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A LOWER BOUND FOR DISCONNECTION BY SIMPLE RANDOM WALK

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We consider simple random walk on $\mathbb{Z}^d$, $d \geq 3$. Motivated by the work of A.-S. Sznitman and the author in [*Probab. Theory Related Fields* **161** (2015) 309–350] and [*Electron. J. Probab.* **19** (2014) 1–26], we investigate the asymptotic behavior of the probability that a large body gets disconnected from infinity by the set of points visited by a simple random walk. We derive asymptotic lower bounds that bring into play random interlacements. Although open at the moment, some of the lower bounds we obtain possibly match the asymptotic upper bounds recently obtained in [Disconnection, random walks, and random interlacements (2014)]. This potentially yields special significance to the tilted walks that we use in this work, and to the strategy that we employ to implement disconnection.

CONTENTS

0. Introduction. How hard is it to disconnect a macroscopic body from infinity by the trace of a simple random walk in $\mathbb{Z}^d$, when $d \geq 3$? In this work, we partially answer this question, motivated by [23] and [22], by deriving an asymptotic lower bound on the probability of such a disconnection. Remarkably, our bounds bring into play random interlacements as well as a suitable strategy to implement disconnection. Although open at the moment, some of the lower bounds we obtain in this work may be sharp, and match the recent upper bounds from [29].

We now describe the model and our results in a more precise fashion. We refer to Section 1 for precise definitions. We consider the continuous-time simple random

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walk on $\mathbb{Z}^d$, $d \geq 3$, and we denote by P_0 the (canonical) law of the walk starting from the origin. We denote by $\mathcal{V} = \mathbb{Z}^d \setminus X_{[0,\infty)}$ the complement of the set of points visited by the walk.

We consider K, a non-empty compact subset of $\mathbb{R}^d$ and for $N \geq 1$ its discrete blow-up:

$$K_N = \{x \in \mathbb{Z}^d; d_\infty(x, NK) \leq 1\}, \tag{0.1}$$

where NK, a non-empty compact subset of $\mathbb{R}^d$, stands for the set homothetic to K with ratio N, and

$$d_\infty(z, NK) = \inf_{y \in NK} |z - y|_\infty \tag{0.2}$$

stands for the sup-norm distance of z to NK. Of central interest for us is the event specifying that K_N is not connected to infinity in $\mathcal{V}$, which we denote by

$$\{K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty\}. \tag{0.3}$$

Our main result brings into play the model of random interlacements. Informally, random interlacements in $\mathbb{Z}^d$ are a Poissonian cloud of doubly-infinite nearest-neighbor paths, with a positive parameter u, which is a multiplicative factor of the intensity of the cloud (we refer to [6] and [9] for further details and references). We denote by $\mathcal{I}^u$ the trace of random interlacements of level u on $\mathbb{Z}^d$, and by $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$ the corresponding vacant set. It is known that there is a critical value $u_{**} \in (0, \infty)$, which can be characterized as the infimum of the levels $u > 0$ for which the probability that the vacant cluster at the origin reaches distance N from the origin has a stretched exponential decay in N; see [28] or [9].

The main result of this article is the following asymptotic lower bound, which confirms the conjecture proposed in Remark 5.1(2) of [22].

THEOREM 0.1.

$$\liminf_{N \to \infty} \frac{1}{N^{d-2}} \log\big(P_0[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty]\big) \geq -\frac{u_{**}}{d} \mathrm{cap}_{\mathbb{R}^d}(K), \tag{0.4}$$

where $\mathrm{cap}_{\mathbb{R}^d}(K)$ *stands for the Brownian capacity of* K.

Actually, the proof of Theorem 0.1 (after minor changes) also shows that for any $M > 1$,

$$\liminf_{N \to \infty} \frac{1}{N^{d-2}} \log\big(P_0[B_N \overset{\mathcal{V}}{\nleftrightarrow} S_N]\big) \geq -\frac{u_{**}}{d} \mathrm{cap}_{\mathbb{R}^d}\big([-1, 1]^d\big), \tag{0.5}$$

where $B_N = \{x \in \mathbb{Z}^d; |x|_\infty \leq N\}$ and $S_N = \{x \in \mathbb{Z}^d; |x|_\infty = [MN]\}$ with $[MN]$ the integer part of MN; see Remark 6.1.

On the other hand, the recent article [29] improves on [33], and shows that for any $M > 1$, the following asymptotic upper bound holds:

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log\big(P_0[B_N \overset{\mathcal{V}}{\nleftrightarrow} S_N]\big) \le -\frac{\overline{u}}{d} \mathrm{cap}_{\mathbb{R}^d}\big([-1,1]^d\big), \tag{0.6}$$

where $\overline{u}$ is a certain critical level introduced in [29], such that $0 < u < \overline{u}$ corresponds to the *strongly percolative* regime of $\mathcal{V}^u$. Precisely, one knows that $0 < \overline{u} \le u_* \le u_{**} < \infty$, where u_* stands for the critical level for the percolation of $\mathcal{V}^u$ (the positivity of $\overline{u}$, for all $d \ge 3$, actually stems from [10] as explained in Section 2 of [29]). It is plausible, but unproven at the moment, that actually $\overline{u} = u_* = u_{**}$. If this is the case, the asymptotic lower bound (0.5) from the present article matches the asymptotic upper bound (0.6) from [29].

In the case of (0.4), one can also wonder whether one actually has the following asymptotics (possibly with some regularity assumption on K)

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log\big(P_0[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty]\big) = -\frac{u_*}{d} \mathrm{cap}_{\mathbb{R}^d}(K). \tag{0.7}$$

Our proof of Theorem 0.1 [and of (0.5)] relies on the change of probability method. The feature that the asymptotic lower bounds, which we derive in this article, are potentially sharp, yields special significance to the strategy that we employ to implement disconnection.

Let us give some comments about the strategy and the proof. We construct through fine-tuned Radon–Nikodym derivatives new measures $\widetilde{P}_N$, corresponding to the "tilted walks". In essence, these walks evolve as recurrent walks with generator $\widetilde{L}g(x) = \frac{1}{2d}\sum_{|x'-x|=1} \frac{h_N(x')}{h_N(x)}(g(x') - g(x))$, up to a deterministic time T_N, and then as the simple random walk afterward, with $h_N(x) = h(\frac{x}{N})$, where h is the solution of (assuming that K is regular)

$$\begin{cases} \Delta h = 0, & \text{on } \mathbb{R}^d \setminus K, \\ h = 1, & \text{on } K, \text{ and } h \text{ tends to } 0 \text{ at } \infty, \end{cases} \tag{0.8}$$

and T_N is chosen so that the expected time spent by the tilted walk up to T_N at any x in K_N is $u_{**}h_N^2(x) = u_{**}$ (by the choice of h). Informally, $\widetilde{P}_N$ achieves this at a "low entropic cost". Quite remarkably, this constraint on the *time* spent at points and low entropic cost induces a local behavior of the trace of the tilted walk which *geometrically* behaves as random interlacements with a slowly space-modulated parameter $u_{**}h_N^2(x)$, at least close to K_N. This creates a "fence" around K_N, where the vacant set left by the tilted walk is locally in a strongly non-percolative mode, so that

$$\lim_{N\to\infty} \widetilde{P}_N[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty] = 1. \tag{0.9}$$

On the other hand, we show that

$$\widetilde{\lim_{N\to\infty}} \frac{1}{N^{d-2}} H(\widetilde{P}_N | P_0) \le \frac{u_{**}}{d} \mathrm{cap}_{\mathbb{R}^d}(K), \tag{0.10}$$

where $\widetilde{\lim}$ refers to a certain limiting procedure, in which N goes first to infinity, and $H(\widetilde{P}_N|P_0)$ stands for the relative entropy of $\widetilde{P}_N$ with respect to P_0 [see (1.15)]. The main claim (0.4), or (0.5) then quickly follow by a classical inequality; see (1.16).

The above lines are of course mainly heuristic, and the actual proof involves several mollifications of the above strategy: K is slightly enlarged, h is replaced by a compactly supported function smoothed near K, we work with $u_{**}(1+\varepsilon)$ in place of u_{**}, and the tilted walk lives in a ball of radius RN up to time T_N, ... These various mollifications naturally enter the limiting procedure alluded to above in (0.10).

Clearly, a substantial part of this work is to make sense of the above heuristics. Observe that unlike what happened in [22], where an asymptotic lower bound was derived for the disconnection of a macroscopic body by random interlacements, in the present set-up, we only have one single trajectory at our disposal. So the titled walk behaves as a recurrent walk up to time T_N in order to implement disconnection. This makes the extraction of the necessary independence implicit to comparison with random interlacements more delicate. This is achieved by several sorts of analysis on the mesoscopic level. More precisely, on all mesoscopic boxes A_1^x with the center x varying in a "fence" around K_N, we bound from above the tilted probability that there is a path in $\mathcal{V}$ that connects x to the (inner) boundary of A_1^x by the probability that there is such a path in the vacant set of random interlacements with level slightly higher than u_{**} (which is itself small due to the strong non-percolative character of $\mathcal{V}^u$ when $u > u_{**}$) and a correction term:

$$\widetilde{P}_N[x \overset{\mathcal{V}}{\longleftrightarrow} \partial_i A_1^x] \le \mathbb{P}[x \overset{\mathcal{V}^{u_{**}(1+\varepsilon/8)}}{\longleftrightarrow} \partial_i A_1^x] + e^{-c\log^2 N} \le e^{-c'\log^2 N}, \tag{0.11}$$

where $\mathbb{P}$ stands for the law of random interlacements, and $\partial_i A_1^x$ for the inner boundary of the box A_1^x. To prove the above claim, we conduct a local comparison at mesoscopic scale between the trace of the tilted walk, and the occupied set of random interlacements, with a level slightly exceeding u_{**}, via a chain of couplings.

There are two crucial steps in this "chain of couplings," namely Propositions 5.2 and 5.7. In Proposition 5.2, we call upon the estimates on hitting times proved in Section 3 and on the results concerning the quasi-stationary measure from Section 4. We construct a coupling between the trace in A_1 of excursions of the confined walk up to time T_N, and the trace in A_1 of the excursions of many independent confined walks from A_1 to the boundary of a larger mesoscopic box. This proposition enables us to cut the confined walk into "almost" independent sections, and compare it to the trace of a suitable Poisson point process of excursions. On the other hand, Proposition 5.7 uses a result proved in [4], coupling the above mentioned Poisson point process of excursions and the trace of random interlacements. Some of the arguments used in this work are similar to those in [34]. However, in our set-up, special care is needed due to the fact that the stationary measure of the tilted walk is massively non-uniform.

We will now explain how this article is organized. In Section 1, we introduce notation and make a brief review of results concerning continuous-time random walk, continuous-time random interlacements, Markov chains, as well as other useful facts and tools. Section 2 is devoted to the construction of the tilted random walk and the confined walk, as well as the proof of various properties concerning them. Most important are a lower bound of the spectral gap of the confined walk in Proposition 2.12, and an asymptotic upper bound on the relative entropy between the tilted walk and the simple random walk, in Proposition 2.14. In Section 3, we prove some estimates on the hitting times of some mesoscopic objects, namely Propositions 3.5 and 3.7 that will be useful in Section 5. In Section 4, we prove some controls (namely Proposition 4.7) on the quasi-stationary measure that will be crucial for the construction of couplings in Section 5. In Section 5, we develop the chain of couplings and prove that the tilted disconnection probability $\widetilde{P}_N[A_N]$ tends to 1, as N tends to infinity. In the short Section 6, we assemble the various pieces and prove the main Theorem 0.1.

Finally, we explain the convention we use concerning constants. We denote by c, c', c'', $\bar{c}, \dots$ positive constants with values changing from place to place. Throughout the article, the constants depend on the dimension d. Dependence on additional constants is stated at the beginning of each section.

1. Some useful facts. Throughout the article, we assume $d \geq 3$ unless otherwise stated. In this section, we will introduce further notation and recall useful facts concerning continuous-time random walk on $\mathbb{Z}^d$ and its potential theory. We also recall the definition of and some results about continuous-time random interlacements. At the end of the section, we state an inequality on relative entropy and review various results about Markov chains.

We start with some notation. We let $\mathbb{Z}^+ = \{0, 1, \dots\}$ stand for the set of positive integers. We write $|\cdot|$ and $|\cdot|_\infty$ for the Euclidean and l^∞-norms on $\mathbb{R}^d$. We denote by $B(x,r) = \{y \in \mathbb{Z}^d; |x-y| \leq r\}$ the closed Euclidean ball of center x and radius $r \geq 0$ intersected with $\mathbb{Z}^d$ and by $B_\infty(x,r) = \{y \in \mathbb{Z}^d, |x-y|_\infty \leq r\}$ the closed l^∞-ball of center x and radius r intersected with $\mathbb{Z}^d$. When U is a subset of $\mathbb{Z}^d$, we write $|U|$ for the cardinality of U, and $U \subset\subset \mathbb{Z}^d$ means that U is a finite subset of $\mathbb{Z}^d$. We denote by ∂U (resp., $\partial_i U$) the boundary (resp., internal boundary) of U, and by $\overline{U}$ its "closure"

$$\begin{aligned} \partial U &= \{x \in U^c; \exists y \in U, |x-y| = 1\}, \\ \partial_i U &= \{y \in U; \exists U^c, |x-y| = 1\} \quad \text{and} \quad \overline{U} = U \cup \partial U. \end{aligned} \tag{1.1}$$

When $U \subset \mathbb{R}^d$, and $\delta > 0$, we write $U^\delta = \{z \in \mathbb{R}^d; d(z,U) \leq \delta\}$ for the closed δ-neighborhood of U, where $d(x,A) = \inf_{y \in A} |x-y|$ is the Euclidean distance of x to A. We define $d_\infty(x,A)$ in a similar fashion, with $|\cdot|_\infty$ in place of $|\cdot|$. To distinguish balls in $\mathbb{R}^d$ from balls in $\mathbb{Z}^d$, we write $B_{\mathbb{R}^d}(x,r) = \{z \in \mathbb{R}^d; |x-z| \leq r\}$ for the closed Euclidean ball of center x and radius r in $\mathbb{R}^d$ and $B^\circ_{\mathbb{R}^d}(x,r) = \{z \in$

$\mathbb{R}^d; |x - z| < r\}$ for the corresponding open Euclidean ball. We also write the N-discrete blow-up of $U \subseteq \mathbb{R}^d$ as

$$U_N = \{x \in \mathbb{Z}^d; d_\infty(x, NU) \le 1\}, \tag{1.2}$$

where we denote by $NU = \{Nz; z \in U\} \subset \mathbb{R}^d$ the set homothetic to U with ratio N.

We will now collect some notation concerning connectivity properties. We write $x \sim y$ if for $x, y \in \mathbb{Z}^d$, $|x - y| = 1$. We call $\pi : \{1, \dots, n\} \to \mathbb{Z}^d$, with $n \ge 1$, a nearest-neighbor path, when $\pi(i) \sim \pi(i-1)$ for $1 < i \le n$. Given K, L, U subsets of $\mathbb{Z}^d$, we say that K and L are connected by U and write $K \overset{U}{\longleftrightarrow} L$, if there exists a finite nearest-neighbor path π in $\mathbb{Z}^d$ such that $\pi(1)$ belongs to K and $\pi(n)$ belongs to L, and for all $k \in \{1, \dots, n\}$, $\pi(k)$ belongs to U. Otherwise, we say that K and L are not connected by U, and write $K \overset{U}{\nleftrightarrow} L$. Similarly, we say that K is connected to infinity by U, if for K, U subsets of $\mathbb{Z}^d$, $K \overset{U}{\longleftrightarrow} B(0, N)^c$ for all N, and write $K \overset{U}{\longleftrightarrow} \infty$. Otherwise, we say K is not connected to infinity by U, and write $K \overset{U}{\nleftrightarrow} \infty$.

We now turn to the definition of some path spaces and of the continuous-time simple random walk. We consider $\widehat{W}_+$ the spaces of infinite $(\mathbb{Z}^d) \times (0, \infty)$-valued sequences such that the first coordinate of the sequence forms an infinite nearest neighbor path in $\mathbb{Z}^d$, spending finite time in any finite set of $\mathbb{Z}^d$, and the sequence of the second coordinate has an infinite sum. The second coordinate describes the duration at each step corresponding to the first coordinate. We denote by $\widehat{\mathcal{W}}_+$ the respective σ-algebra generated by the coordinate maps, $Z_n, \zeta_n, n \ge 0$ [where Z_n is $\mathbb{Z}^d$-valued and ζ_n is $(0, \infty)$-valued]. We denote by P_x the law on $\widehat{W}_+$ under which Z_n, $n \ge 0$, has the law of the simple random walk on $\mathbb{Z}^d$, starting from x, and ζ_n, $n \ge 0$, are i.i.d. exponential variables with parameter 1, independent from $Z_n, n \ge 0$. We denote by E_x the corresponding expectation. Moreover, if α is a measure on $\mathbb{Z}^d$, we denote by P_α and E_α the measure $\sum_{x \in \mathbb{Z}^d} \alpha(x) P_x$ (not necessarily a probability measure) and its corresponding "expectation" (i.e., the integral with respect to the measure P_α).

We attach to $\widehat{w} \in \widehat{W}_+$ a continuous-time process $(X_t)_{t \ge 0}$ and call it the random walk on $\mathbb{Z}^d$ with constant jump rate 1 under P_x, through the following relations:

$$X_t(\widehat{w}) = Z_k(\widehat{w}) \qquad \text{for } t \ge 0, \text{ when } \tau_k \le t < \tau_{k+1}, \tag{1.3}$$

where for l in $\mathbb{Z}^+$, we set (if $l = 0$, the right sum term is understood as 0),

$$\tau_l = \sum_{i=0}^{l-1} \zeta_i. \tag{1.4}$$

We also introduce the filtration

$$\mathcal{F}_t = \sigma(X_s, s \le t). \tag{1.5}$$

For I a Borel subset of $\mathbb{R}^+$, we record the set of points visited by $(X_t)_{t\geq 0}$ during the time set I as X_I. Importantly, we denote by $\mathcal{V}$ the vacant set, namely the complement of the entire trace $X_{[0,\infty)}$ of X.

Given a subset U of $\mathbb{Z}^d$, and $\widehat{w} \in \widehat{W}_+$, we write $H_U(\widehat{w}) = \inf\{t \geq 0; X_t(\widehat{w}) \in U\}$ and $T_U = \inf\{t \geq 0; X_t(\widehat{w}) \notin U\}$ for the entrance time in U and exit time from U. Moreover, we write $\widetilde{H}_U = \inf\{s \geq \zeta_1; X_s \in U\}$ for the hitting time of U. If $U = \{x\}$, we then write H_x, T_x and $\widetilde{H}_x$.

Given a subset U of $\mathbb{Z}^d$, we write $\Gamma(U)$ for the space of all right-continuous, piecewise constant functions from $[0,\infty)$ to U, with finitely many jumps on any compact interval. We will also denote by $(X_t)_{t\geq 0}$ the canonical coordinate process on $\Gamma(U)$, and when an ambiguity arises, we will specify on which space we are working. For $\gamma \in \Gamma(U)$, we denote by $\mathrm{Range}(\gamma)$ the trace of γ.

Now, we recall some facts concerning equilibrium measure and capacity, and refer to Section 2, Chapter 2 of [19] for more details. Given $M \subset\subset \mathbb{Z}^d$, we write e_M for the equilibrium measure of M:

$$e_M(x) = P_x[\widetilde{H}_M = \infty]1_M(x), \qquad x \in \mathbb{Z}^d, \tag{1.6}$$

and $\mathrm{cap}(M)$ for the capacity of M, which is the total mass of e_M:

$$\mathrm{cap}(M) = \sum_{x\in M} e_M(x). \tag{1.7}$$

We denote the normalized equilibrium measure of M by

$$\widetilde{e}_M(x) = \frac{e_M(x)}{\mathrm{cap}(M)}. \tag{1.8}$$

There is also an equivalent definition of capacity through the Dirichlet form:

$$\mathrm{cap}(M) = \inf_f \mathcal{E}_{\mathbb{Z}^d}(f, f), \tag{1.9}$$

where $f : \mathbb{Z}^d \to \mathbb{R}$ is finitely supported and $f \geq 1$ on M, and

$$\mathcal{E}_{\mathbb{Z}^d}(f, f) = \frac{1}{2} \sum_{|x-y|=1} \frac{1}{2d}\big(f(y) - f(x)\big)^2 \tag{1.10}$$

is the discrete Dirichlet form for simple random walk.

