercritical regimes, for which restrictive assumptions are needed.

We use the same approach and for simplicity, restrict the analysis to the relevant dynamics in respect to the targeted vehicle-sharing applications. Though our interest focuses on finite capacity systems, the infinite capacity case—that could equally be derived from [6]—is included, as a reference and for completeness. Similarly, several standard results (as regards the local limit theorem, or generalized Jackson networks) are here stated, under specific assumptions adapted to the present context, for the sake of self-containment.

The paper is organized as follows. Section 2 displays the central limit and local limit theorems—for independent, nonidentically distributed, random variables—that will be used in the sequel. Section 3 introduces the generalized Jackson processes of interest and states the product-forms of the associated stationary distributions. In Section 4, the equivalence of ensembles is proved for two classes of networks (with respectively infinite and finite capacities). Applications to evaluating the performance of bike-sharing systems is finally developed in Section 5.

2. A local limit theorem. It is classical that the central limit theorem holds for any sequence $(S_N)_{N\geq 1}$ of sums of independent square integrable random variables $(X_{j,N})_{1\leq j\leq J(N)}$

$$S_N = \sum_{j=1}^{J(N)} X_{j,N}$$

if the family $(X_{j,N})_{1\leq j\leq J(N)}$ satisfies the following so-called *Lyapunov condition*:

$$\exists \delta > 0 \quad \text{such that} \quad \lim_{N\to\infty} \frac{1}{b_N^{2+\delta}} \sum_{j=1}^{J(N)} \mathbb{E}\big(|X_{j,N} - m_{j,N}|^{2+\delta}\big) = 0. \tag{2.1}$$

Here and in the sequel, for all $N \geq 1$, $J(N)$ is some integer such that $J(N) \geq 1$, and we use the following notation:

$$m_{j,N} = \mathbb{E}(X_{j,N}) \qquad \big(1 \leq j \leq J(N)\big), \qquad a_N = \mathbb{E}(S_N) = \sum_{j=1}^{J(N)} m_{j,N} \quad \text{and}$$

$$\sigma_{j,N}^2 = \mathbb{E}\big([X_{j,N} - m_{j,N}]^2\big), \qquad b_N^2 = \mathbb{E}\big([S_N - a_N]^2\big) = \sum_{j=1}^{J(N)} \sigma_{j,N}^2.$$

This result is usually derived from the Lindeberg central limit theorem, which states that under the following *Lindeberg condition* (2.2), which is weaker than (2.1), the characteristic function (or Fourier transform) of $b_N^{-1}(S_N - a_N)$ converges to that of the standard normal distribution (see [2], or [7]):

$$\forall \varepsilon > 0, \qquad \lim_{N\to\infty} \frac{1}{b_N^2} \sum_{j=1}^{J(N)} \mathbb{E}\big((X_{j,N} - m_{j,N})^2 \cdot \mathbf{1}_{|X_{j,N} - m_{j,N}| > \varepsilon b_N}\big) = 0. \tag{2.2}$$

It is easily seen that, under the Lyapunov condition, the convergence is uniform on some sequence of intervals $[-A_N, A_N]$ which length grows to infinity. This refinement of the central limit theorem is stated below as *Lyapunov central limit theorem*. It will be used for deriving the next *local limit theorem* that is the key to the equivalence of ensembles.

The proofs of both theorems, as well as those of the two next propositions are deferred to the Appendix.

Let $(X_{j,N})_{N\geq 1, 1\leq j\leq J(N)}$ be a family of square integrable random variables such that, for all $N \geq 1$, $X_{1,N}, \dots, X_{J(N),N}$ are independent. Then define S_N, a_N and b_N as above.

THEOREM 2.1 (Lyapunov central limit theorem). *If the Lyapunov condition* (2.1) *is satisfied*, *then the following convergence holds*:

$$\lim_{N\to\infty} \sup_{|t|\leq A_N} \left|\mathbb{E}\left(e^{it\frac{S_N - a_N}{b_N}}\right) - e^{-t^2/2}\right| = 0, \tag{2.3}$$

for some sequence (A_N) *of positive real numbers converging to infinity.*

All the models of interest in this paper will satisfy the Lyapunov condition with $\delta = 1$.

Deriving of the local limit theorem from the central limit theorem for integer valued random variables has become standard since the paper by Gnedenko [11], addressing the case of i.i.d. variables. We here give a version adapted to our context.

THEOREM 2.2 (Local limit theorem). *Assume that the random variables* $X_{j,N}$ *are integer* $\mathbb{Z}$*-valued and that the following conditions hold*:

1. $\lim_{N\to\infty} b_N = +\infty$,
2. *there exists some sequence* (A_N) *of positive real numbers converging to infinity such that* (2.3) *holds*,
3. *there exists some* $\phi \in L^1(\mathbb{R})$ *such that for all* $N \geq 1$ *and all* $t \in [-\pi, \pi]$,

$$\left|\mathbb{E}\left(e^{itS_N}\right)\right| \leq \phi(b_N t).$$

Then the following is true:

$$\lim_{N\to\infty} \sup_{k\in\mathbb{Z}} \left[b_N \sqrt{2\pi}\,\mathbb{P}(S_N = k) - \exp\left(-\frac{(k-a_N)^2}{2b_N^2}\right)\right] = 0. \tag{2.4}$$

We will now consider two examples of interest for our applications of Section 4. Conditions (i) of the following Propositions 2.1 and 2.2 will there mean that the networks are subcritical as concerns their one-server nodes.

EXAMPLE 1. Suppose here that, for all N, the set of indices $\{1, \dots, J(N)\}$ is the union of two disjoint sets $\mathcal{J}_N^1$ and $\mathcal{J}_N^2$, such that:

- for $j \in \mathcal{J}_N^1$, the random variable $X_{j,N}$ is geometric with parameter $\rho_{j,N} \in [0, 1[$,

$$\text{that is,} \qquad \mathbb{P}(X_{j,N} = k) = (1-\rho_{j,N})\rho_{j,N}^k \qquad \text{for } k \in \mathbb{N},$$

- for $j \in \mathcal{J}_N^2$, the random variable $X_{j,N}$ is Poisson with parameter $\lambda_{j,N} > 0$,

$$\text{that is,} \qquad \mathbb{P}(X_{j,N} = k) = e^{-\lambda_{j,N}} \frac{\lambda_{j,N}^k}{k!} \qquad \text{for } k \in \mathbb{N}.$$

The means and variances of the $X_{j,N}$'s and S_N's are then given by

$$m_{j,N} = \begin{cases} \dfrac{\rho_{j,N}}{1-\rho_{j,N}} & \text{for } j \in \mathcal{J}_1^N, \\ \lambda_{j,N} & \text{for } j \in \mathcal{J}_2^N, \end{cases}$$

$$\sigma_{j,N}^2 = \begin{cases} \dfrac{\rho_{j,N}}{(1-\rho_{j,N})^2} & \text{for } j \in \mathcal{J}_1^N, \\ \lambda_{j,N} & \text{for } j \in \mathcal{J}_2^N, \end{cases}$$

$$a_N = \sum_{j \in \mathcal{J}_1^N} \frac{\rho_{j,N}}{1-\rho_{j,N}} + \sum_{j \in \mathcal{J}_2^N} \lambda_{j,N} \quad \text{and}$$

$$b_N^2 = \sum_{j \in \mathcal{J}_1^N} \frac{\rho_{j,N}}{(1-\rho_{j,N})^2} + \sum_{j \in \mathcal{J}_2^N} \lambda_{j,N}.$$

PROPOSITION 2.1. *If the two following conditions are satisfied*:

(i) *there exists some* $\rho < 1$ *such that for all* $N \geq 1$ *and* $j \in \mathcal{J}_N^1$, $\rho_{j,N} \leq \rho$,
(ii) $\lim_{N\to\infty} b_N = +\infty$,

then (2.4) *holds*.

REMARK 2.1. (i) and (ii) of Proposition 2.1 are not necessary conditions: The local limit theorem can still hold beyond the subcritical phase. It can be proved for example that if the number of indices $j \in \mathcal{J}_1^N$ such that $\rho_{j,N} = \max_{j \in \mathcal{J}_1^N} \rho_{j,N}$ goes to infinity and $\liminf \max_{j \in \mathcal{J}_1^N} \rho_{j,N} > 0$, then equations (2.3) and (2.4) hold.

EXAMPLE 2. Suppose now that, for all N, the set $\{1, \ldots, J(N)\}$ is the union of two disjoint sets $\mathcal{J}_N^1$ and $\mathcal{J}_N^2$, such that:

- for $j \in \mathcal{J}_N^1$, the random variable $X_{j,N}$ has truncated geometric distribution with parameters $\rho_{j,N} \geq 0$ and $c_{j,N} \in \mathbb{N}$, that is, for $k \in \{0, \ldots, c_{j,N}\}$,

$$\mathbb{P}(X_{j,N} = k) = \frac{\rho_{j,N}^k}{\sum_{h=0}^{c_{j,N}} \rho_{j,N}^h},$$

- for $j \in \mathcal{J}_N^2$, the random variable $X_{j,N}$ is Poisson with parameter $\lambda_{j,N} > 0$ as in Example 1.

The means and variances of S_N's are here given by

$$a_N = \sum_{j \in \mathcal{J}_1^N} m_{j,N} + \sum_{j \in \mathcal{J}_2^N} \lambda_{j,N} \quad \text{and} \quad b_N^2 = \sum_{j \in \mathcal{J}_1^N} \sigma_{j,N}^2 + \sum_{j \in \mathcal{J}_2^N} \lambda_{j,N},$$

where for $j \in \mathcal{J}_1^N$,

$$m_{j,N} = \begin{cases} \dfrac{\rho_{j,N}}{1-\rho_{j,N}} - (c_{j,N}+1)\dfrac{\rho_{j,N}^{1+c_{j,N}}}{1-\rho_{j,N}^{1+c_{j,N}}} & \text{if } \rho_{j,N} \neq 1, \\ \dfrac{c_{j,N}}{2} & \text{if } \rho_{j,N} = 1 \end{cases} \quad \text{and}$$

$$\sigma_{j,N}^2 = \begin{cases} \dfrac{\rho_{j,N}}{(1-\rho_{j,N})^2} - (c_{j,N}+1)^2\dfrac{\rho_{j,N}^{1+c_{j,N}}}{(1-\rho_{j,N}^{1+c_{j,N}})^2} & \text{if } \rho_{j,N} \neq 1, \\ \dfrac{c_{j,N}(c_{j,N}+2)}{12} & \text{if } \rho_{j,N} = 1. \end{cases}$$

PROPOSITION 2.2. *If the two following conditions are satisfied*:

(i) *there exists some* $C < +\infty$ *such that for all* $N \geq 1$ *and* $j \in \mathcal{J}_N^1$, $c_{j,N} \leq C$,
(ii) $\lim_{N\to\infty} b_N = +\infty$,

then (2.4) *holds*.

The proof of this proposition (see the Appendix) requires uniform domination of the characteristic functions of truncated geometric variables. This is stated as Lemma 2.2, which itself relies on the following lemma, due to Gnedenko [11], pages 192–193.

LEMMA 2.1 (Gnedenko). *Let* X *be an integer-valued random variable. Denote* $p_n = P(X = n)$ *for* $n \in \mathbb{Z}$ *and*

$$s = \sum_{l \in \mathbb{Z}} \frac{p_{2l}\, p_{2l+1}}{p_{2l} + p_{2l+1}},$$

with the convention that $\frac{p_{2l} p_{2l+1}}{p_{2l}+p_{2l+1}} = 0$ *if* $p_{2l} = p_{2l+1} = 0$. *Then the following inequality holds*:

$$\left|\mathbb{E}(e^{itX})\right| \leq e^{-\frac{2}{\pi^2} s t^2} \qquad \text{for } |t| \leq \pi. \tag{2.5}$$

Now for any $\rho > 0$ and $c \in \mathbb{N}$, let $m_{\rho,c}$ and $\sigma_{\rho,c}^2$ denote the mean and variance of a truncated geometric random variable $X_{\rho,c}$ with parameters ρ and c, and let $s_{\rho,c}$ denote the associated parameter as defined in Lemma 2.1. The following lemma is easily derived from Lemma 2.1.

