Electronic Journal of Statistics
Vol. 9 (2015) 2689–2705
ISSN: 1935-7524
DOI: 10.1214/15-EJS1091

Construction of minimum generalized aberration two-level orthogonal arrays

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Abstract: In this paper we explore the problem of constructing two-level Minimum Generalized Aberration (MGA) orthogonal arrays with strength t, n runs and $q > t$ columns, using a method that employs the J-characteristics of a two-level design. General results for the construction of MGA orthogonal arrays with $t+1$, $t+2$ and $t+3$ columns are given, while all MGA designs with strength $t \geq 2$, $n \equiv 0 \bmod 4$ runs and $q \leq 6$ are constructed. Results are also given for two-level orthogonal arrays with $q = 7$ factors, but with strength greater than two. Projection properties of the MGA designs that have been identified, are also discussed.

MSC 2010 subject classifications: Primary 62K15; secondary 05B20.
Keywords and phrases: Orthogonal arrays, J-characteristics, minimum generalized aberration.

Received July 2015.

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1. Introduction

When q two-level factors are studied in an experiment, a response of interest is measured in a collection of runs that are selected out of the 2^q different combinations of the levels of the factors. It is possible for a specific combination to be selected more than once, or not at all. Such a collection of runs is called a fractional factorial design and it is the most popular choice for experimentation in various fields. When all the 2^q combinations are selected the same number of times, a full 2^q factorial design with replications is generated. The use of a full 2^q factorial design guarantees that all the $2^q - 1$ factorial effects are estimated independently from each other and with the same precision, while an independent estimation of the error variance can be accomplished from the possible replications. In this manner, any collection of runs out of the 2^q different combinations of the levels of the factors (i.e. an arbitrary fractional factorial design) does not guarantee an independent estimation of a collection of effects, a phenomenon that is called aliasing in the literature.

In experiments, it is desirable to use two-level fractional factorial designs that guarantee no aliasing between effects of low order. This can be accomplished if the chosen design belongs to the class of two-level orthogonal arrays of a given strength $t \geq 2$. An orthogonal array $OA(n, q, 2, t)$ is a $n \times q$ array with entries from a set of 2 distinct symbols (usually, -1 and $+1$ are selected), arranged so that for any collection of t columns of the array each of the 2^t row vectors appears equally often. In application to factorial designs, each column corresponds to a factor, the two symbols are the levels of each factor and each row represents a combination of the levels of the factors. So, every $OA(n, q, 2, t)$ is a n-run factorial design for q two-level factors. For a nice overview of factorial designs and orthogonal arrays, one may refer to Dey and Mukerjee [7] and Hedayat, Sloane and Stufken [9].

The strength t of the array, provides information on the aliasing of factorial effects. When $t = 1$, main effects are free of aliasing with the mean but not with each other, when $t = 2$ main effects are free of aliasing with the mean and with each other and so on. It is therefore desirable to use an orthogonal array with the highest possible strength t for running an experiment but, such a choice becomes expensive with respect to the number of runs needed, since $n = \lambda 2^t$. This positive integer λ is called the index of the array.

Deng and Tang [5] proposed an effective criterion that can be used to capture the structure of a two-level fractional factorial design $D = \{d_1, d_2, \ldots, d_q\}$ with n rows in $\{\pm 1\}^q$. For every m-subset $S = \{d_{j_1}, d_{j_2}, \ldots, d_{j_m}\}$ of columns of D, they defined the J-characteristics of D to be:

$$J_m(S) \equiv J_m(d_{j_1}, d_{j_2}, \ldots, d_{j_m}) = |\sum_{i=1}^{n} d_{ij_1} \ldots d_{ij_m}|. \tag{1.1}$$

Tang [15] generalized the definition of J-characteristics of Deng and Tang [5] by removing the absolute value, providing a general form of the J-characteristics of a design D as:

$$J_m(S) \equiv J_m(d_{j_1}, d_{j_2}, \ldots, d_{j_m}) = \sum_{i=1}^{n} d_{ij_1} \ldots d_{ij_m}. \tag{1.2}$$

For the calculation of the J-characteristics of a design D, all possible sets S of cardinality m, $m = 1, 2, \ldots, q$, should be formed out of the q columns of the design, in order to apply (1.2). In what follows in this paper, these sets are formed lexicographically for every selection of m, and the corresponding values of (1.2) are reported as a column vector $\mathbf{J}_m$. For example, when $m = 1$,

$\mathbf{J}_1 = [J_1(d_1), J_1(d_2), \ldots, J_1(d_q)]^T$, when $m = 2$, $\mathbf{J}_2 = [J_2(d_1, d_2),\ J_2(d_1, d_3), \ldots,$ $J_2(d_{q-1}, d_q)]^T$ and so on.

The value of $J_m(S)$ shows the level of aliasing between the m-factor interaction that is generated form the specific m columns in the set S, with the mean column. Clearly, $-n \leq J_m(S) \leq n$. If D is a two-level orthogonal array of strength t, it is easy to verify that $J_m(S) = 0$, for $1 \leq m \leq t$. Furthermore, Deng and Tang [5] showed that when n is a multiple of four, then $J_m(S)$ is a multiple of four. The Confounding Frequency Vector (CFV) of D defined by Deng and Tang [5] has the form

$$CFV = [(f_{t+1,1}, \ldots, f_{t+1,k+1}); (f_{t+2,1}, \ldots, f_{t+2,k+1}); \ldots; (f_{q,1}, \ldots, f_{q,k+1})]$$

where $f_{m,j}$ is the frequency of $m > t$ column combinations that give $|J_m(S)| = 4(k + 1 - j)$, for $j = 1, \ldots, k, k + 1$. It is clear that all the $f_{m,j}$ values are zero for $m \leq t$ and $j \leq k$ and therefore not reported in the vector. Moreover,

$$\sum_{j=1}^{k+1} f_{m,j} = q!/[m!(q-m)!]$$

so the $f_{m,k+1}$ values can also be omitted from the vector without any loss on information. Furthermore, Deng and Tang [6] showed that when n is a multiple of 8, then $J_m(S)$ is also a multiple of 8, but when n is not a multiple of 8 but a multiple of 4, then $J_m(S)$ is a multiple of 8 for $m = 4w + 1$ and $4w + 2$, but not a multiple of 8 for $m = 4w + 3$ and $4w + 4$, where $w =$ 0,1, 2, …. Therefore, the length of the CVF vector can be further manipulated.

The Confounding Frequency Vector of a design D provides essential information on how the effects are confounded and, in the way it is structured, takes into consideration the hierarchy principle. This fact led to the justification of a powerful criterion for evaluating competitive designs. Let D_1 and D_2 be two designs under evaluation and $f_p(D_1)$, $f_p(D_2)$ be the p-th entries in their confounding frequency vectors, $p = 1, \ldots, (q-t)(k+1)$. Let ℓ be the smallest integer such that $f_\ell(D_1) \neq f_\ell(D_2)$. If $f_\ell(D_1) < f_\ell(D_2)$ we say that D_1 has less generalized aberration than D_2. If no design has less generalized aberration than D_1, then D_1 is said to have Minimum Generalized Aberration (MGA). Tang and Deng [16] proposed a related criterion, called Minimum G_2 Aberration. For $m = 1, 2, \ldots, q$, the $J_m(S)$ values of a design D with n runs are summarized in a single value, $A_m^g = n^{-2} \sum J_m^2(S)$. The vector $(A_1^g, A_2^g, \ldots, A_q^g)$ of length q, is called the Generalized Wordlength Pattern of D and minimum G_2-aberration designs are those that sequentially minimize A_1^g, A_2^g,…, A_q^g. Ma and Fang [11] and Xu and Wu [19] provided results on the Generalized Wordlength Pattern of designs with factors with more than two levels.

