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Consistency of the drift parameter estimator for the discretized fractional Ornstein–Uhlenbeck process with Hurst index $H \in (0, \frac{1}{2})$

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Abstract: We consider the Langevin equation which contains an unknown drift parameter θ and where the noise is modeled as fractional Brownian motion with Hurst index $H \in (0, \frac{1}{2})$. The solution corresponds to the fractional Ornstein–Uhlenbeck process. We construct an estimator, based on discrete observations in time, of the unknown drift parameter, that is similar in form to the maximum likelihood estimator for the drift parameter in Langevin equation with standard Brownian motion. It is assumed that the interval between observations is n^{-1}, i.e. tends to zero (high-frequency data) and the number of observations increases to infinity as n^m with $m > 1$. It is proved that for strictly positive θ the estimator is strongly consistent for any $m > 1$, while for $\theta \leq 0$ it is consistent when $m > \frac{1}{2H}$.

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1. Introduction and main results

The choice of an appropriate model is one of the crucial problems appearing in the description of various phenomena in physics, high technology, economics, finance etc. For example, semimartingale models, including the simplest diffusion models based on the Wiener process, provide good service in cases where the data demonstrate the Markovian property and the lack of memory. However, starting with the famous Hurst phenomena that clearly showed inapplicability of the Central Limit Theorem to the sequence of data measurements, the need to involve non-Markov and non-semimartingale processes with memory has become apparent. The fractional Brownian motion is the simplest representative of such processes. As well as in the diffusion model with a standard Wiener process, mean-reverting property is very attractive to model processes with memory, and this naturally explains the appearance of fractional Ornstein-Uhlenbeck process in the modeling, e.g., of stochastic volatility (see [11] for various fractional models in stochastic volatility). In all areas of application the most attention has been paid to the models with the so-called long memory for two reasons: on one hand, the phenomena of the long memory appeared more often, and, on the other hand, it is simpler to describe it analytically. However, recent observations of financial markets (see, e.g., [3]) provide evidence in favor of short and varying memory. A much more detailed review of the related literature is given below, now we only say that, without touching multifractionality, in the present paper we consider the fractional Ornstein-Uhlenbeck process with the short memory.

Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a complete probability space. We consider the fractional Brownian motion $B^H = \{B_t^H,\ t \geq 0\}$ on this probability space, that is, the centered Gaussian process with the covariance function

$$R(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

We restrict ourselves to the case $H \in (0, \frac{1}{2})$ and consider the continuous (and even Hölder up to order H) modification that exists due to the Kolmogorov theorem. Let us introduce the Langevin equation,

$$X_t = x_0 + \theta \int_0^t X_s \, ds + B_t^H, \quad t \geq 0, \quad H \in (0, \tfrac{1}{2}). \tag{1}$$

According to Proposition A.1 from [9], this equation has a unique solution, which is called the fractional Ornstein–Uhlenbeck process and can be presented as

$$X_t = x_0 e^{\theta t} + \theta e^{\theta t} \int_0^t e^{-\theta s} B_s^H \, ds + B_t^H,\ t \geq 0. \tag{2}$$

The goal of the paper is to construct a consistent (strongly consistent) estimator of the unknown drift parameter θ using discrete observations of the process X.

The problem of the estimation of the drift parameter θ in the linear equation containing fBm and in the equation (1) when the Hurst index $H \geq \frac{1}{2}$ was investigated in many works. For linear models, it suffices to mention the papers [2] and [17]. Drift parameter estimators for the fractional Ornstein–Uhlenbeck process with continuous time when the whole trajectory of X is observed were studied in [1, 16, 21]. Kleptsyna and Le Breton [21] constructed the maximum likelihood estimator and proved its strong consistency for any $\theta \in \mathbb{R}$. They also investigated the asymptotic behavior of the bias and the mean square error of this estimator. The sequential maximum likelihood estimation was considered in [28]. Hu and Nualart [16] proved that in the ergodic case ($\theta < 0$) the least square estimator

$$\widehat{\theta}_T = \frac{\int_0^T X_t \, dX_t}{\int_0^T X_t^2 \, dt}, \tag{3}$$

is strongly consistent for all $H \geq \frac{1}{2}$ and asymptotically normal for $H \in [\frac{1}{2}, \frac{3}{4})$. Here $\int_0^T X_t \, dX_t$ is a divergence-type integral. They also obtained the strong consistency and asymptotic normality of the estimator

$$\widehat{\theta}_T = \left(\frac{1}{H\Gamma(H)T} \int_0^T X_t^2 \, dt \right)^{-\frac{1}{2H}}. \tag{4}$$

In [1] the corresponding non-ergodic case $\theta > 0$ was considered and the strong consistency of the least square estimator (3) was proved for $H > \frac{1}{2}$. It was also obtained that $e^{\theta t}(\widehat{\theta}_t - \theta)$ converges in law to $2\theta\mathcal{C}(1)$ as $t \to \infty$, where $\mathcal{C}(1)$ is the standard Cauchy distribution. The minimum contrast estimators in the continuous and discrete cases were studied in [4]. The distributional properties of the maximum likelihood, minimum contrast and the least square estimators were explored in [30]. For the two-parameter generalization see [10].

In [8, 13, 14] the discretized version of (3) was considered, namely

$$\widetilde{\theta}_n = \frac{\sum_{i=1}^n X_{t_{i-1}} \left(X_{t_i} - X_{t_{i-1}} \right)}{\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}, \tag{5}$$

where the process X was observed at the points $t_i = i\Delta_n$, $i = 0, \ldots, n$, such that $\Delta_n \to 0$ and $n\Delta_n \to \infty$ as $n \to \infty$. In [8], the ergodic case $\theta < 0$ was studied, the strong consistency of this estimator was proved for $H \geq \frac{1}{2}$, and the almost sure central limit theorem was obtained for $H \in (\frac{1}{2}, \frac{3}{4})$. The non-ergodic case $\theta > 0$ was considered in Es-Sebaiy and Ndiaye [14]. They proved the strong consistency of the estimator (5) for $H \in (\frac{1}{2}, 1)$ assuming that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ as $n \to \infty$ for some $\alpha > 0$. The same result was obtained for the estimator

$$\widehat{\theta}_n = \frac{X_{t_n}^2}{2\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}.$$

In [18, 32] the following discretized version of the estimator (4) was considered

$$\widehat{\theta}_n = -\left(\frac{1}{nH\Gamma(2H)}\sum_{k=1}^{n} X_{k\Delta}^2\right)^{-\frac{1}{2H}},$$

where $\theta < 0$ and the process X was observed at the points $\Delta, 2\Delta, \ldots, n\Delta$ for some fixed $\Delta > 0$. Hu and Song [18] proved the strong consistency of this estimator for $H \geq \frac{1}{2}$ and the asymptotic normality for $\frac{1}{2} \leq H < \frac{3}{4}$.

In [6, 33], a more general situation was studied, where the equation has the form $dX_t = \theta X_t dt + \sigma dB_t^H$, $t > 0$, and $\vartheta = (\theta, \sigma, H)$ is the unknown parameter, $\theta < 0$. Consistent and asymptotically Gaussian estimators of the parameter θ were proposed using the discrete observations of the sample path $(X_{k\Delta_n}, k = 0, \ldots, n)$ for $H \in (\frac{1}{2}, \frac{3}{4})$, where $n\Delta_n^p \to \infty$, $p > 1$, and $\Delta_n \to 0$ as $n \to \infty$. In [33] the strongly consistent estimator is constructed for the scheme when $H > \frac{1}{2}$, the time interval $[0, T]$ is fixed and the process is observed at the points $h_n, 2h_n, \ldots, nh_n$, where $h_n = \frac{T}{n}$.

In [12, 23], the so-called sub-fractional Ornstein–Uhlenbeck process was studied, where the process B_t^H in (1) was replaced with a sub-fractional Brownian motion. In [12], the maximum likelihood estimator for such a process was constructed and in [23] the estimator (3) was investigated in the case $\theta > 0$. The maximum likelihood drift parameter estimators for the fractional Ornstein–Uhlenbeck process and even more general processes involving fBm with Hurst index from the whole interval $(0, 1)$ were constructed and studied in [31]. These estimators involve singular kernels and therefore are more complicated to study and simulate. To the best of our knowledge, it is the only paper where the discretized estimates of the drift parameter are constructed in the case $H < \frac{1}{2}$. However, the observations of the real financial markets demonstrate that the Hurst index often falls below the level of $\frac{1}{2}$, taking values around 0.45–0.49 ([3]). In order to consider the case of $H < \frac{1}{2}$ and to overcome the technical difficulties connected with singular kernels, we construct a comparatively simple estimator that is similar in form to the maximum likelihood estimator for the Langevin equation with the standard Brownian motion. The observations are assumed to be discrete in time and we assume that the interval between the observations is n^{-1}, i.e. tends to zero, so we consider high-frequency data. At the same time, the number of observations increases to infinity with the speed n^m with $m > 1$. Let $n \geq 1$, $t_{k,n} = \frac{k}{n}$, $0 \leq k \leq n^m$, where $m \in \mathbb{N}$ is some fixed integer. Suppose that we observe X at the points $\{t_{k,n}, n \geq 1, 0 \leq k \leq n^m\}$. Consider the estimator

$$\widehat{\theta}_n(m) = \frac{\sum_{k=0}^{n^m-1} X_{k,n}\Delta X_{k,n}}{\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2}, \tag{6}$$

where $X_{k,n} = X_{t_{k,n}}$, $\Delta X_{k,n} = X_{k+1,n} - X_{k,n}$.

