The Annals of Probability
2016, Vol. 44, No. 6, 3879–3892
DOI: 10.1214/15-AOP1063

WIRED CYCLE-BREAKING DYNAMICS FOR UNIFORM SPANNING FORESTS

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We prove that every component of the wired uniform spanning forest (WUSF) is one-ended almost surely in every transient reversible random graph, removing the bounded degree hypothesis required by earlier results. We deduce that every component of the WUSF is one-ended almost surely in every supercritical Galton–Watson tree, answering a question of Benjamini, Lyons, Peres and Schramm [*Ann. Probab.* **29** (2001) 1–65].

Our proof introduces and exploits a family of Markov chains under which the oriented WUSF is stationary, which we call the *wired cycle-breaking dynamics*.

1. Introduction. The *uniform spanning forests* (USFs) of an infinite, locally finite, connected graph G are defined as infinite-volume limits of uniformly chosen random spanning trees of large finite subgraphs of G. These limits can be taken with respect to two extremal boundary conditions, *free* and *wired*, giving the *free uniform spanning forest* (FUSF) and *wired uniform spanning forest* (WUSF), respectively (see Section 2 for detailed definitions). The study of uniform spanning forests was initiated by Pemantle [12], who, in addition to showing that both limits exist, proved that the wired and free forests coincide in $\mathbb{Z}^d$ for all d and that they are almost surely a single tree if and only if $d \le 4$. The question of connectivity of the WUSF was later given a complete answer by Benjamini, Lyons, Peres and Schramm (henceforth referred to as BLPS) in their seminal work [3], in which they proved that the WUSF of a graph is connected if and only if two independent random walks on the graph intersect almost surely [3], Theorem 9.2.

After connectivity, the most basic topological property of a forest is the number of ends its components have. An infinite connected graph G is said to be *k-ended* if, over all finite sets of vertices W, the graph $G \setminus W$ formed by deleting W from G has a maximum of k distinct infinite connected components. In particular, an infinite tree is one-ended if and only if it does not contain any simple bi-infinite paths and is two-ended if and only if it contains a unique simple bi-infinite path.

Components of the WUSF are known to be one-ended for several large classes of graphs. Again, this problem was first studied by Pemantle [12], who proved that the USF on $\mathbb{Z}^d$ has one end for $2 \le d \le 4$ and that every component has at most

Received April 2015; revised September 2015.
MSC2010 subject classifications. 60D05.
Key words and phrases. Spanning forests, unimodular random graphs, reversible random graphs.

two ends for $d \geq 5$. (For $d = 1$, the forest is all of $\mathbb{Z}$ and is therefore two-ended.) A decade later, BLPS [3], Theorem 10.1, completed and extended Pemantle's result, proving in particular that every component of the WUSF of a Cayley graph is one-ended almost surely if and only if the graph is not itself two-ended. Their proof was then adapted to random graphs by Aldous and Lyons [1], Theorem 7.2, who showed that all WUSF components are one-ended almost surely in every transient reversible random rooted graph with bounded vertex degrees. Taking a different approach, Lyons, Morris and Schramm [9] gave an isoperimetric condition for one-endedness, from which they deduced that all WUSF components are one-ended almost surely in every transient transitive graph and every non-amenable graph.

In this paper, we remove the bounded degree assumption from the result of Aldous and Lyons [1]. We state our result in the natural generality of reversible random rooted networks. Recall that a *network* is a locally finite, connected (multi)graph $G = (\mathsf{V}, \mathsf{E})$ together with a function $c : \mathsf{E} \to (0, \infty)$ assigning a positive *conductance* $c(e)$ to each unoriented edge e of G. For each vertex v, the conductance $c(v)$ of v is defined to be the sum of the conductances of the edges adjacent to v, where self-loops are counted twice. Locally finite, connected graphs without specified conductances are considered to be networks by setting $c \equiv 1$. The WUSF of a network is defined in Section 2 and reversible random rooted networks are defined in Section 5.

THEOREM 1.1. *Let (G, ρ) be a transient reversible random rooted network and suppose that $\mathbb{E}[c(\rho)^{-1}] < \infty$. Then every component of the wired uniform spanning forest of G is one-ended almost surely.*

The condition that the expected inverse conductance of the root is finite is always satisfied by graphs, for which $c(\rho) = \deg(\rho) \geq 1$. In Example 5.1, we show that the theorem can fail in the absence of this condition.

Theorem 1.1 applies (indirectly) to supercritical Galton–Watson trees conditioned to survive, answering positively Question 15.4 of BLPS [3].

COROLLARY 1.2. *Let T be a supercritical Galton–Watson tree conditioned to survive. Then every component of the wired uniform spanning forest of T is one-ended almost surely.*

Previously, this was known only for supercritical Galton–Watson trees with offspring distribution either bounded, in which case the result follows as a corollary to the theorem of Aldous and Lyons [1], or supported on a subset of $[2, \infty)$, in which case the tree is non-amenable and we may apply the theorem of Lyons, Morris and Schramm [9].

