

# BEHAVIOR OF THE GENERALIZED ROSENBLATT PROCESS AT EXTREME CRITICAL EXPONENT VALUES<sup>1</sup>

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The generalized Rosenblatt process is obtained by replacing the single critical exponent characterizing the Rosenblatt process by two different exponents living in the interior of a triangular region. What happens to that generalized Rosenblatt process as these critical exponents approach the boundaries of the triangle? We show by two different methods that on each of the two symmetric boundaries, the limit is non-Gaussian. On the third boundary, the limit is Brownian motion. The rates of convergence to these boundaries are also given. The situation is particularly delicate as one approaches the corners of the triangle, because the limit process will depend on how these corners are approached. All limits are in the sense of weak convergence in  $C[0, 1]$ . These limits cannot be strengthened to convergence in  $L^2(\Omega)$ .

**1. Introduction.** Maejima and Tudor [17] considered recently the following process defined through a second-order Wiener–Itô integral:

$$(1) \quad Z_{\gamma_1, \gamma_2}(t) = A \int_{\mathbb{R}^2}' \left[ \int_0^t (s - x_1)_+^{\gamma_1} (s - x_2)_+^{\gamma_2} ds \right] \mathbf{B}(dx_1) \mathbf{B}(dx_2),$$

where  $A \neq 0$  is a constant,  $\mathbf{B}(\cdot)$  is a Brownian random measure, the prime ' indicates the *exclusion* of the diagonals  $x_1 = x_2$  in the double stochastic integral and the exponents  $\gamma_1, \gamma_2$  live in the following open triangular region (see Figure 1):

$$(2) \quad \Delta = \{(\gamma_1, \gamma_2) : -1 < \gamma_1 < -1/2, -1 < \gamma_2 < -1/2, \gamma_1 + \gamma_2 > -3/2\}.$$

This ensures that the integrand in (1) is in  $L^2(\mathbb{R}^2)$ , and hence the process  $Z_{\gamma_1, \gamma_2}(t)$  is well-defined (see Theorem 3.5 and Remark 3.1 of Bai and Taqqu [3]).

We shall call  $Z_{\gamma_1, \gamma_2}(t)$  a *generalized Rosenblatt process*. The *Rosenblatt process*  $Z_\gamma(t)$  (Taqqu [31]) becomes the special case

$$(3) \quad Z_\gamma(t) = Z_{\gamma, \gamma}(t), \quad -3/4 < \gamma < -1/2.$$

Recent studies on the Rosenblatt process  $Z_\gamma(t)$  include Tudor and Viens [32], Bardet and Tudor [7], Arras [1], Maejima and Tudor [18], Veillette and Taqqu [33] and Bojdecki, Gorostiza and Talarczyk [9]. The Rosenblatt and the generalized

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Received September 2014; revised May 2015.

<sup>1</sup>Supported in part by NSF Grants DMS-10-07616 and DMS-13-09009 at Boston University.  
*MSC2010 subject classifications.* 60F05, 60K35.

*Key words and phrases.* Long memory, self-similar processes, Rosenblatt processes, generalized Rosenblatt processes.

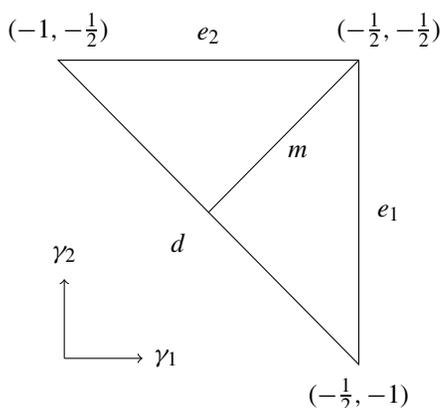


FIG. 1. Region  $\Delta$  defined in (2). The three edges of the triangle are named  $e_1, e_2$  and  $d$  (diagonal), while the middle line segment (symmetric axis) is named  $m$ .

Rosenblatt processes are of interest because they are the simplest extension to the non-Gaussian world of the Gaussian fractional Brownian motion.

Fractional Brownian motion  $B_H(t)$ ,  $1/2 < H < 1$  is defined through a single Wiener-Itô (or Wiener) integral:

$$B_H(t) = C \int_{\mathbb{R}} \left[ \int_0^t (s-x)_+^{H-3/2} ds \right] B(dx),$$

and has covariance

$$(4) \quad \mathbb{E}B_H(s)B_H(t) = \frac{C'}{2} (|s|^{2H} + |t|^{2H} - |s-t|^{2H}),$$

where  $C$  and  $C'$  are two related constants. Fractional Brownian motion reduces to Brownian motion if one sets  $H = 1/2$  in (4). Fractional Brownian motion has stationary increments and, for any  $1/2 < H < 1$ , these increments have a covariance which decreases slowly as the lag increases. This slow decay is often referred to as *long memory* or *long-range dependence*. Fractional Brownian motion is also self-similar with *self-similarity parameter (Hurst index)  $H$* , that is,  $B_H(\lambda t)$  has the same finite-dimensional distributions as  $\lambda^H B_H(t)$  for any  $\lambda > 0$ . It follows from Bai and Taqqu [3] that the generalized Rosenblatt process  $Z_{\gamma_1, \gamma_2}(t)$  is also self-similar with stationary increments with self-similarity parameter

$$(5) \quad H = \gamma_1 + \gamma_2 + 2 \in (1/2, 1).$$

We get  $1/2 < H < 1$  because  $\gamma_1, \gamma_2 < -1/2$  imply  $H < 1$  and  $\gamma_1 + \gamma_2 > -3/2$  implies  $H > 1/2$ .

Fractional Brownian motion and the generalized Rosenblatt process  $Z_{\gamma_1, \gamma_2}(t)$  belong to a broad class of self-similar processes with stationary increments defined on a Wiener chaos called *generalized Hermite processes*. The generalized Hermite

processes appear as limits in various types of non-central limit theorems involving Volterra-type nonlinear process. In particular, the generalized Rosenblatt process  $Z_{\gamma_1, \gamma_2}(t)$  can arise as limit when considering a quadratic form involving two long-memory linear processes with different memory parameters. See Bai and Taqqu [3, 5, 6] for details.

It will be convenient to express the generalized Rosenblatt process as follows:

$$(6) \quad Z_{\gamma_1, \gamma_2}(t) = \frac{A}{2} \int_{\mathbb{R}^2} \left[ \int_0^t [(s - x_1)_+^{\gamma_1} (s - x_2)_+^{\gamma_2} + (s - x_1)_+^{\gamma_2} (s - x_2)_+^{\gamma_1}] ds \right] \mathbf{B}(dx_1) \mathbf{B}(dx_2),$$

where we replaced the kernel  $A \int_0^t (s - x_1)_+^{\gamma_1} (s - x_2)_+^{\gamma_2} ds$  by its symmetrized version. The process  $Z_{\gamma_1, \gamma_2}(t)$  remains invariant under such a modification.

The goal of this paper is to study the distributional behavior of the standardized  $Z_{\gamma_1, \gamma_2}(t)$  (where  $A$  in (6) is chosen so that  $\text{Var}[Z_{\gamma_1, \gamma_2}(1)] = 1$ ), as  $(\gamma_1, \gamma_2)$  approaches the boundaries of the region  $\Delta$  defined in (2).

We show that on the diagonal boundary  $d$ , the limit is Brownian motion. On each of the two symmetric boundaries  $e_1$  and  $e_2$  of  $\Delta$ , the limit is non-Gaussian: it is a fractional Brownian motion times an independent Gaussian random variable. We give two different proofs of this convergence, one based on the method of moments, and one which provides more intuitive insight. We also give the rate of convergence to the marginal distribution in the preceding two cases.

The situation at the corners is particularly delicate. At the corner  $(\gamma_1, \gamma_2) = (-1/2, -1/2)$ , the limit process is a linear combination of two independent degenerate chi-square processes. At the other two corners, the limit is a linear combination of two processes: a Brownian motion and the product of another Brownian motion times an independent Gaussian random variable. These linear combinations, which depend on the direction at which the critical exponents approach the corners, will be given explicitly.

We also show that the convergences mentioned cannot be strengthened from weak convergence to  $L^2(\Omega)$  convergence, nor even to convergence in probability.

The paper is organized as follows. In Section 2, we state the main results with proofs in Section 3. In the following three sections, we provide some additional results: showing that  $L^2(\Omega)$  convergence cannot hold, establishing the rate of marginal convergence on the boundaries  $d$ ,  $e_1$  and  $e_2$ , and giving an alternate proof of the convergence on the boundaries  $e_1$  and  $e_2$ .

**2. Main results.** In the following theorems, we let  $\Rightarrow$  denote weak convergence in the space  $C[0, 1]$  with uniform metric. The multiplicative factor  $A$  in (6) is chosen so that  $\text{Var}[Z_{\gamma_1, \gamma_2}(1)] = 1$ . See (21) below for an explicit expression.

We focus first on results concerning the behavior of  $Z_{\gamma_1, \gamma_2}(t)$  as  $(\gamma_1, \gamma_2)$  approaches the boundary of  $\Delta$  in (2), excluding the corners. Theorem 2.1 involves

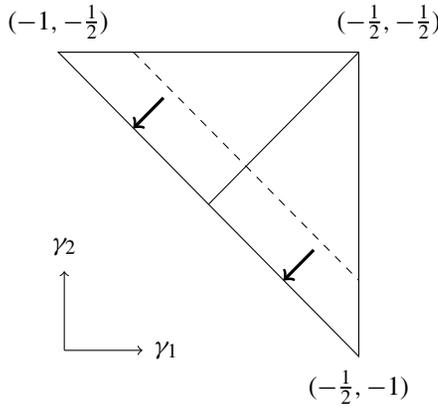


FIG. 2. Illustration of limit taking in Theorem 2.1.

convergence to the diagonal edge  $d$  of  $\Delta$ , where the limit is Brownian motion. See Figure 2.

**THEOREM 2.1.** *Let  $Z_{\gamma_1, \gamma_2}(t)$ ,  $(\gamma_1, \gamma_2) \in \Delta$ , be defined in (6) with  $A = A(\gamma_1, \gamma_2)$  in (21). When  $\gamma_1 + \gamma_2 \rightarrow -3/2$  with  $\gamma_1, \gamma_2 > -1 + \varepsilon$  for arbitrarily fixed  $\varepsilon > 0$ , we have*

$$(7) \quad Z_{\gamma_1, \gamma_2}(t) \Rightarrow B(t),$$

where  $B(t)$  is a standard Brownian motion.

One has  $\gamma_1 + \gamma_2 = -3/2$  all through the diagonal  $d$ . The corners of the triangle are excluded by the requirement  $\gamma_1, \gamma_2 > -1 + \varepsilon$ . Convergence to Brownian motion in (7) is expected heuristically since the self-similarity parameter  $H = \gamma_1 + \gamma_2 + 2 \rightarrow 1/2$  [see (5)], and  $1/2$  is the self-similarity parameter of Brownian motion.

The next Theorem 2.2 involves convergence to either one of the two sides  $e_1$  and  $e_2$  of  $\Delta$ . The vertical side  $e_1$  and the horizontal side  $e_2$  are parameterized respectively by  $(-1/2, \gamma)$  and  $(\gamma, -1/2)$  where  $-1 < \gamma < -1/2$ . See Figure 3.

**THEOREM 2.2.** *Let  $Z_{\gamma_1, \gamma_2}(t)$ ,  $(\gamma_1, \gamma_2) \in \Delta$ , be defined in (6) with  $A = A(\gamma_1, \gamma_2)$  in (21). When  $(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)$  or  $(\gamma_1, \gamma_2) \rightarrow (\gamma, -1/2)$ , where  $-1 < \gamma < -1/2$ , we have*

$$(8) \quad Z_{\gamma_1, \gamma_2}(t) \Rightarrow WB_{\gamma+3/2}(t),$$

where  $B_{\gamma+3/2}(t)$  is a standard fractional Brownian motion with self-similarity parameter  $\gamma + 3/2$ , and  $W$  is a standard normal random variable which is independent of  $B_{\gamma+3/2}(t)$ .

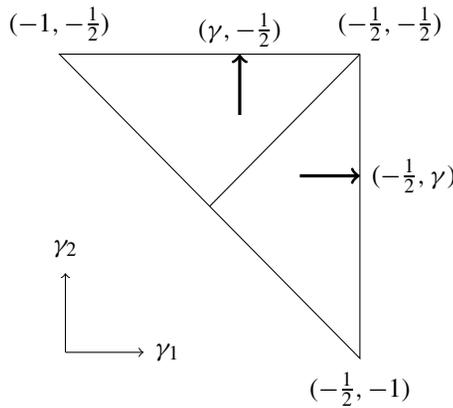


FIG. 3. Illustration of limit taking in Theorem 2.2.

REMARK 2.1. The convergence (8) is more involved since  $WB_{\gamma+3/2}(t)$  is a self-similar process with stationary increments having self-similarity parameter  $H = \gamma + 3/2 \in (1/2, 1)$ , and hence displays long-range dependence. This convergence may be understood heuristically as follows:  $Z_{\gamma_1, \gamma_2}(t)$  in (1) can be regarded as an integrated process of a long-range dependent bilinear moving average of white noise. This bilinear moving average involves a double summation. As the exponent  $\gamma_1 \rightarrow -1/2$ , the corresponding summation yields a term which is extremely persistent, so that it behaves like a frozen Gaussian variable which is independent of the fractional noise defined through the other summation.

REMARK 2.2. Although intuitively the generalized Rosenblatt processes  $Z_{\gamma_1, \gamma_2}(t)$  in (1) form a richer class than the Rosenblatt process  $Z_\gamma(t)$  in (3), they are both self-similar with stationary increments, and hence have the same covariance (4) when  $2\gamma = \gamma_1 + \gamma_2$ . To show that they are different processes, one can compare the higher moments, as was done in Bai and Taquq [4]. The convergence (8) provides another evidence that there are values of  $(\gamma_1, \gamma_2)$  for which  $Z_{\gamma_1, \gamma_2}(t)$  is different from  $Z_\gamma(t)$ . Indeed the limit  $WB_{\gamma+3/2}(t)$  has a symmetric marginal distribution (the so-called product-normal distribution), while the marginal distribution of the Rosenblatt process  $Z_\gamma(t)$  is skewed with a nonzero third cumulant (see (10) and (12) of Veillette and Taquq [33], or set  $\gamma_1 = \gamma_2 = \gamma$  in (20) below).

Note that in Theorems 2.1 and 2.2, we exclude the three corners  $(\gamma_1, \gamma_2) = (-1/2, -1/2)$ ,  $(-1, -1/2)$  and  $(-1/2, -1)$ . It turns out that the limit behavior of  $Z_{\gamma_1, \gamma_2}(t)$  at these corners depends on the direction these corners are approached. Due to the symmetry of  $Z_{\gamma_1, \gamma_2}(t)$  in  $(\gamma_1, \gamma_2)$ , it is sufficient to focus on the case  $\gamma_1 \geq \gamma_2$ , that is, we focus on the subregion of  $\Delta$  in (2) delimited by line segments  $e_1, d$  and  $m$  in Figure 4.

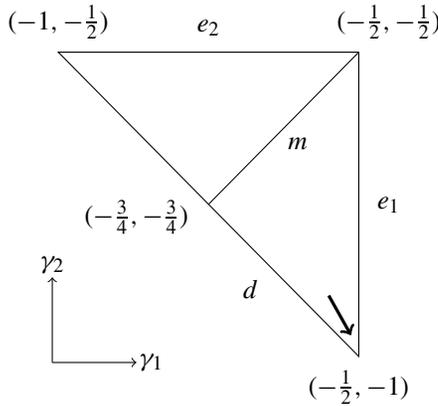


FIG. 4. Illustration of limit taking in Theorem 2.3.

Consider first the corner  $(\gamma_1, \gamma_2) = (-1/2, -1)$ . We will approach it through the line

$$\gamma_2 = \frac{1}{\rho - 1}(\gamma_1 + 1/2) - 1,$$

which can also be expressed as

$$\frac{\gamma_1 + \gamma_2 + 3/2}{\gamma_2 + 1} = \rho.$$

The line passes through the corner  $(-1/2, -1)$  and has a negative slope of  $1/(\rho - 1)$ ,  $0 \leq \rho \leq 1$ . See Figure 4. When  $\rho = 0$ , the line coincides with the diagonal edge  $d$  of the triangle  $\Delta$ , which has slope  $-1$ . When  $\rho = 1$ , the line coincides with the vertical side  $e_1$  of  $\Delta$ , which has slope  $-\infty$ .

**THEOREM 2.3** [The corner  $(\gamma_1, \gamma_2) = (-1/2, -1)$ ]. *Let  $Z_{\gamma_1, \gamma_2}(t)$ ,  $(\gamma_1, \gamma_2) \in \Delta$ , be defined in (6) with  $A = A(\gamma_1, \gamma_2)$  in (21). Suppose that  $\gamma_1 \geq \gamma_2$ . If  $(\gamma_1, \gamma_2) \rightarrow (-1/2, -1)$  in such a way that*

$$(9) \quad \frac{\gamma_1 + \gamma_2 + 3/2}{\gamma_2 + 1} = 1 + \frac{\gamma_1 + 1/2}{\gamma_2 + 1} \rightarrow \rho \in [0, 1],$$

then

$$(10) \quad Z_{\gamma_1, \gamma_2}(t) \Rightarrow X_\rho(t) := \rho^{1/2}WB(t) + (1 - \rho)^{1/2}B'(t),$$

where  $W$  is a standard normal random variable,  $B(t)$  and  $B'(t)$  are standard Brownian motions, and  $W, B(t)$  and  $B'(t)$  are independent.

**REMARK 2.3.** In Theorem 2.3, the limit  $X_\rho(t)$  is an independent linear combination of the two limits obtained in Theorems 2.2 and 2.1 (edges  $e_1$  and  $d$ ), after

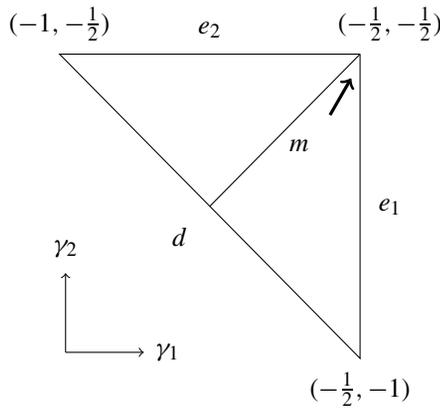


FIG. 5. Illustration of limit taking in Theorem 2.4.

setting  $\gamma = -1$  in Theorem 2.2. Note that since  $\gamma + 3/2 = -1 + 3/2 = 1/2$ , the fractional Brownian motion  $B_{\gamma+3/2}(t)$  in Theorem 2.2 becomes Brownian motion  $B(t)$ .

Consider now the corner  $(\gamma_1, \gamma_2) = (-1/2, -1/2)$ . We will approach it through the line

$$\gamma_2 = \frac{1}{\rho}(\gamma_1 + 1/2) - 1/2,$$

which passes through it and has a positive slope of  $1/\rho$ ,  $0 \leq \rho \leq 1$ . See Figure 5. When  $\rho = 0$ , the line coincides with the vertical side  $e_1$  of  $\Delta$ , which has slope  $+\infty$ . When  $\rho = 1$ , the line coincides with the middle line  $m$ , which has slope 1.