It is well known that (see, e.g., Section 2.2, pages 52–55 of [19])

$$cN^{d-2} \leq \mathrm{cap}\big(B_\infty(0, N)\big) \leq c' N^{d-2}, \tag{1.11}$$

and that

$$e_{B_\infty(0,N)}(x) \geq c_1 N^{-1} \tag{1.12}$$

for x on the inner boundary of $B_\infty(0, N)$.

Now, we turn to random interlacements. We refer to [6, 9, 30] and [31] for more details. Random interlacements are random subsets of $\mathbb{Z}^d$, governed by a nonnegative parameter u (referred to as the "level"), and denoted by $\mathcal{I}^u$. We write $\mathbb{P}$ for the law of $\mathcal{I}^u$. Although the construction of random interlacements is involved, the law of $\mathcal{I}^u$ can be simply characterized by the following relation:

$$\mathbb{P}[\mathcal{I}^u \cap K = \varnothing] = e^{-u\mathrm{cap}(K)} \qquad \text{for all } K \subset\subset \mathbb{Z}^d. \tag{1.13}$$

We denote by $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$ the vacant set of random interlacements at level u.

The connectivity function of the vacant set of random interlacements is known to have a stretched-exponential decay when the level exceeds a certain critical value (see Theorem 4.1 of [32], Theorem 0.1 of [28], or Theorem 3.1 of [24] for recent developments). Namely, there exists a $u_{**} \in (0, \infty)$ which, for our purpose in this article, can be characterized as the smallest positive number such that for all $u > u_{**}$,

$$\mathbb{P}\big[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_\infty(0, N)\big] \le c(u)e^{-c'(u)N^{c'(u)}} \qquad \text{for all } N \ge 0, \tag{1.14}$$

(actually, the exponent of N can be chosen as 1, when $d \ge 4$, and as an arbitrary number in (0, 1) when $d = 3$, see [24]).

We also wish to recall a classical result about relative entropy, which is helpful in Section 2. For $\widetilde{P}$ absolutely continuous with respect to P, the relative entropy of $\widetilde{P}$ with respect to P is defined as

$$H(\widetilde{P}|P) = E^{\widetilde{P}}\left[\log \frac{d\widetilde{P}}{dP}\right] = E^P\left[\frac{d\widetilde{P}}{dP}\log\frac{d\widetilde{P}}{dP}\right] \in [0, \infty]. \tag{1.15}$$

For an event A with positive $\widetilde{P}$-probability, we have the following inequality (see page 76 of [8]):

$$P[A] \ge \widetilde{P}[A]e^{-(H(\widetilde{P}|P)+1/e)/\widetilde{P}[A]}. \tag{1.16}$$

We end this section with some results regarding continuous-time reversible finite Markov chains.

Let L be the generator for an irreducible, reversible continuous-time Markov chain on a finite set V, with jump rates at each state possibly non-constant. Let π be the stationary measure of this Markov chain. Then $-L$ is self-adjoint in $l^2(\pi)$ and has nonnegative eigenvalues $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_{|V|}$. We denote by $\lambda = \lambda_2$ its spectral gap. For any real function f on V, we define its variance with respect to π as $\mathrm{Var}_\pi(f)$. Then the semigroup $H_t = e^{tL}$ satisfies

$$\|H_t f - \pi(f)\|_2 \le e^{-\lambda t}\sqrt{\mathrm{Var}_\pi(f)}. \tag{1.17}$$

One can further show that, for all x and y in V,

$$\big|P_x(X_t = y) - \pi(y)\big| \le \sqrt{\frac{\pi(y)}{\pi(x)}}e^{-\lambda t}, \tag{1.18}$$

see pages 326–328 of [27] for more detail.

We also introduce the so-called "canonical path method" to give a lower bound on the spectral gap λ. We denote by E the edge set

$$\big\{\{x, y\}; x, y \in V, L_{x,y} > 0\big\}, \tag{1.19}$$

where $L_{x,y}$ is the matrix coefficient of L. We investigate the following quantity A

$$A = \max_{e \in E}\left\{\frac{1}{W(e)} \sum_{x,y,\gamma(x,y)\ni e} \operatorname{leng}\big(\gamma(x, y)\big)\pi(x)\pi(y)\right\}, \tag{1.20}$$

where γ is a map, which sends ordered pairs of vertices $(x, y) \in V \times V$ to a finite path $\gamma(x, y)$ between x and y, $\operatorname{leng}(\gamma)$ denotes the length of γ, and

$$W(e) = \pi(x)L_{x,y} = \pi(y)L_{y,x} = (1_x, L1_y)_{l^2(\pi)} = (L1_x, 1_y)_{l^2(\pi)} \tag{1.21}$$

is the edge-weight of $e = \{x, y\} \in E$. Then the proof of Theorem 3.2.1, page 369 of [27] is also valid (note that actually e in [27] is an oriented edge) in the present set-up of possibly nonconstant jump rates and shows that

$$\lambda \geq \frac{1}{A}. \tag{1.22}$$

2. The tilted random walk. In this section, we construct the main protagonists of this work: a time nonhomogenous Markov chain on $\mathbb{Z}^d$, which we will refer to as *the tilted walk*, as well as a continuous-time homogenous Markov chain on a (macroscopic) finite subset of $\mathbb{Z}^d$, which we will refer to as *the confined walk*. The tilted walk coincides with the confined walk up to a certain finite time, which is of order N^d, and then evolves as a simple random walk. We derive a lower bound on the spectral gap of the confined walk in Proposition 2.12. In Proposition 2.14, we prove that with a suitable limiting procedure, the relative entropy between the tilted random walk and the simple random walk has an asymptotic upper bound given by a quantity involving the Brownian capacity of K that appears in Theorem 0.1. In this section, the constants tacitly depend on δ, η, ε and R [see (2.2) and (2.3)].

We recall that K is a compact subset of $\mathbb{R}^d$ as above (0.1). We assume, without loss of generality, that

$$0 \in K. \tag{2.1}$$

Otherwise, as we now explain, we can replace K by $\widetilde{K} = K \cup \{0\}$: on the one hand, by the monotonicity and subadditivity of Brownian capacity (see, e.g., Proposition 1.12, page 60 of [25]), one has $\operatorname{cap}_{\mathbb{R}^d}(K) = \operatorname{cap}_{\mathbb{R}^d}(\widetilde{K})$; on the other hand, since $K \subseteq \widetilde{K}$, it is more difficult to disconnect $\widetilde{K}_N$ than to disconnect K_N, hence $P_0[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty] \geq P_0[\widetilde{K}_N \overset{\mathcal{V}}{\nleftrightarrow} \infty]$. This means that the lower bound (0.4) with K replaced by $\widetilde{K}$ implies (0.4), justifying our claim. From now on, for the sake of simplicity, for any $r > 0$ we write $B_{(r)}$ for the open ball $B^{\circ}_{\mathbb{R}^d}(0, r)$ and B_r for the closed ball $B_{\mathbb{R}^d}(0, r)$. We introduce the three parameters

$$0 < \delta, \eta, \varepsilon < 1, \tag{2.2}$$

where δ will be used as a smoothing radius for K, see (2.4), η will be used as a parameter in the construction of $\widetilde{h}$, the smoothed potential function [see (2.6)] and ε will be used as a parameter in the definition of T_N, the time length of "tilting"; see (2.16). We let $R > 400$ be a large integer (see Remark 2.4 for explanations on why we take R to be an integer) such that

$$K \subset B_{R/100}. \tag{2.3}$$

By definition of R, we always have

$$K^{2\delta} \subset B_{R/50}. \tag{2.4}$$

In the next lemma, we show the existence of a function $\widetilde{h}$ that satisfies various properties (among which the most important is an inequality relating its Dirichlet form to the relative Brownian capacity of $K^{2\delta}$), which, as we will later show, make it the right candidate for the main ingredient in the construction of the tilted walk.

We denote by $\mathcal{E}_{\mathbb{R}^d}(f, f) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2\, dx$ for $f \in H^1(\mathbb{R}^d)$ the usual Dirichlet form on $\mathbb{R}^d$ (see Example 4.2.1, page 167 and (1.2.12), page 11 of [14]). For F and G, respectively, closed and open subsets of $\mathbb{R}^d$ such that $F \subset G$, we define the relative Brownian capacity of F with respect to G by

$$\mathrm{cap}_{\mathbb{R}^d, G}(F) = \inf\{\mathcal{E}_{\mathbb{R}^d}(f, f)\}, \tag{2.5}$$

where the infimum runs over all $f \in H^1(\mathbb{R}^d)$ which are supported in G and satisfy that $f \geq 1$ on F. We write $C^\infty(B_R)$ for the set of functions having all derivatives of every order continuous in $B_{(R)}$, which all have continuous extensions to B_R (see page 10 of [15] for more details).

LEMMA 2.1. *There exists a continuous function $\widetilde{h} : \mathbb{R}^d \to \mathbb{R}$, satisfying the following properties*:

$$\begin{cases} 1.\ \widetilde{h} \text{ is a } C^\infty(B_R) \text{ function when restricted to } B_R, \\ \quad \text{and harmonic on } B_{(R)} \setminus B_{R/2}; \\ 2.\ 0 \leq \widetilde{h}(z) \leq 1 \text{ for all } z \in \mathbb{R}^d, \widetilde{h} = 1 \text{ on } K^{2\delta}, \\ \quad \text{and } \widetilde{h}(z) = 0 \text{ outside } B_{(R)}; \\ 3.\ \mathcal{E}_{\mathbb{R}^d}(\widetilde{h}, \widetilde{h}) \leq (1+\eta)^2 \mathrm{cap}_{\mathbb{R}^d, B_{(R)}}(K^{2\delta}); \\ 4.\ cw_1 \leq \widetilde{h} \leq c' w_2 \text{ where } w_1, w_2 \text{ are defined} \\ \quad \text{respectively in (2.11) and (2.12)}; \\ 5.\ \widetilde{h}(z_1) \geq c\widetilde{h}(z_2) \text{ for all } z_1, z_2 \in \mathbb{R}^d \text{ such that } |z_1| \leq |z_2| \leq R. \end{cases} \tag{2.6}$$

PROOF. We now construct $\widetilde{h}$. We define, with δ as in (2.2),

$$h(z) = W_z[H_{K^{2\delta}} < T_{B_{(R)}}] \qquad \forall z \in \mathbb{R}^d, \tag{2.7}$$

the Brownian relative equilibrium potential function, where W_z stands for the Wiener measure starting from $z \in \mathbb{R}^d$, and $H_{K^{2\delta}}$ and $T_{B_{(R)}}$, respectively, stand for the entrance time of the canonical Brownian motion in $K^{2\delta}$ and its exit time from $B_{(R)}$.

We let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth, nondecreasing and concave function such that $0 \le (\psi)'(z) \le 1$ for all $z \in \mathbb{R}$, $\psi(z) = z$ for $z \in (-\infty, \frac{1}{2}]$, and $\psi(z) = 1$ for $z \in [1 + \frac{\eta}{2}, \infty)$. We consider

$$\widetilde{h} = \psi \circ \big((1+\eta)h\big). \tag{2.8}$$

Now we prove the claims.

We first prove claim 1 in (2.6). It is classical that h is C^∞ on $B_{(R)} \setminus K^{2\delta}$. In addition, h is continuous, equal to 1 on $K^{2\delta}$ and to 0 outside $B_{(R)}$ (note that every point in $K^{2\delta}$ is regular for $K^{2\delta}$). In particular, $(1+\eta)h \ge 1 + \eta/2$ on an open neighborhood of $K^{2\delta}$, which implies that $\widetilde{h}$ is identically equal to 1 on this neighborhood. It follows that $\widetilde{h}$ is C^∞ on $B_{(R)}$. Now we show that $\widetilde{h}$ is C^∞ on B_R. To prove this, it suffices to prove that h is C^∞ on $B_R \setminus B_{R-c}$ for some $c > 0$, where h coincides with $\widetilde{h}$. We then represent h as $G^{B_{(R)}}\mu$, where we denote by $G^{B_{(R)}}$ and μ, respectively, the killed Green function for $B_{(R)}$ and the (Brownian) equilibrium measure of $K^{2\delta}$ relative to $B_{(R)}$. Since μ is supported on $K^{2\delta}$ and $G^{B_{(R)}}(x,y)$ is C^∞ for all $x \in B_R \setminus B_{(R-c)}$ and $y \in K^{2\delta} \subset B_{R/50}$ [by the explicit formula of the killed Green function for a ball (see, e.g., (41) in Section 2.2, page 40 of [13])], we know that h is C^∞ on $B_R \setminus B_{R-c}$, which implies that $\widetilde{h}$ is C^∞ on B_R. This completes the proof of claim 1.

Claim 2 follows directly from the definition of $\widetilde{h}$: for all $z \in \mathbb{R}^d$, $(1+\eta)h(z) \in [0, 1+\eta]$, hence by the definition of ψ, $\widetilde{h}(z) \in [0,1]$; $\widetilde{h} = 1$ on $K^{2\delta}$ is already shown in claim 1 of (2.6); moreover, by (2.7), outside $B_{(R)}$, $h = 0$, hence $\widetilde{h} = 0$.

We now prove claim 3. By $(\mathcal{E}.4)''$, page 5 of [14], an equivalent characterization of Markovian Dirichlet form, one knows that since ψ is a normal contraction,

$$\begin{aligned}\mathcal{E}_{\mathbb{R}^d}(\widetilde{h}, \widetilde{h}) &\le \mathcal{E}_{\mathbb{R}^d}\big((1+\eta)h, (1+\eta)h\big) = (1+\eta)^2 \mathcal{E}_{\mathbb{R}^d}(h,h) \\ &= (1+\eta)^2 \mathrm{cap}_{\mathbb{R}^d, B_{(R)}}\big(K^{2\delta}\big),\end{aligned} \tag{2.9}$$

where the last equality follows from [14], pages 152 and 71.

We now turn to claim 4. Because $B_\delta \subset K^{2\delta} \subset B_{R/50}$ by (2.1), we know that

$$w_1 \le h \le w_2 \qquad \text{on } B_R, \tag{2.10}$$

where

$$w_1(z) = W_z[H_{B_\delta} < T_{B_{(R)}}] = \begin{cases} 1, & |z| \in [0, \delta), \\ \dfrac{|z|^{2-d} - R^{2-d}}{\delta^{2-d} - R^{2-d}}, & |z| \in [\delta, R), \\ 0, & |z| \in [R, \infty) \end{cases} \tag{2.11}$$

and

$$w_2(z) = W_z[H_{B_{R/50}} < T_{B_{(R)}}] \\ = \begin{cases} 1, & |z| \in [0, R/50), \\ \dfrac{|z|^{2-d} - R^{2-d}}{(R/50)^{2-d} - R^{2-d}}, & |z| \in [R/50, R), \\ 0, & |z| \in [R, \infty) \end{cases} \tag{2.12}$$

are respectively the Brownian relative equilibrium potential functions of B_δ and $B_{R/50}$ (see (4) in Section 1.7, page 29 of [11] for the explicit formula of w_1 and w_2). By the definition of ψ, we also know that, $cr \le \psi(r) \le c'r$ for $0 \le r \le 1 + \eta$. Hence by the definition of $\widetilde{h}$, we find that

$$\widetilde{c}w_1 \overset{(2.10)}{\le} ch \le \widetilde{h} \le c'h \overset{(2.10)}{\le} \widetilde{c}'w_2. \tag{2.13}$$

Claim 4 hence follows.

Finally, claim 5 follows by claim 4 and the observation from the explicit formula of w_1 and w_2 that $w_1 \ge cw_2$ uniformly for some positive c on B_R and both w_1 and w_2 are radially symmetric and radially nonincreasing:

$$\widetilde{h}(z_1) \ge cw_1(z_1) \ge c'w_2(z_1) \ge c'w_2(z_2) \ge c''\widetilde{h}(z_2) \\ \text{for } z_1, z_2 \text{ s.t. } |z_1| \le |z_2| \le R. \qquad \square \tag{2.14}$$

We then introduce the restriction to $\mathbb{Z}^d$ of the blow-up of $\widetilde{h}$ and its $l^2(\mathbb{Z}^d)$-normalization as

$$h_N(x) = \widetilde{h}\left(\frac{x}{N}\right) \qquad \text{for } x \in \mathbb{Z}^d \quad \text{and} \quad f(x) = \frac{h_N(x)}{\|h_N\|_2}, \tag{2.15}$$

and also set [see (2.2) for the definition of ε]

$$T_N = u_{**}(1+\varepsilon)\|h_N\|_2^2, \tag{2.16}$$

[recall that u_{**} is the threshold of random interlacements defined above (1.14)]. We define T_N in a way such that the quantity $T_N f^2$ is bigger than u_{**} on K_N^δ, which, roughly speaking, makes the occupational time profile of the tilted random walk (which we will later define) at time T_N on K_N^δ bigger than that of the random interlacement with intensity u_{**}. We also set

$$U^N = B_{(NR)} \cap \mathbb{Z}^d. \tag{2.17}$$

This will be the state space of the confined random walk that we will later define.

In the following lemma, we record some basic properties of f. Intuitively speaking, f is a volcano-shaped function, with maximal value on K_N^δ that vanishes outside U^N. Note that f tacitly depends on δ, η and R.

LEMMA 2.2. *For large N, one has*

$$\begin{cases} 1.\ f \text{ is supported on } U^N \text{ and } f > 0 \text{ on } U^N; \\ 2.\ f^2 \text{ is a probability measure on } \mathbb{Z}^d \text{ supported on } U^N; \\ 3.\ T_N f^2(\cdot) = u_{**}(1+\varepsilon) \text{ on } K_N^\delta. \end{cases} \tag{2.18}$$

PROOF. Claims 1 and 2 follow by the definition of f [see (2.15)] and U^N [see (2.17)], note that by (2.17) $x \in U^N$ implies $\frac{x}{N}$ belongs to the *open* ball $B_{(R)}$. Claim 3 follows from the definition of T_N [see (2.16)] and the fact that $h_N = 1$ on K_N^δ for large N. □

We introduce a subset of U^N (which will be used in Lemma 2.11)

$$\begin{aligned} O^N = {} & \{U^N \setminus (\partial_i U^N \cup B_{NR/2})\} \\ & \cup \{x \in \partial_i U^N; |y| = NR \text{ for all } y \sim x, y \notin U^N\} \end{aligned} \tag{2.19}$$

(note that both N and R are integers). Intuitively speaking, O^N denotes the set of points in U^N which have distance at least $NR/2$ from 0 such that all their neighbors outside U^N (if there exists any) are on the sphere ∂B_{NR}. In the next lemma we collect some properties of h_N and T_N for later use, in particular in the proofs of Lemmas 2.10, 2.11 and Propositions 2.13, 2.14.