The construction or the identification of MGA designs with n runs and q columns has received great attention in the last decades. A popular technique is the evaluation of all non-isomorphic orthogonal arrays for specific values of n and q, using a selected criterion and the identification of the best of them with respect to the criterion used. However, this technique depends heavily on the knowledge of a full list of non-isomorphic orthogonal arrays. For the aforementioned techniques, one may refer to the work of Deng and Tang [6], of Butler [3, 4], of Evangelaras, Koukouvinos and Lappas [8], of Schoen, Eendebak and Nguyen [12], of Lin, Sitter and Tang [10] and of Bulutoglu and Ryan [2] among others. For a nice overview on recent progress to this field, see Xu, Phoa and Wong [18].

Tang and Deng [17] completely solved the problem of constructing two-level MGA designs for any run size $n \equiv 0 \bmod 4$ and up to $q = 5$ factors. In this paper, we address the same problem by using a different method that also captures the projection properties of the constructed designs. In Section 2, we describe the method and give results for the construction of two-level MGA orthogonal arrays with strength t and up to $t+3$ columns. In Section 3, we provide construction results for designs with $n = 4k$ runs and up to $q = 7$ two-level factors. We also discuss the projection properties of these best designs, by taking advantage of their J-characteristics.

2. MGA designs of strength t with up to $t + 3$ columns

For a given q, we make use of the first $2^q - 1$ natural numbers and we take their q dimensional binary representation. We then create 2^q vectors of length q by replacing '0' with '-1'. Let $\mathbf{i}$ be the vector obtained by the i-th natural number, $i = 0, 1, \ldots, 2^q - 1$. Clearly, the $2^q \times q$ matrix

$$F_q = (\mathbf{0}, \mathbf{1}, \ldots, \mathbf{2^q - 1})^T$$

is a full 2^q factorial design.

Let now F_q^m, $j = 1, 2, \ldots, q$, denote the $2^q \times \binom{q}{m}$ matrix that has as columns the element-wise product of all subsets of columns of F_q that have cardinality m. Similar to the calculation of the J-characteristics, these subsets of cardinality m are taken lexicographically and their element-wise product is calculated. Clearly, $F_q^1 = F_q$. Moreover, the matrix

$$H_{2^q} = [\mathbf{I}, F_q, F_q^2, \ldots, F_q^q]$$

is a Hadamard matrix of order 2^q, where $\mathbf{I}$ is a column vector with all its elements equal to 1. Consequently, $H_{2^q} \cdot H_{2^q}^T = 2^q I_{2^q}$.

Let a_i, $i = 0, 1, \ldots, 2^q - 1$ be the number of times the vector $\mathbf{i}$ is selected as a run in the design. Then, the J-characteristics of D are written as linear combinations of the quantities a_i's, as

$$\mathbf{J}_m = (F_q^m)^T \cdot \mathbf{a}.$$

Furthermore, let $\mathbf{J} = [n, \mathbf{J}_1^T, \mathbf{J}_2^T, \ldots, J_q]^T$ be the column vector that summarizes the number of runs n, and all the J-characteristics of D. Then, $H_{2^q}^T \cdot \mathbf{a} = \mathbf{J}$ or equivalently,

$$\mathbf{a} = \frac{1}{2^q} H_{2^q} \cdot \mathbf{J}. \tag{2.1}$$

This connection has been pointed out by Tang [15] and Stufken and Tang [14] in justifying that the vector of J-characteristics uniquely determines a factorial design. Moreover, Stufken and Tang [14] noted that the vector $\mathbf{J}$ is simply the Hadamard transform of the vector $\mathbf{a}$. It is therefore tempting to try to set the values of the vectors $\mathbf{J}_m$ to take their minimum acceptable (absolute) values for every $m = 1, 2, \ldots, q$, in order to construct MGA designs with n runs and q factors. Once such vectors are established, the MGA design can then be constructed using (2.1). The selection of optimal $\mathbf{J}$ vectors can be done by taking into account valuable properties of the J-characteristics. These properties are summarized in the following lemmas. Lemma 2.1 was stated and proven in Stufken and Tang [14] and Lemma 2.2, which generalizes the results of Stufken and Tang [14], was proven in Bulutoglu and Ryan [2].

Lemma 2.1. *In a two-level $OA(\lambda 2^t, q, 2, t)$ of strength t, it holds that $J_m(S) = \mu_m 2^t$ for some integer μ_m.*

- *(i) If λ is even, then μ_m is even.*
- *(ii) If λ is odd and t is even, then μ_m is odd for $m = t+1$ and $t+2$.*
- *(iii) if λ is odd and t is odd, then μ_m is odd for $m = t+1$ and even for $m = t+2$.*

Lemma 2.2. *In a two-level $OA(\lambda 2^t, q, 2, t)$ of strength t and $q \geq t+2$, it holds that $J_m(S) = \mu_m 2^t$ for some integer μ_m.*

- *(i) If λ is even, then μ_m is even.*
- *(ii) If λ is odd then μ_m is odd if $\binom{m-1}{m-t-1} \equiv 1 \bmod 2$ and even otherwise.*

Lemma 2.3. *Let D be a two-level $OA(\lambda 2^t, q, 2, t)$ of strength t and J^m denote the sum of the elements of the $\mathbf{J}_m$ vector, $m = 1, 2, \ldots, q$.*

- *(i) The sum $\lambda 2^t + \sum J^m$ of all $J_m(S)$ values, is a nonnegative multiple of 2^q.*
- *(ii) The sum $\sum\limits_{m\ odd} J^m$ of all $J_m(S)$ values for odd m, is a multiple of 2^{q-1}.*
- *(iii) The sum $\lambda 2^t + \sum\limits_{m\ even} J^m$ of all $J_m(S)$ values for even m, is a multiple of 2^{q-1}.*

Proof. From (2.1) we have that $a_0 = \frac{1}{2^q}(\lambda 2^t + \sum_{m=1}^{q}(-1)^m J^m)$ and $a_{2^q-1} = \frac{1}{2^q}(\lambda 2^t + \sum_{m=1}^{q} J^m)$, where a_0 and a_{2^q-1} are nonnegative integers. Then, it easily comes out that $(\lambda 2^t + \sum\limits_{\text{m even}} J^m) = 2^{q-1}(a_0 + a_{2^q-1})$ and $(\sum\limits_{\text{m odd}} J^m) = 2^{q-1}(a_{2^q-1} - a_0)$. □

Lemmas 2.1, 2.2 and 2.3 can be used to construct MGA $OA(\lambda 2^t, q, 2, t)$ of strength t having $q = t+1$, $q = t+2$ and $q = t+3$ columns, for any choice of t and when λ is odd. This restriction for odd λ is made to guarantee that the strength of the produced MGA array is strictly t and not larger. In what follows in this section we use $\mathbf{1}$ and $\mathbf{0}$ to denote vectors of appropriate length that have all their elements equal to one or zero.

2.1. Construction of MGA designs of strength t, with $t+1$ columns

Two-level orthogonal arrays of strength t having $t+1$ columns have been completely enumerated by Seiden and Zemach [13]. The following result can be utilized to directly construct a MGA orthogonal array $OA(\lambda 2^t, t+1, 2, t)$ for any choice of t and an odd λ.

