Brazilian Journal of Probability and Statistics
2017, Vol. 31, No. 1, 24–40
DOI: 10.1214/15-BJPS301

Strong rate of tamed Euler–Maruyama approximation for stochastic differential equations with Hölder continuous diffusion coefficient

Hoang-Long Ngo and Duc-Trong Luong

Hanoi National University of Education

Abstract. We study the strong rate of convergence of the tamed Euler–Maruyama approximation for one-dimensional stochastic differential equations with superlinearly growing drift and Hölder continuous diffusion coefficients.

1 Introduction

Let consider the following stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s, \qquad x_0 \in \mathbb{R}, t \in [0, T], \tag{1.1}$$

where $(W_t)_{0 \le t \le T}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$.

It is well known that when b and σ are Lipschitz continuous, the standard Euler–Maruyama approximation scheme for (X_t) has a strong rate of convergence of order $1/2$ (see Kloeden and Platen (1992)). Recently, there have been extensive studies on the strong approximations of SDE (1.1) with non-Lipschitz coefficients. The rates of Euler–Maruyama scheme for SDEs with Hölder continuous diffusion coefficients have been investigated by Yan (2002); Gyöngy and Rásonyi (2011); Ngo and Taguchi (2016) (see also Alfonsi (2005); Berkaoui, Bossy and Diop (2008); Dereich, Neuenkirch and Szpruch (2012) for many strong approximation schemes proposed for Cox–Ingersoll–Ross type model). Hairer, Hutzenthaler and Jentzen (2015) have given an example of SDE with globally bounded and smooth coefficients such that the standard Euler–Maruyama approximation converges to the exact solution of the SDE in both strong and weak senses but there is no positive polynomial rate of convergence. Moreover, Hutzenthaler, Jentzen and Kloeden (2011) have showed that if b is superlinear growth then the absolute moments of the standard Euler–Maruyama approximated solution may diverge to infinity while the ones of the true solution are finite. Therefore, the standard Euler–Maruyama scheme may fail to converge in L^p sense. There are basically two methods to overcome this difficulty. The first method named implicit Euler–Maruyama scheme

Key words and phrases. Stochastic differential equation, irregular coefficients, Euler–Maruyama approximation, Hölder continuous diffusion, strong approximation, superlinearly growing drift.
Received January 2015; accepted September 2015.

was introduced by Higham, Mao and Stuart (2002), and Hu (1996). A drawback of the implicit scheme is that at each simulation step, one needs to solve an algebraic equation which may not have an explicit solution. Hutzenthaler, Jentzen and Kloeden (2012) and Sababis (2013) have recently presented the second method named the tamed Euler scheme. It is an explicit scheme in which the drift coefficient is modified so that it is bounded. It has been shown that when the diffusion σ is Lipschitz continuous and the drift b is superlinear growth and one-sided Lipschitz, the tamed Euler scheme has a strong rate of order $1/2$.

In this article, we study the strong convergence rate of the tamed Euler–Maruyama schemes applied to the SDE (1.1) where b is superlinear growth and σ is Hölder continuous. This partly generalizes the results in Gyöngy and Rásonyi (2011); Hutzenthaler, Jentzen and Kloeden (2012) in the one-dimensional framework. The main contributions of the current paper are:

- Establishing a new sufficient condition for the existence and uniqueness of the strong solution for one-dimensional SDEs with locally Hölder continuous diffusion coefficient and superlinearly growing drift coefficient;
- Showing the strong convergence (in L^p sense) of the tamed Euler–Maruyama approximation for these SDEs;
- Obtaining the order of these approximation errors.

The rest of the paper is organized as follows. The next section introduces some notations and assumptions. All main results are presented in Section 3 while the proofs are given in Section 4.

2 Notations and assumptions

For integers $n \geq 1$, we define $\eta_n(s) : [0;T] \to [0;T]$ by $\eta_n(t) = \frac{kT}{n} := t_k^{(n)}$ if $t \in [\frac{kT}{n}; \frac{(k+1)T}{n})$. The tamed Euler–Maruyama approximation of equation (1.1) is defined as follows

$$X_t^{(n)} = x_0 + \int_0^t b_n\big(s, X_{\eta_n(s)}^{(n)}\big)\,ds + \int_0^t \sigma\big(s, X_{\eta_n(s)}^{(n)}\big)\,dW_s, \qquad t \in [0,T] \tag{2.1}$$

with $b_n(t,x) = \frac{b(t,x)}{1+n^{-\lambda}|b(t,x)|}$ for some $\lambda \in (0; \frac{1}{2}]$. Note that if b is replaced by b_n in (2.1) then $X^{(n)}$ is called the standard Euler–Maruyama approximation.

We will make the following assumptions:

A1. There exists a positive constant L such that

$$xb(t,x) \vee \big|\sigma(t,x)\big|^2 \leq L\big(1+|x|^2\big)$$

for any $x \in \mathbb{R}$ and $t \in [0;T]$.

A2. b is one-sided Lipschitz: there exists a positive constant L such that

$$(x-y)\big(b(t,x)-b(t,y)\big) \le L|x-y|^2$$

for any $x, y \in \mathbb{R}$ and $t \in [0; T]$.

A3. There exist positive constants L and l such that

$$\big|b(t,x)-b(t,y)\big| \le L\big(1+|x|^l+|y|^l\big)|x-y|,$$

and

$$\big|b(t,x)\big| \le L\big(1+|x|^{l+1}\big)$$

for any $x, y \in \mathbb{R}$ and $t \in [0; T]$.

A4. σ is $(\alpha+\frac{1}{2})$-Hölder continuous: there exist positive constants L and $\alpha \in (0, \frac{1}{2}]$ such that

$$\big|\sigma(t,x)-\sigma(t,y)\big| \le L|x-y|^{1/2+\alpha}$$

for any $x, y \in \mathbb{R}$ and $t \in [0; T]$.

A5. b is locally Lipschitz and locally bounded: for any $R > 0$, there exists a positive constant $L_R > 0$ such that

$$\big|b(t,x)-b(t,y)\big| \le L_R|x-y|$$

and $|b(t,x)| \le L_R$ for all $|x| \vee |y| \le R$ and $t \in [0,T]$.

A6. σ is locally $(\alpha+\frac{1}{2})$-Hölder continuous: for any $R > 0$, there exist positive constants L_R and $\alpha \in (0, \frac{1}{2}]$ such that

$$\big|\sigma(t,x)-\sigma(t,y)\big| \le L_R|x-y|^{1/2+\alpha}$$

for all $|x| \vee |y| \le R$ and $t \in [0,T]$.

It is clear that the assumptions A3 and A4 imply the assumptions A5 and A6, respectively.

3 Main results

The existence and uniqueness of solution for SDEs with Hölder continuous diffusion coefficient and bounded measurable drift has been established in Veretennikov (1980) (see also Gyöngy and Krylov (1996); Yamada and Watanabe (1971)). In the following, we show the existence and uniqueness of solution to equation (1.1) under assumptions A1, A5 and A6.

