The Annals of Probability
2017, Vol. 45, No. 2, 824–878
DOI: 10.1214/15-AOP1076

MEAN-FIELD STOCHASTIC DIFFERENTIAL EQUATIONS AND ASSOCIATED PDES

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In this paper, we consider a mean-field stochastic differential equation, also called the McKean–Vlasov equation, with initial data $(t,x)\in[0,T]\times\mathbb{R}^d$, whose coefficients depend on both the solution $X_s^{t,x}$ and its law. By considering square integrable random variables ξ as initial condition for this equation, we can easily show the flow property of the solution $X_s^{t,\xi}$ of this new equation. Associating it with a process X_s^{t,x,P_ξ} which coincides with $X_s^{t,\xi}$, when one substitutes ξ for x, but which has the advantage to depend on ξ only through its law P_ξ, we characterize the function $V(t,x,P_\xi)=E[\Phi(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}})]$ under appropriate regularity conditions on the coefficients of the stochastic differential equation as the unique classical solution of a nonlocal partial differential equation of mean-field type, involving the first- and the second-order derivatives of V with respect to its space variable and the probability law. The proof bases heavily on a preliminary study of the first- and second-order derivatives of the solution of the mean-field stochastic differential equation with respect to the probability law and a corresponding Itô formula. In our approach, we use the notion of derivative with respect to a probability measure with finite second moment, introduced by Lions in [Cours au Collège de France: Théorie des jeu à champs moyens (2013)], and we extend it in a direct way to the second-order derivatives.

CONTENTS

Received October 2014; revised October 2015.

[1]Supported by the NSF of P. R. China (Nos. 11071144, 11171187, 11222110), Shandong Province (Nos. BS2011SF010, JQ201202), Program for New Century Excellent Talents in University (No. NCET-12-0331), 111 Project (No. B12023).

MSC2010 subject classifications. Primary 60H10; secondary 60K35.

Key words and phrases. Mean-field stochastic differential equation, McKean–Vlasov equation, value function, PDE of mean-field type.

1. Introduction. Given a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with a Brownian motion $B = (B_t)_{t\in[0,T]}$ and its filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ augmented by all P-null sets and a sufficiently rich sub-σ-algebra $\mathcal{F}_0$ independent of B, we consider the mean-field stochastic differential equation (SDE), also known under the name of McKean–Vlasov SDE,

$$dX_s^{t,x} = \sigma\big(X_s^{t,x}, P_{X_s^{t,x}}\big)\,dB_s + b\big(X_s^{t,x}, P_{X_s^{t,x}}\big)\,ds, \qquad s\in[t,T],\ X_t^{t,x} = x\in\mathbb{R}^d. \tag{1.1}$$

It is well known that under an appropriate Lipschitz assumption on the coefficients this equation possesses for all $(t,x)\in[t,T]\times\mathbb{R}^d$ a unique solution $X_s^{t,x}$, $s\in[0,T]$. For the classical SDE whose coefficients $\sigma(x,\mu) = \sigma(x)$, $b(x,\mu) = b(x)$ depend only on $x\in\mathbb{R}^d$ but not on the probability measure μ, it is well known that the solution $X_s^{t,x}$, $0\le t\le s\le T$, $x\in\mathbb{R}^d$, defines a flow and, if the coefficients are regular enough, the unique classical solution of the partial differential equation (PDE)

$$\begin{aligned}&\partial_t V(t,x) + \tfrac{1}{2}\operatorname{tr}\big(\sigma\sigma^*(x)D_x^2 V(t,x)\big) + b(x)D_x V(t,x) = 0, \qquad (t,x)\in[0,T]\times\mathbb{R}^d,\\ &V(T,x) = \Phi(x), \qquad x\in\mathbb{R}^d,\end{aligned} \tag{1.2}$$

is $V(t,x) = E[\Phi(X_T^{t,x})]$, $(t,x)\in[0,T]\times\mathbb{R}^d$. But how about the above SDE whose coefficients depend on $(x,\mu)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, where $\mathcal{P}_2(\mathbb{R}^d)$ denotes the space of the probability measures over $\mathbb{R}^d$ with finite second moment? Of course, for an SDE with coefficients depending on (x,μ) the solution $X_s^{t,x}$, $0\le s\le t\le T$, $x\in\mathbb{R}^d$, does obviously not define a flow. But we see easily that, if we replace the deterministic initial condition $X_t^{t,x} = x\in\mathbb{R}^d$ by a square integrable random variable $X_t^{t,\xi} = \xi\in L^2(\mathcal{F}_t;\mathbb{R}^d)(:= L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^d))$ and consider the SDE

$$dX_s^{t,\xi} = \sigma\big(X_s^{t,\xi}, P_{X_s^{t,\xi}}\big)\,dB_s + b\big(X_s^{t,\xi}, P_{X_s^{t,\xi}}\big)\,ds, \qquad s\in[t,T],\ X_t^{t,\xi} = \xi\in\mathbb{R}^d \tag{1.3}$$

(where, obviously, in general, $X^{t,\xi}\ne X^{t,x}|_{x=\xi}$), then we have the flow property: For all $0\le t\le s\le T$ and $\xi\in L^2(\mathcal{F}_t;\mathbb{R}^d)$ it holds $X_r^{s,\eta} = X_r^{t,\xi}$, $r\in[s,T]$, for $\eta = X_s^{t,\xi}$. This flow property should give rise to a PDE with a solution $V(t,\xi) = E[\Phi(X_T^{t,\xi}, P_{X_T^{t,\xi}})]$, but the fact that ξ has to belong to $L^2(\mathcal{F}_t;\mathbb{R}^d)$ has the consequence that $V(t,\xi)$ has to be considered over the Hilbert space $L^2(\mathcal{F};\mathbb{R}^d)$, which because of the absence of second-order Fréchet differentiability even in simplest examples makes such PDE difficult to handle. As an alternative, we associate with the above SDE for $X^{t,\xi}$ the equation

$$
(1.4)\qquad \begin{aligned} dX_s^{t,x,\xi} &= \sigma\big(X_s^{t,x,\xi}, P_{X_s^{t,\xi}}\big)\, dB_s + b\big(X_s^{t,x,\xi}, P_{X_s^{t,\xi}}\big)\, ds, \\ &\qquad s \in [t,T],\ X_t^{t,x,\xi} = x \in \mathbb{R}^d. \end{aligned}
$$

It turns out (see Lemma 3.1) that $X_s^{t,x,P_\xi} = X_s^{t,x,\xi}$, $s \in [t,T]$, depends on $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$ only through its law P_ξ, $X_s^{t,\xi} = X_s^{t,x,P_\xi}|_{x=\xi}$, and $(X_s^{t,x,P_\xi}, X_s^{t,\xi})$, $0 \le t \le s \le T$, $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, has the flow property.

The objective of our manuscript is to study under appropriate regularity assumptions on the coefficients the second-order PDE which is associated with this stochastic flow, that is, the PDE whose unique classical solution is given by the function

$$
(1.5)\qquad \begin{aligned} V(t,x,P_\xi) &= E\big[\Phi\big(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\big)\big], \\ &\qquad (t,x) \in [0,T] \times \mathbb{R}^d,\ \xi \in L^2\big(\mathcal{F}_t;\mathbb{R}^d\big). \end{aligned}
$$

The function V is defined over $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, and so the study of the first- and the second-order derivatives with respect to the probability law will play a crucial role. In our work, we have based ourselves on the notion of derivative of a function $f : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ with respect to a probability measure μ, which was studied by Lions in his course at *Collège de France* [17] (the reader is also referred to the notes on this course, redacted by Cardaliaguet [6]). The derivative of f with respect to μ is a function $\partial_\mu f : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ (see Section 2). The main result of our work says that, if the coefficients b and σ are twice differentiable in (x,μ) with bounded Lipschitz derivatives of first and second order, then the function $V(t,x,P_\xi)$ defined above is the unique classical solution of the following nonlocal PDE of mean-field type (see Theorem 7.2):

$$
(1.6)\qquad \begin{aligned} &\partial_t V(t,x,P_\xi) + \sum_{i=1}^d \partial_{x_i} V(t,x,P_\xi) b_i(x,P_\xi) \\ &\quad + \frac{1}{2}\sum_{i,j,k=1}^d \partial^2_{x_i x_j} V(t,x,P_\xi)(\sigma_{i,k}\sigma_{j,k})(x,P_\xi) \\ &\quad + E\Bigg\{\sum_{i=1}^d (\partial_\mu V)_i(t,x,P_\xi,\xi) b_i(\xi,P_\xi) \\ &\quad + \frac{1}{2}\sum_{i,j,k=1}^d \partial_{y_i}(\partial_\mu V)_j(t,x,P_\xi,\xi)(\sigma_{i,k}\sigma_{j,k})(\xi,P_\xi)\Bigg\} = 0, \\ &V(T,x,P_\xi) = \Phi(x,P_\xi), \end{aligned}
$$

with $(t, x, P_\xi) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. We see in particular that, in contrast to the classical case, the derivative $\partial_\mu V(t, x, P_\xi, y)$ and, as a second-order derivative, the derivative of $\partial_\mu V(t, x, P_\xi, y)$ with respect to y are involved.

Mean-field SDEs, also known as McKean–Vlasov equations, were discussed the first time by Kac [13, 14] in the frame of his study of the Boltzman equation for the particle density in diluted monatomic gases as well as in that of the stochastic toy model for the Vlasov kinetic equation of plasma. A by now classical method of solving mean-field SDEs by approximation consists in the use of so-called N-particle systems with weak interaction, formed by N equations driven by independent Brownian motions. The convergence of this system to the mean-field SDE is called in the literature propagation of chaos for the McKean–Vlasov equation.

The pioneering works by Kac, and in the aftermath by other authors, have attracted a lot of researchers interested in the study of the chaos propagation and the limit equations in different frameworks; for getting an impression we refer the reader, for instance, to [1, 3, 10–12, 15, 18–20] and [21] as well as the references therein. With the pioneering works on mean-field stochastic differential games by Lasry and Lions (we refer to [16] and the papers cited therein, but also to [6]), new impulses and new applications for mean-field problems were given. So recently, Buckdahn, Djehiche, Li and Peng [4] studied a special mean field problem by a purely stochastic approach and deduced a new kind of backward SDE (BSDE) which they called mean-field BSDE; they showed that the BSDE can be obtained by an approximation involving N-particle systems with weak interaction. They completed these studies of the approximation with associating a kind of central limit theorem for the approximating systems and obtained as limit some forward-backward SDE of mean-field type, which is not only governed by a Brownian motion but also by an independent Gaussian field. In [5], deepening the investigation of mean-field SDEs and associated mean-field BSDEs, Buckdahn, Li and Peng generalized their previous results on mean-field BSDEs, and in a "Markovian" framework in which the initial data (t, x) were frozen in the law variable of the coefficients; they investigated the associated nonlocal PDE. However, our objective has been to overcome this partial freezing of initial data in the mean-field SDE and to study the associated PDE, and this is done in our present manuscript. Our approach is highly inspired by the courses given by Lions [17] at *Collège de France* (redacted by Cardaliaguet [6]) and by recent works of Bensoussan, Frehse, Yam [2] and Carmona and Delarue, who directly inspired by the works of Lasry, Lions [16] and the courses of Lions [17], translated his rather analytical approach into a stochastic one; let us cite [8, 9] and the references indicated therein.

Let us describe the organization of our manuscript and link this description with the explanation of the novelty in our approach:

In Section 2, "Preliminaries," we introduce the framework of our study. Particular attention is paid to a recall of Lions' definition of the derivative of a function defined over the space of probability measures over $\mathbb{R}^d$ with finite second moment. On the basis of this notion of first-order derivatives, we introduce in exactly the same spirit the second-order derivatives of such a function. The first- and the second-order differentiability of a function with respect to a probability measure allows in the following to derive a second-order Taylor expansion (Lemma 2.1) which is new and turns out to be crucial in our approach. We give an example (Example 2.3) which shows that, in general, even in the most regular cases, the functions lifted from the space of probability measures with finite second moment to the space of square integrable random variables are not twice Fréchet differentiable on L^2, but well twice differentiable with respect to the probability measure. To the best of our knowledge, the passage to higher order derivatives with respect to the measure, the second-order Taylor expansion with respect to the measure and Example 2.3 are new. They constitute the crucial paves in our approach, in particular also for the derivation of the Itô formula for mean-field Itô processes (see Theorem 7.1 in Section 7).

In Section 3, we introduce our mean-field SDE with our standard assumptions on its coefficients [their twice fold differentiability with respect to (x, μ) with bounded Lipschitz derivatives of first and second order], and we study useful properties of the mean-field SDE. A crucial step in our approach constitutes the splitting of mean-field SDE (1.1) into (1.3) and (1.4)—a system of mean-field SDEs which, unlike (1.1), generates a flow. The flow concerns the couple formed by the state and by the law of the process. Such a splitting for the study of mean-field SDEs is novel.

A central property studied in Section 4 is the differentiability of its solution process X^{t,x,P_ξ} with respect to the probability law P_ξ. The identification of the derivative of X^{t,x,P_ξ} with respect to the probability law and the equation satisfied by it are new, to the best of our knowledge. The investigations of Section 4 are completed by Section 5, which is devoted to the study of the second-order derivatives of X^{t,x,P_ξ}, and so namely for that with respect to the probability law. The first- and the second-order derivatives of X^{t,x,P_ξ} are characterized as the unique solution of associated SDEs which on their part allow to get estimates for the derivatives of order 1 and 2 of X^{t,x,P_ξ}. The results obtained for the process X^{t,x,P_ξ} and so also for $X^{t,\xi}$ in the Sections 3–5 are used in Section 6 for the proof of the regularity of the value function $V(t, x, P_\xi)$. Finally, Section 7 is devoted to a novel Itô formula associated with mean-field problems, and it gives our main result, Theorem 7.2, stating that our value function V is the unique classical solution of the PDE (1.6) of mean-field type given above.

2. Preliminaries. Let us begin with introducing some notation and concepts, which we will need in our further computations. We shall in particular introduce the notion of differentiability of a function f defined over the space $\mathcal{P}_2(\mathbb{R}^d)$ of all probability measures μ over $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with finite second moment $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty$ [$\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-field over $\mathbb{R}^d$]. The space

$\mathcal{P}_2(\mathbb{R}^{2d})$ is endowed with the 2-Wasserstein metric

$$W_2(\mu,\nu) := \inf\biggl\{\biggl(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\rho(dx\,dy)\biggr)^{1/2}, \rho\in\mathcal{P}_2(\mathbb{R}^{2d}) \\ \text{with } \rho(\cdot\times\mathbb{R}^d)=\mu, \rho(\mathbb{R}^d\times\cdot)=\nu\biggr\}, \qquad \mu,\nu\in\mathcal{P}_2(\mathbb{R}^d). \tag{2.1}$$

Among the different notions of differentiability of a function f defined over $\mathcal{P}_2(\mathbb{R}^d)$, we adopt for our approach that introduced by Lions [17] in his lectures at *Collège de France* in Paris and revised in the notes by Cardaliaguet [6]; we refer the reader also, for instance, to Carmona and Delarue [9]. Let us consider a probability space $(\Omega,\mathcal{F},P)$ which is "rich enough" (the precise space we will work with will be introduced later). "Rich enough" means that for every $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ there is a random variable $\vartheta\in L^2(\mathcal{F};\mathbb{R}^d)(:=L^2(\Omega,\mathcal{F},P;\mathbb{R}^d))$ such that $P_\vartheta=\mu$. It is well known that the probability space $([0,1],\mathcal{B}([0,1]),dx)$ has this property.

Identifying the random variables in $L^2(\mathcal{F};\mathbb{R}^d)$, which coincide P-a.s., we can regard $L^2(\mathcal{F};\mathbb{R}^d)$ as a Hilbert space with inner product $(\xi,\eta)_{L^2}=E[\xi\cdot\eta]$, $\xi,\eta\in L^2(\mathcal{F};\mathbb{R}^d)$, and norm $|\xi|_{L^2}=(\xi,\xi)_{L^2}^{1/2}$. Recall that, due to the definition made by Lions [6] (see Cardaliaguet [7]), a function $f:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ is said to be differentiable in $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ if, for $\widetilde{f}(\vartheta):=f(P_\vartheta)$, $\vartheta\in L^2(\mathcal{F};\mathbb{R}^d)$, there is some $\vartheta_0\in L^2(\mathcal{F};\mathbb{R}^d)$ with $P_{\vartheta_0}=\mu$, such that the function $\widetilde{f}:L^2(\mathcal{F};\mathbb{R}^d)\to\mathbb{R}$ is differentiable (in Fréchet sense) in ϑ_0, that is, there exists a linear continuous mapping $D\widetilde{f}(\vartheta_0):L^2(\mathcal{F};\mathbb{R}^d)\to\mathbb{R}$ $(D\widetilde{f}(\vartheta_0)\in L(L^2(\mathcal{F};\mathbb{R}^d);\mathbb{R}))$ such that

$$\widetilde{f}(\vartheta_0+\eta)-\widetilde{f}(\vartheta_0)=D\widetilde{f}(\vartheta_0)(\eta)+o(|\eta|_{L^2}), \tag{2.2}$$

with $|\eta|_{L^2}\to 0$ for $\eta\in L^2(\mathcal{F};\mathbb{R}^d)$. Since $D\widetilde{f}(\vartheta_0)\in L(L^2(\mathcal{F};\mathbb{R}^d);\mathbb{R})$, Riesz' representation theorem yields the existence of a (P-a.s.) unique random variable $\theta_0\in L^2(\mathcal{F};\mathbb{R}^d)$ such that $D\widetilde{f}(\vartheta_0)(\eta)=(\theta_0,\eta)_{L^2}=E[\theta_0\eta]$, for all $\eta\in L^2(\mathcal{F};\mathbb{R}^d)$. In [17] (see also [6]), it has been proved that there is a Borel function $h_0:\mathbb{R}^d\to\mathbb{R}^d$ such that $\theta_0=h_0(\vartheta_0)$, P-a.s. The function h_0 only depends on the law P_{ϑ_0} but not on ϑ_0 itself. Taking into account the definition of $\widetilde{f}$, this allows to write

$$f(P_\vartheta)-f(P_{\vartheta_0})=E\bigl[h_0(\vartheta_0)\cdot(\vartheta-\vartheta_0)\bigr]+o(|\vartheta-\vartheta_0|_{L^2}), \tag{2.3}$$

$\vartheta\in L^2(\mathcal{F};\mathbb{R}^d)$.

We call $\partial_\mu f(P_{\vartheta_0},y):=h_0(y)$, $y\in\mathbb{R}^d$, the derivative of $f:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ at P_{ϑ_0}. Note that $\partial_\mu f(P_{\vartheta_0},y)$ is only $P_{\vartheta_0}(dy)$-a.e. uniquely determined.

However, in our approach we have to consider functions $f:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ which are differentiable in all elements of $\mathcal{P}_2(\mathbb{R}^d)$. In order to simplify the argument, we suppose that $\widetilde{f}:L^2(\mathcal{F};\mathbb{R}^d)\to\mathbb{R}$ is Fréchet differential over the whole space $L^2(\mathcal{F};\mathbb{R}^d)$. This corresponds to a large class of important examples. In this case, we have the derivative $\partial_\mu f(P_\vartheta,y)$, defined $P_\vartheta(dy)$-a.e., for all $\vartheta\in L^2(\mathcal{F};\mathbb{R}^d)$. In Lemma 3.2 [9], it is shown that, if, furthermore, the Fréchet derivative $D\widetilde{f}:L^2(\mathcal{F};\mathbb{R}^d)\to L(L^2(\mathcal{F};\mathbb{R}^d),\mathbb{R})$ is Lipschitz continuous (with a Lipschitz constant

$K \in \mathbb{R}_+$), then there is for all $\vartheta \in L^2(\mathcal{F};\mathbb{R}^d)$ a P_ϑ-version of $\partial_\mu f(P_\vartheta,\cdot):\mathbb{R}^d \to \mathbb{R}^d$ such that

$$\left|\partial_\mu f(P_\vartheta, y) - \partial_\mu f\left(P_\vartheta, y'\right)\right| \le K\left|y - y'\right|, \qquad \text{for all } y, y' \in \mathbb{R}^d.$$

This motivates us to make the following definition.

DEFINITION 2.1. We say that $f \in C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$ [continuously differentiable over $\mathcal{P}_2(\mathbb{R}^d)$ with Lipschitz-continuous bounded derivative], if there exists for all $\vartheta \in L^2(\mathcal{F};\mathbb{R}^d)$ a P_ϑ-modification of $\partial_\mu f(P_\vartheta,\cdot)$, again denoted by $\partial_\mu f(P_\vartheta,\cdot)$, such that $\partial_\mu f:\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipschitz continuous, that is, there is some real constant C such that:

(i) $|\partial_\mu f(\mu, x)| \le C$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$;

(ii) $|\partial_\mu f(\mu, x) - \partial_\mu f(\mu', x')| \le C(W_2(\mu,\mu') + |x - x'|)$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, x, $x' \in \mathbb{R}^d$;

we consider this function $\partial_\mu f$ as the derivative of f.

REMARK 2.1. Let us point out that, if $f \in C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$, the version of $\partial_\mu f(P_\vartheta,\cdot)$, $\vartheta \in L^2(\mathcal{F};\mathbb{R}^d)$, indicated in Definition 2.1 is unique. Indeed, given $\vartheta \in L^2(\mathcal{F};\mathbb{R}^d)$, let θ be a d-dimensional vector of independent standard normally distributed random variables, which are independent of ϑ. Then, since $\partial_\mu f(P_{\vartheta+\varepsilon\theta}, \vartheta + \varepsilon\theta)$ is only P-a.s. defined, $\partial_\mu f(P_{\vartheta+\varepsilon\theta}, y)$ is determined only $P_{\vartheta+\varepsilon\theta}(dy)$-a.s. Observing that the random variable $\vartheta+\varepsilon\theta$ possesses a strictly positive density on $\mathbb{R}^d$, it follows that $P_{\vartheta+\varepsilon\theta}$ and the Lebesgue measure over $\mathbb{R}^d$ are equivalent. Consequently, $\partial_\mu f(P_{\vartheta+\varepsilon\theta}, y)$ is defined dy-a.e. on $\mathbb{R}^d$. From the Lipschitz continuity (ii) of $\partial_\mu f$ in Definition 2.1, it then follows that $\partial_\mu f(P_{\vartheta+\varepsilon\theta}, y)$ is defined for all $y \in \mathbb{R}^d$, and taking the limit $0 < \varepsilon \downarrow 0$ yields that $\partial_\mu f(P_\vartheta, y)$ is uniquely defined for all $y \in \mathbb{R}^d$.

Given now $f \in C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$, for fixed $y \in \mathbb{R}^d$ the question of the differentiability of its components $(\partial_\mu f)_j(\cdot, y):\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, $1 \le j \le d$, raises, and it can be discussed in the same way as the first-order derivative $\partial_\mu f$ above. If $(\partial_\mu f)_j(\cdot, y):\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ belongs to $C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$, we have that its derivative $\partial_\mu((\partial_\mu f)_j(\cdot, y))(\cdot,\cdot):\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz-continuous function, for every $y \in \mathbb{R}^d$. Then

$$\partial_\mu^2 f(\mu, x, y) := \left(\partial_\mu\left((\partial_\mu f)_j(\cdot, y)\right)(\mu, x)\right)_{1\le j\le d}, \qquad (\mu, x, y) \in \mathcal{P}_2\left(\mathbb{R}^d\right)\times\mathbb{R}^d\times\mathbb{R}^d, \tag{2.4}$$

defines a function $\partial_\mu^2 f:\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d\otimes\mathbb{R}^d$.

DEFINITION 2.2. We say that $f \in C_b^{2,1}(\mathcal{P}_2(\mathbb{R}^d))$, if $f \in C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$ and:

(i) $(\partial_\mu f)_j(\cdot, y) \in C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$, for all $y \in \mathbb{R}^d, 1 \le j \le d$, and $\partial_\mu^2 f : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is bounded and Lipschitz-continuous;

(ii) $\partial_\mu f(\mu, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is differentiable, for every $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, and its derivative $\partial_y \partial_\mu f : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is bounded and Lipschitz-continuous.

Adopting the above introduced notation, we consider a function $f \in C_b^{2,1}(\mathcal{P}_2(\mathbb{R}^d))$ and discuss its second-order Taylor expansion. For this end, we have still to introduce some notation. Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ be a copy of the probability space $(\Omega, \mathcal{F}, P)$. For any random variable (of arbitrary dimension) ϑ over $(\Omega, \mathcal{F}, P)$, we denote by $\widetilde{\vartheta}$ a copy (of the same law as ϑ, but defined over $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$: $\widetilde{P}_{\widetilde{\vartheta}} = P_\vartheta$. The expectation $\widetilde{E}[\cdot] = \int_{\widetilde{\Omega}} (\cdot)\, d\widetilde{P}$ acts only over the variables endowed with a tilde. This can be made rigorous by working with the product space $(\Omega, \mathcal{F}, P) \otimes (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}) = (\Omega, \mathcal{F}, P) \otimes (\Omega, \mathcal{F}, P)$ and putting $\widetilde{\vartheta}(\widetilde{\omega}, \omega) := \vartheta(\widetilde{\omega})$, $(\widetilde{\omega}, \omega) \in \widetilde{\Omega} \times \Omega = \Omega \times \Omega$, for ϑ random variable defined over $(\Omega, \mathcal{F}, P)$). Of course, this formalism can be easily extended from random variables to stochastic processes.

With the above notation and writing $a \otimes b := (a_i b_j)_{1 \le i, j \le d}$, for $a = (a_i)_{1 \le i \le d}$, $b = (b_j)_{1 \le j \le d} \in \mathbb{R}^d$, we can state now the following result.

LEMMA 2.1. *Let $f \in C_b^{2,1}(\mathcal{P}_2(\mathbb{R}^d))$. Then, for any given $\vartheta_0 \in L^2(\mathcal{F}; \mathbb{R}^d)$ we have the following second-order expansion*:

$$\begin{aligned} & f(P_\vartheta) - f(P_{\vartheta_0}) \\ &\quad = E\big[\partial_\mu f(P_{\vartheta_0}, \vartheta_0) \cdot \eta\big] + \tfrac{1}{2} E\big[\widetilde{E}\big[\operatorname{tr}\big(\partial_\mu^2 f(P_{\vartheta_0}, \widetilde{\vartheta_0}, \vartheta_0) \cdot \widetilde{\eta} \otimes \eta\big)\big]\big] \\ &\qquad + \tfrac{1}{2} E\big[\operatorname{tr}\big(\partial_y \partial_\mu f(P_{\vartheta_0}, \vartheta_0) \cdot \eta \otimes \eta\big)\big] + R(P_\vartheta, P_{\vartheta_0}), \\ &\qquad\qquad \vartheta \in L^2\big(\mathcal{F}; \mathbb{R}^d\big), \end{aligned} \tag{2.5}$$

where $\eta := \vartheta - \vartheta_0$, and for all $\vartheta \in L^2(\mathcal{F}; \mathbb{R}^d)$ the remainder $R(P_\vartheta, P_{\vartheta_0})$ satisfies the estimate

$$\begin{aligned} \big|R(P_\vartheta, P_{\vartheta_0})\big| &\le C\big(\big(E\big[|\vartheta - \vartheta_0|^2\big]\big)^{3/2} + E\big[|\vartheta - \vartheta_0|^3\big]\big) \\ &\le C E\big[|\vartheta - \vartheta_0|^3\big]. \end{aligned} \tag{2.6}$$

The constant $C \in \mathbb{R}_+$ only depends on the Lipschitz constants of $\partial_\mu^2 f$ and $\partial_y \partial_\mu f$.

We observe that the above second-order expansion does not constitute a second-order Taylor expansion for the associated function $\widetilde{f} : L^2(\mathcal{F}; \mathbb{R}^d) \to \mathbb{R}$, since the remainder term $E[|\vartheta - \vartheta_0|^3 \wedge |\vartheta - \vartheta_0|^2]$ is not of order $o(|\vartheta - \vartheta_0|_{L^2}^2)$. Indeed, as the following example shows, in general we only have $E[|\vartheta - \vartheta_0|^3 \wedge |\vartheta - \vartheta_0|^2] = O(|\vartheta - \vartheta_0|_{L^2}^2)$.

