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LOCALITY OF PERCOLATION FOR ABELIAN CAYLEY GRAPHS[1]

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We prove that the value of the critical probability for percolation on an Abelian Cayley graph is determined by its local structure. This is a partial positive answer to a conjecture of Schramm: the function $\mathrm{p_c}$ defined on the set of Cayley graphs of Abelian groups of rank at least 2 is continuous for the Benjamini–Schramm topology. The proof involves group-theoretic tools and a new block argument.

1. Introduction. In the paper [3], Benjamini and Schramm launched the study of percolation in the general setting of transitive graphs. Among the numerous questions that have been studied in this setting stands the question of locality: roughly, "does the value of the critical probability depend only on the local structure of the considered transitive graph?" This question emerged in [2] and is formalized in a conjecture attributed to Oded Schramm. In the same paper, the particular case of (uniformly non-amenable) tree-like graphs is treated.

In the present paper, we study the question of locality in the context of Abelian groups.

- Instead of working in the geometric setting of transitive graphs, we employ the vocabulary of groups—or more precisely of *marked groups*, as presented in Section 2. This allows us to use additional tools of algebraic nature, such as quotient maps, that are crucial to our approach. These tools could be useful to tackle Schramm's conjecture in a more general framework than the one presented in this paper, for example, Cayley graphs of nilpotent groups.
- We extend renormalization techniques developed in [12] by Grimmett and Marstrand for the study of percolation on $\mathbb{Z}^d$ (equipped with its standard graph structure). The Grimmett–Marstrand theorem answers positively the question of locality for the d-dimensional hypercubic lattice. With little extra effort, one can give a positive answer to Schramm's conjecture in the context of Abelian groups, under a symmetry assumption. Our main achievement is to improve the understanding of supercritical bond percolation on general Abelian Cayley graphs: such graphs do not have enough symmetry for Grimmett and Marstrand's arguments to apply directly. The techniques we develop here may be used to extend

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other results of statistical mechanics from symmetric lattices to lattices which are not stable under any reflection.

1.1. *Statement of Schramm's conjecture.* The following paragraph presents the vocabulary needed to state Schramm's conjecture (for more details, see [2]).

Transitive graphs. We recall here some standard definitions from graph theory. A graph is said to be *transitive* if its automorphism group acts transitively on its vertices. Let $\mathfrak{G}$ denote the set of (locally finite, nonempty, connected) transitive graphs considered up to isomorphism. By abuse of notation, we will identify a graph with its isomorphism class. Take $\mathcal{G} \in \mathfrak{G}$ and o any vertex of $\mathcal{G}$. Then consider the *ball* of radius k (for the graph distance) centered at o, equipped with its graph structure and rooted at o. Up to isomorphism of rooted graphs, it is independent of the choice of o, and we denote it by $\mathcal{B}_{\mathcal{G}}(k)$. If $\mathcal{G}, \mathcal{H} \in \mathfrak{G}$, we set the distance between them to be 2^{-n}, where

$$n := \max\{k : B_{\mathcal{G}}(k) \simeq B_{\mathcal{H}}(k)\} \in \mathbb{N} \cup \{\infty\}.$$

This defines the *Benjamini–Schramm distance* on the set $\mathfrak{G}$. It was introduced in [4] and [2].

Locality in percolation theory. We will use the standard definitions from percolation theory and refer to [9] and [13] for background on the subject. To any $\mathcal{G} \in \mathfrak{G}$ corresponds a critical parameter $\mathrm{p_c}(\mathcal{G})$ for i.i.d. bond percolation. One can see $\mathrm{p_c}$ as a function from $\mathfrak{G}$ to $[0, 1]$. The locality question is concerned with the continuity of this function.

QUESTION 1 (Locality of percolation). *Consider a sequence of transitive graphs* $(\mathcal{G}_n)$ *that converges to a limit* $\mathcal{G}$.

$$\textit{Does the convergence } \mathrm{p_c}(\mathcal{G}_n) \underset{n\to\infty}{\longrightarrow} \mathrm{p_c}(\mathcal{G}) \textit{ hold?}$$

With this formulation, the answer is negative. Indeed, for the usual graph structures, the following convergences hold:

- $(\mathbb{Z}/n\mathbb{Z})^2 \underset{n\to\infty}{\longrightarrow} \mathbb{Z}^2$,
- $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z} \underset{n\to\infty}{\longrightarrow} \mathbb{Z}^2$.

In both cases, the critical parameter is constant equal to 1 all along the sequence and jumps to a nontrivial value in the limit. The following conjecture, attributed to Schramm and formulated in [2], states that Question 1 should have a positive answer whenever the previous obstruction is avoided.

CONJECTURE 1.1 (Schramm). *Let* $\mathcal{G}_n \underset{n\to\infty}{\longrightarrow} \mathcal{G}$ *denote a converging sequence of transitive graphs. Assume that* $\sup_n \mathrm{p_c}(\mathcal{G}_n) < 1$. *Then* $\mathrm{p_c}(\mathcal{G}_n) \underset{n\to\infty}{\longrightarrow} \mathrm{p_c}(\mathcal{G})$.

It is unknown whether $\sup_n \mathrm{p_c}(\mathcal{G}_n) < 1$ is equivalent or not to $\mathrm{p_c}(\mathcal{G}_n) < 1$ for all n. In other words, we do not know if 1 is an isolated point in the set of critical probabilities of transitive graphs. Besides, no geometric characterization of the probabilistic condition $\mathrm{p_c}(\mathcal{G}) < 1$ has been established so far, which constitutes part of the difficulty of Schramm's conjecture.

1.2. *The Grimmett–Marstrand theorem.* The following theorem, proved in [12], is an instance of a locality result. It was an important step in the comprehension of the supercritical phase of percolation.

THEOREM 1.2 (Grimmett–Marstrand). *Let $d \geq 2$. For the usual graph structures, the following convergence holds*:

$$\mathrm{p_c}(\mathbb{Z}^2 \times \{-n, \dots, n\}^{d-2}) \underset{n\to\infty}{\longrightarrow} \mathrm{p_c}(\mathbb{Z}^d).$$

REMARK. Grimmett and Marstrand's proof covers more generally the case of edge structures on $\mathbb{Z}^d$ that are invariant under both translation and reflection.

The graph $\mathbb{Z}^2 \times \{-n, \dots, n\}^{d-2}$ is not transitive, so the result does not fit exactly into the framework of the previous subsection. However, as remarked in [2], one can easily deduce from it the following statement:

$$\mathrm{p_c}\left(\mathbb{Z}^2 \times \left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^{d-2}\right) \underset{n\to\infty}{\longrightarrow} \mathrm{p_c}(\mathbb{Z}^d). \tag{1}$$

Actually, after having introduced the space of marked Abelian groups, we will see in Section 2.3 that one can deduce from the Grimmett–Marstrand theorem a statement that is much stronger than convergence (1). We will be able to prove that $\mathrm{p_c}(\mathbb{Z}^d) = \lim \mathrm{p_c}(\mathcal{G}_n)$ for any sequence of Abelian Cayley graphs $\mathcal{G}_n$ converging to $\mathbb{Z}^d$ with respect to the Benjamini–Schramm distance.

1.3. *Main result.* In this paper, we prove the following theorem, which provides a positive answer to Question 1 in the particular case of Cayley graphs of Abelian groups (see definitions in Section 2).

THEOREM 1.3. *Consider a sequence $(\mathcal{G}_n)$ of Cayley graphs of Abelian groups satisfying $\mathrm{p_c}(\mathcal{G}_n) < 1$ for all n. If the sequence converges to a Cayley graph $\mathcal{G}$ of an Abelian group, then*

$$\mathrm{p_c}(\mathcal{G}_n) \underset{n\to\infty}{\longrightarrow} \mathrm{p_c}(\mathcal{G}). \tag{2}$$

We now give three examples of application of this theorem. Let $d \geq 2$, fix a generating set S of $\mathbb{Z}^d$, and denote by $\mathcal{G}$ the associated Cayley graph of $\mathbb{Z}^d$.

EXAMPLE 1. There exists a natural Cayley graph $\mathcal{G}_n$ of $\mathbb{Z}^2 \times (\frac{\mathbb{Z}}{n\mathbb{Z}})^{d-2}$ that is covered by $\mathcal{G}$. For this sequence, the convergence (2) holds, and generalizes (1).

EXAMPLE 2. Consider the generating set of $\mathbb{Z}^d$ obtained by adding to S all the $n \cdot s$, for $s \in S$. The corresponding Cayley graph $\mathcal{H}_n$ converges to the Cartesian product $\mathcal{G} \times \mathcal{G}$, and we get

$$\mathrm{p_c}(\mathcal{H}_n) \underset{n\to\infty}{\longrightarrow} \mathrm{p_c}(\mathcal{G} \times \mathcal{G}).$$

EXAMPLE 3. Consider a sequence of vectors $x_n \in \mathbb{Z}^d$ such that $\lim |x_n| = \infty$, and write $\mathcal{G}_n$ the Cayley graph of $\mathbb{Z}^d$ constructed from the generating set $S \cup \{x_n\}$. Then the following convergence holds:

$$\mathrm{p_c}(\mathcal{G}_n) \underset{n\to\infty}{\longrightarrow} \mathrm{p_c}(\mathcal{G} \times \mathbb{Z}).$$

The content of Example 2 was obtained in [7] when $\mathcal{G}$ is the canonical Cayley graph of $\mathbb{Z}^d$, based on the Grimmett–Marstrand theorem. In the statement above, $\mathcal{G}$ can be any Cayley graph of $\mathbb{Z}^d$, and the Grimmett–Marstrand theorem cannot be applied without additional symmetry assumption.

1.4. *Questions*. In this paper, we work with Abelian groups because their structure is very well understood. An additional important feature is that the net formed by large balls of an Abelian Cayley graph has roughly the same geometric structure as the initial graph. Since nilpotent groups also present these characteristics, the following question appears as a natural step between Theorem 1.3 and Question 1.

QUESTION 2. *Is it possible to extend Theorem 1.3 to nilpotent groups?*

This question can also be asked for other models of statistical mechanics than Bernoulli percolation. In Questions 3 and 4, we mention two other natural contexts where the locality question can be asked.

In [5], Bodineau proves that the critical temperature for the Ising model on the slab $\mathbb{Z}^2 \times \{-n, \dots, n\}^{d-2}$ converges to the critical temperature for the Ising model on $\mathbb{Z}^d$. This locality result, which is the analogous of the Grimmett–Marstrand theorem for Ising model, suggests the following question.

QUESTION 3. *Is it possible to prove Theorem 1.3 for the critical temperature of the Ising model instead of* $\mathrm{p_c}$*?*

Some related locality statements, concerning the Ising model on finite graphs that locally approximate a regular tree, are obtained in [14].

Define c_n as the number of self-avoiding walks starting from a fixed root of a transitive graph $\mathcal{G}$. By sub-multiplicativity, the sequence $c_n^{1/n}$ converges to a limit called the connective constant of $\mathcal{G}$. In this context, the following question was raised by Benjamini [1].

QUESTION 4. *Does the connective constant depend continuously on the considered infinite transitive graph?*

A positive answer for infinite transitive graphs with uniform height functions is given in [10, 11].

1.5. *Organization of the paper.* Section 2 presents the material on marked Abelian groups that will be needed to establish Theorem 1.3. In Section 2.4, we explain the strategy of the proof: it splits into two main lemmas. These lemmas are respectively proved in Sections 3 and 4.

We draw the attention of the interested reader to Lemma 3.6. Together with the uniqueness of the infinite cluster, it allows to avoid the construction of "seeds" in Grimmett and Marstrand's approach.

2. Marked Abelian groups and locality. In this section, we present the space of marked Abelian groups and show how problems of Benjamini–Schramm continuity for Abelian Cayley graphs can be reduced to continuity problems for marked Abelian groups. Then we provide a first example illustrating the use of marked Abelian groups in proofs of Benjamini–Schramm continuity. Finally, Section 2.4 presents the proof of Theorem 2.3, which is the marked group version of our main theorem.