Theorem 3.1.

(i) *Suppose that* A1, A5, A6 *hold, and equation* (1.1) *has a solution* $(X_t)_{t\in[0,T]}$ *then it is the unique solution.*

(ii) *Suppose that* A1, A5, A6 *hold*, *and there exist positive constants* C *and* l *such that*

$$\sup_{t\in[0,T]} \left|b(t,x)\right| \leq C\left(1+|x|^l\right),$$

then equation (1.1) *has a strong solution.*

Example 3.1. For clarity of exposition we consider the following SDE

$$X_t = x_0 + \int_0^t \left(aX_s - bX_s^3\right) ds + \int_0^t \sigma |X_s - K|^{1/2+\alpha}, \tag{3.1}$$

where b is a non-negative constant, $\alpha \in [0, \frac{1}{2}]$ and $x_0, a, \sigma, K \in \mathbb{R}$. It is straightforward to verify that coefficients of this SDE satisfy assumptions A1–A4. Therefore equation (3.1) has a unique strong solution. If $b > 0$, it follows from Theorem 2.1 in Hutzenthaler, Jentzen and Kloeden (2011) that the standard Euler–Maruyama approximated solution of (3.1) does not have a finite moment of any order $p \geq 1$ while X_t's have finite moments of all order (Lemma 4.1). It means that the standard Euler–Maruyama approximation for equation (3.1) does not converge in strong sense.

In the following, we always assume that the equation (1.1) has a unique strong solution. Moreover, we choose $\lambda \in [\alpha, \frac{1}{2}]$ for the tamed Euler–Maruyama approximation (2.1).

Theorem 3.2. *Suppose that* A1–A4 *hold*, *then there is a constant* $C > 0$ *independent of* n *such that*

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\left|X_\tau - X_\tau^{(n)}\right|\right] \leq \begin{cases} \dfrac{C}{n^{\alpha}}, & \text{if } \alpha \in \left(0, \dfrac{1}{2}\right], \\ \dfrac{C}{\log n}, & \text{if } \alpha = 0, \end{cases}$$

where $\mathcal{T}$ *is the set of all stopping times* τ *with respect to the filtration* $(\mathcal{F}_t)$ *satisfying* $\tau \leq T$.

Corollary 3.3. *Suppose that* A1–A4 *hold*, *then there is a constant* $C > 0$ *independent of* n *such that*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} \left|X_t - X_t^{(n)}\right|\Big] \leq \begin{cases} \dfrac{C}{n^{2\alpha^2}}, & \text{if } \alpha \in \left(0, \dfrac{1}{2}\right], \\ \dfrac{C}{\sqrt{\log n}}, & \text{if } \alpha = 0. \end{cases}$$

This proof is similar to the one of Corollary 2.3 in Gyöngy and Rásonyi (2011) and will be omitted.

The following estimates for the moments of strong approximation errors play an important role in designing a Multilevel Monte Carlo scheme to estimate $\mathbb{E}[F(X)]$ for some function F defined on $C[0,T]$ (see Giles (2008)).

Theorem 3.4. *Suppose that* A1–A4 *hold, then for all* $p > 0$, *there is a constant* $C_p > 0$ *depend on* p *and independent of* n *such that*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|X_t - X_t^{(n)}|^p\Big] \le \begin{cases} \dfrac{C_p}{n^{p/2}}, & \text{if } \alpha = \dfrac{1}{2}, \\ \dfrac{C_p}{n^{\alpha}}, & \text{if } \alpha \in \left(0; \dfrac{1}{2}\right) \text{ and } p \ge 2, \\ \dfrac{C_p}{\log n}, & \text{if } \alpha = 0 \text{ and } p \ge 2. \end{cases}$$

We note here that the strong rates of the tamed Euler–Maruyama approximation obtained in Theorems 3.2, 3.4 and Corollary 3.3 are the same as the ones of the standard Euler–Maruyama approximation applied to SDEs with Hölder continuous diffusion coefficient and linear growth drift (see Gyöngy and Rásonyi (2011)).

Finally, we show the convergence of tamed Euler–Maruyama scheme under the local Lipschitz condition on b and local Hölder continuous condition on σ.

Theorem 3.5. *Suppose that* A1, A5 *and* A6 *hold. Then for any* $p > 0$,

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{0\le t\le T}|X_t - X_t^{(n)}|^p\Big] = 0.$$

4 Proofs

In the following we will first prove Theorems 3.2–3.5. The proof of Theorem 3.1 will be given in Sections 4.5 and 4.6.

Throughout this section, C denotes some positive constants which may depend on L, l, T, α and x_0 but independent of n and t. When C depends on p or R, we denote C by C_p or C_R, respectively.

4.1 Some auxiliary estimates

Denote $U_t^{(n)} = X_t^{(n)} - X_{\eta_n(t)}^{(n)}$ and $Y_t^{(n)} = X_t - X_t^{(n)}$. We need the following bounds on moments of X and $X^{(n)}$ which are direct consequences of Lemmas 3.1–3.3 in Sababis (2013).

Lemma 4.1. *Suppose that* A1 *holds, then for any* $p>0$, *there exists a constant* $C_p>0$ *such that*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|X_t|^p\Big]\vee\sup_{n\ge 1}\mathbb{E}\Big[\sup_{0\le t\le T}|X_t^{(n)}|^p\Big]<C_p,$$

and

$$\sup_{0\le t\le T}\mathbb{E}[|U_t^{(n)}|^p]\le\frac{C_p}{n^{p/2}}. \tag{4.1}$$

Lemma 4.2. *Suppose that* A1 *and* A3 *hold, then for any* $p>0$, *there exist a constant* $C_p>0$ *such that*

$$\int_0^T\mathbb{E}[|b(s,X_s^{(n)})-b(s,X_{\eta_n(s)}^{(n)})|^p]\,ds\le\frac{C_p}{n^{p/2}}.$$

Proof. It is enough to prove the lemma for $p>1$. By A3, we have

$$|b(s,X_s^{(n)})-b(s,X_{\eta_n(s)}^{(n)})|^p\le C_p(1+|X_{\eta_n(s)}^{(n)}|^{lp}+|U_s^{(n)}|^{lp})|U_s^{(n)}|^p.$$

According to Lemma 4.1,

$$\mathbb{E}[|X_{\eta_n(s)}^{(n)}|^{lp}|U_s^{(n)}|^p]\le\frac{1}{n^{p/2}}\mathbb{E}[|X_{\eta_n(s)}^{(n)}|^{2lp}]+n^{p/2}\mathbb{E}[|U_s^{(n)}|^{2p}]\le\frac{C_p}{n^{p/2}}.$$

The lemma is proved completely. □

We borrow the following result form Gyöngy and Rásonyi (2011).