**THEOREM 2.4** [The corner  $(\gamma_1, \gamma_2) = (-1/2, -1/2)$ ]. *Let  $Z_{\gamma_1, \gamma_2}(t)$ ,  $(\gamma_1, \gamma_2) \in \Delta$ , be defined in (6) with  $A = A(\gamma_1, \gamma_2)$  in (21). Suppose that  $\gamma_1 \geq \gamma_2$ . If  $(\gamma_1, \gamma_2) \rightarrow (-1/2, -1/2)$  in such a way that*

$$(11) \quad \frac{\gamma_1 + 1/2}{\gamma_2 + 1/2} \rightarrow \rho \in [0, 1],$$

then

$$(12) \quad \begin{aligned} Z_{\gamma_1, \gamma_2}(t) &\Rightarrow Y_\rho(t) \\ &= t \cdot \left[ \frac{(\rho + 1)^{-1} + (2\sqrt{\rho})^{-1}}{\sqrt{2(\rho + 1)^{-2} + (2\rho)^{-1}}} \cdot X_1 \right. \\ &\quad \left. + \frac{(\rho + 1)^{-1} - (2\sqrt{\rho})^{-1}}{\sqrt{2(\rho + 1)^{-2} + (2\rho)^{-1}}} \cdot X_2 \right], \end{aligned}$$

where  $X_1$  and  $X_2$  two independent standardized chi-squared random variables with one degree of freedom (with mean 0 and variance 1). The case  $\rho = 0$  is understood as the limit as  $\rho \rightarrow 0$ .

REMARK 2.4. Since by (5), the self-similarity parameter  $H$  equals  $\gamma_1 + \gamma_2 + 2$ , we get that  $H$  tends to 1 as  $(\gamma_1, \gamma_2) \rightarrow (-1/2, -1/2)$ . It is known (see, e.g., Theorem 3.1.1 of Embrechts and Maejima [12]) that the only self-similar finite-variance processes with stationary increments having  $H = 1$  are degenerate processes. We see this in Theorem 2.4, where the limit is a random variable multiplied by  $t$ .

REMARK 2.5. In Theorem 2.4, if  $\rho = 1$ ,  $Y_\rho(t)$  reduces to  $tX_1$ , where  $X_1$  is a standardized chi-squared random variable with one degree of freedom. Consider now the standardized Rosenblatt process  $Z_\gamma(t)$  in (3). In this case,  $\gamma_1 = \gamma_2 = \gamma$ , and thus  $\rho = 1$ , which corresponds to the middle line  $m$  in Figure 5. From Theorem 2.4, we conclude that if  $\gamma \rightarrow -1/2$ , then the limit is  $tX_1$ . This is consistent with a previous result of Veillette and Taqqu [33], that the limit is a standardized chi-squared random variable when  $t = 1$ .

REMARK 2.6. If  $\rho = 0$ ,  $Y_\rho(t) = \frac{t}{\sqrt{2}}(X_1 - X_2)$ , which has the same distribution as  $t(WB)$ , where  $W$  and  $B$  are two independent standard normal random variables [see (31) below]. This is consistent with Theorem 2.2, where on the edge  $e_1$  the limit is  $WB_{\gamma+3/2}$ . This tends, as  $\gamma \rightarrow -1/2$ , to  $W \cdot B_1(t) = W \cdot B \cdot t = t(WB)$ , where  $B$  is a standard Gaussian random variable.

REMARK 2.7. Theorems 2.1 to 2.4 are consistent with Theorem 3.1 of Nourdin and Poly [22], stating that the limit of a double Wiener–Itô integral can only be a linear combination of a normal and an independent double Wiener–Itô integral.

REMARK 2.8. Theorems 2.3 and 2.4 concern the limit behavior of  $Z_{\gamma_1, \gamma_2}(t)$  as  $(\gamma_1, \gamma_2)$  approaches the corners along some straight-line direction. What happens if one does not approach the corners following a straight-line direction? Then there will be no convergence. To see this, consider the case of Theorem 2.3 (a similar argument can be made for Theorem 2.4). Let

$$\rho(\gamma_1, \gamma_2) = \frac{\gamma_1 + \gamma_2 + 3/2}{\gamma_2 + 1} \in (0, 1)$$

parameterize the straight-line direction. Suppose that  $\rho(\gamma_1, \gamma_2)$  does not converge as  $(\gamma_1, \gamma_2)$  approaches the corner  $(-\frac{1}{2}, -1)$ . Then there are two subsequences of  $(\gamma_1, \gamma_2)$ , such that  $\rho(\gamma_1, \gamma_2)$  of the first subsequence converges to  $\rho_1$  and  $\rho(\gamma_1, \gamma_2)$  of the second subsequence converges to  $\rho_2$ , with  $\rho_1 \neq \rho_2$ . By Theorem 2.3, the corresponding processes  $Z_{\gamma_1, \gamma_2}(t)$  converge to two different limits. Therefore, the original process  $Z_{\gamma_1, \gamma_2}(t)$  does not converge if  $(\gamma_1, \gamma_2)$  does not follow a straight-line direction.

**3. Proof of the main theorems.** Since we will use a method of moments, we state first a cumulant formula for a linear combination of  $Z_{\gamma_1, \gamma_2}(t)$  at finite time points. We let  $\kappa_m(\cdot)$  denote the  $m$ th cumulant. In the following proposition, the constant  $A$  in (6) is arbitrary.

**PROPOSITION 3.1.** *The  $m$ th cumulant ( $m \geq 2$ ) of  $\sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i)$ ,  $c_i \in \mathbb{R}$ ,  $t_i \in [0, \infty)$ , equals*

$$(13) \quad \kappa_m \left( \sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i) \right) = \frac{1}{2} (m-1)! A^m C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c}),$$

where

$$(14) \quad \begin{aligned} & C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c}) \\ &= \sum_{\sigma \in \{1, 2\}^m} \sum_{i_1, \dots, i_m=1}^n c_{i_1} \cdots c_{i_m} \int_0^{t_{i_1}} ds_1 \cdots \int_0^{t_{i_m}} ds_m \\ & \times \prod_{j=1}^m \left[ (s_j - s_{j-1})_+^{\gamma_{\sigma_j} + \gamma_{\sigma'_{j-1}} + 1} \mathbf{B}(\gamma_{\sigma'_{j-1}} + 1, -\gamma_{\sigma_j} - \gamma_{\sigma'_{j-1}} - 1) \right. \\ & \left. + (s_{j-1} - s_j)_+^{\gamma_{\sigma_j} + \gamma_{\sigma'_{j-1}} + 1} \mathbf{B}(\gamma_{\sigma_j} + 1, -\gamma_{\sigma_j} - \gamma_{\sigma'_{j-1}} - 1) \right], \end{aligned}$$

where

$$(15) \quad \mathbf{B}(x, y) = \int_0^1 u^{x-1} (1-u)^{y-1} du = \int_0^\infty w^{x-1} (1+w)^{-x-y} dw, \quad x, y > 0,$$

is the beta function, the sum runs over  $\sigma = (\sigma_1, \dots, \sigma_m)$  with  $\sigma_i = 1$  or  $2$ , and  $\sigma'$  is the complement of  $\sigma$ , namely,  $\sigma'_i = 1$  if  $\sigma_i = 2$  and  $\sigma'_i = 2$  if  $\sigma_i = 1$ ,  $i = 1, \dots, m$ . Moreover,  $\sigma'_0 = \sigma'_m$  and  $s_0 = s_m$ ,  $i = 1, \dots, m$ .

Proposition 3.1 is an extension of Theorem 2.1 of Bai and Taqqu [4]. We shall use the following cumulant formula for a double Wiener–Itô integral (see, e.g., (8.4.3) of Nourdin and Peccati [20]).

**LEMMA 3.1.** *If  $f$  is a symmetric function in  $L^2(\mathbb{R}^2)$ , then the  $m$ th cumulant of the double Wiener–Itô integral  $X = \int_{\mathbb{R}^2}' f(y_1, y_2) \mathbf{B}(dy_1) \mathbf{B}(dy_2)$  is given by the following circular integral:*

$$\begin{aligned} \kappa_m(X) &= 2^{m-1} (m-1)! \int_{\mathbb{R}^m} f(y_1, y_2) f(y_2, y_3) \cdots \\ & \times f(y_{m-1}, y_m) f(y_m, y_1) dy_1 \cdots dy_m. \end{aligned}$$

**PROOF OF PROPOSITION 3.1.** Set

$$g(x, y) = \frac{A}{2} (x_+^{\gamma_1} y_+^{\gamma_2} + x_+^{\gamma_2} y_+^{\gamma_1}).$$

Let

$$h_t(x, y) = \int_0^t g(s - x, s - y) ds,$$

and observe that  $h_t$  is symmetric. So using the linearity of the Wiener–Itô integral and Lemma 3.1, we have

$$\begin{aligned} \kappa_m \left( \sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i) \right) &= \kappa_m \left( \int_{\mathbb{R}^2} \sum_{i=1}^n c_i h_{t_i}(x_1, x_2) \mathbf{B}(dx_1) \mathbf{B}(dx_2) \right) \\ &= 2^{m-1} (m-1)! \int_{\mathbb{R}^m} d\mathbf{x} \prod_{j=1}^m \left[ \sum_{i=1}^n c_i h_{t_i}(x_j, x_{j+1}) \right] \\ &= 2^{m-1} (m-1)! \sum_{i_1, \dots, i_m=1}^n c_{i_1} \cdots c_{i_m} \int_{\mathbb{R}^m} d\mathbf{x} \\ &\quad \times \prod_{j=1}^m \int_0^{t_{i_j}} g(s_j - x_j, s_j - x_{j+1}) ds_j, \end{aligned}$$

and hence

$$\begin{aligned} \kappa_m \left( \sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i) \right) &= \frac{1}{2} (m-1)! A^m \sum_{i_1, \dots, i_m=1}^n c_{i_1} \cdots c_{i_m} \\ (16) \quad &\times \int_0^{t_{i_1}} ds_1 \cdots \int_0^{t_{i_m}} ds_m \left( \int_{\mathbb{R}^m} \prod_{j=1}^m [(s_j - x_j)_+^{\gamma_1} (s_j - x_{j+1})_+^{\gamma_2} \right. \\ &\quad \left. + (s_j - x_j)_+^{\gamma_2} (s_j - x_{j+1})_+^{\gamma_1}] d\mathbf{x} \right), \end{aligned}$$

where we view the index  $j$  as modulo  $m$ , for example,  $x_{m+1} = x_1$ .

Then using the notation in the statement of Proposition 3.1, one has

$$\begin{aligned} I &:= \int_{\mathbb{R}^m} \prod_{j=1}^m [(s_j - x_j)_+^{\gamma_1} (s_j - x_{j+1})_+^{\gamma_2} + (s_j - x_j)_+^{\gamma_2} (s_j - x_{j+1})_+^{\gamma_1}] d\mathbf{x} \\ &= \sum_{\sigma \in \{1, 2\}^m} \int_{\mathbb{R}^m} \prod_{j=1}^m (s_j - x_j)_+^{\gamma_{\sigma_j}} (s_j - x_{j+1})_+^{\gamma_{\sigma'_j}} d\mathbf{x} \\ &= \sum_{\sigma \in \{1, 2\}^m} \int_{\mathbb{R}^m} \prod_{j=1}^m (s_j - x_j)_+^{\gamma_{\sigma_j}} (s_{j-1} - x_j)_+^{\gamma_{\sigma'_{j-1}}} d\mathbf{x}, \end{aligned}$$

and thus

$$\begin{aligned}
 (17) \quad I &= \sum_{\sigma \in \{1,2\}^m} \prod_{j=1}^m [(s_j - s_{j-1})_+^{\gamma_{\sigma_j} + \gamma_{\sigma'_{j-1}} + 1} \\
 &\quad \times \mathbf{B}(\gamma_{\sigma'_{j-1}} + 1, -\gamma_{\sigma_j} - \gamma_{\sigma'_{j-1}} - 1) \\
 &\quad + (s_{j-1} - s_j)_+^{\gamma_{\sigma_j} + \gamma_{\sigma'_{j-1}} + 1} \mathbf{B}(\gamma_{\sigma_j} + 1, -\gamma_{\sigma_j} - \gamma_{\sigma'_{j-1}} - 1)],
 \end{aligned}$$

where we have used the following relation valid for  $a, b \in (-1, -1/2)$ :

$$\begin{aligned}
 (18) \quad \int_{\mathbb{R}} (s_1 - u)_+^a (s_2 - u)_+^b du \\
 &= (s_2 - s_1)_+^{a+b+1} \mathbf{B}(a + 1, -a - b - 1) \\
 &\quad + (s_1 - s_2)_+^{a+b+1} \mathbf{B}(b + 1, -a - b - 1).
 \end{aligned}$$

(See Lemma 3.2 of Bai and Taqqu [4].) Substituting (17) into (16), equation (13) is obtained.  $\square$

Note that  $\mathbb{E}Z_{\gamma_1, \gamma_2}(1) = 0$  by the property of Wiener–Itô integral, and hence the second and the third moments coincide with the second and the third cumulants. As two special cases of Proposition 3.1, one has the following explicit formulas for the second and the third moment of the generalized Rosenblatt distribution (Bai and Taqqu [4], Theorem 2.1).

The second moment of  $Z_{\gamma_1, \gamma_2}(1)$  is

$$\begin{aligned}
 (19) \quad \mu_2(\gamma_1, \gamma_2) &= \frac{A^2}{(\gamma_1 + \gamma_2 + 2)(2(\gamma_1 + \gamma_2) + 3)} \\
 &\quad \times [\mathbf{B}(\gamma_1 + 1, -\gamma_1 - \gamma_2 - 1)\mathbf{B}(\gamma_2 + 1, -\gamma_1 - \gamma_2 - 1) \\
 &\quad + \mathbf{B}(\gamma_1 + 1, -2\gamma_1 - 1)\mathbf{B}(\gamma_2 + 1, -2\gamma_2 - 1)].
 \end{aligned}$$

The third moment of  $Z_{\gamma_1, \gamma_2}(1)$  is

$$\begin{aligned}
 (20) \quad \mu_3(\gamma_1, \gamma_2) &= \frac{2A^3}{(\gamma_1 + \gamma_2 + 2)(3(\gamma_1 + \gamma_2) + 5)} \\
 &\quad \times \left[ \sum_{\sigma \in \{1,2\}^3} \mathbf{B}(\gamma_{\sigma_1} + 1, -\gamma_{\sigma_1} - \gamma_{\sigma'_3} - 1)\mathbf{B}(\gamma_{\sigma'_1} + 1, -\gamma_{\sigma'_1} - \gamma_{\sigma_2} - 1) \right. \\
 &\quad \left. \times \mathbf{B}(\gamma_{\sigma'_2} + 1, -\gamma_{\sigma'_2} - \gamma_{\sigma_3} - 1)\mathbf{B}(\gamma_{\sigma'_1} + \gamma_{\sigma_2} + 2, \gamma_{\sigma'_2} + \gamma_{\sigma_3} + 2) \right].
 \end{aligned}$$

To standardize  $Z_{\gamma_1, \gamma_2}(t)$ , we set  $\mu_2(\gamma_1, \gamma_2) = 1$ . By (19), this determines the constant  $A$  as

$$\begin{aligned}
 A(\gamma_1, \gamma_2) &= [(\gamma_1 + \gamma_2 + 2)(2(\gamma_1 + \gamma_2) + 3)]^{1/2} \\
 (21) \quad &\times [\mathbf{B}(\gamma_1 + 1, -\gamma_1 - \gamma_2 - 1)\mathbf{B}(\gamma_2 + 1, -\gamma_1 - \gamma_2 - 1) \\
 &+ \mathbf{B}(\gamma_1 + 1, -2\gamma_1 - 1)\mathbf{B}(\gamma_2 + 1, -2\gamma_2 - 1)]^{-1/2}.
 \end{aligned}$$

3.1. *Proof of Theorem 2.1.* We will use a result for bounding integral of powers of linear functions in Euclidean space. First, some notation. Let  $L_1(\mathbf{s}) = \langle \mathbf{w}_1, \mathbf{s} \rangle, \dots, L_m(\mathbf{s}) = \langle \mathbf{w}_m, \mathbf{s} \rangle$  be linear functions on  $\mathbb{R}^n$ , where  $\langle \cdot, \cdot \rangle$  denotes the Euclidean inner product. Let

$$P(\mathbf{s}) = \prod_{j=1}^m |L_j(\mathbf{s})|^{\alpha_j}.$$

Set  $T = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ . For any nonempty  $W \subset T$ , define

$$(22) \quad S(W) = T \cap \text{span}\{W\},$$

where  $\text{span}\{W\}$  denotes linear subspace spanned by  $W$ , and define the quantity

$$d(P, W) = |W| + \sum_{j: \mathbf{w}_j \in S(W)} \alpha_j,$$

where  $|W|$  is the cardinality of the set  $W$ . Then we have the following so-called power counting lemma.

LEMMA 3.2 (Theorem 3.1 of Fox and Taquq [13]). *Suppose that*

$$(23) \quad d(P, W) > 0$$

*for any  $W \subset T$  which consists of linearly independent  $\mathbf{w}_j$ 's.<sup>2</sup> Then*

$$\int_{[0,1]^n} P(\mathbf{s}) \, d\mathbf{s} < \infty.$$

LEMMA 3.3. *The function*

$$(24) \quad f(\alpha_1, \dots, \alpha_m) := \int_{[0,1]^m} |s_1 - s_m|^{\alpha_1} |s_2 - s_1|^{\alpha_2} \dots |s_m - s_{m-1}|^{\alpha_m} \, d\mathbf{s}$$

*is finite and continuous on the domain*

$$(25) \quad D = \left\{ (\alpha_1, \dots, \alpha_m) : \alpha_i > -1, \sum_{i=1}^m \alpha_i + m > 1 \right\}.$$

---

<sup>2</sup>Theorem 3.1 of Fox and Taquq [13] states that it is enough to consider  $W \subset T$  consisting of linearly independent  $\mathbf{w}_j$ 's with *negative* exponent  $\alpha_j$ 's. This is because the nonnegative exponents  $\alpha_j$  cannot make the integral  $\int_{[0,1]^n} P(\mathbf{s}) \, d\mathbf{s}$  blow up.

PROOF. We first show that  $f(\alpha_1, \dots, \alpha_m) < \infty$  on  $D$  using Lemma 3.2. Following the notation introduced for the lemma, we have  $L_1(\mathbf{s}) = s_1 - s_m$ ,  $L_2(\mathbf{s}) = s_2 - s_1, \dots, L_m(\mathbf{s}) = s_m - s_{m-1}$ , and hence  $\mathbf{w}_1 = (1, 0, \dots, 0, -1)$ ,  $\mathbf{w}_2 = (-1, 1, 0, \dots, 0), \dots, \mathbf{w}_m = (0, \dots, 0, -1, 1)$  and  $T = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ .

It is easy to see that a subset  $W \subset T$  consists of linearly independent  $\mathbf{w}_j$ 's if and only if  $|W| \leq m - 1$ . When  $|W| \leq m - 2$ , the set  $S(W)$  defined in (22) is equal to  $W$ . The condition (23) is satisfied in this case because each  $\alpha_j > -1$ , and hence

$$D(P, W) = |W| + \sum_{j:\mathbf{w}_j \in S(W)} \alpha_j > |W| + \sum_{j:\mathbf{w}_j \in W} (-1) = |W| - |W| = 0.$$

When  $|W| = m - 1$ , one has  $\text{span}(W) = T$ , and hence  $S(W) = T$ . Thus, the condition (23) in this case becomes

$$D(P, W) = m - 1 + \sum_{i=1}^m \alpha_i > 0,$$

which is satisfied in view of (25). Hence, the integral  $f(\alpha_1, \dots, \alpha_m)$  in (24) is finite by Lemma 3.2.

To verify the continuity of  $f(\alpha_1, \dots, \alpha_m)$ , suppose that as  $n \rightarrow \infty$ ,  $\alpha_n \rightarrow \alpha := (\alpha_1, \dots, \alpha_m)$ . Then for large  $n$ ,  $\alpha_n \geq \alpha_\varepsilon := (\alpha_1 - \varepsilon, \dots, \alpha_m - \varepsilon)$ , where the small  $\varepsilon$  is chosen such that  $\alpha_\varepsilon \in D$ . Denote the integrand in (24) by  $I(\mathbf{s}; \alpha)$ , and recall that  $I(\mathbf{s}; \alpha)$  is decreasing in every component of  $\alpha$ . Hence, when  $n$  is large,  $I(\mathbf{s}; \alpha_n) \leq I(\mathbf{s}; \alpha_\varepsilon)$ . Since  $I(\mathbf{s}; \alpha_\varepsilon)$  is integrable, we can apply the dominated convergence theorem to obtain the convergence  $f(\alpha_n) \rightarrow f(\alpha)$  as  $n \rightarrow \infty$ , proving the continuity.  $\square$

In the following corollary, the exponents are supposed to be away from the boundary of the set  $D$  defined in (25).

COROLLARY 3.1. *Let  $C_1, C_2$  be two fixed constants such that  $C_1 > -1$  and  $C_2 > 1$ . Then the function  $f(\alpha_1, \dots, \alpha_m)$  defined in (24) is bounded on the domain*

$$D(C_1, C_2) = \left\{ (\alpha_1, \dots, \alpha_m) : \alpha_i \geq C_1, \sum_{i=1}^m \alpha_i + m \geq C_2 \right\}.$$

PROOF. Let  $M$  be a large positive constant. Define

$$\begin{aligned} D_M(C_1, C_2) &= D(C_1, C_2) \cap (-\infty, M]^m \\ &= \left\{ (\alpha_1, \dots, \alpha_m) : C_1 \leq \alpha_i \leq M, \sum_{i=1}^m \alpha_i + m \geq C_2 \right\}. \end{aligned}$$

Since  $D_M(C_1, C_2)$  is a compact subset of  $D$  in (25), and  $f(\alpha_1, \dots, \alpha_m)$  is continuous on  $D$  by Lemma 3.3, we deduce that  $f$  is bounded on  $D_M(C_1, C_2)$ . The boundedness on  $D(C_1, C_2)$  follows since  $f$  decreases when any  $\alpha_i$  increases.  $\square$

LEMMA 3.4. *Let  $A(\gamma_1, \gamma_2)$  be as in (21), where  $(\gamma_1, \gamma_2) \in \mathbf{\Delta}$  which is defined in (2). Then there exists a constant  $C > 0$  independent of  $\gamma_1$  and  $\gamma_2$  such that*

$$|A(\gamma_1, \gamma_2)| \leq C[2(\gamma_1 + \gamma_2) + 3]^{1/2}.$$

PROOF. This is immediate by noting that the beta function  $B(x, y)$  defined in (15) is decreasing in  $x$  and in  $y$ . Since in addition  $\mathbf{\Delta}$  is a bounded region, the beta functions in (21) are bounded from below, and hence the factor with negative power  $-1/2$  in (21) is bounded from above.  $\square$

The following hypercontractivity inequality for multiple Wiener–Itô integral (see, e.g., Corollary 5.6 of Major [19] or Theorem 2.7.2 of Nourdin and Peccati [20]) is useful.

LEMMA 3.5. *For any  $m \in \mathbb{Z}_+$ , there exists a constant  $C_m > 0$ , such that*

$$\mathbb{E}|I_k(f)|^{2m} \leq C_m(\mathbb{E}|I_k(f)|^2)^m \quad \text{for all } f \in L^2(\mathbb{R}^k).$$

Tightness of standardized  $Z_{\gamma_1, \gamma_2}(t)$  in  $C[0, 1]$  will follow from the following lemma.

LEMMA 3.6. *Let  $Z_{\gamma_1, \gamma_2}(t)$  be as in (6) with  $A$  as in (21) and  $(\gamma_1, \gamma_2)$  in the region  $\mathbf{\Delta}$  defined in (2). Then there exists a constant  $C > 0$  which does not depend on  $\gamma_1, \gamma_2$ , such that for all  $0 \leq s \leq t \leq 1$ ,*

$$\mathbb{E}|Z_{\gamma_1, \gamma_2}(t) - Z_{\gamma_1, \gamma_2}(s)|^4 \leq C(t - s)^2,$$

which implies that the law of  $\{Z_{\gamma_1, \gamma_2}(t) : (\gamma_1, \gamma_2) \in \mathbf{\Delta}\}$  is tight in  $C[0, 1]$ .

PROOF. Using Lemma 3.5, self-similarity and stationary-increment property of  $Z_{\gamma_1, \gamma_2}(t)$ , one has

$$\begin{aligned} \mathbb{E}|Z_{\gamma_1, \gamma_2}(t) - Z_{\gamma_1, \gamma_2}(s)|^4 &\leq C_2(\mathbb{E}|Z_{\gamma_1, \gamma_2}(t) - Z_{\gamma_1, \gamma_2}(s)|^2)^2 \\ &= C_2(t - s)^{4H} \leq C_2(t - s)^2, \end{aligned}$$

where  $H := \gamma_1 + \gamma_2 + 2 \geq 1/2$  and  $0 \leq t - s \leq 1$ . So  $Z_{\gamma_1, \gamma_2}(t)$  by Kolmogorov’s criterion admits a continuous version. Tightness follows from, for example, Prokhorov [28], Lemma 2.2.  $\square$

We now prove Theorem 2.1. By Lemma 3.6, tightness in  $C[0, 1]$  holds. We are left to show convergence of finite-dimensional distributions ( $\xrightarrow{\text{f.d.d.}}$ ). From here on, we let  $C$  and  $c$  denote constants whose values can change from line to line.

PROOF OF  $\xrightarrow{\text{f.d.d.}}$  THEOREM 2.1. Due to self-similarity and stationary increments, the covariance of the standardized  $Z_{\gamma_1, \gamma_2}(t)$  is

$$\mathbb{E}Z_{\gamma_1, \gamma_2}(s)Z_{\gamma_1, \gamma_2}(t) = \frac{1}{2}(s^{2\gamma_1+2\gamma_2+4} + t^{2\gamma_1+2\gamma_2+4} - |s-t|^{2\gamma_1+2\gamma_2+4}), \quad t, s \geq 0,$$

which converges to the Brownian motion covariance  $\mathbb{E}B(s)B(t) = s \wedge t = \frac{1}{2}(s + t - |s - t|)$  as  $\gamma_1 + \gamma_2 \rightarrow -3/2$ . By using the method of moments, it is sufficient to show that

$$(26) \quad \kappa_m \left( \sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i) \right) \rightarrow 0, \quad m \geq 3.$$

As  $\gamma_1 + \gamma_2 \rightarrow -3/2$ , the factor  $A(\gamma_1, \gamma_2)$  in (21) converges to zero by Lemma 3.4. It is therefore sufficient to show that for  $m \geq 3$ , and  $\gamma_1, \gamma_2 > -1 + \varepsilon$ , the factor  $C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c})$  in (14) is bounded.

Under the constraints  $\gamma_1 + \gamma_2 \geq -3/2$  and  $\gamma_1, \gamma_2 > -1 + \varepsilon$  (or equivalently  $\gamma_1, \gamma_2 < -1/2 - \varepsilon$ ), the factors  $B(\gamma_{\sigma'_{j-1}} + 1, -\gamma_{\sigma_j} - \gamma_{\sigma'_{j-1}} - 1)$  and  $B(\gamma_{\sigma_j} + 1, -\gamma_{\sigma_j} - \gamma_{\sigma'_{j-1}} - 1)$  are bounded by a constant  $C > 0$  for any  $\sigma$  and  $j$ . This is because the beta function  $B(x, y)$  defined in (15) is bounded if both  $x$  and  $y$  stay away from a neighborhood of 0. Choosing  $T \geq \max(t_1, \dots, t_n)$ , one then has

$$\begin{aligned} |C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c})| &\leq C \sum_{\sigma \in \{1, 2\}^m} \int_{[0, T]^m} ds \prod_{j=1}^m |s_j - s_{j-1}|^{\gamma_{\sigma_j} + \gamma_{\sigma'_{j-1}} + 1} \\ &\leq C \sum_{\sigma \in \{1, 2\}^m} \int_{[0, 1]^m} ds \prod_{j=1}^m |s_j - s_{j-1}|^{\gamma_{\sigma_j} + \gamma_{\sigma'_{j-1}} + 1}, \end{aligned}$$

where the last constant  $C$  depends on  $T, m$  and  $\varepsilon$ .

We now want to apply Corollary 3.1 to establish the boundedness of each of the term in the preceding sum. Using the notation in Lemma 3.3, we set

$$\alpha_j = \gamma_{\sigma_j} + \gamma_{\sigma'_{j-1}} + 1.$$

Recall that  $\gamma_{\sigma_j}$  and  $\gamma_{\sigma'_{j-1}}$  are either  $\gamma_1$  or  $\gamma_2$  and  $\gamma_{\sigma_j} + \gamma_{\sigma'_j} = \gamma_1 + \gamma_2$ . Now since  $\gamma_1 + \gamma_2 \geq -3/2$  and  $\gamma_j \geq -1 + \varepsilon$ , we have

$$\alpha_j \geq \begin{cases} 2\gamma_j + 1 \geq -1 + 2\varepsilon, & \text{if } \sigma'_{j-1} = \sigma_j, \\ \gamma_1 + \gamma_2 + 1 \geq -3/2 + 1 = -1/2, & \text{if } \sigma'_{j-1} \neq \sigma_j. \end{cases}$$

We get  $\alpha_j \geq C_1 := -1 + 2\varepsilon > -1$ .

On the other hand, when  $m \geq 3$ ,

$$\sum_{i=1}^m \alpha_i + m = m(\gamma_1 + \gamma_2) + 2m \geq m(-3/2) + 2m = \frac{m}{2} \geq C_2 := \frac{3}{2} > 1.$$

So Corollary 3.1 can be applied to deduce the boundedness of  $|C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c})|$  when  $\gamma_1, \gamma_2 \geq -1 + \varepsilon$ , and the proof is thus concluded.  $\square$

REMARK 3.1. Theorem 2.1 involves convergence to a Gaussian process. In this case, according to the results of Nualart and Peccati [24] and Peccati and Tudor [26], it suffices to show that (26) holds for  $m = 4$  and  $n = 1$ . Focusing on the fourth cumulant, the covariance structure and the one-dimensional distribution, however, does not simplify significantly the proof as can be seen by examining the proof of Theorem 2.1.

3.2. Proof of Theorem 2.2.

LEMMA 3.7. Suppose that  $\alpha > -1$ , then for any  $t_1, t_2 \in \mathbb{R}$ ,

$$\int_0^{t_1} \int_0^{t_2} |x_1 - x_2|^\alpha dx_1 dx_2 = \frac{1}{(\alpha + 1)(\alpha + 2)} (|t_1|^{\alpha+2} + |t_2|^{\alpha+2} - |t_1 - t_2|^{\alpha+2}).$$

PROOF. Suppose  $0 < t_1 \leq t_2$ . The other cases are similar. Then

$$\begin{aligned} & \int_0^{t_1} \int_0^{t_2} |x_1 - x_2|^\alpha dx_1 dx_2 \\ &= \int_0^{t_1} \int_0^{t_1} |x_1 - x_2|^\alpha dx_1 dx_2 + \int_0^{t_1} \int_{t_1}^{t_2} (x_2 - x_1)^\alpha dx_2 dx_1 \\ &= \frac{2}{(\alpha + 1)(\alpha + 2)} t_1^{\alpha+2} + \frac{1}{(\alpha + 1)(\alpha + 2)} [t_2^{\alpha+2} - t_1^{\alpha+2} - (t_2 - t_1)^{\alpha+2}] \\ &= \frac{1}{(\alpha + 1)(\alpha + 2)} [t_1^{\alpha+2} + t_2^{\alpha+2} - (t_2 - t_1)^{\alpha+2}]. \quad \square \end{aligned}$$

Below the notation  $A \sim B$  means asymptotic equivalence, namely, the ratio  $A/B$  converges to 1. We include first a fact about the asymptotics of the beta function  $B(\cdot, \cdot)$  when one of the exponents approaches the boundary.

LEMMA 3.8. Let  $0 < b_0 < b_1 < \infty$ . Then as  $\alpha \rightarrow 0$ , we have

$$\alpha B(\alpha, \beta) \rightarrow 1$$

uniformly in  $\beta \in [b_0, b_1]$ . Since the beta functions is symmetric, we also have  $\alpha B(\beta, \alpha) \rightarrow 1$  as  $\alpha \rightarrow 0$  uniformly in  $\beta \in [b_0, b_1]$ .

PROOF. Assume without loss of generality that  $b_0 \leq 1 \leq b_1$ . Fix any small  $\varepsilon > 0$ . Then

$$\begin{aligned} (27) \quad B(\alpha, \beta) &= \int_0^\varepsilon x^{\alpha-1} (1-x)^{\beta-1} dx + \int_\varepsilon^1 x^{\alpha-1} (1-x)^{\beta-1} dx \\ &=: I_1(\alpha, \beta; \varepsilon) + I_2(\alpha, \beta; \varepsilon). \end{aligned}$$

For  $I_1(\alpha, \beta; \varepsilon)$ , we have

$$\begin{aligned} \alpha^{-1} \varepsilon^\alpha (1 - \varepsilon)^{b_1 - 1} &= \int_0^\varepsilon x^{\alpha - 1} dx (1 - \varepsilon)^{b_1 - 1} \leq I_1(\alpha, \beta; \varepsilon) \\ &\leq \int_0^\varepsilon x^{\alpha - 1} dx (1 - \varepsilon)^{b_0 - 1} = \alpha^{-1} \varepsilon^\alpha (1 - \varepsilon)^{b_0 - 1}. \end{aligned}$$

This yields that

$$\begin{aligned} (1 - \varepsilon)^{b_1 - 1} &\leq \liminf_{\alpha \rightarrow 0, \beta \in [b_0, b_1]} \alpha I_1(\alpha, \beta, \varepsilon) \\ (28) \qquad &\leq \limsup_{\alpha \rightarrow 0, \beta \in [b_0, b_1]} \alpha I_1(\alpha, \beta, \varepsilon) \leq (1 - \varepsilon)^{b_0 - 1}. \end{aligned}$$

For  $I_2(\alpha, \beta; \varepsilon)$ , it is uniformly bounded with respect to  $\alpha \leq 1$  and  $\beta$  as follows:

$$\begin{aligned} I_2(\alpha, \beta; \varepsilon) &\leq \varepsilon^{\alpha - 1} \int_\varepsilon^1 (1 - x)^{\beta - 1} dx = \varepsilon^{\alpha - 1} \beta^{-1} (1 - \varepsilon)^\beta \\ (29) \qquad &\leq \varepsilon^{-1} b_0^{-1} (1 - \varepsilon)^{b_0}. \end{aligned}$$

Combining (27), (28) and (29), we get

$$(1 - \varepsilon)^{b_1 - 1} \leq \liminf_{\alpha \rightarrow 0, \beta \in [b_0, b_1]} \alpha B(\alpha, \beta) \leq \limsup_{\alpha \rightarrow 0, \beta \in [b_0, b_1]} \alpha B(\alpha, \beta) \leq (1 - \varepsilon)^{b_0 - 1}.$$

Since  $\varepsilon$  is arbitrary, we get that  $\alpha B(\alpha, \beta) \rightarrow 1$  as  $\alpha \rightarrow 0$ .  $\square$

The limit  $\alpha B(\alpha, \beta) \rightarrow 1$  as  $\alpha \rightarrow 0$  will be used extensively, mostly in the form

$$B(\alpha, \beta) \sim \alpha^{-1} \rightarrow \infty.$$

LEMMA 3.9. *Let  $WB_{\gamma+3/2}(t)$  be the process given as Theorem 2.2. We also include the case  $\gamma = -1$  where  $B_{\gamma+3/2}(t) = B_{1/2}(t)$  is Brownian motion. Then the  $m$ th cumulant of the linear combination of  $WB_{\gamma+3/2}(t)$  at different time points is given by*

$$\begin{aligned} \kappa_m \left( \sum_{i=1}^n c_i WB_{\gamma+3/2}(t_i) \right) \\ (30) \qquad &= (m - 1)! \left[ \sum_{i_1, i_2=1}^n \frac{c_{i_1} c_{i_2}}{2} (|t_{i_1}|^{2\gamma+3} + |t_{i_2}|^{2\gamma+3} - |t_{i_1} - t_{i_2}|^{2\gamma+3}) \right]^{m/2} \end{aligned}$$

if  $m$  is even, and 0 if  $m$  is odd.

PROOF.

$$\sum_{i=1}^n c_i WB_{\gamma+3/2}(t_i) = W \sum_{i=1}^n c_i B_{\gamma+3/2}(t_i) = \sigma WZ,$$

where  $Z$  is a standard normal random variable which is independent of  $W$ , and

$$\begin{aligned} \sigma &= \left( \text{Var} \left[ \sum_{i=1}^n c_i B_{\gamma+3/2}(t_i) \right] \right)^{1/2} = \left[ \mathbb{E} \sum_{i_1, i_2=1}^n c_{i_1} c_{i_2} B_{\gamma+3/2}(t_{i_1}) B_{\gamma+3/2}(t_{i_2}) \right]^{1/2} \\ &= \left[ \sum_{i_1, i_2=1}^n \frac{c_{i_1} c_{i_2}}{2} (|t_{i_1}|^{2\gamma+3} + |t_{i_2}|^{2\gamma+3} - |t_{i_1} - t_{i_2}|^{2\gamma+3}) \right]^{1/2}, \end{aligned}$$

using the covariance of fractional Brownian motion. Then note that

$$(31) \quad WZ = \frac{1}{2} \left[ \left( \frac{W+Z}{\sqrt{2}} \right)^2 - \left( \frac{W-Z}{\sqrt{2}} \right)^2 \right],$$

where  $Z_1^2 := [\frac{W+Z}{\sqrt{2}}]^2$  and  $Z_2^2 := [\frac{W-Z}{\sqrt{2}}]^2$  are two independent  $\chi_1^2$  (chi-squared random variables with one degree of freedom). The independence is due to the fact that  $Z + W$  and  $Z - W$  are uncorrelated. Since the  $m$ th cumulant of a  $\chi_1^2$  variable is  $2^{m-1}(m-1)!$ , and using the scaling property and the additive property of cumulant under independence, we have

$$\begin{aligned} \kappa_m(\sigma WZ) &= \left( \frac{\sigma}{2} \right)^m [\kappa_m(Z_1^2) + (-1)^m \kappa_m(Z_2^2)] \\ &= \left( \frac{\sigma}{2} \right)^m [2^{m-1}(m-1)! + (-1)^m 2^{m-1}(m-1)!], \end{aligned}$$

which is equal to 0 if  $m$  is odd, and equal to  $\sigma^m(m-1)!$  if  $m$  is even, proving (30). □

REMARK 3.2. Starting with the  $\chi_1^2$  characteristic function  $\phi(t) = (1 - 2it)^{-1/2}$ , it is easy to derive using (31) that the characteristic function of the standard product-normal distribution  $WZ$  is  $\varphi(t) = (1 + t^2)^{-1/2}$ .