LEMMA 2.2. *For any $C \geq 1$, there exists some $\kappa > 0$ such that for any $\rho > 0$ and any $c \in \{0, \dots, C\}$,*

$$\left|\mathbb{E}(e^{itX_{\rho,c}})\right| \leq e^{-\frac{2}{\pi^2}\kappa\sigma_{\rho,c}^2 t^2} \qquad \textit{for } |t| \leq \pi.$$

PROOF. It is enough to consider the case when $0 < \rho \leq 1$, since $\sigma_{\rho^{-1},c}^2 = \sigma_{\rho,c}^2$ and

$$\left|\mathbb{E}(e^{itX_{\rho,c}})\right| = \left|\mathbb{E}(e^{it(c-X_{\rho^{-1},c})})\right| = \left|\mathbb{E}(e^{-itX_{\rho^{-1},c}})\right|$$

due to the following elementary remark.

REMARK 2.2. For all $c \in \mathbb{N}$ and $\rho \in]0, +\infty[$, the random variable $X_{\rho,c}$ has the same distribution as $c - X_{\rho^{-1},c}$. As a consequence, the following symmetries hold for any $c \in \mathbb{N}$ and $\rho > 0$:

$$m_{\rho^{-1},c} = c - m_{\rho,c} \quad \text{and} \quad \sigma_{\rho^{-1},c}^2 = \sigma_{\rho,c}^2.$$

This induces a duality property between occupied and empty rooms in the finite capacity systems to come, known as the *particles-holes duality* (see [14]).

So, it is now enough to prove existence of some positive κ such that

$$s_{\rho,c} \geq \kappa\sigma_{\rho,c}^2$$

for all $c \in \{1, \dots, C\}$ and $0 < \rho \leq 1$. (The case $c = 0$ needs not be considered, since $s_{\rho,0} = \sigma_{\rho,0}^2 = 0$ for all ρ.)

It is clear that for fixed $c \geq 1$, both mappings $\rho \mapsto s_{\rho,c}$ and $\rho \mapsto \sigma_{\rho,c}^2$ are positive valued and continuous on the interval $]0, 1]$. It is then enough to prove that, for $c = 1, \dots, C$,

$$\liminf_{\rho\to 0} s_{\rho,c}/\sigma_{\rho,c}^2 > 0.$$

Now from the definition of $s_{\rho,c}$, one gets (using $\rho \leq 1$ for the second inequality)

$$\begin{aligned} s_{\rho,c} &= \left(\sum_{l=0}^{c}\rho^l\right)^{-1}\left(\sum_{0\leq 2l\leq c-1}\frac{\rho^{4l+1}}{\rho^{2l}+\rho^{2l+1}}\right) \\ &\geq \frac{\rho}{2}\left(\sum_{0\leq 2l\leq c-1}\rho^{2l}\right)\left(\sum_{l=0}^{c}\rho^l\right)^{-1} \\ &\geq \frac{\rho}{2}\left(\sum_{l=0}^{c}\rho^l\right)^{-1}. \end{aligned}$$

Together with $\sigma_{\rho,c}^2 = \frac{\rho}{(1-\rho)^2} - (c+1)^2\frac{\rho^{c+1}}{(1-\rho^{c+1})^2}$ if $\rho < 1$, we obtain

$$\liminf_{\rho\to 0}\frac{s_{\rho,c}}{\sigma_{\rho,c}^2} \geq \liminf_{\rho\to 0}\left[2\left(\sum_{l=0}^{c}\rho^l\right)\left(\frac{1}{(1-\rho)^2} - (c+1)^2\frac{\rho^c}{(1-\rho^{c+1})^2}\right)\right]^{-1} = \frac{1}{2}. \qquad \square$$

3. Generalized closed Jackson networks. In accordance with the terminology of Serfozo [16], a standard Jackson network consists of a finite number of nodes at which customers are successively served, according to possibly different procedures. Nodes operate independently, which means that the rate at which customers leave a node only depends on the current occupation of this node. When a customer completes his service at some node, he instantly moves to another node for another service. New nodes are chosen at random according to a fixed routing matrix. All service processes and routings are independent. For a *closed* Jackson network, which is the case considered in this paper, customers stay forever in the system and there are no external arrivals.

In this description all nodes have unlimited capacity. This dynamics can be generalized to a system including nodes with finite capacities. But the transition rule needs then be state dependent (constrained by the current location of the free space). It can no longer be given by some fixed matrix. Different blocking or rerouting policies can be considered.

3.1. *Infinite capacity model.* First, consider the case when all node capacities are infinite. Denote by N the number of nodes, by M the fixed number of customers and by $P = (p_{ij})_{1\leq i,j\leq N}$ the routing matrix (i.e., p_{ij} is the probability that a customer leaving node i moves to node j). P is assumed irreducible, which ensures uniqueness of its associated invariant distribution $\theta = (\theta_1, \dots, \theta_N)$ on $\{1, \dots, N\}$. For each node i, the departure rate of customers from node i is $g_i(k)$ when k customers are present at i. Here, g_i is some function defined on $\mathbb{N}$ such that $g_i(0) = 0$ and $g_i(k) > 0$ for $k \geq 1$.

The node-occupation process is Markov with state space

$$\mathcal{S}_{N,M} = \left\{(n_1, \dots, n_N) \in \mathbb{N}^N \text{ such that } \sum_{j=1}^{N} n_j = M\right\},$$

and transitions

$$n \longrightarrow n - e_i + e_j \qquad \text{at rate } q(n, n - e_i + e_j) = g_i(n_i)p_{ij},$$

for $n \in \mathcal{S}_{N,M}$ and $1 \leq i, j \leq N$. Here e_i denotes the ith unit vector.

A well-known and remarkable feature of this class of processes is that the joint stationary distribution of the queues lengths $\xi_1, \dots, \xi_N$ is explicitly known and has product form, given by

$$\mathbb{P}(\xi_1 = n_1, \dots, \xi_N = n_N) = \frac{1}{Z} \prod_{j=1}^{N} \frac{\theta_j^{n_j}}{(g_j!)(n_j)} \qquad \text{for } (n_1, \dots, n_N) \in \mathcal{S}_{N,M}, \tag{3.1}$$

where Z is a normalizing constant and for $j = 1, \dots, N$, the function $(g_j!)$ is defined on $\mathbb{N}$ by

$$(g_j!)(0) = 1 \quad \text{and} \quad (g_j!)(n) = \prod_{k=1}^{n} g_j(k) \qquad \text{for } n \geq 1.$$

3.2. *Finite capacity models.* If in the contrary some nodes have finite capacities, denote c_i the capacity of node i ($0 \le c_i \le +\infty$ for $i = 1, \dots, N$) and c the vector $(c_1, \dots, c_N)$. The state space is now changed for

$$\mathcal{S}^c_{N,M} = \left\{(n_1, \dots, n_N) \in \mathbb{N}^N : \sum_{j=1}^N n_j = M \text{ and } n_j \le c_j \text{ for } c_j < +\infty\right\},$$

which is nonempty provided that $\sum_{j=1}^N c_j \ge M$. Several dynamics are considered.

1. Model with blocking.

The dynamics is the same as in the previous infinite capacity case, except that when node j is at capacity, any move from some node i to node j is canceled (i.e., the customer involved renews service at node i).

In other terms, the transitions are given by

$$n \longrightarrow n - e_i + e_j \qquad \text{at rate } q(n, n - e_i + e_j) = g_i(n_i) p_{ij} \mathbf{1}_{n_j < c_j},$$

for $n \in \mathcal{S}^c_{N,M}$ and $1 \le i, j \le N$.

The infinite capacity model can be seen as a particular case of this finite capacity model with blocking. But the explicit formula (3.1) for the invariant distribution does not extend, in general, to the finite capacity model with blocking. It is the case, yet, if the routing matrix P is *reversible* with respect to its invariant measure θ, that is, if

$$\theta_i p_{ij} = \theta_j p_{ji} \qquad \text{for } 1 \le i, j \le N. \tag{3.2}$$

If (3.2) is satisfied, then (3.1) still holds here for $(n_1, \dots, n_N) \in \mathcal{S}^c_{N,M}$ (see [16]).

2. Model with blocking and rerouting.

This is a particular case of a dynamics introduced by Economou and Fakinos [5]. Here, any customer routed—according to the matrix P—to some saturated node will move instantly (i.e., at infinite speed) across the system, still according to P, up to find some nonsaturated node where to settle for a new service. Note that this customer will certainly find such a node, possibly the actual node he has just leaved (where one unit of capacity is thus available). Of course, in this case, no transition occurs.

The corresponding process has the following transitions and rates:

$$n \longrightarrow n - e_i + e_j \qquad \text{at rate } q(n, n - e_i + e_j) = g_i(n_i) p^*_{ij}(n), \tag{3.3}$$

for $n \in \mathcal{S}^c_{N,M}$ and $1 \le i \ne j \le N$, where $p^*_{ij}(n)$ denotes the probability that a Markov chain with transition matrix P initiated at i enters the set $A_i(n)$ at j, for

$$A_i(n) \overset{\text{def}}{=} \{j, 1 \le j \le N, j = i \text{ or } n_j < c_j\}.$$

Contrary to the simple blocking case where P has to be assumed reversible, here the product form (3.1) holds *without any restriction on the routing matrix P*. This dynamics thus appears as the appropriate generalization of the standard—infinite capacity—Jackson dynamics [in which case, $p^*_{ij}(n) = p_{ij}$ for all n, i, j].

THEOREM 3.1 (Economou and Fakinos). *For any irreducible routing matrix P, the Markov process with state space $S^c_{N,M}$ and transition rates given by* (3.3) *has invariant distribution given by* (3.1) *for* $(n_1, \dots, n_N) \in \mathcal{S}^c_{N,M}$.

3. Other state-dependent routings.

The scheme introduced in [5] is slightly more general than the blocking and rerouting dynamics just described. In the general setting of [5], the set $A_i(n)$—of nodes that are below capacity when some customer has just left node i—is split into disjoints blocks. A partition is thus associated with each configuration $n - e_i$ of $M - 1$ customers over the N nodes. The just served customer at i still explores the network at infinite speed according to some Markov chain with transition P, but settles at the first visited node that *belongs to the same block* $B_{n-e_i}(i)$ as i. The same product form stationary distribution then holds, given by (3.1).

This result can be generalized to a larger class of state-dependent routings. We consider the following transitions and rates:

$$
\begin{aligned}
&n \longrightarrow n - e_i + e_j \\
&\qquad \text{at rate } q(n, n - e_i + e_j) = g_i(n_i) p_{ij}(n - e_i) \mathbf{1}_{n_i > 0, n_j < c_j},
\end{aligned}
\tag{3.4}
$$

for $n \in \mathcal{S}^c_{N,M}$ and $1 \le i \ne j \le N$, where a transition matrix $P(m) = (p_{ij}(m))_{i,j \in A(m)}$ on the set

$$
A(m) \stackrel{\text{def}}{=} \{j, 1 \le j \le N, m_j < c_j\}
$$

is associated with each $m \in \mathcal{S}^c_{N,M-1}$.

THEOREM 3.2. *Assume that some positive vector $\theta = (\theta_1, \dots, \theta_N)$ is such that*

$$
\forall m \in \mathcal{S}^c_{N,M-1}, \forall j \in \{1, \dots, N\}, \qquad \sum_{i \in A(m)} \theta_i p_{ij}(m) = \theta_j.
$$

Then the Markov process with state space $\mathcal{S}^c_{N,M}$ and transition rates given by (3.4) *has invariant distribution given by* (3.1) *for* $(n_1, \dots, n_N) \in \mathcal{S}^c_{N,M}$.

The dynamics considered in [5] are of the form described in (3.4), with line i of matrix $P(m)$ then given, for all $m \in \mathcal{S}^c_{N,M-1}$ and $i \in A(m)$, by the distribution of the return point to $B_{n-e_i}(i)$ of a Markov chain with transition matrix P initiated at i. The condition of Theorem 3.2 is then satisfied for the invariant vector θ of P, due to the well-known following result: For any ergodic Markov chain on $\{1, \dots, N\}$ with invariant vector $(\theta_i)_{1 \le i \le N}$, and for any subset B of $\{1, \dots, N\}$, the embedded Markov chain at times of visits to B has invariant vector $(\theta_i)_{i \in B}$.