LEMMA 2.3. *For large N, one has*

$$\begin{cases} 1.\ cN^{-2} \le h_N(x) \le 1 \text{ for all } x \in U^N; \\ 2.\ h_N(x) \le cN^{-1} \text{ for all } x \in \partial_i U^N; \\ 3.\ h_N(x) \ge c' N^{-1} \text{ for all } x \in O^N; \\ 4.\ c' N^d \le \|h_N\|_2^2 \le c'' N^d; \\ 5.\ cN^d \le T_N \le c' N^d. \end{cases} \tag{2.20}$$

PROOF. We first prove claim 1. The right-hand side inequality follows by the definition of h_N [see (2.15)] and $\widetilde{h}$ [see (2.8)]. We now turn to the left-hand side inequality of claim 1. For all $x \in U^N$, one has $|x|^2 < (NR)^2$ by the definition of U^N [see (2.17)]. Since x has integer coordinates, this implies $|x| \le \sqrt{(NR)^2 - 1}$, hence for all $x \in U^N$,

$$|x| \le NR - cN^{-1}. \tag{2.21}$$

Thus, by claim 4 of (2.6) and (2.11) one has

$$\widetilde{h}(z) \ge c' N^{-2} \qquad \text{for all } |z| \le R - \frac{c}{N^2}. \tag{2.22}$$

This implies that for large N, for all $x \in U^N$,

$$h_N(x) = \widetilde{h}\left(\frac{x}{N}\right) \ge c'' N^{-2}. \tag{2.23}$$

Similarly, to prove claims 2, and 3, again by claim 4 of (2.6) and respectively (2.12) and (2.11), it suffices to prove that

$$|x| \geq NR - 1 \qquad \forall x \in \partial_i U^N \tag{2.24}$$

and that

$$|x| \leq NR - c' \qquad \forall x \in O^N. \tag{2.25}$$

To prove (2.24), we observe that, if $x \in \partial_i U^N$, there exists $y \notin U^N$, such that $x \sim y$. Since $|y| \geq NR$ and $|x - y| = 1$, the claim (2.24) follows by triangle inequality. Now we prove (2.25). We consider $x = (a_1, \dots, a_d) \in O^N$. By definition of O^N [see (2.19)], $a_1^2 + \cdots + a_d^2 \geq c(NR)^2$, hence without loss of generality, we assume that $|a_1| \geq cNR$. By the definition of O^N, we also know that $(|a_1| + 1)^2 + a_2^2 + \cdots + a_d^2 \leq (NR)^2$, which implies that $|x| = \sqrt{a_1^2 + \cdots + a_d^2} \leq NR(1 - c'/N)^{1/2} \leq NR - c''$, and hence (2.25).

Claim 4 follows by the observation that by claim 2 of (2.6), on the one hand $h_N \leq 1$ on $\mathbb{Z}^d$ and h_N is supported on U^N, and on the other hand $h_N = 1$ on $(NK^\delta) \cap \mathbb{Z}^d$.

Claim 5 follows as a consequence of claim 4 and the definition of T_N; see (2.16). □

REMARK 2.4. With Lemma 2.3 we reveal the reason for choosing R to be an integer: because we wish that the lattice points are not too close to the boundary of B_{NR} [see (2.21)]. This enables us to show, for example, that h_N is not too small on U^N, as in claim 1 of (2.20).

Now, we introduce a nonnegative martingale that plays an important role in our construction of the tilted random walk. Given a real-valued function g on $\mathbb{Z}^d$, we denote its discrete Laplacian by

$$\Delta_{\text{dis}} g(x) = \frac{1}{2d} \sum_{|e|=1} g(x+e) - g(x). \tag{2.26}$$

For the finitely supported nonnegative f defined in (2.15), for all x in U^N, we introduce under the measure P_x the stochastic process

$$M_t = \frac{f(X_{t \wedge T_{U^N}})}{f(x)} e^{\int_0^{t \wedge T_{U^N}} v(X_s)\, ds}, \qquad t \geq 0, P_x\text{-a.s.}, \tag{2.27}$$

where

$$v = -\frac{\Delta_{\text{dis}} f}{f}. \tag{2.28}$$

We define for all $T \geq 0$, a nonnegative measure $\widehat{P}_{x,T}$ (on $\widehat{W}_+$) with density M_T with respect to P_x,

$$\widehat{P}_{x,T} = M_T P_x. \tag{2.29}$$

In the next lemma, we show that $\widehat{P}_{x,T}$ is the law of a Markov chain and identify its infinitesimal generator.

LEMMA 2.5. *For all $x \in U^N$, one has*

(2.30) $\widehat{P}_{x,T}$ *is the probability measure for a Markov chain up to time T on U^N.*

Its semi-group (*acting on the finite dimensional space of functions on U^N*) *admits a generator given by the bounded operator*:

$$\widetilde{L}g(x) = \frac{1}{2d} \sum_{y \in U^N, y \sim x} \frac{f(y)}{f(x)} \big(g(y) - g(x)\big). \tag{2.31}$$

PROOF. To prove the claims (2.30) and (2.31), we first prove that

$$M_t \text{ is an } (\mathcal{F}_t)\text{-martingale under } P_x. \tag{2.32}$$

For $\zeta \in (0, 1)$, we define $f^{(\zeta)} = f + \zeta$ and $v^{(\zeta)} = -\frac{\Delta_{\mathrm{dis}} f^{(\zeta)}}{f^{(\zeta)}} = -\frac{\Delta_{\mathrm{dis}} f}{f^{(\zeta)}}$. We denote by $M_t^{(\zeta)}$, $t \geq 0$, the stochastic process similarly defined as M_t in (2.27) by

$$M_t^{(\zeta)} = \frac{f^{(\zeta)}(X_{t \wedge T_{U^N}})}{f^{(\zeta)}(x)} e^{\int_0^{t \wedge T_{U^N}} v^{(\zeta)}(X_s)\, ds} \qquad t \geq 0, P_x\text{-a.s.} \tag{2.33}$$

By Lemma 3.2 in Chapter 4, page 174 of [12], $M_t^{(\zeta)}$ is an $(\mathcal{F}_t)$-martingale under P_x. Since N is fixed, $f^{(\zeta)}$ is uniformly for $\zeta \in (0, 1)$ bounded from above and below on U^N, $v^{(\zeta)}$ is uniformly in ζ bounded on U^N as well. Hence, for all $t \geq 0$, $M_t^{(\zeta)}$ is bounded above uniformly for all $\zeta \in (0, 1)$. Therefore, the claim (2.32) follows from the dominated convergence theorem since for all $x \in U^N$, P_x-a.s., $\lim_{\zeta \to 0} M_t^{(\zeta)} = M_t$. To prove the claim (2.30), we just note that

$$E_x[M_T] = M_0 = 1. \tag{2.34}$$

Moreover, for all x in U^N by claim 1 of (2.18)

$$f(X_{T_{U^N}}) = 0, \tag{2.35}$$

thus $\widehat{P}_{x,T}$ vanishes on all paths which exit U^N before T. Then the claim (2.31) follows by Theorem 2.5, page 61 of [7]. □

REMARK 2.6. When we apply the lemma from [12] mentioned in the proof above, we need that $\inf_{x \in \overline{U^N}} f(x) > 0$. However, by claim 1 of (2.18), we know that $f(x) = 0$ for all $x \in \partial U^N$. To cope with this problem, we introduce a perturbation term ζ, and apply the lemma to the perturbated objects instead of the original ones.

We then denote the law of the "tilted random walk" by

$$\widetilde{P}_N = \widehat{P}_{0,T_N}. \tag{2.36}$$

REMARK 2.7. Intuitively speaking, $\widetilde{P}_N$ is the law of a tilted random walk, which restrains itself up to time T_N from exiting U^N and then, after the deterministic time T_N, continues as the simple random walk. It is absolutely continuous with respect to P_0.

It is convenient for us to define $\{\overline{P}_x\}_{x\in U^N}$, a family of finite-space Markov chains on U^N, with generator $\widetilde{L}$ defined in (2.31). We will call this Markov chain "the confined walk," since it is supported on $\Gamma(U^N)$ [see below (1.5) for the definition]. We will also tacitly regard it as a Markov chain on $\mathbb{Z}^d$, when no ambiguity rises. We denote by $\overline{E}_x$ the expectation with respect to $\overline{P}_x$, for all $x \in U^N$.

Thus, the following corollary is immediate.

COROLLARY 2.8.

$$\textit{Up to time } T_N,\ \widetilde{P}_N \textit{ coincides with } \overline{P}_0. \tag{2.37}$$

PROOF. It suffices to identify the finite time marginals of the two measures with the help of the Markov property and (2.31). □

REMARK 2.9. Since the confined walk is time-homogenous, in Sections 3, 4 and 5 we will actually perform the analysis on the confined walk instead of the tilted walk, and transfer the result concerning the time period $[0, T_N)$ back to the tilted walk thanks to the above corollary. See, for instance, (5.20).

We now state and prove some basic estimates about the confined walk.

LEMMA 2.10. *One has*

$$\begin{cases} 1.\ \textit{The measure } \pi(x) = f^2(x), x \in U^N, \textit{is a reversible} \\ \quad \textit{measure for the (irreducible) confined walk } \{\overline{P}_x\}_{x\in U^N}; \\ 2.\ \textit{The Dirichlet form associated with } \{\overline{P}_x\}_{x\in U^N} \textit{ and } \pi \textit{ is} \\ \quad \overline{\mathcal{E}}(g,g) = (-\widetilde{L}g, g)_{l^2(\pi)} = \frac{1}{2}\sum_{x,y\in U^N, x\sim y} \frac{f(x)f(y)}{2d}(g(x)-g(y))^2 \\ \quad \textit{with } g: U^N \to \mathbb{R}^+; \\ 3.\ \textit{If } x, y \in U^N, |x| \le |y|, \textit{ then one has } h_N(x) \ge c h_N(y) \\ \quad \textit{and } \pi(x) \ge c'\pi(y); \\ 4.\ \textit{For all } x \in U^N, cN^{-d-4} \le \pi(x) = f^2(x) \le c'N^{-d}. \end{cases} \tag{2.38}$$

PROOF. Claim 1 follows from claims 1 and 2 of (2.18) and the observation that by (2.31) $\widetilde{L}$ is self-adjoint in $l^2(\pi)$. Claim 2 follows from claim 1 and (2.31).

Claim 3 follows from (2.15) and claim 5 of (2.6). Claim 4 follows from claims 1 and 4 of (2.20) and the definition of f [see (2.15)]. □

In the next lemma, we control the fluctuation of v with a rough lower bound and a more refined upper bound.

LEMMA 2.11. *One has* [*recall* v *is defined in* (2.28)], *for all* x *in* U^N,

$$-cN^2 \le v(x) \le c' N^{-2}. \tag{2.39}$$

PROOF. We first record an identity for later use:

$$v(x) \overset{(2.28)}{=} -\frac{\Delta_{\mathrm{dis}} f(x)}{f(x)} \overset{(2.15)}{=} -\frac{\Delta_{\mathrm{dis}} h_N(x)}{h_N(x)}. \tag{2.40}$$

The inequality on the left-hand side of (2.39) is very rough and follows from

$$v \underset{h_N \ge 0}{\overset{(2.40)}{\ge}} -\frac{\max_{x\in U^N} h_N(x)}{\min_{x\in U^N} h_N(x)} \overset{(2.20)1.}{\ge} -cN^2. \tag{2.41}$$

Next, we prove the inequality on the right-hand side of (2.39). We split U^N into three parts and call them by I^N, O^N and S^N, respectively. Before we go into detail, we describe roughly the division, and what it entails. The region $I^N = B_{NR/2} \cap \mathbb{Z}^d$ is the “inner part” of U^N; the region O^N that already appears in (2.19) is the “outer part” of U^N that does not feel the push of the “hard” boundary, that is, all neighbors of its points belong to B_{NR}; the region $S^N = \partial_i U^N \setminus O^N$ is a subset of the inner boundary of U^N, where all points have a least one neighbor outside $B_{NR} \cap \mathbb{Z}^d$, and thus “feel the hard push” from outside U^N. As we will later see, in the microscopic region that corresponds to I^N, $\widetilde{h}$ is a smooth function; in the region O^N, h_N is at least of order N^{-1} and $|\Delta_{\mathrm{dis}} h_N|$ is at most of order N^{-3}; in the region S^N, one has $\Delta_{\mathrm{dis}} h_N > 0$.

We first record an estimate. Using a Taylor formula at second order with Lagrange remainder (see Theorem 5.16, pages 110–111 of [26]), since for all $x \in U^N \setminus S^N$, all y adjacent to x belongs to B_{NR}, we know from (2.15) that

$$\Delta_{\mathrm{dis}} h_N(x) \ge \frac{1}{N^2}\left(\frac{1}{2d}\Delta \widetilde{h}\left(\frac{x}{N}\right) - cN^{-1}\right) \qquad \text{for all } x \in U^N \setminus S^N. \tag{2.42}$$

We first treat points in $I^N = B_{NR/2} \cap \mathbb{Z}^d$. On $B_{R/2}$, we know that $\widetilde{h} \ge c$ and $\widetilde{h}$ is C^∞ by claim 1 of (2.6). We thus obtain that for all x in I^N,

$$-\frac{\Delta_{\mathrm{dis}} h_N(x)}{h_N(x)} \underset{(2.42)}{\overset{(2.15)}{\le}} -\frac{\Delta \widetilde{h}(\frac{x}{N}) - cN^{-1}}{\widetilde{h}(\frac{x}{N})N^2} \le cN^{-2}. \tag{2.43}$$

We then recall that $O^N = \{U^N \setminus (\partial_i U^N \cup B_{NR/2})\} \cup \{x \in \partial_i U^N; |y| = NR \text{ for all } y \sim x, y \notin U^N\}$, as defined in (2.19). By claim 1 of (2.6), we know that for all $x \in O^N$, $\Delta\widetilde{h}(\frac{x}{N}) = 0$. Hence, we find that

$$v(x) \overset{(2.42)}{\leq} -\frac{\Delta\widetilde{h}(x/N) - cN^{-1}}{h_N(x)N^2} \overset{(2.20)3.}{\leq} \frac{cN^{-1}}{c'N^{-1} \cdot N^2} = c''N^{-2} \quad \text{for all } x \in O^N. \tag{2.44}$$

We finally treat points in $S^N = \partial_i U^N \setminus O^N$. By Lemma 6.37, page 136 of [15], $\widetilde{h}$ can be extended to a C^3 function w on $B_{(R+1)}$ such that $w = \widetilde{h}$ in B_R and all the derivatives of w up to order three are uniformly bounded in $B_{(R+1)}$. Hence, we have for all $x \in S^N$,

$$\begin{aligned} -\Delta_{\text{dis}} h_N(x) &= \left(w\left(\frac{x}{N}\right) - \frac{1}{2d}\sum_{y\sim x} w\left(\frac{y}{N}\right)\right) \\ &\quad + \frac{1}{2d}\sum_{y\sim x, y\notin U^N}\left(w\left(\frac{y}{N}\right) - \widetilde{h}\left(\frac{y}{N}\right)\right) = \mathrm{I} + \mathrm{II}. \end{aligned} \tag{2.45}$$

On the one hand, by a second-order Taylor expansion with Lagrange remainder, and since $\Delta w = 0$ in $B_R \setminus B_{R/2}$, we have

$$\mathrm{I} \leq \frac{1}{N^2}\left(\frac{1}{2d}\Delta w\left(\frac{x}{N}\right) + \frac{c}{N}\right) = \frac{c'}{N^3} \qquad \text{for } x \in S^N. \tag{2.46}$$

On the other hand, we know that by claim 2 of (2.6)

$$\widetilde{h}\left(\frac{y}{N}\right) = 0 \qquad \text{for all } y \notin U^N. \tag{2.47}$$

Moreover, by definition of S^N, there exists a point y in $\mathbb{Z}^d$, adjacent to x, such that $NR < |y| \leq NR + 1$. This implies that

$$\begin{aligned} (NR+1)^2 \geq |y|^2 \geq (NR)^2 + 1 \quad &\text{and hence} \\ R + \frac{1}{N} \geq \frac{|y|}{N} \geq R + c'N^{-2}&. \end{aligned} \tag{2.48}$$

By claim 4 of (2.6), since $\widetilde{h}$ is bounded from above and below by two functions having (constant) negative outer normal derivatives on ∂B_R, we find that

$$\frac{\partial \widetilde{h}}{\partial n}(z) < -c \qquad \text{uniformly for all } z \in \partial B_R, \tag{2.49}$$

where $\frac{\partial\widetilde{h}}{\partial n}$ denotes the outer normal derivative of $\widetilde{h}$. Thus, we find that for large N,

$$w\left(\frac{y}{N}\right) \leq -\bar{c}N^{-2}. \tag{2.50}$$

This implies that

$$\mathrm{II} \overset{(2.47)}{=} \frac{1}{2d}\sum_{y\sim x, y\notin U^N} w\left(\frac{y}{N}\right) \leq -c''N^{-2}. \tag{2.51}$$

Combining (2.46) and (2.51), it follows that for large N and all $x \in S^N$,

$$v(x) \overset{(2.40)}{=} -\frac{\Delta_{\mathrm{dis}} h_N(x)}{h_N(x)} \underset{(2.51)}{\overset{(2.45),(2.46)}{\leq}} \frac{cN^{-3} - c'' N^{-2}}{h_N(x)} < 0. \tag{2.52}$$

Since I^N, O^N and S^N form a partition of U^N, the inequality in the right-hand side of (2.39) follows by collecting (2.43), (2.44) and (2.52). □

We will now derive a lower bound for the spectral gap of the confined walk, which we denote by $\overline{\lambda}$. We use the method introduced at the end of Section 1 and derive an upper bound for the quantity A [recall that A is defined in (1.20)]. However, we first need to specify our choice of paths γ. For $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d) \in U^N$, we assume, without loss of generality, that for some $l \in \{0, \dots, d\}$ we have

$$\begin{cases} |x_i| \geq |y_i|, & \text{for } i = 1, \dots, l \\ |x_i| < |y_i|, & \text{for } i = l+1, \dots, d, \end{cases} \tag{2.53}$$

($l = 0$ means that $|x_i| < |y_i|$ for all $i = 1, \dots, d$, and $l = d$ means that $|x_i| \geq |y_i|$ for all $i = 1, \dots, d$). For $p, q \in \mathbb{Z}^d$, which differ only in one coordinate, we denote by $\beta(p, q)$ the straight (and shortest) path between them. Then $\gamma(x, y)$ is defined as follows:

$$\begin{aligned} \gamma(x, y) = {} & \text{the concatenation of the paths} \\ & \beta\big((y_1, \dots, y_{i-1}, x_i, \dots, x_d), (y_1, \dots, y_i, x_{i+1}, \dots, x_d)\big) \\ & \text{as } i \text{ goes from 1 to } d. \end{aligned} \tag{2.54}$$

Loosely speaking, $\gamma(x, y)$ successively "adjusts" each coordinate of x with the corresponding coordinate of y by first "decreasing" the coordinates where $|x_i|$ is bigger or equal to $|y_i|$ and then "increasing" the coordinates where $|y_i|$ is bigger than $|x_i|$. It is easy to check that this path lies entirely in U^N, since by (2.53), for all $\{p, q\} \in \gamma(x, y)$, one has

$$\max\big(|p|, |q|\big) \leq \max\big(|x|, |y|\big). \tag{2.55}$$

PROPOSITION 2.12. *One has*

$$\overline{\lambda} \geq cN^{-2}. \tag{2.56}$$

PROOF. Recall that the quantity

$$A = \max_{e \in E} \left\{ \frac{1}{W(e)} \sum_{x, y, \gamma(x,y) \ni e} \operatorname{leng}\big(\gamma(x, y)\big) \pi(x) \pi(y) \right\}$$

is defined in (1.20). By (1.22), to prove (2.56), it suffices to prove that

$$A \leq c' N^2. \tag{2.57}$$

On the one hand, by (2.55) and claim 3 of (2.38) one obtains that for all $\{p,q\} \in \gamma(x,y)$,

$$\min(\pi(p),\pi(q)) \geq c\min(\pi(x),\pi(y)). \tag{2.58}$$

This implies that for all $\{p,q\} \in \gamma(x,y)$

$$\begin{aligned} W(\{p,q\}) &\overset{(2.31)}{\underset{(1.21)}{=}} \pi(p)\frac{f(q)}{2df(p)} \overset{(2.38)1.}{=} \frac{1}{2d}f(p)f(q) \\ &\overset{(2.58)}{=} \frac{1}{2d}\sqrt{\pi(p)\pi(q)} \geq c' \min(\pi(x),\pi(y)). \end{aligned} \tag{2.59}$$

On the other hand, for any $x, y \in U^N$, one has

$$\text{leng}(\gamma(x,y)) \leq cN. \tag{2.60}$$

Now we estimate the maximal possible number of paths that could pass through a certain edge. We claim that, for any edge $e \in E^N$, where we denote by E^N the edge set of U^N consisting of unordered pairs of neighboring vertices in U^N:

$$E^N = \{\{x,y\}; x,y \in U^N, |x-y| = 1\}, \tag{2.61}$$

there are at most cN^{d+1} paths passing through e. We now prove the claim. To fix a pair of $\{x,y\}$ such that $e = \{(a_1,\dots,a_k,\dots,a_d),(a_1,\dots,a_k+1,\dots,a_d)\}$ belongs to $\gamma(x,y)$, where $k \in \{1,\dots,d\}$, there are $2d$ coordinates to be chosen. Actually, for $i = 1,\dots,k-1,k+1,\dots,d$ the ith coordinate of either x or y must be a_i. This leaves us at most 2^{d-1} ways of choosing $(d-1)$ coordinates of x and y to be fixed by $a_1,\dots,a_{k-1},a_{k+1},\dots,a_d$. For the other $(d+1)$ coordinates, we have no more than cN choices for each of them, since both x and y must lie in U^N. This implies that there are no more that $c'N^{d+1}$ pairs of $\{x,y\} \subset U^N$, such that $e \in \gamma(x,y)$ is possible.