Theorem 2.1. *Let λ be a nonnegative odd integer. A MGA $OA(\lambda 2^t, t+1, 2, t)$ is constructed using (2.1) and $J_{t+1} = 2^t$.*

Proof. By Lemma 2.1, it is obvious that the value of J_{t+1} that is set, is the minimum acceptable absolute value for a MGA design. It remains to justify that this selection guarantees the production of nonnegative integers, when using equation (2.1). Since the array is of strength t, equation (2.1) relaxes to

$$\mathbf{a} = \frac{1}{2^{t+1}}[\mathbf{I}, F_{t+1}^{t+1}] \cdot [\lambda 2^t, 2^t].$$

Let x denote the number of '-1' in the i-th row of F_{t+1}. Then, the i-th element of vector $\mathbf{a}$ is

$$a_i = [\lambda + (-1)^x]/2.$$

This is a nonnegative integer for every odd value of λ. $\square$

2.2. Construction of MGA designs of strength t, with $t+2$ columns

Two-level orthogonal arrays of strength t having $t+2$ columns have been completely enumerated by Stufken and Tang [14]. Stufken and Tang also provide a systematic way of constructing these arrays. In this section we provide a result that can be applied to directly construct a MGA $OA(\lambda 2^t, t+2, 2, t)$ for any choice of t, without exploiting the full list of non-isomorphic arrays. Lemma 2.4 is used in Theorem 2.2 to verify that the vector $\mathbf{a}$ of the multiplicities of the runs of the full 2^{t+2} factorial design will be integers.

Lemma 2.4. *Let $q = t+2$. The sum of the elements of row i in the matrices F_{t+2}^{t+1} and F_{t+2}^{t+2} is $S_{t+2}^{t+1} = (-1)^x(t+2-2x)$ and $S_{t+2}^{t+2} = (-1)^x$ respectively, where x is the number of '-1's in the i-th row of F_{t+2}.*

Proof. On the i-th row of F_{t+2}^{t+1} there are x $-(-1)^x$s and $(t+2-x)$ $(-1)^x$s. The i-th element of F_{t+2}^{t+2} is equal to $(-1)^x$. $\square$

Theorem 2.2. *A minimum generalized aberration $OA(\lambda 2^t, t+2, 2, t)$ is constructed via (2.1), when*

i) $t \equiv 0 \bmod 4$ and $\lambda \equiv 1 \bmod 4$, or $t \equiv 2 \bmod 4$ and $\lambda \equiv 3 \bmod 4$, for $\lambda \geq t+1$, by using $\mathbf{J}_{t+1} = 2^t\mathbf{1}^T$ and $J_{t+2} = 2^t$.
ii) $t \equiv 0 \bmod 4$ and $\lambda \equiv 3 \bmod 4$ or $t \equiv 2 \bmod 4$ and $\lambda \equiv 1 \bmod 4$, for $\lambda \geq t+3$, by using $\mathbf{J}_{t+1} = 2^t\mathbf{1}^T$ and $J_{t+2} = -2^t$.
iii) $t \equiv 1 \bmod 4$ and $\lambda \equiv 1 \bmod 4$ or $t \equiv 3 \bmod 4$ and $\lambda \equiv 3 \bmod 4$, for $\lambda \geq t$, by using $\mathbf{J}_{t+1} = 2^t\mathbf{1}^T$ and $J_{t+2} = 0$.

iv) $t \equiv 1 \bmod 4$ *and* $\lambda \equiv 3 \bmod 4$ *or* $t \equiv 3 \bmod 4$ *and* $\lambda \equiv 1 \bmod 4$*, for* $\lambda \geq t+2$*, by using* $\mathbf{J}_{t+1} = -2^t \mathbf{1}^T$ *and* $J_{t+2} = 0$*.*

Proof. By Lemma 2.1, the values of J-characteristics that are set for every case considered, are the minimum acceptable absolute values for MGA designs. It remains to be proven that these selections guarantee the production of non-negative integers, when using equation (2.1). Since the array is of strength t, equation (2.1) becomes

$$\mathbf{a} = \frac{1}{2^{t+2}}[\mathbf{I}, F_{t+2}^{t+1}, F_{t+2}^{t+2}] \cdot [\lambda 2^t, \mathbf{J}_{t+1}, J_{t+2}]^T.$$

When $\mathbf{J}_{t+1} = [2^t, 2^t, \ldots, 2^t]^T$ and $J_{t+2} = 2^t$ are used, by Lemma 2.4 the i-th element of vector $\mathbf{a}$ is

$$a_i = [\lambda + S_{t+2}^{t+1} + S_{t+2}^{t+2}]/4 = [\lambda + (-1)^x(t+3-2x)]/4.$$

Similarly, if $\mathbf{J}_{t+1} = [2^t, 2^t, \ldots, 2^t]^T$ and $J_{t+2} = -2^t$, then

$$a_i = [\lambda + S_{t+2}^{t+1} - S_{t+2}^{t+2}]/4 = [\lambda + (-1)^x(t+1-2x)]/4.$$

If $\mathbf{J}_{t+1} = [2^t, 2^t, \ldots, 2^t]^T$ and $J_{t+2} = 0$, then

$$a_i = [\lambda + S_{t+2}^{t+1}]/4 = [\lambda + (-1)^x(t+2-2x)]/4$$

and if $\mathbf{J}_{t+1} = [-2^t, -2^t, \ldots, -2^t]^T$ and $J_{t+2} = 0$, then

$$a_i = [\lambda - S_{t+2}^{t+1}]/4 = [\lambda - (-1)^x(t+2-2x)]/4.$$

The result follows by taking into account the restrictions on t and λ that are given in every case considered. □

Theorem 2.2 can be applied for the construction of a MGA orthogonal array of strength t having $t+2$ columns, for any selection of an odd value of λ since the constraint on the minimum values of λ that are set in the result, satisfies the condition $\lambda \geq t$ that should be true for the existence of an $OA(\lambda 2^t, t+2, 2, t)$ for all the cases we study, see Theorem 2.29, page 30 of Hedayat, Sloane and Stufken [9]. Furthermore, the stated result shows that we can always construct a MGA two-level orthogonal array with strength t and $t+2$ columns that has equal values of J-characteristics in each $\mathbf{J}_m$ vector, $m = t+1$, $t+2$.

2.3. Construction of MGA designs of strength t, with $t+3$ columns

Two-level orthogonal arrays of strength t having $t+3$ columns have not been completely enumerated yet. The construction of MGA $OA(\lambda 2^t, t+3, 2, t)$ for any strength t and odd λ cannot be handled as efficiently as the previous cases, due to the increased complexity of the problem. However, a construction method that directly produces a MGA $OA(\lambda 2^t, t+3, 2, t)$ for various λ and t can be achieved. The following lemma is useful in identifying the minimum absolute value of J_{t+3} when the minimum allowed absolute values of $\mathbf{J}_{t+1}$ are achieved. A more general result that covers also the case we consider, has been given in Bulutoglu and Kaziska [1].

Lemma 2.5. *Let $q = t + 3$ and λ be odd. If a MGA $OA(\lambda 2^t, t + 3, 2, t)$ with $|\mathbf{J}_{t+1}| = 2^t \mathbf{1}^T$ exists, then $J_{t+3} \neq 0$, when $t \equiv 1 \bmod 4$ or $t \equiv 0 \bmod 4$ but J_{t+3} can be equal to zero, when $t \equiv 3 \bmod 4$ or $t \equiv 2 \bmod 4$.*

Proof. The vector $\mathbf{J}_{t+1}$ consists of $(t + 3)(t + 2)/2$ elements. Let y of them be equal to -2^t and the remaining $[(t + 3)(t + 2) - 2y]/2$ are equal to 2^t.

Let t be odd, so $t + 1$ and $t + 3$ are even numbers and assume that $J_{t+3} = 0$. By Lemma 2.3, we have that $\lambda 2^t + \sum_{\text{m even}} J^m \equiv 0 \bmod 2^{t+2}$ and therefore,

$$[\lambda + \frac{(t+3)(t+2) - 4y}{2}]2^t \equiv 0 \mod 2^{t+2}$$

or,

$$[\lambda + \frac{(t+3)(t+2) - 4y}{2}] \equiv 0 \mod 4.$$

Then λ is even when $t \equiv 1 \bmod 4$, a contradiction. When $t \equiv 3 \bmod 4$, the last relation can be achieved for odd values of λ so, in this case J_{t+3} can be zero.