Our proof introduces a new and simple method, outlined as follows. For every transient network, we define a procedure to "update an oriented forest at an edge", in which the edge is added to the forest while another edge is deleted. Updating oriented forests at randomly chosen edges defines a family of Markov chains on oriented spanning forests, which we call the *wired cycle-breaking dynamics*, for which the oriented wired uniform spanning forest measure is stationary (Proposition 3.2). This stationarity allows us to prove the following theorem, from which we show Theorem 1.1 to follow by known methods.

THEOREM 1.3. *Let G be any network. If the wired uniform spanning forest of G contains more than one two-ended component with positive probability, then it contains a component with three or more ends with positive probability.*

The case of *recurrent* reversible random rooted graphs remains open, even under the assumption of bounded degree. In this case, it should be that the single tree of the WUSF has the same number of ends as the graph (this prediction appears in [1]). BLPS proved this for transitive recurrent graphs [3], Theorem 10.6.

1.1. *Consequences*. The one-endedness of WUSF components has consequences of fundamental importance for the *Abelian sandpile model*. Járai and Werning [7] proved that the infinite-volume limit of the sandpile measures exists on every graph for which every component of the WUSF is one-ended almost surely. Furthermore, Járai and Redig [6] proved that, for any graph which is both transient and has one-ended WUSF components, the sandpile configuration obtained by adding a single grain of sand to the infinite-volume random sandpile can be stabilized by finitely many topplings (their proof is given for $\mathbb{Z}^d$ but extends to this setting, see [5]). Thus, a consequence of Theorem 1.1 is that these properties hold for the Abelian sandpile model on transient reversible random graphs of unbounded degree.

Theorem 1.1 also has several interesting consequences for random plane graphs, which we address in upcoming work with Angel, Nachmias and Ray. In particular, we deduce from Theorem 1.1 that every Benjamini–Schramm limit of finite planar graphs is almost surely Liouville, that is, does not admit non-constant bounded harmonic functions.

2. The wired uniform spanning forest.

In this section, we briefly define the wired uniform spanning forest and introduce the properties that we will need. For a comprehensive treatment of uniform spanning trees and forests, as well as a detailed history of the subject, we refer the reader to Chapters 4 and 10 of [11].

Notation and orientation. Throughout this paper, the graphs on which the USFs and USTs are defined will be connected and locally finite unless stated otherwise. We do not distinguish notationally between oriented and unoriented trees, forests or edges. Whether or not a tree, forest or edge is oriented will be clear from context. Edges e are oriented from their tail e^- to their head e^+, and have reversal $-e$. An oriented tree or forest is a tree or forest together with an orientation of its edges.

Given an oriented tree or forest in a graph, we define the *past* of each vertex v to be the set of vertices u for which there is a directed path from u to v in the oriented tree or forest.

For a finite connected graph G, we write UST_G for the uniform measure on the set of spanning trees (i.e., connected cycle-free subgraphs containing every vertex) of G, considered for measure-theoretic purposes to be functions from E to $\{0, 1\}$. More generally, if G is a finite network, we define UST_G to be the probability measure on spanning trees of G for which the measure of a tree t is proportional to the product of the conductances of its edges.

There are two extremal (with respect to stochastic ordering) ways to define infinite volume limits of the uniform spanning tree measures. Let G be an infinite network and let V_n be an increasing sequence of finite connected subsets of V such that $\cup V_n = V$, which we call an *exhaustion* of G. For each n, let the network G_n be the subgraph of G induced by V_n together with the conductances inherited from G. The weak limit of the measures UST_{G_n} is known as the *free uniform spanning forest*: for each finite subset $S \subset \mathsf{E}$,

$$\mathsf{FUSF}_G(S \subseteq F) := \lim_{n\to\infty} \mathsf{UST}_{G_n}(S \subseteq T).$$

Alternatively, at each step of the exhaustion we define a network G_n^* by identifying ("wiring") $V \setminus V_n$ into a single vertex ∂_n and deleting all the self-loops that are created, and define the *wired uniform spanning forest* to be the weak limit

$$\mathsf{WUSF}_G(S \subseteq F) := \lim_{n\to\infty} \mathsf{UST}_{G_n^*}(S \subseteq T).$$

Both limits were shown (implicitly) to exist for every network and every choice of exhaustion by Pemantle [12], although the WUSF was not defined explicitly until the work of Häggström [4]. As a consequence, the limits do not depend on the choice of exhaustion. Both measures are supported on spanning forests (i.e., cycle-free subgraphs containing every vertex) of G for which every connected component is infinite. The WUSF is usually much more tractable, thanks in part to Wilson's algorithm rooted at infinity, which both connects the WUSF to loop-erased random walk and allows us to sample the WUSF of an infinite network directly rather than by passing to an exhaustion.

