The Annals of Applied Probability
2016, Vol. 26, No. 6, 3699–3726
DOI: 10.1214/16-AAP1189

LARGE-TIME OPTION PRICING USING THE DONSKER–VARADHAN LDP—CORRELATED STOCHASTIC VOLATILITY WITH STOCHASTIC INTEREST RATES AND JUMPS

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We establish a large-time large deviation principle (LDP) for a general mean-reverting stochastic volatility model with nonzero correlation and sublinear growth for the volatility coefficient, using the Donsker–Varadhan [*Comm. Pure Appl. Math.* **36** (1983) 183–212] LDP for the occupation measure of a Feller process under mild ergodicity conditions. We verify that these conditions are satisfied when the process driving the volatility is an Ornstein–Uhlenbeck (OU) process with a perturbed (sublinear) drift. We then translate these results into large-time asymptotics for call options and implied volatility and we verify our results numerically using Monte Carlo simulation. Finally, we extend our analysis to include a CIR short rate process and an independent driving Lévy process.

1. Introduction. The last few years has seen the emergence of a number of articles on large-time asymptotics for stochastic volatility models, with and without jumps. Using the Gärtner–Ellis theorem from large deviations theory, [13] compute the asymptotic (i.e., leading order) implied volatility smile for the well-known Heston model in the so-called large-time, large log-moneyness regime, under a mild restriction on the model parameters, and the rate function is computed numerically as a Fenchel–Legendre transform which is just a one-dimensional root-finding exercise. [17] show that the asymptotic smile can actually be computed in closed-form via the SVI parameterization and [14] compute the correction term to this smile using saddlepoint methods; [12] derives a similar result for the Stein–Stein model and [19] derive a similar result for a displaced Heston model (and relax the aforementioned condition on the parameters). [18] extended the results in [13] to a general class of affine stochastic volatility models (with jumps), which includes the Heston, Bates and the Barndorff–Nielsen–Shephard model, and under mild assumptions, they show that the limiting smile necessarily corresponds to the smile generated by an exponential Lévy model. More recently, [15] compute large-time asymptotics for a fractional local-stochastic volatility model and large-time asymptotics for European and barrier options under conventional and

Received November 2014; revised February 2016.
[1] Supported by NSF Grant DMS-12-09363.
MSC2010 subject classifications. 60G99, 60J60, 60J25.
Key words and phrases. Stochastic volatility, large deviations, Donsker–Varadhan large deviation principle, implied volatility asymptotics, ergodic processes, occupation measures.

fractional exponential Lévy models, using the deAcosta LDP for a Lévy process on path space.

[11] derives a large deviation principle for the log stock price under an uncorrelated stochastic volatility model driven by an Ornstein–Uhlenbeck process with a bounded volatility function. For this, we use the fact that the occupation measure for the Ornstein–Uhlenbeck process satisfies an LDP with a good, convex lower semicontinuous rate function under the topology of weak convergence (and under the Prokhorov metric); see Section 7 in Donsker and Varadhan [5] (see also page 178 in Stroock [25] and Proposition 1.3 in [21]), combined with the standard contraction principle and exponential tightness. The large-time regime is also closely related to the small-time, fast mean reverting regime considered in Feng, Fouque and Kumar [8] for a more general stochastic volatility model. The problem then falls into the class of homogenization and averaging problems for nonlinear HJB type equations, where the fast volatility variable lives on a noncompact set.

1.1. *Outline of article.* In this article, we consider a stochastic volatility model for a log stock price process X_t of the form

$$\begin{cases} dX_t = -\frac{1}{2}\sigma(Y_t)^2\,dt + \sigma(Y_t)\big(\sqrt{1-\rho^2}\,dW_t^1 + \rho\,dW_t^2\big), \\ dY_t = \big(-\alpha Y_t + g(Y_t)\big)\,dt + dW_t^2, \end{cases} \tag{1.1}$$

where W^1, W^2 are independent Brownian motions. We first relax the assumptions that σ is bounded and ρ and g are zero that are imposed in [11]. This requires an auxiliary result, namely that the variance of a probability measure on the real line can be bounded in terms of the Donsker–Varadhan rate function of the measure. Using this property, we then establish an LDP for (X_t/t) using the trivial joint LDP for the two independent variables $(W_t^1/t, \mu_t)$ (where μ_t is the occupation measure of Y), combined with the extended contraction principle for noncontinuous functionals given in Theorem 4.2.23 in [3]. This is the same theorem which can be used to prove the Freidlin–Wentzell *small-noise* LDP from Schilder's theorem, despite the lack of continuity of the Itô map in the sup norm topology (see, e.g., proof of Theorem 5.6.7 in [3]), and is also used in rough paths theory to prove the small-noise LDP for a rough differential equation driven by fractional Brownian motion (cf. Section 15.7 and Proposition 19.14 in [16]). The rate function for X_t/t in this article has the variational representation $I(x) = \inf_{\mu\in\mathcal{P}(\mathbb{R})}[\frac{(x-M(\mu))^2}{2\nu(\mu)} + I_Y(\mu)]$, for some linear functionals $M(\mu)$, $\nu(\mu)$ which depend on the correlation ρ between the log stock price process and the Y process. For the at-the-money case $x = 0$ with $\rho = 0$, we find that the rate function reduces to the classical Rayleigh–Ritz formula for the principal eigenvalue λ_1 of an associated Sturm–Liouville equation.

In Section 5, we translate these results into large-time asymptotics for call options and implied volatility; this requires computing the corresponding LDP for the log stock price under the so-called Share measure $\mathbb{P}^*$ associated with using the stock price process as the numéraire, and in Section 6, we compute $I(x)$ numerically, using the Ritz method from the theory of calculus of variations. The Ritz method is described at length in Gelfand and Fomin [10]—we choose an n-dimensional subspace of the space of admissible functions, in this case the Hilbert space $L^2(\mu_\infty)$, and we then minimize the objective function $\frac{(x-M(\mu))^2}{2v(\mu)} + I_Y(\mu)$ by minimizing over the subspace for the n Fourier coefficients.

In Section 7, we enrich the general model with an additional independent CIR short rate process r_t and an independent driving Lévy process Z_t. It is well known that stochastic interest rates make a significant difference to the price of European options at large maturities, but to our knowledge this effect has never been properly quantified using asymptotics; specifically, we show that the log stock price now satisfies the LDP with rate function $I_r(x) = \inf_{a,y,z:a+y+z=x}[I(y) + I_{\mathrm{CIR}}(a) + V_J^*(z)]$ where $I_{\mathrm{CIR}}(a) = \kappa_r^2(a-\theta_r)^2/(2a\sigma_r^2)$ is the rate function for $\frac{1}{t}\int_0^t r_s\,ds$, and $V_J^*(x)$ is the rate function for Z_t/t.

2. The Donsker–Varadhan large deviation principle. Let Ω denote the space of real-valued functions $\omega(\cdot)$ on $-\infty < t < \infty$ with discontinuities of the first kind, normalized to be right continuous, and with convergence induced by the Skorokhod topology on bounded intervals. Let $(Y, \mathbb{P}_y)$ be a Markov process on Ω with invariant distribution $\mu_\infty(dy)$ such that the mapping $y \to \mathbb{P}_y$ is weakly continuous (which implies the Feller property for the process Y). Let $p(t,x,dy)$ denote the transition probability for Y, P_t denote the semigroup associated with Y, and let L denote the infinitesimal generator of P_t and $\mathcal{D} = \mathcal{D}(L) \subset C_b(\mathbb{R})$ its domain. For each $t > 0$ and $A \in \mathcal{B}(\mathbb{R})$, let

$$\mu_t(A) = \frac{1}{t}\int_0^t 1_A(Y_s)\,ds$$

denote the occupation time distribution of Y, that is, the proportion of time that Y spends in the set A. For each $t > 0$ and ω, $\mu_t(\cdot)$ is a probability measure on $\mathbb{R}$. Let $\mathcal{P}(\mathbb{R})$ denote the space of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then from [5] (or page 178 in Stroock [25] or Pinsky [23]), under suitable recurrence and transitivity conditions (see the next subsection for details), $\mu_t(\cdot)$ satisfies the LDP as $t \to \infty$ in the topology of weak convergence, with a convex, lower semicontinuous rate function $I_Y : \mathcal{P}(\mathbb{R}) \mapsto [0,\infty]$ given by

$$I_Y(\mu) = -\inf_{u\in\mathcal{D}^+}\int_{-\infty}^{\infty}\frac{Lu}{u}\,d\mu \tag{2.1}$$

for each $\mu \in \mathcal{P}(\mathbb{R})$, where $\mathcal{D}^+$ is the set of u in the domain $\mathcal{D}$ of L with $u \geq \varepsilon > 0$ for some $\varepsilon > 0$. More precisely, if we define a probability measure $Q_{t,y}$ on $\mathcal{P}(\mathbb{R})$ by $Q_{t,y} = \mathbb{P}_y \circ \mu_t^{-1}$, then for any closed set $C \subset \mathcal{P}(\mathbb{R})$ (weak topology) and for any open set on $G \subset \mathcal{P}(\mathbb{R})$ we have

$$\limsup_{t\to\infty} \frac{1}{t} \log Q_{t,y}(C) \le -\inf_{\mu\in C} I_Y(\mu),$$
(2.2)
$$\liminf_{t\to\infty} \frac{1}{t} \log Q_{t,y}(G) \ge -\inf_{\mu\in G} I_Y(\mu).$$

$I_Y(\cdot)$ is know as the I-function for the process Y.

REMARK 2.1. By the ergodic theorem, $Q_{t,y} \xrightarrow{w} \delta_{\mu_\infty}$ as $t \to \infty$, and it is well known that $I_Y(\mu) = 0$ if and only if $\mu = \mu_\infty$ (see, e.g., the proof of Corollary 1.5 in [23]).

2.1. *Sufficient conditions for the LDP upper and lower bounds.* In [26] (page 34) and [5, 6], it is shown that the following five conditions imply the LDP upper bound in (2.2).

There exists a sequence u_n of functions in $\mathcal{D}(L)$ with the five properties:

1. $u_n(y) \ge c > 0$ for y and n.
2. For all compact sets $K \subset \mathbb{R}$, there exists a constant C_K such that $\sup_{y\in K} \sup_n u_n(y) \le C_K$.
3. $V_n(y) := -(Lu_n/u_n)(y) \ge -C$ for all n and y.
4. There exists a function $V(y)$ such that for all $y \in \mathbb{R}$, $\lim_{n\to\infty} V_n(y) = V(y)$.
5. The set $\{y : V(y) \le \ell\}$ is compact for all $\ell < \infty$.

Moreover, the following two conditions imply the LDP lower bound:

There exists a density function for $p(1, x, dy)$ with respect to a reference measure α on $\mathbb{R}$ such that:

I. $p(1, x, dy) = p(1, x, y)\alpha(dy)$.
II. $p(1, x, \cdot)$ as a mapping from $\mathbb{R} \to L^1(\alpha)$ is continuous.

These two conditions are given in [6, 26], where the LDP is proved as a corollary of a more general LDP on path space in terms of the entropy function (see Theorem 13.1.31 in [26]). These two conditions simplify the more cumbersome conditions for the LDP lower bound given on page 393 in [5].