General marked groups are introduced in [8]. Here, we only define marked groups and Cayley graphs in the *Abelian* setting, since we do not need a higher level of generality.

2.1. *The space of marked Abelian groups.* Let d denote a positive integer. A *(d-)marked Abelian group* is the data of an Abelian group together with a generating d-tuple $(s_1, \dots, s_d)$, up to isomorphism. [We say that $(G; s_1, \dots, s_d)$ and $(G'; s'_1, \dots, s'_d)$ are isomorphic if there exists a group isomorphism from G to G' mapping s_i to s'_i for all i.] We write $\mathbf{G}_d$ for the set of the d-marked Abelian groups. Elements of $\mathbf{G}_d$ will be denoted by $[G; s_1, \dots, s_d]$ or $G^\bullet$, depending on whether we want to insist on the generating system or not. Finally, we write $\mathbf{G}$ the set of all the marked Abelian groups: it is the disjoint union of all the $\mathbf{G}_d$'s.

Quotient of a marked Abelian group. Given a marked Abelian group $G^\bullet = [G; s_1, \dots, s_d]$ and a subgroup Λ of G, we define the *quotient* $G^\bullet/\Lambda$ by

$$G^\bullet/\Lambda = [G/\Lambda; \overline{s}_1, \dots, \overline{s}_d],$$

where $(\overline{s}_1, \dots, \overline{s}_d)$ is the image of $(s_1, \dots, s_d)$ by the canonical surjection from G onto G/Λ. Quotients of marked Abelian groups will be crucial to define and understand the topology of the set of marked Abelian groups. In particular, for the topology defined below, the quotients of a marked Abelian group $G^\bullet$ forms a neighbourhood of it.

The topology. We first define the topology on $\mathbf{G}_d$. Let $\boldsymbol{\delta}$ denote the canonical generating system of $\mathbb{Z}^d$. To each subgroup Γ of $\mathbb{Z}^d$, we can associate an element of $\mathbf{G}_d$ via the mapping

$$\Gamma \longmapsto [\mathbb{Z}^d; \boldsymbol{\delta}]/\Gamma. \tag{3}$$

One can verify that the mapping defined by (3) realizes a bijection from the set of the subgroups of $\mathbb{Z}^d$ onto $\mathbf{G}_d$. This way, $\mathbf{G}_d$ can be seen as a subset of $\{0, 1\}^{\mathbb{Z}^d}$. We consider on $\mathbf{G}_d$ the topology induced by the product topology on $\{0, 1\}^{\mathbb{Z}^d}$. This makes $\mathbf{G}_d$ a Hausdorff compact space, as a closed subset of $\{0, 1\}^{\mathbb{Z}^d}$. Finally, we equip $\mathbf{G}$ with the topology generated by the open subsets of the $\mathbf{G}_d$'s. (In particular, $\mathbf{G}_d$ is an open subset of $\mathbf{G}$.)

Let us illustrate the topology with three examples of converging sequences:

- $[\mathbb{Z}/n\mathbb{Z}; 1]$ converges to $[\mathbb{Z}; 1]$.
- $[\mathbb{Z}; 1, n, \dots, n^{d-1}]$ converges to $[\mathbb{Z}^d; \boldsymbol{\delta}]$.
- $[\mathbb{Z}; 1, n, n+1]$ converges to $[\mathbb{Z}^2; \boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\delta}_1 + \boldsymbol{\delta}_2]$.

These examples show that the strict inequality $\limsup(\mathrm{rank}(G_n)) < \mathrm{rank}(G)$ can happen for some particular converging sequences of Abelian Cayley graphs. In the opposite, we will see the rank cannot decrease in the limit. More precisely, Proposition 2.1 implies that $\limsup(\mathrm{rank}(G_n)) \leq \mathrm{rank}(G)$.

Cayley graphs. Let $G^\bullet = [G; s_1, \dots, s_d]$ be a marked Abelian group. Its Cayley graph, denoted $\mathsf{Cay}(G^\bullet)$, is defined by taking G as vertex-set and declaring a and b to be neighbours if there exists i such that $a = b \pm s_i$. It is uniquely defined up to graph isomorphism. We write $B_{G^\bullet}(k) \subset G$ the ball of radius k in $\mathsf{Cay}(G^\bullet)$, centered at 0.

Converging sequences of marked Abelian groups. In the rest of the paper, we will use the topology of $\mathbf{G}$ through the following proposition, which gives a geometric flavour to the topology. In particular, it will allow us to make the connection with the Benjamini–Schramm topology through Corollary 2.2.

PROPOSITION 2.1. *Let $(G_n^\bullet)$ be a sequence of marked Abelian groups that converges to some $G^\bullet$. Then, for any integer k, the following holds for n large enough*:

1. *$G_n^\bullet$ is of the form $G^\bullet/\Lambda_n$, for some subgroup Λ_n of G, and*
2. *$\Lambda_n \cap B_{G^\bullet}(k) = \{0\}$.*

PROOF. Let d be such that $G^\bullet \in \mathbf{G}_d$. For n large enough, we also have $G_n^\bullet \in \mathbf{G}_d$. Let Γ (resp., Γ_n) denote the unique subgroup of $\mathbb{Z}^d$ that corresponds to $G^\bullet$ (resp., $G_n^\bullet$) via bijection (3). The group Γ is finitely generated: we consider F a finite generating subset of it. Taking n large enough, we can assume that Γ_n contains F, which implies that Γ is a subgroup of Γ_n. We have the following situation:

$$\mathbb{Z}^d \xrightarrow{\varphi} \mathbb{Z}^d/\Gamma \xrightarrow{\psi_n} \mathbb{Z}^d/\Gamma_n.$$

Identifying G with $\mathbb{Z}^d/\Gamma$ and taking $\Lambda_n = \ker \psi_n = \Gamma_n/\Gamma$, we obtain the first point of the proposition.

By definition of the topology, taking n large enough ensures that $\Gamma_n \cap B_{\mathbb{Z}^d}(k) = \Gamma \cap B_{\mathbb{Z}^d}(k)$. We have

$$\begin{aligned} B_{\mathbb{Z}^d/\Gamma}(k) \cap \Lambda_n &= \varphi\big(B_{\mathbb{Z}^d}(k) \cap \Gamma_n\big) \\ &= \varphi\big(B_{\mathbb{Z}^d}(k) \cap \Gamma\big) \\ &= \{0\}. \end{aligned}$$

This completes the proof of the second point. $\square$

COROLLARY 2.2. *The mapping* Cay *from* $\mathbf{G}$ *to* $\mathfrak{G}$ *that associates to a marked Abelian group its Cayley graph is continuous.*

2.2. *Percolation on marked Abelian groups.* Via its Cayley graph, we can associate to each marked Abelian group $G^\bullet$ a critical parameter $\mathrm{p}_\mathrm{c}^\bullet(G^\bullet) := \mathrm{p}_\mathrm{c}(\mathsf{Cay}(G^\bullet))$ for bond percolation. If $G^\bullet$ is a marked Abelian group, then $\mathrm{p}_\mathrm{c}^\bullet(G^\bullet) < 1$ if and only if the rank of G is at least 2. (We commit the abuse of language of calling *rank* of an Abelian group the rank of its torsion-free part.) This motivates the following definition:

$$\tilde{\mathbf{G}} = \{G^\bullet \in \mathbf{G} : \mathsf{rank}(G) \geq 2\}.$$

In the context of marked Abelian groups, we will prove the following theorem:

THEOREM 2.3. *Consider* $G_n^\bullet \longrightarrow G^\bullet$ *a converging sequence in* $\tilde{\mathbf{G}}$. *Then*

$$\mathrm{p}_\mathrm{c}^\bullet(G_n^\bullet) \underset{n\to\infty}{\longrightarrow} \mathrm{p}_\mathrm{c}^\bullet(G^\bullet).$$

Theorem 2.3 above states that $\mathrm{p}_\mathrm{c}^\bullet$ is continuous on $\tilde{\mathbf{G}}$. It seems a priori weaker than Theorem 1.3. Nevertheless, the following lemma allows us to deduce Theorem 1.3 from Theorem 2.3.

LEMMA 2.4. *Let* $G^\bullet$ *be an element of* $\tilde{\mathbf{G}}$. *Assume it is a continuity point of the restricted function*

$$\mathrm{p}_\mathrm{c}^\bullet : \tilde{\mathbf{G}} \longrightarrow (0, 1).$$

Then its associated Cayley graph $\mathsf{Cay}(G^\bullet)$ *is a continuity point of the restricted function*

$$\mathrm{p_c} : \mathsf{Cay}(\tilde{\mathbf{G}}) \longrightarrow (0, 1).$$

Above, $\tilde{\mathbf{G}}$ *is equipped with the marked Abelian group topology, and* $\mathsf{Cay}(\tilde{\mathbf{G}})$ *with the Benjamini–Schramm topology.*

PROOF. Assume by contradiction that there exists a sequence of marked Abelian groups $G_n^\bullet$ in $\tilde{\mathbf{G}}$ such that $\mathsf{Cay}(G_n^\bullet)$ converges to some $\mathsf{Cay}(G^\bullet)$ and $\mathrm{p}_\mathrm{c}^\bullet(G_n^\bullet)$ stays away from $\mathrm{p}_\mathrm{c}^\bullet(G^\bullet)$. Define d to be the degree of $\mathsf{Cay}(G^\bullet)$. Considering n large enough, we can assume that all the $G_n^\bullet$'s lie in the compact set $\bigcup_{d' \le d} \mathbf{G}_{d'}$. Up to extraction, one can then assume that $G_n^\bullet$ converges to some marked Abelian group $G_\infty^\bullet$. By Proposition 2.1, this group must have rank at least 2. Since Cay is continuous, $\mathsf{Cay}(G^\bullet) = \mathsf{Cay}(G_\infty^\bullet)$ and Theorem 2.3 is contradicted by the sequence $(G_n^\bullet)$ that converges to $G_\infty^\bullet$. □

We will also use the following theorem, which is a particular case of Theorem 3.1 in [3].

THEOREM 2.5. *Let* $G^\bullet$ *be a marked Abelian group and* Λ *a subgroup of* G. *Then*

$$\mathrm{p}_\mathrm{c}^\bullet(G^\bullet/\Lambda) \ge \mathrm{p}_\mathrm{c}^\bullet(G^\bullet).$$

2.3. *A first continuity result.* In this section, we will prove Proposition 2.6, which is a particular case of Theorem 1.3. We deem interesting to provide a short separate proof of it, using the Grimmett–Marstrand theorem. This proposition epitomizes the scope of Grimmett–Marstrand results in our context. It also illustrates how marked groups can appear as useful tools to deal with locality questions. More precisely, Lemma 2.4 reduces some questions of continuity in the Benjamini–Schramm space to equivalent questions in the space of marked Abelian groups, where the topology allows to employ methods of algebraic nature.

PROPOSITION 2.6. *Let* $(G_n^\bullet)$ *be a sequence in* $\tilde{\mathbf{G}}$. *Assume that* $G_n^\bullet \underset{n\to\infty}{\longrightarrow} [\mathbb{Z}^d; \boldsymbol{\delta}]$, *where* $\boldsymbol{\delta}$ *stands for the canonical generating system of* $\mathbb{Z}^d$. *Then*

$$\mathrm{p}_\mathrm{c}^\bullet(G_n^\bullet) \underset{n\to\infty}{\longrightarrow} \mathrm{p}_\mathrm{c}^\bullet([\mathbb{Z}^d; \boldsymbol{\delta}]).$$

PROOF. Since $\mathbf{G}_d$ is open, we can assume that $G_n^\bullet$ belongs to it. It is thus a quotient of $[\mathbb{Z}^d; \boldsymbol{\delta}]$, and Theorem 2.5 gives

$$\liminf \mathrm{p}_\mathrm{c}^\bullet(G_n^\bullet) \ge \mathrm{p}_\mathrm{c}^\bullet([\mathbb{Z}^d; \boldsymbol{\delta}]).$$

To establish the other semi-continuity, we will show that the Cayley graph of $G_n^\bullet$ eventually contains $\mathbb{Z}^2 \times \{0, \dots, K\}$ as a subgraph (for K arbitrarily large), and conclude by applying Grimmett–Marstrand theorem.