Lemma 4.3. *Let* $(X_t)_{t\ge 0}$ *be a nonnegative stochastic process and set* $V_t=\sup_{s\le t}X_s$. *Assume that for some* $p>0, q\ge 1, \rho\in[1,q]$ *and constants* K *and* $\Delta\ge 0$, *it holds*

$$\mathbb{E}[V_t^p]\le K\mathbb{E}\Big(\int_0^t V_s\,ds\Big)^p+K\mathbb{E}\Big(\int_0^t X_s^\rho\Big)^{p/q}+\Delta<\infty\qquad\text{for any } t\ge 0.$$

Then for each $T\ge 0$, *the following statements hold.*

(i) *If* $\rho=q$, *then there is a constant* C_T *such that* $\mathbb{E}[V_T^p]\le C_T\Delta$.

(ii) *If* $p\ge q$ *or both* $\rho<q$ *and* $p>q+1-\rho$ *hold, then the exist constants* C_1 *and* C_2 *depending on* K,T,ρ,q *and* p *such that* $\mathbb{E}[V_T^p]\le C_1\Delta+C_2\int_0^T\mathbb{E}[X_s]\,ds$.

We will repeatedly use the approximation technique of Yamada and Watanabe (see Yamada and Watanabe (1971); Gyöngy and Rásonyi (2011)). For each $\delta>1$ and $\varepsilon>0$, there exists a continuous function $\psi_{\delta\varepsilon}:\mathbb{R}\to\mathbb{R}^+$ with supp $\psi_{\delta\varepsilon}\subset[\varepsilon/\delta;\varepsilon]$ such that $\int_{\varepsilon/\delta}^{\varepsilon}\psi_{\delta\varepsilon}(z)\,dz=1$ and $0\le\psi_{\delta\varepsilon}(z)\le\frac{2}{z\log\delta}$ for $z>0$. Define

$$\phi_{\delta\varepsilon}(x):=\int_0^{|x|}\int_0^y\psi_{\delta\varepsilon}(z)\,dz\,dy,\qquad x\in\mathbb{R}.$$

It is easy to verify that $\phi_{\delta\varepsilon}$ has the following useful properties: for any $x\in\mathbb{R}$:

(i) $\phi'_{\delta\varepsilon}(x) = \frac{x}{|x|}\phi'_{\delta\varepsilon}(|x|)$,
(ii) $0 \le |\phi'_{\delta\varepsilon}(x)| \le 1$,
(iii) $|x| \le \varepsilon + \phi_{\delta\varepsilon}(x)$,
(iv) $\frac{\phi'_{\delta\varepsilon}(|x|)}{|x|} \le \frac{\delta}{\varepsilon}$,
(v) $\phi''_{\delta\varepsilon}(|x|) = \psi_{\delta\varepsilon}(|x|) \le \frac{2}{|x|\log\delta} 1_{[\varepsilon/\delta;\varepsilon]}(|x|)$,

where $\phi'_{\delta\varepsilon}$ and $\phi''_{\delta\varepsilon}$ denote the first and second order derivatives of ϕ with respect to x, respectively.

Applying Itô's formula for $\phi_{\delta\varepsilon}(Y_t^{(n)})$ and using (iii), we get

$$\begin{aligned} |Y_t^{(n)}| \le \varepsilon &+ \int_0^t \phi'_{\delta\varepsilon}(Y_s^{(n)})\big[b(s, X_s) - b_n\big(s, X_{\eta_n(s)}^{(n)}\big)\big]\,ds \\ &+ \frac{1}{2}\int_0^t \phi''_{\delta\varepsilon}(Y_s^{(n)})\big[\sigma(s, X_s) - \sigma\big(s, X_{\eta_n(s)}^{(n)}\big)\big]^2\,ds \\ &+ \int_0^t \phi'_{\delta\varepsilon}(Y_s^{(n)})\big[\sigma(s, X_s) - \sigma\big(s, X_{\eta_n(s)}^{(n)}\big)\big]\,dW_s. \end{aligned} \tag{4.2}$$

The estimate (4.2) will play a central role in our argument.

4.2 Proof of Theorem 3.2

We will use the estimate (4.2) to bound $\mathbb{E}[|X_\tau - X_\tau^{(n)}|] = \mathbb{E}[|Y_\tau^{(n)}|]$. Since the expectation of the last integral on the right-hand side of (4.2) is zero, we only need to bound the first and second integrals. We rewrite the first integral in (4.2) as

$$\begin{aligned} S_1 = &\int_0^t \frac{\phi'_{\delta\varepsilon}(|Y_s^{(n)}|)}{|Y_s^{(n)}|}(X_s - X_s^{(n)})\big[b(s, X_s) - b\big(s, X_s^{(n)}\big)\big]\,ds \\ &+ \int_0^t \phi'_{\delta\varepsilon}(Y_s^{(n)})\big[b\big(s, X_s^{(n)}\big) - b\big(s, X_{\eta_n(s)}^{(n)}\big)\big]\,ds \\ &+ \int_0^t \phi'_{\delta\varepsilon}(Y_s^{(n)})\big[b\big(s, X_{\eta_n(s)}^{(n)}\big) - b_n\big(s, X_{\eta_n(s)}^{(n)}\big)\big]\,ds. \end{aligned}$$

It follows from assumptions A2, A3, and the estimate (ii) that

$$\begin{aligned} S_1 \le\ & L\int_0^t \frac{\phi'_{\delta\varepsilon}(|Y_s^{(n)}|)}{|Y_s^{(n)}|}|Y_s^{(n)}|^2\,ds + \int_0^t \big|b\big(s, X_s^{(n)}\big) - b\big(s, X_{\eta_n(s)}^{(n)}\big)\big|\,ds \\ &+ \int_0^t \frac{n^{-\lambda}|b(s, X_{\eta_n(s)}^{(n)})|^2}{1 + n^{-\lambda}|b(s, X_{\eta_n(s)}^{(n)})|}\,ds \\ \le\ & L\int_0^t |Y_s^{(n)}|\,ds + \int_0^T \big|b\big(s, X_s^{(n)}\big) - b\big(s, X_{\eta_n(s)}^{(n)}\big)\big|\,ds \\ &+ \frac{2L^2}{n^\lambda}\int_0^T \big(1 + |X_{\eta_n(s)}^{(n)}|^{2l+2}\big)\,ds. \end{aligned} \tag{4.3}$$

Next, thanks to condition A4 and the estimate (v), the second integral in (4.3) is bounded by

$$\begin{aligned}&\int_0^t \frac{2^{2\alpha}L^2}{|Y_s^{(n)}\log\delta|}1_{[\varepsilon/\delta\le|Y_s^{(n)}|\le\varepsilon]}\big[\big|Y_s^{(n)}\big|^{1+2\alpha}+\big|U_s^{(n)}\big|^{1+2\alpha}\big]\,ds\\&\qquad\le\frac{2^{2\alpha}L^2\varepsilon^{2\alpha}T}{\log\delta}+\int_0^T\frac{2^{2\alpha}L^2\delta}{\varepsilon\log\delta}\big|U_s^{(n)}\big|^{1+2\alpha}\,ds.\end{aligned}\tag{4.4}$$