EXAMPLE 2.1. Let $\vartheta_\ell = I_{A_\ell}$, with $A_\ell \in \mathcal{F}$ such that $P(A_\ell) \to 0$, as $\ell \to \infty$. Then $\vartheta_\ell \to 0$ in $L^2(\ell \to \infty)$, and $E[|\vartheta_\ell|^3 \wedge |\vartheta_\ell|^2] = P(A_\ell) = E[\vartheta_\ell^2] = |\vartheta_\ell|_{L^2}^2 \to 0$ $(\ell \to \infty)$.

If we had in Lemma 2.1 a remainder $R(P_\vartheta, P_{\vartheta_0}) = o(|\vartheta - \vartheta_0|_{L^2}^2)$, this would suggest that $\widetilde{f}(\zeta) = f(P_\zeta)$ is twice Fréchet differentiable at ϑ_0. But however, as the following Example 2.3 shows, even in easiest cases $\widetilde{f}$ is not twice Fréchet differentiable.

However, for our purposes the above expansion is fine.

PROOF OF LEMMA 2.1. Let $\vartheta_0 \in L^2(\mathcal{F}; \mathbb{R}^d)$. Then, for all $\vartheta \in L^2(\mathcal{F}; \mathbb{R}^d)$, putting $\eta := \vartheta - \vartheta_0$ and using the fact that $f \in C_b^{2,1}(\mathcal{P}_2(\mathbb{R}^d))$, we have

$$
\begin{aligned}
f(P_\vartheta) - f(P_{\vartheta_0}) &= \int_0^1 \frac{d}{d\lambda} f(P_{\vartheta_0+\lambda\eta})\, d\lambda \\
&= \int_0^1 E\big[\partial_\mu f(P_{\vartheta_0+\lambda\eta}, \vartheta_0 + \lambda\eta) \cdot \eta\big]\, d\lambda.
\end{aligned} \tag{2.7}
$$

Let us now compute $\frac{d}{d\lambda}\partial_\mu f(P_{\vartheta_0+\lambda\eta}, \vartheta_0 + \lambda\eta)$. Since $f \in C_b^{2,1}(\mathcal{P}_2(\mathbb{R}^d))$, the lifted function $\widetilde{\partial_\mu f}(\xi, y) = \partial_\mu f(P_\xi, y)$, $\xi \in L^2(\mathcal{F}; \mathbb{R}^d)$, is Fréchet differentiable in ξ, and

$$
\begin{aligned}
\frac{d}{d\lambda}\partial_\mu f(P_{\vartheta_0+\lambda\eta}, y) &= \frac{d}{d\lambda}\widetilde{\partial_\mu f}(\vartheta_0 + \lambda\eta, y) \\
&= E\big[\partial_\mu^2 f(P_{\vartheta_0+\lambda\eta}, \vartheta_0 + \lambda\eta, y) \cdot \eta\big],
\end{aligned} \tag{2.8}
$$

$\lambda \in \mathbb{R}$, $y \in \mathbb{R}^d$. Then, choosing an independent copy $(\widetilde{\vartheta}, \widetilde{\vartheta_0})$ of (ϑ, ϑ_0) defined over $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ (i.e., in particular, $\widetilde{P}_{(\widetilde{\vartheta}, \widetilde{\vartheta_0})} = P_{(\vartheta, \vartheta_0)}$), we have for $\widetilde{\eta} := \widetilde{\vartheta} - \widetilde{\vartheta_0}$,

$$
\begin{aligned}
\frac{d}{d\lambda}\partial_\mu f(P_{\vartheta_0+\lambda\eta}, y) &= \widetilde{E}\big[\partial_\mu^2 f(P_{\vartheta_0+\lambda\eta}, \widetilde{\vartheta_0} + \lambda\widetilde{\eta}, y) \cdot \widetilde{\eta}\big], \\
&\qquad \lambda \in \mathbb{R}, y \in \mathbb{R}^d.
\end{aligned} \tag{2.9}
$$

Consequently,

$$
\begin{aligned}
\frac{d}{d\lambda}\partial_\mu f(P_{\vartheta_0+\lambda\eta}, \vartheta_0 + \lambda\eta) &= \widetilde{E}\big[\partial_\mu^2 f(P_{\vartheta_0+\lambda\eta}, \widetilde{\vartheta_0} + \lambda\widetilde{\eta}, \vartheta_0 + \lambda\eta) \cdot \widetilde{\eta}\big] \\
&\quad + \partial_y \partial_\mu f(P_{\vartheta_0+\lambda\eta}, \vartheta_0 + \lambda\eta) \cdot \eta.
\end{aligned} \tag{2.10}
$$

Thus,

$$
\begin{aligned}
& f(P_\vartheta) - f(P_{\vartheta_0}) \\
& \quad = \int_0^1 E\big[\partial_\mu f(P_{\vartheta_0+\lambda\eta}, \vartheta_0 + \lambda\eta) \cdot \eta\big]\, d\lambda
\end{aligned}
$$

$$
\begin{aligned}
&= E\big[\partial_\mu f(P_{\vartheta_0}, \vartheta_0)\cdot\eta\big] \\
&\quad + \int_0^1\int_0^\lambda E\bigg[\frac{d}{d\rho}\partial_\mu f(P_{\vartheta_0+\rho\eta}, \vartheta_0+\rho\eta)\cdot\eta\bigg]\,d\rho\,d\lambda \qquad (2.11)\\
&= E\big[\partial_\mu f(P_{\vartheta_0}, \vartheta_0)\cdot\eta\big] \\
&\quad + \int_0^1\int_0^\lambda E\big[\mathrm{tr}\big(\partial_y\partial_\mu f(P_{\vartheta_0+\rho\eta}, \vartheta_0+\rho\eta)\cdot\eta\otimes\eta\big)\big]\,d\rho\,d\lambda \\
&\quad + \int_0^1\int_0^\lambda E\big[\widetilde{E}\big[\mathrm{tr}\big(\partial_\mu^2 f(P_{\vartheta_0+\rho\eta}, \widetilde{\vartheta_0}+\rho\widetilde{\eta}, \vartheta_0+\rho\eta)\cdot\widetilde{\eta}\otimes\eta\big)\big]\big]\,d\rho\,d\lambda.
\end{aligned}
$$

From this latter relation, we get

$$
\begin{aligned}
&f(P_\vartheta) - f(P_{\vartheta_0}) \\
&\quad = E\big[\partial_\mu f(P_{\vartheta_0}, \vartheta_0)\cdot\eta\big] + \tfrac{1}{2}E\big[\widetilde{E}\big[\mathrm{tr}\big(\partial_\mu^2 f(P_{\vartheta_0}, \widetilde{\vartheta_0}, \vartheta_0)\cdot\widetilde{\eta}\otimes\eta\big)\big]\big] \qquad (2.12)\\
&\qquad + \tfrac{1}{2}E\big[\mathrm{tr}\big(\partial_y\partial_\mu f(P_{\vartheta_0}, \vartheta_0)\cdot\eta\otimes\eta\big)\big] + R_1(P_\vartheta, P_{\vartheta_0}) + R_2(P_\vartheta, P_{\vartheta_0}),
\end{aligned}
$$

with the remainders

$$
\begin{aligned}
R_1(P_\vartheta, P_{\vartheta_0}) = \int_0^1\int_0^\lambda & E\big[\widetilde{E}\big[\mathrm{tr}\big(\big(\partial_\mu^2 f(P_{\vartheta_0+\rho\eta}, \widetilde{\vartheta_0}+\rho\widetilde{\eta}, \vartheta_0+\rho\eta) \\
&- \partial_\mu^2 f(P_{\vartheta_0}, \widetilde{\vartheta_0}, \vartheta_0)\big)\cdot\widetilde{\eta}\otimes\eta\big)\big]\big]\,d\rho\,d\lambda
\end{aligned} \tag{2.13}
$$

and

$$
\begin{aligned}
R_2(P_\vartheta, P_{\vartheta_0}) = \int_0^1\int_0^\lambda & E\big[\mathrm{tr}\big(\big(\partial_y\partial_\mu f(P_{\vartheta_0+\rho\eta}, \vartheta_0+\rho\eta) \\
&- \partial_y\partial_\mu f(P_{\vartheta_0}, \vartheta_0)\big)\cdot\eta\otimes\eta\big)\big]\,d\rho\,d\lambda.
\end{aligned} \tag{2.14}
$$

Finally, from the boundedness and the Lipschitz continuity of the functions $\partial_\mu^2 f$ and $\partial_y\partial_\mu f$ we conclude that, for some $C \in \mathbb{R}_+$ only depending on $\partial_\mu^2 f$, $\partial_y\partial_\mu f$, $|R_1(P_\vartheta, P_{\vartheta_0})| \le C(E[|\eta|^2])^{3/2}$, while $|R_2(P_\vartheta, P_{\vartheta_0})| \le C(E|\eta|^2)^{3/2} + CE|\eta|^3$. This proves the statement. $\square$

Let us finish our preliminary discussion with two illustrating examples.

EXAMPLE 2.2. Given two twice continuously differentiable functions $h : \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ with bounded derivatives, we consider $f(P_\vartheta) := g(E[h(\vartheta)])$, $\vartheta \in L^2(\mathcal{F}; \mathbb{R}^d)$. Then, given any $\vartheta_0 \in L^2(\mathcal{F}; \mathbb{R}^d)$, $\widetilde{f}(\vartheta) := f(P_\vartheta) = g(E[h(\vartheta)])$ is Fréchet differentiable in ϑ_0, and

$$
\begin{aligned}
\widetilde{f}(\vartheta_0+\eta) - \widetilde{f}(\vartheta_0) &= \int_0^1 g'\big(E\big[h(\vartheta_0+s\eta)\big]\big)E\big[h'(\vartheta_0+s\eta)\eta\big]\,ds \\
&= g'\big(E\big[h(\vartheta_0)\big]\big)E\big[h'(\vartheta_0)\eta\big] + o\big(|\eta|_{L^2}\big) \\
&= E\big[g'\big(E\big[h(\vartheta_0)\big]\big)h'(\vartheta_0)\eta\big] + o\big(|\eta|_{L^2}\big).
\end{aligned}
$$

Thus, $D\widetilde{f}(\vartheta_0)(\eta) = E[g'(E[h(\vartheta_0)])\partial_y h(\vartheta_0)\eta]$, $\eta \in L^2(\mathcal{F}; \mathbb{R}^d)$, that is,

$$\partial_\mu f(P_{\vartheta_0}, y) = g'\big(E\big[h(\vartheta_0)\big]\big)(\partial_y h)(y), \qquad y \in \mathbb{R}^d.$$

With the same argument, we see that

$$\partial_\mu^2 f(P_{\vartheta_0}, x, y) = g''\big(E\big[h(\vartheta_0)\big]\big)(\partial_x h)(x) \otimes (\partial_y h)(y)$$

and

$$\partial_y \partial_\mu f(P_{\vartheta_0}, y) = g'\big(E\big[h(\vartheta_0)\big]\big)\big(\partial_y^2 h\big)(y).$$

Consequently, if g and h are three times continuously differentiable with bounded derivatives of all order, then the second-order expansion stated in the above Lemma 2.1 takes for this example the special form

$$\begin{aligned}
&g\big(E\big[h(\vartheta)\big]\big) - g\big(E\big[h(\vartheta_0)\big]\big) \\
&\quad = g'\big(E\big[h(\vartheta_0)\big]\big)E\big[\partial_y h(\vartheta_0) \cdot (\vartheta - \vartheta_0)\big] \\
&\qquad + \tfrac{1}{2} g''\big(E\big[h(\vartheta_0)\big]\big)\big(E\big[\partial_y h(\vartheta_0) \cdot (\vartheta - \vartheta_0)\big]\big)^2 \\
&\qquad + \tfrac{1}{2} g'\big(E\big[h(\vartheta_0)\big]\big)E\big[\mathrm{tr}\big(\partial_y^2 h(\vartheta_0) \cdot (\vartheta - \vartheta_0) \otimes (\vartheta - \vartheta_0)\big)\big] \\
&\qquad + O\big(\big(E\big[|\vartheta - \vartheta_0|^2\big]\big)^{3/2} + E\big[|\vartheta - \vartheta_0|^3\big]\big).
\end{aligned}$$

EXAMPLE 2.3. Let us consider a special case of Example 2.2. For $d = 1$, we choose $g(x) = x$, $x \in \mathbb{R}$, and $h(x) = e^{ix}$, $x \in \mathbb{R}$. Then $\widetilde{f}(\vartheta) = f(P_\vartheta) = \varphi_\vartheta(1) = E[e^{i\vartheta}]$ is just the characteristic function of ϑ at 1. Let $\vartheta_0 \in L^2(\mathcal{F}; \mathbb{R})$ be arbitrary and $A \in \mathcal{F}$ independent of ϑ_0. We put $\eta = I_A$ and $\vartheta = \vartheta_0 + \eta$. Obviously, the first-order Fréchet derivative of $\widetilde{f}$ at ϑ_0 is $D_\vartheta \widetilde{f}(\vartheta_0)(\eta) = iE[e^{i\vartheta_0}\eta]$, and if $\widetilde{f}$ were twice Fréchet differentiable we would have

$$D_\vartheta^2 \widetilde{f}(\vartheta_0)(\eta, \eta) = D_\vartheta\big[D_\vartheta \widetilde{f}(\vartheta)\big](\eta)|_{\vartheta = \vartheta_0} = -E\big[e^{i\vartheta_0}\big]\eta^2.$$

Then

$$\begin{aligned}
R(P_\vartheta, P_{\vartheta_0}) &= \widetilde{f}(\vartheta) - \big[\widetilde{f}(\vartheta_0) + D_\vartheta \widetilde{f}(\vartheta_0)(\eta) + \tfrac{1}{2} D_\vartheta^2 f(\vartheta_0)(\eta, \eta)\big] \\
&= E\big[e^{i\vartheta_0}\big(e^{i\eta} - \big[1 + i\eta - \tfrac{1}{2}\eta^2\big]\big)\big] = E\big[e^{i\vartheta_0}\big(e^i - \big(\tfrac{1}{2} + i\big)\big)I_A\big] \\
&= \big(e^i - \big(\tfrac{1}{2} + i\big)\big)\varphi_{\vartheta_0}(1)P(A) = \big(e^i - \big(\tfrac{1}{2} + i\big)\big)\varphi_{\vartheta_0}(1)|\eta|_{L^2}^2,
\end{aligned}$$

as $|\eta|_{L^2} = P(A)^{1/2} \to 0$. But this means that $\widetilde{f}$ is not twice Fréchet differentiable in ϑ_0, and taking into account the arbitrariness of $\vartheta_0 \in L^2(\mathcal{F}; \mathbb{R})$, we see that $\widetilde{f}$ is nowhere twice Fréchet differentiable.

3. The mean-field stochastic differential equation. Let us now consider a complete probability space $(\Omega, \mathcal{F}, P)$ on which is defined a d-dimensional Brownian motion $B(= (B^1, \dots, B^d)) = (B_t)_{t\in[0,T]}$, and $T > 0$ denotes an arbitrarily fixed time horizon. We suppose that there is a sub-σ-field $\mathcal{F}_0 \subset \mathcal{F}$ such that:

(i) the Brownian motion B is independent of $\mathcal{F}_0$, and
(ii) $\mathcal{F}_0$ is "rich enough," that is, $\mathcal{P}_2(\mathbb{R}^k) = \{P_\vartheta, \vartheta \in L^2(\mathcal{F}_0; \mathbb{R}^k)\}$, $k \ge 1$.

By $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ we denote the filtration generated by B, completed and augmented by $\mathcal{F}_0$.

Given deterministic Lipschitz functions $\sigma : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ and $b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, we consider for the initial data $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)$ the stochastic differential equations (SDEs)

$$X_s^{t,\xi} = \xi + \int_t^s \sigma\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}\big)\, dB_r + \int_t^s b\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}\big)\, dr, \qquad s \in [t, T], \tag{3.1}$$

and

$$X_s^{t,x,\xi} = x + \int_t^s \sigma\big(X_r^{t,x,\xi}, P_{X_r^{t,\xi}}\big)\, dB_r + \int_t^s b\big(X_r^{t,x,\xi}, P_{X_r^{t,\xi}}\big)\, dr, \qquad s \in [t, T]. \tag{3.2}$$

We observe that under our Lipschitz assumption on the coefficients the both SDEs have a unique solution in $\mathcal{S}^2([t, T]; \mathbb{R}^d)$, the space of $\mathbb{F}$-adapted continuous processes $Y = (Y_s)_{s\in[t,T]}$ with the property $E[\sup_{s\in[t,T]} |Y_s|^2] < +\infty$ (see, e.g., Carmona and Delarue [9]). We see, in particular, that the solution $X^{t,\xi}$ of the first equation allows to determine that of the second equation. As SDE standard estimates show, we have for some $C \in \mathbb{R}_+$ depending only on the Lipschitz constants of σ and b,

$$E\Big[\sup_{s\in[t,T]} \big|X_s^{t,x,\xi} - X_s^{t,x',\xi}\big|^2\Big] \le C\big|x - x'\big|^2, \tag{3.3}$$

for all $t \in [0, T]$, $x, x' \in \mathbb{R}^d$, $\xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)$. This allows to substitute in the second SDE for x the random variable ξ and shows that $X^{t,x,\xi}|_{x=\xi}$ solves the same SDE as $X^{t,\xi}$. From the uniqueness of the solution, we conclude

$$X_s^{t,x,\xi}|_{x=\xi} = X_s^{t,\xi}, \qquad s \in [t, T]. \tag{3.4}$$

Moreover, from the uniqueness of the solution of the both equations we deduce the following *flow property*:

$$\big(X_r^{s,X_s^{t,x,\xi},X_s^{t,\xi}}, X_r^{s,X_s^{t,\xi}}\big) = \big(X_r^{t,x,\xi}, X_r^{t,\xi}\big), \qquad r \in [s, T], \tag{3.5}$$

for all $0 \le t \le s \le T, x \in \mathbb{R}^d, \xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$. In fact, putting $\eta = X_s^{t,\xi} \in L^2(\mathcal{F}_s;\mathbb{R}^d)$, and considering the SDEs (3.1) and (3.2) with the initial data (s,y) and (s,η), respectively,

$$X_r^{s,\eta} = \eta + \int_s^r \sigma\big(X_u^{s,\eta}, P_{X_u^{s,\eta}}\big)\,dB_u + \int_s^r b\big(X_u^{s,\eta}, P_{X_u^{s,\eta}}\big)\,du, \qquad r \in [s,T],$$

and

$$X_r^{s,y,\eta} = y + \int_s^r \sigma\big(X_u^{s,y,\eta}, P_{X_u^{s,\eta}}\big)\,dB_u + \int_s^r b\big(X_u^{s,y,\eta}, P_{X_u^{s,\eta}}\big)\,du, \qquad r \in [s,T],$$

we get from the uniqueness of the solution that $X_r^{s,\eta} = X_r^{t,\xi}$, $r \in [s,T]$, and, consequently, $X_r^{s,X_s^{t,x,\xi},\eta} = X_r^{t,x,\xi}$, $r \in [t,T]$, that is, we have (3.5).

Having this flow property, it is natural to define for a sufficiently regular function $\Phi : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ an associated value function

$$V(t,x,\xi) := E\big[\Phi\big(X_T^{t,x,\xi}, P_{X_T^{t,\xi}}\big)\big], \qquad (t,x) \in [0,T] \times \mathbb{R}^d, \xi \in L^2\big(\mathcal{F}_t;\mathbb{R}^d\big), \tag{3.6}$$

and to ask which PDE is satisfied by this function V. In order to be able to answer this question in the frame of the concept, we have introduced above, we have to show that the function $V(t,x,\xi)$ does not depend on ξ itself but only on its law P_ξ, that is, that we have to do with a function $V : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$. For this, the following lemma is useful.

LEMMA 3.1. *For all $p \ge 2$, there is a constant $C_p \in \mathbb{R}_+$ only depending on the Lipschitz constants of σ and b, such that we have the following estimate*:

$$E\Big[\sup_{s\in[t,T]} \big|X_s^{t,x_1,\xi_1} - X_s^{t,x_2,\xi_2}\big|^p\Big] \le C_p\big(|x_1 - x_2|^p + W_2(P_{\xi_1}, P_{\xi_2})^p\big), \tag{3.7}$$

for all $t \in [0,T], x_1, x_2 \in \mathbb{R}^d, \xi_1, \xi_2 \in L^2(\mathcal{F}_t;\mathbb{R}^d)$.

PROOF. Recall that for the 2-Wasserstein metric $W_2(\cdot,\cdot)$ we have

$$\begin{aligned} W_2(P_\vartheta, P_\theta) &= \inf\big\{\big(E\big[|\vartheta' - \theta'|^2\big]\big)^{1/2}, \text{ for all } \vartheta', \theta' \in L^2\big(\mathcal{F}_0;\mathbb{R}^d\big) \\ &\qquad \text{with } P_{\vartheta'} = P_\vartheta, P_{\theta'} = P_\theta\big\} \\ &\le \big(E\big[|\vartheta - \theta|^2\big]\big)^{1/2} \qquad \text{for all } \vartheta, \theta \in L^2\big(\mathcal{F};\mathbb{R}^d\big), \end{aligned} \tag{3.8}$$

because we have chosen $\mathcal{F}_0$ "rich enough". Since our coefficients σ and b are Lipschitz over $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, this allows to get with the help of standard estimates for the SDEs for $X^{t,\xi}$ and $X^{t,x,\xi}$ that, for some constant $C \in \mathbb{R}_+$ only depending on the Lipschitz constants of σ and b,

$$E\Big[\sup_{s\in[t,T]} \big|X_s^{t,\xi_1} - X_s^{t,\xi_2}\big|^2\Big] \le C E\big[|\xi_1 - \xi_2|^2\big], \qquad \xi_1, \xi_2 \in L^2\big(\mathcal{F}_t;\mathbb{R}^d\big), t \in [0,T], \tag{3.9}$$

and, for some $C_p \in \mathbb{R}$ depending only on p and the Lipschitz constants of the coefficients,

$$
\begin{aligned}
&E\Big[\sup_{s\in[t,v]}\big|X_s^{t,x_1,\xi_1}-X_s^{t,x_2,\xi_2}\big|^p\Big] \\
(3.10)\qquad &\quad \le C_p\biggl(|x_1-x_2|^p+\int_t^v W_2(P_{X_r^{t,\xi_1}},P_{X_r^{t,\xi_2}})^p\,dr\biggr),
\end{aligned}
$$

for all $x_1, x_2 \in \mathbb{R}^d$, $\xi_1, \xi_2 \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, $0 \le t \le v \le T$. On the other hand, from the SDE for $X^{t,x,\xi}$ we derive easily that $X^{t,\xi',\xi}(:= X^{t,x,\xi}|_{x=\xi'})$ obeys the same law as $X^{t,\xi}(= X^{t,x,\xi}|_{x=\xi})$, whenever $\xi, \xi' \in L^2(\mathcal{F}_t;\mathbb{R}^d)$ have the same law. This allows to deduce from the latter estimate, for $p = 2$,

$$
\begin{aligned}
&\sup_{s\in[t,v]} W_2(P_{X_s^{t,\xi_1}},P_{X_s^{t,\xi_2}})^2 \\
(3.11)\qquad &\quad \le \sup_{s\in[t,v]} E\big[\big|X_s^{t,\xi_1',\xi_1}-X_s^{t,\xi_2',\xi_2}\big|^2\big] \le E\Big[\sup_{s\in[t,v]}\big|X_s^{t,\xi_1',\xi_1}-X_s^{t,\xi_2',\xi_2}\big|^2\Big] \\
&\quad \le C\biggl(E\big[|\xi_1'-\xi_2'|^2\big]+\int_t^v W_2(P_{X_r^{t,\xi_1}},P_{X_r^{t,\xi_2}})^2\,dr\biggr), \qquad v\in[t,T],
\end{aligned}
$$

for all $\xi_1', \xi_2' \in L^2(\mathcal{F}_t;\mathbb{R}^d)$ with $P_{\xi_1'} = P_{\xi_1}$ and $P_{\xi_2'} = P_{\xi_2}$. Hence, taking at the right-hand side of (3.11) the infimum over all such $\xi_1', \xi_2' \in L^2(\mathcal{F}_t;\mathbb{R}^d)$ and considering the above characterization of the 2-Wasserstein metric, we get

$$
\begin{aligned}
&\sup_{s\in[t,v]} W_2(P_{X_s^{t,\xi_1}},P_{X_s^{t,\xi_2}})^2 \\
(3.12)\qquad &\quad \le C\biggl(W_2(P_{\xi_1},P_{\xi_2})^2+\int_t^v W_2(P_{X_r^{t,\xi_1}},P_{X_r^{t,\xi_2}})^2\,dr\biggr),
\end{aligned}
$$

$v \in [t, T]$. Then Gronwall's inequality implies

$$
\begin{aligned}
&\sup_{s\in[t,T]} W_2(P_{X_s^{t,\xi_1}},P_{X_s^{t,\xi_2}})^2 \le C W_2(P_{\xi_1},P_{\xi_2})^2, \\
(3.13)\qquad &\qquad\qquad t\in[0,T], \xi_1,\xi_2\in L^2(\mathcal{F}_t;\mathbb{R}^d),
\end{aligned}
$$

which allows to deduce from the estimate (3.10)

$$
(3.14)\qquad E\Big[\sup_{s\in[t,T]}\big|X_s^{t,x_1,\xi_1}-X_s^{t,x_2,\xi_2}\big|^p\Big] \le C_p\big(|x_1-x_2|^p+W_2(P_{\xi_1},P_{\xi_2})^p\big),
$$

for all $t \in [0, T]$, $x_1, x_2 \in \mathbb{R}^d$, $\xi_1, \xi_2 \in L^2(\mathcal{F}_t;\mathbb{R}^d)$. The proof is complete now. $\square$

REMARK 3.1. An immediate consequence of the above Lemma 3.1 is that, given $(t, x) \in [0, T] \times \mathbb{R}^d$, the processes X^{t,x,ξ_1} and X^{t,x,ξ_2} are indistinguishable, whenever the laws of $\xi_1 \in L^2(\mathcal{F}_t;\mathbb{R}^d)$ and $\xi_2 \in L^2(\mathcal{F}_t;\mathbb{R}^d)$ are the same. But this means that we can define

$$X^{t,x,P_\xi} := X^{t,x,\xi}, \qquad (t,x) \in [0,T] \times \mathbb{R}^d, \xi \in L^2(\mathcal{F}_t;\mathbb{R}^d), \tag{3.15}$$

and, extending the notation introduced in the preceding section for functions to random variables and processes, we shall consider the lifted process $\widetilde{X}_s^{t,x,\xi} = X_s^{t,x,P_\xi} = X_s^{t,x,\xi}$, $s \in [t,T]$, $(t,x) \in [0,T] \times \mathbb{R}^d$, $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$. However, we prefer to continue to write $X^{t,x,\xi}$ and reserve the notation $\widetilde{X}^{t,x,P_\xi}$ for an independent copy of X^{t,x,P_ξ}, which we will introduce later.

4. First-order derivatives of X^{t,x,P_ξ}. Having now defined by the above relation the process $X^{t,x,\mu}$ for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the question of its differentiability with respect to μ raises; it will be studied through the Fréchet differentiability of the mapping $L^2(\mathcal{F}_t;\mathbb{R}^d) \ni \xi \to X_s^{t,x,\xi} \in L^2(\mathcal{F}_s;\mathbb{R}^d)$, $s \in [t,T]$. For this we suppose the following.

Hypothesis (H.1). The couple of coefficients (σ, b) belongs to $C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d} \times \mathbb{R}^d)$, that is, the components $\sigma_{i,j}, b_j$, $1 \le i,j \le d$, have the following properties:

(i) $\sigma_{i,j}(x,\cdot), b_j(x,\cdot)$ belong to $C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$, for all $x \in \mathbb{R}^d$;

(ii) $\sigma_{i,j}(\cdot,\mu), b_j(\cdot,\mu)$ belong to $C_b^1(\mathbb{R}^d)$, for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$;

(iii) The derivatives $\partial_x\sigma_{i,j}, \partial_x b_j : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ and $\partial_\mu\sigma_{i,j}, \partial_\mu b_j : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ are bounded and Lipschitz continuous.

