

SOBOLEV DIFFERENTIABLE FLOWS OF SDES WITH LOCAL SOBOLEV AND SUPER-LINEAR GROWTH COEFFICIENTS¹

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By establishing a characterization for Sobolev differentiability of random fields, we prove the weak differentiability of solutions to stochastic differential equations with local Sobolev and super-linear growth coefficients with respect to the starting point. Moreover, we also study the strong Feller property and the irreducibility to the associated diffusion semigroup.

1. Introduction and main results. Consider the following stochastic differential equation (SDE) in \mathbb{R}^d :

$$(1.1) \quad dX_t(x) = b(t, X_t(x)) dt + \sigma(t, X_t(x)) dW_t, \quad X_0(x) = x,$$

where $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ are two measurable functions, $(W_t)_{t \geq 0}$ is an m -dimensional standard Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is a classical result that if the coefficients are global Lipschitz continuous and have linear growth in x uniformly with respect to t , then SDE (1.1) admits a unique global strong solution which forms a stochastic flow of homeomorphisms on \mathbb{R}^d (cf. [23]). However, in many applications, the Lipschitz continuity and linear growth condition imposed on the coefficients are broken (see [11, 16, 22, 27] and references therein). Notice that in the deterministic case (i.e., $\sigma \equiv 0$), SDE (1.1) becomes an ordinary differential equation (ODE):

$$(1.2) \quad x'(t) = b(t, x(t)), \quad x(0) = x_0.$$

A unique regular Lagrangian flow was constructed in [11] by DiPerna and Lions for Sobolev vector fields with bounded divergence (see also [6] for a direct argument). This result was later extended by Ambrosio in [2] to BV vector fields with bounded divergence (see also [12, 38] and [4] for stochastic extensions). It is emphasized that the solvability of (1.2) in the DiPerna–Lions theory is only for Lebesgue almost all starting point x_0 . An interesting phenomena is that when $\sigma \neq 0$ is nondegenerate, the noise term will play some regularization effect and SDE (1.1) can be well-posed for quite singular drift b and for every starting point x .

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In the past decades, there is increasing interest in the study of the strong solutions and their properties to SDEs (1.1) with irregular coefficients. Let us briefly recall some well-known results in this direction.

In the additive noise case [i.e., $\sigma_t(x) = \sigma_t$ is nondegenerate], when b is bounded and measurable, Veretennikov [32] first proved that SDE (1.1) has a unique global strong solution $X_t(x)$. Recently, it was shown in [26] that $X_t(\cdot)$ lies in the space $\bigcap_{p \geq 1} L^2(\Omega; W_\rho^{1,p}(\mathbb{R}^d))$, where $W_\rho^{1,p}(\mathbb{R}^d)$ denotes weighted Sobolev space with weight ρ possessing finite p th moment with respect to the Lebesgue measure in \mathbb{R}^d . When $b \in L^q_{\text{loc}}(\mathbb{R}_+; L^p(\mathbb{R}^d))$ for some $p, q \in (1, \infty)$ with $\frac{d}{p} + \frac{2}{q} < 1$, using estimates of solutions to the associated PDE, the existence and uniqueness of a global strong solution $X_t(x)$ for SDE (1.1) were obtained by Krylov and Röckner in [22]. Under the same condition, Fedrizzi and Flandoli [13–15] proved that the map $x \rightarrow X_t(x)$ is α -Hölder continuous for any $\alpha \in (0, 1)$, and is also Sobolev differentiable. We also mention that the Sobolev regularity of the strong solution in spacial variable enables us to study the associated stochastic transport equation since it is closely related to SDE (1.1) by the inverse flow of the strong solution; see [15, 16, 26, 29] and references therein. Moreover, Bismut–Elworthy–Li’s formula was also established in [25] by using the Sobolev and Malliavin differentiability of strong solutions with respect to the initial values and sample paths, respectively.

In the multiplicative noise case, if the SDE is time homogeneous, supposing that $\sigma(x), b(x)$ are in $C^2(\mathbb{R}^d)$, and $\nabla \sigma$ and ∇b have some mild growth at infinity, by investigating the corresponding derivative flow equation, Li [24] studied the strong completeness for SDE (1.1), that is, $(t, x) \mapsto X_t(x)$ admits a bi-continuous version. More recently, this result was extended to the case of Sobolev coefficients in [5] and the Sobolev regularity of solutions with respect to the initial value was also studied. The main argument in [5] is the mollifying approximation to SDE (1.1) and the key point is to prove some uniform estimates of the solution to the corresponding derivative flow equation. It is emphasized that in [5, 24], $\nabla \sigma$ and ∇b are not necessarily bounded. Very recently, Zhang [34, 35, 37] proved under the assumptions that σ is bounded, uniformly elliptic and uniformly continuous in x locally uniformly with respect to t , and $|b|, |\nabla \sigma| \in L^q_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ for $q > d + 2$, there exists a unique strong solution $X_t(x)$ to (1.1) up to the explosion time $\zeta(x)$ for every $x \in \mathbb{R}^d$. Meanwhile, under the *global* assumptions that $|b|, |\nabla \sigma| \in L^q_{\text{loc}}(\mathbb{R}_+; L^q(\mathbb{R}^d))$ for $q > d + 2$, the solution $\{X_t(x)\}$ forms a stochastic flow of homeomorphisms on \mathbb{R}^d , and $x \mapsto X_t(x)$ is Sobolev differentiable.