Wilson's algorithm [13] is a remarkable method of generating the UST on a finite or recurrent network by joining together loop-erased random walks. It was extended to generate the WUSF of transient networks by BLPS [3]. Let G be a network, and let γ be a path in G that is either finite or transient, that is, visits each vertex of G at most finitely many times. The *loop-erasure* $\mathsf{LE}(\gamma)$ is formed by erasing cycles from γ chronologically as they are created. Formally, $\mathsf{LE}(\gamma)_i = \gamma_{t_i}$ where the times t_i are defined recursively by $t_0 = 0$ and $t_i = 1 + \max\{t \geq t_{i-1} : \gamma_t = \gamma_{t_{i-1}}\}$. (In the presence of multiple edges, a path is not determined by its vertex-trajectory. However, the definition of the loop-erasure extends to this setting in the obvious way. Similarly, when performing Wilson's algorithm in the presence of multiple edges, we consider the random walks and their loop-erasures to be random paths in the graph.) Let $\{v_j : j \in \mathbb{N}\}$ be an enumeration of the vertices of G and define a sequence of forests in G as follows:

1. If G is finite or recurrent, choose a root vertex v_0 and let F_0 include v_0 and no edges (in which case we call the algorithm *Wilson's algorithm rooted at* v_0). If G is transient, let $F_0 = \varnothing$ (in which case we call the algorithm *Wilson's algorithm rooted at infinity*).
2. Given F_i, start an independent random walk from v_{i+1} stopped if and when it hits the set of vertices already included in F_i.
3. Form the loop-erasure of this random walk path and let F_{i+1} be the union of F_i with this loop-erased path.
4. Let $F = \cup F_i$.

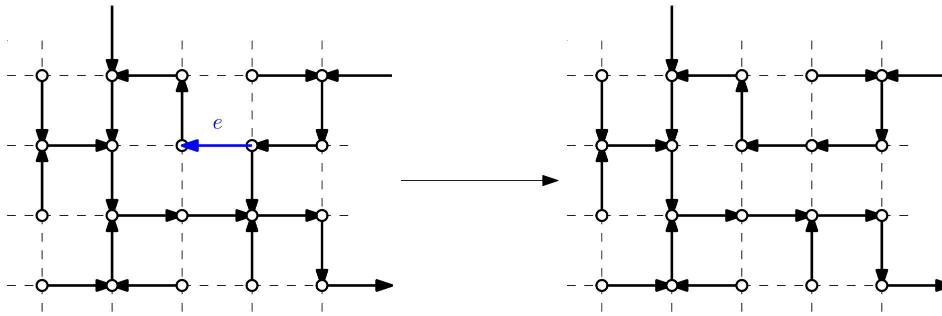
This is Wilson's algorithm: the resulting forest F has law UST_G in the finite case [13] and WUSF_G in the infinite case [3], and is independent of the choice of enumeration.

We also consider oriented spanning trees and forests. Let $\mathsf{OUST}_{G_n^*}$ denote the law of the uniform spanning tree of G_n^* oriented towards the boundary vertex ∂_n, so that every vertex of G_n^* other than ∂_n has exactly one oriented edge emanating from it in the tree, while ∂_n does not have any oriented edges emanating from it. Wilson's algorithm on G_n^* rooted at ∂_n may be modified to produce an oriented tree with law $\mathsf{OUST}_{G_n^*}$ by considering the loop-erased paths in step (2) to be oriented chronologically. If G is transient, making the same modification to Wilson's algorithm rooted at infinity yields a random oriented forest, known as the *oriented wired uniform spanning forest* [3] of G and denoted OWUSF_G. The proof of the correctness of Wilson's algorithm rooted at infinity [3], Theorem 5.1, also shows that, when G_n is an exhaustion of a transient network G, the measures $\mathsf{OUST}_{G_n^*}$ converge weakly to OWUSF_G.

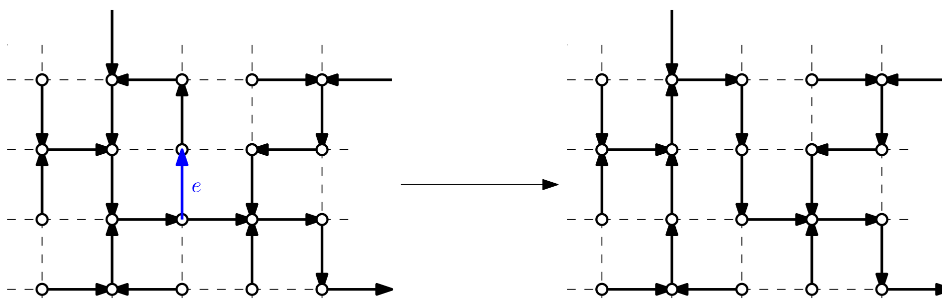
3. Wired cycle-breaking dynamics. Let G be an infinite transient network and let $\mathcal{F}(G)$ denote the set of oriented spanning forests f of G such that every vertex has exactly one oriented edge emanating from it in f. For each $f \in \mathcal{F}(G)$ and oriented edge e of G, the update $U(f,e) \in \mathcal{F}(G)$ of f is defined by the following procedure:

DEFINITION 3.1 (Updating f at e). If e or its reversal $-e$ is already included in f, or is a self-loop, let $U(f,e) = f$. Otherwise:

- If e^+ is in the past of e^- in f, so that there is a directed path $\langle e_1, \dots, e_k, d\rangle$ from e^+ to e^- in f, let
$$U(f,e) = f \cup \{-e, -e_1, \dots, -e_k\} \setminus \{d, e_k, \dots, e_1\}.$$
- Otherwise, if e^+ is not in the past of e^- in f, let d be the unique oriented edge of f with $d^- = e^-$ and let $U(f,e) = f \cup \{e\} \setminus \{d\}$.