In view of Lemma 3.6, we are left to prove the convergence of the finite-dimensional distributions ( $\xrightarrow{\text{f.d.d.}}$ ) in Theorem 2.2.

PROOF OF  $\xrightarrow{\text{f.d.d.}}$  IN THEOREM 2.2. By the Cramér–Wold device, we need to show as  $\gamma_1 \rightarrow -1/2$  and  $\gamma_2 \rightarrow \gamma \in (-1/2, -1)$  that

$$\sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i) \xrightarrow{d} \sum_{i=1}^n c_i WB_{\gamma+3/2}(t_i).$$

Since  $\sum_{i=1}^n c_i WB_{\gamma+3/2}(t_i)$  has an analytic characteristic function (Remark 3.2), its distribution is moment-determinate, and hence we can apply a method of moments here. In fact, by Theorem 3.4 of Nourdin and Poly [22], only a finite number of moments are required to prove convergence in distribution.

The cumulant formula of  $\sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i)$  is given in Proposition 3.1, which involves the factors  $A(\gamma_1, \gamma_2)$  in (21) (recall that  $Z_{\gamma_1, \gamma_2}$  is standardized) and  $C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c})$  in (14). Assume  $m \geq 2$  below.

Examining  $A(\gamma_1, \gamma_2)$ , by Lemma 3.8, one can see that as  $\gamma_1 \rightarrow -1/2$  and  $\gamma_2 \rightarrow \gamma$ ,

$$A(\gamma_1, \gamma_2)^m \sim [(\gamma + 3/2)(2\gamma + 2)]^{m/2} [B(1/2, -\gamma - 1/2)B(\gamma + 1, -\gamma - 1/2) + B(1/2, -2\gamma_1 - 1)B(\gamma + 1, -2\gamma - 1)]^{-m/2}.$$

The first two and the fourth beta functions are bounded but the third blows up since

$$B(1/2, -2\gamma_1 - 1) \sim (-2\gamma_1 - 1)^{-1}$$

as  $\gamma_1 \rightarrow -1/2$  by Lemma 3.8. Hence, as  $\gamma_1 \rightarrow -1/2$ ,

$$\begin{aligned} A(\gamma_1, \gamma_2)^m & \sim [(\gamma + 3/2)(2\gamma + 2)]^{m/2} [B(1/2, -2\gamma_1 - 1)B(\gamma + 1, -2\gamma - 1)]^{-m/2} \\ (32) \quad & \sim (-2\gamma_1 - 1)^{m/2} (2\gamma + 3)^{m/2} (\gamma + 1)^{m/2} B(\gamma + 1, -2\gamma - 1)^{-m/2}, \end{aligned}$$

which converges to zero.

On the other hand, in the expression of  $C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c})$  in (14), the only factors diverging to  $\infty$  as  $\gamma_1 \rightarrow -1/2$  and  $\gamma_2 \rightarrow \gamma$  are  $B(\gamma_{\sigma'_j-1} + 1, -\gamma_{\sigma_j} - \gamma_{\sigma'_j-1} - 1)$  and  $B(\gamma_{\sigma_j} + 1, -\gamma_{\sigma_j} - \gamma_{\sigma'_j-1} - 1)$  and only when  $\sigma_j = \sigma'_j-1 = 1$ , because  $-\gamma_{\sigma_j} - \gamma_{\sigma'_j-1} - 1 = -2\gamma_1 - 1 \rightarrow 0$ , and hence the beta functions each diverge like  $(-2\gamma_1 - 1)^{-1}$  by Lemma 3.8. To get the highest order of divergence to  $\infty$ , one chooses  $\sigma \in \{1, 2\}^m$  such that  $\sigma_j = \sigma'_j-1 = 1$  happens as many times as possible.

In the case  $m$  is odd,

$$\max_{\sigma \in \{1, 2\}^m} \#\{j : \sigma_j = \sigma'_j-1 = 1, j = 1, \dots, m\} = (m - 1)/2,$$

because if  $\sigma_j = \sigma'_j-1 = 1$ , then  $\sigma'_j = 2$ , and we therefore cannot have  $\sigma_{j+1} = \sigma'_j = 1$ . So

$$(33) \quad C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c}) \sim cB(1/2, -2\gamma_1 - 1)^{(m-1)/2} \sim c(-2\gamma_1 - 1)^{-(m-1)/2},$$

which diverges to  $\infty$  as  $\gamma_1 \rightarrow -1/2$ . By (32) and (33), when  $m$  is odd,

$$\begin{aligned} (34) \quad \kappa_m \left( \sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i) \right) & = \frac{1}{2} (m - 1)! A(\gamma_1, \gamma_2)^m C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c}) \\ & \sim c(-2\gamma_1 - 1)^{1/2} \rightarrow 0. \end{aligned}$$

When  $m$  is even, the sequences  $\sigma$  for which one has the greatest number of  $j$ 's such that  $\sigma_j = \sigma'_j-1 = 1$  is

$$\begin{aligned} (35) \quad \operatorname{argmax}_{\sigma \in \{1, 2\}^m} \#\{j : \sigma_j = \sigma'_j-1 = 1, j = 1, \dots, m\} & \\ & = (1, 2, 1, 2, \dots, 1, 2) \quad \text{or} \quad (2, 1, 2, 1, \dots, 2, 1), \end{aligned}$$

and one gets maximally  $m/2$  number of  $j$ 's where  $\sigma_j = \sigma'_{j-1} = 1$ . The product of the  $m/2$  contributing beta factors diverge like  $(-2\gamma_1 - 1)^{m/2}$ . But since the case  $m$  even will yield a nonzero limit, we need to keep track of the multiplicative constants. Because  $\sigma = (1, 2, 1, 2, \dots, 1, 2)$  and  $\sigma = (2, 1, 2, 1, \dots, 2, 1)$  yield the same term, one has as  $\gamma_1 \rightarrow -1/2$  and  $\gamma_2 \rightarrow \gamma$  that

$$\begin{aligned}
 & C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c}) \\
 & \sim 2(-2\gamma_1 - 1)^{-m/2} \left[ \sum_{i_1, \dots, i_n=1}^n c_{i_1} \cdots c_{i_m} \mathbf{B}(\gamma + 1, -2\gamma - 1)^{m/2} \right. \\
 (36) \quad & \times \int_0^{t_{i_1}} \cdots \int_0^{t_{i_m}} |s_1 - s_2|^{2\gamma+1} |s_3 - s_4|^{2\gamma+1} \cdots |s_{m-1} - s_m|^{2\gamma+1} ds \left. \right] \\
 & = 2(-2\gamma_1 - 1)^{-m/2} (2\gamma + 3)^{-m/2} (\gamma + 1)^{-m/2} \mathbf{B}(\gamma + 1, -2\gamma - 1)^{m/2} \\
 & \times \left[ \sum_{i_1, i_2=1}^n \frac{c_{i_1} c_{i_2}}{2} (|t_{i_1}|^{2\gamma+3} + |t_{i_2}|^{2\gamma+3} - |t_{i_1} - t_{i_2}|^{2\gamma+3}) \right]^{m/2},
 \end{aligned}$$

where the asymptotic equivalence  $\sim$  in the first line can be justified by the dominated convergence theorem, and the last equality is due to Lemma 3.7.

Combining (13), (32) and (36), one gets as  $\gamma_1 \rightarrow -1/2$  and  $\gamma_2 \rightarrow \gamma$  that for  $m$  even,

$$\begin{aligned}
 & \kappa_m \left( \sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i) \right) \\
 (37) \quad & \rightarrow (m - 1)! \left[ \sum_{i_1, i_2=1}^n \frac{c_{i_1} c_{i_2}}{2} (|t_{i_1}|^{2\gamma+3} + |t_{i_2}|^{2\gamma+3} - |t_{i_1} - t_{i_2}|^{2\gamma+3}) \right]^{m/2}.
 \end{aligned}$$

The proof is complete by comparing (34) and (37) with Lemma 3.9.  $\square$

We state a byproduct of the preceding proof which will be used in Section 5.

**COROLLARY 3.2.** *Under the condition and the notation of Theorem 2.2, when  $m \geq 4$  is even, we have*

$$\kappa_m(Z_{\gamma_1, \gamma_2}(1)) = (m - 1)! + O(-\gamma_1 - 1/2).$$

**PROOF.** We are focusing here on the marginal distribution, and hence  $t = 1$ ,  $c = 1$  and  $n = 1$  in (14). To get the rate of convergence  $O(-\gamma_1 - 1/2)$ , we need to expand  $C_m(\gamma_1, \gamma_2; 1, 1)$  to a higher order than (36). Following the preceding proof of Theorem 2.2, we need to consider the  $\sigma$ 's with the second most occurrences of

$\sigma'_{j-1} = \sigma_j = 1$ . These  $\sigma$ 's have  $\sigma'_{j-1} = \sigma_j = 1$  occurring  $m/2 - 1$  times instead of  $m/2$  times as in (35). Adding this type of  $\sigma$ 's into (36), we have

$$C_m(\gamma_1, \gamma_2; 1, 1) = c_{\gamma,m}(-\gamma_1 - 1/2)^{-m/2} + O((-\gamma_1 - 1/2)^{-m/2+1}),$$

where  $c_{\gamma,m}$  is the constant given by (36) with  $t = 1, c = 1$  and  $n = 1$ . By Proposition 3.1,

$$\kappa_m(Z_{\gamma_1, \gamma_2}(1)) = \frac{1}{2}(m - 1)!A(\gamma_1, \gamma_2)^m C_m(\gamma_1, \gamma_2; 1, 1).$$

So the conclusion follows in view of the expression  $A(\gamma_1, \gamma_2)^m$  in (32).  $\square$

### 3.3. Proof of Theorem 2.3.

LEMMA 3.10. *Let  $t_1, \dots, t_m > 0$ , and  $m \geq 4$  be an even integer. Consider the function:*

$$(38) \quad f(a, b; \mathbf{t}) = \int_0^{t_1} \cdots \int_0^{t_m} |x_1 - x_m|^a |x_2 - x_1|^b |x_3 - x_2|^a |x_4 - x_3|^b \cdots \\ \times |x_{m-1} - x_{m-2}|^a |x_m - x_{m-1}|^b d\mathbf{x},$$

where  $-1 < a, b < 0$ . Then as  $(a, b) \rightarrow (0, -1)$ , we have that

$$f(a, b; \mathbf{t}) \sim (b + 1)^{-m/2} \prod_{i=2,4,\dots,m} (t_i + t_{i-1} - |t_i - t_{i-1}|).$$

PROOF. First, assume without loss of generality that  $t_1, \dots, t_m < 1$ . Otherwise, one can scale them by a change of variables.

We first derive a lower bound for  $f(a, b; \mathbf{t})$ . Since each  $|x_i - x_{i-1}|^a \geq 1$ , one has by Lemma 3.7 that

$$(39) \quad f(a, b; \mathbf{t}) \geq f(0, b; \mathbf{t}) \\ = \prod_{i=2,4,\dots,m} \int_0^{t_i} \int_0^{t_{i-1}} |x_i - x_{i-1}|^b dx_i dx_{i-1} \\ = (b + 1)^{-m/2} (b + 2)^{-m/2} \prod_{i=2,4,\dots,m} (t_i^{b+2} + t_{i-1}^{b+2} - |t_i - t_{i-1}|^{b+2}) \\ \sim (b + 1)^{-m/2} \prod_{i=2,4,\dots,m} (t_i + t_{i-1} - |t_i - t_{i-1}|) \quad \text{as } b \rightarrow -1.$$

To get an upper bound for  $f(a, b; \mathbf{t})$ , we apply the Cauchy–Schwarz inequality to break the cyclic structure. In particular in (38), view  $|x_1 - x_m|^a |x_3 - x_2|^a$  as the integrand, and treat the other factors as the density of measure. We have

$$(40) \quad f(a, b; \mathbf{t}) \leq \sqrt{f_1(a, b; \mathbf{t}) f_2(a, b; \mathbf{t})},$$

where

$$f_1(a, b; \mathbf{t}) = \int_0^{t_1} dx_1 \cdots \int_0^{t_m} dx_m |x_1 - x_m|^{2a} |x_2 - x_1|^b |x_4 - x_3|^b |x_5 - x_4|^a \cdots \\ \times |x_{m-1} - x_{m-2}|^a |x_m - x_{m-1}|^b$$

and

$$f_2(a, b; \mathbf{t}) = \int_0^{t_1} dx_1 \cdots \int_0^{t_m} dx_m |x_3 - x_2|^{2a} |x_2 - x_1|^b |x_4 - x_3|^b |x_5 - x_4|^a \cdots \\ \times |x_{m-1} - x_{m-2}|^a |x_m - x_{m-1}|^b.$$

Set

$$|x|^a = 1 + h_a(x).$$

Then the integrand in  $f_1$  can be rewritten as

$$[1 + h_{2a}(x_1 - x_m)] |x_2 - x_1|^b |x_4 - x_3|^b [1 + h_a(x_5 - x_4)] \cdots \\ \times [1 + h_a(x_{m-1} - x_{m-2})] |x_m - x_{m-1}|^b.$$

Observe that the product of terms involving neither  $h_a$  nor  $h_{2a}$  equals  $f(0, b; \mathbf{t})$ . Hence, one can write

$$f_1(a, b; \mathbf{t}) = f(0, b; \mathbf{t}) + R(a, b; \mathbf{t}),$$

where the remainder  $R(a, b; \mathbf{t})$  is a sum of terms each involving at least one  $h_a$  or  $h_{2a}$ . We claim that  $|R(a, b; \mathbf{t})| = o((b + 1)^{-m/2})$ . Indeed, let  $R_1(a, b; \mathbf{t})$  be the term of  $R(a, b; \mathbf{t})$  involving only one  $h_{2a}$  and no other  $h_a$ . Using the fact that when  $f$  is a nonnegative function and  $0 < x_1, x_2 < t$ , we have

$$\int_0^t f(x_2 - x_1) dx_2 = \int_{-x_1}^{t-x_1} f(x) dx \leq \int_{-1}^1 f(x) dx.$$

Therefore,

$$\begin{aligned} & |R_1(a, b; \mathbf{t})| \\ &= \int_0^{t_1} dx_1 \cdots \int_0^{t_m} dx_m h_{2a}(x_1 - x_m) |x_2 - x_1|^b |x_4 - x_3|^b \cdots |x_m - x_{m-1}|^b \\ &\leq \int_0^{t_1} dx_1 \int_0^{t_3} dx_3 \cdots \int_0^{t_m} dx_m h_{2a}(x_1 - x_m) \\ &\quad \times \int_{-1}^1 |x_2|^b dx_2 |x_4 - x_3|^b \cdots |x_m - x_{m-1}|^b \\ &\leq 2(b + 1)^{-1} \\ (41) \quad &\times \int_0^{t_3} dx_3 \cdots \int_0^{t_m} dx_m \int_{-1}^1 h_{2a}(x_1) dx_1 |x_4 - x_3|^b \cdots |x_m - x_{m-1}|^b \end{aligned}$$

$$\begin{aligned}
 &\leq 2(b + 1)^{-1} \\
 &\quad \times \int_0^{t_3} dx_3 \cdots \int_0^{t_m} dx_m \int_{-1}^1 (|x_1|^{2a} - 1) dx_1 |x_4 - x_3|^b \cdots |x_m - x_{m-1}|^b \\
 &= 4[(2a + 1)^{-1} - 1](b + 1)^{-1} \int_0^{t_3} dx_3 \cdots \\
 &\quad \times \int_0^{t_m} dx_m |x_4 - x_3|^b |x_6 - x_5|^b \cdots |x_m - x_{m-1}|^b \\
 &\leq \cdots \leq C[(2a + 1)^{-1} - 1](b + 1)^{-m/2} = o(1)(b + 1)^{-m/2}.
 \end{aligned}$$

Similar estimates apply to the other terms of  $R(a, b; \mathbf{t})$ , which may involve a greater number of  $h_a$  or  $h_{2a}$ , and end up converging faster to zero as  $a \rightarrow 0$ . Hence,

$$\begin{aligned}
 f_1(a, b; \mathbf{t}) &\leq f(0, b; \mathbf{t}) + o((b + 1)^{-m/2}) \\
 &\sim (b + 1)^{-m/2} \prod_{i=2,4,\dots,m} (t_i + t_{i-1} - |t_i - t_{i-1}|)
 \end{aligned}$$

using (39). The same estimate holds for  $f_2(a, b; \mathbf{t})$ . Hence, by (40),

$$\begin{aligned}
 (42) \quad f(a, b; \mathbf{t}) &\leq f(0, b; \mathbf{t}) + o((b + 1)^{-m/2}) \\
 &\sim (b + 1)^{-m/2} \prod_{i=2,4,\dots,m} (t_i + t_{i-1} - |t_i - t_{i-1}|).
 \end{aligned}$$

Combining (39) and (42) completes the proof.  $\square$

LEMMA 3.11. *Let  $X_\rho(t)$  be the limit process in (10). For  $m \geq 3$ ,*

$$\kappa_m \left( \sum_{i=1}^n c_i X_\rho(t_i) \right) = \begin{cases} \rho^{m/2} (m - 1)! \left[ \sum_{i,j=1}^n c_i c_j \frac{1}{2} (|t_i| + |t_j| - |t_i - t_j|) \right]^{m/2}, & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

PROOF. Then because  $B_1(t)$ ,  $B_2(t)$  and  $W$  are independent,

$$\kappa_m \left( \sum_{i=1}^n c_i X_\rho(t_i) \right) = \kappa_m \left( \rho^{1/2} \sum_{i=1}^n c_i WB(t_i) \right) + \kappa_m \left( (1 - \rho)^{1/2} \sum_{i=1}^n c_i B'(t_i) \right).$$

Now note that the second term is Gaussian, and thus the cumulants of order higher than 2 are always zero. Applying Lemma 3.9 (with  $\gamma = -1$ ) to the first term completes the proof.  $\square$

Now we proceed to the proof of Theorem 2.3. Again by Lemma 3.6, tightness always holds. We only need to show the convergence of the finite-dimensional distributions.