2.2. *Examples*: *The OU process and the perturbed OU process.*

- For the Ornstein–Uhlenbeck process,

$$dY_t = -\alpha Y_t\, dt + dW_t$$

conditions 1–5 in Section 2.1 are satisfied with $u_n(y) = \cosh(n\theta(y/n))$, if $\theta(y) = y$ for $0 \le y \le 1$ and $\theta, \theta', \theta''$ are uniformly bounded on $\mathbb{R}$ and θ is odd (see Sections 7 in [5] and [6]), and in this case $V(y) = -\frac{Lu}{u}(y) =$

$-\frac{1}{2}+y\alpha\tanh y$, which tends to $|y|$ as $y\to\pm\infty$. In the next bullet point, we will show that the two lower bounds are also satisfied, as a special case of a more general perturbed OU process. Thus, μ_t satisfies the LDP with rate $I_Y(\mu)$ as in (2.1) as $t\to\infty$. The OU process has a unique invariant distribution given by $\mu_\infty(y)=(\frac{\alpha}{\pi})^{\frac{1}{2}}e^{-\alpha y^2}$, that is, $N(0,\frac{1}{2\alpha})$. If μ is absolutely continuous with respect to μ_∞, then [because the OU process is symmetric, that is, its generator is self-adjoint with respect to $\mu_\infty(y)$] we can simplify the rate function I_Y to

$$I_Y(\mu)=\frac{1}{2}\int_{-\infty}^{\infty}\psi'(y)^2\mu_\infty(dy) \tag{2.3}$$

if $\psi'\in L^2(\mu_\infty)$, where $\phi=\frac{d\mu}{d\mu_\infty}$ is the Radon–Nikodým derivative and $\psi=\sqrt{\phi}$ (see page 179 under exercise (8.28) in Stroock [25]). If μ is not absolutely continuous with respect to μ_∞, then $I_Y(\mu)=\infty$. The representation in (2.3) will be used for the numerics in Section 6 using the Ritz method.

- For a perturbed OU process of the form

$$dY_t=\big(-\alpha Y_t+g(Y_t)\big)\,dt+dW_t, \tag{2.4}$$

where g is C^3 with sublinear growth at $\pm\infty$ and continuous bounded derivatives of all orders up and including 3, the $-\alpha y$ drift term swamps the $g(y)$ term as $|y|\to\infty$ and the five conditions 1–5 for the LDP upper bound are still satisfied with the same $u_n(y)$ as above. This includes the case when, for example, $g(y)=\alpha\theta$ for a constant θ which is the mean-reversion level for Y.

LEMMA 2.1. *The perturbed OU process in* (2.4) *satisfies the two lower bound conditions I and II above.*

PROOF. See Appendix B. □

We also note that the process Y in (2.4) has a unique invariant distribution given by

$$\mu_\infty(y)=\frac{e^{-\alpha y^2}e^{2\int_{y_0}^{y}g(u)\,du}}{\int_{-\infty}^{\infty}e^{-\alpha u^2}e^{2\int_{y_0}^{u}g(v)\,dv\,du}\,du}. \tag{2.5}$$

2.3. *The Prokhorov metric on $\mathcal{P}(\mathbb{R})$ and goodness of the rate function $I_Y(\mu)$.* We can also topologize $\mathcal{P}(\mathbb{R})$ with the *Prokhorov metric*, defined as

$$d(\mu,\mu_1)=\inf\{\delta>0:\mu(C)\le\mu_1(C^\delta)+\delta \text{ for all closed } C\in\mathcal{B}(\mathbb{R})\}$$

for $\mu,\mu_1\in\mathcal{P}(\mathbb{R})$, where C^δ is the δ-neighbourhood of C[2] (see page 96 in Ethier and Kurtz [7]). Under this metric, $\mathcal{P}(\mathbb{R})$ is a metric space [note also that

[2]The set of all points which are of distance $\le\delta$ from C.

$d(\mu, \mu_1) \leq 1$ for all μ, μ_1]. Moreover, $\mathbb{R}$ is separable, so convergence of measures in the Prokhorov metric is equivalent to weak convergence of measures (see Theorem 3.1 part (a) and part (b) in [7] for details), so the Donsker–Varadhan LDP for μ_t also holds in the topology induced by the metric d.

REMARK 2.2. By Lemma 7.1 (see also page 461) in [5] μ_t is exponentially tight in the weak topology (and thus also in the Prokhorov topology), and thus (by Lemma 1.2.18 in [3]) $I_Y(\cdot)$ is a *good* rate function.

2.4. *The tail behaviour of probability measures inside the level sets of I_Y.* The following lemma is the main observation on which the article is based, which characterizes the tail behaviour of the measures inside a level set of I_Y.

LEMMA 2.2. *Consider the perturbed OU process in* (2.4). *Then for $\mu \in \mathcal{P}(\mathbb{R})$ we have the following bound for the second moment of μ in terms of $I_Y(\mu)$:*

$$\int_{-\infty}^{\infty} y^2 \mu(dy) \leq K_2(\alpha) I_Y(\mu) + K_3(\alpha)$$

for some constants $K_2(\alpha) > 0$ and $K_3(\alpha)$.

PROOF. The infinitesimal generator $\mathcal{L}$ of Y coincides with the differential operator $\mathcal{L} = (-\alpha y + g(y))\partial_y + \frac{1}{2}\partial_{yy}^2$ on $C_b^2(\mathbb{R})$. Define a function ψ such that

$$\psi(y) := \begin{cases} y, & (0 \leq y \leq 1), \\ 2, & (y \geq 2), \\ \text{smoothly increasing}, & (1 \leq y \leq 2) \end{cases}$$

and ψ is an odd function. Consequently, ψ, ψ', ψ'' are uniformly bounded, $\psi' \geq 0$ and $\psi(u)/u > 0$ and is uniformly bounded when $u \neq 0$. Let $u_n(y) = e^{\frac{c}{2}[n\psi(\frac{y}{n})]^2}$ with $c \in (0, \alpha \wedge (\frac{\alpha}{\sup_{u \neq 0} \frac{\psi(u)}{u}\psi'(u)}))$. Then

$$\begin{aligned} -Lu_n(y) &= -\biggl[(-\alpha y + g(y))\Bigl(cn\psi\Bigl(\frac{y}{n}\Bigr)\psi'\Bigl(\frac{y}{n}\Bigr)\Bigr) \\ &\quad + \frac{1}{2}\Bigl(c^2 n^2 \psi^2\Bigl(\frac{y}{n}\Bigr)\Bigl(\psi'\Bigl(\frac{y}{n}\Bigr)\Bigr)^2 + c\Bigl(\psi'\Bigl(\frac{y}{n}\Bigr)\Bigr)^2 \\ &\quad + c\psi\Bigl(\frac{y}{n}\Bigr)\psi''\Bigl(\frac{y}{n}\Bigr)\Bigr)\biggr] 1_{\{|\frac{y}{n}| \leq 2\}} u_n(y) \\ &= -\biggl[(-\alpha y^2 + yg(y))\Bigl(c\frac{n}{y}\psi\Bigl(\frac{y}{n}\Bigr)\psi'\Bigl(\frac{y}{n}\Bigr)\Bigr) \\ &\quad + \frac{1}{2}\Bigl(c^2 y^2 \frac{n^2}{y^2}\psi^2\Bigl(\frac{y}{n}\Bigr)\Bigl(\psi'\Bigl(\frac{y}{n}\Bigr)\Bigr)^2 + c\Bigl(\psi'\Bigl(\frac{y}{n}\Bigr)\Bigr)^2 \end{aligned}$$

$$
\begin{aligned}
&\qquad + c\psi\left(\frac{y}{n}\right)\psi''\left(\frac{y}{n}\right)\right)\Bigg]u_n(y) \\
-\frac{Lu_n}{u_n}(y) &= c\frac{\psi(y/n)}{y/n}\psi'(y/n)\left[\alpha y^2 - yg(y) - \frac{cy^2}{2}\frac{\psi(y/n)}{y/n}\psi'(y/n)\right] \\
&\quad - \frac{1}{2}\left[c\left(\psi'\left(\frac{y}{n}\right)\right)^2 + c\psi\left(\frac{y}{n}\right)\psi''\left(\frac{y}{n}\right)\right] \\
&= I + II.
\end{aligned}
$$

Observe that the second term II is uniformly bounded as ψ, ψ', ψ'' are uniformly bounded. As for the first term I, note that $\frac{\psi(u)}{u}\psi'(u) > 0$ and is uniformly bounded for $u \neq 0$, hence $-\frac{cy^2}{2}\frac{\psi(u)}{u}\psi'(u) > -\frac{\alpha}{2}y^2$ if $c\frac{\psi(u)\psi'(u)}{u} < \alpha$. Moreover, since $g(y)$ has sublinear growth, there exists a constant $c_1 > 0$ such that $\frac{\alpha}{2}y^2 - yg(y) > -c_1$ for all y. Hence, $-\frac{Lu_n}{u_n}(y)$ is uniformly bounded from below. Since $\psi(y/n) = y/n$ and $\psi'(y/n) = 1$ for $|y| \le n$, it is trivial to check that $-\frac{Lu_n}{u_n}(y) \to (c\alpha - \frac{c^2}{2})y^2 - cyg(y) - \frac{c}{2}$ pointwise as $n \to \infty$ and $Lu_n \in C_b$ because $Lu_n(y) = 0$ for y sufficiently large, so $u_n \in \mathcal{D}^+$. From this, we obtain

$$I_Y(\mu) = \sup_{u \in \mathcal{D}^+} -\int_{-\infty}^{\infty} \frac{Lu}{u}\, d\mu \ge -\int_{-\infty}^{\infty} \frac{Lu_n}{u_n}\, d\mu.$$

Taking the liminf of both sides as $n \to \infty$ and using Fatou's lemma, we obtain

$$I_Y(\mu) \ge \liminf_{n\to\infty} \int_{-\infty}^{\infty} -\frac{Lu_n}{u_n}\, d\mu \ge \int_{-\infty}^{\infty} \left(\left(c\alpha - \frac{c^2}{2}\right)y^2 - cyg(y) - \frac{c}{2}\right)\mu(dy).$$

Since $yg(y)$ is subquadratic, we can find a positive constant $K_1(\alpha)$ such that $(c\alpha - c^2/2)y^2 - cyg(y) = \frac{(c\alpha - c^2/2)}{2}y^2 + \frac{(c\alpha - c^2/2)}{2}y^2 - cyg(y) \ge \frac{(c\alpha - c^2/2)}{2}y^2 - K_1(\alpha)$. Thus, we have that $I_Y(\mu) \ge \frac{(c\alpha - c^2/2)}{2}\int_{-\infty}^{\infty} y^2\mu(dy) - K_1(\alpha) - \frac{1}{2}c$, and the result follows by re-arranging. □

3. The stochastic volatility model.

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ unless otherwise stated, with a filtration $(\mathcal{F}_t)_{t\ge 0}$ and satisfying the usual conditions. We consider the following stochastic volatility model for a log stock price process $X_t = \log S_t$ driven by a perturbed Ornstein–Uhlenbeck process Y:

$$
\begin{cases}
dX_t = -\frac{1}{2}\sigma(Y_t)^2\, dt + \sigma(Y_t)\left(\rho\, dW_t^2 + \bar{\rho}\, dW_t^1\right), \\
dY_t = \left(-\alpha Y_t + g(Y_t)\right) dt + dW_t^2,
\end{cases}
\tag{3.1}
$$

where $\alpha > 0$, $X_0 = x_0$, $Y_0 = y_0$, W^1, W^2 are two independent standard Brownian motions, $\rho \in (-1, 1)$, $\bar{\rho} = \sqrt{1 - \rho^2}$ and *we make the following assumptions on σ and g throughout.*

ASSUMPTION 3.1. $\sigma : \mathbb{R} \mapsto (0, \infty)$ and $g : \mathbb{R} \mapsto \mathbb{R}$ are both continuous and satisfy the sublinear growth conditions[3]

$$\sigma(y) \vee g(y) \le K_1\big(1 + |y|^p\big) \tag{3.2}$$

for some $K_1 > 0$, $p \in (0, 1)$.