(a) In this example, e^+ is not in the past of e^- in the forest.



(b) In this example, e^+ is in the past of e^- in the forest.

FIG. 1. *Updating an oriented spanning forest (left, solid black) of $\mathbb{Z}^2$ (dashed black) at an oriented edge e (left, blue) to obtain a new oriented spanning forest (right, solid black). Arrow heads represent orientations of edges.*

See Figure 1 for examples. Note that in either case, *as unoriented forests*, we have simply that $U(f, e) = f \cup \{e\} \setminus \{d\}$; the change in orientation in the first case ensures that every vertex has exactly one oriented edge emanating from it in $U(f, e)$, so that $U(f, e) \in \mathcal{F}(G)$.

Let v be a vertex of G. We define the *wired cycle-breaking dynamics rooted at* v to be the Markov chain on $\mathcal{F}(G)$ with transition probabilities

$$p^v(f_0, f_1) = \frac{1}{c(v)} c\bigl(\{e : e^- = v \text{ and } U(f_0, e) = f_1\}\bigr).$$

That is, we perform a step of the dynamics by choosing an oriented edge randomly from the set $\{e : e^- = v\}$ with probability proportional to its conductance, and then updating at this edge. Dynamics of this form for the UST on *finite* graphs are well known; see [11], Section 4.4.

To explain our choice of name for these dynamics, as well as our choice to consider oriented forests, let us give a second, equivalent, description of the update rule. If e or its reversal $-e$ is already included in f, or is a self-loop, let $U(f,e) = f$. Otherwise:

- If e^+ and e^- are in the same component of f, then $f \cup e$ contains a (not necessarily oriented) cycle. Break this cycle by deleting the unique edge d of f that is both contained in this cycle and adjacent to e^-, letting $\tilde{U}(f,e) = f \cup \{e\} \setminus \{d\}$.
  - If e^+ was not in the past of e^- in f, let $U(f,e) = \tilde{U}(f,e)$.
  - Otherwise, if e^+ was in the past of e^- in f, then there exists an oriented path from e^- to d^+ in $\tilde{U}(f,e)$. Let $U(f,e)$ be the oriented forest obtained by reversing each edge in this path.
- If e^+ and e^- are not in the same component of f, we consider e together with the two infinite directed paths in f beginning at e^- and e^+ to constitute a *wired cycle*, or "cycle through infinity". Break this wired cycle by deleting the unique edge d in f such that $d^- = e^-$, letting $U(f,e) = f \cup \{e\} \setminus \{d\}$.

The benefit of taking our forests to be oriented is that it allows us to define these wired cycles unambiguously. If every component of the WUSF of G is one-ended almost surely, then there is a unique infinite simple path from each of e^- and e^+ to infinity, so that wired cycles are already defined unambiguously and the update rule may be defined without reference to an orientation.

PROPOSITION 3.2. *Let G be an infinite transient network. Then for each vertex v of G, OWUSF_G is a stationary measure for the wired cycle-breaking dynamics rooted at v, that is, for $p^v(\cdot,\cdot)$.*

PROOF. Let $\langle V_n \rangle_{n\geq 1}$ be an exhaustion of G. We may assume that V_n contains v and all of its neighbours for all $n \geq 1$.

Let $\mathcal{T}(G_n^*)$ denote the set of spanning trees of G_n^* oriented towards the boundary vertex ∂_n. For each $t \in \mathcal{T}(G_n^*)$ and oriented edge e with $e^- = v$, we define the update $U(t,e)$ of t at e by the same procedure (Definition 3.1) as for $f \in \mathcal{F}(G)$.

PROPOSITION 3.3. *$U(T_n, E) \stackrel{d}{=} T_n$ for every $n \geq 1$.*

Proposition 3.3 is a slight variation on the classical *Markov Chain-Tree theorem* [2, 8, 11]: Define a Markov chain on $\mathcal{T}(G_n^*)$, as we did on $\mathcal{F}(G)$, by

$$p^v(t_0,t_1) = \frac{1}{c(v)} c\bigl(\{e : e^- = v \text{ and } U(t_0,e) = t_1\}\bigr).$$

The claimed equality in distribution is equivalent to $\mathsf{OUST}_{G_n^*}$ being a stationary measure for $p^v(\cdot,\cdot)$, and so it suffices to verify that $\mathsf{OUST}_{G_n^*}$ satisfies the detailed balance equations for $p^v(\cdot,\cdot)$. This verification, which is both straightforward and similar to that of the classical Markov Chain-Tree theorem, is omitted.