Before discussing the differentiability of $X^{t,x,\mu}$ with respect to the probability measure μ let us recall in a preparing step its L^2-differentiability with respect to x.

LEMMA 4.1. *Let $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$. Under our above Hypothesis* (H.1), *the process $X^{t,x,P_\xi} = (X_s^{t,x,P_\xi})_{s\in[t,T]}$ is L^2-differentiable with respect to x, for all $s \in [t,T]$. More precisely, there is a* (*unique*) *process $\partial_x X^{t,x,P_\xi} \in \mathcal{S}^2([t,T];\mathbb{R}^{d\times d})$ such that*

$$E\Big[\sup_{s\in[t,T]}\big|X_s^{t,x+h,P_\xi} - X_s^{t,x,P_\xi} - \partial_x X_s^{t,x,P_\xi} h\big|^2\Big] = o\big(|h|^2\big), \qquad \text{as } \mathbb{R}^d \ni h \to 0.$$

[*Recall that $o(|h|)$ stands for an expression which tends quicker to zero than h: $o(|h|)/|h| \to 0$, as $h \to 0$.*] *Moreover, $\partial_x X^{t,x,P_\xi} = (\partial_{x_i} X_{s,j}^{t,x,P_\xi})_{1\le i,j\le d} \in \mathcal{S}^2([t,T];\mathbb{R}^{d\times d})$ is the unique solution of the following SDE*:

$$\begin{aligned}\partial_{x_i} X_{s,j}^{t,x,P_\xi} &= \delta_{i,j} + \sum_{k=1}^d \int_t^s \partial_{x_k} b_j\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_{x_i} X_{r,k}^{t,x,P_\xi}\,dr \\ &\quad + \sum_{k,\ell=1}^d \int_t^s (\partial_{x_k}\sigma_{j,\ell})\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_{x_i} X_{r,k}^{t,x,P_\xi}\,dB_r^\ell, \qquad (4.1)\\ &\qquad s \in [t,T],\end{aligned}$$

$1 \le i, j \le d$ (*here*, $\delta_{i,j}$ *denotes the Kronecker symbol*: *it equals* 1, *if* $i = j$, *and is equal to zero*, *otherwise*). *Furthermore*, *we have the following estimates for the process* $\partial_x X^{t,x,P_\xi}$. *For every* $p \ge 2$, *there is some constant* $C_p \in \mathbb{R}$ *such that*, *for all* $t \in [0,T]$, $x, x' \in \mathbb{R}^d$ *and* $\xi, \xi' \in L^2(\mathcal{F}_t; \mathbb{R}^d)$:

$$
\begin{aligned}
&\text{(i)} \quad E\Big[\sup_{s\in[t,T]} |\partial_x X_s^{t,x,P_\xi}|^p\Big] \le C_p, \\
&\text{(ii)} \quad E\Big[\sup_{s\in[t,T]} |\partial_x X_s^{t,x,P_\xi} - \partial_x X_s^{t,x',P_{\xi'}}|^p\Big] \le C_p\big(|x-x'|^p + W_2(P_\xi, P_{\xi'})^p\big).
\end{aligned}
\tag{4.2}
$$

PROOF. Considering for fixed (t,ξ) the coefficients $\sigma(s,x) := \sigma(x, P_{X_s^{t,\xi}})$ and $b(s,x) := b(x, P_{X_s^{t,\xi}})$, and taking into account that these coefficients are Lipschitz in x, uniformly with respect to s, we get the L^2-differentiability of X^{t,x,P_ξ} and the SDE satisfied by its L^2-derivative directly from the corresponding classical result. The proof for estimates for the L^2-derivative combines standard SDE estimates with the argument developed in the proof for the estimates for X^{t,x,P_ξ}. □

REMARK 4.1. From the above lemma, we see that, for given $(t,\xi) \in [0,T] \times L^2(\mathcal{F}_t; \mathbb{R}^d)$, the process $\partial_x X_s^{t,\xi,P_\xi} := \partial_x X_s^{t,x,P_\xi}|_{x=\xi}$, $s \in [t,T]$ is the unique solution in $\mathcal{S}^2([t,T]; \mathbb{R}^{d\times d})$ of the SDE

$$
\begin{aligned}
\partial_x X_s^{t,\xi,P_\xi} &= I + \int_t^s (\partial_x \sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}\big)\partial_x X_r^{t,\xi,P_\xi}\, dB_r \\
&\quad + \int_t^s (\partial_x b)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}\big)\partial_x X_r^{t,\xi,P_\xi}\, dr, \qquad s \in [t,T].
\end{aligned}
\tag{4.3}
$$

Moreover, it satisfies

$$
E\Big[\sup_{s\in[t,T]} |\partial_x X_s^{t,\xi,P_\xi}|^p | \mathcal{F}_t\Big] \le \sup_{x\in\mathbb{R}} E\Big[\sup_{s\in[t,T]} |\partial_x X_s^{t,x,P_\xi}|^p\Big] \le C_p, \qquad P\text{-a.s.}, \tag{4.4}
$$

for some real constant C_p only depending on $p \ge 2$ and the bounds of $\partial_x \sigma$ and $\partial_x b$.

Before giving the main statement of this section concerning the Fréchet derivative of $L^2(\mathcal{F}_t; \mathbb{R}^d) \ni \xi \to X^{t,x,\xi} = X^{t,x,P_\xi}$, and thus of the differentiability of the process with respect to the probability law P_ξ, let us begin with a heuristic computation for the directional derivative. Subsequently, we will prove that it is indeed the directional derivative and defines a Gâteaux derivative which, on its part, is Lipschitz, and hence, a Fréchet derivative. We will make the computations for dimension $d = 1$; the case $d \ge 1$ can be obtained by an easy extension.

Let $(t,x) \in [0,T] \times \mathbb{R}$, $\xi \in L^2(\mathcal{F}_t) (:= L^2(\mathcal{F}_t; \mathbb{R}))$, and consider an arbitrary "direction" $\eta \in L^2(\mathcal{F}_t)$. Then, supposing in a first attempt that the directional derivative

$$
Y_s^{t,x,\xi}(\eta) = L^2 - \lim_{h\to 0} \frac{1}{h}\big(X_s^{t,x,\xi+h\eta} - X_s^{t,x,\xi}\big), \qquad s \in [t,T], \tag{4.5}
$$

exists, we consider the SDE

$$X_s^{t,x,\xi+h\eta} = x + \int_t^s \sigma\big(X_r^{t,x,P_{\xi+h\eta}}, P_{X_r^{t,\xi+h\eta}}\big)\,dB_r + \int_t^s b\big(X_r^{t,x,P_{\xi+h\eta}}, P_{X_r^{t,\xi+h\eta}}\big)\,dr, \tag{4.6}$$

$s \in [t, T]$, which, after lifting the process X^{t,x,P_ξ}, and the coefficients from the space $\mathbb{R} \times \mathcal{P}_2(\mathbb{R})$ to $\mathbb{R} \times L^2(\mathcal{F})$, takes the form

$$X_s^{t,x,\xi+h\eta} = x + \int_t^s \widetilde{\sigma}\big(X_r^{t,x,\xi+h\eta}, X_r^{t,\xi+h\eta}\big)\,dB_r + \int_t^s \widetilde{b}\big(X_r^{t,x,\xi+h\eta}, X_r^{t,\xi+h\eta}\big)\,dr, \tag{4.7}$$

$s \in [t, T]$, and we derive formally this equation with respect to h at $h = 0$. For this we denote the Fréchet derivatives of $\widetilde{\sigma}$ and $\widetilde{b}$ with respect to their second variable by D_ϑ and we note that, morally, the L^2-derivative of $X_s^{t,\xi+h\eta}$ is given by

$$\partial_h X_s^{t,\xi+h\eta}|_{h=0} = \partial_x X_s^{t,x,P_\xi}|_{x=\xi} \cdot \eta + Y_s^{t,x,\xi}(\eta)|_{x=\xi}. \tag{4.8}$$

Recalling that, for some real C independent of (t, x, P_ξ),

$$E\Big[\sup_{s\in[t,T]} \big|\partial_x X_s^{t,x,P\xi}\big|^2\Big] \le C,$$

we see that for $\partial_x X_s^{t,\xi,P_\xi} := \partial_x X_s^{t,x,P_\xi}|_{x=\xi}$,

$$E\Big[\sup_{s\in[t,T]} \big|\partial_x X_s^{t,\xi,P_\xi}\big|^2 \big| \mathcal{F}_t\Big] \le C, \qquad P\text{-a.s.} \tag{4.9}$$

Using the notation,

$$Y_s^{t,\xi}(\eta) := Y_s^{t,x,\xi}(\eta)|_{x=\xi}, \tag{4.10}$$

we get by formal differentiation of the above equation

$$\begin{aligned} Y_s^{t,x,\xi}(\eta) = {} & \int_t^s (\partial_x \widetilde{\sigma})\big(X_r^{t,x,\xi}, X_r^{t,\xi}\big) Y_s^{t,x,\xi}(\eta)\,dB_r \\ & + \int_t^s (\partial_x \widetilde{b})\big(X_r^{t,x,\xi}, X_r^{t,\xi}\big) Y_s^{t,x,\xi}(\eta)\,dr \\ & + \int_t^s (D_\vartheta \widetilde{\sigma})\big(X_r^{t,x,\xi}, X_r^{t,\xi}\big)\big(\partial_x X_s^{t,\xi,P_\xi}\eta + Y_s^{t,\xi}(\eta)\big)\,dB_r \\ & + \int_t^s (D_\vartheta \widetilde{b})\big(X_r^{t,x,\xi}, X_r^{t,\xi}\big)\big(\partial_x X_s^{t,\xi,P_\xi}\eta + Y_s^{t,\xi}(\eta)\big)\,dr, \qquad s \in [t, T]. \end{aligned} \tag{4.11}$$

As concerns the above term $(D_\vartheta \widetilde{\sigma})(X_r^{t,x,\xi}, X_r^{t,\xi})(\partial_x X_s^{t,\xi,P_\xi}\eta + Y_s^{t,\xi}(\eta))$, we see from the definition of the differentiability of σ with respect to the probability measure, that

$$
\begin{aligned}
(4.12)\qquad & (D_\vartheta\widetilde{\sigma})\big(y, X_r^{t,\xi}\big)\big(\partial_x X_s^{t,\xi,P_\xi}\eta + Y_s^{t,\xi}(\eta)\big) \\
&= E\big[(\partial_\mu\sigma)\big(y, P_{X_s^{t,\xi}}, X_s^{t,\xi}\big)\cdot\big(\partial_x X_s^{t,\xi,P_\xi}\eta + Y_s^{t,\xi}(\eta)\big)\big], \qquad y\in\mathbb{R}.
\end{aligned}
$$

Now, for given $\xi,\eta\in L^2(\mathcal{F}_t)$ we denote by $(\widetilde{\xi},\widetilde{\eta},\widetilde{B})$ a copy of (ξ,η,B) on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{P})$, while $\widetilde{X}^{t,\widetilde{\xi}}$ is the solution of the SDE for $X^{t,\xi}$, but driven by the Brownian motion $\widetilde{B}$ and with initial value $\widetilde{\xi}$. Moreover, $\widetilde{X}^{t,x,\widetilde{P}_{\widetilde{\xi}}}$ denotes the solution of the SDE for X^{t,x,P_ξ}, but governed by $\widetilde{B}$ instead of B:

$$
\begin{aligned}
\widetilde{X}_s^{t,\widetilde{\xi}} &= \widetilde{\xi} + \int_t^s \sigma\big(\widetilde{X}_r^{t,\widetilde{\xi}}, \widetilde{P}_{\widetilde{X}_r^{t,\widetilde{\xi}}}\big)\,d\widetilde{B}_r + \int_t^s b\big(\widetilde{X}_r^{t,\widetilde{\xi}}, \widetilde{P}_{\widetilde{X}_r^{t,\widetilde{\xi}}}\big)\,dr,\\
\widetilde{X}_s^{t,x,\widetilde{P}_{\widetilde{\xi}}} &= x + \int_t^s \sigma\big(\widetilde{X}_r^{t,x,\widetilde{P}_{\widetilde{\xi}}}, \widetilde{P}_{\widetilde{X}_r^{t,\widetilde{\xi}}}\big)\,d\widetilde{B}_r + \int_t^s b\big(\widetilde{X}_r^{t,x,\widetilde{P}_{\widetilde{\xi}}}, \widetilde{P}_{\widetilde{X}_r^{t,\widetilde{\xi}}}\big)\,dr, \qquad s\in[t,T].
\end{aligned}
$$

Obviously, $\widetilde{X}^{t,x,\widetilde{P}_{\widetilde{\xi}}} = \widetilde{X}^{t,x,P_\xi}$, $x\in\mathbb{R}$, and $(\widetilde{\xi},\widetilde{\eta},\widetilde{X}^{t,x,P_\xi},\widetilde{B})$ is an independent copy of $(\xi,\eta,X^{t,x,P_\xi},B)$, defined over $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{P})$. Then, due to the notation already introduced, $\partial_x\widetilde{X}_r^{t,x,P_\xi}$ is the $L^2(\widetilde{P})$-derivative of $\widetilde{X}_r^{t,x,P_\xi}$ with respect to x, $\partial_x\widetilde{X}_r^{t,\widetilde{\xi},P_\xi} := \partial_x\widetilde{X}_r^{t,x,P_\xi}|_{x=\widetilde{\xi}}$, and

$$
(4.13)\qquad \widetilde{Y}_s^{t,x,\widetilde{\xi}}(\widetilde{\eta}) := L^2(\widetilde{P}) - \lim_{h\to 0}\frac{1}{h}\big(\widetilde{X}_s^{t,x,\widetilde{\xi}+h\widetilde{\eta}} - \widetilde{X}_s^{t,x,\widetilde{\xi}}\big), \qquad s\in[t,T].
$$

Having these notation in mind as well as the fact that the expectation $\widetilde{E}[\cdot]$ with respect to $\widetilde{P}$ applies only to variables endowed with a tilde, we see that

$$
\begin{aligned}
(4.14)\qquad & (D_\vartheta\widetilde{\sigma})\big(X_s^{t,x,\xi}, X_r^{t,\xi}\big)\cdot\big(\partial_x X_s^{t,\xi,P_\xi}\eta + Y_s^{t,\xi}(\eta)\big)\\
&= \widetilde{E}\big[(\partial_\mu\sigma)\big(X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}, \widetilde{X}_s^{t,\widetilde{\xi})}\big)\cdot\big(\partial_x\widetilde{X}_s^{t,\widetilde{\xi},P_\xi}\widetilde{\eta} + \widetilde{Y}_s^{t,\widetilde{\xi}}(\widetilde{\eta})\big)\big],\\
&\qquad s\in[t,T].
\end{aligned}
$$

Using also the corresponding formula for $(D_\vartheta\widetilde{b})(X_s^{t,x,\xi}, X_r^{t,\xi})(\partial_x X_s^{t,\xi,P_\xi}\eta + Y_s^{t,\xi}(\eta))$ and taking into account that $\partial_x\widetilde{\sigma}(x,\vartheta) = \partial_x\sigma(x,P_\vartheta)$ and $\partial_x\widetilde{b}(x,\vartheta) = \partial_x b(x,P_\vartheta)$, $(x,\vartheta)\in\mathbb{R}\times L^2(\mathcal{F})$, we obtain

$$
\begin{aligned}
(4.15)\qquad Y_s^{t,x,\xi}(\eta) &= \int_t^s (\partial_x\widetilde{\sigma})\big(X_r^{t,x,\xi}, X_r^{t,\xi}\big)Y_r^{t,x,\xi}(\eta)\,dB_r\\
&\quad + \int_t^s (\partial_x\widetilde{b})\big(X_r^{t,x,\xi}, X_r^{t,\xi}\big)Y_r^{t,x,\xi}(\eta)\,dr\\
&\quad + \int_t^s \widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\cdot\big(\partial_x\widetilde{X}_r^{t,\widetilde{\xi},P_\xi}\widetilde{\eta} + \widetilde{Y}_r^{t,\widetilde{\xi}}(\widetilde{\eta})\big)\big]\,dB_r\\
&\quad + \int_t^s \widetilde{E}\big[(\partial_\mu b)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\cdot\big(\partial_x\widetilde{X}_r^{t,\widetilde{\xi},P_\xi}\widetilde{\eta} + \widetilde{Y}_r^{t,\widetilde{\xi}}(\widetilde{\eta})\big)\big]\,dr,
\end{aligned}
$$

$s \in [t, T]$. Finally, recalling the definition of the process $Y^{t,\xi}(\eta)$, we see that $Y^{t,\xi}(\eta)$ solves the SDE

$$\begin{aligned} Y_s^{t,\xi}(\eta) &= \int_t^s (\partial_x \widetilde{\sigma})\big(X_r^{t,\xi}, X_r^{t,\xi}\big) Y_r^{t,\xi}(\eta)\, dB_r + \int_t^s (\partial_x \widetilde{b})\big(X_r^{t,\xi}, X_r^{t,\xi}\big) Y_r^{t,\xi}(\eta)\, dr \\ &\quad + \int_t^s \widetilde{E}\big[(\partial_\mu \sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \big(\partial_x \widetilde{X}_r^{t,\widetilde{\xi},P_\xi} \widetilde{\eta} + \widetilde{Y}_r^{t,\widetilde{\xi}}(\widetilde{\eta})\big)\big]\, dB_r \\ &\quad + \int_t^s \widetilde{E}\big[(\partial_\mu b)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \big(\partial_x \widetilde{X}_r^{t,\widetilde{\xi},P_\xi} \widetilde{\eta} + \widetilde{Y}_r^{t,\widetilde{\xi}}(\widetilde{\eta})\big)\big]\, dr, \end{aligned} \tag{4.16}$$

$s \in [t, T]$. We remark that, thanks to the boundedness of the first-order derivatives of the coefficients σ and b, the system formed of the both equations above has a unique solution $(Y^{t,x,\xi}(\eta), Y^{t,\xi}(\eta)) \in \mathcal{S}^2([t, T]; \mathbb{R}^2)$. Moreover, both processes are linear in $\eta \in L^2(\mathcal{F}_t)$, and a standard estimate shows that

$$E\Big[\sup_{s\in[t,T]} \big|Y_s^{t,x,\xi}(\eta)\big|^2\Big] \le C E\big[\eta^2\big], \qquad \eta \in L^2(\mathcal{F}_t), \tag{4.17}$$

for some constant C independent of (t, x, P_ξ). This shows, in particular, that $Y_s^{t,x,\xi}(\cdot)$ is a bounded linear operator from $L^2(\mathcal{F}_t)$ to $L^2(\mathcal{F}_s)$:

$$Y_s^{t,x,\xi}(\cdot) \in L\big(L^2(\mathcal{F}_t), L^2(\mathcal{F}_s)\big), \qquad s \in [t, T].$$

We are now able to show in a rigorous manner that our process $Y_s^{t,x,\xi}(\eta)$ is the directional derivative of $X_s^{t,x,\xi}$ in direction η.

LEMMA 4.2. *Under assumption* (H.1), *we have for all* $(t, x, P_\xi) \in [0, T] \times \mathbb{R} \times L^2(\mathcal{F}_t)$,

$$Y_s^{t,x,\xi}(\eta) = L^2 - \lim_{h\to 0} \frac{1}{h}\big(X_s^{t,x,\xi+h\eta} - X_s^{t,x,\xi}\big), \qquad s \in [t, T], \tag{4.18}$$

that is, $Y_s^{t,x,\xi}(\eta)$ *is the directional derivative of* $X_s^{t,x,\xi}$ *in direction* $\eta \in L^2(\mathcal{F}_t)$.

PROOF. The proof uses standard arguments. Let us sketch it, and without restricting the generality of the argument we suppose $b = 0$. First, using the continuous differentiability of σ, we see that

$$\begin{aligned} &\sigma\big(X_s^{t,x,\xi+h\eta}, P_{X_s^{t,\xi+h\eta}}\big) - \sigma\big(X_s^{t,x,\xi}, P_{X_s^{t,\xi}}\big) \\ &\quad = \int_0^1 \partial_\lambda\big(\sigma\big(X_s^{t,x,\xi} + \lambda\big(X_s^{t,x,\xi+h\eta} - X_s^{t,x,\xi}\big), P_{X_s^{t,\xi+h\eta}}\big)\big)\, d\lambda \\ &\quad\quad + \int_0^1 \partial_\lambda\big(\sigma\big(X_s^{t,x,\xi}, P_{X_s^{t,\xi}+\lambda(X_s^{t,\xi+h\eta}-X_s^{t,\xi})}\big)\big)\, d\lambda \\ &\quad = \alpha_s(x, h)\big(X_s^{t,x,\xi+h\eta} - X_s^{t,x,\xi}\big) + \widetilde{E}\big[\beta_s(x, h)\big(\widetilde{X}_s^{t,\widetilde{\xi}+h\widetilde{\eta}} - \widetilde{X}_s^{t,\widetilde{\xi}}\big)\big], \end{aligned} \tag{4.19}$$

where

$$\alpha_s(x,h) = \int_0^1 (\partial_x \sigma)\big(X_s^{t,x,\xi} + \lambda\big(X_s^{t,x,\xi+h\eta} - X_s^{t,x,\xi}\big), P_{X_s^{t,\xi+h\eta}}\big)\,d\lambda,$$

$$\begin{aligned}\beta_s(x,h) = \int_0^1 (\partial_\mu \sigma)\big(&X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}+\lambda(X_s^{t,\xi+h\eta}-X_s^{t,\xi})},\\ &\widetilde{X}_s^{t,\widetilde{\xi}} + \lambda\big(\widetilde{X}_s^{t,\widetilde{\xi}+h\widetilde{\eta}} - \widetilde{X}_s^{t,\widetilde{\xi}}\big)\big)\,d\lambda,\end{aligned}$$

$s \in [t,T]$, $x, h \in \mathbb{R}$, are adapted, uniformly bounded processes which are independent of $\mathcal{F}_t$. Indeed, while for $\alpha(x,h)$ the statement is obvious, concerning $\beta(x,h)$ we have, for $s \in [t,T]$:

$$\begin{aligned}&\partial_\lambda\big(\sigma\big(X_s^{t,x,\xi}, P_{X_s^{t,\xi}+\lambda(X_s^{t,\xi+h\eta}-X_s^{t,\xi})}\big)\big)\\ &\quad= (D_\vartheta \widetilde{\sigma})\big(X_s^{t,x,\xi}, X_s^{t,\xi} + \lambda\big(X_s^{t,\xi+h\eta} - X_s^{t,\xi}\big)\big)\big(X_s^{t,\xi+h\eta} - X_s^{t,\xi}\big)\\ &\quad= \widetilde{E}\big[(\partial_\mu\sigma)\big(X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}+\lambda(X_s^{t,\xi+h\eta}-X_s^{t,\xi})}, \widetilde{X}_s^{t,\widetilde{\xi}} + \lambda\big(\widetilde{X}_s^{t,\widetilde{\xi}+h\widetilde{\eta}} - \widetilde{X}_s^{t,\widetilde{\xi}}\big)\big)\\ &\qquad\times \big(\widetilde{X}_s^{t,\widetilde{\xi}+h\widetilde{\eta}} - \widetilde{X}_s^{t,\widetilde{\xi}}\big)\big].\end{aligned}\tag{4.20}$$

Moreover, using the Lipschitz continuity of $\partial_\mu\sigma$ we see that, for some $C \in \mathbb{R}$, only depending on the Lipschitz constant of $\partial_\mu\sigma$,

$$\begin{aligned}&\widetilde{E}\big[\big|\beta_s(x,h) - (\partial_\mu\sigma)\big(X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}, \widetilde{X}_s^{t,\widetilde{\xi}}\big)\big|^2\big]\\ &\quad\le C\int_0^1 \big(W_2(P_{X_s^{t,\xi}+\lambda(X_s^{t,\xi+h\eta}-X_s^{t,\xi})}, P_{X_s^{t,\xi}})^2 + \widetilde{E}\big[\big|\widetilde{X}_s^{t,\widetilde{\xi}+h\widetilde{\eta}} - \widetilde{X}_s^{t,\widetilde{\xi}}\big|^2\big]\big)\,d\lambda\\ &\quad\le CE\big[\big|X_s^{t,\xi+h\eta} - X_s^{t,\xi}\big|^2\big]\\ &\quad\le CE\big[E\big[\big|X_s^{t,y,P_{\xi+h\eta}} - X_s^{t,y' P_\xi}\big|^2\big]\big|_{y=\xi+h\eta, y'=\xi}\big]\\ &\quad\le CE\big[\big[|y-y'|^2 + W_2(P_{\xi+h\eta}, P_\xi)^2\big]\big|_{y=\xi+h\eta,y'=\xi}\big]\\ &\quad\le Ch^2E\big[\eta^2\big], \qquad s \in [t,T], x, h \in \mathbb{R}, \xi, \eta \in L^2(\mathcal{F}_t).\end{aligned}\tag{4.21}$$

Similarly, for $\partial_x\sigma$, from its Lipschitz continuity and our estimates for the process X^{t,x,P_ξ}, we have for all $p \ge 2$, there is some constant C_p such that

$$\begin{aligned}&E\big[\big|\alpha_s(x,h) - (\partial_x\sigma)\big(X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}\big)\big|^p\big]\\ &\quad\le C_p\big(E\big[\big|X_s^{t,x,P_{\xi+h\eta}} - X_s^{t,x,P_\xi}\big|^p\big] + W_2(P_{X_s^{t,\xi+h\eta}}, P_{X_s^{t,\xi}})^p\big)\\ &\quad\le C_pW_2(P_{\xi+h\eta}, P_\xi)^p + C|h|^pE\big[\eta^2\big]^{p/2}\\ &\quad\le C_p|h|^pE\big[\eta^2\big]^{p/2}, \qquad s \in [t,T], x, h \in \mathbb{R}, \xi, \eta \in L^2(\mathcal{F}_t).\end{aligned}\tag{4.22}$$

Recall that with the processes $\alpha(x,h)$ and $\beta(x,h)$, the difference between $X_s^{t,x,P_{\xi+h\eta}}$ and X_s^{t,x,P_ξ} writes, for all $s\in[t,T]$, as follows:

$$\begin{aligned} X_s^{t,x,P_{\xi+h\eta}}-X_s^{t,x,P_\xi}&=\int_t^s\alpha_r(x,h)\big(X_r^{t,x,P_{\xi+h\eta}}-X^{t,x,P_\xi}\big)\,dB_r\\ &\quad+\int_t^s\widetilde{E}\big[\beta_r(x,h)\big(\widetilde{X}_r^{t,\widetilde{\xi}+h\widetilde{\eta}}-\widetilde{X}_r^{t,\widetilde{\xi}}\big)\big]\,dB_r. \end{aligned}\tag{4.23}$$

Consequently, using the equation for $Y^{t,x,\xi}(\eta)$, we have

$$\begin{aligned} &X_s^{t,x,P_{\xi+h\eta}}-X_s^{t,x,P_\xi}-hY_s^{t,x,P_\xi}(\eta)\\ &\quad=\int_t^s(\partial_x\sigma)\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\big(X_r^{t,x,P_{\xi+h\eta}}-X_r^{t,x,P_\xi}-hY_r^{t,x,\xi}(\eta)\big)\,dB_r\\ &\qquad+\int_t^s\widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\widetilde{X}_r^{t,\widetilde{\xi}}\big)\\ &\qquad\times\big(\widetilde{X}_r^{t,\widetilde{\xi}+h\widetilde{\eta}}-\widetilde{X}_r^{t,\widetilde{\xi}}-h\big(\partial_x\widetilde{X}_r^{t,\widetilde{\xi},P_\xi}\widetilde{\eta}+\widetilde{Y}_r^{t,\widetilde{\xi}}(\widetilde{\eta})\big)\big)\big]\,dB_r\\ &\qquad+R_1(s,x,h), \end{aligned}\tag{4.24}$$

with

$$\begin{aligned} &R_1(s,x,h)\\ &\quad=\int_t^s\big(\alpha_r(x,h)-(\partial_x\sigma)\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\big)\cdot\big(X_r^{t,x,P_{\xi+h\eta}}-X_r^{t,x,P_\xi}\big)\,dB_r\\ &\qquad+\int_t^s\widetilde{E}\big[\big(\beta_r(x,h)-(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\widetilde{X}_r^{t,\widetilde{\xi}}\big)\big)\cdot\big(\widetilde{X}_r^{t,\widetilde{\xi}+h\widetilde{\eta}}-\widetilde{X}_r^{t,\widetilde{\xi}}\big)\big]\,dB_r. \end{aligned}$$