ASSUMPTION 3.2. g has continuous bounded derivatives of all orders order up and including 3, and if $\rho \neq 0$ then σ is differentiable and $|\sigma'(y)|$ is bounded.

REMARK 3.3. Note that for the seemingly more general model:

$$\begin{cases} dX_t = -\frac{1}{2} f(V_t)^2\,dt + f(V_t)\big(\rho\,dW_t^2 + \bar{\rho}\,dW_t^1\big), \\ dV_t = \big[\alpha(m - V_t) + \tilde{h}(V_t)\big]\,dt + \beta\,dW_t^2 \end{cases} \tag{3.3}$$

for $\alpha, \beta > 0$ and f, $\tilde{h}$ satisfying the same conditions as σ and g above, if we set $Y_t = \frac{1}{\beta}(V_t - m)$ and $\sigma(y) = f(\beta y + m)$, $g(y) = \tilde{h}(\beta y + m)$, then we are transformed back to a model of the model in (3.1), so there is no loss of generality in our assumption of zero mean reversion level and vol-of-vol (i.e., diffusion coefficient) equal to 1 in the Y process in (3.1).

We also set $S_0 = 1$ throughout (i.e., $x_0 = 0$) without loss of generality, because $X_t - x_0$ is independent of x_0 as the SDEs have no dependence on x.

3.1. *The integrated variance.* Now let $F : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}^+$ denote the linear functional defined by

$$F(\mu) = \int_{-\infty}^{\infty} \sigma^2(y)\mu(dy). \tag{3.4}$$

Note that F may not be continuous in the weak topology because σ^2 may not be bounded. Define

$$F(\mu_t) = \int_{-\infty}^{\infty} \sigma^2(y)\mu_t(dy) = \frac{1}{t}\int_0^t \sigma^2(Y_s)\,ds,$$

where $\mu_t(dy)$ is the occupation measure of Y; then we see that $F(\mu_t)$ is the time-average of the instantaneous variance for Y. We also define

$$\bar{\sigma}^2 = \int_{-\infty}^{\infty} \sigma^2(y)\mu_\infty(y)\,dy, \tag{3.5}$$

where μ_∞ is defined in (2.5).

[3]The same condition appears in Feng, Fouque and Kumar [8].

4. Large-time asymptotics for the stochastic volatility model.

4.1. *The main result*: *The LDP for the log stock price as* $t \to \infty$. We now state the first main result, which is a large deviation principle for the re-scaled log stock price (X_t/t) as $t \to \infty$.

THEOREM 4.1. *Consider the process* X *defined in* (3.1) *and let* $b(y) = \sigma(y)(\alpha y - g(y)) - \frac{1}{2}\sigma'(y)$, $G(\mu) = \int_{-\infty}^{\infty} b(y)\mu(dy)$. *Then under Assumptions* 3.1 *and* 3.2, X_t/t *satisfies the LDP as* $t \to \infty$ *with a good rate function given by*

$$I(x) = \inf_{\mu \in \mathcal{P}(\mathbb{R})} \left[\frac{(x - M(\mu))^2}{2\nu(\mu)} + I_Y(\mu) \right], \tag{4.1}$$

where $M(\mu) = -\frac{1}{2}F(\mu) + \rho G(\mu)$, $\nu(\mu) = \bar{\rho}^2 F(\mu)$ *and* $I_Y(\mu)$ *is the rate function for the occupation measure of* Y *defined in* (2.1).

PROOF. Integrating (3.1), we see that

$$X_t = -\frac{1}{2}\int_0^t \sigma(Y_s)^2\,ds + \int_0^t \sigma(Y_s)\big(\rho\,dW_s^2 + \bar{\rho}\,dW_s^1\big).$$

If we let $\chi(y) = \int_{y_0}^{y} \sigma(u)\,du$, then

$$\begin{aligned} d\chi(Y_t) &= \sigma(Y_t)\,dY_t + \frac{1}{2}\sigma'(Y_t)\,d\langle Y\rangle_t \\ &= \sigma(Y_t)\big(\big(-\alpha Y_t + g(Y_t)\big)\,dt + dW_t^2\big) + \frac{1}{2}\sigma'(Y_t)\,dt \end{aligned}$$

which we can integrate and re-arrange as follows:

$$\begin{aligned} \int_0^t \sigma(Y_s)\,dW_s^2 &= \chi(Y_t) + \int_0^t \left[\sigma(Y_s)\big(\alpha Y_s - g(Y_s)\big) - \frac{1}{2}\sigma'(Y_s)\right] ds \\ &= \chi(Y_t) + \int_0^t b(Y_s)\,ds. \end{aligned}$$

Now let $Z_t = W_t^1/t$ and $\hat{X}_t = X_t/t$. Conditioning on $(Y_s; 0 \le s \le t)$, we obtain

$$\begin{aligned} \hat{X}_t &\overset{d}{=} -\frac{1}{2}F(\mu_t) + \rho\left[G(\mu_t) + \frac{1}{t}\chi(Y_t)\right] + \frac{\bar{\rho}}{t}W^1_{tF(\mu_t)} \\ &\overset{d}{=} -\frac{1}{2}F(\mu_t) + \rho\left[G(\mu_t) + \frac{1}{t}\chi(Y_t)\right] + \frac{\bar{\rho}\sqrt{F(\mu_t)}}{t}W_t^1 \qquad (4.2) \\ &= M(\mu_t) + \sqrt{\nu(\mu_t)}Z_t + \frac{\rho}{t}\chi(Y_t), \end{aligned}$$

where $M(\mu) = -\frac{1}{2}F(\mu) + \rho G(\mu)$ and $\nu(\mu) = \bar{\rho}^2 F(\mu)$. From the Gärtner–Ellis theorem, we know that Z_t satisfies a large time LDP with good rate function rate

$\frac{1}{2}z^2$, we also know that μ_t satisfies a large time LDP with good rate function $I_Y(\mu)$. Moreover, Z_t and μ_t are independent, so we have

$$\begin{aligned}\mathcal{I}(z,\mu) &= -\lim_{\delta\to 0}\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\big(Z_t\in B_\delta(z),\mu_t\in B_\delta(\mu)\big)\\ &= -\lim_{\delta\to 0}\lim_{t\to\infty}\frac{1}{t}\big[\log\mathbb{P}\big(Z_t\in B_\delta(z)\big)+\log\mathbb{P}\big(\mu_t\in B_\delta(\mu)\big)\big]\\ &= \frac{1}{2}z^2+I_Y(\mu).\end{aligned}\tag{4.3}$$

Thus, (Z_t,μ_t) satisfies the weak LDP with rate $\mathcal{I}(z,\mu)=\frac{1}{2}z^2+I_Y(\mu)$. Since μ_t is exponentially tight (by Remark 2.2), for any $c>0$, there exists a compact set $K_c\subset\mathcal{P}(\mathbb{R})$ such that $\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(\mu_t\notin K_c)\le -c$. Thus, for any $c>0$, there exists a compact set $[-\sqrt{2c},\sqrt{2c}]\times K_c\subset\mathbb{R}\times\mathcal{P}(\mathbb{R})$ such that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\big((Z_t,\mu_t)\notin[-\sqrt{2c},\sqrt{2c}]\times K_c\big)\\ &\quad\le\limsup_{t\to\infty}\frac{1}{t}\log\big[\mathbb{P}\big(|Z_t|>\sqrt{2c}\big)+\mathbb{P}(\mu_t\notin K_c)\big]\\ &\quad\le\max\Big\{\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\big(|Z_t|>\sqrt{2c}\big),\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(\mu_t\notin K_c)\Big\}\\ &\quad\le -c.\end{aligned}$$

Thus, (Z_t,μ_t) is exponentially tight, so (Z_t,μ_t) satisfies the full LDP and (by Lemma 1.2.18b in [3]) the rate function $\mathcal{I}(z,\mu)$ is good. From (4.2), we have

$$\hat{X}_t\stackrel{d}{=}\tilde{X}_t:=\varphi(Z_t,\mu_t)+\frac{\rho}{t}\chi(Y_t),$$

where $\varphi:\mathbb{R}\times\mathcal{P}(\mathbb{R})\mapsto\mathbb{R}$ is given by $\varphi(z,\mu)=M(\mu)+\sqrt{\nu(\mu)}z$. Similarly, define

$$\tilde{X}^m_t=\varphi^m(Z_t,\mu_t),$$

where $\varphi^m(z)=M^m(\mu)+\sqrt{\nu^m(\mu)}z$, where we have truncated the integrands in $M(\mu)$ and $\nu(\mu)$ to get

$$\begin{aligned}M^m(\mu)=\int\Big[&\Big(-\frac{1}{2}\sigma^2(y)+\rho b(y)\Big)1_{\{|y|\le m\}}+\Big(-\frac{1}{2}\sigma^2(m)+\rho b(m)\Big)1_{\{y>m\}}\\ &+\Big(-\frac{1}{2}\sigma^2(-m)+\rho b(-m)\Big)1_{\{y<-m\}}\Big]\mu(dy)\end{aligned}$$

and

$$\nu^m(\mu)=\bar{\rho}^2\int[\sigma^2(y)1_{|y|\le m}+\sigma^2(m)1_{y>m}+\sigma^2(-m)1_{y<-m}]\mu(dy).$$

Since the integrands are bounded and continuous functions of $\mathbb{R}$, $M^m(\mu)$ and $\nu^m(\mu)$ are continuous functionals of μ under the weak topology. Using the Hölder continuity of the square root function: $|\sqrt{x}-\sqrt{y}|\le\sqrt{|x-y|}$, we have

$$|\tilde{X}_t-\tilde{X}_t^m|\le|M(\mu_t)-M^m(\mu_t)|+\sqrt{|\nu(\mu_t)-\nu^m(\mu_t)|}|Z_t|+\left|\frac{\rho}{t}\chi(Y_t)\right|.$$

Then

$$\begin{aligned}
&\mathbb{P}(|\tilde{X}_t-\tilde{X}_t^m|>\delta)\\
&\quad\le\mathbb{P}\left(|M(\mu_t)-M^m(\mu_t)|+\sqrt{|\nu(\mu_t)-\nu^m(\mu_t)|}|Z_t|+\left|\frac{\rho}{t}\chi(Y_t)\right|>\delta\right)\\
&\quad\le\mathbb{P}\left(|M(\mu_t)-M^m(\mu_t)|+\sqrt{|\nu(\mu_t)-\nu^m(\mu_t)|}|Z_t|>\frac{1}{2}\delta\right)\\
&\qquad+\mathbb{P}\left(\left|\frac{\rho}{t}\chi(Y_t)\right|>\frac{1}{2}\delta\right)\\
&\quad\le\mathbb{E}(1_{|M(\mu_t)-M^m(\mu_t)|+\sqrt{|\nu(\mu_t)-\nu^m(\mu_t)|}|Z_t|>\frac{1}{2}\delta}1_{I_Y(\mu_t)\le c})+\mathbb{P}(I_Y(\mu_t)>c)\\
&\qquad+\mathbb{P}\left(\left|\frac{\rho}{t}\chi(Y_t)\right|>\frac{1}{2}\delta\right).
\end{aligned}\tag{4.4}$$

We will use the following lemma in the subsequent proof.