To complete the proof, we show that $U(T_n,E)$ converges to $U(F,E)$ in distribution. It might at first seem that this convergence holds trivially, but in fact some work is required: Updating F or T_n at E requires knowledge of whether or not E^+ is in the past of E^-, which cannot necessarily be obtained by observing the tree or forest only within a finite set. A priori, it is therefore possible that E^+ is in the past of E^- in T_n due to the existence of a very long oriented path from E^+ to E^- in T_n that disappears in the limit, obstructing the claimed convergence in distribution. This behaviour will be ruled out by Lemma 3.4.

By the Skorokhod representation theorem, there exist random variables $\langle T_n \rangle_{n\geq 1}$ and F, defined on some common probability space, such that T_n has law $\mathsf{OUST}_{G_n^*}$ for each n, F has law OWUSF_G, and T_n converges to F almost surely as n tends to infinity. Let E be an oriented edge chosen randomly from the set $\{e : e^- = v\}$ with probability proportional to its conductance, independently of $\langle T_n \rangle_{n\geq 1}$ and F. We write $\mathbb{P}$ for the probability measure under which $\langle T_n \rangle_{n\geq 1}$, F and E are sampled as indicated. It suffices to prove that $U(T_n, E)$ converges to $U(F, E)$ in probability with respect to $\mathbb{P}$.

Given F, let R be the length of the longest finite simple path in F connecting v to one of its neighbours in G that is in the same component as v in F. Since T_n converges to F almost surely, there exists a random N such that T_n and F coincide on the ball $B_R(v)$ of radius R about v in G for all $n \geq N$.

We claim that, with probability tending to one, F and T_n agree about whether or not E^+ is in the past of v.

LEMMA 3.4. *Consider the events*

$$\mathscr{P} = \{E^+ \text{ is in the past of } v \text{ in } F\} \quad \text{and} \quad \mathscr{P}_n = \{E^+ \text{ is in the past of } v \text{ in } T_n\}.$$

The probability of the symmetric difference $\mathscr{P} \triangle \mathscr{P}_n$ converges to zero as $n \to \infty$.

PROOF. Given E, the probability that E^+ is in the past of v in T_n is, by Wilson's algorithm, the probability that v is contained in the loop-erasure of a random walk from E^+ to ∂_n in G_n^*. Since G is transient, this probability converges to the probability that v is contained in the loop-erased random walk from E^+ in G. This probability is exactly the probability that E^+ is in the past of v in F, and so

$$\mathbb{P}(\mathscr{P}_n) \xrightarrow[n\to\infty]{} \mathbb{P}(\mathscr{P}).$$

If $\mathbb{P}(\mathscr{P}) \in \{0, 1\}$, we are done. Otherwise, on the event $\mathscr{P}$, there is by definition a finite directed path from E^+ to v in F. This directed path is also contained in T_n for all $n \geq N$ and so

$$\mathbb{P}(\mathscr{P}_n | \mathscr{P}) \xrightarrow[n\to\infty]{} 1.$$

Combining these two above limits gives

$$\mathbb{P}(\mathscr{P}_n | \neg\mathscr{P}) = \frac{\mathbb{P}(\mathscr{P}_n) - \mathbb{P}(\mathscr{P}_n | \mathscr{P})\mathbb{P}(\mathscr{P})}{\mathbb{P}(\neg\mathscr{P})} \xrightarrow[n\to\infty]{} 0$$

and hence

$$\begin{aligned}\mathbb{P}(\mathscr{P}\triangle\mathscr{P}_n) &= \mathbb{P}(\mathscr{P}) - \mathbb{P}(\mathscr{P}\cap\mathscr{P}_n) + \mathbb{P}(\mathscr{P}_n\cap\neg\mathscr{P}) \\ &= \mathbb{P}(\mathscr{P}) - \mathbb{P}(\mathscr{P}_n|\mathscr{P})\mathbb{P}(\mathscr{P}) + \mathbb{P}(\mathscr{P}_n|\neg\mathscr{P})\mathbb{P}(\neg\mathscr{P}) \\ &\xrightarrow[n\to\infty]{} \mathbb{P}(\mathscr{P}) - \mathbb{P}(\mathscr{P}) + 0 = 0. \qquad \square\end{aligned}$$

Let $r \geq 1$. Observe that on the event

$$\{T_n \text{ and } F \text{ coincide on the ball of radius } \max\{R, r\} \text{ about } v\} \setminus (\mathscr{P}\triangle\mathscr{P}_n),$$

$U(F, E)$ and $U(T_n, E)$ coincide on the ball of radius r about v. By Lemma 3.4 and the definition of $\mathbb{P}$, the probability of this event converges to 1 as $n \to \infty$, and consequently $U(T_n, E)$ converges to $U(F, E)$ in probability with respect to $\mathbb{P}$. $\square$

3.1. *Update-tolerance.* Let G be a transient network and let F be a sample of OWUSF_G. An immediate consequence of Proposition 3.2 is that for each oriented edge e of G, the law of $U(F, e)$ is absolutely continuous with respect to the law of F.