Remark 1. The estimator $\widehat{\theta}_n(m)$ is well-defined for all $0 < H < 1$. However, the case $H \geq 1/2$ has already been studied in many works. In particular, for

$H > 1/2$ the strong consistency of the estimator (6) for $m > 1$ follows directly from the corresponding results for the estimator $\widetilde{\theta}_n$ defined in (5). Indeed, let us consider $\widetilde{\theta}_n$ with $\Delta_n = n^{-1/m}$, $m > 1$. In this case, the conditions $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ as $n \to \infty$ are satisfied, and $\widetilde{\theta}_n \to \theta$ a. s. as $n \to \infty$ by [8] for the ergodic case, and by [14] for the non-ergodic one. Then the estimator $\widehat{\theta}_n(m)$ is also strongly consistent, since $\widehat{\theta}_n(m) = \widetilde{\theta}_{n^m}$. It is worth to mention that in the case $H = 1/2$ the estimators of this type have been known since the mid-seventies ([5]), for the corresponding strong consistency results see, e. g., [19, 20, 24, 27, 29] and the references cited therein. Therefore, in this work we concentrate on the case $H < 1/2$.

Remark 2. Considering the asymptotic behavior of the estimator $\widehat{\theta}_n(m)$, we need in particular to study the asymptotics of its denominator which is a non-linear functional of the integral type (sum, as the discrete analog of the integral) of the fractional Brownian motion. The results in this direction, including the non-ergodic case, were obtained in the paper [7] using Hermite ranks of the functionals. Since our goal is simply to compare the numerator to the denominator, we use another approach, bounding the denominator from below.

According to (1), the estimator $\widehat{\theta}_n(m)$ from (6) can be represented in the following form, which is more convenient for evaluation:

$$\widehat{\theta}_n(m) = \theta + \frac{\theta \sum_{k=0}^{n^m-1} X_{k,n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (X_s - X_{k,n})\, ds + \sum_{k=0}^{n^m-1} X_{k,n} \Delta B_{k,n}^H}{\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2}. \tag{7}$$

It is proved that for strictly positive θ the estimator is strongly consistent for any $m > 1$ and for $\theta \le 0$ it is consistent for $m > \frac{1}{2H}$.

Theorem 1.1. *Let $\theta > 0$. Then for any $m > 1$ the estimator $\widehat{\theta}_n(m)$ is strongly consistent.*

Theorem 1.2. *Let $\theta \le 0$. Then for any $m > \frac{1}{2H}$ the estimator $\widehat{\theta}_n(m)$ is consistent.*

Our paper is organized as follows. Section 2 is devoted to numerics. In Section 3 we consider an auxiliary result, namely, the bounds with probability 1 for the values and increments of the fractional Brownian motion and the fractional Ornstein–Uhlenbeck process. The bounds are factorized to the increasing non-random function and a random variable not depending on time. In Section 4 we get the bounds for the numerator of the estimator, while in Section 5 we relate the discretized integral sum in the denominator of the estimator to the corresponding integral $\int_0^t X_s^2\, ds$. This relation is convenient for some values of the parameters because it is easier to apply the L'Hôpital's rule to the integral $\int_0^t X_s^2\, ds$ than the Stolz–Cesàro theorem to the sum $\sum_{k=0}^{n^m-1} X_{k,n}^2$ with its terms depending on n. In this section, we also provide the convergence of the ratios that appear in the proof of the main result for the case $\theta < 0$, which is obviously more technical. Section 6 contains the proofs of Theorem 1.1 and Theorem 1.2. Section 7 contains some auxiliary statements.

2. Simulations

In this section, we present the results of simulation experiments. We simulate 20 trajectories of the fractional Ornstein–Uhlenbeck process (1) with $x_0 = 1$ for different values of θ and H. Then we compute the values of $\widehat{\theta}_n(m)$. For each combination of θ, H, n and m the mean of the estimator is reported.

In Tables 1–3 the true value of the drift parameter θ equals 2. In this case the behavior of the estimators is almost the same for different values of H. Also, we can see that the value of $\widehat{\theta}_n(m)$ is determined by n and does not depend on m. Further, we consider the case of negative θ. We simulate the process with $H = 0.45$, $\theta = -3$ and $m = 4, 5$. The results are reported in Tables 4–5. One can see that the method works but the rate of convergence to the true value of the parameter is not very high. There are two reasons for this: the estimator is only consistent but not strongly consistent, and moreover, the trajectories are so irregular that even though the length of the interval is small we can not "catch" the trajectory. Simulation results for the process with zero drift are reported in Tables 6–7.

Table 1
$\theta = 2,\ m = 2$

n	5	10	50	100	500	1000
$H = 0.05$	2.45763	2.21281	2.0395	2.01911	2.00300	2.00100
$H = 0.25$	2.45766	2.21281	2.0395	2.01911	2.00300	2.00100
$H = 0.45$	2.45794	2.21281	2.0395	2.01911	2.00300	2.00100

Table 2
$\theta = 2,\ m = 3$

n	5	10	20	25
$H = 0.05$	2.45763	2.21281	2.10231	2.08109
$H = 0.25$	2.45763	2.21281	2.10231	2.08109
$H = 0.45$	2.45763	2.21281	2.10231	2.08109

Table 3
$\theta = 2,\ m = 4$

n	5	8	10	12	15
$H = 0.05$	2.45763	2.27092	2.21281	2.17240	2.13566
$H = 0.25$	2.45763	2.27092	2.21281	2.17240	2.13566
$H = 0.45$	2.45763	2.27092	2.21281	2.17240	2.13566

Table 4
$\theta = -3,\ H = 0.45,\ m = 4$

n	2	4	6	8	10
$\widehat{\theta}_n(4)$	-1.50913	-2.41157	-2.71411	-2.9546	-3.12058

Table 5
$\theta = -3,\ H = 0.45,\ m = 5$

n	2	3	4	5	6
$\widehat{\theta}_n(5)$	-1.63396	-2.04297	-2.38237	-2.5595	-2.72538

TABLE 6
$\theta = 0,\ H = 0.45,\ m = 3$

n	5	10	20	25
$\widehat{\theta}_n(3)$	-0.10060	-0.04206	-0.01149	-0.01008

TABLE 7
$\theta = 0,\ H = 0.45,\ m = 4$

n	4	5	8	10
$\widehat{\theta}_n(4)$	-0.04281	-0.03331	-0.00962	-0.00727

3. Bounds for the values and the increments of the fractional Brownian motion and the fractional Ornstein–Uhlenbeck process

In what follows we shall use auxiliary estimates for the rate of the asymptotic growth with probability 1 of the fractional Brownian motion and its increments. Throughout the paper while considering functions of the form $t^p \log t,\ p > 0$ we suppose that $0 \cdot \infty = 0$.

Proposition 3.1. *(i) For any $p > 1$ and any $H \in (0,1)$ there exists a nonnegative random variable $\xi(p,H)$ such that for all $t \geq 0$,*

$$\sup_{0 \leq s \leq t} \left|B_s^H\right| \leq \left(\left(t^H |\log t|^p\right) \vee 1\right) \xi(p,H), \tag{8}$$

and there exists such a number $c_\xi(p,H) > 0$ that for any $0 < y < c_\xi(p,H)$, $\mathbf{E}\exp\{y\xi^2(p,H)\} < \infty$.

(ii) For any $q > \frac{1}{2}$ and any $H \in (0,1)$ there exists a nonnegative random variable $\eta(q,H)$ such that for any $0 < t_1 < t_2 < \infty$

$$\left|B_{t_2}^H - B_{t_1}^H\right| \leq (t_2 - t_1)^H \left(\left|\log(t_2 - t_1)\right|^{1/2} + 1\right) (\log(t_2 + 2))^q \eta(q,H), \tag{9}$$

and there exists such a number $c_\eta(q,H) > 0$ that for any $0 < y < c_\eta(q,H)$, $\mathbf{E}\exp\{y\eta^2(q,H)\} < \infty$.

Proof. The 1st statement was proved in the paper [22]. The 2nd statement follows immediately from the next relation that can be proved similarly to Theorem 1 from [25], where an even more complicated functional than the increment of fractional Brownian motion, more precisely, the fractional derivative, was considered. Thus, we have from Theorem 1, [25], that for any $q > \frac{1}{2}$ and any $H \in (0,1)$ the random variable

$$\eta(q,H) = \sup_{0 \leq t_1 < t_2 \leq t_1 + 1} \frac{\left|B_{t_1}^H - B_{t_2}^H\right|}{(t_2 - t_1)^H \left(\left|\log(t_2 - t_1)\right|^{1/2} + 1\right) (\log(t_2 + 2))^q}$$

is finite almost surely, whence (ii) follows. □

Now our goal is to estimate the numerator in (7) and to compare it to the denominator. At first, we describe the bounds for the values of X and its increments.

Lemma 3.2. *We have the following bounds for the fractional Ornstein-Uhlenbeck process X in terms of the underlying fractional Brownian motion:*

(i) Let $\theta > 0$. Then for any $t > 0$

$$\sup_{0\le s\le t} |X_s| \le |x_0|\, e^{\theta t} + \theta e^{\theta t} \int_0^t e^{-\theta s} \sup_{0\le u\le s} \left|B_u^H\right| ds + \sup_{0\le s\le t} \left|B_s^H\right| \tag{10}$$

and for any $s \in [\frac{k}{n}, \frac{k+1}{n})$

$$\begin{aligned}\sup_{\frac{k}{n}\le u\le s} |X_u - X_{k,n}| \le \int_{\frac{k}{n}}^{s} \Bigg(e^{\theta u} \left(|x_0| + \theta \int_0^u e^{-\theta v} \sup_{0\le z\le v} \left|B_z^H\right| dv \right) \\ + \sup_{0\le z\le u} \left|B_z^H\right| \Bigg) du + \sup_{\frac{k}{n}\le u\le s} \left|B_u^H - B_{k,n}^H\right|.\end{aligned} \tag{11}$$

(ii) Let $\theta < 0$. Then for any $t > 0$

$$\sup_{0\le s\le t} |X_s| \le |x_0| + 2 \sup_{0\le s\le t} \left|B_s^H\right| \tag{12}$$

and for any $s \in [\frac{k}{n}, \frac{k+1}{n})$

$$\begin{aligned}\sup_{\frac{k}{n}\le u\le s} |X_u - X_{k,n}| \le \frac{|\theta|\,|x_0|}{n} + \frac{2\,|\theta|}{n} \sup_{0\le u\le s} \left|B_u^H\right| \\ + \sup_{\frac{k}{n}\le u\le s} \left|B_u^H - B_{k,n}^H\right|.\end{aligned} \tag{13}$$

Proof. (i) The bound (10) follows immediately from (2), and the bound (11) follows immediately from (10) and (1).