From Hölder's inequality, (4.22) and our estimates for X^{t,x,P_ξ} we obtain

$$\begin{aligned} &E\big[\big|\alpha_r(x,h)-(\partial_x\sigma)\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\big|^2\big|X_r^{t,x,P_{\xi+h\eta}}-X_r^{t,x,P_\xi}\big|^2\big]\\ &\qquad\le Ch^2E[\eta^2]W_2(P_{\xi+h\eta},P_\xi)^2\le Ch^4E[\eta^2]^2, \end{aligned}\tag{4.25}$$

and (4.21) yields

$$\begin{aligned} &E\big[\big|\widetilde{E}\big[\big(\beta_r(h,x)-(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\widetilde{X}_r^{t,\widetilde{\xi}}\big)\big)\cdot\big(\widetilde{X}_r^{t,\widetilde{\xi}+h\widetilde{\eta}}-\widetilde{X}_r^{t,\widetilde{\xi}}\big)\big]\big|^2\big]\\ &\quad\le E\big[\widetilde{E}\big[\big|\beta_r(h,x)-(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\widetilde{X}_r^{t,\widetilde{\xi}}\big)\big|^2\big]\\ &\qquad\times\widetilde{E}\big[\big|\widetilde{X}_r^{t,\widetilde{\xi}+h\widetilde{\eta}}-\widetilde{X}_r^{t,\widetilde{\xi}}\big|^2\big]\big]\\ &\quad\le Ch^4E[\eta^2]^2,\qquad r\in[t,T],h,x\in\mathbb{R},\xi,\eta\in L^2(\mathcal{F}_t). \end{aligned}\tag{4.26}$$

Consequently, the remainder $R_1(s,x,h)$ can be estimated as follows:

$$E\Big[\sup_{s\in[t,T]}\big|R_1(s,x,h)\big|^2\Big]\le Ch^4E[\eta^2]^2,\qquad h,x\in\mathbb{R},\xi,\eta\in L^2(\mathcal{F}_t),$$

and, using Gronwall's inequality, we obtain, for a suitable constant C not depending on (t, x, ξ, η), that for all $u \in [t, T]$,

$$\sup_{x\in\mathbb{R}} E\big[\sup_{s\in[t,u]} \big|X_s^{t,x,P_{\xi+h\eta}} - X_s^{t,x,P_\xi} - hY_s^{t,x,\xi}(\eta)\big|^2\big]$$

$$\begin{aligned}\text{(4.27)}\qquad &\le Ch^4 E[\eta^2]^2 \\ &\quad + C\int_t^u E\big[\widetilde{E}\big[\big|\widetilde{X}_r^{t,\widetilde{\xi}+h\widetilde{\eta}} - \widetilde{X}_r^{t,\widetilde{\xi}} - h\big(\partial_x \widetilde{X}_r^{t,\widetilde{\xi},P_\xi}\widetilde{\eta} + \widetilde{Y}_r^{t,\widetilde{\xi}}(\widetilde{\eta})\big)\big|\big]^2\big]\,dr.\end{aligned}$$

In order to conclude, let us now estimate

$$\begin{aligned}&\widetilde{E}\big[\big|\widetilde{X}_r^{t,\widetilde{\xi}+h\widetilde{\eta}} - \widetilde{X}_r^{t,\widetilde{\xi}} - h\big(\partial_x \widetilde{X}_r^{t,\widetilde{\xi},P_\xi}\widetilde{\eta} + \widetilde{Y}_r^{t,\widetilde{\xi}}(\widetilde{\eta})\big)\big|\big] \\ &\qquad = E\big[\big|X_r^{t,\xi+h\eta} - X_r^{t,\xi} - h\big(\partial_x X_r^{t,\xi,P_\xi}\eta + Y_r^{t,\xi}(\eta)\big)\big|\big].\end{aligned}$$

For this, we observe that

$$X_r^{t,\xi+h\eta} - X_r^{t,\xi} - h\big(\partial_x X_r^{t,\xi,P_\xi}\eta + Y_r^{t,\xi}(\eta)\big) = I_1(r,h) + I_2(r,h),$$

where $I_1(r,h) = \{(X_r^{t,x,P_{\xi+h\eta}} - X_r^{t,x,P_\xi} - hY_r^{t,x,\xi}(\eta))\}|_{x=\xi}$ satisfies

$$E\big[\big|I_1(r,h)\big|\big]^2 \le \sup_{x\in\mathbb{R}} E\big[\big|X_r^{t,x,P_{\xi+h\eta}} - X_r^{t,x,P_\xi} - hY_r^{t,x,\xi}(\eta)\big|^2\big]$$

and for $I_2(r,h) = X_r^{t,\xi+h\eta,P_{\xi+h\eta}} - X_r^{t,\xi,P_{\xi+h\eta}} - h\partial_x X_r^{t,\xi,P_\xi}\eta$ we have

$$\begin{aligned}E\big[\big|I_2(r,h)\big|\big]^2 &\le h^2 E[\eta^2]\int_0^1 E\big[\big|\partial_x X_r^{t,\xi+\lambda h\eta,P_{\xi+h\eta}} - \partial_x X_r^{t,\xi,P_\xi}\big|^2\big]\,d\lambda \\ &\le h^2 E[\eta^2]\int_0^1 E\big[E\big[\big|\partial_x X_r^{t,y,P_{\xi+h\eta}} - \partial_x X_r^{t,y',P_\xi}\big|^2\big]\big|_{y=\xi+\lambda h\eta,y'=\xi}\big]\,d\lambda \\ &\le Ch^4 E[\eta^2]^2.\end{aligned}$$

Therefore, combining these three latter estimates, we obtain

$$E\Big[\sup_{s\in[t,T]} \big|X_s^{t,x,P_{\xi+h\eta}} - X_s^{t,x,P_\xi} - hY_s^{t,x,\xi}(\eta)\big|^2\Big] \le Ch^4 E[\eta^2]^2,$$

for all $h \in \mathbb{R}$, $\eta \in L^2(\mathcal{F}_t)$, for some constant C independent of $t \in [0, T]$, $x \in \mathbb{R}$ and $\xi \in L^2(\mathcal{F}_t)$. The proof is complete now. $\square$

PROPOSITION 4.1. *For any given $t \in [0, T]$, $\xi \in L^2(\mathcal{F}_t)$, $x, y \in \mathbb{R}$, let*

$$\text{(4.28)}\quad \big(U^{t,x,\xi}(y), U^{t,\xi}(y)\big) = \big(U_s^{t,x,\xi}(y), U_s^{t,\xi}(y)\big)_{s\in[t,T]} \in \mathcal{S}\big([t,T];\mathbb{R}^2\big)$$

be the unique solution of the system of (*uncoupled*) *SDEs*

$$
\begin{aligned}
U_s^{t,x,\xi}(y) = {} & \int_t^s \partial_x\sigma\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)U_r^{t,x,\xi}(y)\,dB_r \\
& + \int_t^s \partial_x b\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)U_r^{t,x,\xi}(y)\,dr \\
& + \int_t^s \widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\partial_x\widetilde{X}_r^{t,y,P_\xi} \\
& + (\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\widetilde{U}_r^{t,\widetilde{\xi}}(y)\big]\,dB_r \\
& + \int_t^s \widetilde{E}\big[(\partial_\mu b)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\partial_x\widetilde{X}_r^{t,y,P_\xi} \\
& + (\partial_\mu b)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\widetilde{U}_r^{t,\widetilde{\xi}}(y)\big]\,dr,
\end{aligned} \tag{4.29}
$$

$$
\begin{aligned}
U_s^{t,\xi}(y) = {} & \int_t^s \partial_x\sigma\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}\big)U_r^{t,\xi}(y)\,dB_r \\
& + \int_t^s \partial_x b\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}\big)U_r^{t,\xi}(y)\,dr \\
& + \int_t^s \widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\partial_x\widetilde{X}_r^{t,y,P_\xi} \\
& + (\partial_\mu\sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\cdot\widetilde{U}_r^{t,\widetilde{\xi}}(y)\big]\,dB_r \\
& + \int_t^s \widetilde{E}\big[(\partial_\mu b)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\partial_x\widetilde{X}_r^{t,y,P_\xi} \\
& + (\partial_\mu b)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\cdot\widetilde{U}_r^{t,\widetilde{\xi}}(y)\big]\,dr,
\end{aligned} \tag{4.30}
$$

$s \in [t,T]$, *where* $(\widetilde{U}^{t,\widetilde{\xi}}(y), \widetilde{B})$ *is supposed to follow under* $\widetilde{P}$ *exactly the same law as* $(U^{t,\xi}(y), B)$ *under* P. *Then, for all* $\eta \in L^2(\mathcal{F}_t)$, *the directional derivative* $Y_s^{t,x,\xi}(\eta)$ *of* $\xi \to X_s^{t,x,\xi} = X_s^{t,x,P_\xi}$ *satisfies*

$$
Y_s^{t,x,\xi}(\eta) = \widetilde{E}\big[U_s^{t,x,\xi}(\widetilde{\xi})\cdot\widetilde{\eta}\big], \qquad s \in [t,T],\ P\text{-a.s.} \tag{4.31}
$$

REMARK 4.2. (1) One can consider $\widetilde{U}^{t,\widetilde{\xi}}(y)$ as the unique solution of the SDE for $U^{t,\xi}(y)$, but with the data $(\widetilde{\xi}, \widetilde{B})$ instead of (ξ, B).

(2) Since the derivatives $\partial_x\sigma, \partial_x b, \partial_\mu\sigma$ and $\partial_\mu b$ are bounded and the process $\partial_x X^{t,y,P_\xi}$ is bounded in L^2 by a constant independent of $y \in \mathbb{R}$, it is easy to prove the existence of the solution $U^{t,\xi}(y)$ for the above SDE (4.30) and to show that it is bounded in L^2 by a constant independent of $y \in \mathbb{R}$. Once having the process $U^{t,\xi}(y)$, the existence of the solution $U^{t,x,\xi}(y)$ of (4.29) is immediate.

Before proving the above proposition, let us state the following lemma.

LEMMA 4.3. *Assume* (H.1). *Then, for all $p \geq 2$ there is some constant $C_p \in \mathbb{R}$ such that, for all $t \in [0, T]$, $x, x', y, y' \in \mathbb{R}^d$ and $\xi, \xi' \in L^2(\mathcal{F}_t; \mathbb{R}^d)$*:

$$\text{(i)} \quad E\Big[\sup_{s\in[t,T]} \big|U_s^{t,x,\xi}(y)\big|^p\Big] \leq C_p,$$

$$\text{(ii)} \quad E\Big[\sup_{s\in[t,T]} \big|U_s^{t,x,\xi}(y) - U_s^{t,x',\xi'}(y')\big|^p\Big] \leq C_p\big(|x-x'|^p + |y-y'|^p + W_2(P_\xi, P_{\xi'})^p\big). \tag{4.32}$$

REMARK 4.3. From estimate (ii) of the above lemma, we see that $U^{t,x,\xi}(y)$ depends on $\xi \in L^2(\mathcal{F}_t)$ only through its law P_ξ. This allows, in analogy to X^{t,x,P_ξ}, to write $U^{t,x,P_\xi}(y) := U^{t,x,\xi}(y)$.

Moreover, it is easy to verify that the solution $U^{t,x,P_\xi}(y)$ is $(\sigma\{B_r - B_t, r \in [t,s]\} \vee \mathcal{N}_P)$-adapted and, hence, independent of $\mathcal{F}_t$ and, in particular, of ξ. Substituting in $U^{t,x,P_\xi}(y)$ the random variable ξ for x, we deduce from the uniqueness of the solution of the equation for $U^{t,\xi}(y)$ that

$$U_s^{t,\xi}(y) = U_s^{t,x,P_\xi}(y)|_{x=\xi}, \qquad s \in [t,T], P\text{-a.s.} \tag{4.33}$$

The same argument allows also to substitute $\mathcal{F}_t$-measurable random variables for y in $U_s^{t,\xi}(y)$.

PROOF OF LEMMA 4.3. We continue to restrict ourselves to the one-dimensional case $d = 1$, and to simplify a bit more we suppose also that $b = 0$. By standard arguments already used in the proof of the estimates for $X^{t,x;P_\xi}$, which we combine with our estimates for $X^{t,x;P_\xi}$ and $\partial_x X^{t,x;P_\xi}$ we see that, for all $t \in [0,T]$, $x, x', y, y' \in \mathbb{R}$ and $\xi, \xi' \in L^2(\mathcal{F}_t)$,

$$E\Big[\sup_{s\in[t,T]} \big(\big|U_s^{t,x,\xi}(y)\big|^p + \big|U_s^{t,\xi}(y)\big|^p\big)\Big] \leq C_p. \tag{4.34}$$

As concern the proof of the estimate

$$E\Big[\sup_{s\in[t,T]} \big(\big|U_s^{t,x,\xi}(y) - U_s^{t,x',\xi'}(y')\big|^p + \big|U_s^{t,\xi}(y) - U_s^{t,\xi'}(y')\big|^p\big)\Big] \leq C_p\big(|x-x'|^p + |y-y'|^p + W_2(P_\xi, P_{\xi'})^p\big), \tag{4.35}$$

we notice that its central ingredient is the following estimate, which uses the Lipschitz property of $\partial_\mu \sigma$ with respect to all its variables as well as the boundedness of $\partial_\mu \sigma$:

$$\begin{aligned} &E\big[\big|\widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big) \cdot \partial_x \widetilde{X}_r^{t,y,P_\xi} \\ &\quad - (\partial_\mu\sigma)\big(X_r^{t,x',P_{\xi'}}, P_{X_r^{t,\xi'}}, \widetilde{X}_r^{t,y',P_{\xi'}}\big) \cdot \partial_x \widetilde{X}_r^{t,y',P_{\xi'}}\big]\big|^p\big] \\ &\quad \leq C_p\big(E\big[\big|X_r^{t,x,P_\xi} - X_r^{t,x',P_{\xi'}}\big|^{2p} + \big(\widetilde{E}\big[\big|\widetilde{X}_r^{t,y,P_\xi} - \widetilde{X}_r^{t,y',P_{\xi'}}\big|^2\big]\big)^p\big] \\ &\qquad + W_2(P_{X_r^{t,\xi}}, P_{X_r^{t,\xi'}})^{2p}\big)^{1/2} + C_p\big(\widetilde{E}\big[\big|\partial_x \widetilde{X}_s^{t,y,P_\xi} - \partial_x \widetilde{X}_s^{t,y',P_{\xi'}}\big|^2\big]\big)^{p/2}. \end{aligned} \tag{4.36}$$

Once having the above estimate, we can use our estimates for X^{t,x,P_ξ} and $\partial_x X^{t,x,P_\xi}$, in order to deduce that

$$\begin{aligned}
&E\big[\big|\widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\partial_x\widetilde{X}_r^{t,y,P_\xi}\\
&\quad-(\partial_\mu\sigma)\big(X_r^{t,x',P_{\xi'}}, P_{X_r^{t,\xi'}}, \widetilde{X}_r^{t,y',P_{\xi'}}\big)\cdot\partial_x\widetilde{X}_r^{t,y',P_{\xi'}}\big]\big|^p\big]\\
&\quad\le C_p\big(|x-x'|^p+|y-y'|^p+W_2(P_\xi,P_{\xi'})^p\big)
\end{aligned}\tag{4.37}$$

and

$$\begin{aligned}
&E\big[\big|\widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\partial_x\widetilde{X}_r^{t,y,P_\xi}\\
&\quad-(\partial_\mu\sigma)\big(X_r^{t,\xi'}, P_{X_r^{t,\xi'}}, \widetilde{X}_r^{t,\xi'}\big)\cdot\partial_x\widetilde{X}_r^{t,y',P_{\xi'}}\big]\big|^p\big]\\
&\quad\le C_p\big(|x-x'|^p+|y-y'|^p+W_2(P_\xi,P_{\xi'})^p\big).
\end{aligned}\tag{4.38}$$

Consequently, from the equations for $U^{t,x,\xi}(y)$, $U^{t,x',\xi'}(y')$, the above estimates and Gronwall's inequality we see that for all $p\ge 2$, there is some constant C_p such that, for all $t\in[0,T]$, $x,x',y,y'\in\mathbb{R}$ and $\xi,\xi'\in L^2(\mathcal{F}_t)$,

$$\begin{aligned}
&E\Big[\sup_{r\in[t,s]}\big|U_r^{t,x,\xi}(y)-U_r^{t,x',\xi'}(y')\big|^p\Big]\\
&\quad\le C_p\big(|x-x'|^p+|y-y'|^p+W_2(P_\xi,P_{\xi'})^p\big)\\
&\qquad+E\Big[\int_t^s\big|\widetilde{E}\big[\big|\widetilde{U}_r^{t,\widetilde{\xi}}(y)-\widetilde{U}_r^{t,\widetilde{\xi}'}(y')\big|^2\big]\big|^{p/2}\,dr\Big],\qquad s\in[t,T].
\end{aligned}\tag{4.39}$$

Then, from this estimate, for $p=2$,

$$\begin{aligned}
&E\Big[\sup_{r\in[t,s]}\big|U_r^{t,\xi}(y)-U_r^{t,\xi'}(y')\big|^2\Big]\\
&\quad=E\Big[E\Big[\sup_{r\in[t,s]}\big|U_r^{t,x,\xi}(y)-U_r^{t,x',\xi'}(y')\big|^2\Big]\Big|_{x=\xi,x'=\xi'}\Big]\\
&\quad\le C\big(E\big[|\xi-\xi'|^2\big]+|y-y'|^2\big)\\
&\qquad+E\Big[\int_t^s\widetilde{E}\big[\big|\widetilde{U}_r^{t,\widetilde{\xi}}(y)-\widetilde{U}_r^{t,\widetilde{\xi}'}(y')\big|^2\big]\,dr\Big],
\end{aligned}\tag{4.40}$$

$s\in[t,T]$. Applying Gronwall's inequality to (4.40) and substituting the obtained relation in (4.39), we obtain (ii) of the lemma. This completes the proof. □

We are now able to give the proof of Proposition 4.1.

PROOF PROPOSITION 4.1. Let now $(\widehat{\xi},\widehat{\eta},\widehat{B})$ be a copy of (ξ,η,B), independent of (ξ,η,B) and $(\widetilde{\xi},\widetilde{\eta},\widetilde{B})$, and defined over a new probability space $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P})$

which is different from $(\Omega, \mathcal{F}, P)$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$; the expectation $\widehat{E}[\cdot]$ applies only to random variables over $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$. This extension to $(\widehat{\xi}, \widehat{\eta}, \widehat{E}[\cdot])$ here is done in the same spirit as that from (ξ, η, B) to $(\widetilde{\xi}, \widetilde{\eta}, \widetilde{B})$.

In order to obtain the wished relation, we substitute in the equation for $U^{t,\xi}(y)$ for y the random variable $\widehat{\xi}$ and we multiply both sides of the such obtained equation by $\widehat{\eta}$ [observe that $\widehat{\xi}$ and $\widehat{\eta}$ are independent of all terms in the equation for $U^{t,\xi}(y)$]. Then we take the expectation $\widehat{E}[\cdot]$ on both sides of the new equation. This yields

$$
\begin{aligned}
&\widehat{E}\big[U_s^{t,\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big] \\
&\quad = \int_t^s \partial_x \sigma\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}\big) \widehat{E}\big[U_r^{t,\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big] \, dB_r \\
&\qquad + \int_t^s \partial_x b\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}\big) \widehat{E}\big[U_r^{t,\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big] \, dr \\
&\qquad + \int_t^s \widetilde{E}\big[\widehat{E}\big[(\partial_\mu \sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widehat{\xi},P_\xi}\big) \cdot \partial_x \widetilde{X}_r^{t,\widehat{\xi},P_\xi} \cdot \widehat{\eta}\big]\big] \, dB_r \\
&\qquad + \int_t^s \widetilde{E}\big[(\partial_\mu \sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \widehat{E}\big[\widetilde{U}_r^{t,\widetilde{\xi}}(\widehat{\xi}) \cdot \widehat{\eta}\big]\big] \, dB_r \\
&\qquad + \int_t^s \widetilde{E}\big[\widehat{E}\big[(\partial_\mu b)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widehat{\xi},P_\xi}\big) \cdot \partial_x \widetilde{X}_r^{t,\widehat{\xi},P_\xi} \cdot \widehat{\eta}\big]\big] \, dr \\
&\qquad + \int_t^s \widetilde{E}\big[(\partial_\mu b)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \widehat{E}\big[\widetilde{U}_r^{t,\widetilde{\xi}}(\widehat{\xi}) \cdot \widehat{\eta}\big]\big] \, dr, \qquad s \in [t, T].
\end{aligned}
\tag{4.41}
$$

Taking into account that $(\widehat{\xi}, \widehat{\eta})$ is independent of (ξ, η, B) and $(\widetilde{\xi}, \widetilde{\eta}, \widetilde{B})$, and of the same law under $\widehat{P}$ as $(\widetilde{\xi}, \widetilde{\eta})$ under $\widetilde{P}$, we see that

$$
\begin{aligned}
&\widetilde{E}\big[\widehat{E}\big[(\partial_\mu \sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widehat{\xi},P_\xi}\big) \cdot \partial_x \widetilde{X}_r^{t,\widehat{\xi},P_\xi} \cdot \widehat{\eta}\big]\big] \\
&\quad = \widetilde{E}\big[(\partial_\mu \sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \partial_x \widetilde{X}_r^{t,\widetilde{\xi},P_\xi} \cdot \widetilde{\eta}\big],
\end{aligned}
$$

and the same relation also holds true for $\partial_\mu b$ instead of $\partial_\mu \sigma$. Thus, the above equation takes the form

$$
\begin{aligned}
&\widehat{E}\big[U_s^{t,\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big] \\
&\quad = \int_t^s \partial_x \sigma\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}\big) \widehat{E}\big[U_r^{t,\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big] \, dB_r \\
&\qquad + \int_t^s \partial_x b\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}\big) \widehat{E}\big[U_r^{t,\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big] \, dr \\
&\qquad + \int_t^s \widetilde{E}\big[(\partial_\mu \sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \big(\partial_x \widetilde{X}_r^{t,\widetilde{\xi},P_\xi} \cdot \widetilde{\eta} + \widehat{E}\big[\widetilde{U}_r^{t,\widetilde{\xi}}(\widehat{\xi}) \cdot \widehat{\eta}\big]\big)\big] \, dB_r \\
&\qquad + \int_t^s \widetilde{E}\big[(\partial_\mu b)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \big(\partial_x \widetilde{X}_r^{t,\widetilde{\xi},P_\xi} \cdot \widetilde{\eta} \\
&\qquad + \widehat{E}\big[\widetilde{U}_r^{t,\widetilde{\xi}}(\widehat{\xi}) \cdot \widehat{\eta}\big]\big)\big] \, dr, \qquad s \in [t, T].
\end{aligned}
$$

But this latter SDE is just equation (4.16) for $Y^{t,\xi}(\eta)$, and from the uniqueness of the solution of this equation it follows that

$$Y_s^{t,\xi}(\eta) = \widehat{E}\big[U_s^{t,\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big], \qquad s \in [t,T]. \tag{4.42}$$

Finally, we substitute $\widehat{\xi}$ for y in the SDE for $U^{t,x,P_\xi}(y)$, we multiply both sides of the such obtained equation by $\widehat{\eta}$ and take after the expectation $\widehat{E}[\cdot]$ at both sides of the relation. Using the results of the above discussion, we see that this yields

$$\begin{aligned}
&\widehat{E}\big[U_s^{t,x,P_\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big] \\
&\quad = \int_t^s \partial_x \sigma\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\widehat{E}\big[U_r^{t,x,P_\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big]\,dB_r \\
&\qquad + \int_t^s \partial_x b\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\widehat{E}\big[U_r^{t,x,P_\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big]\,dr \\
&\qquad + \int_t^s \widetilde{E}\big[(\partial_\mu \sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \big(\partial_x \widetilde{X}_r^{t,\widetilde{\xi},P_\xi} \cdot \widetilde{\eta} + \widetilde{Y}_r^{t,\widetilde{\xi}}(\widetilde{\eta})\big)\big]\,dB_r \\
&\qquad + \int_t^s \widetilde{E}\big[(\partial_\mu b)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \big(\partial_x \widetilde{X}_r^{t,\widetilde{\xi},P_\xi} \cdot \widetilde{\eta} + \widetilde{Y}_r^{t,\widetilde{\xi}}(\widetilde{\eta})\big)\big]\,dr, \\
&\qquad\qquad s \in [t,T].
\end{aligned} \tag{4.43}$$

But this latter SDE is just that (4.15) satisfied by $Y^{t,x,P_\xi}(\eta)$. Therefore, from the uniqueness of the solution of this SDE it follows that

$$Y_s^{t,x,P_\xi}(\eta) = \widehat{E}\big[U_s^{t,x,P_\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big], \qquad s \in [t,T], \eta \in L^2(\mathcal{F}_t). \tag{4.44}$$

The proof is complete now. □

The preceding both statements allow to derive our main result. We first derive it for the one-dimensional case before stating it for the general case.

PROPOSITION 4.2. *Under the assumption* (H.1), *for any* $(t,x) \in [0,T] \times \mathbb{R}$, $s \in [t,T]$, *the mapping*

$$L^2(\mathcal{F}_t) \ni \xi \quad \to \quad X_s^{t,x,\xi} := X_s^{t,x,P_\xi} \in L^2(\mathcal{F}_s)$$

is Fréchet differentiable. Its Fréchet derivative $D_\xi X_s^{t,x,\xi}$ *satisfies*

$$D_\xi X_s^{t,x,\xi}(\eta) = Y_s^{t,x,\xi}(\eta) = \widehat{E}\big[U_s^{t,x,P_\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big], \qquad \eta \in L^2(\mathcal{F}_t). \tag{4.45}$$

REMARK 4.4. We observe that the latter relation satisfied by $D_\xi X_s^{t,x,\xi}(\eta)$ extends that for the derivative of deterministic functions with respect to a probability law to stochastic processes. In this sense, it is natural to define the derivative of X_s^{t,x,P_ξ} with respect to the probability law P_ξ by putting

$$\partial_\mu X_s^{t,x,P_\xi}(y) := U_s^{t,x,P_\xi}(y), \qquad s \in [t,T], x, y \in \mathbb{R}.$$

We observe that, with this definition, for any $(t,x)\in[0,T]\times\mathbb{R}$, $s\in[t,T]$,

$$D_\xi X_s^{t,x,\xi}(\eta)=Y_s^{t,x,\xi}(\eta)=\widehat{E}\big[\partial_\mu X_s^{t,x,P_\xi}(\widehat{\xi})\cdot\widehat{\eta}\big],\qquad \eta\in L^2(\mathcal{F}_t). \tag{4.46}$$

PROOF OF PROPOSITION 4.2. Let $(t,x)\in[0,T]\times\mathbb{R}$, $\xi\in L^2(\mathcal{F}_t)$ and $s\in[t,T]$. We recall that the directional derivative $Y_s^{t,x,\xi}(\eta)$ of $L^2(\mathcal{F}_t)\ni\xi\to X_s^{t,x,\xi}\in L^2(\mathcal{F}_s)$ in direction $\eta\in L^2(\mathcal{F}_s)$ has the property that $Y_s^{t,x,\xi}(\cdot)\in L(L^2(\mathcal{F}_t),L^2(\mathcal{F}_s))$. Let us denote the operator norm $\|\cdot\|_{L(L^2,L^2)}$ in $L(L^2(\mathcal{F}_t), L^2(\mathcal{F}_s))$. Then, using the preceding lemma, since

$$\begin{aligned}E\big[\big|Y_s^{t,x,\xi}(\eta)\big|^2\big]&=E\big[\big|\widehat{E}\big[U_s^{t,x,P_\xi}(\widehat{\xi})\cdot\widehat{\eta}\big]\big|^2\big]\\&\le E\big[\widehat{E}\big[U_s^{t,x,P_\xi}(\widehat{\xi})^2\big]\widehat{E}\big[\widehat{\eta}^2\big]\big]\le C^2E\big[\eta^2\big],\qquad \eta\in L^2(\mathcal{F}_t),\end{aligned}$$

for some positive constant C depending only on the coefficients b and σ, we have

$$\big\|Y_s^{t,x,\xi}\big\|^2_{L(L^2,L^2)}=\sup\big\{E\big[\big|Y_s^{t,x,\xi}(\eta)\big|^2\big]:\eta\in L^2(\mathcal{F}_t)\text{ with }E\big[\eta^2\big]\le 1\big\}\le C.$$

Moreover, for all $(t,x)\in[0,T]\times\mathbb{R}$, $\xi,\xi'\in L^2(\mathcal{F}_t)$, $s\in[t,T]$,

$$\begin{aligned}E\big[\big|Y_s^{t,x,\xi}(\eta)-Y_s^{t,x,\xi'}(\eta)\big|^2\big]&=E\big[\big|\widehat{E}\big[\big(U_s^{t,x,P_\xi}(\widehat{\xi})-U_s^{t,x,P_{\xi'}}(\widehat{\xi})\big)\cdot\widehat{\eta}\big]\big|^2\big]\\&\le E\big[\widehat{E}\big[\big|U_s^{t,x,P_\xi}(\widehat{\xi})-U_s^{t,x,P_{\xi'}}(\widehat{\xi})\big|^2\big]\widehat{E}\big[\widehat{\eta}^2\big]\big]\\&\le C^2W_2(P_\xi,P_{\xi'})^2E\big[\eta^2\big],\qquad \eta\in L^2(\mathcal{F}_t),\end{aligned}$$

that is,

$$\begin{aligned}&\big\|Y_s^{t,x,\xi}-Y_s^{t,x,\xi'}\big\|^2_{L(L^2,L^2)}\\&\quad=\sup\big\{E\big[\big|Y_s^{t,x,\xi}(\eta)-Y_s^{t,x,\xi'}(\eta)\big|^2\big]:\eta\in L^2(\mathcal{F}_t)\text{ with }E[\eta^2]\le 1\big\}\\&\quad\le CW_2(P_\xi,P_{\xi'})^2\le CE\big[\big|\xi-\xi'\big|^2\big],\qquad \xi,\xi'\in L^2(\mathcal{F}_t).\end{aligned}$$

This proves that the directional derivative $Y_s^{t,x,\xi}$ is a bounded operator (and, hence, a Gâteaux derivative), which, moreover, is continuous in ξ, which proves that it is even the Fréchet derivative of $X_s^{t,x,\xi}$ with respect to ξ. The proof is complete. □

A direct generalization of our preceding computations from the one-dimensional to the multi-dimensional case allows to establish the following general result.