PROOF OF THEOREM 3.2. We check that the global balance equations

$$\begin{aligned}
&\sum_{\substack{1\le i\ne j\le N\\ n_i>0,n_j<c_j}} g_i(n_i)p_{ij}(n-e_i)\\
(3.5)\qquad &\quad = \sum_{\substack{1\le i\ne j\le N,\\ n_i>0,n_j<c_j}} g_j(n_j+1)p_{ji}(n-e_i)\frac{\pi(n-e_i+e_j)}{\pi(n)},
\end{aligned}$$

are satisfied for $n\in\mathcal{S}^c_{N,M}$, where $\pi(n)=Z^{-1}\prod_{j=1}^N \theta_j^{n_j}/(g_j!)(n_j)$.

Recall that $g_i(n_i)=0$ if $n_i=0$, so that the left-hand side of (3.5) is equal to

$$\sum_{i=1}^N g_i(n_i)\bigl(1-p_{ii}(n-e_i)\bigr),$$

while, using $\pi(n-e_i+e_j)/\pi(n)=\theta_j g_i(n_i)/(\theta_i g_j(n_j+1))$, the right-hand side rewrites

$$\sum_{\substack{1\le i\ne j\le N\\ n_j<c_j}} \frac{\theta_j}{\theta_i} g_i(n_i)p_{ji}(n-e_i)=\sum_{i=1}^N \frac{g_i(n_i)}{\theta_i}\sum_{j\ne i,n_j<c_j}\theta_j p_{ji}(n-e_i).$$

Now equation (3.5) results from the following equality: for all $n\in\mathcal{S}^c_{N,M}$ and $i\in\{1,\dots,N\}$ such that $n_i>0$,

$$\theta_i\bigl(1-p_{ii}(n-e_i)\bigr)=\sum_{j\ne i,\, j\in A(n-e_i)}\theta_j p_{ji}(n-e_i),$$

or equivalently [adding $\theta_i p_{ii}(n-e_i)$ to both members]

$$\theta_i=\sum_{j\in A(n-e_i)}\theta_j p_{ji}(n-e_i),$$

which is satisfied by assumption on vector θ. $\square$

REMARK 3.1. It will next be crucial to note that due to the particular form of the constraint $\sum_{j=1}^N n_j=M$, the expression in (3.1), for $(n_1,\dots,n_N)$ in $\mathcal{S}_{N,M}$ or $\mathcal{S}^c_{N,M}$, is unchanged if the invariant vector θ is replaced by some proportional vector $\gamma\theta=(\gamma\theta_1,\dots,\gamma\theta_N)$, where $\gamma>0$ is arbitrary.

4. Equivalence of ensembles for generalized Jackson networks. The product formula (3.1) can be interpreted as follows: the invariant joint distribution of the N queues is the distribution of N independent random variables conditioned to the constraint that their sum is M.

Moreover, for *arbitrary positive* γ, Remark 3.1 allows one to choose the individual distributions of these independent variables, denoted $\eta_1^\gamma, \dots, \eta_N^\gamma$, as given by

$$\mathbb{P}(\eta_j^\gamma = n) = \frac{1}{Z_j(\gamma)} \frac{(\gamma\theta_j)^n}{(g_j!)(n)} \qquad \text{for } 0 \le n < c_j + 1, j = 1, \dots, N, \tag{4.1}$$

provided that γ belongs to the domain of convergence of each power series

$$Z_j(\gamma) = \sum_{n=0}^{c_j} \frac{\theta_j^n}{(g_j!)(n)} \gamma^n \qquad \text{for } j = 1, \dots, N. \tag{4.2}$$

For each such γ, (3.1) rewrites: for $(n_1, \dots, n_N) \in \mathcal{S}_{N,M}$,

$$\mathbb{P}(\xi_1 = n_1, \dots, \xi_N = n_N) = \mathbb{P}\left(\eta_1^\gamma = n_1, \dots, \eta_N^\gamma = n_N \,\Big|\, \sum_{j=1}^N \eta_j^\gamma = M\right). \tag{4.3}$$

In the physical terminology, the distribution in (3.1) is known as the *canonical ensemble*, while the product of the N distributions in (4.1) is the *grand canonical ensemble*. The factor γ is the so-called *chemical potential*.

If it holds, the *principle of equivalence* of canonical and grand canonical ensembles tells that if γ is rightly chosen, namely, such that for η_j^γ's as in (4.1),

$$\mathbb{E}\left(\sum_{j=1}^N \eta_j^\gamma\right) = M, \tag{4.4}$$

then the finite dimensional marginals of the distribution (3.1) are well approximated, for large N and M, by products of distributions in (4.1). In other terms, for N and M large, one can forget the conditioning $\eta_1^\gamma + \cdots + \eta_N^\gamma = M$ provided that γ is rightly tuned, so that the total mean of the N free variables η_j^γ's is M.

The following lemma proves strict monotonicity of the left-hand side in (4.4), ensuring uniqueness of γ solving this equation. Existence of such a γ holds, by continuity, provided that for at least one j [such that, necessarily, $Z_j(\gamma)$ has radius of convergence γ^*]

$$\lim_{\gamma \to \gamma^*} \mathbb{E}(\eta_j^\gamma) = +\infty,$$

where γ^* is the minimal radius of convergence of all the power series $Z_j(\gamma)$.

LEMMA 4.1. *Let $(X^\gamma)_{0<\gamma<\gamma^*}$ be a family of $\mathbb{N}$-valued random variables with distributions*

$$\mathbb{P}(X^\gamma = n) = \frac{\gamma^n \phi(n)}{Z(\gamma)} \qquad \text{for } n \in \mathbb{N},$$

where ϕ is any nonnegative function on $\mathbb{N}$ that is nonzero for at least two points,

$$Z(\gamma) = \sum_{n=0}^{\infty} \gamma^n \phi(n),$$

and γ^* is the radius of convergence of the series $Z(\gamma)$. Then $\mathbb{E}(X^\gamma) < +\infty$ for all $\gamma \in]0, \gamma^*[$ and the mapping $\gamma \in]0, \gamma^*[\mapsto \mathbb{E}(X^\gamma)$ is increasing.

PROOF. Both series $\sum_{n=0}^{\infty} n\gamma^n \phi(n)$ and $\sum_{n=0}^{\infty} n^2 \gamma^n \phi(n)$ have same radius of convergence γ^* as $Z(\gamma)$, and $\mathbb{E}(X^\gamma)$ is given for all $\gamma \in [0, \gamma^*[$ by

$$\mathbb{E}(X^\gamma) = \frac{\sum_{n=0}^{\infty} n\gamma^n \phi(n)}{Z(\gamma)}.$$

Differentiating with respect to $\gamma \in]0, \gamma^*[$ gives

$$\begin{aligned}\frac{\partial \mathbb{E}(X^\gamma)}{\partial \gamma} &= \gamma^{-1}\left(\frac{\sum_{n=0}^{\infty} n^2\gamma^n \phi(n)}{Z(\gamma)} - \left(\frac{\sum_{n=0}^{\infty} n\gamma^n \phi(n)}{Z(\gamma)}\right)^2\right) \\ &= \gamma^{-1} \cdot \operatorname{Var} X^\gamma > 0\end{aligned}$$

from assumption that $\phi(n) > 0$ for at least two values of n, ensuring that all variables X^γ for $0 < \gamma < \gamma^*$ are non-a.s. constant. □

From now on, we will denote $\eta_1, \dots, \eta_N$ the variables $\eta_1^\gamma, \dots, \eta_N^\gamma$ associated with the γ solving (4.4), if it exists. They will be referred as the "free variables".

The local limit theorem is a classical tool for proving that equivalence of ensembles holds [4, 6, 13, 14]. One can assume without loss of generality that the finite dimensional distribution of interest is that of $(\xi_1, \dots, \xi_K)$ for some $K \geq 1$, and write for any $(n_1, \dots, n_K) \in \mathbb{N}^K$

$$\begin{aligned}&\mathbb{P}(\xi_1 = n_1, \dots, \xi_K = n_K) \\ &\quad = \mathbb{P}\left(\eta_1 = n_1, \dots, \eta_K = n_K \,\Big|\, \sum_{j=1}^{N} \eta_j = M\right) \\ &\quad = \mathbb{P}(\eta_1 = n_1, \dots, \eta_K = n_K) \frac{\mathbb{P}(\sum_{j=K+1}^{N} \eta_j = M - \sum_{j=1}^{K} n_j)}{\mathbb{P}(\sum_{j=1}^{N} \eta_j = M)},\end{aligned} \tag{4.5}$$

where the last equality results from independence of the η_j's.

If the local limit theorem holds for both families of variables $(\eta_j, 1 \leq j \leq N)$ and $(\eta_j, K+1 \leq j \leq N)$, as N goes to infinity, then since by choice of γ, $\mathbb{E}(\eta_1) + \cdots + \mathbb{E}(\eta_N) = M$, one gets, informally: $\mathbb{P}(\sum_{j=1}^{N} \eta_j = M) \approx \frac{1}{b_N\sqrt{2\pi}}$ and

$$\mathbb{P}\left(\sum_{j=K+1}^{N} \eta_j = M - \sum_{j=1}^{K} n_j\right) \approx \frac{\exp(-\frac{1}{2(b_N^2 - \sum_{j=1}^{K}\sigma_j^2)}(\sum_{j=1}^{K}(\mathbb{E}(\eta_j) - n_j))^2)}{\sqrt{2\pi(b_N^2 - \sum_{j=1}^{K}\sigma_j^2)}},$$

where $b_N^2 = \sum_{j=1}^{N} \sigma_j^2$, denoting σ_j the standard deviation of η_j.

Now if both $\sum_{j=1}^{K} \sigma_j^2$ and $(\sum_{j=1}^{K} (\mathbb{E}(\eta_j) - n_j))^2$ are negligible with respect to b_N^2, the right-hand side of the last approximation is close to $(b_N\sqrt{2\pi})^{-1}$ and one gets the expected equivalence

$$\mathbb{P}(\xi_1 = n_1, \dots, \xi_K = n_K) \approx \mathbb{P}(\eta_1 = n_1, \dots, \eta_K = n_K).$$

To make this formal, we must consider a sequence of networks. In what follows, this sequence will be indexed by the number N of nodes, thus adding a subscript N to all parameters and variables.

Remark that if the local limit theorem applies to the complete vector of free variables $(\eta_1, \dots, \eta_N)$, it provides an equivalent for the partition function Z in (3.1), as function of (the implicit) γ. Indeed, using equations (3.1), (4.1) and (4.3),

$$Z = \gamma^{-M} \left(\prod_{i=1}^{N} Z_i(\gamma) \right) \mathbb{P}\left(\sum_{i=1}^{N} \eta_i = M \right) \approx (\gamma^M b_N \sqrt{2\pi})^{-1} \prod_{i=1}^{N} Z_i(\gamma).$$

For application to vehicle-sharing systems which is the object of Section 5, we need consider only two types of nodes: single server nodes (with finite or infinite capacity) and infinite server nodes with infinite capacity. For a single server node j, when n customers are present, the departure rate from j is $g_j(n)$ given by

$$g_j(n) = \mu_j \qquad \text{if } n \neq 0 \quad \text{and} \quad g_j(0) = 0,$$

while for an infinite server node, g_j is given by

$$g_j(n) = \mu_j n \qquad \text{for all } n \in \mathbb{N},$$

where in both cases, $\mu_j > 0$ is the parameter of the exponential services at j.

Equation (4.1) then tells that the variable η_j is:

- geometric with parameter $\gamma \mu_j^{-1} \theta_j$, truncated to $\{0, \dots, c_j\}$ if $c_j < \infty$, for a single server node j,
- Poisson with parameter $\gamma \mu_j^{-1} \theta_j$ if node j has infinitely many servers.