It follows from (4.2), (4.3) and (4.4) that

$$\begin{aligned}\big|Y_t^{(n)}\big|&\le\varepsilon+L\int_0^t\big|Y_s^{(n)}\big|\,ds+\int_0^T\big|b\big(s,X_s^{(n)}\big)-b\big(s,X_{\eta_n(s)}^{(n)}\big)\big|\,ds\\&\quad+\frac{2L^2}{n^\lambda}\int_0^T\big(1+\big|X_{\eta_n(s)}^{(n)}\big|^{2l+2}\big)\,ds\\&\quad+2^{2\alpha+1}L^2\Big[\frac{\varepsilon^{2\alpha}T}{\log\delta}+\int_0^T\frac{\delta}{\varepsilon\log\delta}\big|U_s^{(n)}\big|^{1+2\alpha}\,ds\Big]\\&\quad+\int_0^t\phi_{\delta\varepsilon}'\big(Y_s^{(n)}\big)\big[\sigma(s,X_s)-\sigma\big(s,X_{\eta_n(s)}^{(n)}\big)\big]\,dW_s.\end{aligned}\tag{4.5}$$

Let $Z_t^{(n)}:=|Y_{t\wedge\tau}^{(n)}|$ for any stopping time $\tau\le T$. It implies from (4.5) that

$$\begin{aligned}\mathbb{E}[Z_t^{(n)}]&\le\varepsilon+C\Big[\int_0^t\mathbb{E}[Z_s^{(n)}]\,ds+\int_0^T\mathbb{E}\big[\big|b\big(s,X_s^{(n)}\big)-b\big(s,X_{\eta_n(s)}^{(n)}\big)\big|\big]\,ds\Big]\\&\quad+Cn^{-\lambda}\Big[\int_0^T\big(1+\mathbb{E}\big[\big|X_{\eta_n(s)}^{(n)}\big|^{2l+2}\big]\big)\,ds\Big]\\&\quad+C\Big[\frac{\varepsilon^{2\alpha}}{\log\delta}+\int_0^T\frac{\delta}{\varepsilon\log\delta}\mathbb{E}\big[\big|U_s^{(n)}\big]\big|^{1+2\alpha}\,ds\Big].\end{aligned}$$

Thanks to Lemmas 4.1 and 4.2, we have

$$\mathbb{E}[Z_t^{(n)}]\le\varepsilon+C\Big\{\int_0^t\mathbb{E}[Z_s^{(n)}]\,ds+\frac{1}{n^\lambda}+\frac{\varepsilon^{2\alpha}}{\log\delta}+\frac{\delta}{\varepsilon\log\delta}\frac{1}{n^{1/2+\alpha}}\Big\}.$$

By Gronwall's inequality

$$\mathbb{E}[Z_t^{(n)}]\le C\Big\{\varepsilon+\frac{1}{n^\lambda}+\frac{\varepsilon^{2\alpha}}{\log\delta}+\frac{\delta}{\varepsilon\log\delta}\frac{1}{n^{1/2+\alpha}}\Big\}.$$

Case 1: $\alpha\in(0;\frac12]$. By choosing $\varepsilon=\frac{1}{\sqrt{n}}$ and $\delta=2$, we obtain $\mathbb{E}[Z_t^{(n)}]\le\frac{C}{n^\alpha}$. Let $t\uparrow T$ then

$$\sup_{\tau\in\mathcal{T}}\mathbb{E}\big[\big|X_\tau-X_\tau^{(n)}\big|\big]\le\frac{C}{n^\alpha}.$$

Case 2: $\alpha = 0$. By choosing $\varepsilon = \frac{1}{\log n}$ and $\delta = n^{1/3}$, we obtain $\mathbb{E}[Z_t^{(n)}] \le \frac{C}{\log n}$. Let $t \uparrow T$ then

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}\big[\big|X_\tau - X_\tau^{(n)}\big|\big] \le \frac{C}{\log n},$$

and, thus we get the desired result.

4.3 Proof of Theorem 3.4

The proof of this theorem is similar to the one of Theorem 3.2, except the fact that we need to take care of the supremum of the stochastic integral in (4.2).

It is enough to prove the theorem for $p \ge 2$. By (4.5), Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le u\le t} |Y_u^{(n)}|^p\Big] \\
&\quad \le C_p \varepsilon^p + C_p \bigg[\mathbb{E}\bigg[\bigg(\int_0^t \sup_{0\le u\le s} |Y_u^{(n)}|\,ds\bigg)^p\bigg] \\
&\qquad + \int_0^T \mathbb{E}\big[\big|b\big(s, X_s^{(n)}\big) - b\big(s, X_{\eta_n(s)}^{(n)}\big)\big|^p\big]\,ds\bigg] \\
&\qquad + C_p n^{-p\lambda}\bigg[\int_0^T \big(1 + \mathbb{E}\big[\big|X_{\eta_n(s)}^{(n)}\big|^{p(2l+2)}\big]\big)\,ds\bigg] \\
&\qquad + C_p\bigg[\frac{\varepsilon^{2p\alpha}}{(\log\delta)^p} + \int_0^T \frac{\delta^p}{\varepsilon^p(\log\delta)^p}\mathbb{E}\big[\big|U_s^{(n)}\big|^{p(1+2\alpha)}\big]\,ds\bigg] \\
&\qquad + C_p \mathbb{E}\bigg[\sup_{0\le u\le t}\bigg|\int_0^u \phi'_{\delta\varepsilon}\big(Y_s^{(n)}\big)\big[\sigma(s, X_s) - \sigma\big(s, X_{\eta_n(s)}^{(n)}\big)\big]\,dW_s\bigg|^p\bigg].
\end{aligned} \tag{4.6}$$

Applying Burkholder–Davis–Gundy's inequality, Hölder's inequality and Lemma 4.1, the last expectation in (4.6) is less than

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg|\int_0^t \big[\phi'_{\delta\varepsilon}\big(Y_s^{(n)}\big)\big]^2\big[\sigma(s, X_s) - \sigma\big(s, X_{\eta_n(s)}^{(n)}\big)\big]^2\,ds\bigg|^{p/2}\bigg] \\
&\quad \le C_p\bigg\{\mathbb{E}\bigg[\int_0^t \big|Y_s^{(n)}\big|^{1+2\alpha}\,ds\bigg]^{p/2} + \frac{1}{n^{p(1+2\alpha)/4}}\bigg\}.
\end{aligned}$$