Combining the argument in the paragraph above with (2.59) and (2.60), one has

$$\begin{aligned} A &\overset{(1.20)}{=} \max_{e\in E^N} \frac{1}{W(e)} \sum_{x,y,\gamma(x,y)\ni e} \text{leng}(\gamma(x,y))\pi(x)\pi(y) \\ &\overset{(2.59)}{\underset{(2.60)}{\leq}} \max_{e\in E^N} \sum_{x,y,\gamma(x,y)\ni e} c'N \cdot \max(\pi(x),\pi(y)) \\ &\overset{(2.38)4.}{\leq} c''N^{d+1} \cdot N \cdot N^{-d} = c''N^2. \end{aligned} \tag{2.62}$$

This proves (2.57), and hence (2.56). □

We then define for $\{\overline{P}_x\}_{x\in U^N}$ the regeneration time

$$\overline{t_*} = N^2 \log^2 N. \tag{2.63}$$

In view of above proposition, $\overline{t_*}$ is much larger than the relaxation time $1/\overline{\lambda}$, which is of order $O(N^2)$. Hence, for all x in U^N, $\overline{P}_x[X_t = \cdot]$ becomes very close to the stationary distribution π, when $t \geq \overline{t_*}$. More precisely, by (1.18) and (2.56)

$$\sup_{x,y\in U^N} \left|\overline{P}_x[X_t = y] - \pi(y)\right| \leq \sup_{x,y\in U^N} \sqrt{\frac{\pi(y)}{\pi(x)}} e^{-\overline{\lambda}t} \overset{(2.38)4.}{\underset{(2.56),(2.63)}{\leq}} e^{-c\log^2 N} \quad \forall t \geq \overline{t_*}. \tag{2.64}$$

We now relate the relative entropy between $\widetilde{P}_N$ (which tacitly depends on R, η, δ and ε) and P_0 to the Dirichlet form of h_N and derive an asymptotic upper bound for it by successively letting $N \to \infty$, $\eta \to 0$, $R \to \infty$, $\delta \to 0$ and $\varepsilon \to 0$ in the following Propositions 2.13 and 2.14. The Brownian capacity of K will appear as the limit in the above sense of the properly scaled Dirichlet form of h_N.

PROPOSITION 2.13. *One has*

$$H(\widetilde{P}_N|P_0) \leq u_{**}(1+\varepsilon)\mathcal{E}_{\mathbb{Z}^d}(h_N, h_N) + o(N^{d-2}). \tag{2.65}$$

PROOF. By definition of the relative entropy [see (1.15)], we have

$$\begin{aligned} H(\widetilde{P}_N|P_0) &\overset{(1.15)}{=} E^{\widetilde{P}_N}\left[\log \frac{d\widetilde{P}_N}{dP_0}\right] \overset{(2.29)}{\underset{(2.36)}{=}} E^{\widetilde{P}_N}[\log M_{T_N}] \overset{(2.37)}{=} \overline{E}_0[\log M_{T_N}] \\ &\overset{(2.27)}{=} \overline{E}_0\left[\int_0^{T_N} v(X_s)\,ds + \log f(X_{T_N}) - \log f(X_0)\right] \\ &= \overline{E}_0\left[\int_0^{\overline{t_*}} v(X_s)\,ds\right] + \overline{E}_0\left[\int_{\overline{t_*}}^{T_N} v(X_s)\,ds\right] \\ &\quad + \overline{E}_0[\log f(X_{T_N}) - \log f(X_0)] \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{2.66}$$

For an upper bound of I, by (2.39) and the definition of $\overline{t_*}$ [see (2.63)], we have

$$\mathrm{I} \leq \overline{t_*} \max_{x\in U^N} v(x) \leq c\log^2 N. \tag{2.67}$$

For an upper bound of II, we notice that applying (2.64) for $t \in (\overline{t_*}, T_N)$,

$$\begin{aligned} &\left|\overline{E}_0\left[\int_{\overline{t_*}}^{T_N} v(X_t)\,dt\right] - (T_N - \overline{t_*})\int v\,d\pi\right| \\ &\leq (T_N - \overline{t_*}) \sup_{t\in[\overline{t_*},T_N]} \sup_{y\in U^N} \left|\overline{P}_0[X_t = y] - \int v\,d\pi\right| \cdot \max_{y\in U^N}|v(y)| \\ &\overset{(2.64)}{\leq} e^{-c\log^2 N}(T_N - \overline{t_*}) \max_{y\in U^N}|v(y)| \overset{(2.39)}{\underset{(2.20)5.}{\leq}} e^{-c'\log^2 N}. \end{aligned} \tag{2.68}$$

Since f is supported on U^N by claim 1 of (2.18), we may enlarge the range for summation in the second equality in the following calculation without changing the sum and see that

$$\begin{aligned}\int v\,d\pi &\overset{(2.28)}{\underset{(2.38)1.}{=}} \sum_{x\in U^N}\frac{-\Delta_{\text{dis}}f(x)}{f(x)}f^2(x)\\ &\overset{(2.16)}{=} \frac{u_{**}(1+\varepsilon)}{T_N}\sum_{x\in\mathbb{Z}^d}-f(x)\Delta_{\text{dis}}f(x)\|h_N\|_2^2\\ &\overset{(2.15)}{=} \frac{u_{**}(1+\varepsilon)}{T_N}\sum_{x\in\mathbb{Z}^d}-h_N(x)\Delta_{\text{dis}}h_N(x).\end{aligned}\tag{2.69}$$

By the discrete Green–Gauss theorem and the definition of Dirichlet form, we have

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}-h_N(x)\Delta_{\text{dis}}h_N(x) &= \frac{1}{2}\sum_{\substack{x,x'\in\mathbb{Z}^d\\ x\sim x'}}\frac{1}{2d}\big(h_N(x')-h_N(x)\big)^2\\ &= \mathcal{E}_{\mathbb{Z}^d}(h_N,h_N).\end{aligned}\tag{2.70}$$

Hence by (2.69) and (2.70), we know that

$$(T_N-\overline{t_*})\int v\,d\pi \le u_{**}(1+\varepsilon)\mathcal{E}_{\mathbb{Z}^d}(h_N,h_N).\tag{2.71}$$

Thus, we obtain from (2.71) and (2.68) that

$$\text{II} \le u_{**}(1+\varepsilon)\mathcal{E}_{\mathbb{Z}^d}(h_N,h_N)+e^{-c'\log^2 N}.\tag{2.72}$$

For the calculation of III, we know that

$$\begin{aligned}\overline{E}_0\big[\log f(X_{T_N})-\log f(X_0)\big] &\le \log\max_{x\in U^N}f(x)-\log\min_{x\in U^N}f(x)\\ &\overset{(2.38)4.}{\le} c\log N.\end{aligned}\tag{2.73}$$

Combining (2.67), (2.72) and (2.73), we obtain that

$$H(\widetilde{P}_N|P_0)\le u_{**}(1+\varepsilon)\mathcal{E}_{\mathbb{Z}^d}(h_N,h_N)+o\big(N^{d-2}\big),\tag{2.74}$$

which is (2.65). □

PROPOSITION 2.14. *One has*

$$\begin{aligned}&\limsup_{\varepsilon\to0}\limsup_{\delta\to0}\limsup_{R\to\infty}\limsup_{\eta\to0}\limsup_{N\to\infty}\frac{1}{N^{d-2}}H(\widetilde{P}_N|P_0)\\ &\quad\le\frac{u_{**}}{d}\text{cap}_{\mathbb{R}^d}(K).\end{aligned}\tag{2.75}$$

PROOF. By (2.65), we have

$$\limsup_{N\to\infty}\frac{1}{N^{d-2}}H(\widetilde{P}_N|P_0)\le u_{**}(1+\varepsilon)\limsup_{N\to\infty}\frac{1}{N^{d-2}}\mathcal{E}_{\mathbb{Z}^d}(h_N,h_N). \tag{2.76}$$

By the definition of h_N, we have

$$\begin{aligned}\frac{1}{N^{d-2}}&\mathcal{E}_{\mathbb{Z}^d}(h_N,h_N)\\ &=\frac{1}{4dN^{d-2}}\sum_{x\sim y\in\mathbb{Z}^d}\big(h_N(y)-h_N(x)\big)^2\\ &\overset{(2.15)}{=}\frac{1}{4dN^{d-2}}\sum_{x\sim y\in\mathbb{Z}^d}\left(\widetilde{h}\left(\frac{y}{N}\right)-\widetilde{h}\left(\frac{x}{N}\right)\right)^2.\end{aligned}\tag{2.77}$$

By claim 2 of (2.6), the summation in the right member of the second equality in (2.77) can be reduced to $x,y\in U^N\cup\partial U^N$. Then we split the sum into two parts:

$$\sum_{x\sim y\in\mathbb{Z}^d}\left(\widetilde{h}\left(\frac{y}{N}\right)-\widetilde{h}\left(\frac{x}{N}\right)\right)^2=\Sigma_1+\Sigma_2, \tag{2.78}$$

where

$$\Sigma_1=\sum_{x,y\in U^N,x\sim y}\left(\widetilde{h}\left(\frac{y}{N}\right)-\widetilde{h}\left(\frac{x}{N}\right)\right)^2 \tag{2.79}$$

contains all summands with $x,y\in U^N$, and

$$\Sigma_2=2\sum_{x\in U^N,y\notin U^N,x\sim y}\big(h_N(y)-h_N(x)\big)^2 \tag{2.80}$$

contains all summands with x in U^N and y in ∂U^N. By claim 2 of (2.6), we find that

$$\lim_{N\to\infty}\frac{1}{4dN^{d-2}}\Sigma_1=\frac{1}{2d}\int_{\mathbb{R}^d}s\left|\nabla\widetilde{h}(y)\right|^2dy \tag{2.81}$$

by a Riemann sum argument. While by claim 2 of (2.20), we obtain that

$$\Sigma_2\le c\sum_{x\in\partial_iU^N}h_N(x)^2\le c'N^{d-1}\left(\frac{c}{N}\right)^2=c''N^{d-3}. \tag{2.82}$$

This implies that

$$\lim_{N\to\infty}\frac{1}{N^{d-2}}\Sigma_2=0. \tag{2.83}$$

Therefore, we have

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}} \mathcal{E}_{\mathbb{Z}^d}(h_N, h_N) \le \lim_{N\to\infty} \frac{1}{4dN^{d-2}}(\Sigma_1 + \Sigma_2) = \frac{1}{2d}\int_{\mathbb{R}^d} \left|\nabla \widetilde{h}(y)\right|^2 dy = \frac{1}{d}\mathcal{E}_{\mathbb{R}^d}(\widetilde{h}, \widetilde{h}). \tag{2.84}$$

Therefore, by claim 3 of (2.6) we see that

$$\begin{aligned} &\limsup_{\eta\to 0}\limsup_{N\to\infty} \frac{1}{N^{d-2}} H(\widetilde{P}_N | P_0) \\ &\le \limsup_{\eta\to 0} \frac{u_{**}(1+\varepsilon)}{d}\mathcal{E}_{\mathbb{R}^d}(\widetilde{h}, \widetilde{h}) \\ &\le \frac{u_{**}(1+\varepsilon)}{d}\mathrm{cap}_{\mathbb{R}^d, B_{(R)}}(K^{2\delta}). \end{aligned} \tag{2.85}$$

As $R \to \infty$, the relative capacity converges to the usual Brownian capacity (this follows for instance from the variational characterization of the capacity in Theorem 2.1.5 on pages 71 and 72 of [14]):

$$\mathrm{cap}_{\mathbb{R}^d, B_{(R)}}(K^{2\delta}) \to \mathrm{cap}_{\mathbb{R}^d}(K^{2\delta}) \qquad \text{as } R \to \infty. \tag{2.86}$$

Then, letting $\delta \to 0$, by Proposition 1.13, page 60 of [25], we have

$$\mathrm{cap}_{\mathbb{R}^d}(K^{2\delta}) \to \mathrm{cap}_{\mathbb{R}^d}(K) \qquad \text{as } \delta \to 0. \tag{2.87}$$

Finally, by letting $\varepsilon \to 0$ the claim then follows. □

REMARK 2.15. In this section, guided by the heuristic strategy described below (0.7), we have constructed the tilted random walk. In essence, this continuous-time walk spends up to T_N, chosen in (2.16), at each point $x \in K_N^\delta$ an expected time equal to $u_{**}(1+\varepsilon)h_N^2(x) = u_{**}(1+\varepsilon)$, when started with the stationary measure π of the confined walk. The low entropic cost of the tilted walk with respect to the simple random walk is quantified by the above Proposition 2.14. We will now see in the subsequent sections that in the vicinity of points of K_N^δ, the geometric trace left by the tilted walk by time T_N stochastically dominates random interlacements at a level "close to $u_{**}(1+\varepsilon)$".

3. Hitting time estimates. In this section, we relate the entrance time (of the confined walk) into mesoscopic boxes inside K_N^δ to the capacity of these boxes and T_N [see (2.16)] and establish a pair of asymptotically matching bounds in the Propositions 3.5 and 3.7. It is a key ingredient for the construction of couplings in Section 5. The arguments in this section are similar to those in Section 3 and the Appendix of [34]. However, in our set-up, special care is needed due to the fact that the stationary measure is massively non-uniform. In this section, the constants tacitly depend on δ, η, ε and R [see (2.2) and (2.3)], r_1, r_2, r_3, r_4 and r_5 [see (3.1)].

We start with the precise definition of objects of interest for the current and the subsequent sections. We denote by $\Gamma^N = \partial K_N^{\delta/2}$ the boundary in $\mathbb{Z}^d$ of the discrete blow-up of $K^{\delta/2}$ [we recall (1.2) and (1.2) for the definition of the boundary and of the discrete blow-up]. The above Γ^N will serve as a set "surrounding" K_N. We choose real numbers

$$0 < r_1 < r_2 < r_3 < r_4 < r_5 < 1. \tag{3.1}$$

We define for x_0 in Γ^N six boxes centered at x_0 (when there is ambiguity we add a superscript for their center x_0):

$$A_i = B_\infty\big(x_0, \lfloor N^{r_i} \rfloor\big), \qquad 1 \le i \le 5 \quad \text{and} \quad A_6 = B_\infty\left(x_0, \left\lfloor \frac{\delta}{100} N \right\rfloor\right), \tag{3.2}$$

and we tacitly assume that N is sufficiently large so that for all $x_0 \in \Gamma^N$, the following inclusions hold:

$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset A_5 \subset A_6 \subset B_N^\delta \subset\subset \mathbb{Z}^d. \tag{3.3}$$

In view of (3.3) and claim 3 of (2.18) we find that, by (2.31), for large N and all x in U^N

(3.4) the stopped processes $X_{\cdot \wedge T_{A_6}}$ under P_x and $\overline{P}_x$ have the same law.

REMARK 3.1. Recall that the regeneration time $\overline{t_*}$ is defined in (2.63) as $\overline{t_*} = N^2 \log^2 N$, and for all $k = 1, \dots, 5$, A_k are mesoscopic objects of size $O(N^r)$ where $r \in (0, 1)$. Informally, Propositions 3.5 and 3.7 will imply that for all x "far away" from A_k, with a high $\overline{P}_x$-probability,

$$T_N \gg H_{A_k} \gg \overline{t_*}. \tag{3.5}$$

Given any x_0 in Γ^N, we write

$$D = U^N \setminus A_2, \tag{3.6}$$

and let

$$g(x) = \overline{P}_x[H_{A_1} \le T_{A_2}] \overset{(3.4)}{=} P_x[H_{A_1} \le T_{A_2}], \qquad x \in U^N, \tag{3.7}$$

be the (tilted) potential function of A_1 relative to A_2. We also let

$$f_{A_1}(x) = 1 - \frac{\overline{E}_x[H_{A_1}]}{\overline{E}_\pi[H_{A_1}]} \tag{3.8}$$

be the centered fluctuation of the scaled expected entrance time of A_1 (relative to the stationary measure).

The following lemma shows that the inverse of $\overline{E}_\pi[H_{A_1}]$ is closely related to $\overline{\mathcal{E}}(g, g)$. (Actually we are going to prove that they are approximately equal later in this section; see Propositions 3.7 and 3.5, as well as Remark 3.8.)

LEMMA 3.2. *One has*,

$$\overline{\mathcal{E}}(g,g)\Big(1-2\sup_{x\in D}|f_{A_1}(x)|\Big)\le \frac{1}{\overline{E}_\pi[H_{A_1}]}\le \overline{\mathcal{E}}(g,g)\frac{1}{\pi(D)^2}. \tag{3.9}$$

The proof is omitted due to its similarity to the proof of Lemma 3.2 in [5] (which further calls Proposition 3.41 in [2], which is originally intended for Markov chains with constant jump rate).

In the next lemma, we collect some properties of entrance probabilities for later use, namely Propositions 3.5, 3.7, 4.7 and 5.1.