In this paper, we will establish the Sobolev regularity of strong solutions with respect to the initial value, as well as the strong Feller property and irreducibility, to SDE (1.1) with some *local* Sobolev and super-linear growth coefficients. For this purpose, we first establish a useful characterization for Sobolev differentiability of random fields in terms of their moment estimates, which has independent interest.

THEOREM 1.1. *Let $U \subset \mathbb{R}^d$ be a bounded C^1 -domain and $f \in L^q(U; L^p(\Omega; L^r(T)))$ for some $p \in (1, \infty)$ and $q, r \in (1, \infty]$. Then $f \in \mathbb{W}^{1,q}(U; L^p(\Omega;$*

$L^r(T))$ [see (2.1) below for a definition] if and only if there exists a non-negative measurable function $g \in L^q(U)$ such that for Lebesgue-almost all $x, y \in U$,

$$(1.3) \quad \|f(x, \cdot) - f(y, \cdot)\|_{L^p(\Omega; L^r(T))} \leq |x - y|(g(x) + g(y)).$$

Moreover, if (1.3) holds, then for Lebesgue-almost all $x \in U$,

$$\|\partial_i f(x, \cdot)\|_{L^p(\Omega; L^r(T))} \leq 2g(x), \quad i = 1, \dots, d,$$

where $\partial_i f$ is the weak partial derivative of f with respect to the i th spacial variable.

The advantage of this characterization lies in that, when we want to show the Sobolev regularity of the strong solution $X_t(x)$ to SDE (1.1) with respect to x , we just need to estimate the p th moment of $X_t(x) - X_t(y)$, which is much easier to be handled for SDEs (see a recent work [33] for an application of the above characterization). It should be noticed that in all previous works (see [5, 15, 26]), the argument of mollifying coefficients is used to obtain the Sobolev differentiability of strong solutions. This usually leads to some complicated limiting procedures. Here, an interesting open question is that whether we can extend the above characterization to the infinite dimensional case in somehow so that it can be used to the SDE in Hilbert spaces as studied in [7–10].

Now, we turn to the study of SDE (1.1) and make the following assumptions on σ and b .

(H1) (*Local Sobolev integrability*). σ is locally uniformly continuous in x and locally uniformly with respect to $t \in \mathbb{R}_+$, and for some $q > d + 2$,

$$b \in L^q_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d), \quad \nabla \sigma \in L^q_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d),$$

and for some $C_1, \gamma_1 > 0, \alpha' \in (0, \alpha)$ and for all $t \geq 0, x \in \mathbb{R}^d, \xi \in \mathbb{R}^m$,

$$(1.4) \quad |\sigma(t, x)\xi| \geq |\xi|(1_{\alpha>0} \exp\{-C_1(1 + |x|^2)^{\alpha'}\} + 1_{\alpha=0}C_1(1 + |x|^2)^{-\gamma_1}),$$

where α is the same as in (1.5) below.

(H2) (*Super-linear growth*). For some $\alpha \in [0, 1]$ and for all $\kappa > 0$, there exist a constant $C_\kappa \in \mathbb{R}$ and a non-negative function $F_\kappa(t, x) \in L^{q'}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ with some $q' > d + 1$ such that for all $t \geq 0$ and $x, y \in \mathbb{R}^d$,

$$(1.5) \quad \langle x, b(t, x) \rangle + \kappa(1 + |x|^2)^\alpha \|\sigma(t, x)\|^2 \leq C_\kappa(1 + |x|^2),$$

$$(1.6) \quad \begin{aligned} &\langle x - y, b(t, x) - b(t, y) \rangle + \kappa \|\sigma(t, x) - \sigma(t, y)\|^2 \\ &\leq |x - y|^2(F_\kappa(t, x) + F_\kappa(t, y)), \end{aligned}$$

and there exist $\alpha' \in [0, \alpha), R_0 > 0$ and $C_2, \gamma_2, C_3 > 0$ such that for all $t \geq 0$ and $x \in \mathbb{R}^d$,

$$(1.7) \quad |b(t, x)| + \|\sigma(t, x)\| \leq 1_{\alpha>0} \exp\{C_2(1 + |x|^2)^{\alpha'}\} + 1_{\alpha=0}C_2(1 + |x|^2)^{\gamma_2},$$

and for all $t \geq 0$ and $|x| \geq R_0$,

$$(1.8) \quad F_\kappa(t, x) \leq C_3(1_{\alpha>0}(1 + |x|^2)^{\alpha'} + 1_{\alpha=0} \log(1 + |x|^2)).$$

Here, neither uniformly elliptic nor the global L^q -integrability conditions are assumed on b and σ . Notice that $q > d + 2$ in (H1) is almost optimal due to Krylov and Röckner’s sharp condition $\frac{d}{p} + \frac{2}{q} < 1$. Our first main result of this paper is the following.