This estimate together with Lemmas 4.1 and 4.2 implies

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le u\le t} |Y_u^{(n)}|^p\Big] \\
&\quad \le C_p\bigg\{\varepsilon^p + \mathbb{E}\bigg[\int_0^t \sup_{0\le u\le s}\big|Y_u^{(n)}\big|\,ds\bigg]^p + \frac{1}{n^{p\lambda}} + \frac{\varepsilon^{2p\alpha}}{(\log\delta)^p}
\end{aligned} \tag{4.7}$$

$$+\frac{\delta^p}{\varepsilon^p(\log\delta)^p}\cdot\frac{1}{n^{p(1+2\alpha)/2}}$$
$$+\mathbb{E}\Big[\int_0^t |Y_s^{(n)}|^{1+2\alpha}\,ds\Big]^{p/2}+\frac{1}{n^{p(1+2\alpha)/4}}+\frac{1}{n^{p/2}}\Big\}.$$

By choosing $\varepsilon=\frac{1}{\sqrt{n}}$ and $\delta=2$ when $\alpha\in(0,\frac{1}{2}]$; and $\varepsilon=\frac{1}{\log n}$ and $\delta=n^{1/3}$ when $\alpha=0$ and applying Lemma 4.3 together with Theorem 3.2, we get the desired result.

4.4 Proof of Theorem 3.5

We will use a localization technique. For each $R>0$, we denote $\tau_R=\inf\{t\geq 0:|X_t|\geq R\}$, $\rho_{nR}=\inf\{t\geq 0:|X_t^{(n)}|\geq R\}$ and $\nu_{nR}=\tau_R\wedge\rho_{nR}$, and $\chi^{(n)}(s)=Y^{(n)}_{s\wedge\nu_{nR}}=X_{s\wedge\nu_{nR}}-X^{(n)}_{s\wedge\nu_{nR}}$. We recall the following result form Sababis (2013).

Lemma 4.4. *Suppose that* A1 *holds. Then for any* $R>0, q>p>2$ *and* $\kappa>0$, *one has*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|Y_t^{(n)}|^p\Big]\leq\frac{\kappa p}{q}C_q+\frac{q-p}{q\kappa^{p/(q-p)}R^p}C_p+\mathbb{E}\Big[\sup_{0\leq t\leq T}|\chi_t^{(n)}|^p\Big].$$

Furthermore, suppose that A4 *holds. Then for any* $R>0$, *there is a* $C_p(R)>0$ *such that*

$$\int_0^T\mathbb{E}[|b(s,X_s^{(n)})-b(s,X^{(n)}_{\eta_n(s)})|^p 1_{(s\leq\nu_{nR})}]\,ds\leq\frac{C_pL_R^p}{n^{p/2}}.$$

The proof of Theorem 3.5 is divided into three steps.

Step 1: We will show that $\sup_{0\leq t\leq T}\mathbb{E}[|\chi^{(n)}(t)|]\to 0$ as $n\to\infty$. Indeed, from (4.2) we have

$$\begin{aligned}|\chi_t^{(n)}|&\leq\varepsilon+\int_0^t\phi'_{\delta\varepsilon}(Y_s^{(n)})[b(s,X_s)-b_n(s,X^{(n)}_{\eta_n(s)})]1_{[s\leq\nu_{nR}]}\,ds\\&\quad+\frac{1}{2}\int_0^t\phi''_{\delta\varepsilon}(Y_s^{(n)})[\sigma(s,X_s)-\sigma(s,X^{(n)}_{\eta_n(s)})]^2 1_{[s\leq\nu_{nR}]}\,ds\\&\quad+\int_0^t\phi'_{\delta\varepsilon}(Y_s^{(n)})[\sigma(s,X_s)-\sigma(s,X^{(n)}_{\eta_n(s)})]1_{[s\leq\nu_{nR}]}\,dW_s.\end{aligned}\tag{4.8}$$

A similar argument as in (4.3) and condition A5 imply that

$$\begin{aligned}&\int_0^t\phi'_{\delta\varepsilon}(Y_s^{(n)})[b(s,X_s)-b_n(s,X^{(n)}_{\eta_n(s)})]1_{[s\leq\nu_{nR}]}\,ds\\&\quad\leq L_R\int_0^t|\chi_s^{(n)}|\,ds+\int_0^T|b(s,X_s^{(n)})-b(s,X^{(n)}_{\eta_n(s)})|1_{[s\leq\nu_{nR}]}\,ds\\&\qquad+\frac{C_R}{n^\lambda}.\end{aligned}\tag{4.9}$$

Thanks to condition A6, we have

$$\begin{aligned}&\int_0^t \phi''_{\delta\varepsilon}(Y_s^{(n)})\big[\sigma(s,X_s)-\sigma\big(s,X^{(n)}_{\eta_n(s)}\big)\big]^2 1_{[s\le\nu_{nR}]}\,ds\\&\quad\le \frac{C_R\varepsilon^{2\alpha}}{\log\delta}+\int_0^T \frac{C_R\delta}{\varepsilon\log\delta}\big|U_s^{(n)}\big|^{1+2\alpha}\,ds.\end{aligned}\tag{4.10}$$

It follows from (4.8), (4.9) and (4.10) that

$$\begin{aligned}\big|\chi_t^{(n)}\big|&\le \varepsilon + C_R\bigg[\int_0^t\big|\chi_s^{(n)}\big|\,ds+\int_0^T\big|b\big(s,X_s^{(n)}\big)-b\big(s,X^{(n)}_{\eta_n(s)}\big)\big|1_{[s\le\nu_{nR}]}\,ds\bigg]\\&\quad+C_R\bigg[n^{-\lambda}+\frac{\varepsilon^{2\alpha}}{\log\delta}+\int_0^T\frac{\delta}{\varepsilon\log\delta}\big|U_s^{(n)}\big|^{1+2\alpha}\,ds\bigg]\\&\quad+\int_0^t\phi'_{\delta\varepsilon}(Y_s^{(n)})\big[\sigma(s,X_s)-\sigma\big(s,X^{(n)}_{\eta_n(s)}\big)\big]1_{[s\le\nu_{nR}]}\,dW_s.\end{aligned}\tag{4.11}$$

Taking expectation both sides of (4.11) and applying Lemmas 4.1 and 4.4, we get

$$\mathbb{E}\big[\big|\chi_t^{(n)}\big|\big]\le\varepsilon+C_R\bigg[\int_0^t\mathbb{E}\big[\big|\chi_s^{(n)}\big|\big]\,ds+\frac{1}{n^\lambda}+\frac{\varepsilon^{2\alpha}}{\log\delta}+\frac{\delta}{\varepsilon n^{1/2+\alpha}\log\delta}\bigg].$$

By Gronwall's inequality

$$\mathbb{E}\big[\big|\chi_t^{(n)}\big|\big]\le C_R\bigg[\varepsilon+\frac{1}{n^\lambda}+\frac{\varepsilon^{2\alpha}}{\log\delta}+\frac{\delta}{\varepsilon n^{1/2+\alpha}\log\delta}\bigg].$$

Letting $n\to\infty$ and then $\varepsilon\to 0$, we conclude step 1.