LEMMA 3.3. *For large N, one has*

$$\overline{P}_x[H_{A_1}<\overline{t_*}]\le N^{-c} \qquad \textit{for all } x\in D, \tag{3.10}$$

and similarly

$$\overline{P}_x[H_{A_2}<\overline{t_*}]\le N^{-c'} \qquad \textit{for all } x\in U^N\setminus A_3. \tag{3.11}$$

Uniformly for all $x\in\partial_i A_1$, one has

$$\begin{aligned} e_{A_1}(x)&\le P_x[T_{A_6}<\widetilde{H}_{A_1}]\le P_x[T_{A_3}<\widetilde{H}_{A_1}]\le P_x[T_{A_2}<\widetilde{H}_{A_1}]\\ &\le e_{A_1}(x)\big(1+N^{-c''}\big). \end{aligned} \tag{3.12}$$

PROOF. We start with (3.10). First, we explain that to prove (3.10), it suffices to show that

$$\sup_{x\in\partial A_2}\overline{P}_x[H_{A_1}<\widetilde{t}]\le \sup_{x\in\partial A_2}\overline{P}_x[H_{A_1}<\widetilde{t}-t_\#]+N^{-c'} \qquad \text{for all } 0\le\widetilde{t}\le\overline{t_*}, \tag{3.13}$$

where we write $t_\#$ for $N^2/\log N$. Indeed, with (3.13), the claim (3.10) follows by an induction argument:

$$\begin{aligned} &\sup_{x\in D}\overline{P}_x[H_{A_1}<\overline{t_*}]\\ &\quad\le \sup_{x\in\partial A_2}\overline{P}_x[H_{A_1}<\overline{t_*}]\le \sup_{x\in\partial A_2}\overline{P}_x[H_{A_1}<\overline{t_*}-t_\#]+N^{-c'}\\ &\quad\le\cdots\le \sup_{x\in\partial A_2}\overline{P}_x\left[H_{A_1}<\overline{t_*}-\left\lceil\frac{\overline{t_*}}{t_\#}\right\rceil t_\#\right]+\left\lceil\frac{\overline{t_*}}{t_\#}\right\rceil N^{-c'}\\ &\quad\overset{(2.63)}{\le} 0+c\log^3 N\cdot N^{-c'}\le N^{-c''}. \end{aligned} \tag{3.14}$$

Now we prove (3.13). We pick $\widetilde{t}$ in $[0,\overline{t_*}]$. One has

$$\begin{aligned} &\sup_{x\in\partial A_2}\overline{P}_x[H_{A_1}<\widetilde{t}]\\ &\quad\le \sup_{x\in\partial A_2}\overline{P}_x[H_{A_1}<T_{A_6}]+\sup_{x\in\partial A_2}\overline{P}_x[T_{A_6}<H_{A_1}<\widetilde{t}]. \end{aligned} \tag{3.15}$$

On the one hand, by Proposition 1.5.10, page 36 of [19], one has

$$\sup_{x\in\partial A_2} \overline{P}_x[H_{A_1} < T_{A_6}] \overset{(3.4)}{=} \sup_{x\in\partial A_2} P_x[H_{A_1} < T_{A_6}] \le N^{-c}. \tag{3.16}$$

Now we seek an upper bound for the second term in the right member of (3.15). We write

$$\begin{aligned} &\sup_{x\in\partial A_2} \overline{P}_x[T_{A_6} < H_{A_1} < \widetilde{t}] \\ &\quad \le \sup_{x\in\partial A_2} \overline{P}_x[t_\# < T_{A_6} < H_{A_1} < \widetilde{t}] + \sup_{x\in\partial A_2} \overline{P}_x[T_{A_6} \le t_\#] = \mathrm{I} + \mathrm{II}. \end{aligned} \tag{3.17}$$

To bound I, we can assume that $t_\# < \widetilde{t}$ (otherwise $\mathrm{I} = 0$). Applying Markov property successively (first at time $t_\#$, then at time T_{A_6}, and finally at time H_{A_2}), we find

$$\begin{aligned} \mathrm{I} &\le \sup_{y\in U^N} \overline{P}_y[T_{A_6} < H_{A_1} < \widetilde{t} - t_\#] \le \sup_{x\in\partial A_6} \overline{P}_x[H_{A_1} < \widetilde{t} - t_\#] \\ &\le \sup_{x\in\partial A_2} \overline{P}_x[H_{A_1} < \widetilde{t} - t_\#]. \end{aligned} \tag{3.18}$$

Hence to prove (3.13), it suffices to prove that

$$\mathrm{II} \le N^{-c}. \tag{3.19}$$

Recalling that $d_\infty(\partial A_2, \partial A_6) \ge cN$, we find that

$$\mathrm{II} \overset{(3.4)}{=} \sup_{x\in\partial A_2} P_x[T_{A_6} < t_\#] \le d P[T_{[-mN,mN]} \le t_0], \tag{3.20}$$

where P is the probability law of a one-dimensional random walk started from 0 (and we denote by E the corresponding expectation), $t_0 = t_\#/d$, and $m = \delta/200$. We know that

$$P[T_{[-mN,mN]} \le t_0] = P\Big[\max_{0\le t\le t_0} |X_t| \ge mN\Big]. \tag{3.21}$$

By Doob's inequality, we have for $\lambda > 0$, using symmetry

$$\begin{aligned} P\Big[\max_{0\le t\le t_0} |X_t| \ge mN\Big] &= 2P\Big[\max_{0\le t\le t_0} \exp(\lambda X_t) \ge \exp(\lambda mN)\Big] \\ &\le \frac{2E[\exp(\lambda X_{t_0})]}{\exp(\lambda mN)}. \end{aligned} \tag{3.22}$$

Note that $\exp\{\lambda X_t - t(\cosh\lambda - 1)\}$, $t \ge 0$, is a martingale under P, so

$$E\big[\exp(\lambda X_{t_0})\big] = \exp\big\{t_0(\cosh\lambda - 1)\big\}. \tag{3.23}$$

Hence by taking $\lambda = \frac{mN}{2t_0} = cN^{-1}\log^{-1} N$, we obtain that the right-hand term of (3.22) is bounded from above by

$$2\exp\left\{t_0(\cosh\lambda - 1) - \frac{m^2N^2}{2t_0}\right\} \le 2\exp\left(-c\frac{m^2N^2}{2t_0}\right) \le N^{-c'}. \tag{3.24}$$

This implies that

$$P[T_{[-mN,mN]} \le t_0] \le N^{-c}. \tag{3.25}$$

Thus, one obtains (3.19) by collecting (3.20) and (3.25). This completes the proof of (3.13), and hence of (3.10).

The claim (3.11) follows by a similar argument.

Now we turn to (3.12). All, except the rightmost inequality of (3.12), are immediate. For the rightmost inequality, we first notice that by an estimate similar to the discussion below (3.25) of [34] we have,

$$P_x[T_{A_2} < \widetilde{H}_{A_1} < \infty] \le N^{-c} e_{A_1}(x) \qquad \text{for all } x \in \partial_i A_1. \tag{3.26}$$

And hence we get that for all $x \in \partial_i A_1$,

$$\begin{aligned} P_x[T_{A_2} < \widetilde{H}_{A_1}] &= P_x[\widetilde{H}_{A_1} = \infty] + P_y[T_{A_2} < \widetilde{H}_{A_1} < \infty] \\ &\le (1 + N^{-c}) e_{A_1}(x). \end{aligned} \tag{3.27}$$

This completes the proof of (3.12), and hence of Lemma 3.3. □

Now we make a further calculation of the tilted Dirichlet form of g defined in (3.7).

PROPOSITION 3.4. *For large N, one has*

$$\frac{\mathrm{cap}(A_1)}{T_N} u_{**}(1+\varepsilon) \le \overline{\mathcal{E}}(g,g) \le (1 + N^{-c}) \frac{\mathrm{cap}(A_1)}{T_N} u_{**}(1+\varepsilon). \tag{3.28}$$

PROOF. Combining the fact that $\pi = f^2$ [from claim 1 of (2.38)], and the observation that g is discrete harmonic in $A_2 \setminus A_1$, $g = 1$ on A_1 and $g = 0$ outside A_2, one has [recall that Z_1 is the first step of the discrete chain attached to X_t, $t \ge 0$, see (1.3)]

$$\begin{aligned} \overline{\mathcal{E}}(g,g) &= (g, -\widetilde{L}g)_{l^2(\pi)} = \frac{u_{**}(1+\varepsilon)}{T_N} \sum_{y\in\partial_i A_1} g(y)\Big(g(y) - \sum_{x\sim y} \frac{1}{2d} g(x)\Big) \\ &\overset{(3.7)}{=} \frac{u_{**}(1+\varepsilon)}{T_N} \sum_{y\in\partial_i A_1} \Big(1 - \sum_{x\sim y} P_y[Z_1 = x] P_x[H_{A_1} < T_{A_2}]\Big) \\ &\overset{\text{Markov}}{=} \frac{u_{**}(1+\varepsilon)}{T_N} \sum_{y\in\partial_i A_1} P_y[T_{A_2} < \widetilde{H}_{A_1}]. \end{aligned} \tag{3.29}$$

On the one hand, by the rightmost inequality in (3.12), one has

$$\sum_{y\in\partial_i A_1} P_y[T_{A_2} < \widetilde{H}_{A_1}] \leq (1+N^{-c}) \sum_{y\in\partial_i A_1} e_{A_1}(y) = (1+N^{-c})\mathrm{cap}(A_1). \tag{3.30}$$

On the other hand, one also knows that

$$\mathrm{cap}(A_1) = \sum_{y\in\partial_i A_1} e_{A_1}(y) \overset{(3.12)}{\leq} \sum_{y\in\partial_i A_1} P_y[T_{A_2} < \widetilde{H}_{A_1}]. \tag{3.31}$$

Thus, the claim (3.28) follows by collecting (3.29), (3.30) and (3.31). □

Next, we prove the first half of the main estimate of this section, namely the upper bound on $1/\overline{E}_\pi[H_{A_1}]$. Let us mention that this upper bound will actually be needed in the proof of Lemma 3.6.

PROPOSITION 3.5. *For large N, one has*

$$\frac{1}{\overline{E}_\pi[H_{A_1}]} \leq (1+N^{-c})\frac{\mathrm{cap}(A_1)}{T_N} u_{**}(1+\varepsilon). \tag{3.32}$$

As a consequence, one has

$$\overline{E}_\pi[H_{A_1}] \geq cN^{2+c'}. \tag{3.33}$$

PROOF. We first prove (3.32). We apply the right-hand inequality in (3.28) to the right-hand estimate in (3.9). Note that

$$\pi(D) = 1 - \pi(A_2) \geq 1 - cN^{(r_2-1)d} \qquad \text{for large } N, \tag{3.34}$$

for large N, with the help of (3.28) we thus find that

$$\begin{aligned} \frac{1}{\overline{E}_\pi[H_{A_1}]} &\overset{(3.9)}{\leq} \frac{\overline{\mathcal{E}}(g,g)}{\pi(D)^2} \overset{(3.34)}{\leq} (1-cN^{(r_2-1)d})^{-2}\overline{\mathcal{E}}(g,g) \\ &\overset{(3.28)}{\leq} (1+N^{-c})\frac{\mathrm{cap}(A_1)}{T_N}u_{**}(1+\varepsilon). \end{aligned} \tag{3.35}$$

This yields (3.32). Then the claim (3.33) follows by observing (1.11) and claim 5 of (2.20). □

In the following Lemma 3.6 and Proposition 3.7, we build a corresponding lower bound by controlling the fluctuation function f_{A_1} defined in (3.8).

LEMMA 3.6. *For large N, one has*

$$f_{A_1}(x) \geq -N^{-c} \qquad \text{for all } x \in U^N. \tag{3.36}$$

and in the notation of (3.6)

$$\begin{aligned} \overline{E}_x[H_{A_1}] \geq \overline{E}_\pi[H_{A_1}] - e^{-c'\log^2 N} - \overline{P}_x[H_{A_1} \leq \overline{t_*}](\overline{t_*} + \overline{E}_\pi[H_{A_1}]) \\ \text{for all } x \in D. \end{aligned} \tag{3.37}$$

PROOF. As we now explain, to prove (3.36), it suffices to show that

$$\big|\overline{E}_x\big[\overline{E}_{X_{\overline{t_*}}}[H_{A_1}]\big] - \overline{E}_\pi[H_{A_1}]\big| \le e^{-c'\log^2 N} \qquad \text{for all } x \in U^N. \tag{3.38}$$

Indeed, since $H_{A_1} \le \overline{t_*} + H_{A_1} \circ \theta_{\overline{t_*}}$, the simple Markov property applied at time $\overline{t_*}$ and (3.38) imply that

$$\sup_{x\in U^N} \overline{E}_x[H_{A_1}] \le \overline{t_*} + e^{-c\log^2 N} + \overline{E}_\pi[H_{A_1}]. \tag{3.39}$$

It then follows that

$$\begin{aligned} &\frac{\sup_{x\in U^N}\overline{E}_x[H_{A_1}]}{\overline{E}_\pi[H_{A_1}]} - 1 \\ &\quad \underset{(3.32)}{\overset{(3.39)}{\le}} \big(\overline{t_*} + e^{-c\log^2 N}\big)c'\frac{\operatorname{cap}(A_1)}{T_N} \\ &\quad \underset{(2.16)}{\overset{(1.11)}{\le}} \big(\overline{t_*} + e^{-c\log^2 N}\big)c'' N^{(d-2)r_1 - d} \underset{(3.1)}{\overset{(2.63)}{\le}} N^{-\widetilde{c}}. \end{aligned} \tag{3.40}$$

This proves (3.36). We now prove (3.38). Let us consider the expectation of H_{A_1} when started from $X_{\overline{t_*}}$. We first note that for all $x \in U^N$,

$$\begin{aligned} &\big|\overline{E}_x\big[\overline{E}_{X_{\overline{t_*}}}[H_{A_1}]\big] - \overline{E}_\pi[H_{A_1}]\big| \\ &\quad \le \sum_{y\in U^N} \big|\overline{P}_x[X_{\overline{t_*}} = y] - \pi(y)\big| \sup_{y\in U^N} \overline{E}_y[H_{A_1}]. \end{aligned} \tag{3.41}$$

By the relaxation to equilibrium estimate (2.64), one has

$$\sum_{y\in U^N} \big|\overline{P}_x[X_{\overline{t_*}} = y] - \pi(y)\big| \le e^{-c\log^2 N} \qquad \text{for all } x \in U^N. \tag{3.42}$$

Thus, to prove (3.38) it suffices to obtain a very crude upper bound for the supremum of the expected entrance time in A_1 as the starting point varies in U^N:

$$\overline{E}_y[H_{A_1}] \le cN^{5+d} \qquad \text{for all } y \in U^N. \tag{3.43}$$

This follows, for example, by a corollary of the commute time identity (see Corollary 4.28, page 59 of [3]):

$$\overline{E}_y[H_{A_1}] \le r_{\text{eff}}(y, A_1)\pi\big(U^N\big) \qquad \text{for all } y \in U^N, \tag{3.44}$$

where $r_{\text{eff}}(y, A_1)$ stands for the effective resistance between y and A_1. On the one hand, by the third equality of (2.59) and claim 4 of (2.38), for all $x, y \in U^N$ such that $x \sim y$, we know that

$$W(x, y) = \frac{1}{2d}\sqrt{\pi(x)\pi(y)} \in \big(cN^{-(4+d)}, 1\big], \tag{3.45}$$

hence the resistance on $\{p, q\}$ does not exceed cN^{4+d}. We know that for any y in U^N, for some $x \in \partial_i A_1$, the effective resistance between y and x [which we denote by $r_{\text{eff}}(y, x)$] is less or equal to the effective resistance between y and x on the path $\gamma(y, x)$ [which we denote by $r^{\gamma}_{\text{eff}}(y, x)$] defined above Proposition 2.12 [note that $\gamma(y, x)$ is a subgraph of U^N]. Since by (2.60) $\gamma(y, x)$ is of length no more than cN, $r^{\gamma}_{\text{eff}}(y, x)$ does not exceed $c' N^{5+d}$ by (3.45). Hence, we obtain that

$$r_{\text{eff}}(y, A_1) \le r_{\text{eff}}(y, x) \le r^{\gamma}_{\text{eff}}(y, x) \le cN^{5+d}. \tag{3.46}$$

On the other hand, one has $\pi(U^N) = 1$ [by claim 1 of (2.38)]. Thus, (3.44) and (3.46) yield that

$$\sup_{y \in U^N} \overline{E}_y[H_{A_1}] \le cN^{5+d}. \tag{3.47}$$

This completes the proof of (3.43), and hence of (3.36).

We now turn to (3.37). We consider any $x \in D$. By the simple Markov property applied at time $\overline{t_*}$, we find that

$$\begin{aligned} &\overline{E}_x[H_{A_1}] \\ &\quad \ge \overline{E}_x\big[\mathbf{1}_{\{H_{A_1} > \overline{t_*}\}} \overline{E}_{X_{\overline{t_*}}}[H_{A_1}]\big] \\ &\quad = \overline{E}_x\big[\overline{E}_{X_{\overline{t_*}}}[H_{A_1}]\big] - E_x\big[\mathbf{1}_{\{H_{A_1} \le \overline{t_*}\}} \overline{E}_{X_{\overline{t_*}}}[H_{A_1}]\big] \\ &\quad \overset{(3.38)}{\ge} \overline{E}_\pi[H_{A_1}] - e^{-c \log^2 N} - \overline{P}_x[H_{A_1} \le \overline{t_*}] \sup_{y \in U^N} \overline{E}_y[H_{A_1}] \\ &\quad \overset{(3.39)}{\ge} \overline{E}_\pi[H_{A_1}] - e^{-c' \log^2 N} - \overline{P}_x[H_{A_1} \le \overline{t_*}]\big(\overline{t_*} + \overline{E}_\pi[H_{A_1}]\big). \end{aligned} \tag{3.48}$$

This proves (3.37) and finishes Lemma 3.6. □

We now prove the second main estimate.

PROPOSITION 3.7. *For large N, one has that*

$$\frac{1}{\overline{E}_\pi[H_{A_1}]} \ge \big(1 - N^{-c}\big) \frac{\text{cap}(A_1)}{T_N} u_{**}(1 + \varepsilon). \tag{3.49}$$

PROOF. By applying (3.28) and the left-hand inequality of (3.9), for large N, one has,

$$\frac{1}{\overline{E}_\pi[H_{A_1}]} \ge \Big(1 - 2 \sup_{x \in D} \big|f_{A_1}(x)\big|\Big) \frac{\text{cap}(A_1)}{T_N} u_{**}(1 + \varepsilon). \tag{3.50}$$

Thus, with (3.36) in mind, to prove (3.49), it suffices to show that for large N,

$$\sup_{x \in D} f_{A_1}(x) \le N^{-c}. \tag{3.51}$$

Dividing by $\overline{E}_\pi[H_{A_1}]$ on both sides of (3.37) and taking the infimum over all $x \in D$, one obtains

$$\begin{aligned}
&\inf_{x\in D}\frac{\overline{E}_x[H_{A_1}]}{\overline{E}_\pi[H_{A_1}]}\\
(3.52)\qquad &\overset{(2.63)}{\geq} 1-\frac{e^{-c'\log^2 N}}{\overline{E}_\pi[H_{A_1}]}-\sup_{x\in D}\overline{P}_x[H_{A_1}\leq \overline{t_*}]\left(\frac{N^2\log^2 N}{\overline{E}_\pi[H_{A_1}]}+1\right)\\
&\underset{(3.10)}{\overset{(3.33)}{\geq}} 1-e^{-\overline{c}'\log^2 N}-N^{-\widetilde{c}'}\bigl(c''(\log N)^2N^{-\widetilde{c}}+1\bigr)\geq 1-N^{-c}.
\end{aligned}$$

Together with (3.50), this proves (3.51) as well as (3.49). □

REMARK 3.8. The combination of Propositions 3.5 and 3.7 forms a pair of asymptotically tight bounds on $\overline{E}_\pi[H_{A_1}]$, namely

$$\begin{aligned}
&\bigl(1-N^{-c}\bigr)\frac{\operatorname{cap}(A_1)}{T_N}u_{**}(1+\varepsilon)\leq\frac{1}{\overline{E}_\pi[H_{A_1}]}\\
(3.53)\qquad &\qquad\leq\bigl(1+N^{-c}\bigr)\frac{\operatorname{cap}(A_1)}{T_N}u_{**}(1+\varepsilon).
\end{aligned}$$

4. Quasi-stationary measure. In this section, we introduce the quasi-stationary distribution (abbreviated below as q.s.d.) induced on D [recall that D is defined in (3.6)] and collect some of its properties. This will help us show in the next section that carefully chopped sections of the confined random walk are approximately independent, allowing us to bring into play excursions of random walk and furthermore random interlacements. In Proposition 4.5, we prove that the q.s.d. on D is an appropriate approximation of the stationary distribution of the random walk conditioned to stay in D. In Proposition 4.7, we show that the hitting distribution of A_1 of the confined walk starting from the q.s.d. on D is very close to the normalized equilibrium measure of A_1. In this section, the constants tacitly depend on δ, η, ε and R [see (2.2) and (2.3)], r_1, r_2, r_3, r_4 and r_5 [see (3.1)].

We fix the choice of A_1 and A_2 as in the last section [see (3.2)]. The arguments in Lemma 4.2, Propositions 4.3, 4.5 and 4.7 below are similar to those of Section 3.2 and the Appendix of [34]. However, in our set-up, special care is needed due to the fact that the stationary measure is massively non-uniform in the present context.

We now define the q.s.d. on $D(=U^N\setminus A_2)$. We denote by $\{H_t^D\}_{t\geq 0}$ the semigroup of $\{\overline{P}_x\}_{x\in U^N}$ killed outside D, so that for all $f \in U^N \to \mathbb{R}$

$$H_t^D f(x)=\overline{E}_x\bigl[f(X_t),H_{A_2}>t\bigr]. \tag{4.1}$$

We denote by L^D the generator of $\{H_t^D\}_{t\geq 0}$. It is classical fact that for $f: D\to\mathbb{R}$,

$$L^D f(x)=\widetilde{L}\widetilde{f}(x)\qquad \forall x\in D, \tag{4.2}$$

where $\widetilde{f}$ is the extension of f to U^N vanishing outside D and $\widetilde{L}$ [defined in (2.31)] is the generator for the tilted walk. We denote by π^D the restriction of the measure π onto D. So, $\{H_t^D\}_{t\geq 0}$ and L^D are self-adjoint in $l^2(\pi^D)$ and

$$H_t^D = e^{tL^D}. \tag{4.3}$$

We then denote by $\lambda_i^D, i = 1, \dots, |D|$, with

$$0 \leq \lambda_i^D \leq \lambda_{i+1}^D, \qquad i = 1, \dots, |D| - 1, \tag{4.4}$$

the eigenvalues of $-L^D$ and by $f_i, i = 1, \dots, |D|$, an $l^2(\pi^D)$-orthonormal basis of eigenfunctions associated to λ_i. Because D is connected, by the Perron–Frobenius theorem, all entries of f_1 are positive. The quasi-stationary distribution on D is the probability measure on D with density with respect to π^d proportional to f_1, that is,

$$\sigma(y) = \frac{(f_1, \delta_y)_{l^2(\pi^D)}}{(f_1, \mathbf{1})_{l^2(\pi^D)}}, \qquad x \in D, \tag{4.5}$$

where, for $y \in D$, $\delta_y : D \to \mathbb{R}$ is the point mass function at y. It is known that the q.s.d. on D is the limit distribution of the walk conditioned on never entering A_2, that is, for all $x, y \in D$, one has (see (6.6.3), page 91 of [17]),

$$\sigma(y) = \lim_{t\to\infty} \overline{P}_x[X_t = y | H_{A_2} > t]. \tag{4.6}$$

We now prove a lemma which is useful in the proof of Proposition 4.3 below.