We will first consider a model where all nodes have infinite capacity, and then a model where single server nodes have finite (uniformly bounded) capacity. The equivalence of ensembles for those two models will respectively be derived from the local limit theorems stated in Propositions 2.1 and 2.2 of Section 2.

4.1. *Networks with infinite capacity nodes.* We here consider a sequence of standard closed Jackson networks: all nodes have infinite capacities. The network numbered N has N nodes (labeled by $1, \dots, N$) and M_N customers. Nodes are divided into two disjoint sets $\mathcal{J}_1^N$ and $\mathcal{J}_2^N$. Those in $\mathcal{J}_1^N$ are single server nodes and those in $\mathcal{J}_2^N$ are infinite servers nodes. To avoid trivialities, the set $\mathcal{J}_1^N$ is assumed to be nonempty.

For each network, we add a subscript N to all quantities already defined for a general Jackson network, thus denoting:

- $P_N = (p_{ij,N})_{1\le i,j\le N}$ the (irreducible) routing matrix,
- $\theta_N = (\theta_{1,N}, \dots, \theta_{N,N})$ the invariant probability vector associated to P_N,
- $\mu_{j,N}$ the rate of exponential services at node j for $1 \le j \le N$,
- $(\xi_{j,N})_{1\le j\le N}$ the stationary node-occupation random vector.

The product formula (3.1) here writes

$$\mathbb{P}(\xi_{1,N} = n_1, \dots, \xi_{N,N} = n_N) = \frac{1}{Z_N} \prod_{j\in\mathcal{J}_1^N} r_{j,N}^{n_j} \prod_{j\in\mathcal{J}_2^N} \frac{r_{j,N}^{n_j}}{n_j!}, \tag{4.6}$$

where $(n_1, \dots, n_N) \in \mathcal{S}_{N,M_N}$ and

$$r_{j,N} = \mu_{j,N}^{-1}\theta_{j,N} \qquad \text{for } 1 \le j \le N.$$

The parameter $r_{j,N}$ is usually referred to as the *utilization* of node j. From Remark 3.1, one knows that those utilizations need only be defined up to a constant. For this reason, some authors normalize the $r_{j,N}$'s in such a way that for each N, the maximum $\max\{r_{j,N}, 1 \le j \le N\}$ is equal to 1. This normalization is crucial in [15], where it is fully part of the assumption that the empirical distribution $N^{-1}\sum_{i=1}^N \delta_{r_{j,N}}$ converges to some probability measure. Here, it will not be necessary, since less stringent conditions are needed: The factor γ will in some sense absorb the normalization factor.

For all N, let $\gamma_N \in [0, 1/\max\{r_{j,N}, j \in \mathcal{J}_1^N\}[$ be the (unique, from Lemma 4.1) solution to equation (4.4), here given by

$$\sum_{j\in\mathcal{J}_1^N} \frac{\gamma_N r_{j,N}}{1 - \gamma_N r_{j,N}} + \sum_{j\in\mathcal{J}_2^N} \gamma_N r_{j,N} = M_N. \tag{4.7}$$

(The left-hand side of this equation clearly goes from 0 to $+\infty$ as γ_N varies from 0 to $1/\max\{r_{j,N}, j \in \mathcal{J}_1^N\}$, ensuring existence of the solution.)

The free variables $(\eta_{j,N})_{1\le j\le N}$ associated to $(\xi_{j,N})_{1\le j\le N}$ are independent and:

- for $j \in \mathcal{J}_1^N$, $\eta_{j,N}$ is geometric with parameter $\gamma_N r_{j,N}$,
- for $j \in \mathcal{J}_2^N$, $\eta_{j,N}$ is Poisson with parameter $\gamma_N r_{j,N}$,

so that $\sum_{j=1}^N \eta_{j,N}$ has mean M_N [from equations (4.7)] and variance

$$b_N^2 = \sum_{j\in\mathcal{J}_1^N} \frac{\gamma_N r_{j,N}}{(1 - \gamma_N r_{j,N})^2} + \sum_{j\in\mathcal{J}_2^N} \gamma_N r_{j,N}. \tag{4.8}$$

THEOREM 4.1. *Assume that the following conditions are satisfied*:

1. $\limsup_{N\to\infty}(\gamma_N \max_{j\in\mathcal{J}_1^N}\{r_{j,N}\}) < 1$,
2. $\lim_{N\to\infty} b_N = +\infty$,
3. $K \ge 1$ *is such that* $\lim_{N\to\infty} b_N^{-1} \sum_{j\in\mathcal{J}_2^N\cap\{1,\dots,K\}} \gamma_N r_{j,N} = 0$.

Then for any $(n_1, \dots, n_K) \in \mathbb{N}^K$,

$$\lim_{N\to\infty} \frac{\mathbb{P}(\xi_{1,N} = n_1, \dots, \xi_{K,N} = n_K)}{\mathbb{P}(\eta_{1,N} = n_1) \cdots \mathbb{P}(\eta_{K,N} = n_K)} = 1. \tag{4.9}$$

PROOF. We use (4.5) (here with a second subscript N for all random variables). From equations (4.7) and (4.8), $\sum_{j=1}^N \eta_i$ has mean M_N and variance b_N^2.

Condition (1) together with inequalities $\gamma_N < 1/\max_{j\in\mathcal{J}_1^N} r_{j,N}$ for all N imply existence of some $\rho < 1$ such that

$$\forall N, \forall j \in \mathcal{J}_1^N, \qquad \gamma_N r_{j,N} \le \rho.$$

Condition (2) then allows to apply Proposition 2.1 to both families $(\eta_{j,N})_{1\le j\le N}$ and $(\eta_{K+j,N})_{1\le j\le N-K}$ [with respectively, $J(N) = N$ and $J(N) = N - K$]. This is immediate for the first family. As for the second family, it results from

$$\mathrm{Var}\left(\sum_{j=K+1}^N \eta_{j,N}\right) = b_N^2 - \sum_{j\in\mathcal{J}_1^N\cap\{1,\dots,K\}} \frac{\gamma_N r_{j,N}}{(1-\gamma_N r_{j,N})^2} - \sum_{j\in\mathcal{J}_2^N\cap\{1,\dots,K\}} \gamma_N r_{j,N},$$

that $\mathrm{Var}(\sum_{j=K+1}^N \eta_{j,N}) \sim b_N^2$ as N goes to infinity, since both sums in the right-hand side are negligible with respect to b_N^2: indeed the first sum is bounded above by $K\rho/(1-\rho)^2$, and the second one is negligible with respect to b_N, hence to b_N^2, from assumption (3).

Then using (2.4), $b_N\sqrt{2\pi}\,\mathbb{P}(\sum_{j=1}^N \eta_{j,N} = M_N)$ converges to 1 and

$$\sqrt{2\pi\left(b_N^2 - \sum_{j=1}^K \sigma_{j,N}^2\right)}\,\mathbb{P}\left(\sum_{j=K+1}^N \eta_{j,N} = M_N - \sum_{j=1}^K n_j\right) - \exp\left(-\frac{(\sum_{j=1}^K (m_{j,N} - n_j))^2}{2(b_N^2 - \sum_{j=1}^K \sigma_{j,N}^2)}\right)$$

converges to 0 for any $n_1, \dots, n_K \in \mathbb{N}$, as N goes to infinity, where for $1 \le j \le K$, $m_{j,N} = \mathbb{E}(\eta_{j,N})$ and $\sigma_{j,N}^2 = \mathrm{Var}(\eta_{j,N})$.

Now, on one hand, $b_N^2 - \sum_{j=1}^K \sigma_{j,N}^2 = \mathrm{Var}(\sum_{j=K+1}^N \eta_{j,N}) \sim b_N^2$ as N goes to infinity, as just shown. On the other hand, the argument of the exponential in the last expression goes to zero since

$$\frac{1}{b_N}\sum_{j=1}^K |m_{j,N} - n_j| \le b_N^{-1}\sum_{j=1}^K n_j + Kb_N^{-1}\rho/(1-\rho) + b_N^{-1}\sum_{j\in\mathcal{J}_2^N\cap\{1,\dots,K\}} \gamma_N r_{j,N}$$

which goes to zero as N tends to infinity, using assumptions (2) and (3). Hence,

$$\lim_{N\to\infty} b_N\sqrt{2\pi}\,\mathbb{P}\left(\sum_{j=K+1}^N \eta_{j,N} = M_N - \sum_{j=1}^K n_j\right) = 1.$$

Equation (4.9) follows, using (4.5) together with independence of the $\eta_{j,N}$'s. $\square$

REMARK 4.1. (i) Note that, since $M_N \leq b_N^2$ for all N, as can be seen from (4.7) and (4.8), condition (2) of Theorem 4.1 is satisfied if $\lim_{N\to\infty} M_N = \infty$.

(ii) Assumption (3) is trivially satisfied for $\{1, \dots, K\} \subset \mathcal{J}_1^N$.

4.2. *Networks with both finite and infinite capacity nodes.* The networks now considered differ from the previous ones on two main points. For all N:

1. for $j = 1, \dots, N$, node j has capacity $c_{j,N}$, which is now *finite* for $j \in \mathcal{J}_1^N$, but remains infinite for $j \in \mathcal{J}_2^N$; we will denote $c_N = (c_{j,N}, 1 \leq j \leq N)$,
2. both $\mathcal{J}_1^N$ and $\mathcal{J}_2^N$ are assumed to be nonempty.

Any of the dynamics described in the previous section can be considered: blocking, blocking-rerouting or general state-dependent routing satisfying the setting of Theorem 3.2. In the simple blocking case, we assume that for all N, the *routing matrix is reversible* with respect to its invariant distribution. Thus, in any of these situations, the queue-length process has stationary distribution given by the same formula (4.6) as in the infinite capacity case, here with state space $\mathcal{S}_{N,M_N}^{c_N}$.

We keep the same notation as before: $r_{j,N}$ are the (nonnormalized) utilizations, $\mu_{j,N}$ the exponential service parameters, θ_N the invariant probability vector, either of the fixed routing matrix P_N (for blocking or blocking/rerouting procedures), or common to all state-dependent routings, in the setting of Theorem 3.2.

For all N, γ_N is now uniquely defined in $[0, +\infty[$ by

$$\sum_{j\in\mathcal{J}_1^N} \frac{\sum_{n=0}^{c_{j,N}} n(\gamma_N r_{j,N})^n}{\sum_{n=0}^{c_{j,N}} (\gamma_N r_{j,N})^n} + \sum_{j\in\mathcal{J}_2^N} \gamma_N r_{j,N} = M_N, \tag{4.10}$$

or equivalently by

$$\sum_{j\in\mathcal{J}_1^N} \left(\frac{\gamma_N r_{j,N}}{1-\gamma_N r_{j,N}} - (c_{j,N}+1) \frac{(\gamma_N r_{j,N})^{c_{j,N}+1}}{1-(\gamma_N r_{j,N})^{c_{j,N}+1}} \right) + \sum_{j\in\mathcal{J}_2^N} \gamma_N r_{j,N} = M_N$$

with the following abuse: term j of the first sum has to be replaced by $c_{j,N}/2$ if $\gamma_N r_{j,N} = 1$.

Existence and uniqueness of γ_N again result from Lemma 4.1 and from the fact that the second term in the left-hand side of equation (4.10) increases to infinity with γ_N (recall that $\mathcal{J}_2^N \neq \varnothing$).

The free variables $(\eta_{j,N})_{1\leq j\leq N}$ associated to $(\xi_{j,N})_{1\leq j\leq N}$ are now such that:

- for $j \in \mathcal{J}_1^N$, $\eta_{j,N}$ is geometric with parameter $\gamma_N r_{j,N}$ truncated to $[0, c_{j,N}]$,
- for $j \in \mathcal{J}_2^N$, $\eta_{j,N}$ is Poisson with parameter $\gamma_N r_{j,N}$.