PROOF OF  $\xrightarrow{\text{f.d.d.}}$  IN THEOREM 2.3. The distribution of  $\sum_{i=1}^n c_i X_\rho(t_i)$  is moment-determinate since it is a second-order polynomial in normal random variables (see, e.g., Slud [30]). One can therefore use a method of moments.

We analyze the asymptotics of the cumulants in (13) with  $m \geq 3$  and  $A(\gamma_1, \gamma_2)$  as given in (21) as  $(\gamma_1, \gamma_2) \rightarrow (-1/2, -1)$ . First, by Lemma 3.8,

$$\begin{aligned}
 & A(\gamma_1, \gamma_2)^m \\
 & \sim (\gamma_1 + \gamma_2 + 3/2)^{m/2} \\
 (43) \quad & \times [\mathbf{B}(1/2, 1/2)\mathbf{B}(\gamma_2 + 1, 1/2) + \mathbf{B}(1/2, -2\gamma_1 - 1)\mathbf{B}(\gamma_2 + 1, 1)]^{-m/2} \\
 & \sim (\gamma_1 + \gamma_2 + 3/2)^{m/2} [\mathbf{B}(1/2, -2\gamma_1 - 1)\mathbf{B}(\gamma_2 + 1, 1)]^{-m/2} \\
 & \sim (\gamma_1 + \gamma_2 + 3/2)^{m/2} (-2\gamma_1 - 1)^{m/2} (\gamma_2 + 1)^{m/2},
 \end{aligned}$$

which converges to 0.

Now we analyze the asymptotics of the terms of  $C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c})$  in (14) as  $\sigma$  varies in  $\{1, 2\}^m$ . When  $m$  is even, consider first the two main terms where

$$\sigma = (1, 2, 1, 2, \dots, 1, 2) \quad \text{and} \quad \sigma = (2, 1, 2, 1, \dots, 2, 1),$$

which correspond to  $\#\{j : \sigma_j = \sigma'_{j-1} = 1\} = m/2$ . As in the proof of Theorem 2.2, the corresponding term when  $\sigma = (1, 2, 1, 2, \dots, 1, 2)$  in (14) [it is the same for  $\sigma = (2, 1, 2, 1, \dots, 2, 1)$ ] is

$$\begin{aligned}
 & \sum_{i_1, \dots, i_m=1}^n c_{i_1} \cdots c_{i_m} \mathbf{B}(\gamma_1 + 1, -2\gamma_1 - 1)^{m/2} \mathbf{B}(\gamma_2 + 1, -2\gamma_2 - 1)^{m/2} \\
 & \times \int_0^{t_{i_1}} ds_1 \cdots \int_0^{t_{i_m}} ds_m |s_1 - s_m|^{2\gamma_1+1} |s_2 - s_1|^{2\gamma_2+1} \cdots \\
 (44) \quad & \times |s_{m-1} - s_{m-2}|^{2\gamma_1+1} |s_m - s_{m-1}|^{2\gamma_2+1} \\
 & \sim (-2\gamma_1 - 1)^{-m/2} (\gamma_2 + 1)^{-m} \left[ \sum_{i,j=1}^n c_i c_j \frac{1}{2} (|t_i| + |t_j| - |t_i - t_j|) \right]^{m/2},
 \end{aligned}$$

where the last line is due to Lemmas 3.8 and 3.10.

Any other  $\sigma$  term in (14) is negligible because it is of order  $O((-2\gamma_1 - 1)^{-r} (\gamma_2 + 1)^{-m})$ , where

$$(45) \quad r = \#\{j : \sigma_j = \sigma'_{j-1} = 1\} = \#\{j : \sigma_j = \sigma'_{j-1} = 2\} < m/2.$$

Indeed, let us suppose (45) and examine a corresponding  $\sigma$  term in the expansion of the product  $\prod_{j=1}^m$  in (14). Call this term  $P_m$ . In  $P_m$ , there are  $r$  factors of

$$(46) \quad \mathbf{B}(\gamma_1 + 1, -2\gamma_1 - 1) |s_j - s_{j-1}|^{2\gamma_1+1},$$

and there are  $r$  factors of

$$(47) \quad \mathbf{B}(\gamma_2 + 1, -2\gamma_2 - 1) |s_j - s_{j-1}|^{2\gamma_2 + 1}.$$

Since (45) implies that  $\#\{j : \sigma_j \neq \sigma'_{j-1}\} = m - 2r$ , there are also  $m - 2r$  factors in  $P_m$ , which are either

$$\begin{aligned} & (s_j - s_{j-1})_+^{\gamma_1 + \gamma_2 + 1} \mathbf{B}(\gamma_1 + 1, -\gamma_1 - \gamma_2 - 1) \\ & + (s_{j-1} - s_j)_+^{\gamma_1 + \gamma_2 + 1} \mathbf{B}(\gamma_2 + 1, -\gamma_1 - \gamma_2 - 1) \end{aligned}$$

or

$$\begin{aligned} & (s_j - s_{j-1})_+^{\gamma_1 + \gamma_2 + 1} \mathbf{B}(\gamma_2 + 1, -\gamma_1 - \gamma_2 - 1) \\ & + (s_{j-1} - s_j)_+^{\gamma_1 + \gamma_2 + 1} \mathbf{B}(\gamma_1 + 1, -\gamma_1 - \gamma_2 - 1). \end{aligned}$$

These last two expressions are both bounded by

$$(48) \quad \begin{aligned} & |s_j - s_{j-1}|^{\gamma_1 + \gamma_2 + 1} [\mathbf{B}(\gamma_2 + 1, -\gamma_1 - \gamma_2 - 1) \\ & + \mathbf{B}(\gamma_1 + 1, -\gamma_1 - \gamma_2 - 1)]. \end{aligned}$$

In view of Lemma 3.8, the beta functions in (46), (47) and (48) behave like  $(-2\gamma_1 - 1)^{-1}$ ,  $(\gamma_2 + 1)^{-1}$  and  $(\gamma_2 + 1)^{-1}$ , respectively. Therefore, the beta functions contribute an order

$$(-2\gamma_1 - 1)^{-r} (\gamma_2 + 1)^{-r} (\gamma_2 + 1)^{-(m-2r)} = (-2\gamma_1 - 1)^{-r} (\gamma_2 + 1)^{-(m-r)}.$$

The integrand involving  $|s_{j-1} - s_j|^{2\gamma_2 + 1}$  contribute an order  $(\gamma_2 + 1)^{-r}$ . So the total order is  $(-2\gamma_1 - 1)^{-r} (\gamma_2 + 1)^{-m}$ . These arguments can be rigorously justified by first applying the Cauchy–Schwartz as in (40) to break the cyclic integrand, and then bound as in (41). Therefore in view of (44), and after also including the case  $\sigma = (2, 1, 2, 1, \dots, 2, 1)$ , we conclude that

$$(49) \quad \begin{aligned} & C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c}) \\ & \sim 2(-2\gamma_1 - 1)^{-m/2} (\gamma_2 + 1)^{-m} \left[ \sum_{i,j=1}^n c_i c_j \frac{1}{2} (|t_i| + |t_j| - |t_i - t_j|) \right]^{m/2}, \end{aligned}$$

if  $m$  is even.

When  $m$  is odd, there are at most  $(m - 1)/2$  times of  $\sigma_j = \sigma'_{j-1} = 1$  or  $\sigma_j = \sigma'_{j-1} = 2$ . It can be shown similarly that  $C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c})$  is of the order

$$(50) \quad (-2\gamma_1 - 1)^{-(m-1)/2} (\gamma_2 + 1)^{-m},$$

which is dominated by the order of convergence to 0 of  $A(\gamma_1, \gamma_2)^m$  in (43). Now combining this fact with (9), (13), (43) and (49), we have when  $m$  is even,

$$\begin{aligned}
 & \kappa_m \left( \sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i) \right) \\
 (51) \quad & \sim \left( \frac{\gamma_1 + \gamma_2 + 3/2}{\gamma_2 + 1} \right)^{m/2} (m-1)! \left[ \sum_{i,j=1}^n c_i c_j \frac{1}{2} (|t_i| + |t_j| - |t_i - t_j|) \right]^{m/2} \\
 & \rightarrow \rho^{m/2} (m-1)! \left[ \sum_{i,j=1}^n c_i c_j \frac{1}{2} (|t_i| + |t_j| - |t_i - t_j|) \right]^{m/2},
 \end{aligned}$$

and when  $m$  is odd,

$$\kappa_m \left( \sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i) \right) \rightarrow 0.$$

Now use Lemma 3.11 to identify the limit process.  $\square$

3.4. Proof of Theorem 2.4. We state first a combinatorial result.

LEMMA 3.12. Let  $\sigma = (\sigma_1, \dots, \sigma_m) \in \{1, 2\}^m$ . Let  $\sigma' = (\sigma'_1, \dots, \sigma'_m)$  be the complement of  $\sigma$ , namely,  $\sigma'_i = 1$  if  $\sigma_i = 2$  and  $\sigma'_i = 2$  if  $\sigma_i = 1$ ,  $i = 1, \dots, m$ . Let  $\sigma_0$  be understood as  $\sigma_m$  and let  $\sigma'_0$  be understood as  $\sigma'_m$ . Then for a fixed integer  $0 \leq r \leq m/2$ ,

$$(52) \quad \#\{\sigma \in \{1, 2\}^m : \#\{j : \sigma_j = \sigma'_{j-1} = 1\} = r\} = 2 \binom{m}{2r}.$$

PROOF. If  $\sigma_{j-1} \neq \sigma_j$ , we say that there is an alternation at  $j$ . There are  $\binom{m}{k}$  ways to place  $k$  alternations. The positions of the alternations determine the whole  $\sigma$  up to the replacement of 1's into 2's and vice versa. Hence, there are  $2 \binom{m}{k}$  possible  $\sigma$ 's. To relate  $k$  to  $r$ , note that the relation  $\sigma_j = \sigma'_{j-1}$  holds if and only if  $\sigma_{j-1} \neq \sigma_j$ . Since

$$r = \#\{j : \sigma_j = \sigma'_{j-1} = 1\} = \#\{j : \sigma_j = \sigma'_{j-1} = 2\},$$

we have

$$k = \#\{j : \sigma_j \neq \sigma_{j-1}\} = \#\{j : \sigma_j = \sigma'_{j-1} = 1\} + \#\{j : \sigma_j = \sigma'_{j-1} = 2\} = 2r. \quad \square$$

LEMMA 3.13. Let  $Y_\rho(t)$  be the limit process in (12). For  $m \geq 3$ ,

$$\begin{aligned}
 (53) \quad \kappa_m \left( \sum_{i=1}^n c_i Y_\rho(t_i) \right) &= \frac{[(\rho + 1)^{-1} + (2\sqrt{\rho})^{-1}]^m + [(\rho + 1)^{-1} - (2\sqrt{\rho})^{-1}]^m}{[(\rho + 1)^{-2} + (4\rho)^{-1}]^{m/2}} \\
 &\quad \times \left( \sum_{i=1}^n c_i t_i \right)^m \frac{(m-1)!}{2}.
 \end{aligned}$$

PROOF. Let

$$a_\rho = \frac{(\rho + 1)^{-1} + (2\sqrt{\rho})^{-1}}{\sqrt{2(\rho + 1)^{-2} + (2\rho)^{-1}}}, \quad b_\rho = \frac{(\rho + 1)^{-1} - (2\sqrt{\rho})^{-1}}{\sqrt{2(\rho + 1)^{-2} + (2\rho)^{-1}}}.$$

Because  $X_1$  and  $X_2$  are two independent standardized  $\chi_1^2$  random variables, we have

$$\begin{aligned} \kappa_m \left( \sum_{i=1}^n c_i Y_\rho(t_i) \right) &= \kappa_m \left( \sum_{i=1}^n c_i t_i (a_\rho X_1 + b_\rho X_2) \right) \\ &= \left( \sum_{i=1}^n c_i t_i \right)^m [\kappa_m(a_\rho X_1) + \kappa_m(b_\rho X_2)] \\ &= \left( \sum_{i=1}^n c_i t_i \right)^m (a_\rho^m + b_\rho^m) \kappa(X_1) \\ &= 2^{m/2} (a_\rho^m + b_\rho^m) \left( \sum_{i=1}^n c_i t_i \right)^m \frac{(m-1)!}{2}. \end{aligned}$$

The factor  $2^{m/2}(a_\rho^m + b_\rho^m)$  can be rewritten as the first factor in (53).  $\square$

Note that  $a + b \sim A + B$  for  $a, b, A, B > 0$ , if  $a \sim A, b \sim B$  and  $a/b \sim \lambda$ , where  $\lambda$  is a fixed number from 0 to  $\infty$  (can be  $\infty$ ), as will always be the case under our assumptions.

We now prove Theorem 2.4. In view of Lemma 3.6, we only need to show the convergence of the finite-dimensional distributions.

PROOF OF  $\xrightarrow{\text{f.d.d.}}$  IN THEOREM 2.4. We can use a method of moments again because the limit  $\sum_{i=1}^n c_i Y_\rho(t_i)$  is a second-order polynomial in normal random variables. We analyze the asymptotics of the cumulants in (13) with  $m \geq 3$  and  $A(\gamma_1, \gamma_2)$  in (21) as  $(\gamma_1, \gamma_2) \rightarrow (-1/2, -1/2)$ . Lemma 3.8 yields

$$(54) \quad A(\gamma_1, \gamma_2)^m \sim [(-\gamma_1 - \gamma_2 - 1)^{-2} + (-2\gamma_1 - 1)^{-1}(-2\gamma_2 - 1)^{-1}]^{-m/2},$$

and  $C_m$  in (14) satisfies

$$(55) \quad C_m(\gamma_1, \gamma_2; \mathbf{t}, \mathbf{c}) \sim \left( \sum_{i=1}^n c_i t_i \right)^m \sum_{\sigma \in \{1,2\}^m} \prod_{j=1}^m (-\gamma_{\sigma_j} - \gamma_{\sigma'_j-1} - 1)^{-1},$$

where we get the term  $(\sum_{i=1}^n c_i t_i)^m$  from  $\sum_{i_1, \dots, i_m=1}^n c_{i_1} \cdots c_{i_m} \int_0^{t_{i_1}} ds_1 \cdots \int_0^{t_{i_m}} ds_m$ .

Let  $r = \#\{j : \sigma_j = \sigma'_{j-1} = 1\} = \#\{j : \sigma_j = \sigma'_{j-1} = 2\}$ . Then using Lemma 3.12, we can write

$$\begin{aligned}
 (56) \quad & \sum_{\sigma \in \{1,2\}^m} \prod_{j=1}^m (-\gamma_{\sigma_j} - \gamma_{\sigma'_{j-1}} - 1)^{-1} \\
 &= \sum_{0 \leq r \leq m/2} 2 \binom{m}{2r} (-2\gamma_1 - 1)^{-r} (-2\gamma_2 - 1)^{-r} (-\gamma_1 - \gamma_2 - 1)^{-(m-2r)}.
 \end{aligned}$$

Hence by (13), (54), (55) and (56), one has

$$\begin{aligned}
 (57) \quad & \kappa_m \left( \sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i) \right) \\
 & \sim (m-1)! \left( \sum_{i=1}^n c_i t_i \right)^m \sum_{0 \leq r \leq m/2} \binom{m}{2r} U(\gamma_1, \gamma_2; m, r),
 \end{aligned}$$

where

$$U(\gamma_1, \gamma_2; m, r) := \frac{(-2\gamma_1 - 1)^{-r} (-2\gamma_2 - 1)^{-r} (-\gamma_1 - \gamma_2 - 1)^{-(m-2r)}}{[(-\gamma_1 - \gamma_2 - 1)^{-2} + (-2\gamma_1 - 1)^{-1} (-2\gamma_2 - 1)^{-1}]^{m/2}}.$$

As  $(\gamma_1, \gamma_2) \rightarrow (-1/2, -1/2)$  and  $(\gamma_1 + 1/2)/(\gamma_2 + 1/2) \rightarrow \rho \in [0, 1]$ , in the case  $\rho > 0$ , some elementary calculation shows

$$(58) \quad U(\gamma_1, \gamma_2; m, r) \rightarrow \frac{[1/(2\sqrt{\rho})]^{2r} [1/(\rho + 1)]^{m-2r}}{[(\rho + 1)^{-2} + (4\rho)^{-1}]^{m/2}},$$

and in the case  $\rho = 0$ ,

$$(59) \quad U(\gamma_1, \gamma_2; m, r) \rightarrow \begin{cases} 1, & \text{if } r = m/2 \text{ (} m \text{ must be even in this case),} \\ 0, & \text{if } r < m/2. \end{cases}$$

This expression (59) also coincides with the limit in (58) as  $\rho \rightarrow 0$ . In the argument below, we omit the case  $\rho = 0$ , which can be either treated separately, or obtained by taking the limit as  $\rho \rightarrow 0$ .

Set  $a = 1/(2\sqrt{\rho})$  and  $b = 1/(\rho + 1)$ . Using the identity  $(a + b)^m + (a - b)^m = \sum_{0 \leq r \leq m/2} 2 \binom{m}{2r} a^{2r} b^{m-2r}$ , one can write following (57) and (58) that

$$\kappa_m \left( \sum_{i=1}^n c_i Z_{\gamma_1, \gamma_2}(t_i) \right) \rightarrow \frac{(a + b)^m - (a - b)^m}{(a^2 + b^2)^{m/2}} \left( \sum_{i=1}^n c_i t_i \right)^m \frac{(m-1)!}{2},$$

which is (53). Now use Lemma 3.13 to identify the limit process, completing the proof.  $\square$

*Additional results.* We deal now with the following additional three points:

1. We show that the weak convergence proved in the previous theorems *cannot* be strengthened to convergence in  $L^2(\Omega)$  nor even in probability.
2. We apply the results of Nourdin and Peccati [21] and Eichelsbacher and Thäle [11] to determine the rate of convergence on the boundaries  $d$  and  $e_1$  (or  $e_2$ ).
3. We include an alternate proof of Theorem 2.2 in the spirit of Remark 2.1 which provides further insight on the convergence.

**4. No convergence in  $L^2(\Omega)$ .** The generalized Rosenblatt process  $Z_{\gamma_1, \gamma_2}(t)$  was defined in (1) [see also (6)]. We have shown weak convergence (convergence in distribution) for the generalized Rosenblatt process  $Z_{\gamma_1, \gamma_2}(t)$  in previous theorems. Is it possible that some of these convergences are actually in a stronger mode, say, in probability? We provide a negative answer here.

**THEOREM 4.1.** *In Theorems 2.1, 2.2, 2.3 and 2.4, the weak convergence cannot be extended to convergence in  $L^2(\Omega)$ , nor even to convergence in probability.*

**REMARK 4.1.** In fact, it suffices to show that the convergence cannot be extended to convergence in  $L^2(\Omega)$ . This is because, on a fixed order Wiener chaos, convergence in  $L^2(\Omega)$  and convergence in probability are equivalent. See Schreiber [29]. Alternatively, to verify the equivalence, suppose that  $X_n$  is a sequence on a fixed order Wiener chaos, and  $X_n$  converges in probability to  $X$ . The sequence is therefore tight. Then by, for example, Lemma 2.1(ii) of Nourdin and Rosinski [23],  $\sup_n \mathbb{E}|X_n|^p < \infty$  for any  $p > 0$ , which entails uniform integrability, and hence convergence in  $L^2(\Omega)$ .