COROLLARY 3.5. *Let G be a transient network and let e be an oriented edge of G. Then for every event $\mathscr{A} \subset \mathcal{F}(G)$,*

$$\mathsf{OWUSF}_G(F \in \mathscr{A}) \geq \frac{c(e)}{c(e^-)}\mathsf{OWUSF}_G\big(U(F, e) \in \mathscr{A}\big).$$

PROOF. By Proposition 3.2,

$$\begin{aligned}\mathsf{OWUSF}_G(F \in \mathscr{A}) &= \sum_{\hat{e}^- = e^-} \frac{c(\hat{e})}{c(e^-)}\mathsf{OWUSF}_G\big(U(F, \hat{e}) \in \mathscr{A}\big) \\ &\geq \frac{c(e)}{c(e^-)}\mathsf{OWUSF}_G\big(U(F, e) \in \mathscr{A}\big). \qquad \square\end{aligned}$$

We refer to this property as *update-tolerance* by analogy to the well-established theories of insertion- and deletion-tolerant invariant percolation processes [11], Chapters 7 and 8.

4. Proof of Theorem 1.3.

PROOF. Let G be a network such that the WUSF of G contains at least two two-ended connected components with positive probability. Since G's WUSF is therefore disconnected with positive probability, Wilson's algorithm implies that G is necessarily transient. The *trunk* of a two-ended tree is defined to be the unique bi-infinite simple path contained in the tree, or equivalently the set of vertices and edges in the tree whose removal disconnects the tree into two infinite connected components.

Let F_0 be a sample of OWUSF_G. By assumption, there exists a (non-random) path $\langle \gamma_i \rangle_{i=0}^n$ in G such that, with positive probability, γ_0 and γ_n are in distinct two-ended components of F_0, γ_n is in the trunk of its component, and γ_i is not in the trunk of γ_n's component for $i < n$. Write $\mathscr{A}_\gamma$ for this event.

For each $1 \leq i \leq n$, let e_i be an edge with $e_i^- = \gamma_i$ and $e_i^+ = \gamma_{i-1}$, and let $F_i \in \mathcal{F}(G)$ be defined recursively by

$$F_i = U(F_{i-1}, e_i) \qquad \text{for } 1 \leq i \leq n.$$

We claim that on the event $\mathscr{A}_\gamma$, the component containing γ_n in the updated forest F_n has at least three ends. Applying update-tolerance (Corollary 3.5) iteratively will then imply that the probability of the WUSF containing a component with three or more ends is at least

$$\mathsf{OWUSF}_G(\mathscr{A}_\gamma) \prod_{i=1}^n \frac{c(e_i)}{c(\gamma_i)},$$

which is positive as claimed.

First, notice that γ_i's component in F_i has at least two ends for each $0 \leq i \leq n$. This may be seen by induction on i. The component of γ_0 in F_0 is two-ended by assumption, while for each $0 \leq i < n$:

- If γ_{i+1} is in the same component as γ_i in F_i, then the component containing γ_{i+1} in the updated forest F_{i+1} has the same number of ends and the same vertex set as the component of γ_i in F_i.
- If γ_{i+1} is in a different component to γ_i in F_i, then the component containing γ_{i+1} in F_{i+1} is equal to the union of the component of γ_i in F_i, the edge e_i, and the past of γ_{i+1} in F_i. Thus, the component of γ_{i+1} in F_{i+1} has at least as many ends as the component of γ_i in F_i.

This induction also shows that for every $0 \leq i \leq n$, the component of F_i containing γ_i has vertex set equal to the union of the vertices in the component of F_0 containing γ_0, and the pasts of the vertices γ_j in F_j for $0 \leq j < i$. By definition of the event $\mathscr{A}_\gamma$, the vertex γ_i is not in the trunk of γ_n's component in F_0 for any $i < n$, and so in particular γ_n is not in the past of γ_i in F_{i-1} for any $i < n$, so that γ_{n-1} and γ_n are in different components of F_{n-1}. Furthermore, since neither endpoint of e_i is contained in the trunk of γ_n's component in F_0 for any $0 \leq i \leq n-1$, the trunk of γ_n's component in F_0 is still contained in F_{n-1}. From this, we see that γ_n's component in F_n has at least three ends as claimed. See Figure 2 for an illustration. $\square$