Let us denote by Γ_n the subgroup of $\mathbb{Z}^d$ associated to $G_n^\bullet$ via bijection (3).

LEMMA 2.7. *For any integer K, for n large enough, there exists a subgroup Π of $\mathbb{Z}^d$ generated by two different elements of the canonical generating system of $\mathbb{Z}^d$, satisfying*

$$\big(\Pi + B_{\mathbb{Z}^d}(0, 2K+1)\big) \cap \Gamma_n = \{0\}.$$

PROOF. To establish Lemma 2.7, we proceed by contradiction. Up to extraction, we can assume that there exists some K such that

$$\text{for all } \Pi, \qquad \big(\Pi + B_{\mathbb{Z}^d}(0, 2K+1)\big) \cap \Gamma_n \neq \{0\}. \tag{4}$$

We denote by v_n^Π a nonzero element of $(\Pi + B_{\mathbb{Z}^d}(0, 2K+1)) \cap \Gamma_n$. Up to extraction, we can assume that, for all Π, the sequence $v_n^\Pi / \|v_n^\Pi\|$ converges to some v_Π. (The vector space $\mathbb{R}^d$ is endowed with an arbitrary norm $\|\cdot\|$.) Since Γ_n converges pointwise to $\{0\}$, for any Π, the sequence $\|v_n^\Pi\|$ tends to infinity. This entails, together with equation (4), that v_Π is contained in the real plane spanned by Π. The incomplete basis theorem implies that the vector space spanned by the v_Π's has dimension at least $d-1$. By continuity of the minors, for n large enough, the vector space spanned by Γ_n as dimension at least $d-1$. This entails that, for n large enough, Γ_n has rank at least $d-1$, which contradicts the hypothesis that $\mathbb{Z}^d/\Gamma_n$ has rank at least 2. □

For any K, provided that n is large enough, one can see $\mathbb{Z}^2 \times \{-K, \ldots, K\}^{d-2}$ as a subgraph of $\mathsf{Cay}(G_n^\bullet)$. [Restrict the quotient map from $\mathbb{Z}^d$ to $G_n^\bullet$ to the $\Pi + B_{\mathbb{Z}^d}(0, K)$ given by Lemma 2.7 and notice that it becomes injective.] It results from this that

$$\limsup \mathrm{p}_\mathrm{c}^\bullet(G_n^\bullet) \leq \mathrm{p}_\mathrm{c}\big(\mathbb{Z}^2 \times \{-K, \ldots, K\}^{d-2}\big).$$

The right-hand side goes to $\mathrm{p}_\mathrm{c}^\bullet([\mathbb{Z}^d; \boldsymbol{\delta}])$ as K goes to infinity, by Grimmett–Marstrand theorem. This establishes the second semi-continuity. □

REMARK. Proposition 2.6 states exactly what Grimmett–Marstrand theorem implies in our setting. Together with Lemma 2.4, it entails that the hypercubic lattice is a continuity point of p_c on $\mathsf{Cay}(\tilde{\mathbf{G}})$. Without additional idea, one could go a bit further: the proof of Grimmett and Marstrand adjusts directly to the case of Cayley graphs of $\mathbb{Z}^d$ that are stable under reflections relative to coordinate hyperplanes. This statement also has a counterpart analog to Proposition 2.6. However, we are still far from Theorem 2.3, since the Grimmett–Marstrand theorem relies heavily on the stability under reflection. In the rest of the paper, we solve the locality problem for general Abelian Cayley graphs. We do so directly in the marked Abelian group setting, and do not use a "slab result" analog to Grimmett–Marstrand theorem.

2.4. *Proof of Theorem* 2.3. The purpose of this section is to reduce the proof of Theorem 2.3 to the proof of two lemmas (Lemmas 2.8 and 2.9). These are respectively established in Sections 3 and 4.

As in Section 2.3, it is the upper semi-continuity of $\mathrm{p}_\mathrm{c}^\bullet$ that is hard to establish: given $G^\bullet$ and $p > \mathrm{p}_\mathrm{c}^\bullet(G^\bullet)$, we need to show that the parameter p remains supercritical for any element of $\tilde{\mathbf{G}}$ that is close enough to $G^\bullet$. To do so, we will characterize supercriticality by using *a finite-size criterion*, that is a property of the type "$\mathbf{P}_p[\mathcal{E}_N] > 1 - \eta$" for some event $\mathcal{E}_N$ that depends only on the states of the edges in the ball of radius N. The finite-size criterion we use is denoted by $\mathcal{FC}(p, N, \eta)$ and characterizes supercriticality through Lemmas 2.8 and 2.9. Its definition involving heavy notation, we postpone it to Section 3.4.

First, we work with a fixed marked Abelian group $G^\bullet$. Assuming that $p > \mathrm{p}_\mathrm{c}^\bullet(G^\bullet)$, we construct in its Cayley graph a box that is well-connected, with high probability. This is formalized by Lemma 2.8 below, which will be proved in Section 3.

LEMMA 2.8. *Let $G^\bullet \in \tilde{\mathbf{G}}$. Let $p > \mathrm{p}_\mathrm{c}^\bullet(G^\bullet)$ and $\eta > 0$. Then there exists N such that $G^\bullet$ satisfies the finite-size criterion $\mathcal{FC}(p, N, \eta)$.*

Then take $H^\bullet = G^\bullet/\Lambda$ a marked Abelian group that is close to $G^\bullet$. Since $\mathsf{Cay}(G^\bullet)$ and $\mathsf{Cay}(H^\bullet)$ have the same balls of large radius, the finite criterion is also satisfied by $H^\bullet$. This enables us to prove that there is also percolation in $\mathsf{Cay}(H^\bullet)$. As in Grimmett and Marstrand's approach, we will not be able to prove that percolation occurs in $\mathsf{Cay}(H^\bullet)$ for the same parameter p, but we will have to slightly increase the parameter. Here comes a precise statement, established in Section 4.

LEMMA 2.9. *Let $G^\bullet \in \tilde{\mathbf{G}}$. Let $p > \mathrm{p}_\mathrm{c}^\bullet(G^\bullet)$ and $\delta > 0$. Then there exists $\eta > 0$ such that the following holds: if there exists N such that $G^\bullet$ satisfies the finite-size criterion $\mathcal{FC}(p, N, \eta)$, then $\mathrm{p}_\mathrm{c}(H^\bullet) < p + \delta$ for any marked Abelian group $H^\bullet$ close enough to $G^\bullet$.*

Assuming these two lemmas, let us prove Theorem 2.3.

PROOF OF THEOREM 2.3. Let $G_n^\bullet \underset{n\to\infty}{\longrightarrow} G^\bullet$ denote a converging sequence of elements of $\tilde{\mathbf{G}}$. Our goal is to establish that $\mathrm{p}_\mathrm{c}^\bullet(G_n^\bullet) \underset{n\to\infty}{\longrightarrow} \mathrm{p}_\mathrm{c}^\bullet(G^\bullet)$.

For n large enough, $G_n^\bullet$ is a quotient of $G^\bullet$. (See Proposition 2.1.) By Theorem 2.5, for n large enough, $\mathrm{p}_\mathrm{c}^\bullet(G^\bullet) \le \mathrm{p}_\mathrm{c}^\bullet(G_n^\bullet)$. Hence, we only need to prove that $\limsup \mathrm{p}_\mathrm{c}^\bullet(G_n^\bullet) \le \mathrm{p}_\mathrm{c}^\bullet(G^\bullet)$.

Take $p > \mathrm{p}_\mathrm{c}$ and $\delta > 0$. By Lemma 2.8, we can pick N such that $\mathcal{FC}(p, N, \eta)$ is satisfied. Lemma 2.9 then guarantees that, for n large enough, $\mathrm{p}_\mathrm{c}^\bullet(G_n^\bullet) \le p + \delta$, which completes the proof. $\square$

3. Proof of Lemma 2.8. Through the entire section, we fix:

- $G^\bullet \in \tilde{\mathbf{G}}$ a marked Abelian group of rank greater than two,
- $p \in (\mathrm{p}_c^\bullet(G^\bullet), 1)$,
- $\eta > 0$.

We write $G^\bullet$ in the form $[\mathbb{Z}^r \times T; S]$, where T is a finite Abelian group. Let $\mathcal{G} = (V, E) = (\mathbb{Z}^r \times T, E)$ denote the Cayley graph associated to $G^\bullet$. Paths and percolation will always be considered relative to this graph structure.

3.1. *Setting and notation.*

3.1.1. *Between continuous and discrete.* An element of $\mathbb{Z}^r \times T$ will be written

$$x = (x_{\mathrm{free}}, x_{\mathrm{tor}}).$$

For the geometric reasonings, we will use linear algebra tools. (The vertex set—$\mathbb{Z}^r \times T$—is roughly $\mathbb{R}^r$.) Endow $\mathbb{R}^r$ with its canonical Euclidean structure. We denote by $\|\cdot\|$ the associated norm and $\mathbb{B}(v, R)$ the closed ball of radius R centered at $v \in \mathbb{R}^r$. If the center is 0, this ball may be denoted by $\mathbb{B}(R)$. Set $R_S := \max_{s \in S} \|s_{\mathrm{free}}\|$. In $\mathcal{G}$, we define for $k > 0$

$$\begin{aligned} B(k) &:= \{x : \|x_{\mathrm{free}}\| \leq k R_S\} \\ &= (\mathbb{B}(k R_S) \cap \mathbb{Z}^d) \times T. \end{aligned}$$

Up to Section 3.4, we fix an orthornomal basis $\mathbf{e} = (e_1, \dots, e_d)$ of $\mathbb{R}^r$. Define

$$\begin{aligned} \pi_{\mathbf{e}} : \quad & \mathbb{R}^r \longrightarrow \mathbb{R}^2, \\ & \sum_{i=1}^r x_i e_i \longmapsto (x_1, x_2). \end{aligned}$$

We now define the function Graph, which allows us to move between the continuous space $\mathbb{R}^2$ and the discrete set V. It associates to each subset X of $\mathbb{R}^2$ the subset of V defined by

$$\mathsf{Graph}(X) := \big((\pi_{\mathbf{e}}^{-1}(X) + \mathbb{B}(R_S)) \cap \mathbb{Z}^r \big) \times T. \tag{5}$$

In Section 3.4, we will have to consider different bases. To make the dependence on the basis $\mathbf{e}$ explicit, we will write $\mathsf{Graph}_{\mathbf{e}}$.

If a and b belong to $\mathbb{R}^2$, we will consider the segment $[a, b]$ and the parallelogram $[a, b, -a, -b]$ spanned by a and b in $\mathbb{R}^2$, defined respectively by

$$[a, b] = \{\lambda a + (1 - \lambda) b; 0 \leq \lambda \leq 1\} \quad \text{and}$$

$$[a, b, -a, -b] = \{\lambda a + \mu b; |\lambda| + |\mu| \leq 1\}.$$

Write then $L(a, b) := \mathsf{Graph}([a, b])$ and $R(a, b) := \mathsf{Graph}([3a, 3b, -3a, -3b])$ the corresponding subsets of V.

The following lemma illustrates one important property of the function Graph connecting continuous and discrete.