To prove Theorem 4.1, it suffices to show that any sequence of

$$Z_{\gamma_1, \gamma_2} := Z_{\gamma_1, \gamma_2}(1)$$

as  $(\gamma_1, \gamma_2)$  approach the boundaries is *not* a Cauchy sequence in  $L^2(\Omega)$ . Let  $(\alpha_1, \alpha_2)$  and  $(\gamma_1, \gamma_2)$  be in the region  $\mathbf{\Delta}$  in (2). Then since  $Z_{\gamma_1, \gamma_2}$  is standardized, we have

$$(60) \quad \mathbb{E}(Z_{\alpha_1, \alpha_2} - Z_{\gamma_1, \gamma_2})^2 = 2 - 2\mathbb{E}Z_{\alpha_1, \alpha_2}Z_{\gamma_1, \gamma_2}.$$

If  $(\alpha_1, \alpha_2)$  and  $(\gamma_1, \gamma_2)$  converge to the same point on the boundary, we may expect that  $\mathbb{E}Z_{\alpha_1, \alpha_2}Z_{\gamma_1, \gamma_2} \rightarrow 1$  and hence  $\mathbb{E}(Z_{\alpha_1, \alpha_2} - Z_{\gamma_1, \gamma_2})^2 \rightarrow 0$ , which would prove Cauchy convergence. We will show, however, that

$$(61) \quad \liminf_{(\alpha_1, \alpha_2), (\gamma_1, \gamma_2) \rightarrow \text{boundary point}} \mathbb{E}Z_{\alpha_1, \alpha_2}Z_{\gamma_1, \gamma_2} < 1.$$

In other words, we will show that there is no  $L^2(\Omega)$  continuity at the boundary.

First, we compute the covariance in (60).

LEMMA 4.1.

$$\begin{aligned}
 & \mathbb{E}Z_{\alpha_1, \alpha_2}Z_{\gamma_1, \gamma_2} \\
 &= A(\alpha_1, \alpha_2)A(\gamma_1, \gamma_2)(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + 3)^{-1}(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + 4)^{-1} \\
 & \times [\mathbf{B}(\alpha_1 + 1, -\alpha_1 - \gamma_1 - 1)\mathbf{B}(\alpha_2 + 1, -\alpha_2 - \gamma_2 - 1) \\
 (62) \quad & + \mathbf{B}(\gamma_1 + 1, -\alpha_1 - \gamma_1 - 1)\mathbf{B}(\gamma_2 + 1, -\alpha_2 - \gamma_2 - 1) \\
 & + \mathbf{B}(\alpha_2 + 1, -\alpha_2 - \gamma_1 - 1)\mathbf{B}(\alpha_1 + 1, -\alpha_1 - \gamma_2 - 1) \\
 & + \mathbf{B}(\gamma_1 + 1, -\alpha_2 - \gamma_1 - 1)\mathbf{B}(\gamma_2 + 1, -\alpha_1 - \gamma_2 - 1)].
 \end{aligned}$$

PROOF. We shall use the representation (6) of  $Z_{\gamma_1, \gamma_2}(t)$  in order to apply the formula

$$\mathbb{E}I_2(f)I_2(g) = 2\langle f, g \rangle_{L^2(\mathbb{R}^2)}$$

for symmetric functions  $f$  and  $g$  (see (7.3.39) of Peccati and Taquq [25]). Using (18), we get

$$\begin{aligned}
 & 2A(\alpha_1, \alpha_2)^{-1}A(\gamma_1, \gamma_2)^{-1}\mathbb{E}Z_{\alpha_1, \alpha_2}Z_{\gamma_1, \gamma_2} \\
 &= \int_{[0,1]^2} ds \int_{\mathbb{R}^2} d\mathbf{x} [(s_1 - x_1)_+^{\alpha_1} (s_1 - x_2)_+^{\alpha_2} + (s_1 - x_1)_+^{\alpha_2} (s_1 - x_2)_+^{\alpha_1}] \\
 & \times [(s_2 - x_1)_+^{\gamma_1} (s_2 - x_2)_+^{\gamma_2} + (s_2 - x_1)_+^{\gamma_2} (s_2 - x_2)_+^{\gamma_1}] \\
 &= 2 \int_{[0,1]^2} ds [(s_2 - s_1)_+^{\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + 2} \\
 & \times \mathbf{B}(\alpha_1 + 1, -\alpha_1 - \gamma_1 - 1)\mathbf{B}(\alpha_2 + 1, \alpha_2 - \gamma_2 - 1) \\
 & + (s_1 - s_2)_+^{\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + 2} \mathbf{B}(\gamma_1 + 1, -\alpha_1 - \gamma_1 - 1) \\
 & \times \mathbf{B}(\gamma_2 + 1, -\alpha_2 - \gamma_2 - 1) \\
 & + (s_2 - s_1)_+^{\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + 2} \mathbf{B}(\alpha_2 + 1, -\alpha_2 - \gamma_1 - 1) \\
 & \times \mathbf{B}(\alpha_1 + 1, -\alpha_1 - \gamma_2 - 1) \\
 & + (s_1 - s_2)_+^{\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + 2} \mathbf{B}(\gamma_1 + 1, -\alpha_2 - \gamma_1 - 1) \\
 & \times \mathbf{B}(\gamma_2 + 1, -\alpha_2 - \gamma_2 - 1)].
 \end{aligned}$$

Since  $\alpha_1 + \alpha_2 > -3/2$  and  $\gamma_1 + \gamma_2 > -3/2$ , we have  $\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + 2 > -1$ . Since

$$\int_{[0,1]^2} (s_1 - s_2)_+^u ds = \int_{[0,1]^2} (s_2 - s_1)_+^u ds = (u + 1)^{-1}(u + 2)^{-1}$$

for  $u > -1$ , we get (62).  $\square$

**PROOF OF THEOREM 4.1.** *Case of Theorem 2.1.* By (7), an element of the second chaos converges in distribution to a Gaussian. That this cannot be extended to convergence in  $L^2(\Omega)$  follows from the fact that  $\{I_2(f) : f \in L^2(\mathbb{R}^2)\}$  is a closed subspace in  $L^2(\Omega)$ . Hence the  $L^2(\Omega)$  limit of a double Wiener–Itô integral must still be a double Wiener–Itô integral, which means that it cannot be Gaussian.

*Case of Theorem 2.2.* Let  $(\alpha_1, \alpha_2) \rightarrow (-1/2, \gamma)$  and  $(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)$ , where  $\gamma \in (-1, -1/2)$ . Assume in addition that the convergence speeds are comparable, that is,  $(\alpha_1 + 1/2)/(\gamma_1 + 1/2) \sim r \in (0, 1)$ . Then using (32) with  $m = 1$ , Lemma 3.8, and (62), one has

$$\begin{aligned} & \mathbb{E}Z_{\alpha_1, \alpha_2}Z_{\gamma_1, \gamma_2} \\ & \sim (-2\alpha_1 - 1)^{1/2}(-2\gamma_1 - 1)^{1/2}(2\gamma + 3)(\gamma + 1)\mathbb{B}(\gamma + 1, -2\gamma - 1)^{-1} \\ & \quad \times (2 + 2\gamma)^{-1}(3 + 2\gamma)^{-1}[2\mathbb{B}(\gamma + 1, -2\gamma - 1)(-\alpha_1 - \gamma_1 - 1)^{-1}] \\ & \sim \frac{(-2\alpha_1 - 1)^{1/2}(-2\gamma_1 - 1)^{1/2}}{(-\alpha_1 - \gamma_1 - 1)} \sim 2r^{1/2}/(1 + r) < 1. \end{aligned}$$

*Case of Theorem 2.3.* When  $\rho < 1$ , the limit in (10) involves a Gaussian component, which by the same reason as in “Case of Theorem 2.1” implies that  $L^2(\Omega)$  convergence cannot hold. We only need to consider the case  $\rho = 1$ .

We therefore suppose that  $(\alpha_1, \alpha_2) \rightarrow (-1/2, -1)$  and  $(\gamma_1, \gamma_2) \rightarrow (-1/2, -1)$  and that  $\rho = 1$ , that is by (9), that  $(\alpha_1 + 1/2)/(\alpha_2 + 1) \rightarrow 0$  and  $(\gamma_1 + 1/2)/(\gamma_2 + 1) \rightarrow 0$ . Assume in addition that  $(\alpha_1 + 1/2)/(\gamma_1 + 1/2) \sim (\alpha_2 + 1)/(\gamma_2 + 1) \sim r \in (0, 1)$ . By (43) with  $m = 1$ , Lemma 3.8, and (62), we have

$$\begin{aligned} & \mathbb{E}Z_{\alpha_1, \alpha_2}Z_{\gamma_1, \gamma_2} \\ & \sim (\alpha_1 + \alpha_2 + 3/2)^{1/2}(-2\alpha_1 - 1)^{1/2}(\alpha_2 + 1)^{1/2}(\gamma_1 + \gamma_2 + 3/2)^{1/2} \\ & \quad \times (-2\gamma_1 - 1)^{1/2}(\gamma_2 + 1)^{1/2} \\ & \quad \times (\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + 3)^{-1}(-\alpha_1 - \gamma_1 - 1)^{-1} \\ & \quad \times [(\alpha_2 + 1)^{-1} + (\gamma_2 + 1)^{-1}] \\ & \sim \frac{(\alpha_2 + 1)(-2\alpha_1 - 1)^{1/2}(\gamma_2 + 1)(-2\gamma_1 - 1)^{1/2}}{(\alpha_2 + 1 + \gamma_2 + 1)(-\alpha_1 - \gamma_1 - 1)} \\ & \quad \times [(\alpha_2 + 1)^{-1} + (\gamma_2 + 1)^{-1}] \\ & \sim 2r^{1/2}/(r + 1) < 1. \end{aligned}$$

*Case of Theorem 2.4.* Suppose that  $(\alpha_1, \alpha_2) \rightarrow (-1/2, -1/2)$  and  $(\gamma_1, \gamma_2) \rightarrow (-1/2, -1/2)$  and that  $(\alpha_1 + 1/2)/(\alpha_2 + 1/2) \sim (\gamma_1 + 1/2)/(\gamma_2 + 1/2) \sim \rho$ , where  $\rho \in [0, 1]$ . Assume in addition that  $(\alpha_1 + 1/2)/(\gamma_1 + 1/2) \sim (\alpha_2 + 1/2)/(\gamma_2 + 1/2) \sim r \in (0, 1)$ . We apply (54) with  $m = 1$ , (62) and Lemma 3.8. In this case, all

beta functions in (62) blow up and we get

$$\begin{aligned} & \mathbb{E}Z_{\alpha_1, \alpha_2} Z_{\gamma_1, \gamma_2} \\ & \sim [(-\alpha_1 - \alpha_2 - 1)^{-2} + (-2\alpha_1 - 1)^{-1}(-2\alpha_2 - 1)^{-1}]^{-1/2} \\ & \quad \times [(-\gamma_1 - \gamma_2 - 1)^{-2} + (-2\gamma_1 - 1)^{-1}(-2\gamma_2 - 1)^{-1}]^{-1/2} \times \frac{1}{2} \\ & \quad \times [2(-\alpha_1 - \gamma_1 - 1)^{-1}(-\alpha_2 - \gamma_2 - 1)^{-1} \\ & \quad + 2(-\alpha_2 - \gamma_1 - 1)^{-1}(-\alpha_1 - \gamma_2 - 1)^{-1}] \\ & \sim \frac{4r}{(r+1)^2} \left( \frac{(r+\rho)(1+r\rho) + (r+1)^2\rho}{(1+\rho)^2 + 4\rho} \right) \frac{(1+\rho)^2}{(r+\rho)(1+r\rho)}, \end{aligned}$$

which is close to zero if  $r$  is small. Thus, (61) holds.  $\square$

**5. Convergence rate of marginal distribution on the boundaries.** Rates of convergence of the marginal distribution of multiple Wiener–Itô integrals are available when the limit is Gaussian or is a product of independent Gaussians. We can thus apply these rates when converging to the boundaries of the triangle, with some corners excluded.

First, we consider the convergence rate of the marginal distribution in the case of Theorems 2.1 and 2.3 and the limit being Gaussian. We use the notation  $A \asymp B$ , where  $A$  and  $B$  are two nonnegative quantities, to denote that there exist constants  $c < C$  independent of  $A$  and  $B$  such that  $cB \leq A \leq CB$ . Let  $d_{TV}(X, Y)$  denote the total variation distance between the distributions of random variables  $X$  and  $Y$ , namely

$$d_{TV}(X, Y) = \sup_{S \in \mathcal{B}(\mathbb{R})} |P(X \in S) - P(Y \in S)|,$$

where  $\mathcal{B}(\mathbb{R})$  denotes the Borel sets on  $\mathbb{R}$ .

In Nourdin and Peccati [21] Theorem 1.2, the following result was established.

**LEMMA 5.1.** *Let  $\{F_\gamma : \gamma \in G \subset \mathbb{R}^k\}$  be a family of random variables defined on a fixed-order Wiener chaos satisfying  $\mathbb{E}F_\gamma^2 = 1$ , where  $G$  is an open set of indices. Suppose that the third cumulant  $\kappa_3(F_\gamma)$  and the fourth cumulant  $\kappa_4(F_\gamma)$  converge uniformly to zero as  $\gamma \in G$  approaches a set  $E \subset \overline{G}$  (as the distance between the point  $\gamma$  and the set  $E$  converges to zero). Then there exists a neighborhood  $\mathcal{N}(E)$  of  $E$  in  $\mathbb{R}^k$ , such that when  $\gamma \in \mathcal{N}(E) \cap G$ , we have*

$$(63) \quad d_{TV}(F_\gamma, N) \asymp M(F_\gamma),$$

where  $N$  is a standard normal random variable and

$$(64) \quad M(F_\gamma) = \max(|\mathbb{E}F_\gamma^3|, |\mathbb{E}F_\gamma^4 - 3|) = \max(|\kappa_3(F_\gamma)|, |\kappa_4(F_\gamma)|).$$

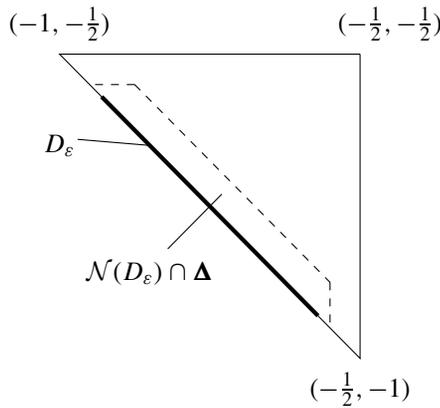


FIG. 6. Illustration of the neighborhood  $\mathcal{N}(D_\varepsilon)$  of  $D_\varepsilon$  in Theorem 5.1.

REMARK 5.1. Though the theorem was originally stated in Nourdin and Peccati [21] for a sequence  $\{F_n\}$  with a discrete parameter  $n$ , examining the proof there one sees that for (63) to hold, one only needs  $\kappa_3(F_\gamma)$  and  $\kappa_4(F_\gamma)$  to converge uniformly to zero, which is implied by our statement of the theorem.

REMARK 5.2. Earlier in [8], the same result (63) was established for the following distributional distance  $d_B(\cdot, \cdot)$ :

$$(65) \quad d_B(X, Y) = \sup_{h \in \mathcal{U}} \{ |\mathbb{E}h(X) - \mathbb{E}h(Y)| \},$$

where  $\mathcal{U}$  is the class of functions that are twice differentiable with continuous derivatives satisfying  $\|h''\|_\infty < \infty$ .

In the case of Theorem 2.1, we considered convergence to the boundary  $d$  through the neighborhood  $\mathcal{N}(D_\varepsilon) \cap \Delta$  illustrated in Figure 6. Applying Lemma 5.1, we get the following.

THEOREM 5.1. Let  $Z_{\gamma_1, \gamma_2} = Z_{\gamma_1, \gamma_2}(1)$ , and let  $N$  be a standard normal random variable. Then under the assumptions of Theorem 2.1, there exists a neighborhood  $\mathcal{N}(D_\varepsilon)$  of the diagonal line segment  $D_\varepsilon := \{\gamma_1 + \gamma_2 + 3/2 = 0 : \gamma_1, \gamma_2 > -1 + \varepsilon\}$ , such that when  $(\gamma_1, \gamma_2) \in \mathcal{N}(D_\varepsilon) \cap \Delta$ , we<sup>3</sup> have

$$(66) \quad d_{TV}(Z_{\gamma_1, \gamma_2}, N) \asymp (\gamma_1 + \gamma_2 + 3/2)^{3/2}.$$

---

<sup>3</sup>Since  $\Delta$  is an open set,  $\mathcal{N}(D_\varepsilon) \cap \Delta$  does not contain the segment  $D_\varepsilon$ .

PROOF. Since  $N$  is Gaussian, we can apply Lemma 5.1. To do so, we need to compute the cumulants  $\kappa_3$  and  $\kappa_4$  which are given in Proposition 3.1. We examine the relation (13) of Proposition 3.1 with  $A = A(\gamma_1, \gamma_2)$  given in (21),  $m = 1$ ,  $t = 1$ , and  $c = 1$ . The factor  $C_m(\gamma_1, \gamma_2, 1, 1)$  in (14) is a positive continuous function with respect to  $(\gamma_1, \gamma_2)$ . This can be shown by the dominated convergence theorem as in Lemma 3.3. Under the assumption of Theorem 2.1, the parameter  $(\gamma_1, \gamma_2)$  is restricted away from boundary. So  $C_m(\gamma_1, \gamma_2, 1, 1)$  is bounded below away from zero and bounded above away from infinity, and so are the factors in (21) except  $[2(\gamma_1 + \gamma_2) + 3]^{1/2}$ , which goes to zero as  $\gamma_1 + \gamma_2 \rightarrow -3/2$ . We get

$$(67) \quad \kappa_m(Z_{\gamma_1, \gamma_2}) \asymp A(\gamma_1, \gamma_2)^m \asymp (\gamma_1 + \gamma_2 + 3/2)^{m/2}, \quad m \geq 3.$$

The maximum in (64) is then  $\kappa_3(F_\gamma)$ . Combining this with (63), we get (66).  $\square$

From (67) and (63), it is the third cumulant that determines the rate of convergence in the case of Theorem 2.1. When  $(\gamma_1, \gamma_2)$  is allowed to be close to the corner  $(-1/2, -1)$ , that is, in the case of Theorem 2.3 when  $\rho = 0$ , we will show that the fourth cumulant may come into play in the rate of convergence.