THEOREM 4.1. *Let* $(\sigma,b)\in C_b^{1,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^{d\times d}\times\mathbb{R}^d)$ *satisfy assumption* (H.1). *Then, for all* $0\le t\le s\le T$ *and* $x\in\mathbb{R}^d$, *the mapping*

$L^2(\mathcal{F}_t;\mathbb{R}^d)\ni\xi\mapsto X_s^{t,x,\xi}=X_s^{t,x,P_\xi}\in L^2(\mathcal{F}_s;\mathbb{R}^d)$ *is Fréchet differentiable*, *with Fréchet derivative*

$$D_\xi X_s^{t,x,\xi}(\eta)=\widetilde{E}\big[U_s^{t,x,P_\xi}(\widetilde{\xi})\cdot\widetilde{\eta}\big]=\Bigg(\widetilde{E}\Bigg[\sum_{j=1}^d U_{s,i,j}^{t,x,P_\xi}(\widetilde{\xi})\cdot\widetilde{\eta}_j\Bigg]\Bigg)_{1\le i\le d},\tag{4.47}$$

for all $\eta=(\eta_1,\dots,\eta_d)\in L^2(\mathcal{F}_t;\mathbb{R}^d)$, *where*, *for all* $y\in\mathbb{R}^d$, $U^{t,x,P_\xi}(y)=((U_{s,i,j}^{t,x,P_\xi}(y))_{s\in[t,T]})_{1\le i,j\le d}\in S_{\mathbb{F}}^2(t,T;\mathbb{R}^{d\times d})$ *is the unique solution of the SDE*

$$\begin{aligned}U_{s,i,j}^{t,x,P_\xi}(y)={}&\sum_{k,\ell=1}^d\int_t^s\partial_{x_k}\sigma_{i,\ell}\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\cdot U_{r,k,j}^{t,x,P_\xi}(y)\,dB_r^\ell\\&+\sum_{k=1}^d\int_t^s\partial_{x_k}b_i\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\cdot U_{r,k,j}^{t,x,P_\xi}(y)\,dr\\&+\sum_{k,\ell=1}^d\int_t^s E\big[(\partial_\mu\sigma_{i,\ell})_k\big(z,P_{X_r^{t,\xi}},X_r^{t,y,P_\xi}\big)\cdot\partial_{x_j}X_{r,k}^{t,y,P_\xi}\\&+(\partial_\mu\sigma_{i,\ell})_k\big(z,P_{X_r^{t,\xi}},X_r^{t,\xi}\big)\cdot U_{r,k,j}^{t,\xi}(y)\big]\Big|_{z=X_r^{t,x,P_\xi}}\,dB_r^\ell\\&+\sum_{k=1}^d\int_t^s E\big[(\partial_\mu b_i)_k\big(z,P_{X_r^{t,\xi}},X_r^{t,y,P_\xi}\big)\cdot\partial_{x_j}X_{r,k}^{t,y,P_\xi}\\&+(\partial_\mu b_i)_k\big(z,P_{X_r^{t,\xi}},X_r^{t,\xi}\big)\cdot U_{r,k,j}^{t,\xi}(y)\big]\Big|_{z=X_r^{t,x,P_\xi}}\,dr,\end{aligned}\tag{4.48}$$

$s\in[t,T]$, $1\le i,j\le d$, *and* $U^{t,\xi}(y)=((U_{s,i,j}^{t,\xi}(y))_{s\in[t,T]})_{1\le i,j\le d}\in S_{\mathbb{F}}^2(t,T;\mathbb{R}^{d\times d})$ *is that of the SDE*

$$\begin{aligned}U_{s,i,j}^{t,\xi}(y)={}&\sum_{k,\ell=1}^d\int_t^s\partial_{x_k}\sigma_{i,\ell}\big(X_r^{t,\xi},P_{X_r^{t,\xi}}\big)\cdot U_{r,k,j}^{t,\xi}(y)\,dB_r^\ell\\&+\sum_{k=1}^d\int_t^s\partial_{x_k}b_i\big(X_r^{t,\xi},P_{X_r^{t,\xi}}\big)\cdot U_{r,k,j}^{t,\xi}(y)\,dr\\&+\sum_{k,\ell=1}^d\int_t^s E\big[(\partial_\mu\sigma_{i,\ell})_k\big(z,P_{X_r^{t,\xi}},X_r^{t,y,P_\xi}\big)\cdot\partial_{x_j}X_{r,k}^{t,y,P_\xi}\\&+(\partial_\mu\sigma_{i,\ell})_k\big(z,P_{X_r^{t,\xi}},X_r^{t,\xi}\big)\cdot U_{r,k,j}^{t,\xi}(y)\big]\Big|_{z=X_r^{t,\xi}}\,dB_r^\ell\\&+\sum_{k=1}^d\int_t^s E\big[(\partial_\mu b_i)_k\big(z,P_{X_r^{t,\xi}},X_r^{t,y,P_\xi}\big)\cdot\partial_{x_j}X_{r,k}^{t,y,P_\xi}\\&+(\partial_\mu b_i)_k\big(z,P_{X_r^{t,\xi}},X_r^{t,\xi}\big)\cdot U_{r,k,j}^{t,\xi}(y)\big]\Big|_{z=X_r^{t,\xi}}\,dr,\end{aligned}\tag{4.49}$$

$s\in[t,T]$, $1\le i,j\le d$.

REMARK 4.5. As for the one-dimensional case, following the definition of the derivative of a function $f:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ explained in Section 2, we consider $U_s^{t,x,P_\xi}(y)=(U_{s,i,j}^{t,x,P_\xi}(y))_{1\le i,j\le d}$ as derivative over $\mathcal{P}_2(\mathbb{R}^d)$ of $X_s^{t,x,P_\xi}=(X_{s,i}^{t,x,P_\xi})_{1\le i\le d}$ with respect to P_ξ. As notation for this derivative we use that already introduced for functions: $s\in[t,T]$,

$$
\begin{aligned}
\partial_\mu X_s^{t,x,P_\xi}(y)&=\big(\partial_\mu X_{s,i,j}^{t,x,P_\xi}(y)\big)_{1\le i,j\le d}:=U_s^{t,x,P_\xi}(y)\\
&=\big(U_{s,i,j}^{t,x,P_\xi}(y)\big)_{1\le i,j\le d}.
\end{aligned}\tag{4.50}
$$

5. Second-order derivatives of X^{t,x,P_ξ}. Let us come now to the study of the second-order derivatives of the process X^{t,x,P_ξ}. For this, we shall suppose the following Hypothesis for the remaining part of the paper:

Hypothesis (H.2). Let (σ,b) belong to $C_b^{2,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^{d\times d}\times\mathbb{R}^d)$, that is, $(\sigma,b)\in C_b^{1,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^{d\times d}\times\mathbb{R}^d)$ [see Hypothesis (H.1)] and the derivatives of the components $\sigma_{i,j},b_j$, $1\le i,j\le d$, have the following properties:

(i) $\partial_k\sigma_{i,j}(\cdot,\cdot),\partial_k b_j(\cdot,\cdot)$ belong to $C^{1,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$, for all $1\le k\le d$;
(ii) $\partial_\mu\sigma_{i,j}(\cdot,\cdot,\cdot),\partial_\mu b_j(\cdot,\cdot,\cdot)$ belong to $C^{1,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d)$;
(iii) All the derivatives of $\sigma_{i,j},b_j$ up to order 2 are bounded and Lipschitz.

REMARK 5.1. With the existence of the second-order mixed derivatives, $\partial_{x_l}(\partial_\mu\sigma_{i,j}(x,\mu,y))$ and $\partial_\mu(\partial_{x_l}\sigma_{i,j}(x,\mu))(y)$, $(x,\mu,y)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d$, the question of their equality raises. Indeed, under Hypothesis (H.2) they coincide, and the same holds true for those for b. More precisely, we have the following statement.

LEMMA 5.1. *Let $g\in C_b^{2,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^{d\times d}\times\mathbb{R}^d)$ [in the sense of Hypothesis* (H.2)]. *Then, for all $1\le l\le d$,*

$$\partial_{x_l}\big(\partial_\mu g(x,\mu,y)\big)=\partial_\mu\big(\partial_{x_l}g(x,\mu)\big)(y),\qquad (x,\mu,y)\in\mathbb{R}^d\times\mathcal{P}_2\big(\mathbb{R}^d\big)\times\mathbb{R}^d.$$

PROOF. Let us restrict to $d=1$. Following the argument of Clairaut's theorem, we have, for all $(x,\xi),(z,\eta)\in\mathbb{R}\times L^2(\mathcal{F})$,

$$
\begin{aligned}
I&:=\big(g(x+z,P_{\xi+\eta})-g(x+z,P_\xi)\big)-\big(g(x,P_{\xi+\eta})-g(x,P_\xi)\big)\\
&=\int_0^1E\big[\big((\partial_\mu g)(x+z,P_{\xi+s\eta},\xi+s\eta)-(\partial_\mu g)(x,P_{\xi+s\eta},\xi+s\eta)\big)\eta\big]\,ds\\
&=\int_0^1\int_0^1E\big[\partial_x(\partial_\mu g)(x+tz,P_{\xi+s\eta},\xi+s\eta)\eta z\big]\,ds\,dt\\
&=E\big[\partial_x(\partial_\mu g)(x,P_\xi,\xi)\eta z\big]+R_1\big((x,\xi),(z,\eta)\big),
\end{aligned}
$$

with $|R_1((x,\xi),(z,\eta))| \le C(|\eta|_{L^2}\cdot|z|^2+|\eta|^2_{L^2}\cdot|z|)$, and at the same time

$$\begin{aligned}
I &= \big(g(x+z,P_{\xi+\eta})-g(x,P_{\xi+\eta})\big)-\big(g(x+z,P_\xi)-g(x,P_\xi)\big)\\
&= \int_0^1 \big((\partial_x g)(x+tz,P_{\xi+\eta})-(\partial_x g)(x+tz,P_\xi)\big)\,dt\cdot z\\
&= \int_0^1\int_0^1 E\big[\partial_\mu\big((\partial_x g)(x+tz,P_{\xi+s\eta})\big)(\xi+s\eta)\eta\big]\,ds\,dt\cdot z\\
&= E\big[\partial_\mu\big((\partial_x g)(x,P_\xi)\big)(\xi)\eta\big]z+R_2\big((x,\xi),(z,\eta)\big),
\end{aligned}$$

with $|R_2((x,\xi),(z,\eta))| \le C(|\eta|_{L^2}\cdot|z|^2+|\eta|^2_{L^2}\cdot|z|)$, where C is the Lipschitz constant of $\partial_\mu(\partial_x g)$ and $\partial_x(\partial_\mu g)$. It follows that $\partial_x(\partial_\mu g)(x,P_\xi,\xi)=\partial_\mu((\partial_x g)(x,P_\xi))(\xi)$, P-a.s., and hence,

$$\partial_x(\partial_\mu g)(x,P_\xi,y)=\partial_\mu\big((\partial_x g)(x,P_\xi)\big)(y)\qquad, P_\xi(dy)\text{-a.e.}$$

Letting $\varepsilon>0$ and θ be a standard normally distributed random variable, which is independent of ξ, and taking $\xi+\varepsilon\theta$ instead of ξ, we have

$$\partial_x(\partial_\mu g)(x,P_{\xi+\varepsilon\theta},y)=\partial_\mu\big((\partial_x g)(x,P_{\xi+\varepsilon\theta})\big)(y),\qquad P_{\xi+\varepsilon\theta}(dy)\text{-a.e.},$$

and thus, dy-a.s. on $\mathbb{R}$. Taking into account that $\partial_x(\partial_\mu g)$, $\partial_\mu(\partial_x g)$ are Lipschitz, this yields

$$\partial_x(\partial_\mu g)(x,P_{\xi+\varepsilon\theta},y)=\partial_\mu\big((\partial_x g)(x,P_{\xi+\varepsilon\theta})\big)(y),\qquad \text{for all } y\in\mathbb{R}.$$

Finally, using $W_2(P_{\xi+\varepsilon\theta},P_\xi)\le\varepsilon(E[\theta^2])^{1/2}=C\varepsilon$ and again the Lipschitz property of $\partial_x(\partial_\mu g)$ and $\partial_\mu(\partial_x g)$, we obtain by taking the limit as $\varepsilon\downarrow 0$,

$$\partial_x(\partial_\mu g)(x,P_\xi,y)=\partial_\mu\big((\partial_x g)(x,P_\xi)\big)(y),\qquad \text{for all } x,y\in\mathbb{R}, \xi\in L^2(\mathcal{F}).$$

The proof is complete. $\square$

After the above preparing discussion, let us study now the second-order derivatives. Following the approach for the first-order derivatives, we restrict here ourselves to the one-dimensional, and to shorten the formulas let us put $b=0$. We emphasize that the general case with dimension $d\ge 1$ and a drift b not identically equal to zero can be obtained with a straight-forward extension. For the purpose of better comprehension, the main result in this section, concerning the general case, will be given only after our computations.

We begin with recalling that the process $\partial_x X^{t,x,P_\xi}\in\mathcal{S}([t,T];\mathbb{R})$ is the unique solution of the SDE

$$\partial_x X_s^{t,x,P_\xi}=1+\int_t^s(\partial_x\sigma)\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\partial_x X_r^{t,x,P_\xi}\,dB_r,\qquad s\in[t,T]. \tag{5.1}$$

(Recall that $b=0$ in our computations.) With the same classical arguments which have shown the existence of this first-order derivative in L^2-sense with respect to x, we can prove the existence of the second-order L^2-derivative $\partial_x^2 X^{t,x,P_\xi}$ with respect to x and we can characterize it as the unique solution in $\mathcal{S}([t,T];\mathbb{R})$ of the SDE

$$
\begin{aligned}
\partial_x^2 X_s^{t,x,P_\xi} &= \int_t^s \big((\partial_x\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_x^2 X_r^{t,x,P_\xi} \\
&\quad + \big(\partial_x^2\sigma\big)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\big(\partial_x X_r^{t,x,P_\xi}\big)^2\big)\, dB_r, \qquad s\in[t,T].
\end{aligned}
\tag{5.2}
$$

We also notice that, with standard arguments we obtain the following lemma.

LEMMA 5.2. *Under Hypothesis* (H.2), *for all* $p\ge 2$, *there is some constant* C_p *only depending on* p *and the coefficients* σ *and* b, *such that*, *for all* $t\in[0,T]$, $x,x'\in\mathbb{R}$, $\xi,\xi'\in L^2(\mathcal{F}_t)$:

$$
\begin{aligned}
&\text{(i)} \quad E\Big[\sup_{s\in[t,T]}\big|\partial_x^2 X_s^{t,x,P_\xi}\big|^p\Big]\le C_p; \\
&\text{(ii)} \quad E\Big[\sup_{s\in[t,T]}\big|\partial_x^2 X_s^{t,x,P_\xi}-\partial_x^2 X_s^{t,x',P_{\xi'}}\big|^p\Big]\le C_p\big(|x-x'|^p+W_2(P_\xi,P_{\xi'})^p\big).
\end{aligned}
\tag{5.3}
$$

In the same manner as one proves the L^2-differentiability of $x\to X_s^{t,x,P_\xi}$, $s\in[t,T]$, one proves that for $\xi\to\partial_\mu X_s^{t,x,P_\xi}(y)$ and $y\to\partial_\mu X_s^{t,x,P_\xi}(y)$, $s\in[t,T]$. Standard arguments give the following result.

LEMMA 5.3. *Under Hypothesis* (H.2), *for all* $t\in[0,T]$, $\xi\in L^2(\mathcal{F}_t)$, *the process* $\partial_\mu X^{t,x,P_\xi}(y)$ *is* L^2*-differentiable in* $x,y\in\mathbb{R}$, *and its derivatives* $\partial_x(\partial_\mu X^{t,x,P_\xi}(y))$, $\partial_y(\partial_\mu X^{t,x,P_\xi}(y))\in\mathcal{S}^2([t,T];\mathbb{R})$ *are the unique solutions of the SDE*

$$
\begin{aligned}
\partial_x\big(\partial_\mu X^{t,x,P_\xi}(y)\big) &= \int_t^s (\partial_x\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_x\big(\partial_\mu X_r^{t,x,P_\xi}(y)\big)\, dB_r \\
&\quad + \int_t^s \big(\partial_x^2\sigma\big)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_\mu X_r^{t,x,P_\xi}(y)\cdot\partial_x X_r^{t,x,P_\xi}\, dB_r \\
&\quad + \int_t^s \widetilde{E}\big[\partial_x(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\partial_x\widetilde{X}_r^{t,y,P_\xi} \\
&\quad + \partial_x(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\widetilde{U}_r^{t,\widetilde{\xi}}(y)\big]\partial_x X_r^{t,x,P_\xi}\, dB_r
\end{aligned}
\tag{5.4}
$$

and

$$
\begin{aligned}
\partial_y\big(\partial_\mu X^{t,x,P_\xi}(y)\big) &= \int_t^s (\partial_x\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_y\big(\partial_\mu X_r^{t,x,P_\xi}(y)\big)\, dB_r \\
&\quad + \int_t^s \widetilde{E}\big[\partial_y(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\big(\partial_x\widetilde{X}_r^{t,y,P_\xi}\big)^2\big]\, dB_r \\
&\quad + \int_t^s \widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\partial_x^2\widetilde{X}_r^{t,y,P_\xi} \\
&\quad + (\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\partial_y\widetilde{U}_r^{t,\widetilde{\xi}}(y)\big]\, dB_r,
\end{aligned}
\tag{5.5}
$$

respectively, where the latter equation is coupled with the SDE

$$
\begin{aligned}
\partial_y U_s^{t,\xi}(y) &= \int_t^s (\partial_x\sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}\big)\partial_y U_r^{t,\xi}(y)\,dB_r \\
&\quad + \int_t^s \widetilde{E}\big[\partial_y(\partial_\mu\sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big) \\
&\quad \times \big(\partial_x \widetilde{X}_r^{t,y,P_\xi}\big)^2\big]\,dB_r \qquad (5.6)\\
&\quad + \int_t^s \widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\partial_x^2 \widetilde{X}_r^{t,y,P_\xi} \\
&\quad + (\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\partial_y \widetilde{U}_r^{t,\widetilde{\xi}}(y)\big]\,dB_r, \qquad s\in[t,T],
\end{aligned}
$$

which solution $\partial_y U^{t,\xi}(y)\in\mathcal{S}([t,T];\mathbb{R})$ is the L^2-derivative with respect to $y\in\mathbb{R}$ of $U^{t,\xi}(y)=\partial_\mu X^{t,x,P_\xi}(y)|_{x=\xi}$. Moreover, the techniques of estimate explained in the preceding section yield that for all $p\ge 2$, there is some $C_p\in\mathbb{R}$ such that, for all $t\in[0,T]$, $x,x',y,y'\in\mathbb{R}$, $\xi,\xi'\in L^2(\mathcal{F}_t)$:

$$
\begin{aligned}
&\text{(i)}\quad E\Big[\sup_{s\in[t,T]}\big(\big|\partial_x\big(\partial_\mu X^{t,x,P_\xi}(y)\big)\big|^p + \big|\partial_y\big(\partial_\mu X^{t,x,P_\xi}(y)\big)\big|^p\big)\Big]\le C_p,\\
&\text{(ii)}\quad E\Big[\sup_{s\in[t,T]}\big(\big|\partial_x\big(\partial_\mu X^{t,x,P_\xi}(y)\big) - \partial_x\big(\partial_\mu X^{t,x',P_{\xi'}}(y')\big)\big|^p \\
&\qquad + \big|\partial_y\big(\partial_\mu X^{t,x,P_\xi}(y)\big) - \partial_y\big(\partial_\mu X^{t,x',P_{\xi'}}(y')\big)\big|^p\big)\Big] \qquad (5.7)\\
&\qquad \le C_p\big(|x-x'|^p + |y-y'|^p + W_2(P_\xi,P_{\xi'})^p\big).
\end{aligned}
$$

Let us now consider the derivative of the process $\partial_x X^{t,x,P_\xi}$ with respect to the probability law. Knowing already that the mixed second-order derivatives $\partial_x\partial_\mu$ and $\partial_\mu\partial_x$ coincide for all functions from $C^{1,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}))$, we guess the following.

LEMMA 5.4. *Under Hypothesis* (H.2), *for all $(t,x)\in[0,T]\times\mathbb{R}$, $\xi\in L^2(\mathcal{F}_t)$,*

$$
\partial_\mu\partial_x X_s^{t,x,P_\xi}(y) = \partial_x\partial_\mu X_s^{t,x,P_\xi}(y), \qquad s\in[t,T].
$$

PROOF. Recall that we have put $b=0$. Then, using the fact that $\partial_x\sigma$ and $\partial_x^2\sigma$ are bounded, a standard estimate involving the SDEs for X^{t,x,P_ξ} and $\partial_x X^{t,x,P_\xi}$ shows that there is some constant $C\in\mathbb{R}$ such that, for all $(t,x)\in[0,T]\times\mathbb{R}$, $\xi\in L^2(\mathcal{F}_t)$ and all $s\in[t,T]$, $h\in\mathbb{R}\setminus\{0\}$,

$$
E\bigg[\bigg|\frac{1}{h}\big(X_s^{t,x+h,P_\xi} - X_s^{t,x,P_\xi}\big) - \partial_x X_s^{t,x,P_\xi}\bigg|^2\bigg]\le Ch^2. \qquad (5.8)
$$

Then, for all direction $\eta \in L^2(\mathcal{F}_t)$ and all $\lambda \in \mathbb{R} \setminus \{0\}$, we have, for arbitrary $h \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}
&E\Big[\Big|\frac{1}{\lambda}\big(\partial_x X_s^{t,x,P_{\xi+\lambda\eta}} - \partial_x X_s^{t,x,P_\xi}\big) - \widetilde{E}\big[\partial_x\big(\partial_\mu X_s^{t,x,P_\xi}(\widetilde{\xi})\big)\cdot\widetilde{\eta}\big]\Big|^2\Big] \\
&\quad \le C\frac{h^2}{\lambda^2} + E\Big[\Big|\frac{1}{h\lambda}\big(\big(X_s^{t,x+h,P_{\xi+\lambda\eta}} - X_s^{t,x+h,P_\xi}\big) - \big(X_s^{t,x,P_{\xi+\lambda\eta}} - X_s^{t,x,P_\xi}\big)\big) \\
&\qquad - \widetilde{E}\big[\partial_x\big(\partial_\mu X_s^{t,x,P_\xi}(\widetilde{\xi})\big)\cdot\widetilde{\eta}\big]\Big|^2\Big] \\
&\quad \le C\frac{h^2}{\lambda^2} + E\Big[\Big|\frac{1}{h\lambda}\int_0^\lambda \widetilde{E}\big[\big(\partial_\mu X_s^{t,x+h,P_{\xi+v\eta}}(\widetilde{\xi}+v\widetilde{\eta}) \\
&\qquad - \partial_\mu X_s^{t,x,P_{\xi+v\eta}}(\widetilde{\xi}+v\widetilde{\eta})\big)\cdot\widetilde{\eta}\big]\,dv - \widetilde{E}\big[\partial_x\big(\partial_\mu X_s^{t,x,P_\xi}(\widetilde{\xi})\big)\cdot\widetilde{\eta}\big]\Big|^2\Big] \\
&\quad \le C\frac{h^2}{\lambda^2} + E\Big[\Big|\frac{1}{h\lambda}\int_0^\lambda\int_0^h \widetilde{E}\big[\partial_x\big(\partial_\mu X_s^{t,x+u,P_{\xi+v\eta}}(\widetilde{\xi}+v\widetilde{\eta})\big)\cdot\widetilde{\eta}\big]\,du\,dv \\
&\qquad - \widetilde{E}\big[\partial_x\big(\partial_\mu X_s^{t,x,P_\xi}(\widetilde{\xi})\big)\cdot\widetilde{\eta}\big]\Big|^2\Big] \\
&\quad \le C\frac{h^2}{\lambda^2} + E\Big[\Big|\frac{1}{h\lambda}\int_0^\lambda\int_0^h \widetilde{E}\big[\big|\partial_x\big(\partial_\mu X_s^{t,x+u,P_{\xi+v\eta}}(\widetilde{\xi}+v\widetilde{\eta})\big) \\
&\qquad - \partial_x\big(\partial_\mu X_s^{t,x,P_\xi}(\widetilde{\xi})\big)\big|\cdot|\widetilde{\eta}|\big]\,du\,dv\Big|^2\Big].
\end{aligned}$$

Thus, taking into account that

$$\begin{aligned}
&\widetilde{E}\big[\big|\partial_x\big(\partial_\mu X_s^{t,x+u,P_{\xi+v\eta}}(\widetilde{\xi}+v\widetilde{\eta})\big) - \partial_x\big(\partial_\mu X_s^{t,x,P_\xi}(\widetilde{\xi})\big)\big|\cdot|\widetilde{\eta}|\big] \\
&\quad \le C\big(|u| + W_2(P_{\xi+v\eta}, P_\xi)\big)E[\eta^2]^{1/2} + C|v|E[\eta^2],
\end{aligned} \tag{5.9}$$

we obtain

$$\begin{aligned}
&E\Big[\Big|\frac{1}{\lambda}\big(\partial_x X_s^{t,x,P_{\xi+\lambda\eta}} - \partial_x X_s^{t,x,P_\xi}\big) - \widetilde{E}\big[\partial_x\big(\partial_\mu X_s^{t,x,P_\xi}(\widetilde{\xi})\big)\cdot\widetilde{\eta}\big]\Big|^2\Big] \\
&\quad \le C\Big(\frac{h^2}{\lambda^2} + \lambda^2 E[\eta^2]^2\Big) \longrightarrow C\lambda^2 E[\eta^2]^2, \qquad \text{as } h \to 0.
\end{aligned} \tag{5.10}$$