Similarly, let t be even, so $t + 1$ and $t + 3$ are odd numbers and assume that $J_{t+3} = 0$. By Lemma 2.3, we have that $\sum_{\text{m odd}} J^m \equiv 0 \bmod 2^{t+2}$ and therefore,

$$[(t + 3)(t + 2) - 4y] \equiv 0 \mod 8,$$

which is false when $t \equiv 0 \bmod 4$, since $t+3$ is odd and $(t+2) \equiv 2 \bmod 4$. When $t \equiv 2 \bmod 4$, $t+1$ and $t+3$ are still odd numbers and if $J_{t+3} = 0$, we have from Lemma 2.3, that $\sum_{\text{m odd}} J^m \equiv 0 \bmod 2^{t+2}$ and therefore,

$$[(t + 3)(t + 2) - 4y] \equiv 0 \mod 4,$$

which can hold true for $t \equiv 2 \bmod 4$. □

In what follows, we investigate the construction of MGA orthogonal arrays $OA(\lambda 2^t, t + 3, 2, t)$, when the elements in each $\mathbf{J}_m$ vector, $m = t + 1$, $t + 2$ and $t + 3$ are identical. Lemma 2.6 is useful to verify that the components of the vector $\mathbf{a}$ are integers, in Theorem 2.3.

Lemma 2.6. *Let $q = t + 3$. The sum of the elements of row i in the matrices F_{t+3}^{t+1}, F_{t+3}^{t+2} and F_{t+3}^{t+3} is $S_{t+3}^{t+1} = \frac{(-1)^x}{2}[(t + 3)(t + 2) - 4x(t + 3 - x)]$, $S_{t+3}^{t+2} = (-1)^x(t + 3 - 2x)$ and $S_{t+3}^{t+3} = (-1)^x$, where x is the number of '-1's in the i-th row of F_{t+3}.*

Proof. On the i-th row of F_{t+3} there are $x(t+3-x)$ $-(-1)^x$s and $\binom{x}{2} + \binom{t+3-x}{2}$ $(-1)^x$s. On the i-th row of F_{t+3}^{t+2} there are x $-(-1)^x$s and $(t + 3 - x)$ $(-1)^x$s. Finally, the i-th element of F_{t+3}^{t+3} is $(-1)^x$. □

Theorem 2.3. *A minimum generalized aberration $OA(\lambda 2^t, t + 3, 2, t)$ is constructed via (2.1), when*

i) $t \equiv 0 \bmod 16$ and $\lambda \equiv 1 \bmod 8$, or $t \equiv 8 \bmod 16$ and $\lambda \equiv 5 \bmod 8$, for $\lambda \geq (t + 1)(t + 2)/2$, by using $\mathbf{J}_{t+1} = 2^t \mathbf{1}^T$, $\mathbf{J}_{t+2} = 2^t \mathbf{1}^t$ and $J_{t+3} = 2^t$.

ii) $t \equiv 0 \bmod 16$ *and* $\lambda \equiv 7 \bmod 8$, *or* $t \equiv 8 \bmod 16$ *and* $\lambda \equiv 3 \bmod 8$, *for* $\lambda \geq (t^2+7t+14)/2$, *by using* $\mathbf{J}_{t+1} = 2^t\mathbf{1}^T$, $\mathbf{J}_{t+2} = -2^t\mathbf{1}^t$ *and* $J_{t+3} = 2^t$.

iii) $t \equiv 4 \bmod 16$ *and* $\lambda \equiv 5 \bmod 8$, *or* $t \equiv 12 \bmod 16$ *and* $\lambda \equiv 1 \bmod 8$, *for* $\lambda \geq (t^2+3t-2)/2$, *by using* $\mathbf{J}_{t+1} = 2^t\mathbf{1}^T$, $\mathbf{J}_{t+2} = 2^t\mathbf{1}^t$ *and* $J_{t+3} = -2^t$.

iv) $t \equiv 4 \bmod 16$ *and* $\lambda \equiv 3 \bmod 8$, *or* $t \equiv 12 \bmod 16$ *and* $\lambda \equiv 7 \bmod 8$, *for* $\lambda \geq (t+2)(t+5)/2$, *by using* $\mathbf{J}_{t+1} = 2^t\mathbf{1}^T$, $\mathbf{J}_{t+2} = -2^t\mathbf{1}^t$ *and* $J_{t+3} = -2^t$.

v) $t \equiv 6 \bmod 16$ *and* $\lambda \equiv 3 \bmod 8$, *or* $t \equiv 14 \bmod 16$ *and* $\lambda \equiv 7 \bmod 8$, *for* $\lambda \geq t(t+3)/2$, *by using* $\mathbf{J}_{t+1} = 2^t\mathbf{1}^T$, $\mathbf{J}_{t+2} = 2^t\mathbf{1}^t$ *and* $J_{t+3} = 0$.

vi) $t \equiv 6 \bmod 16$ *and* $\lambda \equiv 5 \bmod 8$, *or* $t \equiv 14 \bmod 16$ *and* $\lambda \equiv 1 \bmod 8$, *for* $\lambda \geq (t^2+7t+12)/2$, *by using* $\mathbf{J}_{t+1} = 2^t\mathbf{1}^T$, $\mathbf{J}_{t+2} = -2^t\mathbf{1}^t$ *and* $J_{t+3} = 0$.

vii) $t \equiv 5 \bmod 16$ *and* $\lambda \equiv 5 \bmod 8$, *or* $t \equiv 13 \bmod 16$ *and* $\lambda \equiv 1 \bmod 8$, *for* $\lambda \geq (t^2+t-4)/2$, *by using* $\mathbf{J}_{t+1} = 2^t\mathbf{1}^T$, $\mathbf{J}_{t+2} = \mathbf{0}^t$ *and* $J_{t+3} = -2^t$.

viii) $t \equiv 5 \bmod 16$ *and* $\lambda \equiv 3 \bmod 8$, *or* $t \equiv 13 \bmod 16$ *and* $\lambda \equiv 7 \bmod 8$, *for* $\lambda \geq (t+1)(t+4)/2$, *by using* $\mathbf{J}_{t+1} = -2^t\mathbf{1}^T$, $\mathbf{J}_{t+2} = \mathbf{0}^t$ *and* $J_{t+3} = 2^t$.

ix) $t \equiv 7 \bmod 16$ *and* $\lambda \equiv 3 \bmod 8$, *or* $t \equiv 15 \bmod 16$ *and* $\lambda \equiv 7 \bmod 8$, *for* $\lambda \geq (t^2+t-2)/2$ *by using* $\mathbf{J}_{t+1} = 2^t\mathbf{1}^T$, $\mathbf{J}_{t+2} = \mathbf{0}^t$ *and* $J_{t+3} = 0$.

x) $t \equiv 7 \bmod 16$ *and* $\lambda \equiv 5 \bmod 8$, *or* $t \equiv 15 \bmod 16$ *and* $\lambda \equiv 1 \bmod 8$, *for* $\lambda \geq (t^2+5t+6)/2$, *by using* $\mathbf{J}_{t+1} = -2^t\mathbf{1}^T$, $\mathbf{J}_{t+2} = \mathbf{0}^t$ *and* $J_{t+3} = 0$.

xi) $t \equiv 1 \bmod 16$ *and* $\lambda \equiv 1 \bmod 8$, *or* $t \equiv 9 \bmod 16$ *and* $\lambda \equiv 5 \bmod 8$, *for* $\lambda \geq t(t+1)/2$, *by using* $\mathbf{J}_{t+1} = 2^t\mathbf{1}^T$, $\mathbf{J}_{t+2} = \mathbf{0}^t$ *and* $J_{t+3} = 2^t$.

xii) $t \equiv 1 \bmod 16$ *and* $\lambda \equiv 7 \bmod 8$, *or* $t \equiv 9 \bmod 16$ *and* $\lambda \equiv 3 \bmod 8$, *for* $\lambda \geq (t^2+5t+8)/2$, *by using* $\mathbf{J}_{t+1} = -2^t\mathbf{1}^T$, $\mathbf{J}_{t+2} = \mathbf{0}^t$ *and* $J_{t+3} = -2^t$.