LEMMA 3.1. *Let $X \subset \mathbb{R}^2$. Let γ be a finite path of length k in $\mathcal{G}$. Assume that $\gamma_0 \in \mathsf{Graph}(X)$ and $\gamma_k \notin \mathsf{Graph}(X)$. Then the support of γ intersects $\mathsf{Graph}(\partial X)$.*

PROOF. It suffices to show that if x and y are two neighbours in $\mathcal{G}$ such that $x \in \mathsf{Graph}(X)$ and $y \notin \mathsf{Graph}(X)$, then x belongs to $\mathsf{Graph}(\partial X)$. By definition of Graph, we have $x_{\text{free}} \in \pi^{-1}(X) + \mathbb{B}(R_S)$, which can be restated as

$$\pi\big(\mathbb{B}(x_{\text{free}}, R_S)\big) \cap X \neq \varnothing. \tag{6}$$

By definition of R_S, we have $y_{\text{free}} \in \mathbb{B}(x_{\text{free}}, R_S)$ and our assumption on y implies that $\pi(y_{\text{free}}) \notin X$, which gives

$$\pi\big(\mathbb{B}(x_{\text{free}}, R_S)\big) \cap X^c \neq \varnothing. \tag{7}$$

Since $\pi(\mathbb{B}(x_{\text{free}}, R_S))$ is connected, (6) and (7) imply that

$$\pi\big(\mathbb{B}(x_{\text{free}}, R_S)\big) \cap \partial X \neq \varnothing,$$

which proves that x belongs to $\mathsf{Graph}(\partial X)$. $\square$

3.1.2. *Percolation toolbox.*

Probabilistic notation. We denote by $\mathbf{P}_p$ the law of independent bond percolation of parameter $p \in [0, 1]$ on $\mathcal{G}$.

Connections. Let A, B and C denote three subsets of V. The event "there exists an open path intersecting A and B that lies in C" will be denoted by "$A \overset{C}{\leftrightarrow} B$". The event "restricting the configuration to C, there exists a unique component that intersects A and B" will be written "$A \overset{!C!}{\leftrightarrow} B$". The event "there exists an infinite open path that touches A and lies in C" will be denoted by "$A \overset{C}{\leftrightarrow} \infty$". If the superscript C is omitted, it means that C is taken to be the whole vertex set.

This paragraph contains the percolation results that will be needed to prove Theorem 2.3. The following lemma, sometimes called "square root trick", is a straightforward consequence of Harris–FKG inequality.

LEMMA 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two increasing events. Assume that $\mathbf{P}_p[\mathcal{A}] \geq \mathbf{P}_p[\mathcal{B}]$. Then the following inequality holds*:

$$\mathbf{P}_p[\mathcal{A}] \geq 1 - \big(1 - \mathbf{P}_p[\mathcal{A} \cup \mathcal{B}]\big)^{1/2}.$$

The lemma above is often used when $\mathbf{P}_p[\mathcal{A}] = \mathbf{P}_p[\mathcal{B}]$, in a context where the equality of the two probabilities is provided by symmetries of the underlying graph (see [9]). This slightly generalized version allows us to link geometric properties to probabilistic estimates without any symmetry assumption, as illustrated by the following lemma.

LEMMA 3.3. *Let a and b be two points in $\mathbb{R}^2$. Let $A \subset V$ be a subset of vertices of $\mathcal{G}$. Assume that*

$$\mathbf{P}_p\big[A \leftrightarrow L(a,b)\big] > 1 - \varepsilon^2 \qquad \text{for some } \varepsilon > 0. \tag{8}$$

Then there exists $u \in [a,b]$ such that both $\mathbf{P}_p[A \leftrightarrow L(a,u)]$ and $\mathbf{P}_p[A \leftrightarrow L(u,b)]$ exceed $1-\varepsilon$.

REMARK. Notice that the same statement holds when we restrict the open paths to lie in a subset C of V.

The proof of Lemma 3.3 uses the following simple reasoning. By monotonicity, one can always define u such that

$$\mathbf{P}_p\big[A \leftrightarrow L(a,u)\big] \simeq \mathbf{P}_p\big[A \leftrightarrow L(u,b)\big].$$

Then by the square root trick, we get that both probabilities are large. In our approach, this idea to introduce a parameter to split intervals into pieces with comparable connection probabilities is very useful, and allows us to overcome the lack of symmetry in our graph. In a different context, this idea can also be used to introduce parameters suitable for a renormalization procedure (see, e.g., [15]).

PROOF OF LEMMA 3.3. We can approximate the event estimated in inequality (8) and pick k large enough such that

$$\mathbf{P}_p\big[A \leftrightarrow L(a,b) \cap B(k)\big] > 1 - \varepsilon^2.$$

The set $L(a,b) \cap B(k)$ being finite, there are only finitely many different sets of the form $L(a,u) \cap B(k)$ for $u \in [a,b]$. We can thus construct $u_1, u_2, \dots, u_n \in [a,b]$ such that $u_1 = a$ and $u_n = b$, and for all $1 \le i < n$:

1. $[a, u_i]$ is a strict subset of $[a, u_{i+1}]$,
2. $L(a,b) \cap B(k)$ is the union of $L(a,u_i) \cap B(k)$ and $L(u_{i+1}, b) \cap B(k)$.

Assume that for some i, the following inequality holds:

$$\mathbf{P}_p\big[A \leftrightarrow L(a,u_i) \cap B(k)\big] \ge \mathbf{P}_p\big[A \leftrightarrow L(u_{i+1},b) \cap B(k)\big]. \tag{9}$$

Lemma 3.2 then implies that

$$\mathbf{P}_p\big[A \leftrightarrow L(a,u_i) \cap B(k)\big] > 1 - \varepsilon.$$

If inequality (9) never holds (resp., if it holds for all possible i), then A is connected to $L(\{a\})$ [resp., to $L(\{b\})$] with probability exceeding $1-\varepsilon$. In these two cases, the conclusion of the lemma is trivially true. We can assume that we are in none these two situations, and define $j \in \{2, \dots, n-1\}$ to be the smallest possible i such that inequality (9) holds. We will show the conclusion of Lemma 3.3 holds for $u = u_j$. We already have

$$\mathbf{P}_p\big[A \leftrightarrow L(a,u_j) \cap B(k)\big] > 1 - \varepsilon,$$

and inequality (9) does not hold for $i = j - 1$. Once again, Lemma 3.2 implies that

$$\mathbf{P}_p\big[A \leftrightarrow L(u_j, b) \cap B(k)\big] > 1 - \varepsilon. \qquad \square$$

LEMMA 3.4. *Bernoulli percolation on $\mathcal{G}$ at a parameter $p > \mathrm{p_c}(\mathcal{G})$ produces almost surely a unique infinite component. Moreover, any fixed infinite subset of V is intersected almost surely infinitely many times by the infinite component.*

The first part of the lemma is standard (see [6] or [9]). The second part follows from the 0–1 law of Kolmogorov.

3.2. *Geometric constructions.* In this section, we prove that a set that is connected to infinity with high probability also has "good" local connections. The projection π and the function Graph give us a 2D-representation of the problem. In the proof, one should keep in mind the following limits of this 2D-representation:

- two paths that do not intersect in the graph $\mathcal{G}$ may intersect in projection;
- if A and B are two disjoint sets in the plane, the corresponding sets $\mathsf{Graph}(A)$ and $\mathsf{Graph}(B)$ in the graph G may intersect.

To formalize this, we need a few additional definitions. We say that $(a, b, u, v) \in (\mathbb{R}^2)^4$ is a *good quadruple* if:

1. $u = \frac{a+b}{2}$,
2. $v \in [-a, b]$ and
3. $[a, b, -a, -b]$ contains the planar ball of radius R_S.

See Figure 1 for an illustration. Property 3 ensures that the parallelogram $[a, b, -a, -b]$ is not too degenerate.

To each good quadruple (a, b, u, v), we associate the following four subsets of the graph $\mathcal{G}$:

$$\mathcal{Z}(a, b, u, v) = \big\{L(a, u), L(u, b), L(b, v), L(v, -a)\big\}.$$

LEMMA 3.5. *Let A be a finite subset of V containing 0 and such that*

$$-A := \{-x; x \in A\} = A.$$

Let $k \geq 1$ be such that $B := B(k)$ contains A. Assume the following relation to hold for some $\varepsilon \in (0, 1)$:

$$\mathbf{P}_p[A \leftrightarrow \infty] > 1 - \varepsilon^{24}.$$

Then there exists a good quadruple (a, b, u, v) such that for Z in $\mathcal{Z}(a, b, u, v)$:

(i) $B \cap Z = \varnothing$,
(ii) $\mathbf{P}_p[A \overset{R(a,b)}{\longleftrightarrow} Z] > 1 - \varepsilon$.

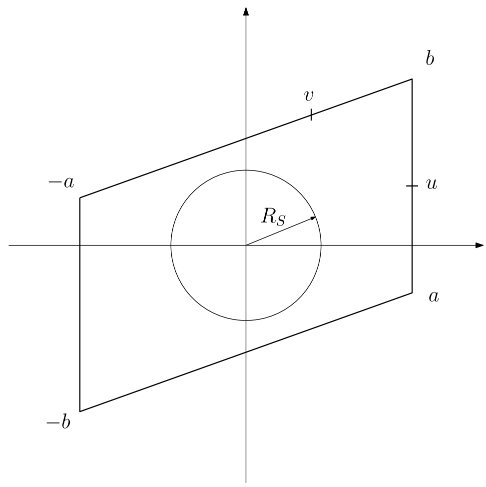

FIG. 1. *A good quadruple.*

PROOF. Let $(n, h, \ell) \in \mathbb{N} \times \mathbb{R} \times \mathbb{R}_+$. Define $a := (n, h - \ell)$, $b := (n, h + \ell)$ and the three following subsets of V illustrated on Figure 2:

$$C(n, h, \ell) := \mathsf{Graph}\big([a, b, -a, -b]\big),$$

$$LR(n, h, \ell) := \mathsf{Graph}\big([a, b] \cup [-a, -b]\big) = L(a, b) \cup L(-a, -b),$$

$$UD(n, h, \ell) := \mathsf{Graph}\big([-a, b] \cup [-b, a]\big) = L(-a, b) \cup L(-b, a).$$

Let us start by focusing on the geometric constraint (i), which we wish to translate into analytic conditions on the triple (n, h, ℓ). We fix $n_B \geq 2$ such that

$$B \cap \mathsf{Graph}\big(\mathbb{R}^2 \setminus (-n_B + 1, n_B - 1)^2\big) = \varnothing. \tag{10}$$

This way, any set defined as the image by the function Graph of a planar set in the complement of $(-n_B + 1, n_B - 1)^2$ will not intersect B. In particular, defining for

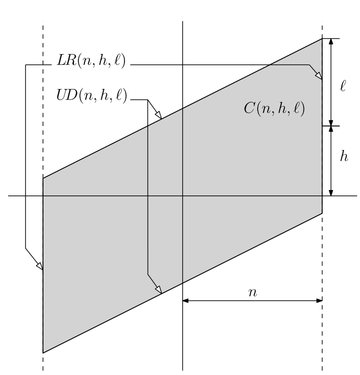

FIG. 2. *Pictures of the planar sets defining $C(n, h, \ell)$, $UD(n, h, \ell)$ and $LR(n, h, \ell)$*

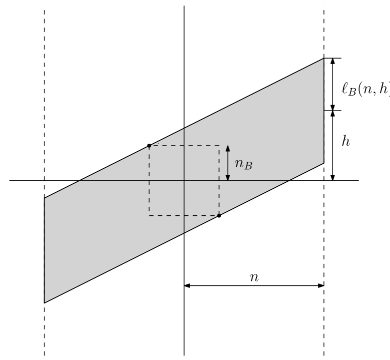

FIG. 3. *Definition of* $\ell_B(n,h)$.