THEOREM 1.2. *Under (H1) and (H2), there exists a unique global strong solution $X_t(x)$ to SDE (1.1) so that $(t, x) \mapsto X_t(x)$ is continuous. Moreover, we have the following conclusions:*

(A) *For each $t > 0$ and almost all ω , the mapping $x \mapsto X_t(x, \omega)$ is Sobolev differentiable, and for any $T > 0$ and $p \geq 1$, there are constants $C, \gamma > 0$ such that for Lebesgue-almost all $x \in \mathbb{R}^d$,*

$$(1.9) \quad \sup_{t \in [0, T]} \mathbb{E} |\nabla X_t(x)|^p \leq C(1_{\alpha>0} e^{(1+|x|^2)^\alpha} + 1_{\alpha=0} (1 + |x|^2)^\gamma),$$

where ∇ denotes the gradient in the distributional sense, and α is the same as in (1.5).

(B) *If in addition, we assume that for some $F_0(t, x) \in L^{q'}_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ with $q' > d + 1$,*

$$(1.10) \quad \|\sigma(t, x) - \sigma(t, y)\|^2 \leq |x - y|^2 (F_0(t, x) + F_0(t, y)),$$

where F_0 also satisfies (1.8), then (1.9) can be strengthened as

$$(1.11) \quad \mathbb{E} \left(\text{ess. sup}_{t \in [0, T]} |\nabla X_t(x)|^p \right) \leq C(1_{\alpha>0} e^{(1+|x|^2)^\alpha} + 1_{\alpha=0} (1 + |x|^2)^\gamma).$$

(C) *For each $t > 0$ and any bounded measurable function f on \mathbb{R}^d ,*

$$x \mapsto \mathbb{E} f(X_t(x)) \quad \text{is continuous.}$$

(D) *For each open set $A \subset \mathbb{R}^d$ and $t > 0, x \in \mathbb{R}^d$,*

$$\mathbb{P}\{\omega : X_t(x, \omega) \in A\} > 0.$$

REMARK 1.3. If b and σ are time independent, then the above (C) means that the semigroup defined by $P_t f(x) := \mathbb{E} f(X_t(x))$ is strong Feller, and the above (D) means that P_t is irreducible. In particular, (C) and (D) imply the uniqueness of the invariant measures associated to $(P_t)_{t \geq 0}$ (if it exists). See [3, 27] for applications.

REMARK 1.4. Assumptions (1.5) and (1.6) are classical coercivity and monotonicity conditions when $\kappa = \frac{1}{2}, \alpha = 0$ and $F_\kappa(t, x) = \text{constant}$ in (1.5) and (1.6). In this case, if, in addition that b and σ are continuous in x , then the existence

and uniqueness of strong solutions to SDE (1.1) are classical (cf. [27]). However, under nondegenerated assumption (1.4), we can drop the continuity assumption on drift b . Moreover, our estimate (1.11) is stronger than the well-known results (cf. [3, 5, 15, 26]) since the essential supremum norm with respect to the time variable is taken in the expectation.

It is well known that under (H1), SDE (1.1) admits a unique local strong solution. We will show in Lemma 3.1 below that SDE (1.1) with coefficients satisfying (1.5) does not explode and the solution has exponential integrability. In view of Theorem 1.1, to show the Sobolev regularity of the strong solution, we will pay our attention on the p th moment estimates of $X_t(x) - X_t(y)$. This is the place where assumptions (1.6)–(1.8) are needed. As in [35, 37], the estimates of Krylov’s type will play an important role throughout this paper. However, since we are assuming only some local integrability conditions and the coefficients may have exponential growth rate at infinity, some new probabilistic estimates are established (see Lemma 2.3 and Lemmas 3.3, 3.4 below).

To illustrate Theorem 1.2, we present below two examples.

EXAMPLE 1.5. Consider the following one-dimensional SDE:

$$dX_t = [(1 - X_t^5)1_{X_t < 0} - (1 + X_t^5)1_{X_t \geq 0}] dt + (1 + |X_t|^2)^\beta dW_t,$$

where $\beta \in [0, 1)$. In this case, $\sigma(x) = (1 + x^2)^\beta$ and $b(x) = (1 - x^5)1_{x < 0} - (1 + x^5)1_{x \geq 0}$ are both of super-linear growth, and the drift b has a jump at 0. Moreover, for any $\kappa > 0$, by Young’s inequality, it is easy to see that

$$\langle x, b(x) \rangle + \kappa(1 + x^2)|\sigma(x)|^2 = -x^6 - |x| + \kappa(1 + x^2)^{1+2\beta} \leq C_{\kappa, \beta}$$

and $\langle x - y, b(x) - b(y) \rangle \leq 0$,

$$\|\sigma(x) - \sigma(y)\|^2 \leq C_\beta |x - y|^2 (1 + |x|^{2(2\beta-1)\vee 0} + |y|^{2(2\beta-1)\vee 0}).$$

Thus, (H1), (H2) and (1.10) hold.