LEMMA 4.1. *For all $x, y \in D$, one has that*

$$\sigma(y) \geq N^{-c}\overline{P}_y[H_x < H_{A_2}]\sigma(x). \tag{4.7}$$

PROOF. By the $l^2(\pi^D)$-self-adjointness of the killed semi-group $(H_t^D)_{t\geq 0}$, we have that for all $x, y \in D$, $t > 0$,

$$\overline{P}_x[X_t = y | H_{A_2} > t] = \overline{P}_y[X_t = x | H_{A_2} > t] \frac{\pi(y)}{\pi(x)} \frac{\overline{P}_y[H_{A_2} > t]}{\overline{P}_x[H_{A_2} > t]}. \tag{4.8}$$

On the one hand, by the strong Markov property applied at time H_x, we know that for all $x, y \in D$,

$$\overline{P}_y[H_{A_2} > t] \geq \overline{P}_y[H_x < H_{A_2}]\overline{P}_x[H_{A_2} > t], \tag{4.9}$$

On the other hand, by claim 4 of (2.38), we know that for all $x, y \in D$, $t > 0$,

$$\frac{\pi(y)}{\pi(x)} \geq cN^{-4}. \tag{4.10}$$

Thus, the claim (4.7) follows by taking limits in t on both sides of (4.8) and incorporating (4.10) and (4.9). □

The next lemma is also a preparation for Proposition 4.3.

LEMMA 4.2. *For all $x \in D \setminus A_4$, one has*

$$\max_{y \in \partial A_3} \overline{P}_y[H_x < H_{A_2}] \geq N^{-c}. \tag{4.11}$$

PROOF. We fix an $x \in D \setminus A_4$ in the proof. Applying the Markov property at time T_{A_3} under $\overline{P}_{y'}$ for $y' \in \partial A_2$, we see that

$$\begin{aligned} \max_{y' \in \partial A_2} \overline{P}_{y'}[H_x < H_{A_2}] &= \max_{y' \in \partial A_2} \overline{P}_{y'}[T_{A_3} < H_x < H_{A_2}] \\ &\leq \max_{y \in \partial A_3} \overline{P}_y[H_x < H_{A_2}]. \end{aligned} \tag{4.12}$$

We now develop a lower bound on the left-hand side of (4.12) via effective resistance estimates. We denote by U^{col} the graph obtained by collapsing A_2 into a single vertex a in U^N. With some abuse of notation, we use the same symbol for the vertices in U^{col} as in U^N except for a. We denote by $W^{\mathrm{col}} : U^{\mathrm{col}} \times U^{\mathrm{col}} \to \mathbb{R}^+$ the induced edge-weight. Let

$$w_a = \sum_{y \in \partial A_2} W^{\mathrm{col}}(a, y) = \sum_{z \in A_2, y \in \partial A_2, z \sim y} W(z, y) \tag{4.13}$$

be the sum of the weights of edges that touch a in U^{col}. We denote by $\{P_z^{\mathrm{col}}\}_{z \in U^{\mathrm{col}}}$ the discrete-time reversible Markov chain with edge-weight W^{col}. The reversible measure of this Markov chain π^{col} is given through

$$\pi^{\mathrm{col}}(z) = \begin{cases} w_a, & z = a, \\ \sum_{y \sim z} W(z, y), & \text{otherwise}. \end{cases} \tag{4.14}$$

Then we have

$$\max_{y' \in \partial A_2} \overline{P}_{y'}[H_x < H_{A_2}] = \max_{y' \in \partial A_2} P_{y'}^{\mathrm{col}}[H_x < H_a] \geq P_a^{\mathrm{col}}[H_x < \widetilde{H}_a], \tag{4.15}$$

By a classical result on electrical networks (see Proposition 3.10, page 69 of [2]), the escape probability in the right-hand side of (4.15) equals

$$P_a^{\mathrm{col}}[H_x < \widetilde{H}_a] = \bigl(w_a r^{\mathrm{col}}(a, x)\bigr)^{-1}, \tag{4.16}$$

where $r^{\mathrm{col}}(a, x)$ is the effective resistance between a and x on U^{col}. We know that $r^{\mathrm{col}}(a, x)$ is smaller or equal to the effective resistance between a and x along a path between a and x of length no more than cN and along this path the edge-weight is no less than N^{-c} by (3.45). Hence, we obtain that

$$r^{\mathrm{col}}(a, x) \leq N^c. \tag{4.17}$$

Moreover, we know that

$$w_a = \sum_{z \in A_2, y \in \partial A_2, z \sim y} W(z, y) \leq N^c \max_{z \in D} W(z, y) \overset{(3.45)}{\leq} N^c. \tag{4.18}$$

Therefore, we conclude from (4.16), (4.17) and (4.18) that

$$P_a^{\mathrm{col}}[H_x < \widetilde{H}_a] \geq N^{-c'}. \tag{4.19}$$

The claim (4.11) follows by collecting (4.12), (4.15) and (4.19). □

The next proposition is a crucial estimate for us, showing that σ is not too small at any point in D. This fact will be used in Proposition 4.5. In the proof, we mainly rely on the reversibility of the confined walk, hitting probability estimates of simple random walk, and the Harnack principle.

PROPOSITION 4.3. *For large N, one has the following lower bound*:

$$\inf_{x \in D} \sigma(x) \geq N^{-c}, \tag{4.20}$$

and for all $x \in D$,

$$N^{c'} \geq f_1(x) \geq N^{-c''}. \tag{4.21}$$

PROOF. We first prove (4.20). The claim (4.21) will then follow. Because σ is a probability measure, and

$$|D| \leq cN^d, \tag{4.22}$$

there must exist some x' in D such that

$$\sigma(x') \geq cN^{-d}. \tag{4.23}$$

By (4.7), to prove (4.20) it suffices to prove that for all $x \in D$,

$$\overline{P}_x[H_{x'} < H_{A_2}] \geq N^{-c'}. \tag{4.24}$$

We now prove (4.24) by treating two cases according to the location of x'.

Case 1: $x' \in A_4 \setminus A_2$ [recall the definition of A_4 in (3.2)]. By (3.4) and a standard hitting estimate (see Proposition 1.5.10, page 36 of [19]) for simple random walk, for all x in D, we have that [recall the definition of A_5 in (3.2)],

$$\overline{P}_x[H_{\partial A_5} < H_{A_2}] \geq N^{-c}, \tag{4.25}$$

(note that the left-hand side equals 1 if $x \notin A_5$). We write

$$l(x) = \overline{P}_x[H_{x'} < H_{A_2}]. \tag{4.26}$$

By the strong Markov property applied at $H_{\partial A_5}$,

$$l(x) \overset{\text{Markov}}{\geq} \overline{P}_x[H_{\partial A_5} < H_{A_2}] \min_{y \in \partial A_5} l(y) \overset{(4.25)}{\geq} N^{-c} \min_{y \in \partial A_5} l(y). \tag{4.27}$$

We now develop a lower bound on the right-hand side of (4.27). Let $S_1 = B_\infty(x_0, 3N^{r_5}) \setminus B_\infty(x_0, \frac{1}{3}N^{r_5})$ and $S_2 = B_\infty(x_0, 2N^{r_5}) \setminus B_\infty(x_0, \frac{1}{2}N^{r_5})$, (we tacitly assume that N is sufficiently large that $S_1 \subset A_6$, and $S_2 \subset D$). It is straightforward to see that $l(x)$ is $\widetilde{L}$-harmonic in $D \setminus \{x'\}$ and that $\widetilde{L}$ coincides with Δ_{dis}

in S_1 [see (3.4)]. By the Harnack inequality (see Theorem 6.3.9, page 131 of [20]), we know that (note that $\partial A_5 \subset S_2$)

$$\min_{y\in\partial A_5} l(y) \geq c' \max_{y\in\partial A_5} l(y). \tag{4.28}$$

This implies by (4.27) that

$$\min_{x\in D} l(x) \geq c' N^{-c} \max_{y\in\partial A_5} l(y). \tag{4.29}$$

We now take any point $y' \in \partial A_5$ of least distance (in the sense of l^∞-norm) to x' on ∂A_5 and sharing $(d-1)$ common coordinates with x' and fix y'. We set $B = B_\infty(y', |y'-x'|_\infty - 1)$. Our way of choosing y' ensures that $x' \in \partial B$. Then by (3.4), we have

$$l(y') = \overline{P}_{y'}[H_{x'} < H_{A_2}] \geq \overline{P}_{y'}[X_{T_B} = x'] \stackrel{(3.4)}{=} P_{y'}[X_{T_B} = x']. \tag{4.30}$$

By a classical estimate (see Lemma 6.3.7, pages 158–159 of [20]), we have

$$P_{y'}[X_{T_B} = x'] \geq c N^{(1-d)r_5}. \tag{4.31}$$

Thus, the claim (4.24) follows by collecting (4.29), (4.30) and (4.31).

Case 2: $x' \in D \setminus A_4$. Since $\partial A_3 \subset A_4 \setminus A_2$, if we can prove that for some $y \in \partial A_3$,

$$\sigma(y) \geq N^{-c}, \tag{4.32}$$

then we are brought back to case 1 by taking the y in (4.32) as the x' in (4.23). Now we show that we can indeed find such y that (4.32) holds. By (4.7) and our assumption that $\sigma(x') \geq N^{-c}$, we have

$$\sigma(y) \stackrel{(4.7)}{\geq} N^{-c}\overline{P}_y[H_{x'} < H_{A_2}]\sigma(x') \stackrel{(4.23)}{\geq} N^{-c'}\overline{P}_y[H_{x'} < H_{A_2}]. \tag{4.33}$$

Hence, we know that by (4.11), if we pick the y that maximizes the probability in the left-hand side of (4.11), the claim (4.32) is indeed true.

With these two cases, we complete the proof of (4.20).

Now we prove (4.21). By the fact that f_1 is a unit vector in $l^2(\pi^D)$ we know that

$$(f_1, f_1)_{l^2(\pi^D)} = 1. \tag{4.34}$$

To prove the first inequality of (4.21), we observe that, thanks to (4.34):

$$1 = (f_1, f_1)_{l^2(\pi^D)} \geq \max_{x\in D} f_1^2(x) \min_{x\in D} \pi^D(x) \stackrel{(2.38)4.}{\geq} N^{-c} \max_{x\in D} f_1^2(x). \tag{4.35}$$

To prove the second inequality of (4.21), we observe that by (4.34)

$$\max_{x\in D} \pi^D(x) f_1^2(x) \geq \frac{1}{|D|}, \tag{4.36}$$

which implies that

$$\max_{x\in D} f_1(x) \geq \sqrt{\frac{1}{|D|\max_{x\in D}\pi^D(x)}} \underset{(4.22)}{\overset{(2.38)4.}{\geq}} N^{-c}. \tag{4.37}$$

This implies that for all $x \in D$,

$$\begin{aligned} f_1(x) &\overset{(4.5)}{=} \frac{1}{\pi^D(x)}\sigma(x)(f_1, \mathbf{1})_{l^2(\pi^D)} \underset{(4.20)}{\overset{(2.38)4.}{\geq}} N^{-c}\max_{x\in D} f_1(x)\min_{x\in D}\pi^D(x) \\ &\underset{(4.37)}{\overset{(2.38)4.}{\geq}} N^{-c'}. \end{aligned} \tag{4.38}$$

This completes the proof of (4.21), and concludes the proof of Proposition 4.3. □

In the following proposition, we show that the spectral gap of L^D is at least of order N^{-2}.

LEMMA 4.4. *One has that for large N*

$$\lambda_2^D - \lambda_1^D \geq cN^{-2}. \tag{4.39}$$

PROOF. Recall that $\overline{\lambda}_2$ stand for the second smallest eigenvalue of $-\widetilde{L}$. By the eigenvalue interlacing inequality (see Theorem 2.1 of [16]), we have

$$\lambda_2^D \geq \overline{\lambda}_2. \tag{4.40}$$

While by the paragraph below equation (12) of [1], we have

$$\lambda_1^D = \frac{1}{\overline{E}_\sigma[H_{A_2}]}. \tag{4.41}$$

By Lemma 10(a) of [1], we have

$$\overline{E}_\sigma[H_{A_2}] \geq \overline{E}_\pi[H_{A_2}] \quad \text{or equivalently} \quad \frac{1}{\overline{E}_\sigma[H_{A_2}]} \leq \frac{1}{\overline{E}_\pi[H_{A_2}]}. \tag{4.42}$$

By an argument similar to the proof of Proposition 3.5 (by replacing A_1, A_2 by A_2, A_3), we find that

$$\frac{1}{\overline{E}_\pi[H_{A_2}]} \leq cN^{-d+(d-2)r_2}. \tag{4.43}$$

This implies by (4.41) and (4.42) that

$$\lambda_1^D \leq cN^{-d+(d-2)r_2}. \tag{4.44}$$

Hence, we obtain that for large N

$$\lambda_2^D - \lambda_1^D \underset{(4.40)}{\overset{(4.44)}{\geq}} \overline{\lambda}_2 - cN^{-d+(d-2)r_2} \overset{(2.56)}{\geq} c'N^{-2}. \tag{4.45}$$

□

The next proposition shows, with the help of the spectral gap estimate obtained in Lemma 4.4, that the q.s.d. on D is very close to the distribution of the confined walk at time $\overline{t_*}$ conditioned on not hitting A_2 [see (2.63) for the definition of $\overline{t_*}$].

PROPOSITION 4.5. *One has that for large* N,

$$\sup_{x,y\in D}\left|\overline{P}_x[X_{\overline{t_*}}=y\,|\,H_{A_2}>\overline{t_*}]-\sigma(y)\right|\le e^{-c\log^2 N}. \tag{4.46}$$

PROOF. The conditional probability in (4.46) is expressed through $H^D_{\overline{t_*}}$ as

$$P_x[X_{\overline{t_*}}=y\,|\,H_{A_2}>\overline{t_*}]=\frac{H^D_{\overline{t_*}}\delta_y(x)}{(H^D_{\overline{t_*}}\mathbf{1})(x)}. \tag{4.47}$$

Now we calculate the numerator in the right-hand side of (4.47). We decompose δ_y in the $l^2(\pi^D)$ base $\{f_i\}_{i=1,\dots,|D|}$:

$$\begin{aligned}&\delta_y=\sum_{i=1}^{|D|}a_i f_i\\ &\text{where } a_i=(\delta_y,f_i)_{l^2(\pi^D)}=f_i(y)\pi^D(y),\text{ for } 1\le i\le |D|.\end{aligned} \tag{4.48}$$

Hence, one can decompose $H^D_{\overline{t_*}}\delta_y(x)$ into linear combinations of $a_i f_i(x)$:

$$H^D_{\overline{t_*}}\delta_y(x)=e^{-\lambda_1^D\overline{t_*}}\left(a_1 f_1(x)+\sum_{i=2}^{|D|}e^{(\lambda_1^D-\lambda_i^D)\overline{t_*}}a_i f_i(x)\right). \tag{4.49}$$

Now we show that the first term inside the brackets on the right-hand side of (4.49) is significantly larger than the other terms. By Proposition 4.3, one has

$$a_1 f_1(x)\overset{(4.48)}{=}\pi^D(y)f_1(y)f_1(x)\underset{(2.38)4.}{\overset{(4.21)}{\ge}}N^{-c}. \tag{4.50}$$

For large N, thanks to Lemma 4.4, the reminder term inside the brackets of (4.49) is bounded by

$$\begin{aligned}\left|\sum_{i=2}^{|D|}e^{(\lambda_1^D-\lambda_i^D)\overline{t_*}}a_i f_i(x)\right| &\underset{(4.48)}{\overset{(4.4)}{\le}}\sum_{i=2}^{|D|}e^{(\lambda_1^D-\lambda_2^D)\overline{t_*}}\left|\pi^D(y)f_i(y)f_i(x)\right|\\ &\underset{(2.63)}{\overset{(4.39)}{\le}}|D|e^{-c\log^2 N}\left|\pi^D(y)f_i(y)f_i(x)\right|\\ &\underset{(2.38)4.}{\overset{(4.22)}{\le}}e^{-c''\log^2 N}.\end{aligned} \tag{4.51}$$

This implies that

$$\left|\frac{H^D_{\overline{t_*}}\delta_y(x)}{e^{-\lambda_1^D\overline{t_*}}a_1 f_1(x)} - 1\right| \overset{(4.49)\text{–}(4.51)}{\leq} e^{-c\log^2 N}. \tag{4.52}$$

We now turn to the denominator of the right-hand side of (4.47). By an argument which is very similar to that leading to (4.52), one can show that

$$\left|\frac{(H^D_{\overline{t_*}}\mathbf{1})(x)}{e^{-\lambda_1^D\overline{t_*}} f_1(x)(f_1, \mathbf{1})_{l^2(\pi^D)}} - 1\right| \leq e^{-c\log^2 N}. \tag{4.53}$$

Combining (4.52) and (4.53), one has that for large N and uniformly for all $x, y \in D$ [remind the definition of $\sigma(\cdot)$ in (4.5)]

$$\begin{aligned} \left|P_x[X_{\overline{t_*}} = y | H_{A_2} > \overline{t_*}] - \sigma(y)\right| &\overset{(4.47)}{=} \left|\frac{H^D_{\overline{t_*}}\delta_y(x)}{(H^D_{\overline{t_*}}\mathbf{1})(x)} - \sigma(y)\right| \\ &\underset{(4.53)}{\overset{(4.52)}{\leq}} e^{-c\log^2 N}\sigma(y) \leq e^{-c\log^2 N}, \end{aligned} \tag{4.54}$$

which is exactly the claim (4.46). □

We define the stopping time V as the first time when the confined random walk has stayed outside A_2 for a consecutive duration of $\overline{t_*}$:

$$V = \inf\{t \geq \overline{t_*} : X_{[t-\overline{t_*}, t]} \cap A_2 = \varnothing\}. \tag{4.55}$$

The next lemma is a preparatory result for Proposition 4.7 below. This lemma shows that the probability $\overline{P}_x[V < \widetilde{H}_{A_1}]$, when normalized by the sum of such probabilities as x varies in the inner boundary of A_1, is approximately equal to $\widetilde{e}_{A_1}(x)$, the normalized equilibrium measure of A_1.