Proof. From Lemmas 2.1, 2.2 and 2.5, it is obvious that the nonzero values of J-characteristics that are set for every case considered, are the minimum acceptable absolute values. It remains to be proven that these selections guarantee the production of nonnegative integers in (2.1). Since the array is of strength t, equation (2.1) becomes

$$\mathbf{a} = \frac{1}{2^{t+3}}[\mathbf{I}, F_{t+3}^{t+1}, F_{t+3}^{t+2}, F_{t+3}^{t+3}] \cdot [\lambda 2^t, \mathbf{J}_{t+1}, \mathbf{J}_{t+2}, J_{t+3}]^T.$$

For case (i), using Lemma 2.6, it follows that the i-th element of vector $\mathbf{a}$ is

$$a_i = [\lambda + S_{t+3}^{t+1} + S_{t+3}^{t+2} + S_{t+3}^{t+3}]/8 = [2\lambda + (-1)^x[(t+4-2x)^2 - (t+2)]]/16.$$

In the second case, the i-th element of vector $\mathbf{a}$ becomes

$$a_i = [\lambda + S_{t+3}^{t+1} - S_{t+3}^{t+2} + S_{t+3}^{t+3}]/8 = [2\lambda + (-1)^x[(t+2-2x)^2 - (t+2)]]/16.$$

In case (iii), it comes out that the i-th element of vector $\mathbf{a}$ is

$$a_i = [\lambda + S_{t+3}^{t+1} + S_{t+3}^{t+2} - S_{t+3}^{t+3}]/8 = [2\lambda + (-1)^x[(t+4-2x)^2 - (t+6)]]/16.$$

In case (iv), the i-th element of vector $\mathbf{a}$ takes the form

$$a_i = [\lambda + S_{t+3}^{t+1} - S_{t+3}^{t+2} - S_{t+3}^{t+3}]/8 = [2\lambda + (-1)^x[(t+2-2x)^2 - (t+6)]]/16.$$

For the case (v), we have that the i-th element of vector $\mathbf{a}$ is

$$a_i = [\lambda + S_{t+3}^{t+1} + S_{t+3}^{t+2}]/8 = [2\lambda + (-1)^x[(t+4-2x)^2 - (t+4)]]/16.$$

Similarly, for the case (vi), the i-th element of vector $\mathbf{a}$ is

$$a_i = [\lambda + S_{t+3}^{t+1} - S_{t+3}^{t+2}]/8 = [2\lambda + (-1)^x[(t+2-2x)^2 - (t+4)]]/16.$$

In case (vii), the i-th element of vector $\mathbf{a}$ becomes

$$a_i = [\lambda + S_{t+3}^{t+1} - S_{t+3}^{t+3}]/8 = [2\lambda + (-1)^x[(t+3-2x)^2 - (t+5)]]/16.$$

For the case (viii), the i-th element of vector $\mathbf{a}$ is

$$a_i = [\lambda - S_{t+3}^{t+1} + S_{t+3}^{t+3}]/8 = [2\lambda + (-1)^x[(t+5) - (t+3-2x)^2]]/16.$$

In case (ix), the i-th element of vector $\mathbf{a}$ becomes

$$a_i = [\lambda + S_{t+3}^{t+1}]/8 = [2\lambda + (-1)^x[(t+3-2x)^2 - (t+3)]]/16.$$

For the case (x), the i-th element of vector $\mathbf{a}$ becomes

$$a_i = [\lambda - S_{t+3}^{t+1}]/8 = [2\lambda + (-1)^x[(t+3) - (t+3-2x)^2]]/16.$$

In case (xi) the i-th element of vector $\mathbf{a}$ is

$$a_i = [\lambda + S_{t+3}^{t+1} + S_{t+3}^{t+3}]/8 = [2\lambda + (-1)^x[(t+3-2x)^2 - (t+1)]]/16.$$

Finally, for the last case, the i-th element of vector $\mathbf{a}$ becomes

$$a_i = [\lambda - S_{t+3}^{t+1} - S_{t+3}^{t+3}]/8 = [2\lambda + (-1)^x[(t+1) - (t+3-2x)^2]]/16.$$

Taking into account the restrictions on t and λ that are given in every case considered, it is easy to verify that a_i are nonnegative integers. □

For the unlisted values of t, a MGA orthogonal array of strength t with $t+3$ columns that has identical elements in each $\mathbf{J}_m$ vector, $m = t+1$, $t+2$ and $t+3$, does not exist and therefore cannot be constructed. However, most likely there are $\mathbf{J}_m$ vectors with the minimum acceptable absolute values that cater for odd values of λ, but further investigation on the distribution of the acceptable negative and positive values over each of the vectors $\mathbf{J}_m$ for these cases is needed.

3. Construction of MGA designs with up to seven factors

In this section, we construct MGA designs with $n \equiv 0 \bmod 4$ runs and with $3 \leq q \leq 7$ factors, by taking advantage of the notable properties of J-characteristics as summarized in Lemmas 2.1, 2.2 and 2.3. These lemmas are taken into consideration for establishing a quick and complete search on acceptable values for the vector $\mathbf{J} = [4k, \mathbf{J}_1^T, \mathbf{J}_2^T, \ldots, J_q]^T$, that lead to the construction of MGA designs with $n = 4k$ runs and $3 \leq q \leq 7$ factors. For $q \leq 6$ the solution to the problem is complete and extends the work of Tang and Deng [17], while for $q = 7$ we construct MGA designs of strength $t \geq 3$. We also discuss similarities of the

Confounding Frequency Vectors of the produced designs, as well as their projection properties. Since all produced arrays are of strength $t \geq 2$, sub-vectors $\mathbf{J}_1^T$ and $\mathbf{J}_2^T$ have zero elements and are not reported in the optimal solutions. In what follows, we use $\mathbf{0}^T$ to denote a column vector with elements equal to zero.

MGA designs with $q = 3$ factors for $n \equiv 0 \bmod 4$, have been extensively studied in the literature. We present the well known results in Proposition 3.1.

Proposition 3.1. *The vector* $\mathbf{J} = [4k, \mathbf{0}^T, \mathbf{0}^T, 0]^T$ *produces a MGA design with* $n = 4k$ *runs and* $q = 3$ *factors, when* $k \equiv 0 \bmod 2$ *while the vector* $\mathbf{J} = [4k, \mathbf{0}^T, \mathbf{0}^T, 4]^T$ *produces a MGA design with* $n = 4k$ *runs and* $q = 3$ *factors, when* $k \equiv 1 \bmod 2$.

MGA designs with $q = 4$ factors can be constructed for $n \equiv 0 \bmod 4$, using the following result.

Proposition 3.2. *The vector* $\mathbf{J} = [4k, \mathbf{0}^T, \mathbf{0}^T, \mathbf{J}_3^T, J_4]^T$ *whose values are listed in Table 1 for every selection of* k*, produces a MGA design with* $n = 4k$ *runs and* $q = 4$ *factors.*

TABLE 1
$\mathbf{J}$ *vectors for MGA designs with* $q = 4$ *factors*

	Sets S (in lexicographical order)				
	$d_1d_2d_3$	$d_1d_2d_4$	$d_1d_3d_4$	$d_2d_3d_4$	$d_1d_2d_3d_4$
k	$\mathbf{J}_3^T$				J_4
0 mod 4	0	0	0	0	0
1 mod 4[1]	4	4	4	4	-4
2 mod 4	0	0	0	0	8
3 mod 4	4	4	4	4	4

[1] $k > 1$

Vectors $\mathbf{J}$ that are listed in Table 1 for designs with $q = 4$ factors makes the proof of Corollary 3.1 evident.