EXAMPLE 1.6. Suppose that for any $\kappa > 0$ and $T > 0$, there is a convex function $F_\kappa(x)$ such that

$$\sup_{|\xi|=1} \langle \xi, \nabla_\xi b(t, x) \rangle + \kappa \|\nabla \sigma(t, x)\|^2 \leq F_\kappa(x),$$

where $\nabla_\xi f := \langle \nabla f, \xi \rangle$ for a C^1 -function $f : \mathbb{R}^d \rightarrow \mathbb{R}$. Under this assumption, (1.6) holds. In fact, by the mean-value formula, we have

$$\begin{aligned} & (\langle x - y, b(t, x) - b(t, y) \rangle + \kappa \|\sigma(t, x) - \sigma(t, y)\|^2) / |x - y|^2 \\ & \leq \int_0^1 \left[\sup_{|\xi|=1} \langle \xi, \nabla_\xi b(t, \theta x + (1 - \theta)y) \rangle + \kappa \|\nabla \sigma(t, \theta x + (1 - \theta)y)\|^2 \right] d\theta \\ & \leq \int_0^1 F_\kappa(\theta x + (1 - \theta)y) d\theta \leq \frac{F_\kappa(x) + F_\kappa(y)}{2}, \end{aligned}$$

where the last step is due to the convexity of F_κ . Compared with [5], our assumptions (H1) and (H2) are significantly weaker.

In Theorem 1.2, the drift b is locally bounded. Our next result allows the drift b to be locally singular and of linear growth at infinity. To this aim, we make the following assumptions:

(H1') (*Local Sobolev integrability*). σ is uniformly continuous in x and locally uniformly with respect to $t \in \mathbb{R}_+$, and for some $q > 2d + 2$,

$$b \in L^q_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d), \quad \nabla \sigma \in L^q_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d),$$

and for any $T > 0$, there is a constant $K \geq 1$ such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$K^{-1}|\xi|^2 \leq |\sigma(t, x)\xi|^2 \leq K|\xi|^2 \quad \forall \xi \in \mathbb{R}^m.$$

(H2') (*Lipschitz continuity outside a ball*). For any $T > 0$, there exist $R_0 \geq 1$, $\alpha' \in [0, 1)$ and constant $C > 0$ such that for all $t \in [0, T]$,

$$|b(t, x)| \leq C(1 + |x|), \quad |x| \geq R_0,$$

and for all $t \in [0, T]$ and $|x|, |y| \geq R_0$,

$$(1.12) \quad \begin{aligned} \|\sigma(t, x) - \sigma(t, y)\| &\leq C|x - y|, \\ |b(t, x) - b(t, y)| &\leq C|x - y|(|x|^{2\alpha'} + |y|^{2\alpha'}). \end{aligned}$$

It should be noticed that conditions in (H2') are assumed to hold only outside a large ball, while b can be singular in the ball. We have the following.

THEOREM 1.7. *Under (H1') and (H2'), there exists a unique global strong solution $X_t(x)$ to SDE (1.1) so that $(t, x) \mapsto X_t(x)$ is continuous. Moreover, the conclusions (A) with (1.11), (C) and (D) in Theorem 1.2 still hold, and the α in (1.11) is taken to be 1.*

We organize this paper as follows: In Section 2, we make some preparations, and give the proof of Theorem 1.1 and a criterion on the existence of exponential moments of a Markov process. In Section 3, we provide some estimates on the solution to equation (1.1) and give the proof of Theorem 1.2. Finally, the proof of Theorem 1.7 is given in Section 4 by using Zvonkin's transformation and Theorem 1.2.

Throughout this paper, we use the following convention: C with or without subscripts will denote a positive constant, whose value may change in different places, and whose dependence on the parameters can be traced from the calculations.

2. Preliminaries. Let U be an open domain in \mathbb{R}^d . For $p \in [1, \infty]$, let $\mathbb{W}^{1,p}(U)$ be the classical first-order Sobolev space:

$$\mathbb{W}^{1,p}(U) := \{f \in L^1_{\text{loc}}(U) : \|f\|_{1,p} := \|f\|_p + \|\nabla f\|_p < +\infty\},$$

where $\|\cdot\|_p$ is the usual $L^p(U)$ -norm and ∇ denotes the gradient in the distributional sense. When U is a bounded C^1 -domain, it was proved in [18] that a function $f \in \mathbb{W}^{1,p}(U)$ if and only if $f \in L^p(U)$ and there exists a nonnegative function $g \in L^p(U)$ such that for Lebesgue-almost all $x, y \in U$,