(ii) The bound (12) follows from (2):

$$|X_t| \le |x_0|\, e^{\theta t} + |\theta|\, e^{\theta t} \sup_{0\le s\le t} \left|B_s^H\right| \cdot \int_0^t e^{-\theta s} ds + \left|B_t^H\right| \le |x_0| + 2 \sup_{0\le s\le t} \left|B_s^H\right|.$$

To establish the bound (13), we substitute (12) into the following inequality that can be easily obtained from (1): for $s \ge \frac{k}{n}$

$$|X_s - X_{k,n}| \le |\theta| \int_{\frac{k}{n}}^{s} |X_u|\, du + \left|B_s^H - B_{k,n}^H\right|. \qquad \square$$

Remark 3. Plugging $p = 2$ and $q = 1$ into the formulae (8)–(9), we get the following bounds:

$$\sup_{0\le s\le t} \left|B_s^H\right| \le \left(t^H \log^2 t + 1\right) \xi(2, H), \tag{14}$$

and for $s \in [\frac{k}{n}, \frac{k+1}{n}]$

$$\begin{aligned}\left|B_s^H - B_{\frac{k}{n}}^H\right| &\le \left(s - \tfrac{k}{n}\right)^H \left(\left|\log\left(s - \tfrac{k}{n}\right)\right|^{1/2} + 1 \right) \log(s+2) \eta(1, H) \\ &\le \left(\left(s - \tfrac{k}{n}\right)^H \left|\log\left(s - \tfrac{k}{n}\right)\right|^{1/2} + \left(s - \tfrac{k}{n}\right)^H \right) \log(s+2) \eta(1, H).\end{aligned} \tag{15}$$

The function $f(x) = x^r \left|\log x\right|^{\frac{1}{2}}$ is bounded on the interval $(0,1]$ for any $r > 0$. Therefore

$$\left(s - \tfrac{k}{n}\right)^H \left|\log\left(s - \tfrac{k}{n}\right)\right|^{1/2} \le C \left(s - \tfrac{k}{n}\right)^{H-r}$$

for any $0 < r < H$. Furthermore, for $s \in [\frac{k}{n}, \frac{k+1}{n}]$ we have that $(s - \frac{k}{n})^H \le (s - \frac{k}{n})^{H-r}$. Therefore, we get from (15) that for any $0 < r < H$ and for $s \in [\frac{k}{n}, \frac{k+1}{n}]$

$$\left|B_s^H - B_{\frac{k}{n}}^H\right| \le C \left(s - \tfrac{k}{n}\right)^{H-r} \log(n^{m-1} + 2)\eta(1, H). \tag{16}$$

It follows immediately from (14) that for $\theta > 0$

$$\int_0^t e^{-\theta s} \sup_{0 \le u \le s} \left|B_u^H\right| ds \le \xi(2, H) \int_0^t e^{-\theta s} \left(s^H \log^2 s + 1\right) ds \le C\xi(2, H),$$

and therefore both integrals $\int_0^\infty e^{-\theta s} B_s^H ds$ and $\int_0^\infty e^{-\theta s} \sup_{0 \le u \le s} \left|B_u^H\right| ds$ exist with probability 1 and admit the same upper bound $C\xi(2, H)$. Combining (10)–(13), (14) and (16), we get that for $\theta > 0$

$$\sup_{0 \le u \le s} |X_u| \le |x_0| e^{\theta s} + C\theta e^{\theta s} \xi(2, H) + \left(s^H \log^2 s + 1\right) \xi(2, H),$$

and for $s \in [\frac{k}{n}, \frac{k+1}{n}]$

$$\begin{aligned}\sup_{\frac{k}{n} \le u \le s} |X_u - X_{k,n}| \le \theta \int_{\frac{k}{n}}^s &\left(e^{\theta u} \left(|x_0| + C\theta\xi(2, H)\right)\right. \\ &\left.+ \left(u^H \log^2 u + 1\right)\xi(2, H)\right) du + \left(n^{-H+r} \log n\right) \eta(1, H),\end{aligned}$$

while for $\theta < 0$

$$\sup_{0 \le u \le s} |X_u| \le |x_0| + 2 \left(s^H \log^2 s + 1\right) \xi(2, H),$$

and for $s \in [\frac{k}{n}, \frac{k+1}{n}]$

$$\begin{aligned}\sup_{\frac{k}{n} \le u \le s} |X_u - X_{k,n}| &\le \frac{|\theta|\,|x_0|}{n} + \frac{2\,|\theta|}{n} \sup_{0 \le u \le s} \left|B_u^H\right| \\ + \sup_{\frac{k}{n} \le u \le s} \left|B_u^H - B_{k,n}^H\right| &\le \frac{|\theta|\,|x_0|}{n} + \frac{2\,|\theta|}{n} \left(s^H \log^2 s + 1\right) \xi(2, H) \\ &\quad + \left(n^{-H+r} \log n\right) \eta(1, H).\end{aligned}$$

To simplify the notations, we denote by C any constant whose value is not important for our bounds. Furthermore, we denote by $\mathfrak{Z}$ the class of nonnegative random variables with the following property: there exists $C > 0$ not depending on n such that $\mathbf{E} \exp\{x\zeta^2\} < \infty$ for any $0 < x < C$. For example, $\xi(2, H) + C$ and $\eta(1, H) + C$, $C\xi(2, H)$ and $C\eta(1, H)$ for any constant C belong to $\mathfrak{Z}$. Also, note that for fixed $m > 1$ and $n > 3$ we have the upper bound $\log(n^{m-1}+3) \le C \log n$. Moreover, for any $\alpha > 0$ there exists such $n(\alpha)$ that for $n \ge n(\alpha)$ we have $\log n < n^{\alpha}$. Taking this into account and using the simplified notations, we get the bounds with the same $\zeta \in \mathfrak{Z}$: for $\theta > 0$ we have for any fixed $\alpha > 0$, starting with $n \ge n(\alpha)$:

$$\sup_{0 \le u \le s} |X_u| \le \left(e^{\theta s} + s^H \log^2 s\right) \zeta, \tag{17}$$

and for $s \in [\frac{k}{n}, \frac{k+1}{n}]$

$$\sup_{\frac{k}{n} \le u \le s} |X_u - X_{k,n}| \le \left(\tfrac{1}{n} e^{\theta s} + \tfrac{1}{n} s^H \log^2 s + n^{-H+\alpha}\right) \zeta, \tag{18}$$

while for $\theta < 0$

$$\sup_{0 \le u \le s} |X_u| \le \left(1 + s^H \log^2 s\right) \zeta, \tag{19}$$

and for $s \in [\frac{k}{n}, \frac{k+1}{n}]$

$$\sup_{\frac{k}{n} \le u \le s} |X_u - X_{k,n}| \le \left(\tfrac{1}{n} + \tfrac{1}{n} s^H \log^2 s + n^{-H+\alpha}\right) \zeta. \tag{20}$$

4. The bounds for the numerator of the estimator

Now we are in the position to bound both terms in the numerator of the right-hand side of (7). First, we present the bound with probability 1 for the 1st term in the numerator of (7). All inequalities claimed in Lemma 4.1 hold for any $\alpha > 0$ starting with some non-random number $n(\alpha)$.

Lemma 4.1. *(i) Let $\theta > 0$. Then for any $m > 1$ there exists such $\zeta \in \mathfrak{Z}$ that*

$$\left| \sum_{k=0}^{n^m-1} X_{k,n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (X_s - X_{k,n})\, ds \right| \le \zeta^2 n^{-1} e^{2\theta n^{m-1}}.$$

(ii) Let $\theta < 0$. Then we have two cases.

(a) Let $1 < m \le \frac{1}{H}$. Then there exists such $\zeta \in \mathfrak{Z}$ that

$$\left| \sum_{k=0}^{n^m-1} X_{k,n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (X_s - X_{k,n})\, ds \right| \le \zeta^2 n^{mH+m-2H-1+\alpha}.$$

(b) Let $m > \frac{1}{H}$. Then there exists such $\zeta \in \mathfrak{Z}$ that

$$\left| \sum_{k=0}^{n^m-1} X_{k,n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (X_s - X_{k,n})\, ds \right| \le \zeta^2 n^{2Hm+m-2H-2+\alpha}.$$

Proof. (i) It follows immediately from (17) that

$$|X_{k,n}| \le \sup_{0\le u\le \frac{k+1}{n}} |X_u| \le \left(e^{\theta\frac{k+1}{n}} + \left(\tfrac{k+1}{n}\right)^H \log^2\left(\tfrac{k+1}{n}\right)\right)\zeta. \tag{21}$$