Corollary 3.1. *MGA designs with* $n = 4k$ *runs and* $q = 4$ *factors share the same Confounding Frequency Vector when* $k \equiv 1 \bmod 4$ *and* $k \equiv 3 \bmod 4$*,* $k > 1$.

For $q = 5$ factors and $n \equiv 0 \bmod 4$, we can construct a MGA design using Proposition 3.3.

Proposition 3.3. *The vector* $\mathbf{J} = [4k, \mathbf{0}^T, \mathbf{0}^T, \mathbf{J}_3^T, \mathbf{J}_4^T, J_5]^T$ *whose values are listed in Table 2 for every selection of* k*, produces a MGA design with* $n = 4k$ *runs and* $q = 5$ *factors.*

Corollary 3.2 for designs with $q = 5$ factors comes out easily by observing the vectors $\mathbf{J}$ that are listed in Table 2.

Corollary 3.2. *Minimum generalized aberration designs with* $n = 4k$ *runs and* $q = 5$ *factors share the same Confounding Frequency Vector when:*

(i) $k \equiv 1 \bmod 8$, $k \equiv 3 \bmod 8$, $k \equiv 5 \bmod 8$ *and* $k \equiv 7 \bmod 8$, $k > 1$.
(ii) $k \equiv 2 \bmod 8$ *and* $k \equiv 6 \bmod 8$, $k > 2$.

TABLE 2
J *vectors for MGA designs with $q = 5$ factors*

	Sets S (in lexicographical order)															
hline	$\mathbf{J}_3^T$										$\mathbf{J}_4^T$					J_5
$k = 1$	An $OA(4, 5, 2, 2)$ does not exist															
$k = 2$	0	0	8	0	0	0	0	0	0	8	8	0	0	0	0	0
$k \equiv 0 \bmod 8$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 1 \bmod 8$[1]	4	4	4	4	4	4	4	4	4	-4	-4	-4	-4	4	4	0
$k \equiv 2 \bmod 8$[2]	0	0	0	0	0	0	0	0	0	0	-8	-8	-8	-8	-8	0
$k \equiv 3 \bmod 8$	4	4	4	4	4	-4	4	-4	4	-4	4	-4	-4	4	4	0
$k \equiv 4 \bmod 8$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16
$k \equiv 5 \bmod 8$	4	4	4	4	4	-4	4	-4	4	-4	-4	4	4	-4	-4	0
$k \equiv 6 \bmod 8$	0	0	0	0	0	0	0	0	0	0	8	8	8	8	8	0
$k \equiv 7 \bmod 8$	4	4	4	4	4	4	4	4	4	-4	4	4	4	-4	-4	0

[1] $k > 1$; [2] $k > 2$

Proposition 3.4 covers the construction of MGA designs with $q = 6$ factors.

Proposition 3.4. *The vector* $\mathbf{J} = [4k, \mathbf{0}^T, \mathbf{0}^T, \mathbf{J}_3^T, \mathbf{J}_4^T, \mathbf{J}_5^T, J_6]^T$ *whose values are listed in Table 3 for every selection of k, produces a MGA design with $n = 4k$ runs and $q = 6$ factors.*

Next, properties of the Confounding Frequency Vector of MGA designs with $q = 6$ factors are established.

Corollary 3.3. *The MGA designs with $n = 4k$ runs and $q = 6$ factors share the same Confounding Frequency Vector when:*

(i) $k \equiv 1$ *mod 16,* $k \equiv 3$ *mod 16,* $k \equiv 13$ *mod 16 and* $k \equiv 15$ *mod 16,* $k > 1$.
(ii) $k \equiv 2$ *mod 16,* $k \equiv 6$ *mod 16,* $k \equiv 10$ *mod 16 and* $k \equiv 14$ *mod 16,* $k > 2$.
(iii) $k \equiv 5$ *mod 16,* $k \equiv 7$ *mod 16,* $k \equiv 9$ *mod 16 and* $k \equiv 11$ *mod 16,* $k > 5$.
(iv) $k \equiv 4$ *mod 16 and* $k \equiv 12$ *mod 16,* $k > 12$.

For $q = 7$ factors, we can apply the following result in order to obtain MGA designs of strength $t \geq 3$.

Proposition 3.5. *The vector* $\mathbf{J} = [4k, \mathbf{0}^T, \mathbf{0}^T, \mathbf{J}_3^T, \mathbf{J}_4^T, \mathbf{J}_5^T, \mathbf{J}_6^T, J_7]^T$ *whose values are listed in Table 4 for every selection of k, produces a MGA design of strength $t \geq 3$ with $n = 4k$ runs and $q = 7$ factors.*

Corollary 3.4 states relations between the Confounding Frequency Vector of MGA designs with $q = 7$ factors.

Corollary 3.4. *The MGA designs with $n = 4k$ runs and $q = 7$ factors share the same Confounding Frequency Vector when:*

(i) $k \equiv 2$ *mod 32,* $k \equiv 6$ *mod 32,* $k \equiv 26$ *mod 32 and* $k \equiv 30$ *mod 32,* $k \neq 2$.
(ii) $k \equiv 10$ *mod 32,* $k \equiv 14$ *mod 32,* $k \equiv 18$ *mod 32 and* $k \equiv 22$ *mod 32,* $k > 18$.
(iii) $k \equiv 4$ *mod 32,* $k \equiv 12$ *mod 32,* $k \equiv 20$ *mod 32 and* $k \equiv 28$ *mod 32,* $k > 28$.
(iv) $k \equiv 8$ *mod 32 and* $k \equiv 24$ *mod 32,* $k > 24$.

Table 3. **J** *vectors for MGA designs with* $q = 6$ *factors*

	Sets of columns in lexicographical order																			
	$\mathbf{J}_3^T$																			
$k=1$	An $OA(4,6,2,2)$ does not exist																			
$k=2$	-8	0	0	0	0	0	0	-8	0	0	0	0	0	0	-8	0	0	0	-8	0
$k=4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k=5$	-4	-4	-4	-4	-4	-4	-4	4	-4	4	-4	4	-4	-4	4	4	-4	4	4	4
$k=12$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 0 \bmod 16$	**0**																			
$k \equiv 1 \bmod 16$[1]	4	4	4	4	-4	-4	-4	-4	-4	4	-4	-4	-4	-4	-4	4	-4	-4	-4	-4
$k \equiv 2 \bmod 16$[2]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 3 \bmod 16$	4	4	4	4	-4	-4	4	4	-4	4	-4	4	-4	-4	4	4	-4	-4	-4	-4
$k \equiv 4 \bmod 16$[3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 5 \bmod 16$[4]	4	4	4	4	-4	-4	4	4	-4	4	-4	4	-4	-4	4	4	-4	-4	-4	-4
$k \equiv 6 \bmod 16$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 7 \bmod 16$	4	4	4	4	-4	-4	-4	-4	-4	4	-4	4	4	4	4	4	-4	-4	-4	-4
$k \equiv 8 \bmod 16$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 9 \bmod 16$	4	4	4	4	-4	-4	-4	-4	-4	4	4	-4	-4	4	4	-4	-4	-4	4	4
$k \equiv 10 \bmod 16$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 11 \bmod 16$	4	4	4	4	-4	-4	4	4	-4	4	-4	4	-4	-4	4	4	-4	-4	-4	-4
$k \equiv 12 \bmod 16$[5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 13 \bmod 16$	4	4	4	4	-4	-4	4	4	-4	4	-4	4	-4	-4	4	4	-4	-4	-4	-4
$k \equiv 14 \bmod 16$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 15 \bmod 16$	4	4	4	4	-4	-4	-4	-4	-4	4	-4	-4	-4	-4	-4	4	-4	-4	-4	-4