But this proves that $\widetilde{E}[\partial_x(\partial_\mu X_s^{t,x,P_\xi}(\widetilde{\xi}))\cdot\widetilde{\eta}]$ is the directional derivative of $\partial_x X^{t,x,P_\xi}$ in direction η. Using the SDE for $\partial_x(\partial_\mu X^{t,x,P_\xi}(y))$, we show like in the preceding section that this directional derivative is in fact a Fréchet derivative:

$$D_\xi\big(\partial_x X^{t,x,\xi}\big)(\eta) = \widetilde{E}\big[\partial_x\big(\partial_\mu X_s^{t,x,P_\xi}(\widetilde{\xi})\big)\cdot\widetilde{\eta}\big]$$

of the lifted mapping $L^2(\mathcal{F}_t) \ni \xi \mapsto \partial_x X_s^{t,x,\xi} := \partial_x X_s^{t,x,P_\xi}$, $s \in [t, T]$. The statement of the lemma follows directly. $\square$

It remains to study the second-order derivative of X^{t,x,P_ξ} with respect to the probability law, that is, the derivative of $\partial_\mu X^{t,x,P_\xi}(y)$ in P_ξ. Recall that, using that $b = 0$, we have that $\partial_\mu X^{t,x,P_\xi}(y) = U^{t,x,P_\xi}(y)$ is the unique solution in $\mathcal{S}^2([t, T]; \mathbb{R})$ of the following (uncoupled) SDE:

$$\begin{aligned} U_s^{t,x,\xi}(y) = {} & \int_t^s \partial_x \sigma\bigl(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\bigr) U_r^{t,x,\xi}(y)\, dB_r \\ & + \int_t^s \widetilde{E}\bigl[(\partial_\mu \sigma)\bigl(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\bigr) \cdot \partial_x \widetilde{X}_r^{t,y,P_\xi} \\ & + (\partial_\mu \sigma)\bigl(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\bigr) \widetilde{U}_r^{t,\widetilde{\xi}}(y)\bigr]\, dB_r, \end{aligned} \tag{5.11}$$

$$\begin{aligned} U_s^{t,\xi}(y) = {} & \int_t^s \partial_x \sigma\bigl(X_r^{t,\xi}, P_{X_r^{t,\xi}}\bigr) U_r^{t,\xi}(y)\, dB_r \\ & + \int_t^s \widetilde{E}\bigl[(\partial_\mu \sigma)\bigl(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\bigr) \cdot \partial_x \widetilde{X}_r^{t,y,P_\xi} \\ & + (\partial_\mu \sigma)\bigl(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\bigr) \cdot \widetilde{U}_r^{t,\widetilde{\xi}}(y)\bigr]\, dB_r. \end{aligned} \tag{5.12}$$

Arguing similarly as in the preceding section in the proof of the Fréchet differentiability of the lifted process $X^{t,x,\xi} := X^{t,x,P_\xi}$, we derive formally the lifted process $U_s^{t,x,\xi}(y) := U_s^{t,x,P_\xi}(y)$ for fixed $(t, x) \in [0, T] \times \mathbb{R}$, $y \in \mathbb{R}$, in direction $\eta \in L^2(\mathcal{F}_t)$. This gives for the formal directional derivative

$$Z_s^{t,x,\xi}(y, \eta) = L^2 - \lim_{h \to 0} \frac{1}{h}\bigl(U_s^{t,x,\xi+h\eta}(y) - U_s^{t,x,\xi}(y)\bigr), \qquad s \in [t, T],$$

the following SDE:

$$\begin{aligned} & Z_s^{t,x,\xi}(y, \eta) \\ & \quad = \int_t^s (\partial_x \sigma)\bigl(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\bigr) Z_r^{t,x,\xi}(y, \eta)\, dB_r \\ & \qquad + \int_t^s \bigl(\partial_x^2 \sigma\bigr)\bigl(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\bigr) U_r^{t,x,P_\xi}(y) \widehat{E}\bigl[U_r^{t,x,P_\xi}(\widehat{\xi}) \cdot \widehat{\eta}\bigr]\, dB_r \\ & \qquad + \int_t^s \widetilde{E}\bigl[\partial_\mu(\partial_x \sigma)\bigl(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\bigr) U_r^{t,x,P_\xi}(y)\bigl(\partial_x \widetilde{X}_r^{t,\widetilde{\xi},P_\xi} \cdot \widetilde{\eta} \\ & \qquad + \widehat{E}\bigl[\widetilde{U}_r^{t,\widetilde{\xi}}(\widehat{\xi}) \cdot \widehat{\eta}\bigr]\bigr)\bigr]\, dB_r \\ & \qquad + \int_t^s \widetilde{E}\bigl[(\partial_\mu \sigma)\bigl(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\bigr) \cdot \widehat{E}\bigl[\partial_\mu \partial_x \widetilde{X}_r^{t,y,P_\xi}(\widehat{\xi}) \cdot \widehat{\eta}\bigr]\bigr]\, dB_r \\ & \qquad + \int_t^s \widetilde{E}\bigl[\partial_x(\partial_\mu \sigma)\bigl(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\bigr) \cdot \partial_x \widetilde{X}_r^{t,y,P_\xi}\bigr] \end{aligned}$$

$$
\begin{aligned}
& \times \widehat{E}\big[U_r^{t,x,P_\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big]\, dB_r \\
& + \int_t^s \widetilde{E}\big[\overline{E}\big[(\partial_\mu^2 \sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}, \overline{X}_r^{t,\overline{\xi}}\big) \cdot \partial_x \widetilde{X}_r^{t,y,P_\xi}\big(\partial_x \overline{X}_r^{t,\overline{\xi},P_\xi} \cdot \overline{\eta} \\
(5.13)\qquad & + \widehat{E}\big[\overline{U}_r^{t,\overline{\xi}}(\widehat{\xi}) \cdot \widehat{\eta}\big]\big)\big]\big]\, dB_r \\
& + \int_t^s \widetilde{E}\big[\partial_y(\partial_\mu \sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big) \cdot \partial_x \widetilde{X}_r^{t,y,P_\xi} \\
& \times \widehat{E}\big[\widetilde{U}_r^{t,y,P_\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big]\big]\, dB_r \\
& + \int_t^s \widetilde{E}\big[(\partial_\mu \sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \big(\partial_x\big(\partial_\mu \widetilde{X}_r^{t,\widetilde{\xi},P_\xi}(y)\big) \cdot \widetilde{\eta} \\
& + \widetilde{Z}_r^{t,\widetilde{\xi}}(y, \widetilde{\eta})\big)\big]\, dB_r \\
& + \int_t^s \widetilde{E}\big[\partial_x(\partial_\mu \sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \widetilde{U}_r^{t,\widetilde{\xi}}(y)\big] \cdot \widehat{E}\big[U_r^{t,x,P_\xi}(\widehat{\xi}) \cdot \widehat{\eta}\big]\, dB_r \\
& + \int_t^s \widetilde{E}\big[\overline{E}\big[(\partial_\mu^2 \sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}, \overline{X}_r^{t,\overline{\xi}}\big) \cdot \widetilde{U}_r^{t,\widetilde{\xi}}(y)\big(\partial_x \overline{X}_r^{t,\overline{\xi},P_\xi} \cdot \overline{\eta} \\
& + \widehat{E}\big[\overline{U}_r^{t,\overline{\xi}}(\widehat{\xi}) \cdot \widehat{\eta}\big]\big)\big]\big]\, dB_r \\
& + \int_t^s \widetilde{E}\big[\partial_y(\partial_\mu \sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \cdot \widetilde{U}_r^{t,\widetilde{\xi}}(y) \cdot \big(\partial_x \widetilde{X}_r^{t,\widetilde{\xi},P_\xi} \cdot \widetilde{\eta} \\
& + \widehat{E}\big[\widetilde{U}_r^{t,\widetilde{\xi}}(\widehat{\xi}) \cdot \widehat{\eta}\big]\big)\big]\, dB_r,
\end{aligned}
$$

$s \in [t, T]$, where $Z^{t,\xi}(y, \eta) := Z^{t,x,P_\xi}(y, \eta)|_{x=\xi} \in \mathcal{S}^2([t, T]; \mathbb{R})$ is the unique solution of the above equation after substituting everywhere $x = \xi$. Let us also point out that in the above equation we have used the notation $(\overline{X}^{t,\overline{\xi}}, \overline{U}^{t,\overline{\xi}}(y))$; it is used in the same sense as the corresponding processes endowed with $\widetilde{\ }$ or $\widehat{\ }$: We consider a copy $(\overline{\xi}, \overline{\eta}, \overline{B})$ independent of (ξ, η, B), $(\widetilde{\xi}, \widetilde{\eta}, \widetilde{B})$ and $(\widehat{\xi}, \widehat{\eta}, \widehat{B})$, and the process $\overline{X}^{t,\overline{\xi}}$ is the solution of the SDE for $X^{t,\xi}$ and $\overline{U}^{t,\overline{\xi}}$ that of the SDE for $U^{t,\xi}$, but both with the data $(\overline{\xi}, \overline{B})$ instead of (ξ, B).

Let us comment also the expression $\partial_\mu^2 \sigma(x, P_\vartheta, y, z) = \partial_\mu(\partial_\mu \sigma(x, P_\vartheta, y))(z)$ in the above formula. Recalling that $\partial_\mu^2 \sigma(x, P_\vartheta, y, z) = \partial_\mu(\partial_\mu \sigma(x, P_\vartheta, y))(z)$ is defined through the relation $D_\vartheta[\widetilde{\partial_\mu \sigma}(x, \vartheta, y)](\theta) = E[\partial_\mu^2 \sigma(x, P_\vartheta, y, \vartheta) \cdot \theta]$, for $\vartheta, \theta \in L^2(\mathcal{F})$, $x, y \in \mathbb{R}$, where D_ϑ denotes the Fréchet derivative with respect to ϑ, we have namely for the Fréchet derivative of $L^2(\mathcal{F}) \ni \vartheta \mapsto \widetilde{\partial_\mu \sigma}(x, \vartheta, y) := (\partial_\mu \sigma)(x, P_\vartheta, y)$ in direction $\theta \in L^2(\mathcal{F})$,

$$
D_\vartheta\big[\widetilde{\partial_\mu \sigma}(x, \vartheta, y)\big](\theta) = \overline{E}\big[(\partial_\mu^2 \sigma)(x, P_\vartheta, y, \overline{\vartheta}) \cdot \overline{\theta}\big].
$$

Then, of course, the Fréchet derivative of $\xi \to \widetilde{\partial_\mu\sigma}(X_r^{t,x,P_\xi}, X_r^{t,\xi}, \widetilde{X}^{t,y,P_\xi})$ is given by

$$
\begin{aligned}
&D_\xi\big[\widetilde{\partial_\mu\sigma}\big(X_r^{t,x,P_\xi}, X_r^{t,\xi}, \widetilde{X}^{t,y,P_\xi}\big)\big](\eta)\\
&\quad= \partial_x(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot \widehat{E}\big[U_r^{t,x,P_\xi}(\widehat{\xi})\cdot\widehat{\eta}\big]\\
&\qquad+\overline{E}\big[\big(\partial_\mu^2\sigma\big)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}^{t,y,P_\xi}, \overline{X}_r^{t,\overline{\xi}}\big)\\
&\qquad\times\big(\partial_x\overline{X}_r^{t,\overline{\xi},P_\xi}\cdot\overline{\eta}+\widehat{E}\big[\overline{U}_r^{t,\overline{\xi}}(\widehat{\xi})\cdot\widehat{\eta}\big]\big)\big]\\
&\qquad+\partial_y(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\widehat{E}\big[\widetilde{U}_r^{t,y,P_\xi}(\widehat{\xi})\cdot\widehat{\eta}\big],\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \eta\in L^2(\mathcal{F}_t),
\end{aligned}
\tag{5.14}
$$

$r\in[t,T]$, but this is just, what has been used for the above formula in combination with arguments already developed in the preceding section.

Let us now compare the solution $Z^{t,x,\xi}(y,\eta)$ of the above SDE with the process $U^{t,x,P_\xi}(y,z)\in\mathcal{S}_{\mathbb{F}}^2(t,T)$ defined as the unique solution of the following SDE:

$$
\begin{aligned}
&U_s^{t,x,P_\xi}(y,z)\\
&\quad=\int_t^s(\partial_x\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)U_r^{t,x,P_\xi}(y,z)\,dB_r\\
&\qquad+\int_t^s\big(\partial_x^2\sigma\big)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)U_r^{t,x,P_\xi}(y)\cdot U_r^{t,x,P_\xi}(z)\,dB_r\\
&\qquad+\int_t^s\widetilde{E}\big[\partial_\mu(\partial_x\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,z,P_\xi}\big)U_r^{t,x,P_\xi}(y)\cdot\partial_x\widetilde{X}_r^{t,z,P_\xi}\big]\,dB_r\\
&\qquad+\int_t^s\widetilde{E}\big[\partial_\mu(\partial_x\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)U_r^{t,x,P_\xi}(y)\widetilde{U}_r^{t,\widetilde{\xi}}(z)\big]\,dB_r\\
&\qquad+\int_t^s\widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\partial_\mu\big(\partial_x\widetilde{X}_r^{t,y,P_\xi}\big)(z)\big]\,dB_r\\
&\qquad+\int_t^s\widetilde{E}\big[\partial_x(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\partial_x\widetilde{X}_r^{t,y,P_\xi}\big]\cdot U_r^{t,x,P_\xi}(z)\,dB_r\\
&\qquad+\int_t^s\widetilde{E}\big[\overline{E}\big[\big(\partial_\mu^2\sigma\big)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}, \overline{X}_r^{t,z,P_\xi}\big)\\
&\qquad\times\partial_x\widetilde{X}_r^{t,y,P_\xi}\cdot\partial_x\overline{X}_r^{t,z,P_\xi}\big]\big]\,dB_r\\
&\qquad+\int_t^s\widetilde{E}\big[\overline{E}\big[\big(\partial_\mu^2\sigma\big)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}, \overline{X}_r^{t,\overline{\xi}}\big)\cdot\partial_x\widetilde{X}_r^{t,y,P_\xi}\overline{U}_r^{t,\overline{\xi}}(z)\big]\big]\,dB_r\\
&\qquad+\int_t^s\widetilde{E}\big[\partial_y(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,y,P_\xi}\big)\cdot\partial_x\widetilde{X}_r^{t,y,P_\xi}\widetilde{U}_r^{t,y,P_\xi}(z)\big]\,dB_r
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
&+\int_t^s \widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\cdot \widetilde{U}_r^{t,\widetilde{\xi}}(y,z)\big]\,dB_r\\
&+\int_t^s \widetilde{E}\big[(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,z,P_\xi}\big)\cdot \partial_x\big(\partial_\mu \widetilde{X}_r^{t,z,P_\xi}(y)\big)\big]\,dB_r\\
&+\int_t^s \widetilde{E}\big[\partial_x(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\cdot \widetilde{U}_r^{t,\widetilde{\xi}}(y)\big]\cdot U_r^{t,x,P_\xi}(z)\,dB_r\\
&+\int_t^s \widetilde{E}\big[\overline{E}\big[(\partial_\mu^2\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}, \overline{X}_r^{t,z,P_\xi}\big)\\
&\times \widetilde{U}_r^{t,\widetilde{\xi}}(y)\cdot \partial_x \overline{X}_r^{t,z,P_\xi}\big]\big]\,dB_r\\
&+\int_t^s \widetilde{E}\big[\overline{E}\big[(\partial_\mu^2\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}, \overline{X}_r^{t,\overline{\xi}}\big)\cdot \widetilde{U}_r^{t,\widetilde{\xi}}(y)\cdot \overline{U}_r^{t,\overline{\xi}}(z)\big]\big]\,dB_r\\
&+\int_t^s \widetilde{E}\big[\partial_y(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,z,P_\xi}\big)\cdot \widetilde{U}_r^{t,\widetilde{\xi}}(y)\cdot \partial_x \widetilde{X}_r^{t,z,P_\xi}\big]\,dB_r\\
&+\int_t^s \widetilde{E}\big[\partial_y(\partial_\mu\sigma)\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\cdot \widetilde{U}_r^{t,\widetilde{\xi}}(y)\cdot \widetilde{U}_r^{t,\widetilde{\xi}}(z)\big]\,dB_r,\\
&\hspace{25em} s\in[0,T],
\end{aligned}
$$

combined with the SDE for $U^{t,\xi}(y,z)=(U_s^{t,\xi}(y,z))_{s\in[t,T]}$, obtained by substituting $x=\xi$ in the equation for $U^{t,x,P_\xi}(y,z)$ (recall namely that $X^{t,\xi}=X^{t,x,P_\xi}|_{x=\xi}, U^{t,\xi}=U^{t,x,P_\xi}|_{x=\xi}$). We consider now the processes $\widehat{E}[U^{t,x,P_\xi}(y,\widehat{\xi})\cdot\widehat{\eta}]$ and $\widehat{E}[U^{t,\xi}(y,\widehat{\xi})\cdot\widehat{\eta}]$. Substituting first $z=\widehat{\xi}$ in the SDE for $U^{t,x,P_\xi}(y,z)$ and that for $U^{t,\xi}(y,z)$, then multiplying the both sides of these SDEs with $\widehat{\eta}$ and taking the expectation $\widehat{E}[\cdot]$ of this product, we get just the SDEs solved by $Z_s^{t,x,P_\xi}(y,\eta)$ and $Z_s^{t,\xi}(y,\eta)$ (see also the corresponding proof for the first-order derivatives in the preceding section), and from the uniqueness of the solution of these SDEs we conclude that

$$
(5.16)\qquad Z_s^{t,x,\xi}(y,\eta)=\widehat{E}\big[U_s^{t,x,P_\xi}(y,\widehat{\xi})\cdot\widehat{\eta}\big],\qquad s\in[t,T], y\in\mathbb{R}.
$$

We also observe that the SDEs for $U^{t,x,P_\xi}(y,z)$ and $U^{t,\xi}(y,z)$ allow to make the following estimates.

LEMMA 5.5. *Under Hypothesis* (H.2), *for all* $p\ge 2$, *there is some constant* $C_p\in\mathbb{R}$ *such that, for all* $t\in[0,T]$, $x,x',y,y',z,z'\in\mathbb{R}$ *and* $\xi,\xi'\in L^2(\mathcal{F}_t)$:

$$
\text{(i)}\qquad E\Big[\sup_{s\in[t,T]}\big|U_s^{t,x,P_\xi}(y,z)\big|^p\Big]\le C_p,
$$

$$
\begin{aligned}
(5.17)\ \text{(ii)}\qquad & E\Big[\sup_{s\in[t,T]}\big|U_s^{t,x,P_\xi}(y,z)-U_s^{t,x',P_{\xi'}}(y',z')\big|^p\Big]\\
&\le C_p\big(|x-x'|^p+|y-y'|^p+|z-z'|^p+W_2(P_\xi,P_{\xi'})^p\big).
\end{aligned}
$$

The estimates of Lemma 5.5 allow to show in analogy to the argument developed for the proof of the Fréchet differentiability of $\xi \mapsto X_s^{t,x,\xi} := X_s^{t,x,P_\xi}$ that the mappings $L^2(\mathcal{F}_t) \ni \xi \mapsto U_s^{t,x,P_\xi}(y) \in L^2(\mathcal{F}_s)$ and $L^2(\mathcal{F}_t) \ni \xi \mapsto U_s^{t,\xi}(y) \in L^2(\mathcal{F}_s)$ are Fréchet differentiable, and

$$\begin{aligned} D_\xi\big[\partial_\mu X_s^{t,x,P_\xi}(y)\big](\eta) &= D_\xi\big[U_s^{t,x,P_\xi}(y)\big](\eta) = \widehat{E}\big[U_s^{t,x,P_\xi}(y,\widehat{\xi}) \cdot \widehat{\eta}\big], \\ D_\xi\big[\partial_\mu X_s^{t,\xi}(y)\big](\eta) &= D_\xi\big[U_s^{t,\xi}(y)\big](\eta) \\ &= \partial_x U_s^{t,x,P_\xi}(y)|_{x=\xi} \cdot \eta + \widehat{E}\big[U_s^{t,\xi}(y,\widehat{\xi}) \cdot \widehat{\eta}\big]. \end{aligned}$$

But this means that

$$\partial_\mu^2 X_s^{t,x,P_\xi}(y,z) = U_s^{t,x,P_\xi}(y,z), \qquad s \in [0,T], \tag{5.18}$$

for all $t \in [0,T], x, y, z \in \mathbb{R}, \xi \in L^2(\mathcal{F}_t)$.

Let us now summarize the above results concerning the second-order derivatives and formulate the main result concerning them, but for dimension $d \geq 1$ and b not necessarily equal to zero. It can be proved by a straight forward extension of the preceding computations.

THEOREM 5.1. *Under Hypothesis* (H.2) *the first-order derivatives* $\partial_{x_i} X_s^{t,x,P_\xi}$, $1 \leq i \leq d$ *and* $\partial_\mu X_s^{t,x,P_\xi}(y)$ *are in* L^2*-sense differentiable with respect to* x *and* y, *and interpreted as functional of* $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$ *they are also Fréchet differentiable with respect to* ξ. *Moreover, for all* $t \in [0,T], x, y, z \in \mathbb{R}$ *and* $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, *there are stochastic processes* $\partial_\mu(\partial_{x_i} X^{t,x,P_\xi})(y) \in \mathcal{S}^2([t,T];\mathbb{R}^d)$, $1 \leq i \leq d$, *and* $\partial_\mu^2 X^{t,x,P_\xi}(y,z) = \partial_\mu(\partial_\mu X^{t,x,P_\xi}(y))(z)$ *in* $\mathcal{S}^2([t,T];\mathbb{R}^{d\times d})$ *such that*, *for all* $\eta \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, *the Fréchet derivatives in* ξ, $D_\xi[\partial_x X_s^{t,x,P_\xi}](\cdot)$ *and* $D_\xi[\partial_\mu X_s^{t,x,P_\xi}(y)](\cdot)$ *satisfy*

$$\begin{aligned} &\text{(i)} \quad D_\xi\big[\partial_{x_i} X_s^{t,x,P_\xi}\big](\eta) = \widehat{E}\big[\partial_\mu\big(\partial_{x_i} X_s^{t,x,P_\xi}\big)(\widehat{\xi}) \cdot \widehat{\eta}\big], \\ &\text{(ii)} \quad D_\xi\big[\partial_\mu X_s^{t,x,P_\xi}(y)\big](\eta) = \widehat{E}\big[\partial_\mu^2 X_s^{t,x,P_\xi}(y,\widehat{\xi}) \cdot \widehat{\eta}\big], \\ &\qquad\qquad s \in [t,T], y \in \mathbb{R}^d. \end{aligned} \tag{5.19}$$

Furthermore, the mixed derivatives $\partial_{x_i}(\partial_\mu X^{t,x,P_\xi}(y))$ *and* $\partial_\mu(\partial_{x_i} X^{t,x,P_\xi})(y)$ *coincide*, *that is*,

$$\partial_{x_i}\big(\partial_\mu X_s^{t,x,P_\xi}(y)\big) = \partial_\mu\big(\partial_{x_i} X_s^{t,x,P_\xi}\big)(y), \qquad s \in [t,T],\ P\text{-a.s.},$$

and for

$$\begin{aligned} &M_s^{t,x,P_\xi}(y,z) \\ &\quad := \big(\partial_{x_i x_j}^2 X_s^{t,x,P_\xi}, \partial_\mu\big(\partial_{x_i} X_s^{t,x,P_\xi}\big)(y), \partial_\mu^2 X_s^{t,x,P_\xi}(y,z), \partial_{y_i}\big(\partial_\mu X_s^{t,x,P_\xi}(y)\big)\big), \end{aligned}$$

$1 \le i \le d$, *we have that, for all* $p \ge 2$, *there is some constant* $C_p \in \mathbb{R}$, *such that, for all* $t \in [0,T]$, x, x', y, y', z, z' *and* $\xi, \xi' \in L^2(\mathcal{F}_t; \mathbb{R}^d)$:

$$\text{(iii)} \quad E\Big[\sup_{s\in[t,T]} \big|M_s^{t,x,P_\xi}(y,z)\big|^p\Big] \le C_p,$$

$$\text{(iv)} \quad E\Big[\sup_{s\in[t,T]} \big|M_s^{t,x,P_\xi}(y,z) - M_s^{t,x',P_{\xi'}}(y',z')\big|^p\Big] \le C_p\big(|x-x'|^p + |y-y'|^p + |z-z'|^p + W_2(P_\xi, P_{\xi'})^p\big). \tag{5.20}$$

6. Regularity of the value function. Given a function $\Phi \in C_b^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, the objective of this section is to study the regularity of the function $V : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$,

$$V(t,x,P_\xi) = E\big[\Phi\big(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\big)\big], \quad (t,x,\xi) \in [0,T] \times \mathbb{R}^d \times L^2(\mathcal{F}_t; \mathbb{R}^d). \tag{6.1}$$

LEMMA 6.1. *Suppose that* $\Phi \in C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$. *Then, under our Hypothesis* (H.1), $V(t,\cdot,\cdot) \in C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, *for all* $t \in [0,T]$, *and the derivatives* $\partial_x V(t,x,P_\xi) = (\partial_{x_i} V(t,x,P_\xi,y))_{1\le i\le d}$ *and* $\partial_\mu V(t,x,P_\xi,y) = ((\partial_\mu V)_i(t,x,P_\xi,y))_{1\le i\le d}$ *are of the form*

$$\partial_{x_i} V(t,x,P_\xi) = \sum_{j=1}^d E\big[(\partial_{x_j}\Phi)\big(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\big) \cdot \partial_{x_i} X_{T,j}^{t,x,P_\xi}\big], \tag{6.2}$$

$$\begin{aligned}(\partial_\mu V)_i(t,x,P_\xi,y) = \sum_{j=1}^d & E\big[(\partial_{x_j}\Phi)\big(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\big)\big(\partial_\mu X_{T,j}^{t,x,P_\xi}\big)_i(y) \\ & + \widetilde{E}\big[(\partial_\mu\Phi)_j\big(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}, \widetilde{X}_T^{t,y,P_\xi}\big) \cdot \partial_{x_i} \widetilde{X}_{T,j}^{t,y,P_\xi} \\ & + (\partial_\mu\Phi)_j\big(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}, \widetilde{X}_T^{t,\widetilde{\xi}}\big) \cdot \big(\partial_\mu \widetilde{X}_{T,j}^{t,\widetilde{\xi},P_\xi}\big)_i(y)\big]\big].\end{aligned} \tag{6.3}$$

Moreover, there is some constant $C \in \mathbb{R}$ *such that, for all* $t, t' \in [0,T]$, $x, x', y, y' \in \mathbb{R}^d$, *and* $\xi, \xi' \in L^2(\mathcal{F}_t; \mathbb{R}^d)$:

$$\begin{aligned}
&\text{(i)} \quad \big|\partial_{x_i} V(t,x,P_\xi)\big| + \big|(\partial_\mu V)_i(t,x,P_\xi,y)\big| \le C, \\
&\text{(ii)} \quad \big|\partial_{x_i} V(t,x,P_\xi) - \partial_{x_i} V\big(t,x',P_{\xi'}\big)\big| \\
&\qquad + \big|(\partial_\mu V)_i(t,x,P_\xi,y) - (\partial_\mu V)_i\big(t,x',P_{\xi'},y'\big)\big| \\
&\qquad \le C\big(|x-x'| + |y-y'| + W_2(P_\xi,P_{\xi'})\big), \\
&\text{(iii)} \quad \big|V(t,x,P_\xi) - V\big(t',x,P_\xi\big)\big| + \big|\partial_{x_i} V(t,x,P_\xi) - \partial_{x_i} V\big(t',x,P_\xi\big)\big| \\
&\qquad + \big|(\partial_\mu V)_i(t,x,P_\xi,y) - (\partial_\mu V)_i\big(t',x,P_\xi,y\big)\big| \le C|t-t'|^{1/2}.
\end{aligned} \tag{6.4}$$

PROOF. In order to simplify the presentation, we consider again the case of dimension $d = 1$, but without restricting the generality of the argument we use.