Now we take into account (21), substitute $\frac{k+1}{n}$ instead of s into (18) and apply Lemma 7.1 to get the following relations for any $\alpha > 0$:

$$\begin{aligned}
\left|\sum_{k=0}^{n^m-1} X_{k,n}\int_{\frac{k}{n}}^{\frac{k+1}{n}} (X_s - X_{k,n})\,ds\right| &\le \zeta^2 \frac{1}{n}\sum_{k=0}^{n^m-1}\left(e^{\theta\frac{k+1}{n}} + \left(\tfrac{k+1}{n}\right)^H \log^2\left(\tfrac{k+1}{n}\right)\right)\\
&\quad\times\left(\frac{1}{n}e^{\theta\frac{k+1}{n}} + \frac{1}{n}\left(\tfrac{k+1}{n}\right)^H\log^2\left(\tfrac{k+1}{n}\right) + n^{-H+\alpha}\log n\right)\\
&= \zeta^2\left(\frac{1}{n^2}\sum_{k=0}^{n^m-1} e^{2\theta\frac{k+1}{n}} + \frac{2}{n^2}\sum_{k=0}^{n^m-1} e^{\theta\frac{k+1}{n}}\left(\tfrac{k+1}{n}\right)^H\log^2\left(\tfrac{k+1}{n}\right)\right.\\
&\quad + \frac{1}{n^2}\sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n}\right)^{2H}\log^4\left(\tfrac{k+1}{n}\right) + n^{-1-H+\alpha}\log n\\
&\quad\left.\times\left(\sum_{k=0}^{n^m-1} e^{\theta\frac{k+1}{n}} + \sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n}\right)^H\log^2\tfrac{k+1}{n}\right)\right)\\
&\le \zeta^2\left(\frac{1}{n}e^{2\theta n^{m-1}} + e^{\theta n^{m-1}}\left(n^{H(m-1)+m-2+\alpha} + n^{-H+\alpha}\right)\right.\\
&\quad\left. + n^{m-1-H+(m-1)H+\alpha}\right).
\end{aligned}\tag{22}$$

Evidently, the term $\frac{1}{n}e^{2\theta n^{m-1}}$ dominates and the other terms are negligible, whence the proof of (i) follows.

(ii) According to (19),

$$|X_{k,n}| \le \sup_{0\le u\le\frac{k+1}{n}}|X_u| \le \left(1 + \left(\tfrac{k+1}{n}\right)^H\log^2\tfrac{k+1}{n}\right)\zeta.$$

Substituting $\frac{k+1}{n}$ instead of s into (20), we get the following relations:

$$\begin{aligned}
\left|\sum_{k=0}^{n^m-1} X_{k,n}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(X_s - X_{k,n})\,ds\right| &\le \zeta^2\frac{1}{n}\sum_{k=0}^{n^m-1}\left(1 + \left(\tfrac{k+1}{n}\right)^H\log^2\tfrac{k+1}{n}\right)\\
&\quad\times\left(\tfrac{1}{n} + \tfrac{1}{n}\left(\tfrac{k+1}{n}\right)^H\log^2\left(\tfrac{k+1}{n}\right) + n^{-H+\alpha}\right)\\
= \zeta^2\left(n^{m-2} + \frac{2}{n^2}\sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n}\right)^H\log^2\left(\tfrac{k+1}{n}\right) + \frac{1}{n^2}\sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n}\right)^{2H}\log^4\tfrac{k+1}{n}\right.&\\
\left. + n^{m-1-H+\alpha} + n^{-1-H+\alpha}\sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n}\right)^H\log^2\tfrac{k+1}{n}\right).&
\end{aligned}\tag{23}$$

Substituting the bounds from Lemma 7.1 into the right-hand side of (23), we obtain

$$\begin{aligned}\left|\sum_{k=0}^{n^m-1} X_{k,n}\int_{\frac{k}{n}}^{\frac{k+1}{n}} (X_s - X_{k,n})\,ds\right| &\le \zeta^2\Big(n^{m-2} + n^{(m-1)H-2+m}\log^2 n \\ &+ n^{2H(m-1)-2+m}\log^4 n + n^{m-1-H+\alpha} + n^{(m-2)H-1+m+\alpha}\log^2 n\Big).\end{aligned}$$

We take into account that $\log n = o(n^\alpha)$ as $n\to\infty$, for any $\alpha > 0$. So, it is necessary to compare the exponents $m-2$, $(m-1)H-2+m$, $2H(m-1)-2+m$, $m-1-H$ and $(m-2)H-1+m$. We get that the exponent $2H(m-1)-2+m$ is the largest under the condition $m > \frac{1}{H}$ while the exponent $(m-2)H-1+m$ is the largest under the condition $m \le \frac{1}{H}$ whence the proof of (ii) follows. $\square$

Now we establish the moment bounds for the 2nd term in the numerator of the right-hand side of (7). In order to do this, we apply the well-known Isserlis' formula to calculate the higher moments of the Gaussian distribution: let $\{\chi_1,\chi_2,\chi_3,\chi_4\}$ be a Gaussian vector, then

$$\mathbf{E}(\chi_1\chi_2\chi_3\chi_4) = \mathbf{E}(\chi_1\chi_2)\mathbf{E}(\chi_3\chi_4) + \mathbf{E}(\chi_1\chi_3)\mathbf{E}(\chi_2\chi_4) + \mathbf{E}(\chi_1\chi_4)\mathbf{E}(\chi_2\chi_3).$$

Therefore, we can calculate the mathematical expectations $\mathbf{E}B_s^H B_u^H \Delta B_k^H \Delta B_j^H$ for $k \ne j$ as

$$\begin{aligned}\mathbf{E}B_s^H B_u^H \Delta B_k^H \Delta B_j^H &= \mathbf{E}B_u^H \Delta B_k^H \mathbf{E}B_s^H \Delta B_j^H + \mathbf{E}B_s^H \Delta B_k^H \mathbf{E}B_u^H \Delta B_j^H \\ &\quad + \mathbf{E}B_s^H B_u^H \mathbf{E}\Delta B_k^H \Delta B_j^H \le \mathbf{E}B_u^H \Delta B_k^H \mathbf{E}B_s^H \Delta B_j^H \\ &\quad + \mathbf{E}B_s^H \Delta B_k^H \mathbf{E}B_u^H \Delta B_j^H \qquad (24)\end{aligned}$$

because for $H \in (0, \frac{1}{2})$ the increments of fBm B^H are negatively correlated and so $\mathbf{E}\Delta B_k^H \Delta B_j^H < 0$. Similarly,

$$\mathbf{E}B_s^H B_u^H \left(\Delta B_{k,n}^H\right)^2 = 2\mathbf{E}B_u^H \Delta B_{k,n}^H \mathbf{E}B_s^H \Delta B_{k,n}^H + n^{-2H}\mathbf{E}B_s^H B_u^H. \qquad (25)$$

Lemma 4.2. (i) *Let $\theta > 0$. Then for any $m > 1$ we have the following moment bound*

$$\mathbf{E}\left(\sum_{k=0}^{n^m-1} X_{k,n}\Delta B_{k,n}^H\right)^2 \le Cn^{2-4H}e^{2\theta n^{m-1}}.$$

(ii) *Let $\theta < 0$. Then for any $m > 1$ we have the following moment bound*

$$\mathbf{E}\left(\sum_{k=0}^{n^m-1} X_{k,n}\Delta B_{k,n}^H\right)^2 \le Cn^{2m-4H}.$$

Proof. (i) It follows from (2) that

$$\sum_{k=0}^{n^m-1} X_{k,n}\Delta B^H_{k,n} = x_0 \sum_{k=0}^{n^m-1} e^{\theta\frac{k}{n}}\Delta B^H_{k,n}$$
$$+\,\theta \sum_{k=0}^{n^m-1} e^{\theta\frac{k}{n}} \int_0^{\frac{k}{n}} e^{-\theta s} B^H_s\, ds \cdot \Delta B^H_{k,n} + \sum_{k=0}^{n^m-1} B^H_{k,n}\Delta B^H_{k,n} =: I^1_n + I^2_n + I^3_n.$$

Note that $\mathbf{E}\Delta B^H_{k,n}\Delta B^H_{j,n} < 0$ for $k \neq j$. Therefore

$$0 \le \mathbf{E}\left(\sum_{k=0}^{n^m-1} e^{\theta\frac{k}{n}}\Delta B^H_{k,n}\right)^2$$
$$\le n^{-2H} \sum_{k=0}^{n^m-1} e^{2\theta\frac{k}{n}} \le n^{1-2H} \int_0^{n^{m-1}} e^{2\theta s} ds = Cn^{1-2H} e^{2\theta n^{m-1}}.$$

So, $\mathbf{E}(I^1_n)^2 \le Cn^{1-2H}e^{2\theta n^{m-1}}$, $n \ge 1$.

Consider I^3_n. It is well known (see, e. g., the relation (1.8) from [26], but it can be easily deduced from the ergodic properties of the quadratic variation of the fractional Brownian motion) that for $H \in (0, \frac{1}{2})$

$$n^{2H-1} \sum_{k=0}^{n-1} B^H_{k,n}\Delta B^H_{k,n} \xrightarrow{L^2(\mathbf{P})} c_H,$$

where c_H is some constant. Therefore, there exists $C > 0$ such that

$$\mathbf{E}\left(n^{2H-1} \sum_{k=0}^{n-1} B^H_{k,n}\Delta B^H_{k,n}\right)^2 \le C, \quad n \ge 1.$$

Now we can use the self-similarity property of B^H, namely,

$$\left(B^H_{at}, t \ge 0\right) \stackrel{d}{=} a^H \left(B^H_t, t \ge 0\right)$$

and get

$$\begin{aligned}\mathbf{E}\left(I^3_n\right)^2 &= \mathbf{E}\left(\sum_{k=0}^{n^m-1} B^H_{k,n}\Delta B^H_{k,n}\right)^2 \\ &= n^{4(m-1)H}\mathbf{E}\left(\sum_{k=0}^{n^m-1} B^H_{\frac{k}{n^m}}\left(B^H_{\frac{k+1}{n^m}} - B^H_{\frac{k}{n^m}}\right)\right)^2 \\ &\le n^{4(m-1)H}\cdot n^{2m(1-2H)}\mathbf{E}\left(n^{m(2H-1)} \sum_{k=0}^{n^m-1} B^H_{\frac{k}{n^m}}\left(B^H_{\frac{k+1}{n^m}} - B^H_{\frac{k}{n^m}}\right)\right)^2 \\ &\le Cn^{2m-4H}.\end{aligned} \tag{26}$$