LEMMA 4.1. *Consider the Y process defined in* (3.1). *Then for $\upsilon>0$ we have* $\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(|\frac{1}{t}\chi(Y_t)|>\upsilon)=-\infty$, *where $\chi(\cdot)$ is defined as in the proof of Theorem* 4.1.

PROOF. See Appendix C. □

Recall that $M(\mu)=-\frac{1}{2}F(\mu)+\rho G(\mu)$ and $\nu(\mu)=\bar{\rho}^2F(\mu)$. Then if $I_Y(\mu)\le c$, from Lemmas A.1 to A.4 in Appendix A we obtain

$$|M(\mu)-M^m(\mu)|\vee|\nu(\mu)-\nu^m(\mu)|\le\gamma_m(c),\tag{4.5}$$

where $\gamma_m(c)=2A(\frac{1}{m^2}+\frac{1}{m^{2-q}})(K_2(\alpha)c+K_3(\alpha))$ for some $A>0$ and $q=1+p$, and $\gamma_m(c)\to0$ as $m\to\infty$. Now let $\zeta=\mathbb{P}(I_Y(\mu_t)>c)+\mathbb{P}(|\frac{\rho}{t}\chi(Y_t)|>\frac{1}{2}\delta)$. Then using (4.5), we can now further bound the right-hand side of (4.4) as follows:

$$\begin{aligned}
\mathbb{P}(|\tilde{X}_t-\tilde{X}_t^m|>\delta)&\le\mathbb{E}(1_{\gamma_m+\sqrt{\gamma_m}|Z_t|>\frac{1}{2}\delta}1_{I_Y(\mu_t)\le c})+\zeta\\
&\le\mathbb{P}\left(\gamma_m(c)+\sqrt{\gamma_m(c)}|Z_t|>\frac{1}{2}\delta\right)+\zeta\\
&\le\mathbb{P}\left(|Z_t|>\frac{\frac{1}{2}\delta-\gamma_m(c)}{\sqrt{\gamma_m(c)}}\right)+\zeta.
\end{aligned}$$

Letting $t \to \infty$ and using the LDP for μ_t and the LDP for Z_t and Lemma 4.1 we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big(\big|\tilde{X}_t - \tilde{X}_t^m\big| > \delta\big) \leq -\frac{(\frac{1}{2}\delta - \gamma_m(c))^2}{2\gamma_m(c)} \wedge c.$$

Now set $c = c(m) = m^\beta$ where $\beta \in (0, 2-q)$. Then $\gamma_m^* = \gamma_m(c(m)) = 2A(\frac{1}{m^2} + \frac{1}{m^{2-q}})(K_2(\alpha)m^\beta + K_3(\alpha)) \to 0$ as $m \to \infty$ and

$$\lim_{m\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big(\big|\tilde{X}_t - \tilde{X}_t^m\big| > \delta\big) \leq -\infty.$$

Thus, X_t^m is an *exponentially good approximation* to X_t in the sense of Definition 4.2.14 in [3]. From the analysis above, we have

$$\begin{aligned}
&\limsup_{m\to\infty} \sup_{(z,\mu):\mathcal{I}(z,\mu)\leq R} \big|\varphi(z,\mu) - \varphi^m(z,\mu)\big| \\
&\quad \leq \limsup_{m\to\infty} \sup_{(z,\mu):\frac{1}{2}z^2 + I_Y(\mu)\leq R} \big|\big|M(\mu) - M^m(\mu)\big| + \sqrt{\nu(\mu) - \nu^m(\mu)}|z|\big| \\
&\quad \leq \limsup_{m\to\infty} \sup_{(z,\mu):\frac{1}{2}z^2 + I_Y(\mu)\leq R} \big|\gamma_m(R) + \sqrt{\gamma_m}(R)\sqrt{2R}\big| \\
&\quad = 0.
\end{aligned}$$

Thus, by Theorem 4.2.23 in [3], $\hat{X}_t$ satisfies the LDP with good rate function

$$I(x) = \inf_{(z,\mu):M(\mu)+\sqrt{\nu(\mu)}z = x} \left[\frac{1}{2}z^2 + I_Y(\mu)\right]. \tag{4.6}$$

But $\nu(\mu) = \bar{\rho}^2 \int_{-\infty}^{\infty} \sigma^2(y)\mu(dy) > 0$ because σ^2 is strictly positive. Thus, we can re-write the right-hand side of (4.6) as $\inf_{\mu\in\mathcal{P}(\mathbb{R})}[\frac{(x-M(\mu))^2}{2\nu(\mu)} + I_Y(\mu)]$. □

4.2. *Properties of the rate function $I(x)$.* The following two corollaries establish some basic properties of $I(x)$.

COROLLARY 4.2. *The infimum of $I(x)$ in (4.1) is attained uniquely at*

$$x_{\min} = M(\mu_\infty) = -\frac{1}{2}\bar{\sigma}^2, \tag{4.7}$$

where $M(\cdot)$ is defined as in Theorem 4.1 and $\bar{\sigma}$ is defined in (3.5).

PROOF. Let $I(x,\mu) = \frac{(x-M(\mu))^2}{2\nu(\mu)} + I_Y(\mu)$. Then, by (4.1), $I(x) = \inf_{\mu\in\mathcal{P}(\mathbb{R})} I(x,\mu)$. Setting $\mu = \mu_\infty$ we have $I(x_{\min},\mu_\infty) = \frac{(x_{\min}-M(\mu_\infty))^2}{2\nu(\mu_\infty)^2} + I_Y(\mu_\infty) = 0$. Therefore,

$$0 \leq I(x_{\min}) = \inf_{\mu\in\mathcal{P}(\mathbb{R})} I(x_{\min},\mu) \leq I(x_{\min},\mu_\infty) = 0$$

so $I(x_{\min}) = 0$.

We show that $x_{\min}$ is the unique minimum by contradiction. Suppose there exists an $x \neq x_{\min}$ such that $\lim_{n\to\infty} I(x,\mu_n) = 0$ for some sequence (μ_n) with $\mu_n \in \mathcal{P}(\mathbb{R})$. If $I(x,\mu_n) \to 0$ as $n \to \infty$ then $I_Y(\mu_n) \to 0$ and $M(\mu_n) \to x$ as $n \to \infty$. We first show that (μ_n) is a tight sequence. For any $k > 0$,

$$k^2 \mu_n[-k,k]^c \le \int_{[-k,k]^c} y^2 \mu_n(dy) \le K_2(\alpha) I_Y(\mu_n) + K_3(\alpha),$$

where we have used Lemma 2.2 for the last inequality. Since $I_Y(\mu_n) \to 0$ as $n \to \infty$, we can find a $C < \infty$ such that $\sup_n I_Y(\mu_n) \le C$. Hence, $k^2 \mu_n[-k,k]^c \le K_2(\alpha) C + K_3(\alpha)$ for all n. Thus, given $\varepsilon > 0$, we can choose k large enough such that

$$\sup_n \mu_n[-k,k]^c \le \frac{K_2(\alpha)C + K_3(\alpha)}{k^2} < \varepsilon,$$

so (μ_n) is tight as required.

Hence, (μ_n) has a convergent subsequence. Without loss of generality, we denote the convergent subsequence by (μ_n) and let μ denote the limit point. Then $I_Y(\mu) = 0$ by lower semicontinuity of I_Y [i.e., $I_Y(\mu) \le \liminf_{\mu_n \to \mu} I_Y(\mu_n) = 0$] and by uniqueness of minimizer of I_Y we obtain $\mu = \mu_\infty$. We will next show that $M(\mu_n) \to M(\mu_\infty) = x_{\min}$ which gives the contradiction.

Let $m > 0$. Then

$$\begin{aligned} |M(\mu_n) - M(\mu_\infty)| &\le |M_m(\mu_n) - M_m(\mu_\infty)| + |M(\mu_n) - M_m(\mu_n)| \\ &\quad + |M(\mu_\infty) - M_m(\mu_\infty)| \\ &\le |M_m(\mu_n) - M_m(\mu_\infty)| + c(m)\big(C_1 I_Y(\mu_n) + C_2\big) \\ &\quad + c(m)\big(C_1 I_Y(\mu_\infty) + C_2\big) \end{aligned}$$

[where we have applied Lemma A.1, and C_1, C_2 are constants and $c(m) = 1/m^2 + 1/m^{2-q}$ for some $q \in (0,2)$]

$$= |M_m(\mu_n) - M_m(\mu_\infty)| + c(m)\big(C_1 I_Y(\mu_n) + C_2\big) + c(m) C_2.$$

Taking $n \to \infty$, we see that

$$\lim_{n\to\infty} |M(\mu_n) - M(\mu_\infty)| \le 0 + 2c(m) C_2 \tag{4.8}$$

because M_m is a continuous functional. Since this holds for any arbitrary $m > 0$, taking $m \to \infty$ and noting that $c(m) \to 0$ as $m \to \infty$, we get $M(\mu_n) \to M(\mu_\infty) = x_{\min}$.

Finally, using the definition of $M(\cdot)$ in Theorem 4.1, we find that $M(\mu_\infty) = -\frac{1}{2}\bar{\sigma}^2 + \rho \bar{b}$ where $\bar{b} = \int_{-\infty}^{\infty} b(y) \mu_\infty(y)\, dy$. Recall that $b(y)$ is defined in Theorem 4.1 as $b(y) = (\alpha y - g(y))\sigma(y) - \frac{1}{2}\sigma'(y)$. Then we have

$$
\begin{aligned}
&\int b(y)\mu_\infty(dy) \\
&\quad = \text{const.} \times \biggl[\int_{-\infty}^{\infty} [\sigma(y)(\alpha y - g(y))] e^{2\int_{y_0}^{y} g(u)\,du} e^{-\alpha y^2}\,dy \\
&\qquad - \int_{-\infty}^{\infty} \frac{1}{2}\sigma'(y) e^{-\alpha y^2} e^{2\int_{y_0}^{y} g(u)\,du}\,dy\biggr] \\
&\quad = \text{const.} \times \biggl[\int_{-\infty}^{\infty} [\sigma(y)(\alpha y - g(y))] e^{2\int_{y_0}^{y} g(u)\,du} e^{-\alpha y^2}\,dy \\
&\qquad + \int_{-\infty}^{\infty} \frac{1}{2}\sigma(y) e^{-\alpha y^2} e^{2\int_{y_0}^{y} g(u)\,du} (-2\alpha y + 2g(y))\,dy\biggr] \\
&\quad = 0,
\end{aligned}
$$

where we have integrated by parts in the second expression of the last line. Thus, we see that $x_{\min} = -\frac{1}{2}\bar{\sigma}^2$. □

COROLLARY 4.3. *$I(x)$ in (4.1) is continuous.*

PROOF. Let $I(x,\mu)$ be as defined in Corollary 4.2. Then $I(x,\mu)$ is upper semicontinuous in x for μ fixed, and $I(x) = \inf_\mu I(x,\mu)$. The pointwise supremum of a family of LSC functions is LSC (see, e.g., Lemma 2.41 on page 43 in [1]), hence the pointwise infimum of a family of USC functions is USC, so $I(x)$ is USC. But $I(x)$ is also a rate function, hence I is also LSC. □

4.3. *The case $x = 0$ with $\rho = 0$—the Rayleigh–Ritz formula.*

COROLLARY 4.4. *For $x = 0$, $\rho = 0$, $I(0)$ reduces to*

$$
\begin{aligned}
I(0) = \lambda_1 &= \inf_{\mu \in \mathcal{P}(\mathbb{R})} \biggl[\frac{1}{8}F(\mu) + I_Y(\mu)\biggr] \\
&= \inf_{\psi \in L^2(\mu_\infty):\|\psi\|_2 = 1} \int_{-\infty}^{\infty} \biggl[\frac{1}{8}\sigma^2(y)\psi(y)^2 + \frac{1}{2}\psi'(y)^2\biggr]\mu_\infty(y)\,dy.
\end{aligned}
\tag{4.9}
$$

PROOF. *The first equality in (4.9) just follows by setting $x = 0$ in (4.1) and simplifying. The second equality just follows by re-writing μ in terms of ψ.* □

REMARK 4.5. (4.9) is the classical Rayleigh–Ritz formula for the lowest eigenvalue λ_1 for the Sturm–Liouville problem $(-\alpha y + g(y))u' + \frac{1}{2}u'' - \frac{1}{8}\sigma^2(y)u = -\lambda_1 u$ (see page 2 in [4] for more details).