In accordance with the notation introduced in Section 2, we put $\widetilde{V}(t, x, \xi) := V(t, x, P_\xi)$, and $\widetilde{\Phi}(z, \vartheta) := \Phi(z, P_\vartheta)$, $(z, \vartheta) \in \mathbb{R} \times L^2(\mathcal{F})$. Recall also that, in the same sense, $X^{t,x,\xi} = X^{t,x,P_\xi}$. Then $\widetilde{\Phi}(X_T^{t,x,\xi}, X_T^{t,\xi}) = \Phi(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}})$.

As $\Phi \in C_b^{1,1}(\mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$, its first-order derivatives are bounded and Lipschitz continuous. Thus, standard arguments combined with the results from the preceding section show the existence of the Fréchet derivative $D_\xi(\widetilde{\Phi}(X_T^{t,x,\xi}, X_T^{t,\xi}))$ of the mapping $L^2(\mathcal{F}_t) \ni \xi \to \widetilde{\Phi}(X_T^{t,x,\xi}, X_T^{t,\xi}) \in L^2(\mathcal{F}_T)$, and for all $\eta \in L^2(\mathcal{F}_t)$ we have

$$\begin{aligned}
&D_\xi\bigl(\widetilde{\Phi}\bigl(X_T^{t,x,\xi}, X_T^{t,\xi}\bigr)\bigr)(\eta)\\
&\quad= \partial_x \widetilde{\Phi}\bigl(X_T^{t,x,\xi}, X_T^{t,\xi}\bigr) D_\xi X_T^{t,x,\xi}(\eta) + (D\widetilde{\Phi})\bigl(X_T^{t,x,\xi}, X_T^{t,\xi}\bigr)\bigl(D_\xi\bigl[X_T^{t,\xi}\bigr](\eta)\bigr)\\
&\quad= \partial_x \Phi\bigl(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\bigr)\widehat{E}\bigl[\partial_\mu X_T^{t,x,P_\xi}(\widehat{\xi})\cdot\widehat{\eta}\bigr]\\
&\qquad+ \widetilde{E}\bigl[\partial_\mu \Phi\bigl(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}, \widetilde{X}_T^{t,\widetilde{\xi}}\bigr) D_{\widetilde{\xi}}\bigl[\widetilde{X}_T^{t,\widetilde{\xi}}(\widetilde{\eta})\bigr]\bigr]\\
&\quad= \partial_x \Phi\bigl(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\bigr)\widehat{E}\bigl[\partial_\mu X_T^{t,x,P_\xi}(\widehat{\xi})\cdot\widehat{\eta}\bigr]\\
&\qquad+ \widetilde{E}\bigl[\partial_\mu \Phi\bigl(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}, \widetilde{X}_T^{t,\widetilde{\xi}}\bigr)\cdot\bigl(\partial_x \widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\cdot\widetilde{\eta} + \widehat{E}\bigl[\partial_\mu \widetilde{X}_T^{t,\widetilde{\xi}}(\widehat{\xi})\cdot\widehat{\eta}\bigr]\bigr)\bigr].
\end{aligned}\tag{6.5}$$

(For the notation used here, the reader is referred to the previous sections.) With the argument developed in the study of the first-order derivatives for X^{t,x,P_ξ}, we conclude that the derivative of $\Phi(X_T^{t,x,\xi}, P_{X_T^{t,\xi}})$ with respect to the measure in P_ξ is given by

$$\begin{aligned}
&\partial_\mu\bigl(\Phi\bigl(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\bigr)\bigr)(y)\\
&\quad= \partial_x \Phi\bigl(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\bigr)\partial_\mu X_T^{t,x,P_\xi}(y)\\
&\qquad+ \widetilde{E}\bigl[\partial_\mu \Phi\bigl(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}, \widetilde{X}_T^{t,y,P_\xi}\bigr)\cdot\partial_x \widetilde{X}_T^{t,y,P_\xi}\\
&\qquad+ \partial_\mu \Phi\bigl(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}, \widetilde{X}_T^{t,\widetilde{\xi}}\bigr)\cdot\partial_\mu \widetilde{X}_T^{t,\widetilde{\xi},P_\xi}(y)\bigr], \qquad y \in \mathbb{R}.
\end{aligned}\tag{6.6}$$

In particular, we can deduce from this latter formula and the estimates from the preceding section that, for all $p \geq 2$, there is a constant $C_p \in \mathbb{R}$ such that, for all $x, x', y, y' \in \mathbb{R}$, $\xi, \xi' \in L^2(\mathcal{F}_t)$,

$$\begin{aligned}
&E\bigl[\bigl|\partial_\mu\bigl(\Phi\bigl(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\bigr)\bigr)(y)\bigr|^p\bigr] \leq C_p,\\
&E\bigl[\bigl|\partial_\mu\bigl(\Phi\bigl(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\bigr)\bigr)(y) - \partial_\mu\bigl(\Phi\bigl(X_T^{t,x',P_{\xi'}}, P_{X_T^{t,\xi'}}\bigr)\bigr)(y')\bigr|^p\bigr]\\
&\qquad\leq C_p\bigl(|x-x'|^p + |y-y'|^p + W_2(P_\xi, P_{\xi'})^p\bigr).
\end{aligned}\tag{6.7}$$

As the expectation $E[\cdot] : L^2(\mathcal{F}) \to \mathbb{R}$ is a bounded linear operator, it follows from (6.5) and (6.6) that $L^2(\mathcal{F}_t) \ni \xi \mapsto \widetilde{V}(t,x,\xi) := V(t,x,P_\xi) = E[\Phi(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}})]$ is Fréchet differentiable and, for all $\eta \in L^2(\mathcal{F}_t)$,

$$
\begin{aligned}
D_\xi\big[\widetilde{V}(t,x,\xi)\big](\eta) &= E\big[D_\xi\big(\widetilde{\Phi}\big(X_T^{t,x,\xi}, X_T^{t,\xi}\big)\big)(\eta)\big] \\
&= E\big[\widetilde{E}\big[\partial_\mu\big(\Phi\big(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\big)\big)(\widetilde{\xi}) \cdot \widetilde{\eta}\big]\big] \\
&= \widetilde{E}\big[E\big[\partial_\mu\big(\Phi\big(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\big)\big)(\widetilde{\xi})\big] \cdot \widetilde{\eta}\big],
\end{aligned}
\tag{6.8}
$$

that is,

$$
\partial_\mu V(t,x,P_\xi,y) = E\big[\partial_\mu\big(\Phi\big(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\big)\big)(y)\big], \qquad y \in \mathbb{R}. \tag{6.9}
$$

But then from (6.7), we obtain (6.4)(i) and (ii) for $\partial_\mu V(t,x,P_\xi,y)$.

As concerns the derivative of $V(t,x,P_\xi) = E[\Phi(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}})]$ with respect to x, since $z \mapsto \Phi(z, P_{X_T^{t,\xi}})$, is a (deterministic) function with a bounded, Lipschitz continuous derivative of first order, the computation of $\partial_x V(t,x,P_\xi)$ is standard. Concerning the estimates (i) and (ii) for the derivative $\partial_x V$ stated in Lemma 6.1, they are a direct consequence of the assumption on Φ as well as the estimates for the involved processes, studied in the preceding sections.

In order to complete the proof, it remains still to prove (iii). For this end, we observe that due to Lemma 3.1, for arbitrarily given $(t,x) \in [0,T] \times \mathbb{R}$ and $\xi \in L^2(\mathcal{F}_t)$, $X^{t,x,P_\xi} = X^{t,x,P_{\xi'}}$, for all $\xi' \in L^2(\mathcal{F}_t)$ with $P_{\xi'} = P_\xi$. Since due to our assumption $L^2(\mathcal{F}_0)$ is rich enough, we can find some $\xi' \in L^2(\mathcal{F}_0)$ with $P_{\xi'} = P_\xi$, which is independent of the driving Brownian motion B. Using the time-shifted Brownian motion $B_s^t := B_{t+s} - B_t$, $s \geq 0$ (where we consider the Brownian motion B extended beyond the time horizon T), we see that $X^{t,x,P_{\xi'}}$ and $X^{t,\xi'}$ solve the following SDEs (for simplicity, we put $b = 0$ again):

$$
X_{s+t}^{t,\xi'} = \xi' + \int_0^s \sigma\big(X_{r+t}^{t,\xi'}, P_{X_{r+t}^{t,\xi'}}\big)\, dB_r^t, \tag{6.10}
$$

$$
X_{s+t}^{t,x,P_{\xi'}} = x + \int_0^s \sigma\big(X_{r+t}^{t,x,P_{\xi'}}, P_{X_{r+t}^{t,\xi'}}\big)\, dB_r^t, \qquad s \in [0, T-t]. \tag{6.11}
$$

Consequently, $(X_{\cdot+t}^{t,x,P_{\xi'}}, X_{\cdot+t}^{t,\xi'})$ and $(X^{0,x,P_{\xi'}}, X^{0,\xi'})$ are solutions of the same system of SDEs, only driven by different Brownian motions, B^t and B, respectively, both independent of ξ'. It follows that the laws of $(X_{\cdot+t}^{t,x,P_{\xi'}}, X_{\cdot+t}^{t,\xi'})$ and $(X^{0,x,P_{\xi'}}, X^{0,\xi'})$ coincide, and hence,

$$
\begin{aligned}
V(t,x,P_\xi) = V(t,x,P_{\xi'}) &= E\big[\Phi\big(X_T^{t,x,P_{\xi'}}, P_{X_T^{t,\xi'}}\big)\big] \\
&= E\big[\Phi\big(X_{T-t}^{0,x,P_{\xi'}}, P_{X_{T-t}^{0,\xi'}}\big)\big].
\end{aligned}
\tag{6.12}
$$

Thus, for two different initial times $t, t' \in [0, T]$, using the fact that the derivatives of Φ are bounded, that is, Φ is Lipschitz over $\mathbb{R} \times \mathcal{P}_2(\mathbb{R})$, we obtain

$$
\begin{aligned}
&\big|V(t, x, P_\xi) - V\big(t', x, P_\xi\big)\big| \\
&\quad \le E\big[\big|\Phi\big(X_{T-t}^{0,x,P_{\xi'}}, P_{X_{T-t}^{0,\xi'}}\big) - \Phi\big(X_{T-t'}^{0,x,P_{\xi'}}, P_{X_{T-t'}^{0,\xi'}}\big)\big|\big] \\
&\quad \le C\big(E\big[\big|X_{T-t}^{0,x,P_{\xi'}} - X_{T-t'}^{0,x,P_{\xi'}}\big|\big] + W_2\big(P_{X_{T-t}^{0,\xi'}}, P_{X_{T-t'}^{0,\xi'}}\big)\big) \\
&\quad \le C\big(E\big[\big|X_{T-t}^{0,x,P_{\xi'}} - X_{T-t'}^{0,x,P_{\xi'}}\big|^2 + \big|X_{T-t}^{0,\xi'} - X_{T-t'}^{0,\xi'}\big|^2\big]\big)^{1/2}.
\end{aligned}
\tag{6.13}
$$

But, taking into account the boundedness of the coefficient σ of the SDEs for $X^{0,x,P_{\xi'}}$ and $X^{0,\xi'}$, we get

$$
\big|V(t, x, P_\xi) - V\big(t', x, P_\xi\big)\big| \le C\big|t - t'\big|^{1/2}. \tag{6.14}
$$

The proof of the remaining estimate (iii) for the derivatives of V is carried out by using the same kind of argument. Indeed, considering $\xi' \in L^2(\mathcal{F}_0)$ the system of equations for $N^{t,x,P_{\xi'}}(y) := (X^{t,x,P_{\xi'}}, \partial_x X^{t,x,P_{\xi'}}, U^{t,x,P_{\xi'}}(y))$, $x, y \in \mathbb{R}$, and $N^{t,\xi'}(y) := (X^{t,\xi'}, \partial_x X^{t,\xi'}, U^{t,\xi'}(y))$, $y \in \mathbb{R}$ [see (3.2), (5.1), (4.29)], we see again that

$$
\big(\big(N_{\cdot+t}^{t,x,P_{\xi'}}(y)\big)_{x,y\in\mathbb{R}}, \big(N_{\cdot+t}^{t,\xi'}(y)_{y\in\mathbb{R}}\big)\big)
$$

and

$$
\big(\big(N^{0,x,P_{\xi'}}(y)\big)_{x,y\in\mathbb{R}}, \big(N^{0,\xi'}(y)_{y\in\mathbb{R}}\big)\big)
$$

are equal in law.

Hence, from (6.9) we deduce

$$
\begin{aligned}
\partial_\mu V(t, x, P_\xi, y) &= E\big[\partial_\mu\big(\Phi\big(X_T^{t,x,P_{\xi'}}, P_{X_T^{t,\xi'}}\big)\big)(y)\big] \\
&= E\big[\partial_\mu\big(\Phi\big(X_{T-t}^{0,x,P_{\xi'}}, P_{X_{T-t}^{0,\xi'}}\big)\big)(y)\big].
\end{aligned}
\tag{6.15}
$$

Consequently, using the Lipschitz continuity and the boundedness of $\partial_x\Phi : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ and $\partial_\mu\Phi : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ as well as the uniform boundedness in L^p ($p \ge 2$) of the first-order derivatives $\partial_x X^{0,x,P_{\xi'}}$, $\partial_\mu X^{0,x,P_{\xi'}}(y)$, we get from (6.6) with $(0, x, \xi', T - t)$ and $(0, x, \xi', T - t')$ instead of (t, x, ξ, T),

$$
\begin{aligned}
&\big|\partial_\mu V(t, x, P_\xi, y) - \partial_\mu V\big(t', x, P_\xi, y\big)\big| \\
&\quad \le C\big(E\big[\big|X_{T-t}^{0,x,P_{\xi'}} - X_{T-t'}^{0,x,P_{\xi'}}\big| + \big|X_{T-t}^{0,y,P_{\xi'}} - X_{T-t'}^{0,y,P_{\xi'}}\big| + \big|X_{T-t}^{0,\xi'} - X_{T-t'}^{0,\xi'}\big| \\
&\qquad + \big|\partial_x X_{T-t}^{0,y,P_{\xi'}} - \partial_x X_{T-t'}^{0,y,P_{\xi'}}\big| + \big|\partial_\mu X_{T-t}^{0,x,P_{\xi'}}(y) - \partial_\mu X_{T-t'}^{0,x,P_{\xi'}}(y)\big| \\
&\qquad + \big|\partial_\mu X_{T-t}^{0,\xi',P_{\xi'}}(y) - \partial_\mu X_{T-t'}^{0,\xi',P_{\xi'}}(y)\big|\big] + W_2\big(P_{X_{T-t}^{0,\xi'}}, P_{X_{T-t'}^{0,\xi'}}\big)\big).
\end{aligned}
\tag{6.16}
$$

Thus, since ξ' is independent of $X^{0,x,P_{\xi'}}$, $\partial_x X^{0,y,P_{\xi'}}$ and $\partial_\mu X^{0,x',P_{\xi'}}(y)$,

$$\begin{aligned}&\big|\partial_\mu V(t,x,P_\xi,y)-\partial_\mu V\big(t',x,P_\xi,y\big)\big|\\ &\quad\le C\cdot\sup_{x,y\in\mathbb{R}}\big(E\big[\big|X_{T-t}^{0,x,P_{\xi'}}-X_{T-t'}^{0,x,P_{\xi'}}\big|^2+\big|\partial_x X_{T-t}^{0,x,P_{\xi'}}-\partial_x X_{T-t'}^{0,x,P_{\xi'}}(y)\big|^2\\ &\qquad+\big|\partial_\mu X_{T-t}^{0,x,P_{\xi'}}(y)-\partial_\mu X_{T-t'}^{0,x,P_{\xi'}}(y)\big|^2\big]\big)^{1/2},\end{aligned}\tag{6.17}$$

and the uniform boundedness in L^2 of the derivatives of $X^{0,x,P_{\xi'}}$ allows to deduce from the SDEs for $X^{0,x,P_{\xi'}}$, $\partial_x X^{0,x,P_{\xi'}}$ and $\partial_\mu X_{T-t'}^{0,x,P_{\xi'}}(y)$ [(3.2), (5.1), (4.29)] that

$$\big|\partial_\mu V(t,x,P_\xi,y)-\partial_\mu V\big(t',x,P_\xi,y\big)\big|\le C\big|t-t'\big|^{1/2}.\tag{6.18}$$

The proof of the corresponding estimate for $\partial_x V(t,x,P_\xi)$ is similar, and hence, omitted here. □

Let us come now to the discussion of the second-order derivatives of our value function $V(t,x,P_\xi)$.

LEMMA 6.2. *We suppose that Hypothesis* (H.2) *is satisfied by the coefficients* σ *and* b, *and we suppose that* $\Phi\in C_b^{2,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$. *Then, for all* $t\in[0,T]$, $V(t,\cdot,\cdot)\in C_b^{2,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$, *and the mixed second-order derivatives are symmetric*:

$$\begin{aligned}&\partial_{x_i}\big(\partial_\mu V(t,x,P_\xi,y)\big)=\partial_\mu\big(\partial_{x_i}V(t,x,P_\xi)\big)(y),\\ &\qquad (t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d,\xi\in L^2\big(\mathcal{F}_t;\mathbb{R}^d\big),1\le i\le d\end{aligned}$$

and, for

$$\begin{aligned}U\big(t,x,P_\xi,y,z\big)=\big(&\partial^2_{x_ix_j}V(t,x,P_\xi),\partial_{x_i}\big(\partial_\mu V(t,x,P_\xi,y)\big),\\ &\partial_\mu^2 V(t,x,P_\xi,y,z),\partial_y\big(\partial_\mu V(t,x,P_\xi,y)\big)\big),\end{aligned}$$

there is some constant $C\in\mathbb{R}$ *such that, for all* $t,t'\in[0,T]$, $x,x',y,y',z,z'\in\mathbb{R}^d$, $\xi,\xi'\in L^2(\mathcal{F}_t,\mathbb{R}^d)$:

$$\begin{aligned}&\text{(i)}\quad \big|U(t,x,P_\xi,y,z)\big|\le C,\\ &\text{(ii)}\quad \big|U(t,x,P_\xi,y,z)-U\big(t,x',P_{\xi'},y',z'\big)\big|\\ &\qquad\qquad\le C\big(\big|x-x'\big|+\big|y-y'\big|+\big|z-z'\big|+W_2(P_\xi,P_{\xi'})\big),\\ &\text{(iii)}\quad \big|U(t,x,P_\xi,y,z)-U\big(t',x,P_\xi,y,z\big)\big|\le C\big|t-t'\big|^{1/2}.\end{aligned}\tag{6.19}$$

PROOF. As in the preceding proofs, we make our computations for the case of dimension $d=1$. Moreover, in our proof we concentrate on the computation for the second-order derivative with respect to the measure $\partial_\mu^2 V(t,x,P_\xi,y,z)$ and to its estimates; using the preceding lemma on the derivatives of first order, the computation of the second-order derivatives $\partial_x^2 V(t,x,P_\xi)$, $\partial_x(\partial_\mu V(t,x,P_\xi)(y))$, $\partial_\mu(\partial_x V(t,x,P_\xi))(y)$, $\partial_y(\partial_\mu V(t,x,P_\xi)(y))$ and their estimates are rather direct and left to the interested reader. On the other hand, a direct computation based on (6.6) and (6.9) and using the symmetry of the mixed second-order derivatives of Φ and of the processes X^{t,x,P_ξ} and $X^{t,\xi}$ shows that

$$\partial_x\big(\partial_\mu V(t,x,P_\xi,y)\big)=\partial_\mu\big(\partial_x V(t,x,P_\xi)\big)(y),\qquad (t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R},\ \xi\in L^2(\mathcal{F}_t).$$

For the computation of $\partial_\mu^2 V(t,x,P_\xi,y,z)$, we use the formula for $\partial_\mu V(t,x,P_\xi,y)$ in Lemma 6.1 as well as (6.9) and (6.6). We observe that

$$\partial_\mu V(t,x,P_\xi,y)=V_1(t,x,P_\xi,y)+V_2(t,x,P_\xi,y)+V_3(t,x,P_\xi,y),\tag{6.20}$$

with

$$\begin{aligned}
V_1(t,x,P_\xi,y)&=E\big[(\partial_x\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}}\big)\cdot\big(\partial_\mu X_T^{t,x,P_\xi}\big)(y)\big],\\
V_2(t,x,P_\xi,y)&=E\big[\widetilde{E}\big[(\partial_\mu\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,y,P_\xi}\big)\cdot\partial_x\widetilde{X}_T^{t,y,P_\xi}\big]\big],\\
V_3(t,x,P_\xi,y)&=E\big[\widetilde{E}\big[(\partial_\mu\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}}\big)\cdot\big(\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\big)(y)\big]\big].
\end{aligned}$$

Let us consider $V_3(t,x,P_\xi,y)$, the discussion for $V_1(t,x,P_\xi,y)$ and $V_2(t,x,P_\xi,y)$ is analogous. Using the Fréchet differentiability of the terms involved in the definition of V_3, we obtain for the Fréchet derivative of

$$\xi\mapsto\widetilde{V}_3(t,x,\xi,y):=V_3(t,x,P_\xi,y),\qquad (t,x,\xi)\in[0,T]\times\mathbb{R}\times L^2(\mathcal{F}_t),\tag{6.21}$$

that, for all $\eta\in L^2(\mathcal{F}_t)$,

$$\begin{aligned}
&D_\xi\widetilde{V}_3(t,x,\xi,y)(\eta)\\
&\quad=E\big[\widetilde{E}\big[(\partial_\mu\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}}\big)\cdot D_{\widetilde{\xi}}\big[\big(\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\big)(y)\big](\widetilde{\eta})\\
&\qquad+\partial_x(\partial_\mu\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}}\big)\cdot\big(\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\big)(y)\cdot D_\xi\big[X_T^{t,x,P_\xi}\big](\eta)\\
&\qquad+\overline{E}\big[(\partial_\mu^2\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}},\overline{X}_T^{t,\overline{\xi}}\big)\cdot D_{\overline{\xi}}\big[\overline{X}_T^{t,\overline{\xi}}\big](\overline{\eta})\big]\cdot\big(\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\big)(y)\\
&\qquad+\partial_y(\partial_\mu\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}}\big)\cdot\big(\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\big)(y)\cdot D_{\widetilde{\xi}}\big[\widetilde{X}_T^{t,\widetilde{\xi}}\big](\widetilde{\eta})\big]\big]
\end{aligned}\tag{6.22}$$

(recall the notation introduced in Section 4). On the other hand, we know already that:

$$
\text{(6.23)}\quad
\begin{aligned}
&\text{(i)}\quad D_\xi[X_T^{t,x,P_\xi}](\eta)=\widehat{E}[\partial_\mu X_T^{t,x,P_\xi}(\widehat{\xi})\cdot\widehat{\eta}],\\
&\text{(ii)}\quad D_{\overline{\xi}}[\overline{X}_T^{t,\overline{\xi}}](\overline{\eta})=\partial_x\overline{X}_T^{t,\overline{\xi},P_\xi}\cdot\overline{\eta}+\widehat{E}[\partial_\mu\overline{X}_T^{t,\overline{\xi},P_\xi}(\widehat{\xi})\cdot\widehat{\eta}],\\
&\text{(iii)}\quad D_{\widetilde{\xi}}[\widetilde{X}_T^{t,\widetilde{\xi}}](\widetilde{\eta})=\partial_x\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\cdot\widetilde{\eta}+\widehat{E}[\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}(\widehat{\xi})\cdot\widehat{\eta}],\\
&\text{(iv)}\quad D_{\widetilde{\xi}}[(\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi})(y)](\widetilde{\eta})\\
&\qquad\qquad=\partial_x(\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}(y))\cdot\widetilde{\eta}+\widehat{E}[(\partial_\mu^2\widetilde{X}_T^{t,\widetilde{\xi},P_\xi})(y,\widehat{\xi})\cdot\widehat{\eta}].
\end{aligned}
$$

Consequently, we have

$$
\text{(6.24)}\quad
\begin{aligned}
&D_\xi\widetilde{V}_3(t,x,\xi,y)(\eta)\\
&\quad=\widehat{E}\big[E\big[\widetilde{E}\big[(\partial_\mu\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}}\big)\cdot\big(\partial_x\big(\partial_\mu\widetilde{X}_T^{t,\widehat{\xi},P_\xi}(y)\big)\\
&\qquad+\big(\partial_\mu^2\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\big)(y,\widehat{\xi})\big)\\
&\qquad+\partial_x(\partial_\mu\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}}\big)\cdot\big(\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\big)(y)\cdot\partial_\mu X_T^{t,x,P_\xi}(\widehat{\xi})\\
&\qquad+\overline{E}\big[\big(\partial_\mu^2\Phi\big)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}},\overline{X}_T^{t,\overline{\xi}}\big)\cdot\big(\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\big)(y)\\
&\qquad\times\big(\partial_x\overline{X}_T^{t,\widehat{\xi},P_\xi}+\partial_\mu\overline{X}_T^{t,\overline{\xi},P_\xi}(\widehat{\xi})\big)\big]\\
&\qquad+\partial_y(\partial_\mu\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}}\big)\cdot\big(\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\big)(y)\\
&\qquad\times\big(\partial_x\widetilde{X}_T^{t,\widehat{\xi},P_\xi}+\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}(\widehat{\xi})\big)\big]\big]\cdot\widehat{\eta}\big].
\end{aligned}
$$

Therefore,

$$
\text{(6.25)}\quad
\begin{aligned}
&\partial_\mu V_3(t,x,P_\xi,y,z)\\
&\quad=E\big[\widetilde{E}\big[(\partial_\mu\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}}\big)\cdot\big(\partial_x\big(\partial_\mu\widetilde{X}_T^{t,z,P_\xi}(y)\big)+\big(\partial_\mu^2\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\big)(y,z)\big)\\
&\qquad+\partial_x(\partial_\mu\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}}\big)\cdot\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}(y)\cdot\partial_\mu X_T^{t,x,P_\xi}(z)\\
&\qquad+\overline{E}\big[\big(\partial_\mu^2\Phi\big)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}},\overline{X}_T^{t,\overline{\xi}}\big)\cdot\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}(y)\\
&\qquad\times\big(\partial_x\overline{X}_T^{t,z,P_\xi}+\partial_\mu\overline{X}_T^{t,\overline{\xi},P_\xi}(z)\big)\big]\\
&\qquad+\partial_y(\partial_\mu\Phi)\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\widetilde{X}_T^{t,\widetilde{\xi}}\big)\cdot\big(\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}\big)(y)\\
&\qquad\times\big(\partial_x\widetilde{X}_T^{t,z,P_\xi}+\partial_\mu\widetilde{X}_T^{t,\widetilde{\xi},P_\xi}(z)\big)\big]\big],
\end{aligned}
$$

$t \in [0, T], x, y, z \in \mathbb{R}, \xi \in L^2(\mathcal{F}_t)$, satisfies the relation

$$D\widetilde{V}_3(t, x, \xi, y)(\eta) = \widehat{E}\big[\partial_\mu V_3(t, x, P_\xi, y, \widehat{\xi}) \cdot \widehat{\eta}\big], \tag{6.26}$$

that is, the function $\partial_\mu V_3(t, x, P_\xi, y, z)$ is the derivative of $V_3(t, x, P_\xi, y)$ with respect to the measure at P_ξ. Moreover, the above expression for $\partial_\mu V_3(t, x, P_\xi, y, z)$ combined with the estimates for the process X^{t,x,P_ξ} and those of its first- and second-order derivatives studied in the preceding sections allows to obtain after a direct computation

$$\begin{aligned} &\big|\partial_\mu V_3(t, x, P_\xi, y, z) - \partial_\mu V_3\big(t, x', P'_\xi, y', z'\big)\big| \\ &\quad \le C\big(|x - x'| + |y - y'| + |z - z'| + W_2(P_\xi, P_{\xi'})\big), \end{aligned} \tag{6.27}$$

for all $t \in [0, T], x, x', y, y', z, z' \in \mathbb{R}$ and $\xi, \xi' \in L^2(\mathcal{F}_t)$. Furthermore, extending in a direct way the corresponding argument for the estimate of the difference for the first-order derivatives of $V(t, x, P_\xi)$ at different time points [see (6.16)], we deduce from the explicit expression for $\partial_\mu V_3(t, x, P_\xi, y, z)$ that, for some real $C \in \mathbb{R}$,

$$\big|\partial_\mu V_3(t, x, P_\xi, y, z) - \partial_\mu V_3\big(t', x, P_\xi, y, z\big)\big| \le C\big|t - t'\big|^{1/2}, \tag{6.28}$$

for all $t, t' \in [0, T], x, y, z \in \mathbb{R}$ and $\xi \in L^2(\mathcal{F}_t)$.