At last,

$$0 \le \mathbf{E}\left(I_n^2\right)^2 = \theta^2 \sum_{j,k=0}^{n^m-1} e^{\theta\frac{k}{n}+\theta\frac{j}{n}} \int_0^{\frac{k}{n}}\int_0^{\frac{j}{n}} e^{-\theta s-\theta u}\,\mathbf{E}B_s^H B_u^H \Delta B_{k,n}^H \Delta B_{j,n}^H\,du\,ds$$
$$= \theta^2 \sum_{k=0}^{n^m-1} e^{2\theta\frac{k}{n}} \int_0^{\frac{k}{n}}\int_0^{\frac{k}{n}} e^{-\theta s-\theta u}\,\mathbf{E}B_s^H B_u^H \left(\Delta B_{k,n}^H\right)^2 du\,ds$$
$$+\theta^2 \sum_{k\neq j} e^{\theta\frac{k}{n}+\theta\frac{j}{n}} \int_0^{\frac{k}{n}}\int_0^{\frac{j}{n}} e^{-\theta s-\theta u}\,\mathbf{E}B_s^H B_u^H \Delta B_{k,n}^H \Delta B_{j,n}^H\,du\,ds =: J_1^n + J_2^n. \tag{27}$$

We get from (24) and (25) that

$$\begin{aligned}\mathbf{E}B_s^H B_u^H \left(\Delta B_{k,n}^H\right)^2 &= \frac{1}{2}\left(\left(\tfrac{k+1}{n}\right)^{2H} - \left|u-\tfrac{k+1}{n}\right|^{2H} - \left(\tfrac{k}{n}\right)^{2H} + \left|u-\tfrac{k}{n}\right|^{2H}\right)\\ &\times\left(\left(\tfrac{k+1}{n}\right)^{2H} - \left|s-\tfrac{k+1}{n}\right|^{2H} - \left(\tfrac{k}{n}\right)^{2H} + \left|s-\tfrac{k}{n}\right|^{2H}\right) + n^{-2H}\mathbf{E}B_s^H B_u^H\\ &\le Cn^{-4H} + Cn^{-2H}\left(s^{2H}+u^{2H}-|s-u|^{2H}\right),\end{aligned} \tag{28}$$

and

$$\begin{aligned}\mathbf{E}B_s^H B_u^H \Delta B_k^H \Delta B_j^H &\le \frac{1}{2}\left(\left(\tfrac{k+1}{n}\right)^{2H} - \left(\tfrac{k}{n}\right)^{2H} - \left|u-\tfrac{k+1}{n}\right|^{2H} + \left|u-\tfrac{k}{n}\right|^{2H}\right)\\ &\times\left(\left(\tfrac{k+1}{n}\right)^{2H} - \left(\tfrac{k}{n}\right)^{2H} - \left|s-\tfrac{k+1}{n}\right|^{2H} + \left|s-\tfrac{k}{n}\right|^{2H}\right) \le Cn^{-4H}.\end{aligned} \tag{29}$$

Substituting the above bounds into (27), we get that

$$J_1^n \le Cn^{1-2H}e^{2\theta n^{m-1}} \quad \text{and} \quad J_2^n \le Cn^{2-4H}e^{2\theta n^{m-1}}.$$

Evidently, for $m > 1$ the largest contribution is from the bound $Cn^{2-4H}e^{2\theta n^{m-1}}$ whence the proof follows.

(ii) Let $\theta < 0$. In this case

$$0 \le \mathbf{E}\left(\sum_{k=0}^{n^m-1} e^{\theta\frac{k}{n}}\Delta B_{k,n}^H\right)^2 \le \sum_{k=0}^{n^m-1} e^{2\theta\frac{k}{n}}\mathbf{E}\left(\Delta B_{k,n}^H\right)^2$$
$$\le \sum_{k=0}^{n^m-1}\mathbf{E}\left(\Delta B_{k,n}^H\right)^2 = n^m n^{-2H} = n^{m-2H}.$$

So, $\mathbf{E}(I_n^1)^2 \le Cn^{m-2H}$, $n \ge 1$.

The term I_n^3 is estimated as before, and

$$0 \le \mathbf{E}\left(I_n^2\right)^2 = \theta^2 \sum_{j,k=0}^{n^m-1} e^{\theta\frac{k}{n}+\theta\frac{j}{n}} \int_0^{\frac{k}{n}}\int_0^{\frac{j}{n}} e^{-\theta s-\theta u}\,\mathbf{E}B_s^H B_u^H \Delta B_{k,n}^H \Delta B_{j,n}^H\,du\,ds$$

$$= \theta^2 \sum_{k=0}^{n^m-1} e^{2\theta\frac{k}{n}} \int_0^{\frac{k}{n}} \int_0^{\frac{k}{n}} e^{-\theta s - \theta u}\, \mathbf{E} B_s^H B_u^H \left(\Delta B_{k,n}^H\right)^2 du\, ds$$

$$+ \theta^2 \sum_{k\neq j} e^{\theta\frac{k}{n}+\theta\frac{j}{n}} \int_0^{\frac{k}{n}} \int_0^{\frac{j}{n}} e^{-\theta s - \theta u}\, \mathbf{E} B_s^H B_u^H \Delta B_{k,n}^H \Delta B_{j,n}^H\, du\, ds =: J_1^n + J_2^n. \tag{30}$$

Substituting bounds (28) and (29) into (30), we get that

$$J_1^n \le Cn^{-4H} \sum_{k=0}^{n^m-1} e^{2\theta\frac{k}{n}} \int_0^{\frac{k}{n}} \int_0^{\frac{k}{n}} e^{-\theta s - \theta u} ds\, du$$

$$+ Cn^{-2H} \sum_{k=0}^{n^m-1} e^{2\theta\frac{k}{n}} \int_0^{\frac{k}{n}} \int_0^{\frac{k}{n}} e^{-\theta s - \theta u} \left(s^{2H} + u^{2H} - |s-u|^{2H}\right) ds\, du$$

$$\le Cn^{2m-4H} + Cn^{-2H} \sum_{k=0}^{n^m-1} e^{\theta\frac{k}{n}} \int_0^{\frac{k}{n}} e^{-\theta s} s^{2H} ds$$

$$\le Cn^{2m-4H} + Cn^{1-2H} \int_0^{n^{m-1}} e^{\theta s} \int_0^s e^{-\theta u} u^{2H} du\, ds$$

$$\le Cn^{2m-4H} + Cn^{1-2H} e^{\theta n^{m-1}} \int_0^{n^{m-1}} e^{-\theta u} u^{2H} du + Cn^{1-2H} \int_0^{n^{m-1}} u^{2H} du$$

$$\le Cn^{2m-4H} + Cn^{1-2H} \int_0^{n^{m-1}} u^{2H} du$$

$$\le Cn^{2m-4H} + Cn^{2Hm+m-4H} \le Cn^{2m-4H}, \quad \text{and} \quad J_2^n \le Cn^{2m-4H}.$$

Comparing the exponents $2m-4H$, $4H+2m-8Hm$ and $m-2H$, we get that for $m>1$ $2m-4H$ is the largest one, whence the proof follows. □

Corollary 4.3. (i) *Let* $\theta > 0$. *Then*

$$\mathbf{E}\left(n^{4H-2} e^{-2\theta n^{m-1}} \sum_{k=0}^{n^m-1} X_{k,n} \Delta B_{k,n}^H\right)^2 \le C.$$

If we denote $\xi_n = n^{2H-1} e^{-\theta n^{m-1}} \sum_{k=0}^{n^m-1} X_{k,n} \Delta B_{k,n}^H$ *then* $\sup_{n\ge1} \mathbf{E}\xi_n^2 < \infty$. *It means that for any* $m>1$ *the numerator of* (7) *can be bounded by the sum*

$$\zeta^2 n^{-1} e^{2\theta n^{m-1}} + n^{1-2H} e^{\theta n^{m-1}} \xi_n,$$

where $\sup_{n\ge1} \mathbf{E}\xi_n^2 < \infty$.

(ii) *Let* $\theta < 0$. *Then we have two cases.*

(a) *Let* $1 < m \le \frac{1}{H}$. *Then for any* $\alpha > 0$ *the numerator of* (7) *can be bounded by the sum*

$$\zeta^2 n^{(m-2)H+m-1+\alpha} + n^{m-2H} \xi_n,$$

where $\sup_{n\ge1} \mathbf{E}\xi_n^2 < \infty$.

(*b*) *Let* $m > \frac{1}{H}$. *Then for any* $\alpha > 0$ *the numerator of* (7) *can be bounded by the sum*

$$\zeta^2 n^{(2H+1)m-2H-2+\alpha} + n^{m-2H}\xi_n,$$

where $\sup_{n\geq 1} \mathbf{E}\xi_n^2 < \infty$.

5. How to deal with the denominator and with the ratios

Now our goal is to present the denominator of (7) in a more convenient form. First, we compare the sum $\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2$ to the corresponding integral $\int_0^{n^{m-1}} X_s^2\, ds$. The reason to replace the sum with the corresponding integral is that for some values of H and m we can prove the consistency with the help of some kind of L'Hôpital's rule, however, the application of the L'Hôpital's rule or the Stolz—Cesàro theorem to the sum $\sum_{k=0}^{n^m-1} X_{k,n}^2$ is problematic because not only the upper bound but also the terms in the sum depend on n.