$n > n_B$ and $h \in \mathbb{R}$

$$\ell_B(n,h) = n_B\left(1 + \frac{|h|}{n}\right),$$

the set $UD(n,h,\ell)$ does not intersect B whenever $\ell \geq \ell_B - 1$ (see Figure 3). Suppose that A intersects the infinite cluster. By Lemma 3.4, $V \setminus C(n,h,\ell)$—which is infinite—intersects the infinite cluster almost surely. Thus, there exists an open path from A to $V \setminus C(n,h,\ell)$. By Lemma 3.1, A is connected to $UD(n,h,\ell) \cup LR(n,h,\ell)$ within $C(n,h,\ell)$, which gives the following inequality:

$$\mathbf{P}_p\big[\big(A \overset{C(n,h,\ell)}{\longleftrightarrow} LR(n,h,\ell)\big) \cup \big(A \overset{C(n,h,\ell)}{\longleftrightarrow} UD(n,h,\ell)\big)\big] > 1 - \varepsilon^{24}. \tag{11}$$

The strategy of the proof is to work with some sets $C(n,h,\ell)$ that are balanced in the sense that

$$\mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} LR(n,h,\ell)\big] \quad \text{and} \quad \mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} UD(n,h,\ell)\big]$$

are close, and conclude with Lemma 3.2. We shall now prove two facts, which ensure that the inequality between the two aforementioned probabilities reverses for some ℓ between $\ell_B(n,h)$ and infinity.

FACT 1. *There exists $n > n_B$ such that, for all $h \in \mathbb{R}$, when $\ell = \ell_B(n,h)$*

$$\mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} LR(n,h,\ell)\big] < \mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} UD(n,h,\ell)\big].$$

PROOF. For $n > n_B + R_S$, define the following sets, illustrated on Figure 4:

$$X := \mathsf{Graph}\big(\big((-\infty,n_B) \times \mathbb{R}\big) \cup \big(\mathbb{R} \times [-n_B,\infty)\big)\big),$$

$$\partial X := \mathsf{Graph}\big(\big(\{n_B\} \times (-\infty,-n_B]\big) \cup \big([n_B,\infty) \times \{-n_B\}\big)\big),$$

$$X_n := \mathsf{Graph}\big(\big([-n,n_B) \times \mathbb{R}\big) \cup \big([-n,n] \times [-n_B,\infty)\big)\big),$$

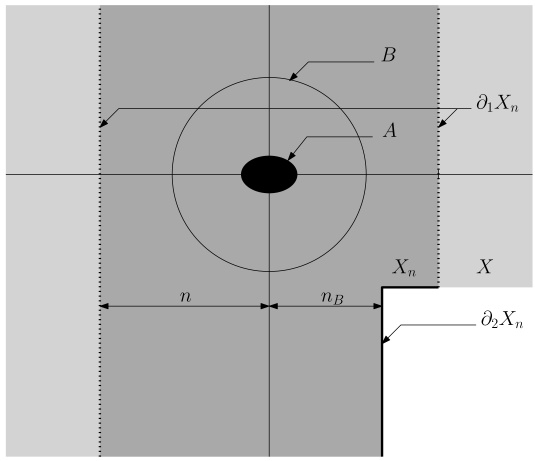

FIG. 4. *Planar pictures corresponding to X, X_n, $\partial_1 X_n$ and $\partial_2 X_n$.*

$$\partial_1 X_n := \mathsf{Graph}\big(\{-n\} \times \mathbb{R} \cup \{n\} \times [-n_B, \infty)\big),$$
$$\partial_2 X_n := \mathsf{Graph}\big(\{n_B\} \times (-\infty, -n_B] \cup [n_B, n] \times \{-n_B\}\big).$$

Since the sequence of events $(A \xleftrightarrow{X_n} \partial_1 X_n)_{n>n_B+R_S}$ is decreasing, we have

$$\begin{aligned}\lim_{n\to\infty} \mathbf{P}_p[A \xleftrightarrow{X_n} \partial_1 X_n] &= \mathbf{P}_p\Big[\bigcap_{n>n_B+R_S} (A \xleftrightarrow{X_n} \partial_1 X_n)\Big] \\ &\le \mathbf{P}_p[A \overset{X}{\leftrightarrow} \infty] \\ &= \mathbf{P}_p\big[(A \overset{X}{\leftrightarrow} \infty) \cap (A \overset{X}{\leftrightarrow} \partial X)\big].\end{aligned} \tag{12}$$

(The last equality results from the fact that the infinite set $V \setminus X$ intersects the infinite cluster almost surely.)

The sequence $(A \xleftrightarrow{X_n} \partial_2 X_n)_{n>n_B+R_S}$ is increasing, hence we have

$$\begin{aligned}\lim_{n\to\infty} \mathbf{P}_p[A \xleftrightarrow{X_n} \partial_2 X_n] &= \mathbf{P}_p\Big[\bigcup_{n>n_B+R_S} (A \xleftrightarrow{X_n} \partial_2 X_n)\Big] \\ &= \mathbf{P}_p[A \leftrightarrow \partial X].\end{aligned} \tag{13}$$

Since $p \in (0, 1)$ and A is finite, the probability that A is connected to ∂X but intersects only finite clusters is positive. Thus, the following strict inequality holds:

$$\mathbf{P}_p\big[(A \overset{X}{\leftrightarrow} \infty) \cap (A \overset{X}{\leftrightarrow} \partial X)\big] < \mathbf{P}_p[A \leftrightarrow \partial X]. \tag{14}$$

From (12), (13) and (14), we can pick $n_1 > n_B + R_S$ large enough such that, for all $n \geq n_1$,

$$\mathbf{P}_p[A \overset{X_n}{\longleftrightarrow} \partial_1 X_n] < \mathbf{P}_p[A \overset{X_n}{\longleftrightarrow} \partial_2 X_n].$$

Fix $n \geq n_1$ and $h \geq 0$, then define $\ell = \ell_B(n,h)$. For these parameters, we have $A \subset C(n,h,\ell) \subset X_n$ and $LR(n,h,\ell) \subset \partial_1 X_n$, which gives

$$\begin{aligned}\mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} LR(n,h,\ell)\big] &\leq \mathbf{P}_p[A \overset{X_n}{\longleftrightarrow} \partial_1 X_n] \\ &< \mathbf{P}_p[A \overset{X_n}{\longleftrightarrow} \partial_2 X_n] \\ &\leq \mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} UD(n,h,\ell)\big].\end{aligned}$$

The last inequality follows from the observation that each path connecting A to $\partial_2 X_n$ inside X_n has to cross $UD(n,h,\ell)$.

The computation above shows that the following strict inequality holds for $n \geq n_1$, $h \geq 0$, and $\ell = \ell_B(n,h)$:

$$\mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} LR(n,h,\ell)\big] < \mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} UD(n,h,\ell)\big]. \tag{15}$$

In the same way, we find n_2 such that for all $n \geq n_2$ and $h \leq 0$, equation (15) holds for $\ell = \ell_B(n,h)$. Taking $n = \max(n_1, n_2)$ completes the proof of the fact. $\square$

In the rest of the proof, we fix n as in the previous fact. For $h \in \mathbb{R}$, define

$$\begin{aligned}\ell_{\mathrm{eq}}(h) = \sup\big\{\ell \geq \ell_B(n,h) - 1 : \mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} UD(n,h,\ell)\big] \\ \geq \mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} LR(n,h,\ell)\big]\big\}.\end{aligned}$$

FACT 2. *For all $h \in \mathbb{R}$, the quantity $\ell_{\mathrm{eq}}(h)$ is finite.*

PROOF. We fix $h \in \mathbb{R}$ and use the same technique as developed in the proof of the fact 1. Define

$$Y := \mathsf{Graph}\big([-n,n] \times \mathbb{R}\big),$$
$$\partial Y := \mathsf{Graph}\big(\{-n,n\} \times \mathbb{R}\big).$$

In the same way, we proved equations (12), (13) and (14) we have here

$$\lim_{\ell\to\infty} \mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} UD(n,h,\ell)\big] \leq \mathbf{P}_p\big[(A \overset{Y}{\leftrightarrow} \infty) \cap (A \overset{Y}{\leftrightarrow} \partial Y)\big],$$

$$\lim_{\ell\to\infty} \mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} LR(n,h,\ell)\big] = \mathbf{P}_p[A \leftrightarrow \partial Y],$$

$$\text{and} \quad \mathbf{P}_p\big[(A \overset{Y}{\leftrightarrow} \infty) \cap (A \overset{Y}{\leftrightarrow} \partial Y)\big] < \mathbf{P}_p[A \leftrightarrow \partial Y].$$

Thus, we can find a finite ℓ large enough such that

$$\mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} UD(n,h,\ell)\big] < \mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} LR(n,h,\ell)\big]. \qquad \square$$

The quantity ℓ_{eq} plays a central role in our proof, linking geometric and probabilistic estimates. We can apply Lemma 3.2 with the two events appearing in inequality (11), to obtain the following alternative:

$$\text{if } \ell < \ell_{\mathrm{eq}}(h), \quad \text{then } \mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} UD(n,h,\ell)\big] > 1 - \varepsilon^{12}, \tag{16a}$$

$$\text{if } \ell > \ell_{\mathrm{eq}}(h), \quad \text{then } \mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} LR(n,h,\ell)\big] > 1 - \varepsilon^{12}. \tag{16b}$$

Fix $(h_{\mathrm{opt}}, \ell_0) \in \mathbb{R} \times \mathbb{R}_+$ such that

$$\ell_{\mathrm{eq}}(h_{\mathrm{opt}}) < \ell_0 < \inf_{h\in\mathbb{R}}\big(\ell_{\mathrm{eq}}(h)\big) + \frac{1}{6}. \tag{17}$$

With such notation, we derive from (16b)

$$\mathbf{P}_p\big[A \overset{C(n,h_{\mathrm{opt}},\ell_0)}{\longleftrightarrow} LR(n,h_{\mathrm{opt}},\ell_0)\big] > 1 - \varepsilon^{12}.$$

Another application of Lemma 3.2 ensures then the existence of a real number h_0 of the form $h_0 = h_{\mathrm{opt}} + \sigma\ell_0/3$ (for $\sigma \in \{-2, 0, +2\}$) such that

$$\mathbf{P}_p\big[A \overset{C(n,h_{\mathrm{opt}},\ell_0)}{\longleftrightarrow} LR(n,h_0,\ell_0/3)\big] > 1 - \varepsilon^{4}.$$

Recall that $LR(n,h_0,\ell_0/3) = L(a_0,b_0) \cup L(-a_0,-b_0)$ with $a_0 = (n, h_0 - \ell_0/3)$ and $b_0 = (n, h_0 + \ell_0/3)$. By symmetry, the set A is connected inside $C(n,h_0,\ell_0/3)$ to $L(a_0,b_0)$ and to $L(-a_0,-b_0)$ with equal probabilities. Applying again Lemma 3.2 gives

$$\mathbf{P}_p\big[A \overset{C(n,h_{\mathrm{opt}},\ell_0)}{\longleftrightarrow} L(a_0,b_0)\big] > 1 - \varepsilon^{2}.$$

(The event estimated above is illustrated on Figure 5 in the case $h_0 = h_{\mathrm{opt}} + 2\ell_0/3$.)

Then use Lemma 3.3 to split $L(a_0,b_0)$ into two parts that both have a high probability to be connected to A: we can pick $u = (n,h) \in [a_0,b_0]$ such that both

$$\mathbf{P}_p\big[A \overset{C(n,h_{\mathrm{opt}},\ell_0)}{\longleftrightarrow} L(a_0,u)\big] \quad \text{and} \quad \mathbf{P}_p\big[A \overset{C(n,h_{\mathrm{opt}},\ell_0)}{\longleftrightarrow} L(u,b_0)\big]$$

exceed $1 - \varepsilon$. Finally, pick ℓ such that $\ell_{\mathrm{eq}}(h) - 1/6 < \ell < \ell_{\mathrm{eq}}(h)$. Define $a = u + (0,-\ell)$ and $b = u + (0,\ell)$. In particular, we have $u = (a+b)/2$. Our choice of ℓ_0 [see equation (17)] implies that $\ell > \ell_0 - 1/3 \geq 2\ell_0/3$, and the following inclusions hold:

$$L(a_0,u) \subset L(a,u),$$

$$L(u,b_0) \subset L(u,b),$$

$$C(n,h_{\mathrm{opt}},\ell_0) \subset R(a,b).$$

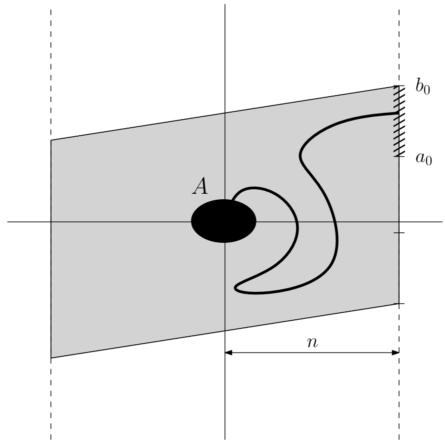

FIG. 5. *Illustration of the event* $A \overset{C(n,h_{\mathrm{opt}},\ell_0)}{\longleftrightarrow} L(a_0,b_0)$ *in the case* $h_0 = h_{\mathrm{opt}} + 2\ell_0/3$.