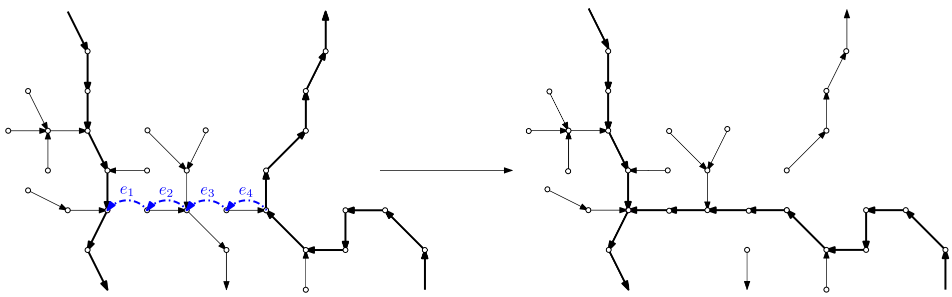


FIG. 2. *When we update along a path* (*blue arcs*) *connecting a two-ended component to the trunk of another two-ended component* (*with each edge oriented backwards*), *a three-ended component is created. Edges whose removal disconnects their component into two infinite connected components are bold.*

5. Reversible random networks and the proof of Theorem 1.1. A *rooted network* (G, ρ) is a network G together with a distinguished vertex ρ, the root. An isomorphism of graphs is an isomorphism of rooted networks if it preserves the conductances and the root. A *random rooted network* (G, ρ) is a random variable taking values in the space of isomorphism classes of random rooted networks (see [1] for precise definitions, including that of the topology on this space). Similarly, we define *doubly-rooted networks* to be networks together with an ordered pair of distinguished vertices. Let (G, ρ) be a random rooted network and let $\langle X_n \rangle_{n \geq 0}$ be simple random walk on G started at ρ. We say that (G, ρ) is *reversible* if the random doubly-rooted networks (G, ρ, X_n) and (G, X_n, ρ) have the same distribution

$$(G, \rho, X_n) \stackrel{d}{=} (G, X_n, \rho)$$

for every n, or equivalently for $n = 1$. Be careful to note that this is not the same as the reversibility of the random walk on G, which holds for any network. Reversibility is essentialy equivalent to the related property of *unimodularity*. We refer the reader to [1] for a systematic development and overview of the beautiful theory of reversible and unimodular random rooted graphs and networks, as well as many examples.

We now deduce Theorem 1.1 from Theorem 1.3. Our proof that the WUSF cannot have a unique two-ended component is adapted closely from Theorem 10.3 of [3].

PROOF OF THEOREM 1.1. Let (G, ρ) be a reversible random rooted network such that $\mathbb{E}[c(\rho)^{-1}] < \infty$. Biasing the law of (G, ρ) by the inverse conductance $c(\rho)^{-1}$ (that is, reweighting the law of (G, ρ) by the Radon–Nikodym derivative $c(\rho)^{-1}/\mathbb{E}[c(\rho)^{-1}]$) gives an equivalent unimodular random rooted network, as can be seen by checking involution invariance of the biased measure [1], Proposition 2.2. This allows us to apply Theorem 6.2 and Proposition 7.1 of [1] to deduce that every component of the WUSF of G has at most two ends almost surely.

Theorem 1.3 then implies that the WUSF of G contains at most one two-ended component almost surely.

Suppose for contradiction that the WUSF contains a single two-ended component with positive probability. Recall that the trunk of this component is defined to be the unique bi-infinite path in the component, which consists exactly of those edges and vertices whose removal disconnects the component into two infinite connected components.

Let $\langle X_n \rangle_{n\geq 0}$ be a random walk on G started at ρ, and let F be an independent random spanning forest of G with law WUSF_G, so that (since WUSF_G does not depend on the choice of exhaustion of G) the sequence $\langle (G, X_n, F)\rangle_{n\geq 0}$ is stationary. If the trunk of F is at some distance r from ρ, then X_r is in the trunk with positive probability, and it follows by stationarity that ρ is in the trunk of F with positive probability. We will show for contradiction that in fact the probability that the root is in the trunk must be zero.

Recall that, for each n, the forest F may be sampled by running Wilson's algorithm rooted at infinity, starting with the vertices ρ and X_n. If we sample F in this way and find that both ρ and X_n are contained in F's unique trunk, we must have had either that the random walk started from ρ hit X_n, or that the random walk started from X_n hit ρ. Taking a union bound,

$$\begin{aligned}\mathbb{P}(\rho \text{ and } X_n \text{ in trunk}) \leq\ & \mathbb{P}(\text{random walk started at } X_n \text{ hits } \rho)\\ &+ \mathbb{P}(\text{random walk started at } \rho \text{ hits } X_n)\end{aligned}$$

By reversibility, the two terms on the right-hand side are equal and hence

$$\mathbb{P}(\rho \text{ and } X_n \text{ in trunk}) \leq 2\mathbb{P}(\text{random walk started at } X_n \text{ hits } \rho).$$

The probability on the right-hand side is now exactly the probability that simple random walk started at ρ returns to ρ at time n or greater, and by transience this converges to zero. Thus,

$$\mathbb{P}(\rho \text{ and } X_n \text{ in trunk}) = \mathbb{E}\big[\mathbb{1}(\rho \text{ in trunk})\mathbb{1}(X_n \text{ in trunk})\big] \xrightarrow[n\to\infty]{} 0$$

and so

$$(\star) \qquad \mathbb{E}\left[\mathbb{1}(\rho \text{ in trunk})\frac{1}{n}\sum_{1}^{n}\mathbb{1}(X_i \text{ in trunk})\right] \xrightarrow[n\to\infty]{} 0.$$