Denoting by $m_{j,N}$ and $\sigma^2_{j,N}$ the mean and variance of $\eta_{j,N}$ (for $1 \le j \le N$), equation (4.10) rewrites $M_N = \sum_{j=1}^N m_{j,N}$, while $b_N^2 = \sum_{j=1}^N \sigma^2_{j,N}$ or equivalently

$$b_N^2 = \sum_{j \in \mathcal{J}_1^N} \left(\frac{\gamma_N r_{j,N}}{(1-\gamma_N r_{j,N})^2} - (c_{j,N}+1)^2 \frac{(\gamma_N r_{j,N})^{c_{j,N}+1}}{(1-(\gamma_N r_{j,N})^{c_{j,N}+1})^2} \right) + \sum_{j \in \mathcal{J}_2^N} \gamma_N r_{j,N}.$$

Here again, term j of the first sum must be replaced by $c_{j,N}(c_{j,N}+2)/12$ when $\gamma_N r_{j,N} = 1$.

THEOREM 4.2. *If the following conditions are satisfied*:

1. *there exists some* $C < +\infty$ *such that for all* $N \ge 1$ *and* $j \in \mathcal{J}_N^1$, $c_{j,N} \le C$,
2. $\lim_{N\to\infty} b_N = +\infty$,
3. $K \ge 1$ *is such that* $\lim_{N\to\infty} b_N^{-1} \sum_{j \in \mathcal{J}_2^N \cap \{1,\dots,K\}} \gamma_N r_{j,N} = 0$,

then (4.9) *holds for all* $(n_1, \dots, n_K) \in \mathbb{N}^K$.

PROOF. The proof is the same as for Theorem 4.1, here using Proposition 2.2. Condition (1) ensures that all $m_{j,N}$ and $\sigma_{j,N}$ for $j \in \mathcal{J}_1^N$ are bounded by some constant (not depending on N), namely, by C^2 (since the corresponding $\eta_{j,N}$'s have values in $\{0, \dots, C\}$). $\square$

Note that (ii) of Remark 4.1 still holds. But this is not the case for (i): if $m_{\rho,c}$ and $\sigma^2_{\rho,c}$ denote the mean and variance of some truncated geometric random variable $X_{\rho,c}$ with parameters ρ and c, the ratio $\sigma_{\rho,c}/m_{\rho,c}$ approaches zero as ρ goes to zero or to infinity, for fixed $c \ge 1$.

5. Application to bike-sharing systems. The purpose of this section is to derive practical results on the performance of bike-sharing systems such as the Velib' network in Paris. Here, a (large) set of bikes are distributed over a (large) number of docking stations and offered for use to the population of the city. Any user can take a bike at some station, and then ride to another station where he returns the bike. The payment process requires that each bike be locked to a terminal, so that each station can accommodate only a given number of bikes.

5.1. *Infinite capacity approach.* It is easily seen that such a system can be modeled—at least as long as all stations remain below capacity—by a closed Jackson network with a fixed number of customers (here the bikes), and both single server nodes (the stations) and infinite servers nodes (the routes from one station to another). Indeed, assuming that the arrival process of users at any station a is Poisson with parameter μ_a, then bikes leave station a at rate μ_a (here, inter-arrival intervals between users act as service durations for the waiting bikes). Next, assuming that ride durations are independent and exponentially distributed with parameter μ_r for route r, the number of bikes on route r decreases from n to $n-1$ at rate $\mu_r n$.

These remarks lead to a model with infinite node capacities proposed in [6] and [10] that is considered now. Denote by $\mathcal{J}_1$ the set of stations, with $J_1 = |\mathcal{J}_1|$, and $\mathcal{J}_2 \subset \{[ij] : i, j \in \mathcal{J}_1\}$, with $J_2 = |\mathcal{J}_2| \leq J_1(J_1 - 1)$, the set of possible routes: Here, $[ij]$ denotes the route from i to j. It is assumed that the bike moves obey to some statistics which is constant in time: that is, any user taking a bike at station i has probability q_{ij} to put it back at station j. Stations here have unlimited parking capacity, as well as routes can accommodate as many riders as necessary. Then bikes/customers alternatively occupy nodes in $\mathcal{J}_1$ and in $\mathcal{J}_2$ and move according to the following routing matrix P on $\mathcal{J}_1 \cup \mathcal{J}_2$:

$$
\begin{aligned}
&\forall i, j \in \mathcal{J}_1, \qquad p_{i[ij]} = q_{ij} \quad \text{and} \quad p_{[ij]j} = 1; \\
&p_{ab} = 0 \qquad \text{for other } a, b \in \mathcal{J}_1 \cup \mathcal{J}_2.
\end{aligned}
\tag{5.1}
$$

Here, $\mathcal{J}_2$ is assumed to include all $[ij]$ such that $q_{ij} > 0$, so that (5.1) well-defines a routing process on $\mathcal{J}_1 \cup \mathcal{J}_2$. All services (inter-arrival times between users at stations or trip durations) are independent, exponentially distributed with parameter μ_a at node a ($a = j$ or $[ij]$ for some $i, j \in \mathcal{J}_1$) and independent of the routing processes. This model thus constitutes a standard closed Jackson network with M customers and $N = J_1 + J_2$ nodes.

Note that P is irreducible if and only if the following two conditions are satisfied:

1. the matrix $Q = (q_{ij})_{i,j \in \mathcal{J}_1}$ is irreducible,
2. $\forall i, j \in \mathcal{J}_1, [ij] \in \mathcal{J}_2 \Longleftrightarrow q_{[ij]} > 0$.

We assume that this is the case. Then, denoting by $\nu = (\nu_j)_{j \in \mathcal{J}_1}$ the unique Q-invariant distribution on $\mathcal{J}_1$, the routing matrix P has invariant distribution $\theta = (\theta_a)_{a \in \mathcal{J}_1 \cup \mathcal{J}_2}$ given by

$$
\theta_j = \frac{1}{2}\nu_j \qquad \text{for } j \in \mathcal{J}_1 \quad \text{and} \quad \theta_{[ij]} = \frac{1}{2}\nu_i q_{ij} \qquad \text{for } [ij] \in \mathcal{J}_2. \tag{5.2}
$$

The process so defined fits the frame of Section 3.1 and has stationary state given by

$$
\mathbb{P}(\xi_1 = n_1, \dots, \xi_N = n_N) = \frac{1}{Z} \prod_{j \in \mathcal{J}_1} r_j^{n_j} \prod_{[ij] \in \mathcal{J}_2} \frac{r_{[ij]}^{n_{[ij]}}}{n_{[ij]}!}, \tag{5.3}
$$

where $(n_1, \dots, n_N) \in \mathcal{S}_{N,M}$ and

$$
\text{for } j \in \mathcal{J}_1, \qquad r_j = \frac{\nu_j}{2\mu_j} \quad \text{and} \quad \text{for } [ij] \in \mathcal{J}_2, \qquad r_{[ij]} = \frac{\nu_i q_{ij}}{2\mu_{[ij]}}.
$$

This model does not wholly convey the complexity of real bike-sharing systems, since it ignores the blocking mechanism at saturated stations (which is the main problem for these networks). Still, it describes the system up to the first blocking time. In [10], asymptotic results are provided for M large, as N is fixed. Here, Theorem 4.1 is used to describe this network as both M and N are large.

The practical problem addressed is the following. For a network with given stations and routes, how many bikes and parking places should be offered? It can be assumed that from observation-based estimations, the demand of the users is well known: that is, the frequencies at which users arrive at the different stations (μ_j's), the popularities of the different routes (q_{ij}'s) and their mean ride durations ($\mu_{[ij]}$'s). In other terms, utilizations $r_a = \mu_a^{-1}\theta_a$, are known. Theorem 4.1 then helps evaluating, through approximation by geometric variables, the probability that some station is empty, or exceeds some level of occupation, indicating how to adapt the stations capacities and/to the number M of bikes offered.

In order to derive results from Theorem 4.1, we again consider a sequence of networks indexed by N, and as in Section 4, add a subscript N to all parameters defined above, rewriting equation (5.3) as equation (4.6).

A simple way to ensure that condition (1) of Theorem 4.1 is satisfied is to choose some small positive δ and set

$$\gamma_N = \frac{1-\delta}{\max_{j\in\mathcal{J}_1^N} r_{j,N}} \quad \text{and} \quad M_N = \sum_{j\in\mathcal{J}_1^N} \frac{\gamma_N r_{j,N}}{1-\gamma_N r_{j,N}} + \sum_{[ij]\in\mathcal{J}_2^N} \gamma_N r_{[ij],N}, \tag{5.4}$$

so that γ_N solves equation (4.7). Theorem 4.1 then has the following corollary.

COROLLARY 5.1. *Let $\delta \in [0, 1]$ be fixed and for all N, define γ_N and M_N by* (5.4):

(i) *If all $r_{j,N}$ for $j \in \mathcal{J}_1^N$ are of the same order, that is if*

$$\liminf_{N\to\infty} \frac{\min_{j\in\mathcal{J}_1^N} r_{j,N}}{\max_{j\in\mathcal{J}_1^N} r_{j,N}} > 0,$$

then as N goes to infinity, the stationary queue-lengths at stations get asymptotically independent, and approximately geometric with parameters $\gamma_N r_{j,N} = (1-\delta) r_{j,N}/\max_{i\in\mathcal{J}_1^N} r_{i,N}$ ($j \in \mathcal{J}_1^N$).

(ii) *If moreover,*

$$\lim_{N\to\infty} \frac{\max_{[ij]\in\mathcal{J}_2^N} r_{[ij],N}}{\sqrt{J_1^N}\max_{k\in\mathcal{J}_1^N} r_{k,N}} = 0,$$

then at stationarity, both stations and routes become asymptotically independent, and the level of occupation of route $[ij]$ is approximately Poisson with parameter $\gamma_N r_{[ij],N} = (1-\delta) r_{[ij],N}/\max_{k\in\mathcal{J}_1^N} r_{k,N}$.

REMARK 5.1. This corollary applies in particular under the following natural set of assumptions, for which γ_N and M_N are both of the order of J_1^N:

- all $r_{j,N}$ for $j \in \mathcal{J}_1^N$ are of the order of $1/J_1^N$,
- all $r_{[ij],N}$ for $j \in \mathcal{J}_2^N$ are of the order of $1/J_2^N$

(as, e.g., if all $\mu_{j,N}$ and $\mu_{[ij],N}$ are of the order of 1, $\nu_{j,N}$ of the order of $1/J_1^N$ and $\nu_{i,N}q_{ij,N}$ the order of $1/J_2^N$. Recall that $\sum_j \nu_{j,N} = \sum_{[ij]} \nu_{i,N}q_{[ij],N} = 1$). Indeed, $\sqrt{J_1^N}/J_2^N \leq 1/\sqrt{J_1^N} \to 0$ as $N \to \infty$, since $J_1^N \leq J_2^N$ by irreductibility of (q_{ij}^N), and $N = J_1^N + J_2^N \leq (J_1^N)^2$.

PROOF OF COROLLARY 5.1. (i) Due to (ii) of Remark 4.1, we need only check that condition (2) of Theorem 4.1 is satisfied [condition (1) holds by choice of γ_N]. But here

$$b_N^2 \geq M_N \geq \sum_{j \in \mathcal{J}_1^N} \gamma_N r_{j,N} = (1-\delta)\frac{\sum_{j\in\mathcal{J}_1^N} r_{j,N}}{\max_{j\in\mathcal{J}_1} r_{j,N}} \geq (1-\delta)J_1^N \frac{\min_{j\in\mathcal{J}_1^N} r_{j,N}}{\max_{j\in\mathcal{J}_1^N} r_{j,N}}$$

which goes to infinity with N, since $J_1^N \geq \sqrt{N}$ and $\liminf_{N\to\infty} \frac{\min_{j\in\mathcal{J}_1^N} r_{j,N}}{\max_{j\in\mathcal{J}_1^N} r_{j,N}} > 0$.