$$|f(x) - f(y)| \leq |x - y|(g(x) + g(y)).$$

Let us now extend the above characterization to the case of random fields. For $p, q, r \in [1, \infty]$ and $T > 0$, let $L^r(T) := L^r([0, T])$ and define

$$(2.1) \quad \begin{aligned} &\mathbb{W}^{1,q}(U; L^p(\Omega; L^r(T))) \\ &:= \{f \in L^1_{\text{loc}}(U \times [0, T]; L^1(\Omega)) : f, \nabla f \in L^q(U; L^p(\Omega; L^r(T)))\} \end{aligned}$$

and

$$\|f\|_{\mathbb{W}^{1,q}(U; L^p(\Omega; L^r(T)))} := \|f\|_{L^q(U; L^p(\Omega; L^r(T)))} + \|\nabla f\|_{L^q(U; L^p(\Omega; L^r(T)))}.$$

Notice that, by Fubini's theorem,

$$L^p(U; L^p([0, T] \times \Omega)) = L^p([0, T] \times \Omega; L^p(U)),$$

and hence,

$$(2.2) \quad \mathbb{W}^{1,p}(U; L^p([0, T] \times \Omega)) = L^p([0, T] \times \Omega; \mathbb{W}^{1,p}(U)).$$

In what follows, we write $B_r := \{x \in \mathbb{R}^d : |x| < r\}$.

LEMMA 2.1. *Let $p \in (1, \infty)$, $q, r \in (1, \infty]$ and $f \in L^q(U; L^p(\Omega; L^r(T)))$. Assume that there exists a nonnegative measurable function $g \in L^q(U)$ such that for Lebesgue-almost all $x, y \in U$,*

$$(2.3) \quad \|f(x, \cdot) - f(y, \cdot)\|_{L^p(\Omega; L^r(T))} \leq |x - y|(g(x) + g(y)),$$

then $f \in \mathbb{W}^{1,q}(U; L^p(\Omega; L^r(T)))$, and for Lebesgue-almost all $x \in U$,

$$(2.4) \quad \|\partial_i f(x, \cdot)\|_{L^p(\Omega; L^r(T))} \leq 2g(x), \quad i = 1, \dots, d.$$

PROOF. Below, we always extend a function f defined on U to \mathbb{R}^d by setting $f(x, \cdot) \equiv 0$ for $x \notin U$. Let $\varrho : \mathbb{R}^d \rightarrow [0, 1]$ be a smooth function with support in B_1 and $\int \varrho \, dx = 1$. For $n \in \mathbb{N}$, define a family of mollifiers $\varrho_n(x)$ as follows:

$$(2.5) \quad \varrho_n(x) := n^d \varrho(nx).$$

Define the mollifying approximations of f and g by

$$(2.6) \quad f_n(x, t, \omega) := f(\cdot, t, \omega) * \varrho_n(x), \quad g_n(x) := g * \varrho_n(x).$$

For $\varepsilon \in (0, 1]$, set

$$U_\varepsilon := \{x \in U : \mathbf{d}(x, \partial U) > \varepsilon\},$$

where $\mathbf{d}(x, \partial U)$ denotes the distance between x and the boundary ∂U . By (2.3), it is easy to see that for any $x, y \in U_\varepsilon$ and $n > 2/\varepsilon$,

$$\begin{aligned} \|f_n(x) - f_n(y)\|_{L^p(\Omega; L^r(T))} &\leq \int_{\mathbb{R}^d} \|f(x-z) - f(y-z)\|_{L^p(\Omega; L^r(T))} \varrho_n(z) \, dz \\ (2.7) \qquad \qquad \qquad &\leq |x-y| \int_{\mathbb{R}^d} (g(x-z) + g(y-z)) \varrho_n(z) \, dz \\ &= |x-y|(g_n(x) + g_n(y)). \end{aligned}$$

Let $\{e_i, i = 1, \dots, d\}$ be the canonical basis of \mathbb{R}^d . For all $x \in U_\varepsilon$ and $n > 2/\varepsilon$, by Fatou’s lemma and (2.7), we have

$$\begin{aligned} \|\partial_i f_n(x)\|_{L^p(\Omega; L^r(T))} &= \left\| \lim_{\delta \rightarrow 0} \frac{|f_n(x + \delta e_i) - f_n(x)|}{\delta} \right\|_{L^p(\Omega; L^r(T))} \\ (2.8) \qquad \qquad \qquad &\leq \lim_{\delta \rightarrow 0} \frac{\|f_n(x + \delta e_i) - f_n(x)\|_{L^p(\Omega; L^r(T))}}{\delta} \\ &\leq \lim_{\delta \rightarrow 0} (g_n(x + \delta e_i) + g_n(x)) = 2g_n(x). \end{aligned}$$