Lemma 5.1. (*i*) *Let* $\theta > 0$. *Then there exists such* $\zeta_1 \in \mathfrak{Z}$ *that*

$$\left| \int_0^{n^{m-1}} X_s^2\, ds - \frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2 \right| \leq \frac{\zeta_1^2}{n} e^{2\theta n^{m-1}}.$$

(*ii*) *Let* $\theta < 0$. *Then we have two cases.*

(*a*) *Let* $1 < m \leq \frac{1}{H}$. *Then there exists such* $\zeta_1 \in \mathfrak{Z}$ *that for any* $\beta > 0$ *we have the following bound*

$$\left| \int_0^{n^{m-1}} X_s^2\, ds - \frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2 \right| \leq \zeta_1^2 n^{mH+m-2H-1+\beta}.$$

(*b*) *Let* $m > \frac{1}{H}$. *Then there exists such* $\zeta_1 \in \mathfrak{Z}$ *that for any* $\beta > 0$ *we have the following bound*

$$\left| \int_0^{n^{m-1}} X_s^2\, ds - \frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2 \right| \leq \zeta_1^2 n^{2mH+m-2H-2+\beta}.$$

Proof. Evidently, the difference between the integral and the corresponding integral sum can be bounded as

$$\left| \int_0^{n^{m-1}} X_s^2\, ds - \frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2 \right| \leq \int_0^{n^{m-1}} |\varphi_n(s)|\, ds,$$

where the integrand has the form

$$\varphi_n(s) = \left(X_s^2 - X_{k,n}^2\right) \mathbf{1}_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right)}, \quad 0 \leq k \leq n^m - 1.$$

Furthermore, the integrand can be bound as

$$\begin{aligned}|\varphi_n(s)| &\le |X_s - X_{k,n}| \left(|X_s| + |X_{k,n}|\right) \mathbf{1}_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right)}\\ &\le 2\,|X_s - X_{k,n}| \sup_{0\le u\le s} |X_u|\, \mathbf{1}_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right)}.\end{aligned}$$

(i) Let $\theta > 0$. Then from (17), (18) and similarly to (22),

$$\begin{aligned}|X_s - X_{k,n}| \sup_{0\le u\le s} |X_u|\, \mathbf{1}_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right)} \le \Big(&n^{-1}e^{2\theta s} + 2n^{-1}e^{\theta s}s^H \log^2 s\\ &+ e^{\theta s}n^{-H+\alpha} + n^{-1}s^{2H}\log^4 s + n^{-H+r}s^H \log^2 s\Big)\zeta_1^2.\end{aligned} \tag{31}$$

Integrating over $[0, n^{m-1}]$, we see that the integral of the first term in the right-hand side of (31) dominates, whence the proof follows.

(ii) Let $\theta < 0$. Then according to (19)–(20),

$$\begin{aligned}|X_s - X_{k,n}| \sup_{0\le u\le s} |X_u|\, \mathbf{1}_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right)} \le \Big(&\tfrac{1}{n} + \tfrac{1}{n}\left(\tfrac{k+1}{n}\right)^H \log^2\left(\tfrac{k+1}{n}\right)\\ &+ n^{-H+\alpha}\Big)\left(1 + \left(\tfrac{k+1}{n}\right)^H \log^2\left(\tfrac{k+1}{n}\right)\right)\zeta_1^2,\end{aligned}$$

therefore

$$\begin{aligned}\int_0^{n^{m-1}} |\varphi_n(s)|\, ds \le \Bigg(&n^{m-2} + \frac{2}{n^2}\sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n}\right)^H \log^2 \tfrac{k+1}{n}\\ &+ \frac{1}{n^2}\sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n}\right)^{2H}\log^4\tfrac{k+1}{n} + n^{m-H-1+\alpha}\\ &+ n^{-H-1+\alpha}\sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n}\right)^H\log^2\tfrac{k+1}{n}\Bigg)\zeta_1^2.\end{aligned} \tag{32}$$

To get rid of logarithms, we apply Lemma 7.1 to (32) and obtain that for any $\beta > 0$

$$\begin{aligned}\int_0^{n^{m-1}} |\varphi_n(s)|\, ds \le \big(&n^{m-2} + n^{mH+m-H-2+\beta} + n^{m-H-1+\beta}\\ &+ n^{2Hm+m-2H-2+\beta} + n^{mH+m-2H-1+\beta}\big)\zeta_1^2.\end{aligned}$$

Comparing the exponents $m-2$, $mH+m-H-2$, $m-H-1$, $2Hm+m-2H-2$ and $mH+m-2H-1$, we deduce that for $1 < m \le \frac{1}{H}$ the largest exponent equals $mH+m-2H-1$, and for $m > \frac{1}{H}$ the largest exponent equals $2Hm+m-2H-2$, whence the proof follows. ☐

Corollary 5.2. (i) *Let $\theta > 0$. Then there exists such $\zeta_1 \in \mathfrak{Z}$ that*

$$\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2 = \int_0^{n^{m-1}} X_s^2\, ds + \vartheta_n,$$

where

$$|\vartheta_n| \le \frac{\zeta_1^2}{n} e^{2\theta n^{m-1}}.$$

(*ii*) *Let* $\theta < 0$. *Then we have two cases.*

(*a*) *Let* $1 < m \le \frac{1}{H}$. *Then there exists such* $\zeta_1 \in \mathfrak{Z}$ *that for any* $\beta > 0$

$$\frac{1}{n} \sum_{k=0}^{n^m-1} X_{k,n}^2 = \int_0^{n^{m-1}} X_s^2\, ds + \vartheta_n(\beta), \tag{33}$$

where

$$|\vartheta_n(\beta)| \le \zeta_1^2 n^{mH+m-2H-1+\beta}. \tag{34}$$

(*b*) *Let* $m > \frac{1}{H}$. *Then for any* $\beta > 0$ *the representation* (33) *holds with*

$$|\vartheta_n(\beta)| \le \zeta_1^2 n^{2mH+m-2H-2+\beta}.$$

Now, in order to adequately treat the case $\theta < 0$ that is more technically complicated, we additionally bound the following ratios:

$$K_n^1(m,\alpha,\beta) := \frac{n^{mH+m-2H-1+\alpha}}{\int_0^{n^{m-1}} X_s^2\, ds + \vartheta_n}\ ,\ \widetilde{K}_n^1(m,\alpha,\beta) := \frac{n^{2Hm+m-2H-2+\alpha}}{\int_0^{n^{m-1}} X_s^2\, ds + \vartheta_n}$$

and $K_n^2 := \frac{n^{m-2H}\xi_n}{\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2}$.

Lemma 5.3. *Let* $\theta < 0$.

(*i*) *For any* $1 < m < \frac{2H+1}{H+1} < \frac{1}{H}$ *there exist such* $\alpha > 0$ *and* $\beta > 0$ *that*

$$K_n^1(m,\alpha,\beta) \to 0$$

a.s. as $n \to \infty$.

(*ii*) *For any* $\frac{2H+1}{H+1} \le m \le \frac{1}{H}$ *there exist such* $\alpha > 0$ *and* $\beta > 0$ *that*

$$K_n^1(m,\alpha,\beta) \to 0$$

in probability as $n \to \infty$.

(*iii*) *There exist such* $\alpha > 0$ *and* $\beta > 0$ *that for any* $m > \frac{1}{H}$

$$\widetilde{K}_n^1(m,\alpha,\beta) \to 0$$

in probability as $n \to \infty$.

Proof. (*i*) Let $1 < m < \frac{2H+1}{H+1} < \frac{1}{H}$. Then the exponent $mH + m - 2H - 1$ is negative. Indeed, for $H < \frac{1}{2}$ we have the inequality $\frac{2H+1}{H+1} < \frac{1}{H}$. First, choose $\alpha > 0$ so that $mH + m - 2H - 1 + \alpha < 0$ and put $\beta = \alpha$. Then it is sufficient to note that $n^{mH+m-2H-1+\alpha} \to 0$ and $\vartheta_n \to 0$ a.s. as $n \to \infty$ while the integral $\int_0^{n^{m-1}} X_s^2\, ds$ is increasing with probability 1 and tends to a nonzero with probability 1 random variable as $n \to \infty$.

(ii) Let $\frac{2H+1}{H+1} \leq m \leq \frac{1}{H}$. Then the exponent $mH + m - 2H - 1$ is positive. Choose $\alpha = \beta$. It is sufficient to prove that there exists such $\alpha > 0$ that

$$n^{-mH-m+2H+1-\alpha} \int_0^{n^{m-1}} X_s^2\, ds + n^{-mH-m+2H+1-\alpha} \vartheta_n(\alpha) \to \infty$$

in probability as $n \to \infty$. In view of (34) it is equivalent to

$$n^{-mH-m+2H+1-\alpha} \int_0^{n^{m-1}} X_s^2\, ds \to \infty \quad \text{in probability as } n \to \infty.$$

To establish this convergence note that it follows from the Cauchy–Schwarz inequality that

$$n^{-mH-m+2H+1-\alpha} \int_0^{n^{m-1}} X_s^2\, ds \geq n^{-mH-2m+2H+2-\alpha} \left(\int_0^{n^{m-1}} X_s\, ds \right)^2.$$

Denote $\gamma = -mH - 2m + 2H + 2 - \alpha < 0$. Without loss of generality suppose that $x_0 > 0$. Note that $\int_0^{n^{m-1}} X_s\, ds$ is a Gaussian process with the mean

$$e_n = \frac{x_0}{\theta} \left(e^{\theta n^{m-1}} - 1 \right) \in (0, -\frac{x_0}{\theta})$$

and variance

$$\begin{aligned} \sigma_n^2 = \int_0^{n^{m-1}} \int_0^{n^{m-1}} \mathbf{E} X_s X_t\, ds\, dt - e_n^2 \\ = \int_0^{n^{m-1}} \int_0^{n^{m-1}} \mathbf{E} \left(\theta e^{\theta s} \int_0^s e^{-\theta u} B_u^H\, du + B_s^H \right) \\ \times \left(\theta e^{\theta t} \int_0^t e^{-\theta z} B_z^H\, dz + B_t^H \right) ds\, dt. \end{aligned} \tag{35}$$