These three inclusions together with the estimates above conclude the point (ii) of Lemma 3.5 for $Z = L(a,u)$ and $Z = L(u,b)$.

Now, let us construct a suitable vector $v \in [-a,b]$ such that the point (ii) of Lemma 3.5 is satisfied for $Z = L(-a,v)$ and $Z = L(v,b)$. To do so, we consider the connection probabilities inside the set $C(n,h,\ell)$ illustrated in Figure 6. Equation (16a) implies that

$$\mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} UD(n,h,\ell)\big] > 1 - \varepsilon^{12}.$$

As above, using $UD(n,h,\ell) = L(-a,b) \cup L(-b,a)$, symmetries and Lemma 3.2, we obtain

$$\mathbf{P}_p\big[A \overset{C(n,h,\ell)}{\longleftrightarrow} L(-a,b)\big] > 1 - \varepsilon^{6}.$$

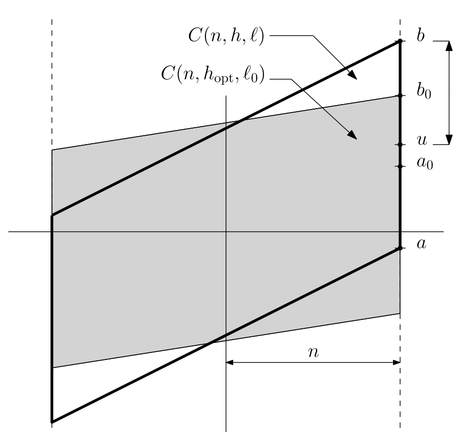

FIG. 6. *Illustration of the sets* $C(n,h_{\mathrm{opt}},\ell_0)$ *and* $C(n,h,\ell)$.

By Lemma 3.3, we can pick $v \in [-a, b]$ such that the following estimate holds for $Z = L(-a, v), L(v, b)$:

$$\mathbf{P}_p[A \overset{C(n,h,\ell)}{\longleftrightarrow} Z] > 1 - \varepsilon^3 \geq 1 - \varepsilon.$$

It remains to verify the point (i). For $Z = L(a, u), L(u, b)$, it follows from $n > n_B$ and the definition of n_B; see equation (10). For $Z = L(-a, v), L(v, b)$, it follows from $\ell > \ell_B(n, h) - 1$ (see Fact 1) and the definition of $\ell_B(n, h)$. $\square$

3.3. *Construction of good blocks.* In this section, we will define a finite block together with a local event that "characterize" supercritical percolation—in the sense that the event happening on this block with high probability will guarantee supercriticality. This block will be used in Section 4 for a coarse graining argument.

In Grimmett and Marstrand's proof of Theorem 1.2, the coarse graining argument uses "seeds" (big balls, all the edges of which are open) in order to propagate an infinite cluster from local connections. More precisely, they define an exploration process of the infinite cluster: at each step, the exploration is successful if it creates a new seed in a suitable place, from which one can iterate the process. If the probability of success at each step is large enough, then with positive probability, the exploration process does not stop and an infinite cluster is created.

In their proof, the seeds grow in the unexplored region. Since we cannot control this region, we use the explored region to produce seeds instead. Formally, long finite self-avoiding paths will play the role of the seeds in the proof of Grimmett and Marstrand. The idea is the following: if a point is reached at some step of the exploration process, it must be connected to a long self-avoiding path, which is enough to iterate the process.

LEMMA 3.6. *For all $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that, for any fixed self-avoiding path γ of length m,*

$$\mathbf{P}_p[\gamma \leftrightarrow \infty] > 1 - \varepsilon.$$

PROOF. By translation invariance, we can restrict ourselves to self-avoiding paths starting at the origin 0. Fix $\varepsilon > 0$. For all $k \in \mathbb{N}$, we consider one self-avoiding path $\gamma^{(k)}$ starting at the origin that minimizes the probability to intersect the infinite cluster among all the self-avoiding paths of length k:

$$\mathbf{P}_p[\gamma^{(k)} \leftrightarrow \infty] = \min_{\gamma:\text{length}(\gamma)=k} \mathbf{P}_p[\gamma \leftrightarrow \infty].$$

By diagonal extraction, we can consider an infinite self-avoiding path $\gamma^{(\infty)}$ such that, for any $k_0 \in \mathbb{N}$, $(\gamma_0^{(\infty)}, \gamma_1^{(\infty)}, \dots, \gamma_{k_0}^{(\infty)})$ is the beginning of infinitely many

$\gamma^{(k)}$'s. By Lemma 3.4, $\gamma^{(\infty)}$ intersects almost surely the infinite cluster of a p-percolation. Thus, there exists an integer k_0 such that

$$\mathbf{P}_p[\{\gamma_0^{(\infty)}, \gamma_1^{(\infty)}, \dots, \gamma_{k_0}^{(\infty)}\} \leftrightarrow \infty] > 1 - \varepsilon.$$

Finally, there exists m such that γ_m begins with the sequence

$$(\gamma_0^{(\infty)}, \gamma_1^{(\infty)}, \dots, \gamma_{k_0}^{(\infty)}),$$

thus it intersects the infinite cluster of a p-percolation with probability exceeding $1 - \varepsilon$. By choice of $\gamma^{(m)}$, it holds for any other self-avoiding path γ of length m that

$$\mathbf{P}_p[\gamma \leftrightarrow \infty] > 1 - \varepsilon. \qquad \square$$

We will focus on paths that start close to the origin. Let us define $\mathcal{S}(m)$ to be the set of self-avoiding paths of length m that start in $B(1)$.

LEMMA 3.7. *For any $\eta > 0$, there exist two integers $1 \le m < N$ and a good quadruple (a, b, u, v) such that*

$$\forall \gamma \in \mathcal{S}(m), \forall Z \in \mathcal{Z}(a, b, u, v), \qquad \mathbf{P}_p\big[\gamma \overset{R(a,b) \cap B(N)}{\longleftrightarrow} Z \cap B(N)\big] > 1 - 3\eta.$$

PROOF. By Lemma 3.6, we can pick m such that any self-avoiding path $\gamma \in \mathcal{S}(m)$ satisfies

$$\mathbf{P}_p[\gamma \leftrightarrow \infty] > 1 - \eta.$$

Pick $k \ge m + 1$ such that

$$\mathbf{P}_p\big[B(k) \leftrightarrow \infty\big] > 1 - \eta^{24}.$$

The number of disjoint clusters [for the configuration restricted to $B(n+1)$] connecting $B(k)$ to $B(n)^c$ converges when n tends to infinity to the number of infinite clusters intersecting $B(k)$. The infinite cluster being unique, we can pick n such that

$$\mathbf{P}_p\big[B(k) \overset{!B(n+1)!}{\longleftrightarrow} B(n)^c\big] > 1 - \eta. \tag{18}$$

Applying Lemma 3.5 with $A = B(k)$ and $B = B(n+1)$ provides a good quadruple (a, b, u, v) such that the following two properties hold for any $Z \in \mathcal{Z}(a, b, u, v)$:

(i) $B(n+1) \cap Z = \varnothing$,

(ii) $\mathbf{P}_p[B(k) \overset{R(a,b)}{\longleftrightarrow} Z] > 1 - \eta$.

Note that condition (i) implies in particular that $B(n+1)$ is a subset of $R(a, b)$. Equation (18) provides with high probability a "uniqueness zone" between $B(k)$ and $B(n)^c$: any pair of open paths crossing this region must be connected inside

$B(n+1)$. In particular, when γ is connected to infinity, and $B(k)$ is connected to Z inside $R(a,b)$, this "uniqueness zone" ensures that γ is connected to Z by an open path lying inside $R(a,b)$:

$$\begin{aligned}
&\mathbf{P}_p[\gamma \overset{R(a,b)}{\longleftrightarrow} Z] \\
&\quad \geq \mathbf{P}_p\big[\{\gamma \leftrightarrow \infty\} \cap \big\{B(k) \overset{!B(n+1)!}{\longleftrightarrow} B(n)^c\big\} \cap \big\{B(k) \overset{R(a,b)}{\longleftrightarrow} Z\big\}\big] \\
&\quad > 1 - 3\eta.
\end{aligned}$$

The identity

$$\mathbf{P}_p[\gamma \overset{R(a,b)}{\longleftrightarrow} Z] = \lim_{N\to\infty} \mathbf{P}_p\big[\gamma \overset{R(a,b)\cap B(N)}{\longleftrightarrow} Z \cap B(N)\big]$$

completes the proof of Lemma 3.7. $\square$

3.4. *Construction of a finite-size criterion.* In this section, we give a precise definition of the finite-size criterion $\mathcal{FC}(p, N, \eta)$ used in Lemmas 2.8 and 2.9. Its construction is based on Lemma 3.7.

Recall that, up to now, we worked with a fixed orthonormal basis $\mathbf{e}$, which was hidden in the definition of $\mathsf{Graph} = \mathsf{Graph}_{\mathbf{e}}$; see equation (5). In order to perform the coarse graining argument in any marked group $G^\bullet/\Lambda$ close to $G^\bullet$, we will need to have the conclusion of Lemma 3.7 for all the orthonormal bases simultaneously.

Denote by $\mathcal{B}$ the set of the orthonormal basis of $\mathbb{R}^r$. It is a compact subset of $\mathbb{R}^{r\times r}$. If we fix $X \subset \mathbb{R}^2$, a positive integer N and $\mathbf{e} \in \mathcal{B}$ then the following inclusion holds for any orthonormal basis $\mathbf{f}$ close enough to $\mathbf{e}$ in $\mathcal{B}$:

$$\mathsf{Graph}_{\mathbf{e}}(X) \cap B(N) \subset \big(\mathsf{Graph}_{\mathbf{f}}(X) + B(1)\big) \cap B(N). \tag{19}$$

We define $\mathcal{N}(\mathbf{e}, N) \subset \mathcal{B}$ to be the neighbourhood of $\mathbf{e}$ formed by the orthonormal bases $\mathbf{f}$ for which the inclusion above holds. A slight modification of the orthonormal basis in Lemma 3.7 keeps its conclusion with the same integer N and the same vectors a, b, u, v, but with:

- $Z + B(1)$ in place of Z
- and $R(a,b) + B(1)$ instead of $R(a,b)$.

In order to state this result properly, let us define

$$\begin{aligned}
\mathcal{Z}_{N,\mathbf{e}}(a,b,u,v) &:= \big\{\big(Z + B(1)\big) \cap B(N) : Z \in \mathcal{Z}_{\mathbf{e}}(a,b,u,v)\big\}, \\
R_{N,\mathbf{e}}(a,b) &:= \big(R(a,b) + B(1)\big) \cap B(N).
\end{aligned}$$

Note that we add the subscript $\mathbf{e}$ here to insist on the dependence on the basis $\mathbf{e}$. This dependence was implicit for the sets Z and $R(a,b)$ which were defined via the function Graph.

We are ready to define the finite size criterion $\mathcal{FC}(p, N, \eta)$ that appears in Lemmas 2.8 and 2.9.