LEMMA 4.6. *For large N, one has that for all $x \in D$,*

$$\left|\frac{\overline{P}_x[V < \widetilde{H}_{A_1}]}{\sum_{y\in\partial_i A_1}\overline{P}_y[V < \widetilde{H}_{A_1}]\widetilde{e}_{A_1}(x)} - 1\right| \leq N^{-c}. \tag{4.56}$$

PROOF. For any $y \in \partial_i A_1$, by (3.11) and the strong Markov property applied at time T_{A_3}, we obtain that

$$\begin{aligned} \overline{P}_y[V < \widetilde{H}_{A_1}] &\underset{(4.55)}{\overset{\text{Markov}}{\geq}} \overline{P}_y[T_{A_3} < \widetilde{H}_{A_1}] \inf_{x\in U^N\setminus A_3} \overline{P}_x[H_{A_2} > \overline{t_*}] \\ &\underset{(3.4)}{\overset{(3.11)}{\geq}} P_y[T_{A_3} < \widetilde{H}_{A_1}](1 - N^{-c}) \overset{(3.12)}{\geq} e_{A_1}(y)(1 - N^{-c}). \end{aligned} \tag{4.57}$$

On the other hand, $\overline{P}_y[V < \widetilde{H}_{A_1}]$ is bounded from above by [recall that $V > T_{A_2}$ by definition of V, see (4.55)]

$$\overline{P}_y[V < \widetilde{H}_{A_1}] \le \overline{P}_y[T_{A_2} < \widetilde{H}_{A_1}] \overset{(3.4)}{=} P_y[T_{A_2} < \widetilde{H}_{A_1}] \overset{(3.12)}{\le} e_{A_1}(y)\bigl(1 + N^{-c}\bigr). \tag{4.58}$$

Together with (4.57), we find that

$$\bigl(1 - N^{-c}\bigr)e_{A_1}(y) \le \overline{P}_y[V < \widetilde{H}_{A_1}] \le \bigl(1 + N^{-c'}\bigr)e_{A_1}(y) \quad \text{for any } y \in \partial_i A. \tag{4.59}$$

Summing over $y \in \partial_i A_1$ we obtain that

$$\bigl(1 - N^{-c}\bigr) \sum_{y\in\partial_i A_1} \overline{P}_y[V < \widetilde{H}_{A_1}] \le \operatorname{cap}(A_1) \le \bigl(1 + N^{-c'}\bigr) \sum_{y\in\partial_i A_1} \overline{P}_y[V < \widetilde{H}_{A_1}]. \tag{4.60}$$

The claim (4.56) follows by combining (4.59) and (4.60), recalling that by the definition of normalized equilibrium measure, $\widetilde{e}_{A_1}(x) = e_{A_1(x)}/\operatorname{cap}(A_1)$. □

The following proposition shows that the hitting distribution of the confined walk on A_1 started from the q.s.d. on D is very close to the normalized equilibrium measure of A_1. The proof of the next proposition is close to the proof of Lemma 3.10 of [34], and can be found in the Appendix at the end of this article.

PROPOSITION 4.7. *For large N and any $x_0 \in \Gamma^N$ (recall that A_1 tacitly depends on x_0), one has*

$$\sup_{x\in\partial_i A_1} \left| \frac{\overline{P}_\sigma[X_{H_{A_1}} = x]}{\widetilde{e}_{A_1}(x)} - 1 \right| \le N^{-c}. \tag{4.61}$$

5. Chain coupling of excursions. In this section, we prove in Theorem 5.9 that the tilted random walk disconnects K_N from infinity with a probability, which tends to 1 as N tends to infinity. For this purpose, we show that the confined random walk visits the mesoscopic boxes A_1 centered at Γ^N[defined in (3.2)] sufficiently often so that at time T_N the trace of the walk "locally" dominates (via a chain of couplings) random interlacements with intensity higher than u_{**}. Hence, it disconnects in each such box the center from its boundary with very high probability. Some arguments in this section are based on Section 4 of [34], with necessary adaptations. In this section, the constants tacitly depend on δ, η, ε and R [see (2.2) and (2.3)], r_1, r_2, r_3, r_4 and r_5 [see (3.1)].

Throughout this section, we fix $x_0 \in \Gamma^N$, the center of the boxes A_1 through A_6, except in Proposition 5.8 and Theorem 5.9.

We recall the definition of V in (4.55). For a path in $\Gamma(U^N)$, we denote by R_k and V_k the successive entrance times H_{A_1} and stopping times V:

$$R_1 = H_{A_1}; \qquad V_1 = R_1 + V \circ \theta_{R_1}; \quad \text{and for } i \geq 2,$$
$$R_i = V_{i-1} + H_{A_1} \circ \theta_{V_{i-1}}; \qquad V_i = R_i + V \circ \theta_{R_i}.$$

Colloquially, we call such sections $X_{[R_i, V_i)}$ "long excursions" in contrast to the "short excursions" we will later define [see above (5.16)]. We set

$$J = \lfloor (1 + \varepsilon/2) u_{**} \mathrm{cap}(A_1) \rfloor. \tag{5.1}$$

The next proposition shows that, with high probability, the confined random walk has already made at least J "long excursions" before time T_N.

PROPOSITION 5.1. *For large N, one has*

$$\overline{P}_0[R_J \geq T_N] \leq e^{-N^c}. \tag{5.2}$$

The proof is deferred to the Appendix at the end of this article because it is rather technical and similar to the proof of Lemma 4.3 of [34].

Next, we construct a chain of couplings. Simply speaking, it is a sequence of couplings involving multiple random sets, in which the preceding set stochastically dominate the following set with probability close (or sometimes equal) to 1.

We start with the first coupling. The following proposition shows that one can construct a probability space where $(J-1)$ "long excursions" (counted from the second excursion) coincide with high probability with $(J-1)$ independent "long excursions" started from the q.s.d. We write $\overline{P_2^J} = \bigotimes_{i=2}^J \overline{P}_\sigma^i$ for the product of $(J-1)$ independent copies of $\overline{P}_\sigma$. We denote by $\mathcal{A}_i$ the random set $X_{[R_i, V_i)} \cap A_1$ and set $\mathcal{A} = \bigcup_{i=2}^J \mathcal{A}_i$.

PROPOSITION 5.2. *For large N, there exists a probability space $(\Omega_0, \mathcal{B}_0, Q_0)$, endowed with a random set $\mathcal{A}$ with the same law as $\mathcal{A}$ under $\overline{P}_0$ and random sets $\check{\mathcal{A}}_i$, $i = 2, \dots, J$, distributed as $\check{X}^i_{[0,V_1)} \cap A_1$ where for $i \geq 2$, $\check{X}^i$'s are i.i.d. distributed as X under $\overline{P}_\sigma$, such that*

$$Q_0[\mathcal{A} \neq \check{\mathcal{A}}] \leq e^{-c'' \log^2 N}, \tag{5.3}$$

where $\check{\mathcal{A}} = \bigcup_{i=2}^J \check{\mathcal{A}}_i$.

PROOF. For each $x \in D$, we use Proposition 4.7, page 50 in [21] and Proposition 4.5 to construct a coupling q_x of random variables Ξ with the law of $X_{\overline{t_*}}$ under $\overline{P}_x[\cdot | H_{A_2} > \overline{t_*}]$ and Σ with the law of σ such that

$$\max_{x \in D} q_x[\Xi \neq \Sigma] \leq |D| e^{-c \log^2 N} \leq e^{-c' \log^2 N}. \tag{5.4}$$

We introduce L, the index of last "step" of the path in A_2 before time V [see (1.4) and the paragraph above (1.3) for the definition of τ_l and Z_l, resp.]:

$$L = \sup\{l \geq 0 : \tau_l \leq V, Z_l \in A_2\}. \tag{5.5}$$

We then introduce $L_i = L \circ \theta_{R_i} + l_i$, where l_i satisfies $\tau_{l_i} = R_i$ for $i \geq 1$ as the last step at which the i-th excursion is in A_2.

We now construct Q_0 with the help of (5.4) in a similar fashion to the proof of Lemma 4.2 in [34]. The procedure goes inductively. We start by choosing $x_1^+ \in \partial A_2$ according to $\overline{P}_0[Z_{L_1+1} = \cdot]$. For $i \geq 1$, if x_i^+ is chosen, we choose x_{i+1} and $\check{x}_{i+1}$ points in $D = U^N \setminus A_2$ according to $q_{x_i^+}[\Xi = \cdot, \Sigma = \cdot]$. If x_{i+1} and $\check{x}_{i+1}$ coincide (which is the typical case, that is, if the coupling is successful at step $i+1$), we choose $\mathcal{A}_{i+1} = \check{\mathcal{A}}_{i+1}$ subsets of A_1 and $x_{i+1}^+ = \check{x}_{i+1}^+$ points in ∂A_2 according to $\overline{P}_{x_{i+1}}[\mathcal{A}_1 = \cdot, Z_{L_1+1} = \cdot]$. Otherwise, if x_{i+1} differs from $\check{x}_{i+1}$ (which means that the coupling fails at step $i+1$), then we choose independently $\mathcal{A}_{i+1}, x_{i+1}^+$ according to $\overline{P}_{x_{i+1}}[\mathcal{A}_1 = \cdot, Z_{L_1+1} = \cdot]$ and $\check{\mathcal{A}}_{i+1}, \check{x}_{i+1}^+$ according to $\overline{P}_{\check{x}_{i+1}}[\mathcal{A}_1 = \cdot, Z_{L_1+1} = \cdot]$. In both cases, we repeat the above procedure until step J. Then we write $\mathcal{A} = \bigcup_{i=2}^J \mathcal{A}_i$ and $\check{\mathcal{A}} = \bigcup_{i=2}^J \check{\mathcal{A}}_i$.

By a procedure as in the proof of Lemma 4.2 in [34], (we replace A by A_1, B by A_2, t_* by $\overline{t_*}$, $\mathbb{T}$ by U^N, X_i by Z_i, Y_t by X_t, k by J, U_1 by V_1, $\bar{x}_i$ and $\bar{x}_i^+$ by $\check{x}_i$ and $\check{x}_i^+$), we can check that Q_0 is a coupling of $\mathcal{A}$ and $\check{\mathcal{A}}$, and the probability that the coupling fails has an upper bound

$$\begin{aligned} Q_0[\mathcal{A} \neq \check{\mathcal{A}}] &\leq (J-1) \max_{x \in D} q_x[\Xi \neq \Sigma] \underset{(5.4)}{\overset{(5.1)}{\leq}} c' N^{d-2} e^{-c \log^2 N} \\ &\leq e^{-c'' \log^2 N}, \end{aligned} \tag{5.6}$$

which is exactly what we want. $\square$

On an auxiliary probability space $(\mathcal{O}_1, \mathcal{F}_1, \mathcal{P}^{\mathcal{I}_1})$, we denote by η_1 the Poisson point process on $\Gamma(U^N)$ with intensity $(1+\varepsilon/3)u_{**}\mathrm{cap}(A_1)\kappa_1$, where κ_1 is defined as the law of the stopped process $X_{(H_{A_1}+\cdot)\wedge V_1}$ under $\overline{P}_\sigma$. In other words, κ_1 is the law of "long excursions" started from σ and recorded from the first time it enters A_1. We denote by

$$\mathcal{I}_1 = \bigcup_{\gamma \in \mathrm{supp}(\eta_1)} \mathrm{Range}(\gamma) \cap A_1 \tag{5.7}$$

the trace of η_1 on A_1. In the next proposition, we construct a second coupling such that $\check{\mathcal{A}}$ dominates $\mathcal{I}_1$ with high probability.

PROPOSITION 5.3. *There exists a probability space $(\Omega_1, \mathcal{B}_1, Q_1)$, endowed with random sets $\mathcal{I}_1$ with the same law as $\mathcal{I}_1$ under $\mathcal{P}^{\mathcal{I}_1}$ and $\check{\mathcal{A}}$ with the same law as $\check{\mathcal{A}}$ under $\overline{P_2^J}$, such that*

$$Q_1[\check{\mathcal{A}} \supseteq \mathcal{I}_1] \geq 1 - e^{-N^c}. \tag{5.8}$$

PROOF. We pick a Poisson random variable ξ with parameter $(1+\varepsilon/3)u_{**} \times \mathrm{cap}(A_1)$. Then we generate (independently from ξ) an infinite sequence $\{\check{X}^i\}_{i\geq 1}$ of i.i.d. confined walks under $\overline{P}_\sigma$. We then let $\mathcal{I}_1 \sim \bigcup_{i=2}^{\xi+1} \check{X}^i_{[0,V_1)} \cap A_1$ and $\check{\mathcal{A}} = \bigcup_{i=2}^{J} \check{X}^i_{[0,V_1)} \cap A_1$, both having the respective required laws. Moreover $\{\check{\mathcal{A}} \supseteq \mathcal{I}_1\} = \{J \geq \xi + 1\}$, by the definition of J [see (5.1)] and a standard estimate on the deviation of Poisson random variables, we have

$$Q_1[\check{\mathcal{A}} \supseteq \mathcal{I}_1] = Q_1[J \geq \xi + 1] \geq 1 - e^{-N^c}, \tag{5.9}$$

which is exactly (5.8). □

Now on another auxiliary probability space $(\mathcal{O}_2, \mathcal{F}_2, \mathcal{P}^{\mathcal{I}_2})$, we denote by η_2 the Poisson point process on $\Gamma(U^N)$ with intensity $(1+\varepsilon/4)u_{**}\mathrm{cap}(A_1)\kappa_2$, where κ_2 is defined as the law of the stopped process $X_{\cdot\wedge V_1}$ under $\overline{P}_{\widetilde{e}_{A_1}}$. In other words, it is the law of "long excursions" started from the normalized equilibrium measure of A_1 (note that, since in this case the excursions start from inside A_1, we start recording directly from time 0). We denote by

$$\mathcal{I}_2 = \bigcup_{\gamma \in \mathrm{supp}(\eta_2)} \mathrm{Range}(\gamma) \cap A_1 \tag{5.10}$$

the trace of η_2 on A_1. The next proposition and corollary construct the third coupling so that $\mathcal{I}_1$ dominates $\mathcal{I}_2$ almost surely. This is shown by proving that the intensity measure of $\mathcal{I}_1$ is bigger than that of $\mathcal{I}_2$ with the help of Proposition 4.7.

PROPOSITION 5.4. *For large N, one has*

$$\left(1+\frac{\varepsilon}{3}\right)\kappa_1 \geq \left(1+\frac{\varepsilon}{4}\right)\kappa_2. \tag{5.11}$$

PROOF. By the definition of κ_1 and κ_2, and the strong Markov property applied at time H_{A_1}, we can represent the Radon–Nikodym derivative of κ_1 and κ_2 through a function of the starting point of the trajectory

$$\frac{d\kappa_1}{d\kappa_2} = \phi(X_0) \quad \text{where } \phi(x) = \frac{\overline{P}_\sigma[X_{H_{A_1}} = x]}{\widetilde{e}_{A_1}(x)} \text{ for all } x \in \partial_i A_1 \text{ and 0 otherwise.} \tag{5.12}$$

Hence we obtain, via (4.61), that for large N,

$$\frac{d(\kappa_1 - \kappa_2)}{d\kappa_2} = \phi(X_0) - 1 \geq -N^{-c} \geq \frac{-\varepsilon/12}{(1+\varepsilon/3)}, \qquad \kappa_2\text{-a.s.} \tag{5.13}$$

This implies (5.11) after rearrangement. □

As a consequence, we have the following corollary.

COROLLARY 5.5. *For large N, there exists a probability space $(\Sigma_2, \mathcal{B}_2, Q_2)$ endowed with random sets $\mathcal{I}_1$ with the same law as $\mathcal{I}_1$ under $P^{\mathcal{I}_1}$ and $\mathcal{I}_2$ with the same law as $\mathcal{I}_2$ under $P^{\mathcal{I}_2}$, such that*

$$\mathcal{I}_1 \supseteq \mathcal{I}_2, \qquad Q_2\text{-a.s.} \tag{5.14}$$

PROOF. This follows immediately from the domination of measures. Indeed, we first construct $\mathcal{I}_2$ on some probability space. Then we consider the positive measure on $\Gamma(U^N)$

$$\alpha = (1+\varepsilon/3)\kappa_1 - (1+\varepsilon/4)\kappa_2, \tag{5.15}$$

and construct (independently from $\mathcal{I}_2$) a Poisson point process $\widehat{\eta}$ on $\Gamma(U^N)$ with intensity measure α. Then $\mathcal{I}_1 = (\bigcup_{\gamma \in \mathrm{supp}(\widehat{\eta})} \mathrm{Range}(\gamma) \cap A_1) \cup \mathcal{I}_2$ has the required law. □

On another auxiliary probability space $(\mathcal{O}_2', \mathcal{F}_2', \mathcal{P}^{\mathcal{I}_2'})$, we denote by η_2' the law of the Poisson point process on $\Gamma(U^N)$ with intensity $(1+\varepsilon/4)u_{**}\mathrm{cap}(A_1)$ κ_2', where κ_2' is defined as the stopped process $X_{\cdot \wedge T_{A_2}}$ under $P_{\widetilde{e}_{A_1}}$, or equivalently $\overline{P}_{\widetilde{e}_{A_1}}$. Contrary to the definition of a "long excursion," we would like to call $X_{[H_{A_1}, T_{A_2})}$ a "short excursion," since we stop the excursion earlier than a "long excursion" (this is because $T_{A_2} < V_1$). In other words, κ_2' is the measure of "short excursions" started from the normalized equilibrium measure of A_1. We denote by

$$\mathcal{I}_2' = \bigcup_{\gamma \in \mathrm{supp}(\eta_2')} \mathrm{Range}(\gamma) \cap A_1 \tag{5.16}$$

the trace of η_2' in A_1. Hence, we can naturally construct the fourth coupling such that $\mathcal{I}_2$ dominates $\mathcal{I}_2'$ almost surely, which is stated in the corollary below.

COROLLARY 5.6. *When N is large, there exists a probability space $(\Sigma_2', \mathcal{B}_2', zQ_2')$, endowed with random sets $\mathcal{I}_2'$ with the same law as $\mathcal{I}_2'$ under $\mathcal{P}^{\mathcal{I}_2'}$, and $\mathcal{I}_2$ with the same law as $\mathcal{I}_2$ under $\mathcal{P}^{\mathcal{I}_2}$ such that*

$$\mathcal{I}_2 \supseteq \mathcal{I}_2', \qquad Q_2'\text{-a.s.} \tag{5.17}$$

The fifth coupling establishes the stochastic domination of $\mathcal{I}_2'$ on the trace of $\mathcal{I}^{(1+\varepsilon/8)u_{**}}$ in A_1. It is reproduced from [4].

PROPOSITION 5.7. *When N is large, there exists a probability space $(\Sigma_3, \mathcal{B}_3, Q_3)$ endowed with random sets $\mathcal{I}$ with the same law as $\mathcal{I}^{u_{**}(1+\varepsilon/8)} \cap A_1$ under $\mathbb{P}$ and $\mathcal{I}_2'$ with the same law as $\mathcal{I}_2'$ under $P^{\mathcal{I}_2'}$, such that*

$$Q_3[\mathcal{I}_2' \supseteq \mathcal{I}] \geq 1 - e^{-N^c}. \tag{5.18}$$

We refer the readers to Proposition 5.4 of [4] and to Section 9 of [4] for its proof.

The next proposition links together the above couplings from Propositions 5.2, 5.3, Corollaries 5.5, 5.6, and Proposition 5.7. We prove that for any x_0 in the "strip" Γ^N, the probability that it is connected in $\mathcal{V}$ [i.e., the vacant set of the random walk, see below (1.5)] to the (inner) boundary of $A_1^{x_0}$ is small.

PROPOSITION 5.8. *For large N and all $x_0 \in \Gamma^N$, one has*

$$\widetilde{P}_N[x_0 \overset{\mathcal{V}}{\longleftrightarrow} \partial_i A_1^{x_0}] \le e^{-c\log^2 N}. \tag{5.19}$$

PROOF. First, by Corollary 2.8, Proposition 5.1 and the first two couplings [namely Proposition 5.2 (see ibid. for notation) and Corollary 5.3], one knows that for large N,

$$\begin{aligned}\widetilde{P}_N[x_0 \overset{\mathcal{V}}{\longleftrightarrow} \partial_i A_1^{x_0}] &\overset{(2.37)}{\le} \overline{P}_0[x_0 \overset{(X_{[R_2,T_N)})^c}{\longleftrightarrow} \partial_i A_1^{x_0}] \\ &\underset{(5.3)}{\overset{(5.2)}{\le}} \overline{P_2^J}[x_0 \overset{\check{A}^c}{\longleftrightarrow} \partial_i A_1^{x_0}] + e^{-c\log^2 N} \\ &\overset{(5.8)}{\le} P^{\mathcal{I}_1}[x_0 \overset{\mathcal{I}_1^c}{\longleftrightarrow} \partial_i A_1^{x_0}] + e^{-c'\log^2 N}.\end{aligned} \tag{5.20}$$

Then, by the third, fourth and fifth couplings, namely Corollaries 5.5, 5.6 and Proposition 5.7, and the strong super-criticality of random interlacements [see (1.14)], for large N, one obtains the following inequalities:

$$\begin{aligned}\mathcal{P}^{\mathcal{I}_1}[x_0 \overset{\mathcal{I}_1^c}{\longleftrightarrow} \partial_i A_1^{x_0}] &\overset{(5.14)}{\le} \mathcal{P}^{\mathcal{I}_2}[x_0 \overset{\mathcal{I}_2^c}{\longleftrightarrow} \partial_i A_1^{x_0}] \overset{(5.17)}{\le} \mathcal{P}^{\mathcal{I}_2'}[x_0 \overset{\mathcal{I}_2'^c}{\longleftrightarrow} \partial_i A_1^{x_0}] \\ &\overset{(5.18)}{\le} Q_3[x_0 \overset{\mathcal{I}^c}{\longleftrightarrow} \partial_i A_1^{x_0}] + e^{-N^c} \overset{(1.14)}{\le} e^{-N^{c'}},\end{aligned} \tag{5.21}$$

which show that the first term to the right of the last inequality in (5.20) has a stretched exponential decay in N. The claim (5.19) hence follows by inserting (5.21) into (5.20). □

We are ready now to state and prove the main result of this section, namely that the tilted disconnection probability tends to 1 as N tends to infinity.