4.4. *A general vol-of-vol coefficient.* For a more general model of the form,

$$\begin{cases} dX_t = -\frac{1}{2}\sigma(V_t)^2\,dt + \sigma(V_t)\big(\rho\,dW_t^2 + \bar{\rho}\,dW_t^1\big), \\ dV_t = \big(-\alpha V_t + g(V_t)\big)\,dt + \beta(V_t)\,dW_t^2 \end{cases}$$

for g, σ satisfying the same conditions as before, $\beta \in C^4$ with bounded first derivative (so β is Lipschitz), $0 < \underline{\beta} \le \beta(v) \le \bar{\beta} < \infty$, $\beta(v) \to \beta_\infty$ as $|v| \to \infty$ and $\frac{1}{\beta(v)} - \frac{1}{\beta_\infty} = O(1/1 + |v|^\gamma)$ for some $\gamma > 0$, then making the transformation $Y_t = U(V_t)$, where $U(v) = \int_0^v \frac{dz}{\beta(z)}$, we find that

$$\begin{aligned} dY_t &= U'(V_t)\,dV_t + \frac{1}{2}U''(V_t)\,d\langle V\rangle_t \\ &= U'(V_t)\big[\big(-\alpha V_t + g(V_t)\big)\,dt + \beta(V_t)\,dW_t^2\big] + \frac{1}{2}U''(V_t)\beta(V_t)^2\,dt \\ &= \frac{1}{\beta(V_t)}\big[\big(-\alpha V_t + g(V_t)\big)\,dt\big] + dW_t^2 - \frac{1}{2}\beta'(V_t)\,dt \\ &= \left[-\frac{\alpha}{\beta(V_t)}V_t + \frac{g(V_t)}{\beta(V_t)} - \frac{1}{2}\beta'(V_t)\right]dt + dW_t^2 \qquad (4.10) \\ &= \left[-\alpha Y_t + \left(\alpha Y_t - \frac{\alpha}{\beta(V_t)}V_t\right) + \left(\frac{g(V_t)}{\beta(V_t)} - \frac{1}{2}\beta'(V_t)\right)\right]dt + dW_t^2 \\ &= \left[-\alpha Y_t + \left(\alpha Y_t - \frac{\alpha}{\beta(U^{-1}(Y_t))}U^{-1}(Y_t)\right)\right. \\ &\qquad \left. + \left(\frac{g(U^{-1}(Y_t))}{\beta(U^{-1}(Y_t))} - \frac{1}{2}\beta'\big(U^{-1}(Y_t)\big)\right)\right]dt + dW_t^2. \end{aligned}$$

We need to show that the terms $\alpha Y_t - \frac{\alpha}{\beta(U^{-1}(Y_t))}U^{-1}(Y_t)$ and $\frac{g(U^{-1}(Y_t))}{\beta(U^{-1}(Y_t))} - \frac{1}{2}\beta'(U^{-1}(Y_t))$ satisfy the sublinear growth condition in Assumption 3.1. Henceforth, "sublinear growth" will mean that equation (3.2) is satisfied.

We first look at the term $\frac{g(U^{-1}(Y_t))}{\beta(U^{-1}(Y_t))} - \frac{1}{2}\beta'(U^{-1}(Y_t))$. Since $1/\beta(\cdot)$ and $\beta'(\cdot)$ are bounded functions, it is sufficient to show that $g(U^{-1}(Y_t))$ has sublinear growth in Y_t. By the definition of Y and bounds on $\beta(\cdot)$, we get $V_t/\bar{\beta} \le Y_t \le V_t/\underline{\beta}$ which then gives us the inequality $\underline{\beta}Y_t \le V_t = U^{-1}(Y_t) \le \bar{\beta}Y_t$. Since g has sublinear growth and V_t grows linearly with Y_t, we get that $g(U^{-1}(Y_t))$ is a sub linear function of Y_t.

We next show that $|y - \frac{U^{-1}(y)}{\beta(U^{-1}(y))}| \le \text{constant} \times (1 + |y|)^\delta$ for some $\delta \in (0, 1)$. By definition of Y and properties of $\beta(\cdot)$, we get

$$y = U(v) = \int_0^v \frac{1}{\beta(z)}\,dz = \frac{v}{\beta_\infty} + \int_0^v \left(\frac{1}{\beta(z)} - \frac{1}{\beta_\infty}\right)dz = \frac{v}{\beta_\infty} + O\big(1 + |v|^{1-\gamma}\big)$$

and

$$\frac{v}{\beta(v)} = \frac{v}{\beta_\infty} + O\big(1 + |v|^{1-\gamma}\big).$$

Putting this together, we get

$$y - \frac{U^{-1}(y)}{\beta(U^{-1}(y))} = U(v) - \frac{v}{\beta(v)} = O\big(1 + |v|^{1-\gamma}\big).$$

Since V_t grows linearly with Y_t, we get $|y - \frac{U^{-1}(y)}{\beta(U^{-1}(y))}| = O(1 + |y|^{1-\gamma})$. So $|y - \frac{U^{-1}(y)}{\beta(U^{-1}(y))}| \le$ constant if $\gamma > 1$ and $|y - \frac{U^{-1}(y)}{\beta(U^{-1}(y))}| \le \text{constant} \times (1 + |y|^{1-\gamma})$ if $\gamma \in (0, 1)$. Thus,

$$\begin{cases} dX_t = -\frac{1}{2}\tilde{\sigma}(Y_t)^2\,dt + \tilde{\sigma}(Y_t)\big(\rho\,dW_t^2 + \bar{\rho}\,dW_t^1\big), \\ dY_t = \big(-\alpha Y_t + \tilde{g}(Y_t)\big)\,dt + dW_t^2 \end{cases}$$

for some $\tilde{\sigma}$, $\tilde{g}$ which satisfy Assumptions 3.1 and 3.2, so we are back to a model of the form in (3.1), and thus the main result in Theorem 4.1 still holds. If we want to impose less stringent conditions on β, we would have to manually verify the upper bound conditions 1–5 and the lower bound conditions A, B in Section 2.1.

5. Call options and implied volatility. We now verify the martingale property for $S_t = e^{X_t}$. This will be used to define the *Share measure* $\mathbb{P}^*$ below.

PROPOSITION 5.1. *$(S_t)_{0 \le t < \infty}$ defined in (3.1) is a martingale.*

PROOF. See Appendix D. □

We consider the family of probability measures $\mathbb{P}_T^S(A) := \frac{1}{S_0}\mathbb{E}(S_T 1_A)$ defined for each $T > 0$, for $A \in \mathcal{F}_t$ and $t \le T$ [$\mathbb{P}_T^S$ is a probability measure on $\mathcal{F}_T$ because $(S_t)_{0 \le t \le T}$ is a martingale by Proposition 5.1]. From Girsanov's theorem, we have that

$$\begin{cases} dX_t = \frac{1}{2}\sigma(Y_t)^2\,dt + \sigma(Y_t)\big(\rho\,dW_t^{*2} + \bar{\rho}\,dW_t^{*1}\big), \\ dY_t = \big(-\alpha Y_t + g(Y_t) + \rho\sigma(Y_t)\big)\,dt + dW_t^{*2}, \end{cases} \tag{5.1}$$

where W_t^{*1}, W_t^{*2} are independent $\mathbb{P}^S$-Brownian motions. Let $\mathbb{P}^*$ be a probability measure under which (X, Y) satisfies (5.1) for *all* $t > 0$ with $X_0 = 0$ and $Y_0 = y_0$.

PROPOSITION 5.2. *X_t/t satisfies the LDP under $\mathbb{P}^*$ as $t \to \infty$ with a good rate function given by*

$$I^*(x) = \inf_{\mu \in \mathcal{P}(\mathbb{R})}\bigg[\frac{(x - M^*(\mu))^2}{2\nu(\mu)} + I_Y(\mu)\bigg],$$

where $M^*(\mu) = \frac{1}{2}F(\mu) + \rho G^*(\mu)$, *where* $G^*(\mu) = \int_{-\infty}^{\infty}[(\alpha y - g(y) - \rho\sigma(y))\sigma(y) - \frac{1}{2}\sigma'(y)]\mu(dy)$, *and the minimum of* $I^*(x)$ *is attained uniquely at* $x^*_{\min} = M^*(\mu^*_\infty)$; *where* μ^*_∞ *is the invariant distribution of the* Y *process under* $\mathbb{P}^*$.

PROOF. If we let $\tilde{g}(y) = g(y) + \rho\sigma(y)$, then $\tilde{g}$ also has sublinear growth, and the proof then just follows by an almost identical argument to the proofs of Theorem 4.1 and Corollary 4.2. $\square$

COROLLARY 5.3. *The unique minimizers* $x_{\min}$ *and* $x^*_{\min}$ *of the rate functions* I *and* I^*, *respectively* (*defined in Corollary* 4.2 *and Proposition* 5.2, *resp.*), *satisfy the inequality* $x^*_{\min} > x_{\min}$.