In the same manner as we obtained the wished results for $\partial_\mu V_3(t, x, P_\xi, y, z)$, we can investigate $\partial_\mu V_1(t, x, P_\xi, y, z)$ and $\partial_\mu V_2(t, x, P_\xi, y, z)$. This yields the wished results for $\partial^2_\mu V(t, x, P_\xi, y, z)$. The proof is complete. □

7. Itô formula and PDE associated with mean-field SDE. Let us begin with establishing the Itô formula which will be applied after for the study of the PDE associated with our mean-field SDE.

THEOREM 7.1. *Let $F : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ be such that $F(t, \cdot, \cdot) \in C_b^{2,1}(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d))$, for all $t \in [0, T]$, $F(\cdot, x, \mu) \in C^1([0, T])$, for all $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, and all derivatives, with respect to t of first order, and with respect to (x, μ) of first and of second order, are uniformly bounded over $[0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ (for short: $F \in C_b^{1,(2,1)}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$). Then, under Hypothesis* (H.2), *for all $0 \le t \le s \le T, x \in \mathbb{R}^d, \xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)$, the following Itô formula is satisfied*:

$$\begin{aligned} &F\big(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}\big) - F(t, x, P_\xi) \\ &\quad = \int_t^s \Bigg(\partial_r F\big(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big) + \sum_{i=1}^d \partial_{x_i} F\big(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big) b_i\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big) \\ &\qquad + \frac{1}{2} \sum_{i,j,k=1}^d \partial^2_{x_i x_j} F\big(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)(\sigma_{i,k}\sigma_{j,k})\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big) \end{aligned} \tag{7.1}$$

$$+ \widetilde{E}\Bigg[\sum_{i=1}^{d}(\partial_\mu F)_i\big(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)b_i\big(\widetilde{X}_r^{t,\widetilde{\xi}}, P_{X_r^{t,\xi}}\big)$$

$$+ \frac{1}{2}\sum_{i,j,k=1}^{d}\partial_{y_i}(\partial_\mu F)_j\big(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)(\sigma_{i,k}\sigma_{j,k})\big(\widetilde{X}_r^{t,\widetilde{\xi}}, P_{X_r^{t,\xi}}\big)\Bigg]\Bigg)\,dr$$

$$+ \int_t^s \sum_{i,j=1}^{d}\partial_{x_i}F\big(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\sigma_{i,j}\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\,dB_r^j, \qquad s \in [t,T].$$

PROOF. As already before in other proofs, let us again restrict ourselves to dimension $d = 1$. The general case is got by a straight-forward extension.

Step 1. Let us begin with considering a function $f \in C_b^{2,1}(\mathcal{P}_2(\mathbb{R}))$, let $\xi \in L^2(\mathcal{F}_t)$ and $0 \le t < sz \le T$. We put $t_i^n := t + i(s-t)2^{-n}, 0 \le i \le 2^n, n \ge 1$. Due to Lemma 2.1, we have

$$\begin{aligned}
&f(P_{X_s^{t,\xi}}) - f(P_\xi)\\
&\quad = \sum_{i=0}^{2^n-1}\big(f(P_{X_{t_{i+1}^n}^{t,\xi}}) - f(P_{X_{t_i^n}^{t,\xi}})\big)\\
&\quad = \sum_{i=0}^{2^n-1}\Big(\widetilde{E}\big[\partial_\mu f\big(P_{X_{t_i^n}^{t,\xi}}, \widetilde{X}_{t_i^n}^{t,\widetilde{\xi}}\big)\big(\widetilde{X}_{t_{i+1}^n}^{t,\widetilde{\xi}} - \widetilde{X}_{t_i^n}^{t,\widetilde{\xi}}\big)\big]\\
&\qquad + \frac{1}{2}\widetilde{E}\big[\overline{E}\big[\partial_\mu^2 f\big(P_{X_{t_i^n}^{t,\xi}}, \widetilde{X}_{t_i^n}^{t,\widetilde{\xi}}, \overline{X}_{t_i^n}^{t,\overline{\xi}}\big)\big(\widetilde{X}_{t_{i+1}^n}^{t,\widetilde{\xi}} - \widetilde{X}_{t_i^n}^{t,\widetilde{\xi}}\big)\big(\overline{X}_{t_{i+1}^n}^{t,\overline{\xi}} - \overline{X}_{t_i^n}^{t,\overline{\xi}}\big)\big]\big]\\
&\qquad + \frac{1}{2}\widetilde{E}\big[\partial_y\partial_\mu f\big(P_{X_{t_i^n}^{t,\xi}}, \widetilde{X}_{t_i^n}^{t,\widetilde{\xi}}\big)\big(\widetilde{X}_{t_{i+1}^n}^{t,\widetilde{\xi}} - \widetilde{X}_{t_i^n}^{t,\widetilde{\xi}}\big)^2\big] + R_{t_i^n}\big(P_{X_{t_{i+1}^n}^{t,\xi}}, P_{X_{t_i^n}^{t,\xi}}\big)\Big)
\end{aligned} \tag{7.2}$$

(for the notation we refer to the preceding sections), where, for some $C \in \mathbb{R}_+$ depending only on the Lipschitz constants of $\partial_\mu^2 f$ and $\partial_y\partial_\mu f$,

$$\begin{aligned}
\big|R_{t_i^n}\big(P_{X_{t_{i+1}^n}^{t,\xi}}, P_{X_{t_i^n}^{t,\xi}}\big)\big| &\le CE\big[\big|X_{t_{i+1}^n}^{t,\xi} - X_{t_i^n}^{t,\xi}\big|^3\big]\\
&\le C\bigg(E\bigg[\bigg(\int_{t_i^n}^{t_{i+1}^n}\big|b\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\big|\,dr\bigg)^3\bigg]\\
&\qquad + E\bigg[\bigg(\int_{t_i^n}^{t_{i+1}^n}\big|\sigma\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\big|^2\,dr\bigg)^{3/2}\bigg]\bigg)\\
&\le C\big(t_{i+1}^n - t_i^n\big)^{3/2}, \qquad 0 \le i \le 2^n - 1.
\end{aligned} \tag{7.3}$$

Thus, taking into account the relations (recall that B and $\widetilde{B}$ are independent Brownian motions)

$$
\begin{aligned}
\text{(i)}\quad & \widetilde{E}\big[\partial_\mu f\big(P_{X^{t,\xi}_{t_i^n}}, \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}\big)\big(\widetilde{X}^{t,\widetilde{\xi}}_{t_{i+1}^n} - \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}\big)\big] \\
&= \widetilde{E}\bigg[\partial_\mu f\big(P_{X^{t,\xi}_{t_i^n}}, \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}\big)\cdot\bigg(\int_{t_i^n}^{t_{i+1}^n}\sigma\big(\widetilde{X}^{t,\widetilde{\xi}}_r, P_{X^{t,\xi}_r}\big)\,d\widetilde{B}_r \\
&\qquad + \int_{t_i^n}^{t_{i+1}^n} b\big(\widetilde{X}^{t,\widetilde{\xi}}_r, P_{X^{t,\xi}_r}\big)\,dr\bigg)\bigg] \\
&= \widetilde{E}\bigg[\partial_\mu f\big(P_{X^{t,\xi}_{t_i^n}}, \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}\big)\cdot\int_{t_i^n}^{t_{i+1}^n} b\big(\widetilde{X}^{t,\widetilde{\xi}}_r, P_{X^{t,\xi}_r}\big)\,dr\bigg], \\
\text{(ii)}\quad & \widetilde{E}\big[\overline{E}\big[\partial^2_\mu f\big(P_{X^{t,\xi}_{t_i^n}}, \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}, \overline{X}^{t,\overline{\xi}}_{t_i^n}\big)\big(\widetilde{X}^{t,\widetilde{\xi}}_{t_{i+1}^n} - \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}\big)\big(\overline{X}^{t,\overline{\xi}}_{t_{i+1}^n} - \overline{X}^{t,\overline{\xi}}_{t_i^n}\big)\big]\big] \\
&= \widetilde{E}\bigg[\overline{E}\bigg[\partial^2_\mu f\big(P_{X^{t,\xi}_{t_i^n}}, \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}, \overline{X}^{t,\overline{\xi}}_{t_i^n}\big)\cdot\bigg(\int_{t_i^n}^{t_{i+1}^n}\sigma\big(\widetilde{X}^{t,\widetilde{\xi}}_r, P_{X^{t,\xi}_r}\big)\,d\widetilde{B}_r \\
&\qquad + \int_{t_i^n}^{t_{i+1}^n} b\big(\widetilde{X}^{t,\widetilde{\xi}}_r, P_{X^{t,\xi}_r}\big)\,dr\bigg) \\
&\qquad \times\bigg(\int_{t_i^n}^{t_{i+1}^n}\sigma\big(\overline{X}^{t,\overline{\xi}}_r, P_{X^{t,\xi}_r}\big)\,d\overline{B}_r + \int_{t_i^n}^{t_{i+1}^n} b\big(\overline{X}^{t,\overline{\xi}}_r, P_{X^{t,\xi}_r}\big)\,dr\bigg)\bigg]\bigg] \\
&= \widetilde{E}\bigg[\overline{E}\bigg[\partial^2_\mu f\big(P_{X^{t,\xi}_{t_i^n}}, \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}, \overline{X}^{t,\overline{\xi}}_{t_i^n}\big)\cdot\bigg(\int_{t_i^n}^{t_{i+1}^n} b\big(\widetilde{X}^{t,\widetilde{\xi}}_r, P_{X^{t,\xi}_r}\big)\,dr\bigg) \\
&\qquad \times\bigg(\int_{t_i^n}^{t_{i+1}^n} b\big(\overline{X}^{t,\overline{\xi}}_r, P_{X^{t,\xi}_r}\big)\,dr\bigg)\bigg]\bigg], \\
\text{(iii)}\quad & \widetilde{E}\big[\partial_y\partial_\mu f\big(P_{X^{t,\xi}_{t_i^n}}, \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}\big)\big(\widetilde{X}^{t,\widetilde{\xi}}_{t_{i+1}^n} - \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}\big)^2\big] \\
&= \widetilde{E}\bigg[\partial_y\partial_\mu f\big(P_{X^{t,\xi}_{t_i^n}}, \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}\big)\bigg(\int_{t_i^n}^{t_{i+1}^n}\sigma\big(\widetilde{X}^{t,\widetilde{\xi}}_r, P_{X^{t,\xi}_r}\big)\,d\widetilde{B}_r \\
&\qquad + \int_{t_i^n}^{t_{i+1}^n} b\big(\widetilde{X}^{t,\widetilde{\xi}}_r, P_{X^{t,\xi}_r}\big)\,dr\bigg)^2\bigg] \\
&= \widetilde{E}\bigg[\partial_y\partial_\mu f\big(P_{X^{t,\xi}_{t_i^n}}, \widetilde{X}^{t,\widetilde{\xi}}_{t_i^n}\big)\cdot\int_{t_i^n}^{t_{i+1}^n}\big|\sigma\big(\widetilde{X}^{t,\widetilde{\xi}}_r, P_{X^{t,\xi}_r}\big)\big|^2\,dr\bigg] + Q_{t_i^n},
\end{aligned}
$$

with $|Q_{t_i^n}| \le C(t_{i+1}^n - t_i^n)^{3/2}$, $0 \le i \le 2^n - 1$, as well as the continuity of $r \mapsto (X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \in L^2(\mathcal{F}; \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$, we get from the above sum over the second-order Taylor expansion, as $n \to +\infty$,

$$
\begin{aligned}
&f(P_{X_s^{t,\xi}}) - f(P_\xi) \\
&\quad = \int_t^s \widetilde{E}\big[b\big(\widetilde{X}_r^{t,\widetilde{\xi}}, P_{X_r^{t,\xi}}\big)\partial_\mu f\big(P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\big]\,dr \\
&\qquad + \frac{1}{2}\int_t^s \widetilde{E}\big[\sigma^2\big(\widetilde{X}_r^{t,\widetilde{\xi}}, P_{X_r^{t,\xi}}\big)(\partial_y\partial_\mu f)\big(P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\big]\,dr, \qquad s \in [t,T].
\end{aligned} \tag{7.4}
$$

From this latter relation, we see that, for fixed (t,ξ), the function $s \mapsto f(P_{X_s^{t,\xi}})$, $s \in [t,T]$, is continuously differentiable in s and twice continuously differentiable, and in particular,

$$
\begin{aligned}
\partial_s f(P_{X_s^{t,\xi}}) &= \widetilde{E}\big[b\big(\widetilde{X}_s^{t,\widetilde{\xi}}, P_{X_s^{t,\xi}}\big)\partial_\mu f\big(P_{X_s^{t,\xi}}, \widetilde{X}_s^{t,\widetilde{\xi}}\big) \\
&\quad + \tfrac{1}{2}\sigma^2\big(\widetilde{X}_s^{t,\widetilde{\xi}}, P_{X_s^{t,\xi}}\big)(\partial_y\partial_\mu f)\big(P_{X_s^{t,\xi}}, \widetilde{X}_s^{t,\widetilde{\xi}}\big)\big], \qquad s \in [t,T].
\end{aligned} \tag{7.5}
$$

Step 2. From the preceding step, we can derive that, for $F \in C_b^{1,(2,1)}([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$ also $\Psi(s,x) := F(s,x,P_{X_s^{t,\xi}})$ is continuously differentiable,

$$
\begin{aligned}
\partial_s\Psi(s,x) &= (\partial_s F)(s,x,P_{X_s^{t,\xi}}) + \widetilde{E}\big[b\big(\widetilde{X}_s^{t,\widetilde{\xi}}, P_{X_s^{t,\xi}}\big)\partial_\mu F\big(s,x,P_{X_s^{t,\xi}}, \widetilde{X}_s^{t,\widetilde{\xi}}\big) \\
&\quad + \tfrac{1}{2}\sigma^2\big(\widetilde{X}_s^{t,\widetilde{\xi}}, P_{X_s^{t,\xi}}\big)(\partial_y\partial_\mu F)\big(s,x,P_{X_s^{t,\xi}}, \widetilde{X}_s^{t,\widetilde{\xi}}\big)\big],
\end{aligned}
$$

and twice continuously differentiable with respect to x. Hence, we can apply to $\Psi(s, X_s^{t,x,P_\xi})(= F(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}))$ the classical Itô formula. This yields

$$
\begin{aligned}
&F\big(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}\big) - F(t,x,P_\xi) \\
&\quad = \Psi\big(s, X_s^{t,x,P_\xi}\big) - \Psi(t,x) \\
&\quad = \int_t^s \Big(\partial_r\Psi\big(r, X_r^{t,x,P_\xi}\big) + b\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_x\Psi\big(r, X_r^{t,x,P_\xi}\big) \\
&\qquad + \frac{1}{2}\sigma^2\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_x^2\Psi\big(r, X_r^{t,x,P_\xi}\big)\Big)\,dr \\
&\qquad + \int_t^s \sigma\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_x\Psi\big(r, X_r^{t,x,P_\xi}\big)\,dB_r \\
&\quad = \int_t^s \Big(\partial_r F\big(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big) \\
&\qquad + \widetilde{E}\Big[b\big(\widetilde{X}_r^{t,\widetilde{\xi}}, P_{X_r^{t,\xi}}\big)\partial_\mu F\big(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big) \\
&\qquad + \frac{1}{2}\sigma^2\big(\widetilde{X}_r^{t,\widetilde{\xi}}, P_{X_r^{t,\xi}}\big)(\partial_y\partial_\mu F)\big(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \widetilde{X}_r^{t,\widetilde{\xi}}\big)\Big]
\end{aligned} \tag{7.6}
$$

$$
\begin{aligned}
&+ b\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_x \Phi\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big) \\
&+ \frac{1}{2}\sigma^2\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_x^2 F\big(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\Big) dr \\
&+ \int_t^s \sigma\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\partial_x \Phi\big(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}\big)\, dB_r, \qquad s \in [t,T].
\end{aligned}
$$

The proof is complete. $\square$

The above Itô formula allows now to show that our value function $V(t,x,P_\xi)$ is continuously differentiable with respect to t, with a derivative $\partial_t V$ bounded over $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$.

LEMMA 7.1. *Assume that $\Phi \in C^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$. Then, under Hypothesis* (H.2), *$V \in C^{1,(2,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$, and its derivative $\partial_t V(t,x,P_\xi)$ with respect to t verifies, for some constant $C \in \mathbb{R}$*:

$$
\begin{aligned}
&\text{(i)} \quad \big|\partial_t V(t,x,P_\xi)\big| \le C, \\
&\text{(ii)} \quad \big|\partial_t V(t,x,P_\xi) - \partial_t V\big(t,x',P_{\xi'}\big)\big| \le C\big(|x-x'| + W_2(P_\xi,P_{\xi'})\big), \\
&\text{(iii)} \quad \big|\partial_t V(t,x,P_\xi) - \partial_t V\big(t',x,P_\xi\big)\big| \le C|t-t'|^{1/2},
\end{aligned}
\tag{7.7}
$$

for all $t,t' \in [0,T]$, $x,x' \in \mathbb{R}^d$, $\xi,\xi' \in L^2(\mathcal{F}_t;\mathbb{R}^d)$.

PROOF. Recall that, for $t \in [0,T]$, $x \in \mathbb{R}^d$, and ξ [which can be supposed without loss of generality to belong to $L^2(\mathcal{F}_0)$; see our previous discussion in the proof of Lemma 6.1], we have

$$
V(t,x,P_\xi) = E\big[\Phi\big(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}}\big)\big] = E\big[\Phi\big(X_{T-t}^{0,x,P_\xi}, P_{X_{T-t}^{0,\xi}}\big)\big]. \tag{7.8}
$$

Hence, taking the expectation over the Itô formula in the preceding theorem for $F(s,x,P_\xi) = \Phi(x,P_\xi)$, $s = T-t$ and initial time 0, we get with, for simplicity, $d=1$ again,

$$
\begin{aligned}
&V(t,x,P_\xi) - V(T,x,P_\xi) \\
&\quad = \int_0^{T-t} E\Big[\Big(\partial_x \Phi\big(X_r^{0,x,P_\xi}, P_{X_r^{0,\xi}}\big) b\big(X_r^{0,x,P_\xi}, P_{X_r^{0,\xi}}\big) \\
&\qquad + \frac{1}{2}\partial_x^2 \Phi\big(X_r^{0,x,P_\xi}, P_{X_r^{0,\xi}}\big)\sigma^2\big(X_r^{0,x,P_\xi}, P_{X_r^{0,\xi}}\big) \\
&\qquad + \widetilde{E}\Big[(\partial_\mu \Phi)\big(X_r^{0,x,P_\xi}, P_{X_s^{0,\xi}}, \widetilde{X}_r^{0,\widetilde{\xi}}\big) b\big(\widetilde{X}_r^{0,\widetilde{\xi}}, P_{X_r^{0,\xi}}\big) \\
&\qquad + \frac{1}{2}\partial_y(\partial_\mu \Phi)\big(X_r^{0,x,P_\xi}, P_{X_r^{0,\xi}}, \widetilde{X}_r^{0,\widetilde{\xi}}\big)\sigma^2\big(\widetilde{X}_r^{0,\widetilde{\xi}}, P_{X_r^{0,\xi}}\big)\Big]\Big)\Big] dr.
\end{aligned}
\tag{7.9}
$$

Then it is evident that $V(t,x,P_\xi)$ is continuously differentiable with respect to t,

$$\begin{aligned}\partial_t V(t,x,P_\xi) = -E\big[&\partial_x\Phi\big(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}}\big)b\big(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}}\big)\\ &+\tfrac{1}{2}\partial_x^2\Phi\big(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}}\big)\sigma^2\big(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}}\big)\\ &+\widetilde{E}\big[(\partial_\mu\Phi)\big(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}},\widetilde{X}_{T-t}^{0,\widetilde{\xi}}\big)b\big(\widetilde{X}_{T-t}^{0,\widetilde{\xi}},P_{X_{T-t}^{0,\xi}}\big)\\ &+\tfrac{1}{2}\partial_y(\partial_\mu\Phi)\big(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}},\widetilde{X}_{T-t}^{0,\widetilde{\xi}}\big)\sigma^2\big(\widetilde{X}_{T-t}^{0,\widetilde{\xi}},P_{X_{T-t}^{0,\xi}}\big)\big]\big].\end{aligned}\tag{7.10}$$

Moreover, using this latter formula, we can now prove in analogy to the other derivatives of V that $\partial_t V$ satisfies the estimates stated in this lemma. The proof is complete. □

Now we are able to establish and to prove our main result.

THEOREM 7.2. *We suppose that* $\Phi\in C_b^{2,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$. *Then, under Hypothesis* (H.2), *the function* $V(t,x,P_\xi)=E[\Phi(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}})]$, $(t,x,\xi)\in[0,T]\times\mathbb{R}^d\times L^2(\mathcal{F}_t;\mathbb{R}^d)$, *is the unique solution in* $C^{1,(2,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ *of the PDE*

$$\begin{aligned}0 = {}&\partial_t V(t,x,P_\xi)+\sum_{i=1}^d\partial_{x_i}V(t,x,P_\xi)b_i(x,P_\xi)\\ &+\frac{1}{2}\sum_{i,j,k=1}^d\partial^2_{x_ix_j}V(t,x,P_\xi)(\sigma_{i,k}\sigma_{j,k})(x,P_\xi)\\ &+E\Bigg[\sum_{i=1}^d(\partial_\mu V)_i(t,x,P_\xi,\xi)b_i(\xi,P_\xi)\\ &+\frac{1}{2}\sum_{i,j,k=1}^d\partial_{y_i}(\partial_\mu V)_j(t,x,P_\xi,\xi)(\sigma_{i,k}\sigma_{j,k})(\xi,P_\xi)\Bigg],\\ &\qquad (t,x,\xi)\in[0,T]\times\mathbb{R}^d\times L^2(\mathcal{F}_t;\mathbb{R}^d),\\ V(T,x,P_\xi) = {}&\Phi(x,P_\xi),\ (x,\xi)\in\mathbb{R}^d\times L^2(\mathcal{F};\mathbb{R}^d).\end{aligned}\tag{7.11}$$

PROOF. As before we restrict ourselves in this proof to the one-dimensional case $d=1$. Recalling the flow property

$$\begin{aligned}&\big(X_r^{s,X_s^{t,x,P_\xi},P_{X_s^{t,\xi}}},X_r^{s,X_s^{t,\xi}}\big)=\big(X_r^{t,x,P_\xi},X_r^{t,\xi}\big),\\ &\qquad 0\le t\le s\le r\le T,\ x\in\mathbb{R},\ \xi\in L^2(\mathcal{F}_t),\end{aligned}\tag{7.12}$$

of our dynamics as well as

$$V(s,y,P_\vartheta)=E\big[\Phi\big(X_T^{s,y,P_\vartheta},P_{X_T^{s,\vartheta}}\big)\big]=E\big[\Phi\big(X_T^{s,y,P_\vartheta},P_{X_T^{s,\vartheta}}\big)|\mathcal{F}_s\big], \tag{7.13}$$

$s\in[0,T]$, $y\in\mathbb{R}$, $\vartheta\in L^2(\mathcal{F}_s)$, we deduce that

$$\begin{aligned}V\big(s,X_s^{t,x,P_\xi},P_{X_s^{t,\xi}}\big)&=E\big[\Phi\big(X_T^{s,y,P_\vartheta},P_{X_T^{s,\vartheta}}\big)|\mathcal{F}_s\big]\big|_{(y,\vartheta)=(X_s^{t,x,P_\xi},X_s^{t,\xi})}\\&=E\big[\Phi\big(X_T^{s,X_s^{t,x,P_\xi},P_{X_s^{t,\xi}}},P_{X_T^{s,X_s^{t,\xi}}}\big)|\mathcal{F}_s\big]\\&=E\big[\Phi\big(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}}\big)|\mathcal{F}_s\big],\qquad s\in[t,T],\end{aligned} \tag{7.14}$$

that is, $V(s,X_s^{t,x,P_\xi},P_{X_s^{t,\xi}})$, $s\in[t,T]$, is a martingale. On the other hand, since due to Lemma 7.1 the function $V\in C_b^{1,(2,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ satisfies the regularity assumptions for the Itô formula, we know that

$$\begin{aligned}&V\big(s,X_s^{t,x,P_\xi},P_{X_s^{t,\xi}}\big)-V(t,x,P_\xi)\\&\quad=\int_t^s\bigg(\partial_rV\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)+\partial_xV\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)b\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\\&\qquad+\frac12\partial_x^2V\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\sigma^2\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\\&\qquad+\widetilde{E}\Big[\partial_\mu V\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\widetilde{X}_r^{t,\widetilde{\xi}}\big)b\big(\widetilde{X}_r^{t,\widetilde{\xi}},P_{X_r^{t,\xi}}\big)\\&\qquad+\frac12\partial_y(\partial_\mu V)\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\widetilde{X}_r^{t,\widetilde{\xi}}\big)\sigma^2\big(\widetilde{X}_r^{t,\widetilde{\xi}},P_{X_r^{t,\xi}}\big)\Big]\bigg)\,dr\\&\qquad+\int_t^s\partial_xV\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\sigma\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\,dB_r,\qquad s\in[t,T].\end{aligned} \tag{7.15}$$

Consequently,

$$\begin{aligned}&V\big(s,X_s^{t,x,P_\xi},P_{X_s^{t,\xi}}\big)-V(t,x,P_\xi)\\&\quad=\int_t^s\partial_xV\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\sigma\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\,dB_r,\qquad s\in[t,T],\end{aligned} \tag{7.16}$$

and

$$\begin{aligned}0&=\int_t^s\bigg(\partial_rV\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)+\partial_xV\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)b\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\\&\quad+\frac12\partial_x^2V\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\sigma^2\big(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}}\big)\\&\quad+\widetilde{E}\Big[\partial_\mu V\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\widetilde{X}_r^{t,\widetilde{\xi}}\big)b\big(\widetilde{X}_r^{t,\widetilde{\xi}},P_{X_r^{t,\xi}}\big)\\&\quad+\frac12\partial_y(\partial_\mu V)\big(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\widetilde{X}_r^{t,\widetilde{\xi}}\big)\sigma^2\big(\widetilde{X}_r^{t,\widetilde{\xi}},P_{X_r^{t,\xi}}\big)\Big]\bigg)\,dr,\qquad s\in[t,T],\end{aligned} \tag{7.17}$$

from where we obtain easily the wished PDE.

Thus, it only still remains to prove the uniqueness of the solution of the PDE in the class $C_b^{1,(2,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$. Let $U\in C_b^{1,(2,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ be a solution of PDE (7.11). Then, from the Itô formula we have that

$$\begin{aligned}&U(s,X_s^{t,x,P_\xi},P_{X_s^{t,\xi}})-U(t,x,P_\xi)\\ &\quad=\int_t^s\partial_xU(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})\sigma(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})\,dB_r,\end{aligned}\tag{7.18}$$

$s\in[t,T]$, is a martingale. Thus, for all $t\in[0,T]$, $x\in\mathbb{R}$ and $\xi\in L^2(\mathcal{F}_t)$,

$$\begin{aligned}U(t,x,P_\xi)&=E[U(T,X_T^{t,x,P_\xi},P_{X_T^{t,\xi}})|\mathcal{F}_t]\\ &=E[\Phi(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}})]=V(t,x,P_\xi).\end{aligned}\tag{7.19}$$

This proves that the functions U and V coincide, that is, the solution is unique in $C_b^{1,(2,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$. The proof is complete. □

Acknowledgments. The authors would like to thank the anonymous Associate Editor and the anonymous referees for their very helpful comments and suggestions from which the manuscript greatly has benefited.

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