THEOREM 5.2. *Let  $Z_{\gamma_1, \gamma_2} = Z_{\gamma_1, \gamma_2}(1)$ , and let  $N$  be a standard normal random variable. Then under the assumptions of Theorem 2.3 when  $\rho = 0$ , that is when*

$$(68) \quad -\gamma_1 - 1/2 \sim \gamma_2 + 1,$$

*there exists a neighborhood  $\mathcal{N}$  of  $(-1/2, -1)$ , such that when  $(\gamma_1, \gamma_2) \in \mathcal{N} \cap \Delta$ , we have<sup>4</sup>*

$$(69) \quad d_{TV}(Z_{\gamma_1, \gamma_2}, N) \asymp (\gamma_1 + \gamma_2 + 3/2)^{3/2}(\gamma_2 + 1)^{-1}(1 + L(\gamma_1, \gamma_2)),$$

*as  $(\gamma_1, \gamma_2) \rightarrow (-1/2, -1)$ , where*

$$(70) \quad \begin{aligned} L(\gamma_1, \gamma_2) &= \sqrt{(-\gamma_1 - 1/2)^{-1} - (\gamma_2 + 1)^{-1}} \\ &= o((-\gamma_1 - 1/2)^{-1/2}) \quad \text{or} \quad o((\gamma_2 + 1)^{-1/2}). \end{aligned}$$

PROOF. First in view of (9) with  $\rho = 0$ , we have

$$V(\gamma_1, \gamma_2) := (\gamma_1 + \gamma_2 + 3/2)^{3/2}(\gamma_2 + 1)^{-1} \rightarrow 0 \quad \text{as } (\gamma_1, \gamma_2) \rightarrow (-1/2, -1).$$

By (13), (43), (50) with  $m = 3$ , and (68), we get for the third cumulant

$$(71) \quad \begin{aligned} \kappa_3(Z_{\gamma_1, \gamma_2}) &\asymp (-\gamma_1 - 1/2)^{1/2}(\gamma_1 + \gamma_2 + 3/2)^{3/2}(\gamma_2 + 1)^{-3/2} \\ &\sim V(\gamma_1, \gamma_2). \end{aligned}$$

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<sup>4</sup>As before, since  $\Delta$  is an open set,  $\mathcal{N} \cap \Delta$  does not contain the limit point  $(-1/2, -1)$ .

By (51) with  $m = 4$  and also (68), we have for the fourth cumulant

$$\begin{aligned}
 \kappa_4(Z_{\gamma_1, \gamma_2}) &\asymp \left( \frac{\gamma_1 + \gamma_2 + 3/2}{\gamma_2 + 1} \right)^2 \sim V(\gamma_1, \gamma_2) \left( \frac{\gamma_1 + \gamma_2 + 3/2}{(-\gamma_1 - 1/2)(\gamma_2 + 1)} \right)^{1/2} \\
 (72) \qquad &= V(\gamma_1, \gamma_2)L(\gamma_1, \gamma_2).
 \end{aligned}$$

Since  $\max(x, y) \asymp x + y$  for  $x, y \geq 0$ , we get

$$\max[\kappa_3(\gamma_1, \gamma_2), \kappa_4(\gamma_1, \gamma_2)] \asymp V(\gamma_1, \gamma_2)[1 + L(\gamma_1, \gamma_2)].$$

We thus apply Lemma 5.1 to get (69). At last, note that (68) entails that

$$\begin{aligned}
 L(\gamma_1, \gamma_2) &= (-\gamma_1 - 1/2)^{-1/2} \sqrt{1 - \frac{-\gamma_1 - 1/2}{\gamma_2 + 1}} \\
 &= o((-\gamma_1 - 1/2)^{-1/2}) \quad \text{or} \quad o((\gamma_2 + 1)^{-1/2}). \quad \square
 \end{aligned}$$

REMARK 5.3. In view of Remark 5.2, Theorems 5.1 and 5.2 also hold if the distance  $d_{TV}(\cdot, \cdot)$  is replaced by the distance  $d_B(\cdot, \cdot)$  defined by (65).

REMARK 5.4. The rate of convergence to zero in (69) is always slower than that of (66), which is expected since the corner  $(-1/2, -1)$  also belongs to the non-Gaussian boundary.

REMARK 5.5. From (71) and (72), one has

$$\frac{\kappa_4(Z_{\gamma_1, \gamma_2})}{\kappa_3(Z_{\gamma_1, \gamma_2})} \asymp \sqrt{(-\gamma_1 - 1/2)^{-1} - (\gamma_2 + 1)^{-1}} = L(\gamma_1, \gamma_2),$$

which is the term (70) appearing in (69). Note that  $(-\gamma_1 - 1/2)^{-1} > (\gamma_2 + 1)^{-1}$  when  $(\gamma_1, \gamma_2) \in \Delta$ . Therefore, in the case of Theorem 2.3, the fourth cumulant plays a role in determining the rate of convergence as follows: if the fourth cumulant converges much slower compared with the third cumulant, that is, if  $L(\gamma_1, \gamma_2) \rightarrow \infty$ , then this will slow the rate of convergence in (69); if  $L(\gamma_1, \gamma_2)$  is asymptotically bounded, then both the third and fourth cumulants behave like  $V(\gamma_1, \gamma_2)$ .

Now we consider the marginal convergence rate in the case of Theorem 2.2 (see Figure 3). This theorem involves a non-Gaussian limit. For two random variables  $X$  and  $Y$ , we define the Wasserstein distance between their distributions to be

$$d_W(X, Y) = \sup_{h \in \mathcal{L}} \{ |\mathbb{E}h(X) - \mathbb{E}h(Y)| \},$$

where  $\mathcal{L}$  is the class of 1-Lipschitz functions [ $h \in \mathcal{L}$  if  $|h(x) - h(y)| \leq |x - y|$ ]. The following result follows from Eichelsbacher and Thäle [11].

LEMMA 5.2. *Let  $Y = Z_1 Z_2$  where  $Z_i$ 's are two independent standard normal variables and let  $F = I_2(f)$  be an element on the second-order Wiener chaos with  $\mathbb{E}F^2 = 1$ . Then there exists a constant  $C > 0$  such that*

$$(73) \quad d_W(F, Y) \leq C(1 + \frac{1}{6}\kappa_3(F)^2 - \frac{1}{3}\kappa_4(F) + \frac{1}{120}\kappa_6(F))^{1/2}.$$

PROOF. By Proposition 1.2(iii) of Gaunt [14], the distribution of  $Z_1 Z_2$  is the symmetric Variance-Gamma  $VG(1, 0, 1, 0)$ , that is,  $VG(2r, 0, 1/\lambda, 0)$  with  $r = 1/2$  and  $\lambda = 1$ . Inserting these values of  $r$  and  $\lambda$  in Theorem 5.10(b) of Eichelsbacher and Thäle [11] gives (73).  $\square$

Using the preceding result, we get the following bound for the convergence rate as  $(\gamma_1, \gamma_2)$  approaches the boundary  $e_1$ .

THEOREM 5.3. *Let  $Z_{\gamma_1, \gamma_2} = Z_{\gamma_1, \gamma_2}(1)$ , and let  $Y = Z_1 Z_2$  be as in Lemma 5.2. As*

$$(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma), \quad -1 < \gamma < -1/2,$$

we have

$$(74) \quad d_W(Z_{\gamma_1, \gamma_2}, Y) = O((- \gamma_1 - 1/2)^{1/2}).$$

PROOF. Following the proof of Theorem 2.2, one has by (34) that as  $(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)$ ,

$$(75) \quad \kappa_3(Z_{\gamma_1, \gamma_2}) = O((- \gamma_1 - 1/2)^{1/2}).$$

On the other hand by (37), we have the convergence  $\kappa_m(Z_{\gamma_1, \gamma_2}) \rightarrow (m - 1)!$  for  $m$  even. So  $\kappa_4(Z_{\gamma_1, \gamma_2}) \rightarrow 6$  and  $\kappa_6(Z_{\gamma_1, \gamma_2}) \rightarrow 120$ , and hence

$$1 + \frac{1}{6}\kappa_3(Z_{\gamma_1, \gamma_2})^2 - \frac{1}{3}\kappa_4(Z_{\gamma_1, \gamma_2}) + \frac{1}{120}\kappa_6(Z_{\gamma_1, \gamma_2}) \rightarrow 1 + 0 - 2 + 1 = 0.$$

We thus need to study the rate of convergence of the even-order cumulants  $\kappa_4$  and  $\kappa_6$ . It follows from Corollary 3.2 that

$$(76) \quad \begin{aligned} \kappa_4(Z_{\gamma_1, \gamma_2}) &= 6 + O(-\gamma_1 - 1/2), \\ \kappa_6(Z_{\gamma_1, \gamma_2}) &= 120 + O(-\gamma_1 - 1/2). \end{aligned}$$

The proof is complete by plugging (75) and (76) in (73).  $\square$

Recently, Arras et al. [2] obtained the rate of convergence when the limit is  $\sum_{i=1}^q \alpha_i X_i$  where  $X_i$ 's are standardized chi-square random variables with one degree of freedom. Applying this result (Theorem 3.1 of Arras et al. [2]) to the convergence of  $(\gamma_1, \gamma_2) \in \mathbf{\Delta}$  to the corner  $(-1/2, -1/2)$  in the context of Theorem 2.4, they obtained as  $\gamma_1 \rightarrow -1/2$  that

$$d_W(Z_{\gamma_1, \gamma_2}, Y_\rho(1)) = O((- \gamma_1 - 1/2)^{1/2}),$$

where  $Y_\rho(1)$  is as in Theorem 2.4. See Example 3.2 of Arras et al. [2].

**6. A constructive proof of Theorem 2.2.** The method-of-moments proof of Theorem 2.2 gives little intuitive insight of the convergence. Motivated by the observation made in Remark 2.1, we give an alternate proof of Theorem 2.2. The proof is based on discretization which removes the singularities at  $s = x_1$  and  $s = x_2$  of the integrand in (1), so that one is able to interchange the integration orders between  $\int_{\mathbb{R}^2}' \cdot B(dx_1)B(dx_2)$  and  $\int_0^t \cdot ds$ . Then one uses the triangular approximation described at the end of the proof.

The proof is based on several lemmas. We use below the notation  $(s, x)_N^\gamma$  to denote:

$$(77) \quad (s, x)_N^\gamma := \left( \frac{[Ns] - [Nx] + 1}{N} \right)^\gamma I\{[Ns] > [Nx]\}, \quad \gamma < 0.$$

Define also

$$(78) \quad \begin{aligned} [s - x]_N^\gamma &:= (s - x + 2/N)^\gamma I\{s > x + 1/N\} \\ &\leq (s, x)_N^\gamma \leq (s - x)^\gamma I\{s > x\} = (s - x)_+^\gamma. \end{aligned}$$

Let  $Z_{\gamma_1, \gamma_2}(t)$  be as in (1), and let

$$(79) \quad Z_{\gamma_1, \gamma_2}^N(t) = A_N(\gamma_1, \gamma_2) \int_{\mathbb{R}^2}' \int_0^t (s, x_1)_N^{\gamma_1}(s, x_2)_N^{\gamma_2} ds B(dx_1)B(dx_2),$$

where the Brownian measure  $B(\cdot)$  is the same as the one defining  $Z_{\gamma_1, \gamma_2}(t)$ , and where  $A_N(\gamma_1, \gamma_2)$  is chosen such that  $\mathbb{E}Z_{\gamma_1, \gamma_2}^N(1)^2 = 1$ .

LEMMA 6.1. *For any  $t > 0$ , we have*

$$(80) \quad \lim_{N \rightarrow \infty} \limsup_{(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)} \mathbb{E}|Z_{\gamma_1, \gamma_2}(t) - Z_{\gamma_1, \gamma_2}^N(t)|^2 = 0.$$

PROOF. We take for simplicity that  $t = 1$ , while the other cases can be proved similarly. Note that

$$\mathbb{E}|Z_{\gamma_1, \gamma_2}(1) - Z_{\gamma_1, \gamma_2}^N(1)|^2 = 2 - 2\mathbb{E}Z_{\gamma_1, \gamma_2}(1)Z_{\gamma_1, \gamma_2}^N(1).$$

So we need to show that

$$(81) \quad \lim_{N \rightarrow \infty} \liminf_{(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)} \mathbb{E}Z_{\gamma_1, \gamma_2}(1)Z_{\gamma_1, \gamma_2}^N(1) \geq 1.$$

Indeed, using the symmetrized kernel in (6), we have

$$(82) \quad \begin{aligned} \mathbb{E}Z_{\gamma_1, \gamma_2}(1)Z_{\gamma_1, \gamma_2}^N(1) &= \frac{1}{2}A(\gamma_1, \gamma_2) \frac{1}{2}A_N(\gamma_1, \gamma_2)2! \int_{\mathbb{R}^2} dx_1 dx_2 \int_0^1 \int_0^1 ds_1 ds_2 \\ &\times [(s_1 - x_1)_+^{\gamma_1}(s_1 - x_2)_+^{\gamma_2} + (s_1 - x_1)_+^{\gamma_2}(s_1 - x_2)_+^{\gamma_1}] \\ &\times [(s_2, x_1)_N^{\gamma_1}(s_2, x_2)_N^{\gamma_2} + (s_2, x_1)_N^{\gamma_2}(s_2, x_2)_N^{\gamma_1}]. \end{aligned}$$

By definition,

$$\begin{aligned}
 & A_N(\gamma_1, \gamma_2)^{-2} \\
 &= \frac{1}{2} \int_0^1 \int_0^1 ds_1 ds_2 \int_{\mathbb{R}^2} dx_1 dx_2 [(s_1, x_1)_N^{\gamma_1} (s_1, x_2)_N^{\gamma_2} + (s_1, x_1)_N^{\gamma_2} (s_1, x_2)_N^{\gamma_1}] \\
 &\quad \times [(s_2, x_1)_N^{\gamma_1} (s_2, x_2)_N^{\gamma_2} + (s_2, x_1)_N^{\gamma_2} (s_2, x_2)_N^{\gamma_1}].
 \end{aligned}$$

Applying the second inequality of (78) to (82), and using the normalization  $A_N(\gamma_1, \gamma_2)$ , we have

$$\mathbb{E}Z_{\gamma_1, \gamma_2}(1)Z_{\gamma_1, \gamma_2}^N(1) \geq \frac{1}{2}A(\gamma_1, \gamma_2)A_N(\gamma_1, \gamma_2)2A_N(\gamma_1, \gamma_2)^{-2} = \frac{A(\gamma_1, \gamma_2)}{A_N(\gamma_1, \gamma_2)}.$$

So (81) follows from the next lemma.  $\square$

LEMMA 6.2. *Let the normalizations  $A(\gamma_1, \gamma_2)$  and  $A_N(\gamma_1, \gamma_2)$  be as in (21) and (79). Then*

$$(83) \quad \lim_{N \rightarrow \infty} \lim_{(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)} \frac{A(\gamma_1, \gamma_2)}{A_N(\gamma_1, \gamma_2)} = 1,$$

where  $-1 < \gamma_1, \gamma_2 < -1/2$ .

PROOF. By the second inequality of (78), we have

$$(84) \quad A_N(\gamma_1, \gamma_2)^{-2} \leq A(\gamma_1, \gamma_2)^{-2}.$$

By the first inequality of (78), we have

$$\begin{aligned}
 & A_N(\gamma_1, \gamma_2)^{-2} \\
 & \geq \frac{1}{2} \int_0^1 \int_0^1 ds_1 ds_2 \\
 (85) \quad & \times \int_{\mathbb{R}^2} dx_1 dx_2 ([s_1 - x_1]_N^{\gamma_1} [s_1 - x_2]_N^{\gamma_2} + [s_1 - x_1]_N^{\gamma_2} [s_1 - x_2]_N^{\gamma_1}) \\
 & \times ([s_2 - x_1]_N^{\gamma_1} [s_2 - x_2]_N^{\gamma_2} + [s_2 - x_1]_N^{\gamma_2} [s_2 - x_2]_N^{\gamma_1}) \\
 & = P_N(\gamma_1, \gamma_2) + Q_N(\gamma_1, \gamma_2),
 \end{aligned}$$

where

$$\begin{aligned}
 P_N(\gamma_1, \gamma_2) &= 2 \int_{0 < s_1 < s_2 < 1} ds_1 ds_2 \int_{\mathbb{R}} [s_1 - x_1]_N^{\gamma_1} [s_2 - x_1]_N^{\gamma_1} dx_1 \\
 &\quad \times \int_{\mathbb{R}} [s_1 - x_2]_N^{\gamma_2} [s_2 - x_2]_N^{\gamma_2} dx_2
 \end{aligned}$$

and

$$Q_N(\gamma_1, \gamma_2) = 2 \int_{0 < s_1 < s_2 < 1} ds_1 ds_2 \int_{\mathbb{R}} [s_1 - x_1]_N^{\gamma_1} [s_2 - x_1]_N^{\gamma_2} dx_1 \\ \times \int_{\mathbb{R}} [s_1 - x_2]_N^{\gamma_2} [s_2 - x_2]_N^{\gamma_1} dx_2.$$

In the integrals over  $\mathbb{R}$ , the exponents of  $Q_N$  alternate where as those of  $P_N$  are the same. Note that for  $\alpha, \beta \in (-1, -1/2)$  and  $0 < s_1 < s_2 < 1$ , we have

$$\int_{\mathbb{R}} [s_1 - x]_N^\alpha [s_2 - x]_N^\beta dx \\ = \int_{-\infty}^{s_1 - 1/N} (s_1 - x + 2/N)^\alpha (s_2 - x + 2/N)^\beta dx \\ (86) \quad = \int_0^\infty (u + 3/N)^\alpha (s_2 - s_1 + u + 3/N)^\beta du \\ \leq \int_0^\infty u^\alpha (u + s_2 - s_1)^\beta du \\ = (s_2 - s_1)^{\alpha + \beta + 1} B(\alpha + 1, -\alpha - \beta - 1),$$

after setting  $u = s_1 - x - 1/N$ . Thus, the term  $Q_N$  from (85) satisfies

$$Q_N(\gamma_1, \gamma_2) \leq 2(2\gamma_1 + 2\gamma_2 + 3)^{-1} (2\gamma_1 + 2\gamma_2 + 4)^{-1} \\ (87) \quad \times B(\gamma_1 + 1, -\gamma_1 - \gamma_2 - 1) B(\gamma_2 + 1, -\gamma_1 - \gamma_2 - 1) \\ = O(1)$$

as  $(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)$ . The other term  $P_N$  in view of (78) and (86) becomes

$$P_N(\gamma_1, \gamma_2) \\ = 2 \int_{0 < s_1 < s_2 < 1} ds_1 ds_2 \int_0^\infty (u + 3/N)^{\gamma_1} (s_2 - s_1 + u + 3/N)^{\gamma_1} du \\ \times \int_0^\infty (u + 3/N)^{\gamma_2} (s_2 - s_1 + u + 3/N)^{\gamma_2} du.$$

Now in the second integral, use  $(u + 3/N)^{\gamma_2} \geq (s_2 - s_1 + u + 3/N)^{\gamma_2}$ , and in the third integral, replace  $u$  by  $u(s_2 - s_1)$  and then factor  $s_2 - s_1$ . One gets

$$P_N(\gamma_1, \gamma_2) \\ \geq 2 \int_{0 < s_1 < s_2 < 1} ds_1 ds_2 \int_0^\infty (s_2 - s_1 + u + 3/N)^{2\gamma_1} du \\ \times (s_2 - s_1)^{2\gamma_2 + 1} \int_0^\infty \left(u + \frac{3}{N(s_2 - s_1)}\right)^{\gamma_2} \left(1 + u + \frac{3}{N(s_2 - s_1)}\right)^{\gamma_2} du.$$