(ii) We need to prove that $\max_{[ij]\in\mathcal{J}_2^N} \gamma_N r_{[ij],N}/b_N$ goes to zero as N goes to infinity, in order that (3) of Theorem 4.1 holds. But as just shown, $b_N^2 \geq cJ_1^N$ for some positive c and N sufficiently large. Hence, for large N,

$$\begin{aligned}\forall [ij] \in \mathcal{J}_2^N, \qquad \gamma_N r_{[ij],N}/b_N &= (1-\delta)\frac{r_{[ij],N}}{b_N \max_{k\in\mathcal{J}_1^N} r_{k,N}} \\ &\leq \frac{1-\delta}{\sqrt{cJ_1^N}} \frac{r_{[ij],N}}{\max_{k\in\mathcal{J}_1^N} r_{k,N}},\end{aligned}$$

which maximum over $[ij]$ goes to zero by assumption. □

As a practical consequence, under condition (i) of Corollary 5.1, the probability:

- for station j to be empty is approximately $1-(1-\delta)r_{j,N}/\max_{i\in\mathcal{J}_1^N} r_{i,N}$, which is at least δ, and exactly δ at the most loaded station,
- that station j has more than n vehicles is $[(1-\delta)r_{j,N}/\max_{i\in\mathcal{J}_1^N} r_{i,N}]^{n+1}$.

Capacity of station j can then be fixed as the smallest n such that this probability of exceeding level n is less than some given ϵ.

In the ideal situation when all utilizations $r_{j,N}$ for $j \in \mathcal{J}_1^N$ are equal, the above probabilities are the same for all stations. The probability for any station to be empty, δ, is determined by the total number M_N of bikes, given by equation (5.4). And the smallest capacity that sets the blocking probability (at any station) below some given level ϵ is given by the integer part of $\log \epsilon/\log(1-\delta)$.

5.2. *Finite capacity approach.* We now directly consider networks with finite parking capacity (and still infinite capacity routes). Remark first that the routing matrix of the model of [10], described in Section 5.1, is not reversible ($p_{[ij]i} = 0$ for $i \neq j$ while $p_{i[ij]} = q_{ij}$). Thus, the derived model with finite capacity stations and *simple blocking* procedure (as defined in Section 3.2) is not tractable, its stationary distribution being unknown. (Anyway, this procedure would not be realistic, making blocked users perform again the same ride as the one just completed.)

Still, two options are open for modeling such finite parking capacity networks. One is to consider the same routing as in [10] together with the *blocking and rerouting* policy described in Section 3. This choice turns out to be relevant for bike-sharing systems. Indeed, since transitions occur only from stations to routes and from routes to stations, any user blocked at his end-of-route station i will choose some route at random among the routes issuing from i, which well describes the reality. Note that in the standard blocking and rerouting procedure, the new route is chosen according to the same probabilities q_{ij}'s as for new arrived users, which may seem unrealistic, or approximative (the q_{ij}'s need then combine behaviors of new arrived and redirected users). We will discuss existence of other state-dependent routings that may be tractable (i.e., with product-form stationary state) and credible for bike-sharing.

The other option is to use pure blocking dynamics for a simpler model, in which all the routes are aggregated into one unique node, having infinite many servers and infinite capacity. The transition probability from any station to this unique route is then equal to 1, while the transition from this route to any station j is given by some probability q_j, called the *popularity* of station j. This basic model is a simple, easy to handle, approximation of the network in [10], which does not make account of the detailed movements of the vehicles. It has been introduced and analyzed in [9]. Note that the associate routing matrix P is reversible, so that the product form (3.1) holds. Indeed, numbering the N nodes ($N-1$ stations and one route) so that the set of stations is $\mathcal{J}_1 = \{1, \dots, N-1\}$ and the unique route is N, then P is given by

$$p_{jN} = 1 \quad \text{and} \quad p_{Nj} = q_j \qquad \text{for } j \in \mathcal{J}_1; \qquad p_{ij} = 0 \qquad \text{otherwise,} \tag{5.5}$$

and the invariant probability $\theta = (\theta_1, \dots, \theta_N)$ by

$$\theta_N = \frac{1}{2} \quad \text{and} \quad \theta_j = \frac{q_j}{2} \qquad \text{for } j = 1, \dots, N-1,$$

which clearly satisfies the reversibility condition (3.2).

Denote by c_j, for $1 \leq j < N$, the finite capacity of station j. The stationary state of the network is here given by

$$\mathbb{P}(\xi_1 = n_1, \dots, \xi_N = n_N) = \frac{1}{Z_N} \frac{r_N^{n_N}}{n_N!} \prod_{j=1}^{N-1} r_j^{n_j}$$

for $n \in \mathbb{N}^N$ such that $n_1 + \cdots + n_N = M$ and $n_j \le c_j$ for $1 \le j < N$, where

$$\text{for } j \in \{1, \dots, N-1\}, \qquad r_j = \frac{q_j}{2\mu_j} \quad \text{and} \quad r_N = \frac{1}{2\mu_N}.$$

Here, since there is one unique route, the simple *blocking* and the *blocking and rerouting* procedures are actually undistinguishable, both leading to the above product form.

REMARK 5.2. These simplified set of nodes and transition matrix can also be used in the infinite capacity case, instead of those of [10].

In modeling a bike-sharing system by any of these two models, it seems realistic to suppose that all the capacities c_j stay bounded as N is large. Theorem 4.2 can then be applied.

For the last model with one unique route, parameters μ_i of services (inter-arrivals of users and durations of rides) should be assumed to be of the order of 1, while the q_i's should be of the order of N^{-1}. One gets r_j's of the order of N^{-1} for $1 \le j < N$ and r_N of the order of 1.

Equation (4.10) here rewrites

$$M = \sum_{j=1}^{N-1} \left(\frac{\gamma r_j}{1 - \gamma r_j} - (c_j + 1) \frac{(\gamma r_j)^{c_j+1}}{1 - (\gamma r_j)^{c_j+1}} \right) + \gamma r_N$$

(with the same abuse as in Section 4, for terms $j \in \{1, \dots, N-1\}$ such that $\gamma r_j = 1$). It can be used, as in the infinite capacity case, to fix M by choosing γ. The appropriate scale is here $\gamma = \kappa N$ for some positive κ, which gives asymptotically independent stations, with truncated-geometric approximate distributions with parameters $\kappa N r_j$ and c_j $(1 \le j < N)$. Indeed, since $b_N^2 \ge \gamma r_N$ which is of the order of N, the approximation (4.9) of Theorem 4.2 is here valid for $\{1, \dots, K\} \subset \mathcal{J}_1 = \{1, \dots, N-1\}$.

Note that nothing can be said about node N, since b_N is of order $\sqrt{N}$ $[\gamma r_N \le b_N^2 \le \gamma r_N + (N-1) \max c_j]$ so that $b_N^{-1} \gamma r_N$ gets large with N. The unique route is thus not proved to be independent from the stations, nor to have some identified approximate distribution: In physical terms, this scaling of M (through $\gamma = \kappa N$) corresponds to a supercritical regime, with condensation at node N (that accommodates as many vehicles as the order of M).

As regards stations: for $1 \le j < N$,

- $P(\xi_j = 0) \approx 1 / \sum_{n=0}^{c_j} (\kappa N r_j)^n$, which decreases with respect to κ,
- $P(\xi_j = c_j) \approx (\kappa N r_j)^{c_j} / \sum_{n=0}^{c_j} (\kappa N r_j)^n$, which increases with respect to κ.

If capacities are given, one can look for some κ that gives a reasonable trade-off between the previous probabilities. These estimations can also help defining the capacities. Indeed for fixed κ, the approximation $1 / \sum_{n=0}^{c_j} (\kappa N r_j)^n$ of $P(\xi_j = 0)$

decreases as c_j increases. This is also the case for $(\kappa N r_j)^{c_j} / \sum_{n=0}^{c_j} (\kappa N r_j)^n$ that approximates $P(\xi_j = c_j)$, due to Remark 2.2. So c_j can be chosen as the smallest value that sets both quantities below some given level ϵ. However, the first quantity cannot be decreased below $1/\sum_{n=0}^{\infty}(\kappa N r_j)^n$ (which is positive if $\kappa N r_j < 1$). Similarly, the second quantity cannot get below $1/\sum_{n=0}^{\infty}(\kappa N r_j)^{-n}$ (positive if $\kappa N r_j > 1$). Here again, the ideal situation is when all r_j are equal. This indeed allows to select κ such that $\kappa N r_j = 1$ for all $j = 1, \ldots, N-1$; so that the approximate values for $P(\xi_j = 0)$ and $P(\xi_j = c_j)$ approach zero for large c_j's. But real networks generally do not satisfy this condition, so that κ can only be fixed such that $\kappa N r_j = 1$ for a group of stations with equal utilizations. Both probabilities—that a station is empty, or saturated—can then be arbitrarily reduced only for these given stations. No choice of κ and c_j can be globally satisfactory.

We will now consider models derived from that of [10], here limiting station capacities. We first investigate possible alternatives to the standard blocking and rerouting procedure—in which rerouting is ruled by the same matrix Q that rules choice of destination. The setup is the same as described in Section 5.1, except that each node j in $\mathcal{J}_1$ has finite capacity c_j. The Jackson network dynamics is then modified, to avoid overflow of capacity, through the following transitions and rates: For $n \in \mathcal{S}^c_{N,M}$, where $c = (c_a)_{a \in \mathcal{J}_1 \cup \mathcal{J}_2}$, setting $c_{[ij]} = \infty$ for $[ij] \in \mathcal{J}_2$,

$$
\begin{aligned}
& n \longrightarrow n - e_i + e_{[ij]} \qquad \text{at rate } \mu_i q_{ij} \mathbf{1}_{n_i > 0}, \\
(5.6) \quad & n \longrightarrow n - e_{[ij]} + e_j \qquad \text{at rate } \mu_{[ij]} n_{[ij]} \mathbf{1}_{n_j < c_j}, \\
& n \longrightarrow n - e_{[ij]} + e_{[jk]} \qquad \text{at rate } \mu_{[ij]} n_{[ij]} w^{(j)}_{ik}(n - e_{[ij]}) \mathbf{1}_{n_j = c_j, n_{[ij]} > 0}.
\end{aligned}
$$

Here, for all $m \in \mathcal{S}^c_{N,M-1}$ and $j \in \mathcal{J}_1$, a Markovian transition matrix $W^{(j)}(m) = (w^{(j)}_{ik}(m))_{i,k \in \mathcal{J}_1 \setminus \{j\}}$ on the set $\mathcal{J}_1 \setminus \{j\}$ is given. The following result is a consequence of Theorem 3.2.

PROPOSITION 5.1. *If for each $j \in \mathcal{J}_1$, all matrices $W^{(j)}(m)$ for $m \in \mathcal{S}^c_{N,M-1}$ solve the following set of equations, with unknown variable $W = (w_{ik})_{i,k \in \mathcal{J}_1 \setminus \{j\}}$:*

$$
(5.7) \qquad \forall k \in \mathcal{J}_1 \text{ such that } k \neq j, \qquad \nu_j q_{jk} = \sum_{i \in \mathcal{J}_1 \setminus \{j\}} \nu_i q_{ij} w_{ik},
$$

then the process with state space $\mathcal{S}^c_{N,M}$ and transitions defined in (5.6) has stationary distribution given by (5.3) for $(n_1, \ldots, n_N) \in \mathcal{S}^c_{N,M}$.

PROOF. It is easily seen that the rates in (5.6) have the form in (3.4) with

$$
\begin{aligned}
\text{for } h \in \mathbb{N}, \qquad & g_i(h) = \mu_i \mathbf{1}_{h>0} \qquad \text{for } i \in \mathcal{J}_1 \quad \text{and} \\
& g_{[ij]}(h) = \mu_{[ij]} h \qquad \text{for } [ij] \in \mathcal{J}_2
\end{aligned}
$$

and Markovian transition matrix $P(m)$ on the set $A(m) = \{j \in \mathcal{J}_1, m_j < c_j\} \cup \mathcal{J}_2$ given, for $m \in \mathcal{S}^c_{N,M-1}$, by

$$p_{j,[ij]}(m) = q_{ij} \qquad \text{for } [ij] \in \mathcal{J}_2 \text{ such that } i \in A(m),$$

$$p_{[ij],j}(m) = 1 \qquad \text{for } [ij] \in \mathcal{J}_2 \text{ such that } j \in A(m),$$

$$p_{[ij],[jk]}(m) = w^{(j)}_{ik}(m) \qquad \text{for } i, j, k \in \mathcal{J}_1 \text{ such that } [ij], [jk] \in \mathcal{J}_2 \text{ and } j \notin A(m),$$

all other transitions having null rates.