Step 2: We will show that $\lim_{n\to\infty}\mathbb{E}[\sup_{0\le t\le T}|\chi_t^{(n)}|^p]=0$. It follows from (4.11), Lemmas 4.1, 4.4 and Burkholder–Davis–Gundy's inequality that $\mathbb{E}[\sup_{0\le s\le t}|\chi_s^{(n)}|^p]$ is less than

$$\begin{aligned}&C_{R,p}\bigg[\varepsilon+\int_0^t\sup_{0\le u\le s}\mathbb{E}\big[\big|\chi_u^{(n)}\big|^p\big]\,ds+\frac{1}{n^{p\lambda}}+\frac{\varepsilon^{2p\alpha}}{(\log\delta)^p}+\frac{\delta^p}{\varepsilon^p(\log\delta)^p n^{p(1/2+\alpha)}}\bigg]\\&\quad+C_{R,p}\mathbb{E}\bigg[\bigg|\int_0^t\big[\phi'_{\delta\varepsilon}(Y_s^{(n)})\big[\sigma(s,X_s)-\sigma\big(s,X^{(n)}_{\eta_n(s)}\big)\big]1_{[s\le\nu_{nR}]}\big]^2\,ds\bigg|^{p/2}\bigg].\end{aligned}\tag{4.12}$$

Thanks to assumption A6, the last expectation is less than

$$\begin{aligned}&\mathbb{E}\bigg[\bigg|\int_0^t\big|X_s-X^{(n)}_{\eta_n(s)}\big|^{1+2\alpha}1_{[s\le\nu_{nR}]}\,ds\bigg|^{p/2}\bigg]\\&\quad\le C_{R,p}\mathbb{E}\bigg[\bigg|\int_0^t\big|Y_s^{(n)}\big|^{1+2\alpha}1_{[s\le\nu_{nR}]}\,ds\bigg|^{p/2}\bigg]\\&\qquad+C_{R,p}\int_0^t\mathbb{E}\big[\big|U_s^{(n)}\big|^{p(1/2+\alpha)}\,ds\big]\\&\quad\le C_{R,p}\mathbb{E}\bigg[\bigg|\int_0^t\big|\chi_s^{(n)}\big|^{1+2\alpha}\,ds\bigg|^{p/2}\bigg]+\frac{C_{R,p}}{n^{p(1+2\alpha)/4}}.\end{aligned}\tag{4.13}$$

It follows from (4.12) and (4.13) that $\mathbb{E}[\sup_{0\le s\le t}|\chi_s^{(n)}|^p]$ is bounded by

$$C_{R,p}\Big(\varepsilon+\int_0^t \sup_{0\le u\le s}\mathbb{E}[|\chi_u^{(n)}|^p\,ds]+\mathbb{E}\Big[\Big|\int_0^t|\chi_s^{(n)}|^{1+2\alpha}\,ds\Big|^{p/2}\Big]$$
$$+\frac{1}{n^{p\lambda}}+\frac{\varepsilon^{2p\alpha}}{(\log\delta)^p}+\frac{\delta^p}{\varepsilon^p(\log\delta)^p n^{p(1/2+\alpha)}}+\frac{C_{R,p}}{n^{p(1+2\alpha)/4}}\Big).$$

By applying Lemma 4.3 and following a similar argument as in Step 1, we finish Step 2.

Step 3: We will show that $\lim_{n\to\infty}\mathbb{E}[\sup_{0\le t\le T}|Y_t^{(n)}|^p]=0$. It follows from Lemma 4.4 and Step 2 that

$$\limsup_{n\to\infty}\mathbb{E}\Big[\sup_{0\le t\le T}|Y_t^{(n)}|^p\Big]\le\frac{\kappa p}{q}C_q+\frac{q-p}{q\kappa^{p/(q-p)}R^p}C_p.$$

Let $R\to\infty$, then $\limsup_{n\to\infty}\mathbb{E}[\sup_{0\le t\le T}|Y_t^{(n)}|^p]\le\frac{\kappa p}{q}C_q$. Finally, let $\kappa\downarrow 0$, we have $\lim_{n\to\infty}\mathbb{E}[\sup_{0\le t\le T}|Y_t^{(n)}|^p]=0$. The theorem is completely proved.

4.5 Proof of Theorem 3.1(i)

Assume that X'_t is another solution of equation (1.1), we will show that $\mathbb{E}[|X_t-X'_t|]=0$ for all $t\in[0,T]$, which implies the uniqueness of solution.

By following Lemma 4.1, we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}|X_t|^p\Big]\wedge\mathbb{E}\Big[\sup_{0\le t\le T}|X'_t|^p\Big]\le C_p, \tag{4.14}$$

where $p>0$ and C_p is a positive constant. For any $R>0$, we put $\theta_R=\inf\{t\ge 0: |X_t|\vee|X'_t|\ge R\}$. Since X_t, X'_t are solutions of (1.1),

$$X_t-X'_t=\int_0^t\big[b(s,X_s)-b\big(s,X'_s\big)\big]\,ds+\int_0^t\big[\sigma(s,X_s)-\sigma\big(s,X'_s\big)\big]\,dW_s.$$

Apply Itô's formula for $\phi_{\delta\varepsilon}(X_t-X'_t)$ and use property (iii), we have

$$\begin{aligned}
&\big|X_{t\wedge\theta_R}-X'_{t\wedge\theta_R}\big|\\
&\quad\le\varepsilon+\int_0^t\phi'_{\delta\varepsilon}(X_s-X'_s)\big[b(s,X_s)-b\big(s,X'_s\big)\big]1_{\{s\le\theta_R\}}\,ds\\
&\qquad+\frac12\int_0^t\phi''_{\delta\varepsilon}(X_s-X'_s)\big[\sigma(s,X_s)-\sigma\big(s,X'_s\big)\big]^2 1_{\{s\le\theta_R\}}\,ds\\
&\qquad+\int_0^{t\wedge\theta_R}\phi'_{\delta\varepsilon}(X_s-X'_s)\big[\sigma(s,X_s)-\sigma\big(s,X'_s\big)\big]\,dW_s.
\end{aligned}\tag{4.15}$$

Use assumption A5 and property (ii) of $\phi_{\delta\varepsilon}(x)$ then

$$\begin{aligned}&\int_0^t \phi'_{\delta\varepsilon}(X_s - X'_s)[b(s, X_s) - b(s, X'_s)]1_{\{s\le\theta_R\}}\, ds \\ &\quad\le L_R \int_0^t |X_{s\wedge\theta_R} - X'_{s\wedge\theta_R}|\, ds.\end{aligned} \tag{4.16}$$