PROOF. Recall the formula of the invariant density for the perturbed OU process given in (2.5). Then $\mu^*_\infty(y) = \frac{e^{-\alpha y^2} e^{2\int_{y_0}^{y} \tilde{g}(u)\,du}}{\int_{-\infty}^{\infty} e^{-\alpha u^2} e^{2\int_{y_0}^{u} \tilde{g}(v)\,dv\,du}\,du}$, where $\tilde{g}(y) = g(y) + \rho\sigma(y)$ and $\mu_\infty(y) = \frac{e^{-\alpha y^2} e^{2\int_{y_0}^{y} g(u)\,du}}{\int_{-\infty}^{\infty} e^{-\alpha u^2} e^{2\int_{y_0}^{u} g(v)\,dv\,du}\,du}$. Observe that

$$G^*(\mu^*_\infty) = \text{const} \cdot \int_{-\infty}^{\infty}\left[(\alpha y - \tilde{g}(y))\sigma(y) - \frac{1}{2}\sigma'(y)\right]e^{-\alpha y^2} e^{2\int_{y_0}^{y} \tilde{g}(u)\,du}\,dy$$

$$= \text{const} \cdot \int_{-\infty}^{\infty} -\frac{1}{2}\left[\frac{d}{dy}\left(\sigma(y)e^{-\alpha y^2} e^{2\int_{y_0}^{y} \tilde{g}(u)\,du}\right)\right]dy = 0.$$

Similarly, $G(\mu_\infty) = 0$. Thus,

$$x_{\min} = -\frac{1}{2}F(\mu_\infty) + \rho G(\mu_\infty) = -\frac{1}{2}F(\mu_\infty) < F(\mu^*_\infty)$$

$$= \frac{1}{2}F(\mu^*_\infty) + \rho G^*(\mu^*_\infty) = x^*_{\min}. \qquad \square$$

By Proposition 5.2, that is, the LDP for (X_t/t) under $\mathbb{P}^*$, and the continuity of the rate function I^*, we obtain the following corollary, which will be used to characterize the large-time behaviour of call option prices.

COROLLARY 5.4. *For the model in* (3.1), *we have the following large-time behaviour for digital call options*:

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}^*(X_t > xt) = -\Lambda^*(x) \qquad (x > x^*_{\min}),$$

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}^*(X_t < xt) = -\Lambda^*(x) \qquad (x < x^*_{\min}),$$

where

$$\Lambda^*(x) = \begin{cases} \inf_{y>x} I^*(y), & \text{if } x \geq x^*_{\min}, \\ \inf_{y<x} I^*(y), & \text{if } x \leq x^*_{\min}. \end{cases}$$

REMARK 5.5. From the definition of Λ^*, we see that Λ^* is nonincreasing for $x < x^*_{\min}$ and nondecreasing for $x > x^*_{\min}$, and [by the continuity of $I^*(x)$, which can be proved by a similar argument to Corollary 4.3] Λ^* is continuous.

Recall that the payoff of a European call option of strike K is $\mathbb{E}(S_t - K)^+$, and the payoff of a European put option with strike K is $\mathbb{E}(K - S_t)^+$.

COROLLARY 5.6. *For the model in* (3.1), *we have the following large-time asymptotic behaviour for put/call options in the large-time*, *large log-moneyness regime*:

$$(5.2)\quad \begin{aligned} -\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}(S_t - S_0e^{xt})^+ &= \Lambda^*(x) \qquad (x \ge x^*_{\min}),\\ -\lim_{t\to\infty}\frac{1}{t}\log\bigl[S_0 - \mathbb{E}(S_t - S_0e^{xt})^+\bigr] &= \Lambda^*(x) \qquad (x_{\min} \le x \le x^*_{\min}),\\ -\lim_{t\to\infty}\frac{1}{t}\log\bigl(\mathbb{E}(S_0e^{xt} - S_t)^+\bigr) &= \Lambda^*(x) \qquad (x \le x_{\min}),\end{aligned}$$

PROOF. This is now a standard result; see, for example, Corollary 2.4 in [13]. □

5.1. *Implied volatility.* Using the same proofs as in Corollary 1.7 and Corollary 2.17 in [13] for the Heston model (see also Theorem 14 in [18] for a general affine model), we have the following asymptotic behaviour in the large-time, large log-moneyness regime, where $\hat{\sigma}_t(xt)$ is the implied volatility of a put/call option with strike S_0e^{xt} for model in (3.1):

$$\begin{aligned}\hat{\sigma}_\infty(x)^2 &= \lim_{t\to\infty}\hat{\sigma}_t^2(xt)\\ &= \begin{cases} 2\bigl(2\Lambda^*(x) + x - 2\sqrt{\Lambda^*(x)^2 + \Lambda^*(x)x}\bigr) & (x \notin (x_{\min}, x^*_{\min})),\\ 2\bigl(2\Lambda^*(x) + x + 2\sqrt{\Lambda^*(x)^2 + \Lambda^*(x)x}\bigr) & (x \in (x_{\min}, x^*_{\min}))\end{cases}\end{aligned}$$

and we see that $\hat{\sigma}_\infty(0)^2 = 8\Lambda^*(0)$. We omit the details for the sake of brevity.

6. Numerical implementation and results. Recall that the rate function $I(x)$ for X_t/t under the model in (3.1) is given by $I(x) = \inf_{\mu\in\mathcal{P}(\mathbb{R})}[\frac{(x-M(\mu))^2}{2\nu(\mu)} + I_Y(\mu)]$. If $g(y) \equiv 0$, then using the simpler representation for the rate function in (2.3), we can re-write $I(x)$ as

$$I(x) = \inf_{\psi\in L^2(\mu_\infty):\|\psi\|_2=1}\biggl[\frac{(x-M)^2}{2\nu} + \frac{1}{2}\int_{-\infty}^{\infty}\psi'(y)^2\mu_\infty(y)\,dy\biggr]$$

where now $M = M(\psi) = \int_{-\infty}^{\infty} m(y)\psi(y)^2\mu_\infty(y)\,dy$, $m(y) = -\frac{1}{2}\sigma^2(y) + \rho b(y)$ and $\nu = \nu(\psi) = \bar{\rho}^2\int_{-\infty}^{\infty}\sigma(y)^2\psi(y)^2\mu_\infty(y)\,dy$ [the constraint under the inf is just shorthand for $\int_{-\infty}^{\infty}\psi(y)^2\mu_\infty(y)\,dy = 1$].

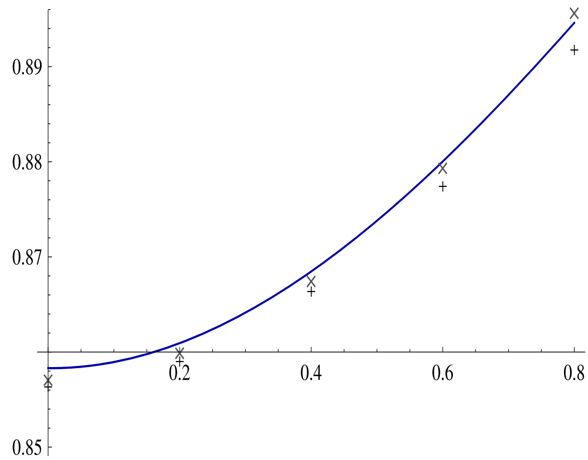

FIG. 1. *Here, we have plotted the right half of the (symmetric) asymptotic implied volatility* $\hat{\sigma}_\infty(x)$ *for the Ornstein–Uhlenbeck model with* $\rho = 0$, $\alpha = 1$ *and* $\sigma(y) = \sqrt{\log(1+e^y)}$ *(solid blue line) using the Ritz method with the NMinimize command in Mathematica and* $n = 7$, *and the values obtained using Monte Carlo simulation for* $t = 75$ *years (grey diagonal crosses) and* $t = 30$ *years (black crosses). For the Monte Carlo, we use* 1,000,000 *simulations and* 1000 *time steps and we use the usual conditioning trick for* $\rho = 0$ *by simulating the integrated variance* $\int_0^t \sigma(Y_s)^2\,ds$ *and then plugging this into the Black–Scholes formula. In this case,* $x^*_{\min} = 0.376131$ *and* $x_{\min} = -x^*_{\min}$. *Note that* $\sigma(y) \sim \sqrt{y}$ *as* $y \to \infty$, *and thus satisfies the sublinear growth condition.*

6.1. *The Ritz method.* We can use the *Ritz method* described in Gelfand and Fomin [10] to provide an approximate numerical solution to this problem in terms of ψ, by considering a $\psi = \alpha_0\varphi_0 + \cdots + \alpha_n\varphi_n$, where $\varphi_0, \varphi_2, \ldots, \varphi_n$ are the first $n+1$ eigenfunctions for the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mu_\infty)$ of square integrable functions with respect to $\mu_\infty(y)$, which can be computed in closed form as $\varphi_n(y) = H_n(\sqrt{\alpha}y)$ (see Section 6.2.1 in Linetsky [22]). We then optimize for the Fourier coefficients $(\alpha_0, \ldots, \alpha_n)$ (see Table 1 and Figure 1).

6.2. *Numerical results.*

TABLE 1
Here we have computed the large-time asymptotic implied volatility $\hat{\sigma}_\infty(x)$ *using the Ritz method, and compared to the answers obtained using Monte Carlo for* $t = 75$ *yrs and* $t = 30$ *yrs*

x	$\hat{\sigma}_\infty(x)$	**Monte Carlo** $t = 75$	**Monte Carlo** $t = 30$
0.0	0.858305	0.857258	0.856479
0.2	0.860926	0.859900	0.859147
0.4	0.868463	0.867634	0.866509
0.6	0.880036	0.879524	0.877567
0.8	0.894606	0.894606	0.891904

7. Stochastic interest rates and jumps. We now consider the following three-factor model for the log stock price X_t under $\mathbb{P}$, which incorporates stochastic volatility and a stochastic short rate driven by a CIR square root process:

$$(7.1)\quad \begin{cases} dX_t = \left(r_t - \frac{1}{2}\sigma(Y_t)^2\right)dt + \sigma(Y_t)\big(\rho\,dW_t^2 + \bar{\rho}\,dW_t^1\big) + dZ_t, \\ dY_t = \big(-\alpha Y_t + g(Y_t)\big)dt + dW_t^2, \\ dr_t = \kappa_r(\theta_r - r_t)\,dt + \sigma_r\sqrt{r_t}\,dW_t^3, \end{cases}$$

where W^1, W^2, W^3 are independent Brownian motions, $x_0, \kappa_r, \theta_r, \sigma_r > 0$, $|\rho| < 1$ and $2\kappa_r\theta_r > \sigma_r^2$,[4] and Z_t is a Lévy process independent of W^1, W^2, W^3 such that e^{Z_t} is a martingale, with cumulant generating function (cgf) $V_J(p)$ so that $\mathbb{E}(e^{pZ_t}) = e^{V_J(p)t}$, and g, σ satisfy Assumptions 3.1 and 3.2.

Assume that $V_J''(p) > 0$ and $V_J(p)$ is essentially smooth on some interval (p_-, p_+) [i.e., $|V_J'(p)| \to \infty$ as $p \nearrow p_+$ and $p \searrow p_-$] with $p_- < 0 < 1 < p_+$. If we let $x_- = V_J'(0)$ and $V_J^*(x) = \sup_p[px - V_J(p)]$ denote the Legendre transform of V_J, then by the Gärtner–Ellis theorem, Z_t/t satisfies the LDP with rate function $V_J^*(x)$, and x_- is the unique minimum of $V_J^*(x)$ where $V_J^*(x_-) = 0$ (see [9] for details).

We will need the following result.