Since for any $u, z > 0$ we have $\mathbf{E} B_u^H B_z^H > 0$, the variance can be bounded from below by the value

$$\begin{aligned} \sigma_n^2 \geq \int_0^{n^{m-1}} \int_0^{n^{m-1}} \mathbf{E} B_s^H B_t^H\, ds\, dt \\ = \frac{1}{2} n^{(m-1)(2H+2)} \int_0^1 \int_0^1 \left(s^{2H} + t^{2H} - |s-t|^{2H} \right) ds\, dt = C n^{(m-1)(2H+2)}. \end{aligned} \tag{36}$$

Note that the other terms in (35) are of the same order so bound (36) is exact. Now, denoting as $\mathcal{N}(0,1)$ the standard Gaussian random variable and $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$, we can deduce that for any $A > 0$ and sufficiently large n

$$\mathbf{P} \left\{ n^\gamma \left(\int_0^{n^{m-1}} X_s\, ds \right)^2 \leq A^2 \right\} = \mathbf{P} \left\{ n^{\frac{\gamma}{2}} \left| \int_0^{n^{m-1}} X_s\, ds \right| \leq A \right\}$$

$$= \mathbf{P}\left\{n^{\frac{\gamma}{2}}\left|\sigma_n\mathcal{N}(0,1)+e_n\right| \le A\right\} = \Phi\left(\frac{A}{\sigma_n n^{\frac{\gamma}{2}}}-\frac{e_n}{\sigma_n}\right)-\Phi\left(-\frac{A}{\sigma_n n^{\frac{\gamma}{2}}}-\frac{e_n}{\sigma_n}\right)$$

$$\le \left|\Phi\left(\frac{A}{\sigma_n n^{\frac{\gamma}{2}}}-\frac{e_n}{\sigma_n}\right)-\frac{1}{2}\right| + \left|\Phi\left(-\frac{A}{\sigma_n n^{\frac{\gamma}{2}}}-\frac{e_n}{\sigma_n}\right)-\frac{1}{2}\right| \le 2\left(\frac{A}{\sigma_n n^{\frac{\gamma}{2}}}-\frac{e_n}{\sigma_n}\right)$$

$$\le \frac{C}{\sigma_n n^{\frac{\gamma}{2}}} \le \frac{C}{n^{\frac{mH}{2}-\frac{\alpha}{2}}}. \tag{37}$$

Choosing $0<\alpha<mH$ we get the proof of (ii).

(iii) For $m>\frac{1}{H}$ the exponent $2Hm+m-2H-2$ is positive. Therefore, we repeat the proof of (ii) with the same σ_n and with $\widetilde{\gamma}=-2Hm-2m+2H+3-\alpha$ instead of γ. So, in the inequality similar to (36), we get in the right-hand side the upper bound

$$\frac{C}{\sigma_n n^{\widetilde{\gamma}/2}} \le \frac{C}{n^{\frac{1}{2}-\frac{\alpha}{2}}}.$$

Choosing $0<\alpha<\frac{1}{2}$ we get the proof of (iii). □

Remark 4. We can prove more than it was mentioned in (i), namely, to establish that

$$\int_0^{n^{m-1}} X_s^2\,ds \to \infty$$

a.s. as $n\to\infty$ (see Lemma 7.2 in Section 7).

Lemma 5.4. *Let $\theta<0$, $m>\frac{1}{2H}$. Then $K_n^2\to 0$ in probability as $n\to\infty$.*

Proof. We apply the same method as in the proof of Lemma 5.3, but to the sum $\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2$ instead of the integral $\int_0^{n^{m-1}} X_s^2\,ds$. As before, suppose that $x_0>0$. According to the Cauchy–Schwarz inequality,

$$\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2 \ge n^{-m-1}\left(\sum_{k=0}^{n^m-1} X_{k,n}\right)^2, \tag{38}$$

where $\sum_{k=0}^{n^m-1} X_{k,n}$ is a Gaussian random variable with the mean

$$0<\widetilde{e}_n = x_0\sum_{k=0}^{n^m-1} e^{\theta\frac{k}{n}} \le nx_0\int_0^{n^{m-1}} e^{\theta s}\,ds \le -\frac{nx_0}{\theta}$$

and variance

$$\widetilde{\sigma}_n^2 = \sum_{k,j=0}^{n^m-1}\mathbf{E}X_{k,n}X_{j,n} \ge \sum_{k,j=0}^{n^m-1}\mathbf{E}B_{k,n}^H B_{j,n}^H$$

$$= \frac{1}{2}\sum_{k,j=0}^{n^m-1}\left(\left(\tfrac{k}{n}\right)^{2H}+\left(\tfrac{j}{n}\right)^{2H}-\left|\tfrac{j}{n}-\tfrac{k}{n}\right|^{2H}\right)$$

$$= n^m\sum_{k=0}^{n^m-1}\left(\tfrac{k}{n}\right)^{2H}-\sum_{l=0}^{n^m-1}\left(\tfrac{l}{n}\right)^{2H}(n^m-l) \tag{39}$$

$$= n^{-2H} \sum_{l=0}^{n^m-1} l^{2H+1} = n^{-2H+m(2H+1)+m} \sum_{l=0}^{n^m-1} \left(\tfrac{l}{n^m}\right)^{2H+1} \tfrac{1}{n^m}$$
$$\geq C n^{-2H+m(2H+1)+m}.$$

Therefore, for any $\varepsilon > 0$ and $x_n > 0$

$$\mathbf{P}\left\{K_n^2 \geq \varepsilon\right\} \leq \mathbf{P}\left\{\frac{|\xi_n|\, n^{m-2H}}{\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2} \geq \varepsilon\right\} \leq \mathbf{P}\left\{\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2 < x_n\right\}$$
$$+ \mathbf{P}\left\{|\xi_n| > x_n \varepsilon n^{2H-m}\right\} \leq \mathbf{P}\left\{\frac{1}{n^{m+1}}\left(\sum_{k=0}^{n^m-1} X_{k,n}\right)^2 < x_n\right\} + \frac{\mathbf{E}\xi_n^2}{x_n^2 \varepsilon^2 n^{4H-2m}}$$
$$\leq \mathbf{P}\left\{\frac{1}{n^{m+1}}\left(\sum_{k=0}^{n^m-1} X_{k,n}\right)^2 < x_n\right\} + \frac{C}{x_n^2 \varepsilon^2 n^{4H-2m}}.$$

Similarly to (37),

$$\mathbf{P}\left\{\frac{1}{n^{m+1}}\left(\sum_{k=0}^{n^m-1} X_{k,n}\right)^2 < x_n\right\} = \mathbf{P}\left\{|\widetilde{\sigma}_n \mathcal{N}(0,1) + \widetilde{e}_n| < x_n^{\frac{1}{2}} n^{\frac{m+1}{2}}\right\}$$
$$= \Phi\left(-\frac{\widetilde{e}_n}{\widetilde{\sigma}_n} + \frac{x_n^{\frac{1}{2}} n^{\frac{m+1}{2}}}{\widetilde{\sigma}_n}\right) - \Phi\left(-\frac{\widetilde{e}_n}{\widetilde{\sigma}_n} - \frac{x_n^{\frac{1}{2}} n^{\frac{m+1}{2}}}{\widetilde{\sigma}_n}\right) \leq C\left(\frac{x_n^{\frac{1}{2}} n^{\frac{m+1}{2}}}{\widetilde{\sigma}_n} + \frac{n}{\widetilde{\sigma}_n}\right).$$

Evidently, for any $m > 1$, $\frac{n}{\widetilde{\sigma}_n} \to 0$ as $n \to \infty$. Therefore, to supply the convergence $K_n^2 \to 0$ in probability, we need to choose x_n in such a way that $\frac{1}{x_n^2 n^{4H-2m}} \to 0$ and $\frac{x_n^{\frac{1}{2}} n^{\frac{m+1}{2}}}{\widetilde{\sigma}_n} \to 0$ as $n \to \infty$. Put $x_n = n^r$. Then r must satisfy the double inequality

$$m - 2H < r < m + 2Hm - 2H - 1.$$

This inequality can be satisfied only for $m > \frac{1}{2H}$ whence the proof follows. $\square$

6. Proofs of the main consistency results

6.1. Proof of Theorem 1.1

According to Corollaries 4.3 and 5.2, it is sufficient to prove that

$$\psi_n := \frac{\zeta^2 n^{-1} e^{2\theta n^{m-1}} + n^{1-2H} e^{\theta n^{m-1}} \xi_n}{\int_0^{n^{m-1}} X_s^2\, ds + \vartheta_n} \to 0$$

a.s. as $n \to \infty$, where $\sup_{n\geq 1} \mathbf{E}\xi_n^2 < \infty$ and $|\vartheta_n| \leq \frac{\zeta_1^2}{n} e^{2\theta n^{m-1}}$. Rewrite ψ_n as

$$\psi_n := \frac{\zeta^2 n^{-1} + n^{1-2H} e^{-\theta n^{m-1}} \xi_n}{e^{-2\theta n^{m-1}} \int_0^{n^{m-1}} X_s^2\, ds + e^{-2\theta n^{m-1}} \vartheta_n}.$$

Evidently, $\zeta^2 n^{-1} \to 0$ a.s. as $n \to \infty$. Furthermore, for any $x > 0$

$$\mathbf{P}\left\{n^{1-2H} e^{-\theta n^{m-1}} \xi_n > x\right\} \le \frac{\mathbf{E}\xi_n^2}{x^2 n^{4H-2} e^{2\theta n^{m-1}}} \le \frac{C}{x^2 n^{4H-2} e^{2\theta n^{m-1}}},$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^{4H-2} e^{2\theta n^{m-1}}}$ converges. It means by the Borel–Cantelli lemma that $n^{1-2H} e^{-\theta n^{m-1}} \xi_n \to 0$ a.s. as $n \to \infty$. Evidently, $e^{-2\theta n^{m-1}} |\vartheta_n| \le \frac{\zeta_1^2}{n} \to 0$ a.s. as $n \to \infty$. At last, according to (2) and the L'Hôpital's rule,

$$\lim_{T \to \infty} \frac{\int_0^T X_s^2\, ds}{e^{2\theta T}} = \lim_{T \to \infty} \frac{X_T^2}{2\theta e^{2\theta T}} = (2\theta)^{-1} \left(x_0 + \int_0^{\infty} e^{-\theta s} B_s^H\, ds\right)^2,$$

and the limit random variable is positive a.s. as the square of a Gaussian variable, whence the proof follows.