From assumption A6 and property (v) of $\phi_{\delta\varepsilon}(x)$, we have

$$\int_0^t \phi''_{\delta\varepsilon}(X_s - X'_s)[\sigma(s, X_s) - \sigma(s, X'_s)]^2 1_{[s\le\theta_R]}\, ds \le \frac{C_R\varepsilon^{2\alpha}}{\log\delta}. \tag{4.17}$$

From (4.15), (4.16) and (4.17), we have

$$\begin{aligned}|X_{t\wedge\theta_R} - X'_{t\wedge\theta_R}| &\le \varepsilon + L_R \int_0^t |X_{s\wedge\theta_R} - X'_{s\wedge\theta_R}|\, ds + \frac{C_R\varepsilon^{2\alpha}}{\log\delta} \\ &\quad + \int_0^{t\wedge\theta_R} \phi'_{\delta\varepsilon}(X_s - X'_s)[\sigma(s, X_s) - \sigma(s, X'_s)]\, dW_s.\end{aligned}$$

Taking the expectation of both sides, we get

$$\mathbb{E}[|X_{t\wedge\theta_R} - X'_{t\wedge\theta_R}|] \le \varepsilon + L_R \int_0^t \mathbb{E}[|X_{s\wedge\theta_R} - X'_{s\wedge\theta_R}|]\, ds + \frac{C_R\varepsilon^{2\alpha}}{\log\delta}.$$

By choosing $\delta = 2$ and letting $\varepsilon \to 0$, we get

$$\mathbb{E}[|X_{t\wedge\theta_R} - X'_{t\wedge\theta_R}|] \le L_R \int_0^t \mathbb{E}[|X_{s\wedge\theta_R} - X'_{s\wedge\theta_R}|]\, ds.$$

By Gronwall's inequality, $\mathbb{E}[|X_{t\wedge\theta_R} - X'_{t\wedge\theta_R}|] = 0$. It means $X_{t\wedge\theta_R} = X'_{t\wedge\theta_R}$ almost surely. This leads to $\mathbb{E}[|X_t - X'_t|] = \mathbb{E}[|X_t - X'_t|1_{[\theta_R\le t]}]$. Applying Young's inequality for $q = 2$ and (4.14), we obtain

$$\begin{aligned}\mathbb{E}[|X_t - X'_t|] &\le \frac{1}{2R}\mathbb{E}|X_t - X'_t|^2 + \frac{R}{2}\mathbb{P}[\theta_R \le T] \\ &\le \frac{1}{2R}\mathbb{E}|X_t - X'_t|^2 + \frac{1}{2R}\Big(\mathbb{E}\Big[\sup_{0\le t\le T}|X_t|^2\Big] + \mathbb{E}\Big[\sup_{0\le t\le T}|X'_t|^2\Big]\Big) \\ &\le \frac{1}{2R}\mathbb{E}|X_t - X'_t|^2 + \frac{C}{R}.\end{aligned}$$

Let $R \to \infty$ we obtain $\mathbb{E}[|X_t - X'_t|] = 0$. It means $X_t = X'_t$ almost surely. The proof is complete.

4.6 Proof of Theorem 3.1(ii)

We will use the following result.

Lemma 4.5 (Gyöngy and Krylov (1996); Gyöngy and Rásonyi (2011)). *Assume that $b(t,x)$ is Lipschitz in x and $\sigma(t,x)$ is $(1/2+\alpha)$-Hölder continuous in x for some $\alpha \in [0,\frac{1}{2}]$, and $b(t,0), \sigma(t,0)$ are bounded on $[0,T]$. Then there exists a unique strong solution of equation (1.1).*

For each $N > 0$, set

$$b_N(t,x) = \begin{cases} b(t,x), & \text{if } |x| \le N, \\ b\Big(t, \dfrac{Nx}{|x|}\Big)\big(N+1-|x|\big), & \text{if } N < |x| < N+1, \\ 0, & \text{if } |x| \ge N+1, \end{cases}$$

and

$$\sigma_N(t,x) = \begin{cases} \sigma(t,x), & \text{if } |x| \le N, \\ \sigma\Big(t, \dfrac{Nx}{|x|}\Big)\big(N+1-|x|\big), & \text{if } N < |x| < N+1, \\ 0, & \text{if } |x| \ge N+1. \end{cases}$$

It is straightforward to verify that b_N and σ_N satisfying the assumptions of Lemma 4.5. Thus, the equation

$$X_N(t) = x_0 + \int_0^t b_N\big(s, X_N(s)\big)\,ds + \int_0^t \sigma_N\big(s, X_N(s)\big)\,dW_s \tag{4.18}$$

has a unique strong solution $X_N(t)$. We will show that when $N \to \infty$, X_N will converge in probability to a process X which satisfies equation (1.1).

For each $N > 0$, put

$$\theta_N = T \wedge \inf\big\{t \in [0;T] : \big|X_N(t)\big| \ge N\big\}.$$

Because the uniqueness of equation (4.18), $X_N(t) = X_M(t)$ almost surely for any $t < \theta_N$ and $N < M$. Next, we will show that $\theta_N = T$ almost surely for all N large enough. Indeed, because of assumption A1,

$$x b_N(t,x) \vee \big|\sigma_N(t,x)\big|^2 \le 2L\big(1+|x|^2\big) \qquad \text{for any } x \in \mathbb{R}.$$

Therefore, $b_N(x)$ and $\sigma_N(t,x)$ also satisfy assumption A1. Apply Lemma 4.1, there exists a constant $C_p > 0$ which does not depend neither on N nor on n, such that

$$\mathbb{E}\Big[\sup_{0\le t\le T} \big|X_N(t)\big|^p\Big] \le C_p \qquad \text{for any } N > 0.$$

Therefore,

$$C_p \ge \mathbb{E}\Big[\sup_{0\le t\le T} \big|X_N(t)\big|^2\Big] \ge \mathbb{E}\Big[\sup_{0\le t\le T} \big|X_N(t)\big|^2 1_{[\theta_N<T]}\Big] \ge N^2\mathbb{P}[\theta_N < T].$$

It leads to $\sum_{N=1}^{\infty} \mathbb{P}(\theta_N < T) < \infty$. By Borel–Cantelli lemma,

$$\mathbb{P}\Big[\limsup_{N} \{\theta_N < T\}\Big] = 0.$$

Because $(\theta_N)_N$ is increasing, so $\theta_N = T$ for all N large enough. It means $\lim_{N\to\infty} X_N(t) = X(t)$ exists almost surely and $X(t) = X_M(t)$ almost surely for any $t < \theta_N$ and $M \geq N$. On the other hand, for any $\kappa > 0, q > k > 1$, by Young's inequality

$$\begin{aligned}
&\mathbb{E}\big[\big|X_{N+k}(t\wedge\theta_{N+k}) - X_N(t\wedge\theta_N)\big|^p\big] \\
&\quad \leq \mathbb{E}\big[\big[\big|X_{N+k}(t)\big|^p + \big|X_N(t)\big|^p\big] 1_{\{\theta_N < T\}}\big] \\
&\quad \leq C_{p,q}\bigg\{\kappa \mathbb{E}\big[\big|X_{N+k}(t)\big|^q + \big|X_N(t)\big|^q\big] + \frac{\mathbb{P}[\theta_N < T]}{\kappa^{p/(q-p)}}\bigg\}.
\end{aligned}$$