LEMMA 7.1. *For the model in* (7.1), $\Gamma_t = \frac{1}{t}\int_0^t r_s\,ds$ *satisfies the LDP as* $t \to \infty$ *with good rate function given by the Fenchel–Legendre transform of* V_{CIR}:

$$I_{\mathrm{CIR}}(a) = \sup_{a>0}\{pa - V_{\mathrm{CIR}}(p)\} = \frac{\kappa_r^2(a-\theta_r)^2}{2a\sigma_r^2},$$

where

$$(7.2)\quad \begin{aligned} V_{\mathrm{CIR}}(p) &= \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\big(e^{p\int_0^t r_s\,ds}\big) \\ &= \begin{cases} \dfrac{\kappa_r\theta_r}{\sigma_r^2}\Big[\kappa_r - \sqrt{\kappa_r^2 - 2\sigma_r^2 p}\Big], & \text{for } p \in (-\infty, p_+], \\ \infty, & \text{for } p \notin (-\infty, p_+], \end{cases} \end{aligned}$$

and $p_+ = \frac{\kappa_r^2}{2\sigma_r^2}$. I_{CIR} *clearly attains its minimum value of zero at* $a = \theta_r$.

PROOF. Just follows from the known closed-form expression for the moment generating function of Γ_t given in, for example, Section 3 in [2] and the Gärtner–Ellis theorem from large deviations theory, using a similar argument to Theorem 2.1 in Forde and Jacquier [13]. □

From the contraction principle, we now have the following.

[4] Which ensures that $r = 0$ is an unattainable boundary.

COROLLARY 7.1. *X_t/t satisfies the LDP as $t \to \infty$ with rate function $I_r(x) = \inf_{a,y,z:a+y+z=x}[I(y) + I_{\mathrm{CIR}}(a) + V_J^*(z)] = \inf_{a,y}[I(y) + I_{\mathrm{CIR}}(a) + V_J^*(x-a-y)]$, where $I(x)$ is defined as in Theorem 4.1.*

REMARK 7.2. For the model in (7.1), if there is no Lévy process component, by conditioning on $\Gamma_t = \frac{1}{t}\int_0^t r_s\,ds$, we can prove the following asymptotic behaviour for the price of a digital call option in the large-time, large log-moneyness regime:

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big(e^{-\int_0^t r_s\,ds} 1_{X_t > xt}\big) = -\inf_{y>x} I_r(y),$$

where $I_r(x) = \inf_{a\in\mathbb{R}^+}[a + I(x-a) + I_{\mathrm{CIR}}(a)]$.

REMARK 7.3. We can adapt this result to compute, for example, large-time asymptotics for European call options.

APPENDIX A: LINEAR FUNCTIONALS OF THE OCCUPATION MEASURE

LEMMA A.1. *Consider a linear functional $\Lambda : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}$ defined by $\Lambda(\mu) = \int_{-\infty}^{\infty} \lambda(y)\mu(dy)$, where λ satisfies the growth condition:*

$$|\lambda(y)| \le A\big(1 + |y|^q\big) \tag{A.1}$$

for $q \in (0,2)$, $A > 0$. Then

$$\big|\Lambda(\mu) - \Lambda^m(\mu)\big| \le 2A\left(\frac{1}{m^2} + \frac{1}{m^{2-q}}\right)\big(K_2(\alpha)I_Y(\mu) + K_3(\alpha)\big),$$

where $\Lambda^m(\mu) = \int[\lambda(y)1_{\{|y|\le m\}} + \lambda(m)1_{\{y>m\}} + \lambda(-m)1_{\{y<-m\}}]\mu(dy)$ and $K_2(\alpha) > 0$ and $K_3(\alpha)$ are the constants introduced in Lemma 2.2.

PROOF. For $I_Y(\mu) \le c$, using the growth condition on λ we obtain

$$\begin{aligned}
&\big|\Lambda(\mu) - \Lambda^m(\mu)\big| \\
&\quad = \int_{|y|>m} \big[\big(\lambda(y) - \lambda(m)\big)1_{\{y>m\}} + \big(\lambda(y) - \lambda(-m)\big)1_{\{y<-m\}}\big]\mu(dy) \\
&\quad \le \int_{|y|>m} \big[\big(|\lambda(y)| + |\lambda(m)|\big)1_{\{y>m\}} + \big(|\lambda(y)| + |\lambda(-m)|\big)1_{\{y<-m\}}\big]\mu(dy) \\
&\quad \le 4\int_{|y|>m} A\big(1 + |y|^q\big)\mu(dy) \\
&\quad \le 4A\left(\frac{1}{m^2} + \frac{1}{m^{2-q}}\right)\int_{-\infty}^{\infty} y^2\mu(dy) \\
&\quad \le 4A\left(\frac{1}{m^2} + \frac{1}{m^{2-q}}\right)\big(K_2(\alpha)I_Y(\mu) + K_3(\alpha)\big),
\end{aligned}$$

where we have used Lemma 2.2 in the final line. □

LEMMA A.2. *$\sigma^2(y)$ satisfies the sub-quadratic growth condition*

$$\sigma^2(y) \le A_1\big(1+|y|^{2p}\big), \tag{A.2}$$

where $A_1 = 3K_1^2$; thus F [as defined in (3.5)] satisfies the conditions of Lemma A.1 with $\lambda(y) = \sigma^2(y)$, $A = A_1$ and $q = 2p \in (0,2)$.

PROOF. From the sublinear growth condition $\sigma(y) \le K_1(1+|y|^p)$, we see that

$$\sigma(y)^2 \le A^2\big(1+|y|^p\big)^2 = A^2\big(1+2|y|^p+|y|^{2p}\big) \le 3A^2\big(1+|y|^{2p}\big),$$

where the final inequality just comes from the inequality $|y|^p \le 1+|y|^{2p}$. □

LEMMA A.3. *b satisfies the growth condition*

$$\big|b(y)\big| \le A_2\big(1+|y|^{1+p}\big) \tag{A.3}$$

for some $A_2 > 0$; hence the functional G defined in Theorem 4.1 satisfies the conditions in Lemma A.1 with $\lambda(y) = b(y)$, $A = A_2$ and $q = 1+p \in (0,2)$.

PROOF. Using the sublinear growth condition (3.2) and the boundedness of σ', we see that

$$\begin{aligned}\big|b(y)\big| &\le \alpha|y|K_1\big(1+|y|^p\big) + \frac{1}{2}\|\sigma'\| = \alpha K_1|y| + \alpha K_1|y|^{1+p} + \frac{1}{2}\|\sigma'\| \\ &\le \alpha K_1\big(1+|y|^{1+p}\big) + \alpha K_1|y|^{1+p} + \frac{1}{2}\|\sigma'\| \\ &\le A_2\big(1+|y|^{1+p}\big)\end{aligned}$$

for some $A_2 > 0$. □

LEMMA A.4. *Let $m(y) = -\frac{1}{2}\sigma^2(y) + \rho b(y)$. Then m satisfies the growth condition*

$$\big|m(y)\big| \le A_3\big(1+|y|^{1+p}\big)$$

for some $A_3 > 0$; thus M satisfies the conditions in Lemma A.1 with $\lambda(y) = m(y)$, $A = A_3$ and $q = 1+p \in (0,2)$.

PROOF. Using (A.2) and (A.3),

$$\begin{aligned}m(y) = \left|-\frac{1}{2}\sigma^2(y) + \rho b(y)\right| &\le \frac{1}{2}A_1\big(1+|y|^{2p}\big) + \rho A_2\big(1+|y|^{1+p}\big) \\ &\le A_3\big(1+|y|^{1+p}\big)\end{aligned}$$

for some $A_3 > 0$. □

APPENDIX B: PROOF OF LEMMA 2.1

To verify the lower bound conditions I and II, we have to show that $p(1,x,dy)$ admits a density $p(1,x,y)$ and that

$$\lim_{x_2\to x_1}\int_{-\infty}^{\infty}\big|p(1,x_2,y)-p(1,x_1,y)\big|\,dy=0. \tag{B.1}$$

For the rest of the proof, we assume that $Y_0=x$. Let $\bar{G}(y)=\int_x^y g(u)\,du$; then $\bar{G}$ has sub-quadratic growth and recall that $|g'|$ is bounded by assumption. Let $h(y):=-\frac{\alpha}{2}y^2+\bar{G}(y)$. Then the perturbed OU process Y in (2.4) satisfies $dY_t=h'(Y_t)\,dt+dW_t$ and

$$dh(Y_t)=h'(Y_t)\big(h'(Y_t)\,dt+dW_t\big)+\frac{1}{2}h''(Y_t)\,dt. \tag{B.2}$$

We now define a measure $\mathbb{Q}$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t}:=e^{-\frac{1}{2}\int_0^t h'(Y_s)^2\,ds-\int_0^t h'(Y_s)\,dW_s}=e^{h(x)-h(Y_t)+\frac{1}{2}\int_0^t\tilde{g}(Y_s)\,ds}, \tag{B.3}$$

where $\tilde{g}(y)=h''(y)+(h'(y))^2=(-\alpha+g'(y))+(-\alpha y+g(y))^2$ and we have used (B.2) to remove the stochastic integral term in (B.3). To check that the right-hand side in (B.3) is a $\mathbb{P}$-martingale, we first define an intermediate change of measure $\frac{d\mathbb{P}^{\mathrm{OU}}}{d\mathbb{P}}|_{\mathcal{F}_t}:=M_1(t)$, where

$$M_1(t)=e^{-\frac{1}{2}\int_0^t g^2(Y_s)\,ds-\int_0^t g(Y_s)\,dW_s}. \tag{B.4}$$

Then M_1 is a $\mathbb{P}$-martingale since the Novikov condition is satisfied following the same argument as Appendix D, and

$$dY_t=-\alpha Y_t\,dt+d\tilde{W}_t,$$

where $\tilde{W}_t=W_t-\int_0^t g(Y_s)\,ds$ is a Brownian motion under $\mathbb{P}^{\mathrm{OU}}$, that is, Y is an (unperturbed) OU process under $\mathbb{P}^{\mathrm{OU}}$. Now define $\frac{d\mathbb{Q}}{d\mathbb{P}^{\mathrm{OU}}}|_{\mathcal{F}_t}:=M_2(t)$, where

$$M_2(t)=e^{-\frac{1}{2}\int_0^t(\alpha^2Y_s^2)\,ds-\int_0^t\alpha Y_s\,d\tilde{W}_s}.$$

If $M_2(t)$ is a $\mathbb{P}^{\mathrm{OU}}$-martingale, then we can go straight from $\mathbb{P}$ to $\mathbb{Q}$ and define as in (B.3):

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t}:=e^{-\frac{1}{2}\int_0^t h'(Y_s)^2\,ds-\int_0^t h'(Y_s)\,dW_s^{(2)}}=M_1(t)M_2(t)$$

and M_1M_2 will be a $\mathbb{P}$-martingale. To check that M_2 is a $\mathbb{P}^{\mathrm{OU}}$ martingale, we verify the Novikov condition

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^{\mathrm{OU}}}\big[e^{\frac{1}{2}\int_s^{s+\varepsilon}(\alpha^2Y_u^2)\,du}\big]&=\mathbb{E}^{\mathbb{P}^{\mathrm{OU}}}\big[e^{\frac{1}{2}\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\varepsilon\alpha^2Y_u^2\,du}\big]\\&\le\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\mathbb{E}^{\mathbb{P}^{\mathrm{OU}}}\big[e^{\frac{1}{2}\varepsilon\alpha^2Y_u^2}\big]\,du\end{aligned}$$

by Jensen's inequality. Under $\mathbb{P}^{\mathrm{OU}}$, $Y_u \sim N(y_0 e^{-\alpha u}, \frac{1-e^{-2\alpha u}}{2\alpha})$, so taking ε small enough (say $\varepsilon = \frac{1}{4\alpha}$), we get $\mathbb{E}^{\mathbb{P}^{\mathrm{OU}}}[e^{\frac{1}{2}\int_s^{s+\varepsilon}(\alpha^2 Y_u^2)\,du}] < \infty$, for any $s > 0$. Thus, by Corollary 5.5.14 on page 199 in [20], M_2 is a $\mathbb{P}^{\mathrm{OU}}$-martingale.