6.2. Proof of Theorem 1.2

Let $\theta < 0$. According to Corollaries 4.3 and 5.2, for $1 < m \le \frac{1}{H}$ we need to bound $K_n^1(m, \alpha, \beta)$ and K_n^2, and for $m > \frac{1}{H}$ we need to bound $\widetilde{K}_n^1(m, \alpha, \beta)$ and the same K_n^2. However, we can establish the convergence of K_n^2 in probability to 0 only for $m > \frac{1}{2H}$, see Lemma 5.4, while $\widetilde{K}_n^1(m, \alpha, \beta)$ tends in probability to 0 for $m > \frac{1}{H}$, see Lemma 5.3. Hence the proof follows.

Let $\theta = 0$. In this case $X_t = x_0 + B_t^H,\ t \ge 0$ and

$$\widehat{\theta}_n(m) = \frac{x_0 B_{n^m}^H + \sum_{k=0}^{n^m-1} B_{k,n}^H \Delta B_{k,n}^H}{\frac{1}{n} \sum_{k=0}^{n^m-1} (x_0 + B_{k,n}^H)^2}.$$

Similarly to (38),

$$\frac{1}{n} \sum_{k=0}^{n^m-1} (x_0 + B_{k,n}^H)^2 \ge n^{-m-1} \left(n^m x_0 + \sum_{k=0}^{n^m-1} B_{k,n}^H\right)^2$$
$$= n^{m-1} \left(x_0 + n^{-m} \sum_{k=0}^{n^m-1} B_{k,n}^H\right)^2 = n^{m-1} \left(x_0 + \widehat{\sigma}_n \mathcal{N}(0,1)\right)^2,$$

where, according to (39),

$$\widehat{\sigma}_n^2 \ge n^{-2m} \sum_{k,j=0}^{n^m-1} \mathbf{E} B_{k,n}^H B_{j,n}^H \ge C n^{2H(m-1)} \to \infty$$

as $n \to \infty$. Since for any $\alpha > 0$ $|B_{n^m}^H| \le n^{Hm+\alpha}\zeta$, then in order to establish that

$$\frac{x_0 B_{n^m}^H}{\frac{1}{n} \sum_{k=0}^{n^m-1} (x_0 + B_{k,n}^H)^2} \to 0$$

in probability, it is sufficient to prove that $\frac{n^{Hm+\alpha-m+1}}{(x_0 + \widehat{\sigma}_n \mathcal{N}(0,1))^2} \to 0$ in probability, as $n \to \infty$. But this is the case when $m \ge 2$ since for any $\varepsilon > 0$

$$\mathbf{P}\left\{\frac{n^{Hm+\alpha-m+1}}{\left(x_0+\widehat{\sigma}_n\mathcal{N}(0,1)\right)^2}>\varepsilon^2\right\}=\mathbf{P}\left\{\left|\frac{x_0}{\widehat{\sigma}_n}+\mathcal{N}(0,1)\right|<\frac{n^{1/2(Hm+\alpha-m+1)}}{\varepsilon\widehat{\sigma}_n}\right\}$$
$$\leq \mathbf{P}\left\{\left|\frac{x_0}{\widehat{\sigma}_n}+\mathcal{N}(0,1)\right|<\frac{n^{-1/2(Hm-\alpha+m-1-H)}}{\varepsilon}\right\}\to 0 \tag{40}$$

as $n\to\infty$, if we choose α sufficiently small. Furthermore, according to (26),

$$\mathbf{E}(\widehat{\xi}_n)^2 := \mathbf{E}\left(\sum_{k=0}^{n^m-1} B^H_{k,n}\Delta B^H_{k,n}\right)^2 \leq Cn^{2m-4H}.$$

Therefore, for any $\varepsilon>0$ and for the sequence $\widehat{x}_n=n^{m-2H+\beta}$

$$\mathbf{P}\left\{\frac{\widehat{\xi}_n}{n^{m-1}\left(x_0+\widehat{\sigma}_n\mathcal{N}(0,1)\right)^2}>\varepsilon^2\right\}$$
$$\leq \frac{Cn^{2m-4H}}{\widehat{x}_n^2}+\mathbf{P}\left\{\left|\frac{x_0}{\widehat{\sigma}_n}+\mathcal{N}(0,1)\right|<\frac{\sqrt{\widehat{x}_n}}{\varepsilon\widehat{\sigma}_n n^{\frac{m-1}{2}}}\right\}$$
$$\leq \frac{C}{n^{2\beta}}+\mathbf{P}\left\{\left|\frac{x_0}{\widehat{\sigma}_n}+\mathcal{N}(0,1)\right|<\frac{n^{1/2-Hm+\beta}}{\varepsilon}\right\}\to 0$$

as $n\to\infty$ for $m>\frac{1}{2H}$ and $0<\beta<Hm-\frac{1}{2}$. Theorem is proved.

7. Auxiliary results

At first we establish an auxiliary result concerning the bounds for several sums of integral type that will participate in the bounds for the numerator of (7).

Lemma 7.1. *For any $m>1$ and $n\geq 2$ there exists $C>0$ not depending on n such that*

(i)
$$\sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n}\right)^H\log^2\tfrac{k+1}{n}\leq Cn^{(m-1)H+m}\log^2 n,$$

(ii)
$$\sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n}\right)^{2H}\log^4\tfrac{k+1}{n}\leq Cn^{2H(m-1)+m}\log^4 n.$$

Proof. We base the proof of both statements on the following evident relation: for any function $f:[0,1]\to\mathbb{R}$ that is Riemann integrable on $[0,1]$, and for any $m\geq 1$ the integral sums $S(f(x),n^m):=\frac{1}{n^m}\sum_{k=0}^{n^m-1}f(\frac{k+1}{n^m})$ tend to the integral $\int_0^1 f(x)dx$ as $n\to\infty$. In particular, these integral sums are bounded. Consider the statement (i). Evidently,

$$\sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n}\right)^H\log^2\tfrac{k+1}{n}=n^{(m-1)H}\sum_{k=0}^{n^m-1}\left(\tfrac{k+1}{n^m}\right)^H\log^2\left(\tfrac{k+1}{n^m}n^{m-1}\right)$$

$$\leq 2n^{(m-1)H} \sum_{k=0}^{n^m-1} \left(\tfrac{k+1}{n^m}\right)^H \left(\log^2\left(\tfrac{k+1}{n^m}\right) + (m-1)^2 \log^2 n\right)$$

$$= 2n^{(m-1)H+m} S(x^H \log^2 x, n^m) + 2(m-1)^2 n^{(m-1)H+m} \log^2 n \cdot S\left(x^H, n^m\right)$$

$$\leq C n^{(m-1)H+m} + C n^{(m-1)H+m} \log^2 n \leq C n^{(m-1)H+m} \log^2 n$$

for $n \geq 2$. Statement (ii) is established similarly. □

The next auxiliary result establishes the asymptotic behavior of the integral $\int_0^T X_s^2\, ds$ as $T \to \infty$.

Lemma 7.2. *Let a process X satisfy the equation (1). Then $\int_0^T X_s^2\, ds \to \infty$ with probability 1 as $T \to \infty$.*

Proof. The result is obvious for $\theta > 0$, therefore we consider only the case $\theta < 0$. Since $\int_0^T X_s^2\, ds$ is nondecreasing in $T > 0$, it is sufficient to prove that $\int_0^T X_s^2\, ds \to \infty$ in probability. For any $\lambda > 0$ consider the moment generation function $\Theta_T(\lambda) = \mathbf{E} \exp\{-\lambda \int_0^T X_s^2\, ds\}$ and $\Theta_\infty(\lambda) = \mathbf{E} \exp\{-\lambda \int_0^\infty X_s^2\, ds\}$ so that

$$\Theta_\infty(\lambda) = \lim_{T\to\infty} \Theta_T(\lambda).$$

Evidently,

$$\int_0^T X_s^2\, ds \geq T^{-1} \left(\int_0^T X_s\, ds\right)^2,$$

whence

$$\Theta_T(\lambda) \leq \Theta_T^{(1)}(\lambda) := \mathbf{E} \exp\left\{-\frac{\lambda}{T} \left(\int_0^T X_s\, ds\right)^2\right\}.$$

The random variable $T^{-\frac{1}{2}} \int_0^T X_s\, ds$ is Gaussian with the mean $m(T)$ and variance $\sigma^2(T)$. Note that for a Gaussian random variable $\xi = m + \sigma \mathcal{N}(0,1)$ we have that

$$\mathbf{E} \exp\left\{-\lambda \xi^2\right\} = \left(2\lambda\sigma^2 + 1\right)^{-\frac{1}{2}} \exp\left\{-\frac{\lambda m^2}{2a\sigma^2 + 1}\right\} \leq \left(2\lambda\sigma^2 + 1\right)^{-\frac{1}{2}}.$$

Therefore, it is sufficient to prove that

$$\lim_{T\to\infty} \sigma^2(T) = \infty.$$

Similarly to (36).

$$\sigma^2(T) \geq T^{2H+1} \int_0^1 \int_0^1 \left(s^{2H} + t^{2H} - |s-t|^{2H}\right) ds\, dt \to \infty$$

as $T \to \infty$, whence the proof follows. □

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