	$\mathbf{J}_4^T$														
$k=2$	0	0	0	0	0	8	0	8	0	0	8	0	0	0	0
$k=4$	16	0	0	0	0	16	0	0	0	0	0	0	0	0	16
$k=5$	-4	4	-4	4	4	-4	4	4	-4	4	4	-4	4	4	4
$k=12$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16
$k \equiv 0 \bmod 16$	**0**														
$k \equiv 1 \bmod 16$[1]	-4	-4	-4	-4	-4	-4	-4	-4	4	4	-4	-4	4	4	-4
$k \equiv 2 \bmod 16$[2]	8	8	8	8	8	8	8	8	8	-8	-8	-8	-8	8	8
$k \equiv 3 \bmod 16$	4	-4	-4	-4	-4	4	-4	-4	4	4	-4	-4	4	4	4
$k \equiv 4 \bmod 16$[3]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 5 \bmod 16$[4]	-4	4	4	4	4	-4	4	4	-4	-4	4	4	-4	-4	-4
$k \equiv 6 \bmod 16$	8	8	8	8	8	-8	8	-8	8	-8	8	-8	-8	8	8
$k \equiv 7 \bmod 16$	4	-4	-4	-4	-4	4	4	4	-4	-4	4	4	-4	-4	4
$k \equiv 8 \bmod 16$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 9 \bmod 16$	4	-4	-4	4	4	4	-4	-4	-4	-4	-4	-4	4	4	-4
$k \equiv 10 \bmod 16$	-8	-8	-8	-8	-8	8	-8	8	-8	8	-8	8	8	-8	-8
$k \equiv 11 \bmod 16$	4	-4	-4	-4	-4	4	-4	-4	4	4	-4	-4	4	4	4
$k \equiv 12 \bmod 16$[5]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k \equiv 13 \bmod 16$	-4	4	4	4	4	-4	4	4	-4	-4	4	4	-4	-4	-4
$k \equiv 14 \bmod 16$	-8	-8	-8	-8	-8	-8	-8	-8	-8	8	8	8	8	-8	-8
$k \equiv 15 \bmod 16$	4	4	4	4	4	4	4	4	-4	-4	4	4	-4	-4	4

	$\mathbf{J}_5^T$						J_6
$k=2$	0	0	0	0	0	0	0
$k=4$	0	0	0	0	0	0	0
$k=5$	-8	8	-8	0	0	-8	-8
$k=12$	16	16	16	16	0	0	0
$k \equiv 0 \bmod 16$	**0**						0
$k \equiv 1 \bmod 16$[1]	0	0	0	0	0	0	-8
$k \equiv 2 \bmod 16$[2]	0	0	0	0	0	0	0
$k \equiv 3 \bmod 16$	0	0	0	0	0	0	-8
$k \equiv 4 \bmod 16$[3]	16	16	16	16	16	16	16
$k \equiv 5 \bmod 16$[4]	0	0	0	0	0	0	-24
$k \equiv 6 \bmod 16$	0	0	0	0	0	0	0
$k \equiv 7 \bmod 16$	0	0	0	0	0	0	-24
$k \equiv 8 \bmod 16$	0	0	0	0	0	0	32
$k \equiv 9 \bmod 16$	0	0	0	0	0	0	-24
$k \equiv 10 \bmod 16$	0	0	0	0	0	0	0
$k \equiv 11 \bmod 16$	0	0	0	0	0	0	24
$k \equiv 12 \bmod 16$[5]	16	16	16	16	16	16	-16
$k \equiv 13 \bmod 16$	0	0	0	0	0	0	8
$k \equiv 14 \bmod 16$	0	0	0	0	0	0	0
$k \equiv 15 \bmod 16$	0	0	0	0	0	0	8

[1] $k > 1$; [2] $k > 2$; [3] $k > 4$; [4] $k > 5$; [5] $k > 12$

Table 4. **J** *vectors for MGA designs of strength $t \geq 3$ with $q = 7$ factors*

k	$\mathbf{J}_3^T$	$\mathbf{J}_4^T$	$\mathbf{J}_5^T$	$\mathbf{J}_6^T$	J_7
$k \equiv 0 \bmod 32$	**0**	**0**	**0**	**0**	0
$k \equiv 8 \bmod 32$[1]	**0**	**0**	**0**	32 32 32 32 32 32 32	0
$k \equiv 16 \bmod 32$	**0**	**0**	**0**	**0**	64
$k \equiv 24 \bmod 32$[2]	**0**	**0**	**0**	-32 -32 -32 -32 -32 -32 -32	0

(Columns $\mathbf{J}_3^T$ to $\mathbf{J}_6^T$: Sets of columns in lexicographical order)

[1] $k > 8$; [2] $k > 24$

	$\mathbf{J}_3^T$	$\mathbf{J}_4^T$	$\mathbf{J}_5^T$	$\mathbf{J}_6^T$	J_7
$k \equiv 4 \bmod 32$[1]	**0**	**0**	-16 -16 -16 -16 -16 -16 -16 -16 -16 16 -16 16 16 16 16 16 16 16 -16 16 16	-16 16 16 16 -16 -16 -16	16
$k \equiv 12 \bmod 32$[2]	**0**	**0**	-16 -16 -16 -16 -16 -16 -16 -16 -16 16 -16 16 16 -16 16 16 16 16 16 16 16	16 -16 -16 -16 16 -16 -16	16
$k \equiv 20 \bmod 32$[3]	**0**	**0**	-16 -16 -16 -16 -16 -16 -16 -16 -16 -16 -16 -16 -16 -16 -16 -16 -16 -16 -16 -16 -16	16 16 16 16 16 16 16	16
$k \equiv 28 \bmod 32$[4]	**0**	**0**	-16 -16 -16 -16 -16 -16 -16 -16 -16 16 -16 -16 16 16 16 16 16 16 16 16 16	16 16 -16 16 16 -16 -16	16

(Columns $\mathbf{J}_3^T$ to $\mathbf{J}_6^T$: Sets of columns in lexicographical order)