DEFINITION OF THE FINITE-SIZE CRITERION. Let $N \geq 1$ and $\eta > 0$. We say that the finite size criterion $\mathcal{FC}(p, N, \eta)$ is satisfied if for any $\mathbf{e} \in \mathcal{B}$, there exist $1 \leq m < N$ and a good quadruple (a, b, u, v) such that

$$\forall \gamma \in \mathcal{S}(m), \forall Z \in \mathcal{Z}_{N,\mathbf{e}}(a, b, u, v), \qquad \mathbf{P}_p[\gamma \overset{R_{N,\mathbf{e}}(a,b)}{\longleftrightarrow} Z] > 1 - \eta. \tag{20}$$

PROOF OF LEMMA 2.8. Let $\eta > 0$. By Lemma 3.7, we can associate to every orthonormal basis $\mathbf{e}$ two integers $m_\mathbf{e}, N_\mathbf{e} \in \mathbb{N}$, and a good quadruple $(a_\mathbf{e}, b_\mathbf{e}, u_\mathbf{e}, v_\mathbf{e})$ such that the following holds (we omit the subscript for the parameters m, a, b, u, v):

$$\forall \gamma \in \mathcal{S}(m), \forall Z \in \mathcal{Z}_\mathbf{e}(a, b, u, v), \qquad \mathbf{P}_p\big[\gamma \overset{R_\mathbf{e}(a,b) \cap B(N_\mathbf{e})}{\longleftrightarrow} Z \cap B(N_\mathbf{e})\big] > 1 - \eta.$$

For any $\mathbf{f} \in \mathcal{N}(\mathbf{e}, N_\mathbf{e})$, we can use the inclusion (19) to derive from the estimate above that for all $\gamma \in \mathcal{S}(m)$ and $Z \in \mathcal{Z}_\mathbf{f}(a, b, u, v)$,

$$\mathbf{P}_p\big[\gamma \overset{(R_\mathbf{f}(a,b)+B(1)) \cap B(N_\mathbf{e})}{\longleftrightarrow} \big(Z + B(1)\big) \cap B(N_\mathbf{e})\big] > 1 - \eta.$$

By compactness of $\mathcal{B}$, we can find a finite subset $\mathcal{F} \subset \mathcal{B}$ of bases such that

$$\mathcal{B} = \bigcup_{\mathbf{e} \in \mathcal{F}} \mathcal{N}(\mathbf{e}, N_\mathbf{e}).$$

For $N := \max_{\mathbf{e} \in \mathcal{F}} N_\mathbf{e}$, the finite-size criterion $\mathcal{FC}(p, N, \eta)$ is satisfied. □

4. Proof of Lemma 2.9.

Through the entire section, we fix:

- $G^\bullet \in \tilde{\mathbf{G}}$ a marked Abelian group of rank greater than two,
- $p \in (\mathrm{p}_c^\bullet(G^\bullet), 1)$,
- $\delta > 0$.

Let $\mathcal{G} = (V, E)$ denote the Cayley graph associated to $G^\bullet$.

4.1. *Hypotheses and notation.* Let us start by an observation that follows from the definition of good quadruple at the beginning of Section 3.2: there exists an absolute constant κ such that for any good quadruple (a, b, u, v) and any $w \in \mathbb{R}^2$,

$$\mathrm{Card}\big\{z \in \mathbb{Z}^2 : w + z_1 u + z_2 v \in [5a, 5b, -5a, -5b]\big\} \leq \kappa.$$

We fix κ as above and choose $\eta > 0$ such that

$$p_0 := \sup_{t \in \mathbb{N}} \big\{1 - (1 - \delta/\kappa)^t - \eta(1 - p)^{-t}\big\} > \mathrm{p}_c^{\mathrm{site}}\big(\mathbb{Z}^2\big). \tag{21}$$

We will prove that this choice of η provides the conclusion of Lemma 2.9. We assume that $G^\bullet$ satisfies $\mathcal{FC}(p, N, \eta)$ for some positive integer N (which will be fixed throughout this section). Let us consider a marked Abelian group $H^\bullet = G^\bullet/\Lambda$ of rank at least 2 and such that

$$\Lambda \cap B(2N + 1) = \{0\}.$$

(Notice that such $H^\bullet$'s form a neighbourhood of $G^\bullet$ in $\tilde{\mathcal{G}}$ by Proposition 2.1.) Under these hypotheses, we will prove that $\mathrm{p_c}(H^\bullet) < p+\delta$, providing the conclusion of Lemma 2.9.

The Cayley graph of $H^\bullet = G^\bullet/\Lambda$ is denoted by $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$. For $x \in V$, we write $\bar{x}$ for the image of x by the quotient map $G \to G/\Lambda$. This quotient map naturally extends to subsets of V and we write $\overline{A}$ for the image of a set $A \subset V$.

4.2. *Sketch of proof.* Under the hypotheses above, we show that percolation occurs in $\overline{\mathcal{G}}$ at parameter $p+\delta$. The proof goes proceeds as follows.

Step 1: Geometric construction. We construct a renormalized graph, that is a family of big boxes (living in $\overline{\mathcal{G}}$) arranged as a square lattice. This gives rise to a notion of neighbour boxes, and the occurrence of the finite-size criterion $\mathcal{FC}(p, N, \eta)$ will imply good connection probabilities between neighbouring boxes. This is the object of Lemma 4.2.

Step 2: Construction of an infinite cluster. Relying on the renormalized graph constructed in the first step, we couple a $(p+\delta)$-percolation on $\overline{\mathcal{G}}$ and a percolation on $\mathbb{Z}^2$ in such a way that the existence of an infinite component in $\mathbb{Z}^2$ would imply an infinite component in $\overline{\mathcal{G}}$. This event will happen with positive probability. The introduction of the parameter δ will allow us to apply a "sprinkling" technique in the coupling argument developed in the proof of Lemma 4.4.

4.3. *Geometric setting*: *Boxes and corridors*. Since Λ has a co-rank of at least 2, we can fix an orthonormal basis $\mathbf{e} \in \mathcal{B}$ such that

$$\Lambda \subset \operatorname{Ker}(\pi_{\mathbf{e}}) \times T. \tag{22}$$

Condition (22) ensures that sets defined in $\mathcal{G}$ via the function $\mathsf{Graph}_{\mathbf{e}}$ have a suitable image in the quotient $\overline{\mathcal{G}}$. More precisely, for any $x \in V$ and any planar set $X \subset \mathbb{R}^2$, we have

$$x \in \mathsf{Graph}_{\mathbf{e}}(X) \quad \Longleftrightarrow \quad \overline{x} \in \overline{\mathsf{Graph}_{\mathbf{e}}(X)}. \tag{23}$$

According to $\mathcal{FC}(p, N, \eta)$, there exists $m < N$ and a good quadruple (a, b, u, v) such that

$$\forall \gamma \in \mathcal{S}(m), \forall Z \in \mathcal{Z}_{N,\mathbf{e}}(a, b, u, v), \qquad \mathbf{P}_p\big[\gamma \overset{R_{N,\mathbf{e}}(a,b)}{\longleftrightarrow} Z\big] > 1-\eta.$$

We introduce here some subsets of $\overline{\mathcal{G}}$, that will play the role of vertices and edges in the renormalized graph.

Box. For z in $\mathbb{Z}^2$, define

$$B_z := \overline{\mathsf{Graph}\big(z_1 u + z_2 v + [a, b, -a, -b]\big)}.$$

When z and z' are neighbours in $\mathbb{Z}^2$ for the standard graph structure, we write $z \sim z'$. In this case, we say that the two boxes B_z and $B_{z'}$ are *neighbours*.

Corridor. For z in $\mathbb{Z}^2$, define

$$C_z := \overline{\mathsf{Graph}\big(z_1 u + z_2 v + [4a, 4b, -4a, -4b]\big)}.$$

We will explore the cluster of the origin in $\overline{\mathcal{G}}$. If the cluster reaches a box B_z, we will try to spread it to the neighbouring boxes ($B_{z'}$ for $z' \sim z$) by creating paths that lie in their respective corridors $C_{z'}$. For this strategy to work, we need the boxes to have good connection probabilities and the corridors to be "sufficiently disjoint": if the exploration is guaranteed to visit each corridor at most $\kappa + 1$ times, then we do not need more than κ "sprinkling operations". These two properties are formalized by the following two lemmas.

LEMMA 4.1. *For all $\bar{x} \in V$,*

$$\operatorname{Card}\{z \in \mathbb{Z}^2 / \bar{x} \in C_z\} \leq \kappa. \tag{24}$$

PROOF. By choice of the basis, equivalence (23) holds and implies, for any $z = (z_1, z_2) \in \mathbb{Z}^2$,

$$\bar{x} \in C_z \quad \iff \quad x \in \mathsf{Graph}_{\mathbf{e}}\big(z_1 u + z_2 v + [4a, 4b, -4a, -4b]\big).$$

By the last condition defining a good quadruple,

$$\bar{x} \in C_z \implies \pi(x) \in z_1 u + z_2 v + [5a, 5b, -5a, -5b].$$

The choice of κ at the beginning of the section [see equation (24)] completes the proof. □

LEMMA 4.2. *For any pair of neighbouring boxes $(B_z, B_{z'})$,*

$$\forall \bar{x} \in B_z, \forall \gamma \in \mathcal{S}(m), \qquad \mathbf{P}_p\big[\bar{x} + \overline{\gamma} \overset{C_{z'}}{\longleftrightarrow} B_{z'} + \overline{B(1)}\big] > 1 - \eta. \tag{25}$$

PROOF. We assume that $z' = z + (0, 1)$. The cases of $z + (1, 0)$, $z + (0, -1)$ and $z + (-1, 0)$ are treated the same way.

The assumption $\Lambda \cap B(2N + 1) = \{0\}$ implies that $\overline{R_{N,\mathbf{e}}(a, b)}$ is isomorphic (as a graph) to $R_{N,\mathbf{e}}(a, b)$. It allows us to derive from estimate (20) that

$$\mathbf{P}_p[\overline{\gamma} \overset{\overline{R_{N,\mathbf{e}}(a,b)}}{\longleftrightarrow} \overline{Z}] > 1 - \eta. \tag{26}$$

Now let B_z and $B_{z'}$ be two neighbouring boxes. Let $\bar{x}$ be any vertex of B_z. By translation invariance, we get from (26) that

$$\mathbf{P}_p[x + \overline{\gamma} \overset{\bar{x} + \overline{R_{N\cdot\mathbf{e}}(a,b)}}{\longleftrightarrow} \bar{x} + \overline{Z}] > 1 - \eta.$$

Here comes the key geometric observation: there exists $Z \in \mathcal{Z}_{N,\mathbf{e}}(a, b, u, v)$ such that

$$\bar{x} + \overline{Z} \subset B_{z'} + \overline{B(1)}.$$

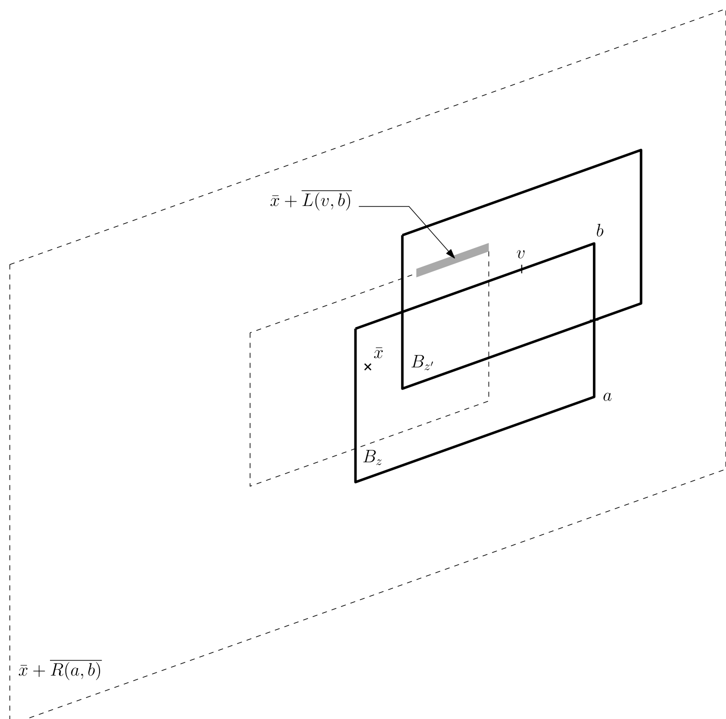

FIG. 7. *If $\bar{x}$ is in the left part of the box B_z, then $\bar{x} + \overline{L(v,b)} \subset B_{z'}$.*

This is illustrated on Figures 7 and 8 when $z = (0,0)$ and $z' = (0,1)$. Besides, $\bar{x} + \overline{R_N(a,b)} \subset C_{z'}$. Hence, by monotonicity, we obtain that

$$\mathbf{P}_p\big[\bar{x} + \overline{\gamma} \overset{C_{z'}}{\longleftrightarrow} B_{z'} + \overline{B(1)}\big] > 1 - \eta. \qquad \square$$

4.4. *Probabilistic setting.* Let ω_0 be Bernoulli percolation of parameter p on $\overline{\mathcal{G}}$. In order to apply a "sprinkling argument", we define for every $z \in \mathbb{Z}^2$ a sequence $(\xi^z(e))_{e \text{ edges in } C_z}$ of independent Bernoulli variables of parameter $\frac{\delta}{\kappa}$. In other words, ξ^z is a $\frac{\delta}{\kappa}$-percolation on C_z. We assume that ω_0 and all the ξ^z's are independent. Lemma 4.1 implies that at most $\kappa + 1$ Bernoulli variables are associated to a given edge e: $\omega_0(e)$ and the $\xi^z(e)$'s for z such that $e \subset C_z$.