Integrating both sides on U_ε , we obtain

$$(2.9) \quad \int_{U_\varepsilon} \|\partial_i f_n(x)\|_{L^p(\Omega; L^r(T))}^q \, dx \leq 2^q \int_{U_\varepsilon} g_n(x)^q \, dx \leq 2^q \|g\|_{L^q(U)}^q.$$

In particular, if we let $\gamma = p \wedge q \wedge r$ and $U_\varepsilon^R := U_\varepsilon \cap B_R$ for $R > 0$, then by (2.2), we have for any $R \in \mathbb{N}$,

$$\sup_n \|f_n\|_{L^\gamma([0, T] \times \Omega; \mathbb{W}^{1, \gamma}(U_\varepsilon^R))} = \sup_n \|f_n\|_{\mathbb{W}^{1, \gamma}(U_\varepsilon^R; L^\gamma([0, T] \times \Omega))} < \infty.$$

Since $\gamma \in (1, \infty)$ and $L^\gamma([0, T] \times \Omega; \mathbb{W}^{1, \gamma}(U_\varepsilon^R))$ is weakly compact and $f_n \rightarrow f$ in $L^\gamma([0, T] \times \Omega; L^\gamma(U))$, we have

$$(2.10) \quad f \in L^\gamma([0, T] \times \Omega; \mathbb{W}^{1, \gamma}(U_\varepsilon^R)).$$

By the arbitrariness of ε and R , one sees that for $dt \times \mathbb{P}(d\omega)$ -almost all (t, ω) , $x \mapsto f(x, t, \omega)$ is weakly differentiable in U , and for all $x \in U_\varepsilon$ and $n > 2/\varepsilon$,

$$\partial_i f_n(x, t, \omega) = \partial_i f(\cdot, t, \omega) * \varrho_n(x),$$

which, by (2.10) and the property of convolutions, then implies that

$$\lim_{n \rightarrow \infty} \|\partial_i f_n - \partial_i f\|_{L^\gamma(U_\varepsilon^R \times [0, T] \times \Omega)} = 0.$$

Thus, for some subsequence n_k and $dx \times dt \times \mathbb{P}(d\omega)$ -almost all $(x, t, \omega) \in U_\varepsilon^R \times [0, T] \times \Omega$,

$$\partial_i f_{n_k}(x, t, \omega) \rightarrow \partial_i f(x, t, \omega).$$

Now, by (2.8) and Fatou’s lemma, we obtain that for Lebesgue-almost all $x \in U_\varepsilon^R$,

$$\|\partial_i f(x)\|_{L^p(\Omega; L^r(T))} \leq \liminf_{k \rightarrow \infty} \|\partial_i f_{n_k}(x)\|_{L^p(\Omega; L^r(T))} \leq 2 \lim_{k \rightarrow \infty} g_{n_k}(x) = 2g(x).$$

The proof is complete by the arbitrariness of ε and R . \square

We also have the following converse result.

LEMMA 2.2. *Let $f \in \mathbb{W}_{loc}^{1,q}(\mathbb{R}^d; L^p(\Omega; L^r(T)))$ for some $p, q, r \in (1, \infty]$. For any $R > 0$, there exists a measurable function $g_R \in L^q_{loc}(\mathbb{R}^d)$ such that for Lebesgue-almost all $x, y \in \mathbb{R}^d$ with $|x - y| < R$,*

$$(2.11) \quad \|f(x, \cdot) - f(y, \cdot)\|_{L^p(\Omega; L^r(T))} \leq |x - y|(g_R(x) + g_R(y)).$$

Moreover, if $f \in \mathbb{W}^{1,q}(\mathbb{R}^d; L^p(\Omega; L^r(T)))$, then R can be ∞ and $g_\infty \in L^q(\mathbb{R}^d)$.

PROOF. Let f_n be the mollifying approximation of f as in (2.6). By [38], Lemma 3.5, we have

$$\begin{aligned} |f_n(x, t, \omega) - f_n(y, t, \omega)| &\leq 2^d \int_0^{|x-y|} \int_{B_s} |\nabla f_n(x+z, t, \omega)| \, dz \, ds \\ &\quad + 2^d \int_0^{|x-y|} \int_{B_s} |\nabla f_n(y+z, t, \omega)| \, dz \, ds. \end{aligned}$$

Hence, for all $x, y \in \mathbb{R}^d$ with $|x - y| < R$,

$$\begin{aligned} \|f_n(x) - f_n(y)\|_{L^p(\Omega; L^r(T))} &\leq 2^d \int_0^{|x-y|} \int_{B_s} \|\nabla f_n(x+z)\|_{L^p(\Omega; L^r(T))} \, dz \, ds \\ (2.12) \quad &\quad + 2^d \int_0^{|x-y|} \int_{B_s} \|\nabla f_n(y+z)\|_{L^p(\Omega; L^r(T))} \, dz \, ds \\ &\leq 2^d |x - y|(g_R^n(x) + g_R^n(y)), \end{aligned}$$