[1] $k > 4$; [2] $k > 12$; [3] $k > 20$; [4] $k > 28$

	$\mathbf{J}_3^T$	$\mathbf{J}_4^T$	$\mathbf{J}_5^T$	$\mathbf{J}_6^T$	J_7
$k = 2$	The unique $OA(8,7,2,2)$				
$k = 10$	**0**	8 8 8 8 8 8 8 -8 8 -8 8 -8 8 -8 -8 8 -8 -8 8 -8 -8 8 -8 -8 8 8 -8 8 8 8 8 -8 8 8 8	**0**	16 -16 16 0 0 16 -16	0
$k = 14$	**0**	8 8 8 8 8 8 8 -8 8 8 8 8 8 8 -8 8 8 8 -8 8 8 8 8 -8 -8 8 -8 8 -8 -8 8 -8 -8 -8 -8	**0**	0 0 0 -16 -16 -16 32	0
$k = 18$	**0**	8 8 8 8 -8 -8 8 -8 -8 -8 -8 8 -8 8 -8 8 -8 8 8 8 -8 8 8 8 -8 8 -8 -8 -8 8 -8 8 8 8 8	**0**	0 0 0 16 16 16 -32	0
$k \equiv 2 \bmod 32$[1]	**0**	8 8 8 8 -8 -8 -8 -8 -8 8 8 -8 -8 -8 -8 -8 8 8 8 8 8 -8 -8 -8 -8 8 8 8 -8 -8 -8 -8 8 8 -8	**0**	0 0 0 0 0 0 16	0
$k \equiv 6 \bmod 32$	**0**	8 8 8 8 -8 -8 8 8 -8 8 8 -8 8 8 -8 -8 8 8 8 8 -8 8 8 8 8 -8 8 8 -8 -8 -8 -8 8 8 8	**0**	0 0 0 0 0 0 16	0
$k \equiv 10 \bmod 32$[2]	**0**	8 8 8 8 8 8 -8 -8 8 -8 8 -8 8 8 -8 -8 -8 -8 -8 -8 -8 8 8 8 8 -8 -8 -8 8 8 -8 -8 8 8 -8	**0**	0 0 0 0 0 0 16	64
$k \equiv 14 \bmod 32$[3]	**0**	8 8 8 8 8 8 8 8 8 -8 8 8 8 8 8 -8 -8 -8 -8 -8 8 8 8 8 8 8 -8 -8 8 8 -8 -8 8 8 8	**0**	0 0 0 0 0 0 16	64
$k \equiv 18 \bmod 32$[4]	**0**	-8 -8 -8 -8 -8 -8 -8 -8 -8 8 -8 -8 -8 -8 -8 8 8 8 8 8 -8 -8 -8 -8 -8 -8 8 8 -8 -8 8 8 -8 -8 -8	**0**	0 0 0 0 0 0 -16	64
$k \equiv 22 \bmod 32$	**0**	-8 -8 -8 -8 -8 -8 8 8 -8 8 -8 8 -8 -8 8 8 8 8 8 8 8 -8 -8 -8 -8 8 8 8 -8 -8 8 8 -8 -8 8	**0**	0 0 0 0 0 0 -16	64
$k \equiv 26 \bmod 32$	**0**	-8 -8 -8 -8 8 8 -8 -8 8 -8 -8 8 -8 -8 8 8 -8 -8 -8 -8 8 -8 -8 -8 -8 8 -8 -8 8 8 8 8 -8 -8 -8	**0**	0 0 0 0 0 0 -16	0
$k \equiv 30 \bmod 32$	**0**	-8 -8 -8 -8 8 8 8 8 8 -8 -8 8 8 8 8 8 -8 -8 -8 -8 -8 8 8 8 8 -8 -8 -8 8 8 8 8 -8 -8 8	**0**	0 0 0 0 0 0 -16	0

(Columns $\mathbf{J}_3^T$ to $\mathbf{J}_6^T$: Sets of columns in lexicographical order)

[1] $k > 2$; [2] $k > 10$; [3] $k > 14$; [4] $k > 18$

3.1. Projection properties of MGA designs with $q \leq 7$ factors

It is well known (Deng and Tang [5], Tang [15]) that the J-characteristics capture the projection properties of a two-level design D. Therefore, by examining the values given in Tables 1, 2, 3 and 4 for MGA two-level designs with $q \leq 7$ factors, it is easy to verify the following result.

Proposition 3.6. *The MGA designs with $n = 4k$ runs and:*

a. $q \leq 6$ two-level factors provide MGA designs when projected onto less than q factors, for $k \neq 4$, 5 and 12.
b. $q = 7$ two-level factors provide MGA designs when projected onto less than seven factors, for $k = 16$ or $k \equiv 0$ mod 4, $k > 28$.
c. $q = 7$ two-level factors provide MGA designs when projected onto less than six factors, for $k \equiv 2$ mod 4.

4. Concluding Remarks

At this point, we briefly describe the procedure that was followed for implementing the exhaustive search for GMA designs with $n = 4k$ runs and $q \leq 7$ factors, for those cases that remain unsolved by Theorems 2.2 and 2.3. From equation (2.1) it follows that $a_i = (c_i + 4k)/2^q$, $i = 0, 1, \ldots, 2^q - 1$, where c_i is a certain linear combination of the J-characteristics, with coefficients ± 1. Taking into account that $c_i \equiv 0 \bmod 4$, we have that $c_i = 4d_i$, where d_i is a certain linear combination of the J-characteristics when divided by four. So, $a_i = (d_i + k)/2^{q-2}$, $i = 0, 2, \ldots, 2^q - 1$. Let $k \equiv x \bmod 2^{q-2}$. Since all a_i should be integers, it follows that all the values of the d_i's should meet the condition $d_i \equiv (2^{q-2} - x) \bmod 2^{q-2}$, $i = 0, 1, \ldots, 2^q - 1$. The search for solutions is performed by checking whether all the d_i values that are produced when certain $\mathbf{J}$ vectors are used, leave the same specific remainder when divided by 2^{q-2}. Conditions for the minimum value of k that this can happen, are then obtained in order for all the a_i to be nonnegative integers.

As an example, we describe in detail the most interesting and computationally expensive case we exhaustively explored, which is the search for MGA designs with $q = 7$ factors and $t = 3$. The results of this case are listed in the last seven rows of the third block of Table 4. Clearly, the values of the vectors $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ are all zero, the 35 elements of the vector $\mathbf{J}_4$ should be either 8 or -8, while the elements of the vectors $\mathbf{J}_5$ and $\mathbf{J}_6$ as well as J_7 are even multiples of 2^3, as Lemma 2.2 indicates. Let $k \equiv x \bmod 32$, where x is even but not a multiple of 4 and assume that the vectors $\mathbf{J}_5$ and $\mathbf{J}_6$ consist of zeros. In such a case, using Lemma 2.3, it follows that the value of J_7 should be a multiple of 64. From equation (2.1), it follows that $a_i + a_{127-i} = (2e_i + 2k)/32$, where e_i, $i = 0, 1, \ldots, 63$, are 64 specific linear combinations of the 35 values of the vector $\mathbf{J}_4$ divided by 4. Therefore, since $a_i + a_{127-i}$ should be an integer and k is constant, it follows that the value of each e_i should leave the same remainder when divided by 16. An exhaustive search of the 2^{35} possible $\mathbf{J}_4$ vectors is needed to find out if there is an acceptable solution. This exhaustive search on the remainder of each of the values of e_i, $i = 0, 1, \dots, 63$, as produced from every $\mathbf{J}_4$ vector tested, gave no solutions and therefore, the vectors $\mathbf{J}_5$ and $\mathbf{J}_6$ cannot consist only of zeros. The next step, in order to obtain a MGA design, is to consider $\mathbf{J}_6$ vectors with one nonzero value that, according to Lemma 2.2 should be equal to 16 or -16. Without loss of generality and since the properties of an OA remain unaffected when permuting its columns, we may assume that the last entry of $\mathbf{J}_6$ is nonzero while, from Lemma 2.3, J_7 should be a multiple of 64. The exhaustive search of the $2^{35} \cdot 2$ possible $\mathbf{J}$ vectors with $J_7 = 0$ provides a solution for $k \equiv 2$, 6, 26 and 30 mod 32 and, the search of the $2^{35} \cdot 2 \cdot 2$ possible $\mathbf{J}$ vectors with $J_7 = 64$ or $J_7 = -64$ gives a solution for the rest of the cases.

Finally, we note that for each of the cases we study, several optimal $\mathbf{J}$ vectors were produced, a fact that was expected, since all designs that belong to the isomorphism class of the MGA design will produce an acceptable $\mathbf{J}$ vector. However, there may be $\mathbf{J}$ vectors that provide an optimal solution and lead to the construction of a MGA design that is non-isomorphic to the MGA design given in the results. This search was beyond the scope of this article and may be addressed in the future, since an efficient screening of the acceptable optimal solutions is needed.

Acknowledgements

We would like to thank the Editor, the Associate Editor and the referees for raising constructive comments that have led to a much improved version of this paper.

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