To state Lemma 4.3, we also need the notion of edge-boundary. The *edge-boundary* of a set A of vertices is the set of the edges of $\mathcal{G}$ with exactly one endpoint in A. It is denoted by ΔA.

LEMMA 4.3. *Let B_z and $B_{z'}$ be two neighbouring boxes. Let H be a subset of $\overline{V}$. Let $(\omega(e))_{e \in E}$ be a family of independent Bernoulli variables of parameter*

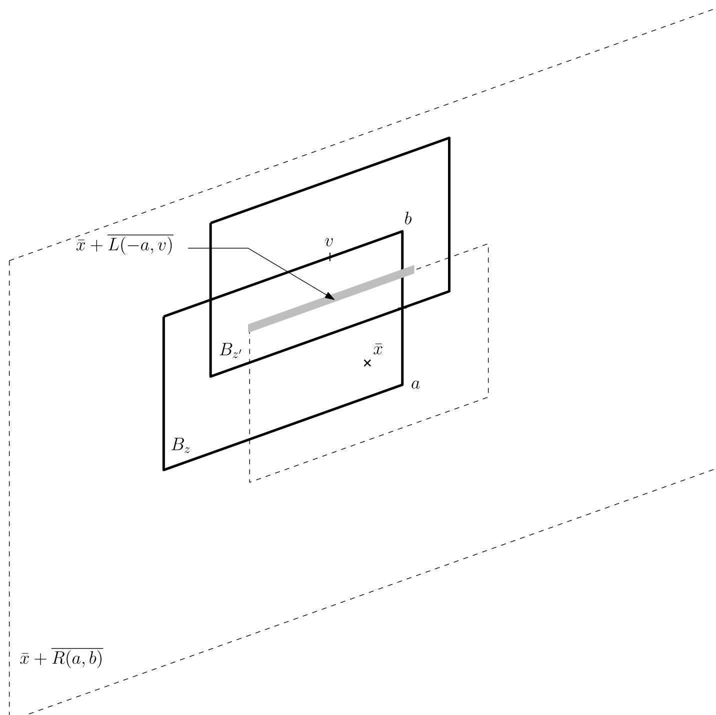

FIG. 8. *If $\bar{x}$ is in the right part of the box B_z, then $\bar{x} + \overline{L(-a,v)} \subset B_{z'}$.*

$\mathbf{P}[\omega(e) = 1] \in [p, 1)$, *independent of $\xi^{z'}$. If there exists $\bar{x} \in B_z$ and $\gamma \in \mathcal{S}(m)$ such that $\bar{x} + \overline{\gamma} \subset H$, then*

$$\mathbf{P}\big[H \underset{\omega \vee \xi^{z'}}{\overset{C_{z'}}{\longleftrightarrow}} B_{z'} + \overline{B(1)} \mid \forall e \in \Delta H, \omega(e) = 0\big] \geq p_0.$$

PROOF. In all of this proof, the marginals of ω are assumed to be Bernoulli random variables of parameter p. The more general statement of Lemma 4.3 follows by a stochastic domination argument. The case $H \cap (B_{z'} + \overline{B(1)}) \neq \varnothing$ being trivial, we assume that $H \cap (B_{z'} + \overline{B(1)}) = \varnothing$.

Let $W \subset \Delta H$ be the (random) set of edges $\{\bar{x}, \bar{y}\} \subset C_{z'}$ such that:

(i) $\bar{x} \in H$, $\bar{y} \in C_{z'} \setminus H$ and
(ii) there is an ω-open path joining $\bar{y}$ to $B_{z'} + \overline{B(1)}$, lying in $C_{z'}$, but using no edge with an endpoint in H.

In a first step, we want to say that $|W|$ cannot be too small. The inclusions $\bar{x} + \overline{\gamma} \subset H \subset (B_{z'} + \overline{B(1)})^c$ imply that any ω-open path from $\bar{x} + \overline{\gamma}$ to $B_{z'} + \overline{B(1)}$ must contain at least one edge of W. Thus, there is no ω-open path connecting $\bar{x} + \overline{\gamma}$ to $B_{z'} + \overline{B(1)}$ in $C_{z'}$ when all the edges of W are ω-closed. Consequently, for any

$t \in \mathbb{N}$, we have

$$\mathbf{P}\big[\big(\bar{x} + \overline{\gamma} \overset{C_{z'}}{\underset{\omega}{\longleftrightarrow}} B_{z'} + \overline{B(1)}\big)^c\big] \geq \mathbf{P}[\text{all edges in } W \text{ are } \omega\text{-closed}]$$
$$\geq (1-p)^t \mathbf{P}\big[|W| \leq t\big].$$

To get the last inequality above, remark that the random set W is independent from the ω-state of the edges in ΔH. Using estimate (25), it can be rewritten as

$$\mathbf{P}\big[|W| \leq t\big] \leq \eta(1-p)^{-t}. \tag{27}$$

We distinguish two cases. Either W is small, which has a probability estimated by equation (27) above; or W is large, and we use in that case that $B_{z'} + \overline{B(1)}$ is connected to H as soon as one edge of W is $\xi^{z'}$-open. The following computation makes this quantitative:

$$\begin{aligned}
&\mathbf{P}\big[H \overset{C_{z'}}{\underset{\omega \vee \xi^{z'}}{\longleftrightarrow}} B_{z'} + \overline{B(1)} \,|\, \forall e \in \Delta H, \omega(e) = 0\big] \\
&\quad \geq \mathbf{P}\big[\text{at least one edge of } W \text{ is } \xi^{z'}\text{-open} \,|\, \forall e \in \Delta H, \omega(e) = 0\big] \\
&\quad = \mathbf{P}\big[\text{at least one edge of } W \text{ is } \xi^{z'}\text{-open}\big] \\
&\quad \geq \mathbf{P}\big[\text{at least one edge of } W \text{ is } \xi^{z'}\text{-open and } |W| > t\big] \\
&\quad \geq 1 - \mathbf{P}\big[\text{all the edges of } W \text{ are } \xi^{z'}\text{-closed} \,|\, |W| > t\big] - \mathbf{P}\big[|W| \leq t\big].
\end{aligned}$$

Using equation (27), we conclude that, for any t,

$$\mathbf{P}\big[H \overset{C_{z'}}{\underset{\omega \vee \xi^{z'}}{\longleftrightarrow}} A \,|\, \forall e \in \Delta H, \xi^{z'}(e) = 0\big] \geq 1 - (1 - \delta/\kappa)^t - \eta(1-p)^{-t}. \tag{28}$$

Our choice of η in (21) makes the right-hand side of (28) larger than p_0. □

LEMMA 4.4. *With positive probability, the origin is connected to infinity in the configuration*

$$\omega_{\text{total}} := \omega_0 \vee \bigvee_{z \in \mathbb{Z}^2} \xi^z.$$

Lemma 4.4 completes the proof of Lemma 2.9 because ω_{total} is stochastically dominated by a $(p+\delta)$-percolation. Indeed, $(\omega_{\text{total}}(e))_e$ is an independent sequence of Bernoulli variables such that, for any edge e,

$$\mathbf{P}\big[\omega_{\text{total}}(e) = 1\big] \leq 1 - (1-p)(1-\delta/\kappa)^\kappa \leq p + \delta.$$

PROOF OF LEMMA 4.4. The strategy of the proof is similar to the one described in the original paper of Grimmett and Marstrand: we explore the Bernoulli variables one after the other in an order prescribed by the algorithm hereafter. During the exploration, we define simultaneously random variables on the graph $\overline{\mathcal{G}}$ and on the square lattice $\mathbb{Z}^2$.

Algorithm.

(0) Set $z(0) = (0, 0) \in \mathbb{Z}^2$. Explore the connected component H_0 of the origin in $\mathcal{G}$ in the configuration ω_0. Notice that only the edges of $H_0 \cup \Delta H_0$ have been explored in order to determine H_0.

- If H_0 contains a path of $\mathcal{S}(m)$, set $X((0, 0)) = 1$ and $(U_0, V_0) = (\{0\}, \varnothing)$ and move to $(t = 1)$.
- Else, set $X((0, 0)) = 0$ and $(U_0, V_0) = (\varnothing, \{0\})$ and move to $(t = 1)$.

(t) Call *unexplored* the vertices in $\mathbb{Z}^2 \setminus (U_t \cup V_t)$. Examine the set of unexplored vertices neighbouring an element of U_t. If this set is empty, define $(U_{t+1}, V_{t+1}) = (U_t, V_t)$ and move to $(t + 1)$. Otherwise, choose such an unexplored vertex z_t. In the configuration $\omega_{t+1} := \omega_t \vee \xi^{z_t}$, explore the connected component H_{t+1} of the origin.

- If $H_{t+1} \cap B_{z_t} \neq \varnothing$, which means in particular that B_{z_t} is connected to 0 by an ω_{t+1}-open path, then set $X(z_t) = 1$ and $(U_{t+1}, V_{t+1}) = (U_t \cup \{z_t\}, V_t)$ and move to $(t + 1)$.
- Else set $X(z_t) = 0$ and $(U_{t+1}, V_{t+1}) = (U_t, V_t \cup \{z_t\})$ and move to $(t + 1)$.

This algorithm defines in particular:

- a random process growing in the lattice $\mathbb{Z}^2$,
$$S_0 = (U_0, V_0), S_1 = (U_1, V_1), \ldots,$$
- a random sequence $(X(z_t))_{t\geq 0}$.

Lemma 4.3 ensures that for all $t \geq 1$, whenever z_t is defined,

$$\mathbf{P}\big[X(z_t) = 1 | S_0, S_1, \ldots, S_{t-1}\big] \geq p_0 > \mathrm{p}_\mathrm{c}^{\mathrm{site}}(\mathbb{Z}^2). \tag{29}$$

Estimate (29) states that each time we explore a new site z_t, whatever the past of the exploration is, we have a sufficiently high probability of success: together with Lemma 1 of [12], it ensures that

$$\mathbf{P}\big[|U| = \infty\big] > 0,$$

where $U := \bigcup_{t\geq 0} U_t$ is the set of z_t's such that $X(z_t)$ equals 1. For such z_t's, we know that B_{z_t} is connected to the origin of $\overline{\mathcal{G}}$ by an ω_{t+1}-open path. Hence, when U is infinite, there must exist an infinite open connected component in the configuration

$$\omega_0 \vee \bigvee_{t\geq 0} \xi^{z_t},$$

which is a sub-configuration of ω_{total}, and Lemma 4.4 is established. $\square$

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