where

$$g_R^n(x) := \mathcal{M}_R \|\nabla f_n\|_{L^p(\Omega; L^r(T))}(x) := \sup_{s \in (0, R)} \int_{B_s} \|\nabla f_n(x+z)\|_{L^p(\Omega; L^r(T))} \, dz$$

is the local maximal function of $x \mapsto \|\nabla f_n(x)\|_{L^p(\Omega; L^r(T))}$. Notice that

$$\begin{aligned} \mathcal{M}_R \|\nabla f_n\|_{L^p(\Omega; L^r(T))}(x) &\leq \sup_{s \in (0, R)} \int_{B_s} \|\nabla f\|_{L^p(\Omega; L^r(T))} * \varrho_n(x+z) \, dz \\ &\leq \mathcal{M}_R \|\nabla f\|_{L^p(\Omega; L^r(T))} * \varrho_n(x), \end{aligned}$$

and $\mathcal{M}_R \|\nabla f\|_{L^p(\Omega; L^r(T))} \in L^q_{\text{loc}}(\mathbb{R}^d)$ by the property of maximal functions (cf. [30]). By taking limits for both sides of

$$\begin{aligned} & \|f_n(x) - f_n(y)\|_{L^p(\Omega; L^r(T))} \\ & \leq 2^d |x - y| (\mathcal{M}_R \|\nabla f\|_{L^p(\Omega; L^r(T))} * \varrho_n(x) \\ & \quad + \mathcal{M}_R \|\nabla f\|_{L^p(\Omega; L^r(T))} * \varrho_n(y)), \end{aligned}$$

we obtain that for Lebesgue-almost all $x, y \in \mathbb{R}^d$ with $|x - y| < R$,

$$(2.13) \quad \begin{aligned} & \|f(x) - f(y)\|_{L^p(\Omega; L^r(T))} \\ & \leq 2^d |x - y| (\mathcal{M}_R \|\nabla f\|_{L^p(\Omega; L^r(T))}(x) + \mathcal{M}_R \|\nabla f\|_{L^p(\Omega; L^r(T))}(y)). \end{aligned}$$

The proof is complete. \square

Combining Lemma 2.1 with Lemma 2.2, we can give the following.

PROOF OF THEOREM 1.1. The sufficiency follows by Lemma 2.1. Let

$$f \in \mathbb{W}^{1,q}(U; L^p(\Omega; L^r(T))).$$

Since U is a bounded C^1 -domain, there exists an extension operator (cf. [1], page 151, Theorem 5.22 or [30], Chapter VI)

$$\mathbb{T} : \mathbb{W}^{1,q}(U; L^p(\Omega; L^r(T))) \rightarrow \mathbb{W}^{1,q}(\mathbb{R}^d; L^p(\Omega; L^r(T)))$$

such that $\mathbb{T}f = f$ restricted on U and

$$\|\mathbb{T}f\|_{\mathbb{W}^{1,q}(\mathbb{R}^d; L^p(\Omega; L^r(T)))} \leq C \|f\|_{\mathbb{W}^{1,q}(U; L^p(\Omega; L^r(T)))}.$$

Thus, (1.3) follows by (2.11). \square

We also need the following local version of Khasminskii’s estimate (see [31], Lemma 2.1).

LEMMA 2.3. *Let $(\Omega, \mathcal{F}, (\mathbb{P}_x)_{x \in \mathbb{R}^d}; (X_t)_{t \geq 0})$ be a family of \mathbb{R}^d -valued time-homogenous Markov process. Let f be a nonnegative measurable function over \mathbb{R}^d . For given $T, R > 0$, if*

$$(2.14) \quad \sup_{|x| \leq R} \mathbb{E}_x \left(\int_0^T f(X_t) 1_{|X_t| \leq R} dt \right) =: c < 1,$$

where \mathbb{E}_x denotes the expectation with respect to \mathbb{P}_x , then for all $x \in \mathbb{R}^d$,

$$(2.15) \quad \mathbb{E}_x \exp \left\{ \int_0^T f(X_t) 1_{|X_t| \leq R} dt \right\} \leq 1 + \frac{1}{1-c} \mathbb{E}_x \left(\int_0^T f(X_t) 1_{|X_t| \leq R} dt \right).$$

PROOF. Set $f_R(x) := f(x)1_{|x| \leq R}$. By Taylor’s expansion, we can write

$$\mathbb{E}_x \exp \left\{ \int_0^T f_R(X_t) dt \right\} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_x \left(\int_0^T f_R(X_t) dt \right)^n.$$