First, let $N \to \infty$ and then let $\eta \to 0$, we get

$$\mathbb{E}\big[\big|X_{N+p}(t\wedge\theta_{N+p}) - X_N(t\wedge\theta_N)\big|^p\big] \to 0 \qquad \text{as } N \to \infty.$$

It means

$$X_N(t\wedge\theta_N) \xrightarrow{L^p} X(t) \qquad \text{as } N \to \infty. \tag{4.19}$$

Furthermore, for all $p > 0$ there exists a constant $C'_p > 0$ such that

$$\sup_{0\leq t\leq T} \mathbb{E}\big[\big|X(t)\big|^p\big] \leq C'_p.$$

From the definition of $b_N(t,x)$, we have

$$\mathbb{E}\bigg|\int_0^{t\wedge\theta_N} \big[b_N\big(s, X_N(s)\big) - b\big(s, X(s)\big)\big]\, ds\bigg|^2 = 0.$$

Moreover, since b is polynomially bounded,

$$\begin{aligned}
\mathbb{E}\bigg[\bigg|\int_{t\wedge\theta_N}^t b\big(s, X(s)\big)\, ds\bigg|^2\bigg] &\leq C\int_0^t \mathbb{E}\big[\big[1 + \big|X(s)\big|^{2l}\big] 1_{\{s\geq\theta_N\}}\big]\, ds \\
&\leq \kappa C\int_0^T \mathbb{E}\big[1 + \big|X(s)\big|^{2l}\big]^2\, ds + \frac{C\mathbb{P}[\theta_N < T]}{\kappa} \\
&\leq \kappa C + \frac{C\mathbb{P}[\theta_N < T]}{\kappa}.
\end{aligned}$$

First let $N \to \infty$ and then let $\kappa \to 0$, we have

$$\lim_{N\to\infty} \mathbb{E}\bigg[\bigg|\int_{t\wedge\theta_N}^t b\big(s, X(s)\big)\, ds\bigg|^2\bigg] = 0.$$

Therefore,

$$\int_0^{t\wedge\theta_N} b_N\big(s, X_N(s)\big)\, ds \xrightarrow{L^2} \int_0^t b\big(s, X(s)\big)\, ds \qquad \text{as } N \to \infty. \tag{4.20}$$

In the same manner, we can see that

$$\int_0^{t\wedge\theta_N} \sigma_N\big(s, X_N(s)\big)\,dW_s \xrightarrow{L^2} \int_0^t \sigma\big(s, X(s)\big)\,dW_s \qquad \text{as } N\to\infty. \tag{4.21}$$

Combining (4.18)–(4.21), we get

$$X_t = x_0 + \int_0^t b\big(s, X(s)\big)\,ds + \int_0^t \sigma\big(s, X(s)\big)\,dW_s,$$

almost surely for all $t\in[0,T]$. It means $(X_t)_{t\in[0,T]}$ is a solution of equation (1.1). The proof is complete.

Acknowledgments

This work was completed while the first author was staying at Vietnam Institute for Advanced Study in Mathematics (VIASM). The author would like to thank the institute for support. Both authors thank an anonymous referee for his/her helpful comments. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.03-2014.14.

References

Alfonsi, A. (2005). On the discretization schemes for the CIR (and Bessel squared) processes. *Monte Carlo Methods Applications* **11**, 355–384. MR2186814

Berkaoui, B., Bossy, M. and Diop, A. (2008). Euler scheme for SDEs with non-Lipschitz diffusion coefficient: Strong convergence. *ESAIM: Probability and Statistics* **12**, 1–11. MR2367990

Dereich, S., Neuenkirch, A. and Szpruch, L. (2012). An Euler-type method for the strong approximation of the Cox–Ingersoll–Ross process. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science* **468**, 1105–1115. MR2898556

Giles, M. B. (2008). Multilevel Monte Carlo path simulation. *Operational Research* **56**, 607–617. MR2436856

Gyöngy, I. and Krylov, N. V. (1996). Existence of strong solutions for Itô's stochastic equations via approximations. *Probability Theory and Related Fields* **105**, 143–158. MR1392450

Hairer, M., Hutzenthaler, M. and Jentzen, A. (2015). Loss of regularity for Kolmogorov equations. *Ann. Probab.* **43**, 468–527. MR3305998

Gyöngy, I. and Rásonyi, M. (2011). A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients. *Stochastic Processes and Their Applications* **121**, 2189–2200. MR2822773

Higham, D. J., Mao, X. and Stuart, A. M. (2002). Strong convergence of Euler type methods for nonlinear stochastic differential equations. *SIAM Journal on Numerical Analysis* **40**, 1041–1063. MR1949404

Hu, Y. (1996). Semi-implicit Euler–Maruyama scheme for stiff stochastic equations. In *Stochastic Analysis and Related Topics V: The Silvri Workshop* (H. Koerezlioglu, ed.), *Progr. Probab.* **38**, 183–202. Boston: Birkhauser. MR1396331

Hutzenthaler, M., Jentzen, A. and Kloeden, P. E. (2011). Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proceedings—Royal Society. Mathematical, Physical and Engineering Sciences* **467**, 1563–1576. MR2795791

Hutzenthaler, M., Jentzen, A. and Kloeden, P. E. (2012). Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients. *The Annals of Applied Probability* **22**, 1611–1641. MR2985171

Kloeden, P. and Platen, E. (1992). *Numerical Solution of Stochastic Differential Equations*. *Applications of Mathematics (New York)* **23**, xxxvi+632 pp. Springer: Berlin. MR1214374

Ngo, H.-L. and Taguchi, D. (2016). Strong rate of convergence for the Euler-Maruyama approximation of stochastic differential equations with irregular coefficients. *Math. Comp.* **85**, 1793–1819. MR3471108

Sababis, S. (2013). A note on tamed Euler approximations. *Electronic Communications in Probability* **18**, 1–10. MR3070913

Veretennikov A. J. (1980). Strong solutions and explicit formulas for solutions of stochastic integral equations (Russian). *Mat. Sb.* **111**, 434–452. MR0568986

Yamada, T. and Watanabe, S. (1971). On the uniqueness of solutions of stochastic differential equations. *Journal of Mathematics of Kyoto University* **11**, 155–167. MR0278420

Yan, B. L. (2002). The Euler scheme with irregular coefficients. *Annals of Probability* **30**, 1172–1194. MR1920104

Hanoi National University of Education
136 Xuan Thuy
Cau Giay
Ha Noi
Vietnam
E-mail: ngolong@hnue.edu.vn
trongsphn@gmail.com