Moreover, it is straightforward to check that for all $m \in \mathcal{S}^c_{N,M-1}$, the restriction of vector θ of (5.2) to the set $A(m)$ is invariant with respect to $P(m)$. This results from equations (5.7) together with invariance of ν with respect to Q.

The proposition then results from Theorem 3.2. $\square$

Equations (5.7) are clearly satisfied if $w^{(j)}_{ik} = q_{jk}$ for all i, j, k, which is the only solution such that $w^{(j)}_{ik}$ depends only on (j, k). In other terms, the above dynamics generalize the standard blocking and rerouting—for the Jackson network in [10]—in the sense that redirection of users blocked at end of route $[ij]$ may now take into account their original station i. Note that this excludes natural reroutings to stations in the neighborhood of j. Still, for each j, equations (5.7) have infinitely many solutions, as it appears by rewriting them as

$$q_{jk} = \sum_{i \in \mathcal{J}_1 \setminus \{j\}} \widetilde{q}_{ji} w_{ik} \quad \text{or else} \quad q_{j\cdot} = \widetilde{q}_{j\cdot} W,$$

where $\widetilde{Q} = (\widetilde{q}_{ij}) = (\nu_i q_{ij} / \nu_j)$ is the time-reversed matrix of Q under its invariant vector ν, and $q_{j\cdot}$, $\widetilde{q}_{j\cdot}$ denote the jth lines of Q and $\widetilde{Q}$. No general explicit solution is available, save for $W^{(j)} = (q_{jk})_{i \neq j, k \neq j}$. But in the special case when Q *is reversible under* ν, the last equation means that W has invariant vector $q_{j\cdot}$; so that $W = \mathrm{Id}$ is a solution for all j. Choosing this solution: $w^{(j)}_{ik}(m) = \mathbf{1}_{k=i}$ $(i, k \neq j)$ for all j and m, means that blocked users are sent back to their original station. It can be interesting to note that for each j the set of solutions of (5.7) is convex, so that one can take convex combinations—with coefficient possibly depending on m—of different solutions. In the reversible Q case, the following rerouting is then possible: blocked users at end of route $[ij]$ flip a coin, which probability of giving “head” may depend on the current distribution m of the other $M - 1$ bikes, and according to the result, either take route $[ji]$, or choose new route $[jk]$ with probability q_{jk}.

Other solutions can be given in particular cases, as if for example Q is uniform, that is, $q_{ij} = (J_1 - 1)^{-1}$ for all i, j with $i \neq j$. Here, variants of the deterministic rerouting from $[ij]$ to $[ji]$ can be mentioned, such as rerouting from $[ij]$ to $[ji^*_{j,m}]$, where m is the state of the network excluding the blocked bike, and for all j, m:

- either $i^*_{j,m}$ is deterministic, and $i \mapsto i^*_{j,m}$ is a one-to-one mapping on $\mathcal{J}_1 \setminus \{j\}$,
- or $i^*_{j,m}$ is the first step of some symmetric random walk—which distribution may depend on j and m—on some graph with vertex set $\mathcal{J}_1$ and constant degree.

We will now consider any dynamics that leads to stationary distribution given by (5.3), here on state space $S^c_{N,M}$. In order to use the asymptotic results of Section 4.2, we here again consider a sequence of networks indexed by N, and use the same notations as in the infinite capacity case of Section 5.1.

A set of natural hypotheses is for example:

(H0) $\exists C \in \mathbb{N}$ such that $c_{j,N} \leq C$ for all $j \in \mathcal{J}_1^N$,
(H1) $\exists \mu_+, \mu_- > 0$ such that $\mu_- \leq \mu_{a,N} \leq \mu_+$ for all N and $a \in \mathcal{J}_1^N \cup \mathcal{J}_2^N$,
(H2) $\exists A > 0$ such that $\max_{[ij]\in\mathcal{J}_2^N} \nu_{i,N} q_{ij,N} \leq A/J_2^N$ for all N.

(Recall that $\sum_{[ij]\in\mathcal{J}_2^N} \nu_{i,N} q_{ij,N} = 1/2$.) Theorem 4.2 then has the following corollary.

COROLLARY 5.2. *Assume that* (H0), (H1) *and* (H2) *hold and that* (M_N) *satisfies*

$$\lim_{N\to\infty} M_N = +\infty \quad \textit{and} \quad \lim_{N\to\infty} (J_2^N)^{-2} M_N = 0.$$

Then as N goes to infinity, at stationarity, the different queue-lengths at stations and routes get asymptotically independent, with respective approximate distributions:

- *truncated-geometric with parameters $c_{j,N}$ and $\gamma_N r_{j,N}$ for station j*,
- *Poisson with parameter $\gamma_N r_{[ij],N}$ for route $[ij]$*,

where γ_N solves equation (4.10). *Moreover, γ_N has same order of magnitude as M_N.*

PROOF. Equation (4.10) together with the elementary relations

$$m_{\rho,c} = \frac{\sum_{n=1}^{c} n\rho^n}{\sum_{n=0}^{c} \rho^n} \leq c\rho \frac{\sum_{n=0}^{c-1} \rho^n}{\sum_{n=0}^{c} \rho^n} \leq c\rho \qquad \text{for } \rho > 0 \text{ and } c \in \mathbb{N},$$

imply the following inequalities, using (H0) and (H1):

$$\begin{aligned}\frac{\gamma_N}{2\mu_+} &\leq \sum_{[ij]\in\mathcal{J}_2^N} \gamma_N r_{[ij],N} \\ &\leq M_N \leq C \sum_{j\in\mathcal{J}_1^N} \gamma_N r_{j,N} + \sum_{[ij]\in\mathcal{J}_2^N} \gamma_N r_{[ij],N} \leq \frac{\gamma_N}{2\mu_-}(C+1),\end{aligned}$$

that show that γ_N is of the same order as M_N.

Now assumptions (1), (2) and (3) of Theorem 4.2 are satisfied, for any $K \geq 1$. Indeed, (1) is equivalent to (H0). As for (2),

$$b_N^2 \geq \sum_{[ij]\in\mathcal{J}_2^N} \gamma_N r_{[ij],N} \geq \frac{\gamma_N}{2\mu_+},$$

where $\gamma_N \geq 2\mu_- M_N/(C+1)$ goes to infinity with N. And for (3), using the previous lower bound on b_N, together with (H2) and inequality $\gamma_N \leq 2\mu_+ M_N$,

$$b_N^{-1} \max_{[ij]\in\mathcal{J}_2^N} \gamma_N r_{[ij],N} \leq \frac{\sqrt{2\mu_+\gamma_N}}{2\mu_-} \max_{[ij]\in\mathcal{J}_2^N} \nu_{i,N} q_{ij,N} \leq A \frac{\mu_+}{\mu_-} \frac{\sqrt{M_N}}{J_2^N},$$

which tends to zero as N goes to infinity by assumption.

The proof is then complete, using Theorem 4.2. $\square$

Corollary 5.2 provides simple explicit approximations that may help measuring the performance of a real network. As an example, the *total stationary rate of failure* is given (here removing index N) by

$$\tau = \sum_{j\in\mathcal{J}_1} \mu_j \mathbb{P}(\xi_j = 0) + \sum_{[ij]\in\mathcal{J}_2} \mu_{[ij]} \mathbb{E}(\xi_{[ij]} \mathbf{1}_{\xi_j = c_j}).$$

Using asymptotic independence and approximate distributions as stated in Corollary 5.2, one gets

$$\tau \approx \sum_{j\in\mathcal{J}_1} \frac{\mu_j}{\sum_{n=0}^{c_j} (\gamma r_j)^n} + \sum_{[ij]\in\mathcal{J}_2} \mu_{[ij]} \gamma r_{[ij]} \frac{(\gamma r_j)^{c_j}}{\sum_{n=0}^{c_j} (\gamma r_j)^n} = \sum_{j\in\mathcal{J}_1} \mu_j \frac{1 + (\gamma r_j)^{c_j+1}}{\sum_{n=0}^{c_j} (\gamma r_j)^n},$$

where we have used $\sum_i \mu_{[ij]} r_{[ij]} = \sum_i \nu_i q_{ij}/2 = \nu_j/2 = \mu_j r_j$.

If all traffic parameters are known, one can then minimize τ over γ, which amounts to choosing the best possible M, since M and γ are related by the one-to-one relation (4.10). Since the optimal γ, and hence M, should be of order J_1 (if r_j's are of order $1/J_1$), conditions of Corollary 5.2 are satisfied.

APPENDIX

PROOF OF THEOREM 2.1. Following the lines of the proof of the Lindeberg central limit theorem given in [2] (Theorem 27.2, page 359), it is not difficult to show that (2.3) is satisfied, for some sequence (A_N) converging to infinity, provided that the following reinforcement of the Lindeberg condition is satisfied: There exists some sequence of positive real numbers ε_N such that $\lim_{N\to\infty} \varepsilon_N = 0$ and

$$\lim_{N\to\infty} \frac{1}{b_N^2} \sum_{j=1}^{J(N)} \mathbb{E}\big((X_{j,N} - m_{j,N})^2 \mathbf{1}_{|X_{j,N} - m_{j,N}| > \varepsilon_N b_N}\big) = 0. \tag{A.1}$$

Now it is easily proved that the Lyapunov condition (2.1) implies existence of such a sequence (ε_N). Indeed, assuming that (2.1) is satisfied, then the following inequality holds for any positive ε:

$$\begin{aligned}\frac{1}{b_N^2}&\sum_{j=1}^{J(N)}\mathbb{E}\big((X_{j,N}-m_{j,N})^2\mathbf{1}_{|X_{j,N}-m_{j,N}|>\varepsilon b_N}\big)\\&\leq\frac{1}{\varepsilon^\delta b_N^{2+\delta}}\sum_{j=1}^{J(N)}\mathbb{E}\big(|X_{j,N}-m_{j,N}|^{2+\delta}\big)=\frac{\alpha_N}{\varepsilon^\delta},\end{aligned}$$

where δ is as in (2.1) and $\alpha_N\stackrel{\text{def}}{=}b_N^{-(2+\delta)}\sum_{j=1}^{J(N)}\mathbb{E}(|X_{j,N}-m_{j,N}|^{2+\delta})$, so that $\lim\alpha_N=0$. It results that (A.1) holds for ε_N defined as $\varepsilon_N=\alpha_N^{1/(2\delta)}$, which clearly satisfies $\lim\varepsilon_N=0$. $\square$

PROOF OF THEOREM 2.2. First note that (2.3) implies the following, apparently stronger, property:

$$\lim_{N\to\infty}\sup_{|t|\leq B_N}\big|e^{\frac{t^2}{2}}\mathbb{E}\big(e^{it\frac{S_N-a_N}{b_N}}\big)-1\big|=0, \tag{A.2}$$

for some sequence (B_N) of positive real numbers converging to infinity. Indeed, assuming that (2.3) holds and setting

$$\beta_N=\sup_{|t|\leq A_N}\big|\mathbb{E}\big(e^{it\frac{S_N-a_N}{b_N}}\big)-e^{-t^2/2}\big|$$

so that $\lim\beta_N=0$, equation (A.2) is then satisfied by any sequence (B_N) such that

$$0<B_N\leq A_N\quad\text{and}\quad e^{B_N^2/2}\leq\beta_N^{-1/2}\qquad\text{for all }N,$$

since, if these inequalities hold, then

$$\sup_{|t|\leq B_N}\big|e^{\frac{t^2}{2}}\mathbb{E}\big(e^{it\frac{S_N-a_N}{b_N}}\big)-1\big|\leq e^{\frac{B_N^2}{2}}\sup_{|t|\leq A_N}\big|\mathbb{E}\big(e^{it\frac{S_N-a_N}{b_N}}\big)-e^{-\frac{t^2}{2}}\big|\leq\beta_N^{1/2}.$$

The numbers $B_N=\min(A_N,\sqrt{-\log\beta_N})$ thus satisfy (A.2) together with $\lim B_N=\infty$.