Let $\mathcal{I}$ be the invariant σ-algebra of the stationary sequence $\langle (G, X_n, F)\rangle_{n\geq 0}$. The Ergodic theorem implies that

$$\frac{1}{n}\sum_{1}^{n}\mathbb{1}(X_i \text{ in trunk}) \xrightarrow[n\to\infty]{\text{a.s.}} \mathbb{P}(\rho \text{ in trunk}|\mathcal{I}).$$

Finally, combining this with ($\star$) and the Dominated Convergence theorem gives

$$\mathbb{E}\big[\mathbb{1}(\rho \text{ in trunk}) \cdot \mathbb{P}(\rho \text{ in trunk}|\mathcal{I})\big] = \mathbb{E}\big[\mathbb{P}(\rho \text{ in trunk}|\mathcal{I})^2\big] = 0.$$

It follows that $\mathbb{P}(\rho \text{ in trunk}) = 0$, contradicting our assumption that F had a unique two-ended component with positive probability. $\square$

PROOF OF COROLLARY 1.2. Given a probability distribution $\langle p_k; k \geq 0\rangle$ on $\mathbb{N}$, the *augmented Galton–Watson tree* T with offspring distribution $\langle p_k\rangle$ is defined by taking two independent Galton–Watson trees T_1 and T_2, both with offspring distribution $\langle p_k\rangle$, and then joining them by a single edge between their roots. Lyons, Pemantle and Peres [10] proved that T is reversible when rooted at the root of the first tree T_1; See also [1], Example 1.1.

If the distribution $\langle p_k\rangle$ is supercritical (i.e., has expectation greater than 1), then the associated Galton–Watson tree is infinite with positive probability and on this event is almost surely transient [11], Chapter 16. Thus, Theorem 1.1 implies that every component of T's WUSF is one-ended almost surely on the event that either T_1 or T_2 is infinite.

Recall that for every connected graph G and every edge e of G which has a positive probability of not being included in G's WUSF, the law of G's WUSF conditioned not to contain e is equal to $\mathsf{WUSF}_{G\setminus\{e\}}$ [3], Proposition 4.2, where, if $G \setminus \{e\}$ is disconnected, $\mathsf{WUSF}_{G\setminus\{e\}}$ is defined to be the union of independent samples of WUSFs of the two connected components of $G \setminus \{e\}$. Let e be the edge between the roots of T_1 and T_2 that was added to form the augmented tree T. On the positive probability event that T_1 and T_2 are both infinite, running Wilson's algorithm on T started from the roots of T_1 and T_2 shows, by transience of T_1 and T_2, that e has positive probability not to be included in T's WUSF. On this event, T's WUSF is distributed as the union of independent samples of WUSF_{T_1} and WUSF_{T_2}. It follows that every component of T_1's WUSF is one-ended almost surely on the event that T_1 is infinite. $\square$

EXAMPLE 5.1 ($\mathbb{E}[c(\rho)^{-1}] < \infty$ is necessary). Let (T, o) be a 3-regular tree with unit conductances rooted at an arbitrary vertex o. Form a network G by adjoining to each vertex v of T an infinite path, and setting the conductance of the nth edge in each of these paths to be 2^{-n-1}. Let o_n be the nth vertex in the added path at o. Define a random vertex ρ of G which is equal to o with probability $4/7$ and equal to the nth vertex in the path at o with probability $3/(7 \cdot 2^n)$ for each $n \geq 1$. The only possible isomorphism classes of (G, ρ, X_1) are of the form (G, o_n, o_{n+1}), (G, o_{n+1}, o_n), (G, o, o_1), (G, o_1, o), or (G, o, o'), where o' is a neighbour of o in T. This allows us to easily verify that (G, ρ) is a reversible random rooted network:

$$\mathbb{P}\big((G, \rho, X_1) = (G, o_n, o_{n+1})\big) = \mathbb{P}\big((G, \rho, X_1) = (G, o_{n+1}, o_n)\big) = \frac{1}{7 \cdot 2^n}$$

for all $n \geq 1$ and

$$\mathbb{P}\big((G, \rho, X_1) = (G, o, o_1)\big) = \mathbb{P}\big((G, X_1, \rho) = (G, o, o_1)\big) = \tfrac{1}{7}.$$

When we run Wilson's algorithm on G started from a vertex of T, every excursion of the random walk into one of the added paths is erased almost surely. It follows that the WUSF of G is simply the union of the WUSF of T with each of the added paths, and hence every component has infinitely many ends almost surely.

Acknowledgements. We thank Tel Aviv University for its hospitality while this work was completed. We also thank Omer Angel and Asaf Nachmias for many valuable comments on earlier versions of the paper, and thank the referee and Associate Editor for their comments and suggestions.

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