By Girsanov's theorem, Y is standard Brownian motion under $\mathbb{Q}$ and for any $f \in C_b(\mathbb{R})$,

$$\begin{aligned}\int_{-\infty}^{\infty} f(y)p(t,x,dy) = \mathbb{E}^{\mathbb{P}}\big(f(Y_t)\big) &= \mathbb{E}^{\mathbb{Q}}\big[f(Y_t)e^{h(Y_t)-h(x)-\frac{1}{2}\int_0^t \tilde{g}(Y_s)\,ds}\big] \\ &= \int_{-\infty}^{\infty} f(y)e^{h(y)-h(x)}\mathbb{E}^{\mathbb{Q}}\big(e^{-\frac{1}{2}\int_0^t \tilde{g}(Y_s)\,ds}|Y_t = y\big)\gamma(t,x,y)\,dy \\ &= \int_{-\infty}^{\infty} f(y)e^{h(y)-h(x)}\phi(t,x,y)\gamma(t,x,y)\,dy\end{aligned}$$

(see also equations (6)–(8) in [24]), where $\gamma(t,x,y) = \frac{1}{\sqrt{2\pi t}}e^{-(y-x)^2/2t}$, $h(y) = \int_x^y(-\alpha u + g(u))\,du = -\frac{\alpha}{2}y^2 + \bar{G}(y) + \frac{\alpha}{2}x^2 - \bar{G}(x)$ and $\bar{G}(y) = \int_x^y g(u)\,du$ and

$$\phi(t,x,y) = \mathbb{E}^{\hat{\mathbb{P}}_{x,y}}\big(e^{-\frac{1}{2}\int_0^t \tilde{g}(Y_s)\,ds}\big),$$

where $\hat{\mathbb{P}}_{x,y}$ is a probability measure under which Y is a Brownian bridge with $Y_0 = x$ and $Y_t = y$. Thus, Y has a transition density given by

$$p(t,x,y) = \gamma(t,x,y)e^{h(y)-h(x)}\phi(t,x,y).$$

$\bar{G}(y)$ is sub-quadratic so we can choose a constant $c > 0$ such that $\bar{G}(y) \le c + \frac{\alpha y^2}{4}$. Then we see that

$$e^{h(Y_t)-h(x)} = e^{-\frac{\alpha Y_t^2}{2}+\bar{G}(Y_t)+\frac{\alpha x^2}{2}-\bar{G}(x)} \le e^{\frac{\alpha x^2}{2}-\bar{G}(x)}e^{c-\frac{\alpha Y_t^2}{4}},$$

and

$$\begin{aligned}\phi(t,x,y) &= \mathbb{E}^{\hat{\mathbb{P}}_{x,y}}\big(e^{-\frac{1}{2}\int_0^t \tilde{g}(Y_s)\,ds}\big) \\ &= \mathbb{E}^{\hat{\mathbb{P}}_{x,y}}\big(e^{\int_0^t[-\frac{1}{2}(-\alpha Y_s+g(Y_s))^2-\frac{1}{2}(-\alpha+g'(Y_s))]\,ds}\big) \\ &\le e^{\frac{\alpha+\|g'\|}{2}t}.\end{aligned}$$

Thus, we have

$$p(1,x,y) \le \frac{1}{\sqrt{2\pi}}e^{\frac{\alpha x^2}{2}-\bar{G}(x)}e^{c-\frac{\alpha y^2}{4}}e^{\frac{\alpha+\|g'\|}{2}} = C_1 e^{-\frac{\alpha}{4}y^2+C_2}$$

for some constants C_1, C_2 which are independent of y. Thus, $\sup_{x\in K} p(1,x,y) \le c_1 e^{-cy^2}$ for any compact set $K \subset \mathbb{R}$. From the main theorem in [24], we also know that $p(t,x,y)$ is continuous in x. Hence, we can apply the dominated convergence theorem to establish (B.1).

APPENDIX C: PROOF OF LEMMA 4.1

Using the sublinear growth condition on σ, we have

$$|\chi(y)| = \left|\int_{y_0}^{y} \sigma(u)\,du\right| \le \left|\int_{y_0}^{y} K_1(1+|u|^p)\right| du \le K_1|y-y_0| + K_1\int_{y_0}^{y} |u|^p\,du$$
$$\le K_1|y-y_0| + \frac{K_1}{1+p}(|y|^{1+p}+|y_0|^{1+p}).$$

Thus, $\limsup_{|y|\to\infty} \frac{|\chi(y)|}{|y|^{1+p}} \le \frac{K_1}{1+p}$ which implies that

$$\liminf_{|y|\to\infty} \frac{|\chi^{-1}(y)|}{|y|^r} \ge \tilde{K} \tag{C.1}$$

for some $\tilde{K} > 0$, where $r = \frac{1}{1+p} \in (\frac{1}{2}, 1)$ [note that $\chi^{-1}(\cdot)$ is well defined because $\chi'(y) = \sigma(y) > 0$]. Then from the analysis in the previous Appendix, we have

$$\mathbb{P}(\chi(Y_t) > tv)$$
$$= \mathbb{E}^{\mathbb{Q}}[e^{h(Y_t)-h(y_0)-\frac{1}{2}\int_0^t (h'(Y_s))^2\,ds - \frac{1}{2}\int_0^t h''(Y_s)\,ds} 1_{Y_t > \chi^{-1}(tv)})]$$
$$= \mathbb{E}^{\mathbb{Q}}[e^{-\frac{\alpha Y_t^2}{2}+\bar{G}(Y_t)+\frac{\alpha y_0^2}{2}-\bar{G}(y_0)-\int_0^t(\frac{1}{2}(-\alpha Y_s+g(Y_s))^2+\frac{1}{2}(-\alpha+g'(Y_s)))\,ds} 1_{Y_t>\chi^{-1}(tv)}] \tag{C.2}$$
$$\le e^{\frac{\alpha y_0^2}{2}-\bar{G}(y_0)+\frac{\alpha+\|g'\|}{2}t+c-\frac{\alpha(\tilde{K}(tv)^r)^2}{4}}\mathbb{Q}(Y_t > \tilde{K}(tv)^r)$$
$$\le c_1 e^{-c_2 t^{2r}}$$

for t sufficiently large, for some constants $c_1, c_2 > 0$, where we have used (C.1) in the penultimate line and that $Y_t \sim N(y_0, t)$ under $\mathbb{Q}$.

APPENDIX D: PROOF OF PROPOSITION 5.1

To show that $S_t = e^{-\frac{1}{2}\int_0^t \sigma^2(Y_s)\,ds + \int_0^t \sigma(Y_s)(\bar{\rho}\,dW_s^1+\rho\,dW_s^2)}$ is a martingale, by Corollary 5.13, page 199 in [20], it is sufficient to check the Novikov condition:

$$\mathbb{E}(e^{\frac{1}{2}\int_0^t \sigma^2(Y_s)\,dt}) < \infty; \qquad 0 \le t < \infty.$$

Fix $0 < t < \infty$. Define u_n as in the proof of Lemma 2.2. Then, as in the proof of Lemma 2.2, $-\frac{Lu_n}{u_n}(y) \to c_0 y^2 - c_1 y g(y) - c_2$ pointwise as $n \to \infty$, where $c_0 > 0$. Thus, by Fatou's lemma we have

$$\int_{-\infty}^{\infty} (c_0 y^2 - c_1 y g(y) - c_2)\mu_t(dy)$$
$$\le \liminf_{n\to\infty} \int_{-\infty}^{\infty} -\frac{Lu_n}{u_n}(y)\mu_t(dy) \qquad \text{a.s.}$$

and

$$\mathbb{E}\big[e^{t\int_{-\infty}^{\infty}(c_0y^2-c_1yg(y)-c_2))\mu_t(dy)}\big] \le \mathbb{E}\big(e^{t\liminf_{n\to\infty}\int_{-\infty}^{\infty}-\frac{Lu_n}{u_n}(y)\mu_t(dy)}\big)$$
$$= \mathbb{E}\Big(\liminf_{n\to\infty} e^{t\int_{-\infty}^{\infty}-\frac{Lu_n}{u_n}(y)\mu_t(dy)}\Big)$$
$$\le \liminf_{n\to\infty}\mathbb{E}\big[e^{t\int_{-\infty}^{\infty}-\frac{Lu_n}{u_n}(y)\mu_t(dy)}\big].$$

As in the proof of Lemma 2.2, using the sublinear growth of g, we can find a constant C_1 such that $c_0y^2 - c_1yg(y) - c_2) \ge \frac{c_0}{2}y^2 - C_1$. From this, we see that

$$\mathbb{E}\big[e^{-C_1t}e^{\frac{1}{2}c_0\int_0^t Y_s^2\,ds}\big] \le \liminf_{n\to\infty}\mathbb{E}\big[e^{t\int_{-\infty}^{\infty}-\frac{Lu_n}{u_n}(y)\mu_t(dy)}\big]. \tag{D.1}$$

The right-hand side in (D.1) can be bounded as

$$\begin{aligned}\mathbb{E}\big[e^{-\int_0^t\frac{Lu_n}{u_n}(Y_s)\,ds}\big] &\le e^{\log u_n(Y_0)}\mathbb{E}\big[e^{\log u_n(Y_t)-\log u_n(Y_0)-\int_0^t\frac{Lu_n}{u_n}(Y_s)\,ds}\big]\\ &\le u_n(y_0),\end{aligned} \tag{D.2}$$

where the inequality follows because $\log u_n(y) = \frac{c}{2}[n\psi(\frac{y}{n})]^2 \ge 0$, and the last equality follows because $M_t = e^{\log u_n(Y_t)-\log u_n(Y_0)\int_0^t\frac{Lu_n}{u_n}(Y_s)\,ds}$ is a local martingale with $M_0 = 1$. Applying this to (D.1) and using the definition of $u_n(y)$, we get

$$\mathbb{E}\big[e^{-C_1t}e^{\frac{1}{2}c_0\int_0^t Y_s^2\,ds}\big] \le e^{\frac{c}{2}y_0^2} < \infty. \tag{D.3}$$

From Assumption 3.1, we know that $\sigma^2(y)$ has sub-quadratic growth, and hence there exists a constant C_2 such that $\frac{1}{2}\sigma^2(y) \le \frac{c_0}{2}y^2 + C_2$. Therefore,

$$\mathbb{E}\big[e^{\frac{1}{2}\int_0^t\sigma^2(Y_s)\,ds}\big] \le \mathbb{E}\big[e^{C_2t}e^{\frac{c_0}{2}\int_0^t Y_s^2\,ds}\big] \le e^{\frac{c}{2}y_0^2+C_2t+C_1t} < \infty$$

from (D.3).

Acknowledgments. The authors would like to thank Jin Feng, Lingfei Li, Vadim Linetsky, Matt Lorig, Josef Teichmann, Srinivasa Varadhan and Guanlin Zhang for helpful discussions.

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