The proof is now standard: Using inverse Fourier transform, it results from

$$\mathbb{E}\big(e^{itS_N}\big)=\sum_{k\in\mathbb{Z}}\mathbb{P}(S_N=k)e^{itk}\qquad(t\in\mathbb{R}),$$

that for any $k\in\mathbb{Z}$,

$$\mathbb{P}(S_N=k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-itk}\mathbb{E}\big(e^{itS_N}\big)\,dt=\frac{1}{2\pi b_N}\int_{-\pi b_N}^{\pi b_N}e^{-itz_N(k)}\mathbb{E}\big(e^{it\frac{S_N-a_N}{b_N}}\big)\,dt,$$

where we have set $z_N(k)=(k-a_N)/b_N$.

It can be assumed without loss of generality that $B_N \le \pi b_N$ for all $N \ge 1$. Otherwise, replace the sequence (B_N) by $(B_N \wedge \pi b_N)$, which still satisfies (A.2) and goes to infinity, since $\lim_N b_N = \infty$ by assumption (1). Then for all N and k,

$$
\begin{aligned}
&\bigl|2\pi b_N \mathbb{P}(S_N = k) - \sqrt{2\pi} e^{-z_N^2(k)/2}\bigr| \\
&\quad = \left| \int_{-\pi b_N}^{\pi b_N} e^{-itz_N(k)} \mathbb{E}\bigl(e^{it\frac{S_N - a_N}{b_N}}\bigr)\, dt - \int_{-\infty}^{\infty} e^{-itz_N(k) - t^2/2}\, dt \right| \\
&\quad = \left| \int_{-B_N}^{B_N} e^{-itz_N(k) - t^2/2} \bigl(e^{t^2/2} \mathbb{E}\bigl(e^{it\frac{S_N - a_N}{b_N}}\bigr) - 1\bigr)\, dt \right. \\
&\qquad \left. + \int_{B_N \le |t| \le \pi b_N} e^{-itz_N(k)} \mathbb{E}\bigl(e^{it\frac{S_N - a_N}{b_N}}\bigr)\, dt - \int_{|t| \ge B_N} e^{-itz_N(k) - t^2/2}\, dt \right| \\
&\quad \le \Bigl(\sup_{|t| \le B_N} \bigl| e^{t^2/2} \mathbb{E}\bigl(e^{it\frac{S_N - a_N}{b_N}}\bigr) - 1 \bigr| \Bigr) \int_{-\infty}^{\infty} e^{-t^2/2}\, dt \\
&\qquad + \int_{|t| \ge B_N} \phi(t)\, dt + \int_{|t| \ge B_N} e^{-t^2/2}\, dt.
\end{aligned}
$$

Here, assumption (3) has been used to dominate the second term. Since the three terms of the last sum do not depend on k and converge to zero as N goes to infinity, the theorem is proved. $\square$

PROOF OF PROPOSITION 2.1. One can assume without loss of generality that all the Poisson parameters $\lambda_{j,N}$ ($N \ge 1$, $j \in \mathcal{J}_N^2$) are bounded above by some fixed number, say 1 for example. Indeed, if this is not the case, one can replace each Poisson variable $X_{j,N}$ with $\lambda_{j,N} > 1$ by the sum of $\lfloor \lambda_{j,N} \rfloor + 1$ independent Poisson variables with the same parameter $\lambda_{j,N}(\lfloor \lambda_{j,N} \rfloor + 1)^{-1} < 1$. This change does not affect S_N, a_N and b_N [only $J(N)$ is increased]. Thus, the proposition will be true in the general case if it is proved to hold when all $\lambda_{j,N}$ are less than 1.

Let us now show that conditions (2) and (3) of Theorem 2.2 are satisfied [note that (1) coincides with (ii) of the proposition]. To prove that (2) holds, we use Theorem 2.1 and check that the Lyapunov condition is satisfied with $\delta = 1$. The following domination of the centered third moment of any positive random variable X by its noncentered third moment is useful:

$$\mathbb{E}\bigl(|X - \mathbb{E}(X)|^3\bigr) \le \mathbb{E}\bigl(X^3\bigr). \tag{A.3}$$

This results from the inequality, due to positivity of X, and hence of $m = \mathbb{E}(X)$, $|X - m|^3 \le (X + m)(X - m)^2 = X^3 - mX^2 - m^2X + m^3$, so that $\mathbb{E}(|X - m|^3) \le \mathbb{E}(X^3) - m\mathbb{E}(X^2) \le \mathbb{E}(X^3)$.

Recall that the third moment of a geometric random variable with parameter ρ is given by $(\rho + 4\rho^2 + \rho^3)/(1 - \rho)^3$, while that of a Poisson variable with parameter λ is $\lambda^3 + 3\lambda^2 + \lambda$. Using the fact that all $\rho_{j,N}$'s and $\lambda_{j,N}$'s are less than 1, one gets

$$
\begin{aligned}
&\frac{1}{b_N^3}\sum_{j=1}^{J(N)} \mathbb{E}\big(|X_{j,N} - m_{j,N}|^3\big) \\
&\quad \le \frac{1}{b_N^3}\Big(\max_{1\le j\le J(N)} \frac{\mathbb{E}(|X_{j,N} - m_{j,N}|^3)}{\sigma_{j,N}^2}\Big)\sum_{j=1}^{J(N)} \sigma_{j,N}^2 \\
&\quad \le \frac{1}{b_N}\max_{1\le j\le J(N)} \frac{\mathbb{E}(X_{j,N}^3)}{\sigma_{j,N}^2} \\
&\quad \le \frac{1}{b_N}\Big(\max_{j\in\mathcal{J}_1^N} \frac{1+4\rho_{j,N}+\rho_{j,N}^2}{1-\rho_{j,N}} + \max_{j\in\mathcal{J}_1^N}\big(\lambda_{j,N}^2 + 3\lambda_{j,N} + 1\big)\Big) \\
&\quad \le \frac{1}{b_N}\Big(\frac{6}{1-\rho} + 5\Big),
\end{aligned}
$$

where for the last step, assumption (i) is used. Then by (ii), the Lyapunov condition is satisfied with $\delta = 1$.

Now to check (3) of Theorem 2.2, recall that the characteristic function of a geometric random variable X with parameter ρ is given by

$$\mathbb{E}\big(e^{itX}\big) = \frac{1-\rho}{1-\rho e^{it}}$$

while if X is Poisson with parameter λ, then

$$\mathbb{E}\big(e^{itX}\big) = e^{\lambda(e^{it}-1)}.$$

One easily derives that

$$
\begin{aligned}
\big|\mathbb{E}\big(e^{itX}\big)\big| &= \frac{1-\rho}{\sqrt{1-2\rho\cos t+\rho^2}} \\
&= \frac{1-\rho}{\sqrt{(1-\rho)^2 + 2\rho(1-\cos t)}} \\
&= \Big(1 + 4\frac{\rho}{(1-\rho)^2}\sin^2(t/2)\Big)^{-\frac{1}{2}}
\end{aligned}
$$

in the first case, while in the second

$$\big|\mathbb{E}\big(e^{itX}\big)\big| = e^{-\lambda(1-\cos t)} = e^{-2\lambda\sin^2(t/2)}.$$

Then, using the convexity inequality $|\sin(t/2)| \geq |t|/\pi$ for $|t| \leq \pi$,

$$\begin{aligned}
|\mathbb{E}(e^{itS_N})| &= \prod_{j=1}^{J(N)} |\mathbb{E}(e^{itX_{j,N}})| \\
&\leq \biggl(\prod_{j\in\mathcal{J}_1^N}\biggl(1+4\frac{t^2}{\pi^2}\frac{\rho_{j,N}}{(1-\rho_{j,N})^2}\biggr)^{-\frac{1}{2}}\biggr)\exp\biggl(-2\frac{t^2}{\pi^2}\sum_{j\in\mathcal{J}_2^N}\lambda_{j,N}\biggr) \\
&= \exp\biggl[-\frac{1}{2}\sum_{j\in\mathcal{J}_1^N}\log\biggl(1+4\frac{t^2}{\pi^2}\frac{\rho_{j,N}}{(1-\rho_{j,N})^2}\biggr)-2\frac{t^2}{\pi^2}\sum_{j\in\mathcal{J}_2^N}\lambda_{j,N}\biggr] \\
&\qquad\qquad \text{for } |t| \leq \pi.
\end{aligned}$$

Due to concavity of the log, the inequality

$$4\frac{t^2}{\pi^2}\frac{\rho_{j,N}}{(1-\rho_{j,N})^2} \leq 4\frac{\rho}{(1-\rho)^2} \stackrel{\text{def}}{=} \tau$$

that holds for all $t \in [-\pi, \pi]$ and all $j \in \mathcal{J}_1^N$ gives

$$\log\biggl(1+4\frac{t^2}{\pi^2}\frac{\rho_{j,N}}{(1-\rho_{j,N})^2}\biggr) \geq 4\frac{\log(1+\tau)}{\tau}\frac{t^2}{\pi^2}\frac{\rho_{j,N}}{(1-\rho_{j,N})^2}.$$

One obtains

$$\begin{aligned}
|\mathbb{E}(e^{itS_N})| &\leq \exp\biggl(-2\frac{\log(1+\tau)}{\tau}\frac{t^2}{\pi^2}\sum_{j\in\mathcal{J}_1^N}\frac{\rho_{j,N}}{(1-\rho_{j,N})^2}-2\frac{t^2}{\pi^2}\sum_{j\in\mathcal{J}_2^N}\lambda_{j,N}\biggr) \\
&\leq \exp\biggl(-2\frac{\log(1+\tau)}{\tau}\frac{b_N^2 t^2}{\pi^2}\biggr) \qquad \text{for } |t| \leq \pi,
\end{aligned}$$

since $\frac{\log(1+\tau)}{\tau} \leq 1$. The condition (3) of Theorem 2.2 is thus satisfied with integrable ϕ given by $\phi(t) = \exp(-2\frac{\log(1+\tau)}{\tau}\frac{t^2}{\pi^2})$. $\square$

PROOF OF PROPOSITION 2.2. As in the preceding proof, one can assume that all $\lambda_{j,N}$ are less than one, and then proceed to check conditions (2) and (3) of Theorem 2.2. Condition (2) is obtained, here again, by proving that the Lyapunov condition holds with $\delta = 1$. This goes exactly as in Proposition 2.1, except for the truncated geometric variables $X_{j,N}$ for $j \in \mathcal{J}_N^1$. Here, instead of using (A.3), we use the following inequality (since $|X_{j,N} - m_{j,N}|$ is dominated by $c_{j,N}$):

$$\mathbb{E}(|X_{j,N}-m_{j,N}|^3) \leq c_{j,N}\mathbb{E}([X_{j,N}-m_{j,N}]^2) = c_{j,N}\sigma_{j,N}^2 \leq C\sigma_{j,N}^2.$$

This gives

$$\frac{1}{b_N^3}\sum_{j=1}^{J(N)}\mathbb{E}\big(|X_{j,N}-m_{j,N}|^3\big)\le\frac{1}{b_N}\bigg(\max_{1\le j\le J(N)}\frac{\mathbb{E}(|X_{j,N}-m_{j,N}|^3)}{\sigma_{j,N}^2}\bigg)$$
$$\le\frac{1}{b_N}\Big(C+\max_{j\in\mathcal{J}_1^N}\big(\lambda_{j,N}^2+3\lambda_{j,N}+1\big)\Big)\le\frac{1}{b_N}(C+5)$$

and by (ii), the last quantity goes to zero as N goes to infinity. The Lyapunov condition, and hence (2), is satisfied.

As for condition (3), using Lemma 2.2 gives

$$\big|\mathbb{E}(e^{itS_N})\big|\le\exp\bigg(-2\kappa\frac{t^2}{\pi^2}\sum_{j\in\mathcal{J}_1^N}\sigma_{j,N}^2-2\frac{t^2}{\pi^2}\sum_{j\in\mathcal{J}_2^N}\lambda_{j,N}\bigg)$$
$$\le\exp\bigg(-2\min(1,\kappa)\frac{b_N^2t^2}{\pi^2}\bigg)\qquad\text{for } |t|\le\pi.\qquad\square$$

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