Since  $\int_0^\infty (s_2 - s_1 + u + 3/N)^{2\gamma_1} du = (-2\gamma_1 - 1)^{-1}(s_2 - s_1 + 3/N)^{2\gamma_1+1}$ , one has

$$\begin{aligned}
 &P_N(\gamma_1, \gamma_2) \\
 &\geq 2(-2\gamma_1 - 1)^{-1} \int_{0 < s_1 < s_2 < 1} ds_1 ds_2 (s_2 - s_1 + 3/N)^{2\gamma_1+1} (s_2 - s_1)^{2\gamma_2+1} \\
 (88) \quad &\times \int_0^\infty \left(u + \frac{3}{N(s_2 - s_1)}\right)^{\gamma_2} \left(u + \frac{3}{N(s_2 - s_1)} + 1\right)^{\gamma_2} du \\
 &=: R_N(\gamma_1, \gamma_2).
 \end{aligned}$$

As  $(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)$ , we have

$$\begin{aligned}
 (-2\gamma_1 - 1)R_N(\gamma_1, \gamma_2) &\rightarrow 2 \int_{0 < s_1 < s_2 < 1} ds_1 ds_2 (s_2 - s_1)^{2\gamma+1} \\
 &\times \int_0^\infty \left(u + \frac{3}{N(s_2 - s_1)}\right)^\gamma \left(u + \frac{3}{N(s_2 - s_1)} + 1\right)^\gamma du.
 \end{aligned}$$

As  $N \rightarrow \infty$ , by the monotone convergence theorem, the right-hand side of the preceding line converges to

$$\begin{aligned}
 &2 \int_{0 < s_1 < s_2 < 1} ds_1 ds_2 (s_2 - s_1)^{2\gamma+1} \int_0^\infty u^\gamma (u + 1)^\gamma du \\
 &= (2\gamma + 3)^{-1}(\gamma + 1)^{-1}B(\gamma + 1, -2\gamma - 1).
 \end{aligned}$$

On the other hand, from (32) with  $m = 2$  we have

$$(89) \quad A(\gamma_1, \gamma_2)^2 \sim (-2\gamma_1 - 1)(2\gamma + 3)(\gamma + 1)B(\gamma + 1, -2\gamma - 1)^{-1}.$$

Hence,

$$(90) \quad \lim_{N \rightarrow \infty} \lim_{(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)} A(\gamma_1, \gamma_2)^2 R_N(\gamma_1, \gamma_2) = 1.$$

Combining (85), (87), (88) and (90) yields

$$\liminf_{N \rightarrow \infty} \liminf_{(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)} \frac{A(\gamma_1, \gamma_2)^2}{A_N(\gamma_1, \gamma_2)^2} \geq 1.$$

This with (84) yields (83).  $\square$

We will now interchange the integrals  $\int_0^t \cdot ds$  and  $\int_{\mathbb{R}^2} \cdot dx_1 dx_2$ , and write

$$\begin{aligned}
 (91) \quad Z_{\gamma_1, \gamma_2}^N(t) &= A_N(\gamma_1, \gamma_2) \int_{\mathbb{R}^2}' \left[ \int_0^t (s, x_1)_N^{\gamma_1} (s, x_2)_N^{\gamma_2} B(dx_1) B(dx_2) ds \right] \\
 &= A_N(\gamma_1, \gamma_2) \int_0^t \left[ \int_{\mathbb{R}^2}' (s, x_1)_N^{\gamma_1} (s, x_2)_N^{\gamma_2} B(dx_1) B(dx_2) \right] ds \quad \text{a.s.},
 \end{aligned}$$

by the stochastic Fubini theorem (see Pipiras and Taqqu [27], Theorem 2.1). It applies since

$$(92) \quad \int_0^t \int_{\mathbb{R}^2} [(s, x_1)_{N}^{\gamma_1} (s, x_2)_{N}^{\gamma_2}]^2 dx_1 dx_2 ds < \infty.$$

Relation (92) follows from the following lemma.

LEMMA 6.3. *For any  $\gamma \in (-1, -1/2)$ ,  $t > 0$  and  $N \in \mathbb{Z}_+$ , we have*

$$\sup_{s \in [0, t]} \int_{\mathbb{R}} (s, x)_{N}^{2\gamma} dx < \infty.$$

PROOF. In view of (77),

$$\begin{aligned} \int_{\mathbb{R}} (s, x)_{N}^{2\gamma} dx &= \frac{1}{N} \int_{\mathbb{R}} \left( \frac{[Ns] - [Nx] + 1}{N} \right)^{2\gamma} I\{[Ns] > [Nx]\} d(Nx) \\ &= N^{-2\gamma-1} \sum_{-\infty < i < [Ns]} ([Ns] - i + 1)^{2\gamma} = N^{2\gamma-1} \sum_{k=2}^{\infty} k^{-2\gamma} < \infty \end{aligned}$$

since  $\gamma < -1/2$ , where we set  $k = [Ns] - i + 1$ . Since the last expression does not depend on  $s$ , the conclusion of the lemma holds.  $\square$

By the product formula of Wiener–Itô integrals (see, e.g., Nourdin and Peccati [20], Theorem 2.7.10), the process  $Z_{\gamma_1, \gamma_2}^N(t)$  in (91) can be rewritten as follows:

$$\begin{aligned} Z_{\gamma_1, \gamma_2}^N(t) &= A_N(\gamma_1, \gamma_2) \\ &\quad \times \int_0^t \left[ \int_{\mathbb{R}} (s, x_1)_{N}^{\gamma_1} \mathbf{B}(dx_1) \int_{\mathbb{R}} (s, x_2)_{N}^{\gamma_2} \mathbf{B}(dx_2) \right. \\ &\quad \left. - \mathbb{E} \int_{\mathbb{R}} (s, x_1)_{N}^{\gamma_1} \mathbf{B}(dx_1) \int_{\mathbb{R}} (s, x_2)_{N}^{\gamma_2} \mathbf{B}(dx_2) \right] ds. \end{aligned}$$

Note that by the scaling property of Brownian motion, for  $j = 1, 2$ ,

$$\begin{aligned} X_{\gamma_j}^N(s) &:= \int_{\mathbb{R}} (s, x)_{N}^{\gamma_j} \mathbf{B}(dx) \\ &= \int_{\mathbb{R}} \left( \frac{[Ns] - [Nx] + 1}{N} \right)^{\gamma_j} I\{[Ns] > [Nx]\} \mathbf{B}(dx) \\ &\stackrel{\text{f.d.d.}}{=} N^{-\gamma_j-1/2} \sum_{-\infty < i < [Ns]} ([Ns] - i + 1)^{\gamma_j} \varepsilon_i, \end{aligned}$$

where  $\varepsilon_i$ 's are i.i.d. standard normal random variables, and  $\stackrel{\text{f.d.d.}}{=}$  means equal in finite-dimensional distributions. Hence (recall that the Hurst index

$$H = \gamma_1 + \gamma_2 + 2),$$

$$\begin{aligned} Z_{\gamma_1, \gamma_2}^N(t) &\stackrel{\text{f.d.d.}}{=} A_N(\gamma_1, \gamma_2) \int_0^t [X_{\gamma_1}^N(s)X_{\gamma_2}^N(s) - \mathbb{E}X_{\gamma_1}^N(s)X_{\gamma_2}^N(s)] ds \\ (93) \quad &= A_N(\gamma_1, \gamma_2)N^{-H} \sum_{n=1}^{[Nt]} [Y_{\gamma_1}(n)Y_{\gamma_2}(n) - \mathbb{E}Y_{\gamma_1}(n)Y_{\gamma_2}(n)] \\ &\quad + R_N(t, \gamma_1, \gamma_2), \end{aligned}$$

where

$$(94) \quad Y_\gamma(n) = \sum_{-\infty < i < n-1} (n-i)^\gamma \varepsilon_i = \sum_{i=2}^\infty i^\gamma \varepsilon_{n-i}$$

is a linear stationary sequence and

$$\begin{aligned} (95) \quad R_N(t, \gamma_1, \gamma_2) &= A_N(\gamma_1, \gamma_2)N^{-H}(Nt - [Nt]) \\ &\quad \times (Y_{\gamma_1}([Nt] + 1)Y_{\gamma_2}([Nt] + 1) - \mathbb{E}Y_{\gamma_1}([Nt] + 1)Y_{\gamma_2}([Nt] + 1)). \end{aligned}$$

We first show that this preceding remainder term is negligible.

LEMMA 6.4.

$$(96) \quad \lim_{N \rightarrow \infty} \limsup_{(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)} \mathbb{E}R_N(t, \gamma_1, \gamma_2)^2 = 0.$$

PROOF. Since  $Nt - [Nt] \leq 1$  and  $Y_\gamma(n)$  is stationary, we can write

$$\begin{aligned} \mathbb{E}R_N(t, \gamma_1, \gamma_2)^2 &\leq N^{-2H} A_N(\gamma_1, \gamma_2)^2 [\mathbb{E}Y_{\gamma_1}(0)^2 Y_{\gamma_2}(0)^2 - (\mathbb{E}Y_{\gamma_1}(0)Y_{\gamma_2}(0))^2]. \end{aligned}$$

We have

$$(97) \quad \mathbb{E}Y_{\gamma_1}(0)Y_{\gamma_2}(0) = \sum_{i=2}^\infty i^{\gamma_1 + \gamma_2}, \quad \mathbb{E}Y_{\gamma_j}(0)^2 = \sum_{i=2}^\infty i^{2\gamma_j}, \quad j = 1, 2.$$

By the diagram formula (see, e.g., Janson [16], Theorem 1.36), we have for jointly centered Gaussian variables  $(Y_1, Y_2)$  that  $\mathbb{E}Y_1^2 Y_2^2 = 2(\mathbb{E}Y_1 Y_2)^2 + \mathbb{E}Y_1^2 \mathbb{E}Y_2^2$ . Expressing this as  $\mathbb{E}Y_1^2 Y_2^2 - (\mathbb{E}Y_1 Y_2)^2 = (\mathbb{E}Y_1 Y_2)^2 + \mathbb{E}Y_1^2 \mathbb{E}Y_2^2$ , one gets

$$\begin{aligned} (98) \quad \mathbb{E}R_N(t, \gamma_1, \gamma_2)^2 &\leq N^{-2H} A_N(\gamma_1, \gamma_2)^2 \left[ \left( \sum_{i=2}^\infty i^{\gamma_1 + \gamma_2} \right)^2 + \left( \sum_{i=2}^\infty i^{2\gamma_1} \right) \left( \sum_{i=2}^\infty i^{2\gamma_2} \right) \right]. \end{aligned}$$

The first and last sums remain bounded as  $(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)$ , but this is not the case for the second sum. Since the function  $x^{2\gamma_1}$  is decreasing, we have for any integer  $k \geq 0$ ,

$$\begin{aligned}
 (-2\gamma_1 - 1)^{-1}(k + 2)^{2\gamma_1+1} &= \int_2^\infty (x + k)^{2\gamma_1} dx \leq \int_2^\infty (x + k)^{\gamma_1} x^{\gamma_1} dx \\
 (99) \qquad \qquad \qquad &\leq \sum_{i=2}^\infty (i + k)^{\gamma_1} i^{\gamma_1} \leq \sum_{i=2}^\infty i^{2\gamma_1} \leq \int_1^\infty x^{2\gamma_1} dx \\
 &= (-2\gamma_1 - 1)^{-1}.
 \end{aligned}$$

In particular,  $\sum_{i=2}^\infty i^{2\gamma_1}$  explodes like  $(-2\gamma_1 - 1)^{-1}$  as  $\gamma_1 \rightarrow -1/2$ . This, however, will be compensated by  $A_N(\gamma_1, \gamma_2)^2$ , since by (83) and (89), we have  $A_N(\gamma_1, \gamma_2) \sim A(\gamma_1, \gamma_2) \asymp (-2\gamma_1 - 1)$  as  $(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)$ . Hence, (98) implies

$$\limsup_{(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)} N^{2H} \mathbb{E}R_N(t, \gamma_1, \gamma_2)^2 < \infty,$$

which entails (96).  $\square$

The following lemma is key.

LEMMA 6.5. *Let  $Y_\gamma(n)$  be as in (94). As  $(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)$ , one has the following joint convergence in distribution:*

$$(A(\gamma_1, \gamma_2)Y_{\gamma_1}(n), Y_{\gamma_2}(n))_{n=1}^N \xrightarrow{d} (\sigma_\gamma W, Y_\gamma(n))_{n=1}^N,$$

for any  $N \in \mathbb{Z}_+$ , where  $W$  is a standard normal random variable which is independent of  $Y_\gamma(n)$ , and

$$(100) \qquad \sigma_\gamma = (2\gamma + 3)^{1/2}(\gamma + 1)^{1/2}\mathbf{B}(\gamma + 1, -2\gamma - 1)^{-1/2}.$$

PROOF. Since  $(A(\gamma_1, \gamma_2)Y_{\gamma_1}(n), Y_{\gamma_2}(n))_{n=1}^N$  is always a centered and jointly Gaussian vector, we only need to show that its covariance structure converges to that of  $(\sigma_\gamma W, Y_\gamma(n))_{n=1}^N$ . Let us first compute the covariance of  $A(\gamma_1, \gamma_2)Y_{\gamma_1}$ . By (89) and (99), we have for  $m \geq n$  (similarly for  $m < n$ )

$$\begin{aligned}
 &\mathbb{E}[A(\gamma_1, \gamma_2)Y_{\gamma_1}(n)A(\gamma_1, \gamma_2)Y_{\gamma_1}(m)] \\
 &= A(\gamma_1, \gamma_2)^2 \mathbb{E}[Y_{\gamma_1}(n)Y_{\gamma_1}(m)] \\
 &\sim (2\gamma + 3)(\gamma + 1)\mathbf{B}(\gamma + 1, -2\gamma - 1)^{-1}(-2\gamma_1 - 1) \sum_{i=2}^\infty (i + m - n)^{\gamma_1} i^{\gamma_1} \\
 &\sim (2\gamma + 3)(\gamma + 1)\mathbf{B}(\gamma + 1, -2\gamma - 1)^{-1} = \sigma_\gamma^2.
 \end{aligned}$$

Since the limit is independent of  $n$ , the limit process is indeed a fixed Gaussian random variable, say  $\sigma_\gamma W$ .

We now focus on the cross-covariance between  $A(\gamma_1, \gamma_2)Y_{\gamma_1}$  and  $Y_{\gamma_2}$ . We have for  $m \geq n$  (similarly for  $m < n$ ) that

$$(101) \quad \begin{aligned} & \mathbb{E}[A(\gamma_1, \gamma_2)Y_{\gamma_1}(n)Y_{\gamma_2}(m)] \\ & \sim [(2\gamma + 3)(\gamma + 1)B(\gamma + 1, -2\gamma - 1)^{-1}(-2\gamma_1 - 1)]^{1/2} \\ & \quad \times \sum_{i=2}^{\infty} (i + m - n)^{\gamma_1} i^\gamma \rightarrow 0, \end{aligned}$$

because  $\sum_{i=2}^{\infty} i^{-1/2+\gamma} < \infty$ . Thus, we have asymptotic independence. Finally, as  $\gamma_2 \rightarrow \gamma$ , the covariance structure of the second term  $Y_{\gamma_2}$  converges to that of  $Y_\gamma$ . The proof is then complete.  $\square$

The following convergence of normalized sum of long-memory linear process to fractional Brownian motion can be found in Giraitis, Koul and Surgailis [15], Corollary 4.4.1, which was originally due to Davydov [10].

LEMMA 6.6. *Let  $Y_\gamma(n)$  be as in (94). Then as  $N \rightarrow \infty$*

$$Z_\gamma^N(t) := N^{-\gamma-2/3} \sum_{n=1}^{[Nt]} Y_\gamma(n) \xrightarrow{\text{f.d.d.}} \sigma_\gamma^{-1} B_{\gamma+3/2}(t),$$

where  $\sigma_\gamma$  is as in (100) and  $B_{\gamma+3/2}(t)$  is a standard fractional Brownian motion with Hurst index  $\gamma + 3/2$ .

We are now ready to combine the last few lemmas into an alternate proof of Theorem 2.2.

PROOF OF THEOREM 2.2. Tightness still follows from Lemma 3.6. To prove the convergence of the finite-dimensional distributions, namely, to prove that

$$Z_{\gamma_1, \gamma_2}(t) \xrightarrow{\text{f.d.d.}} WB_{\gamma+3/2} \quad \text{as } (\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma),$$

it is sufficient to show that the following triangular approximation relations hold (see, e.g., Lemma 4.2.1 of Giraitis, Koul and Surgailis [15]):

$$(102) \quad \begin{aligned} & \lim_{N \rightarrow \infty} \limsup_{(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)} \mathbb{E} \left| Z_{\gamma_1, \gamma_2}(t) \right. \\ & \quad \left. - \frac{A(\gamma_1, \gamma_2)}{A_N(\gamma_1, \gamma_2)} [Z_{\gamma_1, \gamma_2}^N(t) - R_N(t, \gamma_1, \gamma_2)] \right|^2 = 0, \end{aligned}$$

$$(103) \quad \frac{A(\gamma_1, \gamma_2)}{A_N(\gamma_1, \gamma_2)} [Z_{\gamma_1, \gamma_2}^N(t) - R_N(t, \gamma_1, \gamma_2)] \xrightarrow{\text{f.d.d.}} \sigma_\gamma W Z_\gamma^N(t)$$

as  $(\gamma_1, \gamma_2) \rightarrow (-1/2, \gamma)$ ,

$$(104) \quad \sigma_\gamma W Z_\gamma^N(t) \xrightarrow{\text{f.d.d.}} W B_{\gamma+3/2}(t) \quad \text{as } N \rightarrow \infty.$$

The convergence (102) follows from Lemmas 6.1, 6.2 and 6.4. For the convergence (103), we have by (93), Lemma 6.5 and (101) that

$$\begin{aligned} & \frac{A(\gamma_1, \gamma_2)}{A_N(\gamma_1, \gamma_2)} [Z_{\gamma_1, \gamma_2}^N(t) - R_N(t, \gamma_1, \gamma_2)] \\ &= N^{-H} \sum_{n=1}^{[Nt]} [A(\gamma_1, \gamma_2) Y_{\gamma_1}(n) Y_{\gamma_2}(n) - \mathbb{E} A(\gamma_1, \gamma_2) Y_{\gamma_1}(n) Y_{\gamma_2}(n)] \\ & \xrightarrow{\text{f.d.d.}} N^{-\gamma-3/2} \sum_{n=1}^{[Nt]} [\sigma_\gamma W Y_\gamma(n) - 0] = \sigma_\gamma W Z_\gamma^N(t). \end{aligned}$$

Finally, (104) follows from Lemma 6.6.  $\square$

**Acknowledgments.** We would like to thank Yin Huang for a suggestion. We also thank the referees for their helpful comments and suggestions.

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