THEOREM 5.9.

$$\lim_{N\to\infty} \widetilde{P}_N[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty] = 1. \tag{5.22}$$

PROOF. Note that for large N, if a nearest-neighbor path connects K_N and infinity, it must go through the set Γ^N at some point x_0 [see above (3.1) for the definition of Γ^N]. Hence, it connects x_0 to the inner boundary of $A_1^{x_0}$, so that

$$\{K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty\}^c \subset \bigcup_{x_0\in\Gamma^N} \{x_0 \overset{\mathcal{V}}{\longleftrightarrow} \partial_i A_1^{x_0}\}. \tag{5.23}$$

Thus, we see that for large N,

$$\widetilde{P}_N[\{K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty\}^c] \le \sum_{x_0 \in \Gamma^N} \widetilde{P}_N[x_0 \overset{\mathcal{V}}{\longleftrightarrow} \partial_i A_1^{x_0}]. \tag{5.24}$$

By Proposition 5.8, we find that for large N, uniformly for each $x_0 \in \Gamma^N$, we can bound each term on right-hand side of (5.24), and find

$$\widetilde{P}_N[\{K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty\}^c] \le |\Gamma^N| e^{-c \log^2 N} \underset{N\to\infty}{\longrightarrow} 0. \tag{5.25}$$

This completes the proof of Theorem 5.9. □

6. Denouement and epilogue. In this section, we combine the main ingredients, namely Theorem 5.9 and Proposition 2.14 and prove Theorem 0.1.

PROOF OF THEOREM 0.1. We recall the entropy inequality [see (1.16)], and apply it to P_0 and $\widetilde{P}_N$ (which is defined in Section 2). By Theorem 5.9, one has

$$\lim_{N\to\infty} \widetilde{P}_N[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty] = 1, \tag{6.1}$$

thus the relative entropy inequality (1.16) yields that

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log\bigl(P_0[K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty]\bigr) \ge -\limsup_{N\to\infty} \frac{1}{N^{d-2}} H(\widetilde{P}_N|P_0). \tag{6.2}$$

Then, as in the proof of Proposition 2.14, taking consecutively the lim sup as $\eta \to 0$, $R \to \infty$, $\delta \to 0$ and $\varepsilon \to 0$, one has

$$\limsup_{\varepsilon\to 0} \limsup_{\delta\to 0} \limsup_{R\to\infty} \limsup_{\eta\to 0} \limsup_{N\to\infty} \frac{1}{N^{d-2}} H(\widetilde{P}_N|P_0) \le \frac{u_{**}}{d} \mathrm{cap}_{\mathbb{R}^d}(K), \tag{6.3}$$

proving Theorem 0.1. □

REMARK 6.1. Assume for simplicity that the compact K is regular. Notice that unlike what happens for $d \ge 5$, when $d = 3, 4$, the function h defined in (0.8) is not in $L^2(\mathbb{R}^d)$, and $h_N(x) = h(\frac{x}{N})$ is not in $l^2(\mathbb{Z}^d)$. This fact affects T_N defined in (2.16) (which diverges if $R \to \infty$ when $d = 3, 4$, but not when $d \ge 5$). One can wonder whether this feature reflects different qualitative behaviors of the random walk path under the conditional measure $P_0[\cdot|K_N \overset{\mathcal{V}}{\nleftrightarrow} \infty]$ when N becomes large.

APPENDIX

In the appendix, we include the proof of Propositions 4.7 and 5.1.

PROOF OF PROPOSITION 4.7. We first prove that for $x \in \partial_i A_1$

$$\left| \overline{P}_x[V < \widetilde{H}_{A_1}] - \overline{P}_\sigma[X_{H_{A_1}} = x] \sum_{y \in \partial_i A_1} \overline{P}_y[V < \widetilde{H}_{A_1}] \right| \le e^{-c \log^2 N}, \tag{A.1}$$

and, as we will see, the claim (4.61) will then follow. We consider in the left-hand side of (A.2) the probability that the random walk started from $x \in \partial_i A_1$ stays in D for a time interval of length $\overline{t_*}$ before returning to A_1, and then returns to A_1 through some vertex other than x. By reversibility of the confined walk, and the fact that by claim 3 of (2.18) and claim 1 of (2.38), $\pi(y) = \pi(x)$ for all $y \in \partial_i A_1$, this probability can be written as

$$\sum_{y \in \partial_i A_1 \setminus \{x\}} \overline{P}_x[V < \widetilde{H}_{A_1}, X_{H_{A_1}} = y] = \sum_{y \in \partial_i A_1 \setminus \{x\}} \overline{P}_y[V < \widetilde{H}_{A_1}, X_{H_{A_1}} = x]. \tag{A.2}$$

As in (5.5), we consider L defined by

$$L = \sup\{l : \tau_l \le V, Z_l \in A_2\}. \tag{A.3}$$

We consider the summands from (A.2): for all $x, y \in \partial_i A_1$, we sum over all possible values of L and $X_{\tau_L} = Z_L$ [recall the definition of τ_l in (1.4) and the relation between X_{τ_l} and Z_l in (1.3)], and apply Markov property at the times τ_{l+1} and $\tau_{l+1} + \overline{t_*}$:

$$\begin{aligned}
&\overline{P}_x[V < \widetilde{H}_{A_1}, X_{\widetilde{H}_{A_1}} = y] \\
&\quad = \sum_{l \ge 0, x' \in \partial_i A_2} \overline{P}_x\big[L = l, Z_l = x', V < \widetilde{H}_{A_1}, X_{\widetilde{H}_{A_1}} = y\big] \\
&\quad = \sum_{l \ge 0, x' \in \partial_i A_2} \overline{P}_x\big[Z_l = x', \tau_l < \widetilde{H}_{A_1} \wedge V, H_{A_2} \circ \theta_{\tau_{l+1}} > \overline{t_*}, X_{\widetilde{H}_{A_1}} = y\big] \\
&\quad = \sum_{\substack{l \ge 0, x'' \in D \\ x' \in \partial_i A_2}} \overline{P}_{x''}[X_{H_{A_1}} = y] \overline{E}_x\big[Z_l = x', \tau_l < \widetilde{H}_{A_1} \wedge V, \\
&\qquad \overline{P}_{Z_{l+1}}[H_{A_2} > \overline{t_*}] \overline{P}_{Z_{l+1}}\big[X_{\overline{t_*}} = x'' | H_{A_2} > \overline{t_*}\big]\big],
\end{aligned} \tag{A.4}$$

[we will soon use the fact that the conditioned probability in the last expression is close to $\sigma(x'')$ by Proposition 4.5]. Similarly, we have

$$\begin{aligned}
&\overline{P}_x[V < \widetilde{H}_{A_1}] \\
&\qquad = \sum_{l \ge 0, x' \in \partial_i A_2} \overline{E}_x\big[Z_l = x', \tau_l < \widetilde{H}_{A_1} \wedge V, \overline{P}_{Z_{l+1}}[H_{A_2} > \overline{t_*}]\big].
\end{aligned} \tag{A.5}$$

This implies that

$$\overline{P}_x[V < \widetilde{H}_{A_1}]\overline{P}_\sigma[X_{H_{A_1}} = y] \tag{A.6}$$
$$= \sum_{\substack{l\geq 0, x''\in D\\ x'\in\partial_i A_2}} \overline{E}_x[Z_l = x', \tau_l < \widetilde{H}_{A_1} \wedge V, \overline{P}_{Z_{l+1}}[H_{A_2} > \overline{t_*}]]\sigma(x'')$$
$$\times \overline{P}_{x''}[X_{H_{A_1}} = y].$$

Hence, by combining (A.4) and (A.6) we have

$$\big|\overline{P}_x[V < \widetilde{H}_{A_1}, X_{\widetilde{H}_{A_1}} = y] - \overline{P}_x[V < \widetilde{H}_{A_1}]\overline{P}_\sigma[X_{H_{A_1}} = y]\big| \tag{A.7}$$
$$\overset{(4.46)}{\leq} e^{-c\log^2 N}.$$

Applying this estimate in both sides in (A.2), we obtain that

$$\left|\overline{P}_x[V < \widetilde{H}_{A_1}]\overline{P}_\sigma[X_{H_{A_1}} \neq x] - \sum_{y\in\partial_i A_1\setminus\{x\}} \overline{P}_y[V < \widetilde{H}_{A_1}]\overline{P}_\sigma[X_{H_{A_1}} = x]\right| \tag{A.8}$$
$$\leq e^{-c\log^2 N}.$$

Finally, by adding and subtracting $\overline{P}_x[V < \widetilde{H}_{A_1}]\overline{P}_\sigma[X_{H_{A_1}} = x]$, we obtain (A.1) as desired.

Now we prove (4.61). By (1.12) and (4.60) one has that

$$\sum_{y\in\partial_i A_1} \overline{P}_y[V < \widetilde{H}_{A_1}]\widetilde{e}_{A_1}(x) \geq N^{-c'}. \tag{A.9}$$

Hence dividing (A.1) by the left-hand term of (A.9), one obtains

$$\left|\frac{\overline{P}_x[V < \widetilde{H}_{A_1}]}{\sum_{y\in\partial_i A_1} \overline{P}_y[V < \widetilde{H}_{A_1}]\widetilde{e}_{A_1}(x)} - \frac{\overline{P}_\sigma[X_{H_{A_1}} = x]}{\widetilde{e}_{A_1}(x)}\right| \leq e^{-c'\log^2 N}, \tag{A.10}$$

and together with (4.56) the proof of (4.61) is complete. □

PROOF OF PROPOSITION 5.1. In this proof, we always assume that N is sufficiently large. We recall the definition of T_N in (2.16) and the choice of ε in (2.2). In order to prove (5.2), we observe that, $\overline{P}_0$-a.s.,

$$\{R_J \geq T_N\} \subseteq \left\{H_{A_1} + H_{A_1}\circ\theta_{V_1} + \cdots + H_{A_1}\circ\theta_{V_{J-1}} \geq \left(1 - \frac{\varepsilon}{100}\right)T_N\right\} \tag{A.11}$$
$$\cup \left\{V\circ\theta_{R_1} + \cdots + V\circ\theta_{R_{J-1}} \geq \frac{\varepsilon}{100}T_N\right\},$$

that is, the (unlikely) event $\{R_J \geq T_N\}$ happens only when either the sum of H_{A_1}'s exceeds a quantity close to T_N or the sum of shifted V's exceeds a small quantity (but still of order T_N). Now we give an upper bound to their respective probabilities. We define

$$t_N = \sup_{y \in U^N} \overline{E}_y[H_{A_1}], \tag{A.12}$$

which is the maximum of the expected entrance time in A_1 starting from an arbitrary point in U^N (it is not much bigger than $\overline{E}_\pi[H_{A_1}]$ by (3.40)). By the exponential Chebychev inequality and the strong Markov property applied inductively at $V_1, \dots, V_{J-1}$ and $R_1, \dots, R_{J-1}$, we deduce from (A.11) that, for any $\theta > 0$,

$$\begin{aligned} &\overline{P}_0[R_J \geq T_N] \\ &\quad \leq \exp\left(-\theta\left(1 - \frac{\varepsilon}{100}\right)\frac{T_N}{t_N}\right)\left(\sup_{x \in U^N} \overline{E}_x\left[\exp\left(\theta \frac{H_{A_1}}{t_N}\right)\right]\right)^J \\ &\qquad + \exp\left(-\frac{\varepsilon}{100}\frac{T_N}{t_N}\right)\left(\sup_{x \in A_1} \overline{E}_x[e^{V/t_N}]\right)^J. \end{aligned} \tag{A.13}$$

We now treat the first term on the right-hand side of (A.13). Khas'minskii's lemma (see (4) and (6) in [18]) states that for all B subset of U^N and $n \geq 1$,

$$\sup_{x \in U^N} \overline{E}_x[H_B^n] \leq n! \sup_{y \in U^N} \overline{E}_y[H_B]^n. \tag{A.14}$$

Hence, we have

$$\begin{aligned} \sup_{x \in U^N} \overline{E}_x\left[\exp\left(\theta \frac{H_{A_1}}{t_N}\right)\right] &\leq \sum_{j=0}^{\infty} \frac{\theta^j}{j!\,t_N^j} \sup_{x \in U^N} \overline{E}_x[H_{A_1}^j] \overset{\text{(A.14)}}{\underset{\text{(A.12)}}{\leq}} \sum_{j=0}^{\infty} \theta^j = \frac{1}{1-\theta} \\ &\qquad \text{for } \theta \in \left(0, \frac{1}{2}\right). \end{aligned} \tag{A.15}$$

Now, we derive an upper bound for $\sup_{x \in A_2} \overline{E}_x[\exp(\frac{V}{t_N})]$ and treat the second term on the right-hand side of (A.13). We first note that, $\overline{P}_x$-a.s. for any $x \in A_2$,

$$\begin{aligned} V &\leq (T_{A_3} + \overline{t_*}) 1_{\{H_{A_2} \circ \theta_{T_{A_3}} > \overline{t_*}\}} \\ &\quad + (T_{A_3} + \overline{t_*} + V \circ \theta_{H_{A_2}} \circ \theta_{T_{A_3}}) 1_{\{H_{A_2} \circ \theta_{T_{A_3}} \leq \overline{t_*}\}} \\ &= T_{A_3} + \overline{t_*} + V \circ \theta_{H_{A_2}} \circ \theta_{T_{A_3}} 1_{\{H_{A_2} \circ \theta_{T_{A_3}} \leq \overline{t_*}\}}. \end{aligned} \tag{A.16}$$

By the strong Markov property applied at $H_{A_2} \circ \theta_{T_{A_3}} + T_{A_3}$ and T_{A_3}, we have

$$\begin{aligned} &\sup_{x \in A_2} \overline{E}_x[e^{V/t_N}] \\ &\overset{\text{(A.16)}}{\leq} \sup_{x \in A_2} \overline{E}_x[e^{(T_{A_3} + \overline{t_*})/t_N}]\left(1 + \sup_{y \in U^N \setminus A_3} \overline{P}_y[H_{A_2} \leq \overline{t_*}] \sup_{x \in A_2} \overline{E}_x[e^{V/t_N}]\right) \\ &\overset{\text{(3.11)}}{\leq} \sup_{x \in A_2} \overline{E}_x[e^{(T_{A_3} + \overline{t_*})/t_N}]\left(1 + N^{-c} \sup_{x \in A_2} \overline{E}_x[e^{V/t_N}]\right). \end{aligned} \tag{A.17}$$

By Proposition 3.5, we have

$$\text{(A.18)}\qquad \begin{aligned}\frac{1}{t_N} &\overset{\text{(A.12)}}{\leq} \frac{1}{\overline{\overline{E}}_\pi[H_{A_1}]} \overset{\text{(3.32)}}{\leq} (1+N^{-c})\frac{\text{cap}(A_1)}{T_N}u_{**}(1+\varepsilon)\\ &\underset{\text{(1.11)}}{\overset{\text{(2.20)5.}}{\leq}} cN^{-d+r_1(d-2)}.\end{aligned}$$

By an elementary estimate on simple random walk and the observation that the diameter of A_3 is smaller than cN^{r_3}, we have

$$\text{(A.19)}\qquad \overline{E}_x[T_{A_3}] \overset{\text{(3.4)}}{=} E_x[T_{A_3}] \leq cN^{2r_3} \qquad \text{for all } x \in A_3,$$

therefore we obtain that

$$\text{(A.20)}\qquad \frac{\sup_{x\in A_3}\overline{E}_x[T_{A_3}]}{t_N} \leq cN^{-d+2r_3+(d-2)r_1} \leq N^{-c'}.$$

By an argument like (A.15), again with the help of Khasminskii's lemma [see (A.14)], we obtain that

$$\text{(A.21)}\qquad \sup_{x\in A_2} \overline{E}_x\left[\exp\left(\frac{T_{A_3}}{t_N}\right)\right] \leq \frac{1}{1-N^{-c}} \leq e^{N^{-c'}} \qquad \text{for large } N.$$

Moreover, we obtain from (A.18) that

$$\text{(A.22)}\qquad \frac{\overline{t_*}}{t_N} \overset{\text{(2.63)}}{\leq} cN^{-c'}.$$

We apply (A.21) and (A.22) to the right-hand side of (A.17), and conclude after rearrangement [and with an implicit truncation argument where V in (A.16) and (A.17) is replaced by $V \wedge M$] that

$$\text{(A.23)}\qquad \sup_{x\in A_2} \overline{E}_x[e^{V/t_N}] \leq e^{N^{-c}}.$$

We now return to (A.13). Substituting (A.15) and (A.23) into (A.13) and using the fact that for $0 \leq \theta \leq \frac{1}{2}$,

$$\text{(A.24)}\qquad (1-\theta)^{-1} \leq 1+\theta+2\theta^2 \leq e^{\theta+2\theta^2},$$

we deduce that

$$\text{(A.25)}\qquad \begin{aligned}&\overline{P}_0[R_J \geq T_N]\\ &\quad\leq \exp\left(-\theta\left(1-\frac{\varepsilon}{100}\right)\frac{T_N}{t_N} + (\theta+2\theta^2)J\right) + \exp\left(-\frac{\varepsilon}{100}\frac{T_N}{t_N} + N^{-c}J\right)\\ &\quad\underset{\text{(A.24)}}{\overset{\text{(5.1)}}{\leq}} \exp\left(-\theta\left(1-\frac{\varepsilon}{100}\right)\frac{T_N}{t_N} + (\theta+2\theta^2)\lfloor(1+\varepsilon/2)u_{**}\text{cap}(A_1)\rfloor\right)\\ &\qquad\quad + \exp\left(-\frac{\varepsilon}{100}\frac{T_N}{t_N} + N^{-c}\lfloor(1+\varepsilon/2)u_{**}\text{cap}(A_1)\rfloor\right).\end{aligned}$$

Recall the definition of f_{A_1} in (3.8). Using Lemma 3.6, we know that

$$\frac{\overline{E}_x[H_{A_1}]}{\overline{E}_\pi[H_{A_1}]} = 1 - f_{A_1}(x) \overset{(3.36)}{\leq} 1 + N^{-c} \leq \left(1 - \frac{\varepsilon}{100}\right)^{-1} \quad \text{for all } x \in U^N. \tag{A.26}$$

Hence, by Proposition 3.7 we obtain that

$$\frac{T_N}{t_N} \geq \left(1 - \frac{\varepsilon}{100}\right)\frac{T_N}{\overline{E}_\pi[H_{A_1}]} \overset{(3.49)}{\geq} \left(1 - \frac{\varepsilon}{50}\right)(1+\varepsilon)u_{**}\mathrm{cap}(A_1). \tag{A.27}$$

By choosing an appropriately small θ and applying (A.27) we know that

$$-\theta\left(1 - \frac{\varepsilon}{100}\right)\frac{T_N}{t_N} + (\theta + 2\theta^2)\lfloor(1+\varepsilon/2)u_{**}\mathrm{cap}(A_1)\rfloor \leq -N^c \quad \text{for large } N, \tag{A.28}$$

moreover, we also know that

$$-\frac{\varepsilon}{100}\frac{T_N}{t_N} + N^{-c}\lfloor(1+\varepsilon/2)u_{**}\mathrm{cap}(A_1)\rfloor \leq -N^{c'} \qquad \text{for large } N. \tag{A.29}$$

Inserting (A.28) and (A.29) into (A.25), we obtain (5.2) as desired. □

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