For $n \in \mathbb{N}$, noticing that

$$\left(\int_0^T g(t) dt \right)^n = n! \int \cdots \int_{\Delta_T^n} g(t_1) \cdots g(t_n) dt_1 \cdots dt_n,$$

where

$$\Delta_T^n := \{(t_1, \dots, t_n) : 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T\},$$

we further have

$$\begin{aligned} & \mathbb{E}_x \exp \left\{ \int_0^T f_R(X_t) dt \right\} \\ &= 1 + \sum_{n=1}^{\infty} \mathbb{E}_x \left(\int \cdots \int_{\Delta_T^n} f_R(X_{t_1}) \cdots f_R(X_{t_n}) dt_1 \cdots dt_n \right) \\ &= 1 + \sum_{n=1}^{\infty} \mathbb{E}_x \left(\int \cdots \int_{\Delta_T^{n-1}} f_R(X_{t_1}) \cdots f_R(X_{t_{n-1}}) \right. \\ &\quad \left. \times \mathbb{E}_{X_{t_{n-1}}} \int_0^{T-t_{n-1}} f_R(X_{t_n}) dt_n dt_1 \cdots dt_{n-1} \right) \\ &\stackrel{(2.14)}{\leq} 1 + \sum_{n=1}^{\infty} c \mathbb{E}_x \left(\int \cdots \int_{\Delta_T^{n-1}} f_R(X_{t_1}) \cdots f_R(X_{t_{n-1}}) dt_1 \cdots dt_{n-1} \right) \\ &\leq \cdots \\ &\leq 1 + \sum_{n=1}^{\infty} c^{n-1} \mathbb{E}_x \left(\int_0^T f_R(X_{t_1}) dt_1 \right) = 1 + \frac{1}{1-c} \mathbb{E}_x \left(\int_0^T f_R(X_t) dt \right), \end{aligned}$$

where the second equality is due to the Markov property of X_t . \square

Finally, we recall the following Krylov estimate about the distributions of continuous semimartingales (cf. [20] or [17], Lemma 3.1).

LEMMA 2.4. *Let $m = m_t$ be a continuous \mathbb{R}^d -valued local martingale, and $V = V_t$ a continuous \mathbb{R}^d -valued process with finite variation on finite time intervals. Suppose that*

$$m(0) = V(0) = 0, \quad d\langle m \rangle_t \ll dt,$$

and set

$$a(t) := \frac{d\langle m \rangle_t}{2 dt}, \quad X(t) := m(t) + V(t).$$

For any $\lambda > 0$, stopping time τ and nonnegative Borel function $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$, we have

$$(2.16) \quad \begin{aligned} & \mathbb{E} \int_0^\tau e^{-\lambda t} (\det a(t))^{1/(d+1)} f(t, X_t) dt \\ & \leq N_{d,\lambda} (\mathbb{V}^2 + \mathbb{A})^{d/(2(d+1))} \left(\int_0^\infty \int_{\mathbb{R}^d} f^{d+1}(t, x) dx dt \right)^{1/(d+1)}, \end{aligned}$$

where

$$(2.17) \quad \mathbb{V} := \mathbb{E} \int_0^\tau e^{-\lambda t} |dV(t)|, \quad \mathbb{A} := \mathbb{E} \int_0^\tau e^{-\lambda t} \text{tr} a(t) dt,$$

and $N_{d,\lambda}$ is a constant depending only on d and λ .

3. Proof of Theorem 1.2. Below we write

$$\mathcal{L}_s^{\sigma,b} f(x) := \frac{1}{2} \sum_{ijk} \sigma_{ik}(s, x) \sigma_{jk}(s, x) \partial_i \partial_j f(x) + \sum_i b_i(s, x) \partial_i f(x).$$

Under (H1), it has been proven in [34, 37] that SDE (1.1) admits a unique local strong solution. The following lemma gives the nonexplosion and the exponential integrability of $X_t(x)$ under (1.5) (see also [36]).

LEMMA 3.1. *Let $X_t(x)$ be the unique local solution of (1.1) with starting point x . Under (H1) and (1.5), there is a unique global solution $X_t(x)$ to a SDE (1.1). Moreover, let $\alpha \in [0, 1]$ and $\kappa \mapsto C_\kappa \in \mathbb{R}$ be as in (1.5).*

- ($\alpha > 0$) For any $\lambda \geq 2\alpha C_{\alpha+1}$ and for all $t \geq 0$ and $x \in \mathbb{R}^d$, we have

$$(3.1) \quad \mathbb{E} \exp\{e^{-\lambda t} (1 + |X_t(x)|^2)^\alpha\} \leq \exp\{(1 + |x|^2)^\alpha\}.$$

- ($\alpha = 0$) For any $p \geq 1$ and $\lambda \geq C_p$ and for all $t \geq 0$ and $x \in \mathbb{R}^d$, we have

$$(3.2) \quad \mathbb{E}(1 + |X_t(x)|^2)^p \leq e^{\lambda t} (1 + |x|^2)^p.$$

PROOF. We only consider the case of $\alpha > 0$. For $\alpha = 0$, it is similar. For $R > 0$, define

$$(3.3) \quad \tau_R := \inf\{t \geq 0 : |X_t(x)| \geq R\},$$

and for $\lambda \in \mathbb{R}$,

$$(3.4) \quad f(t, x) := \exp\{e^{-\lambda t} (1 + |x|^2)^\alpha\}.$$

By Itô's formula, we have

$$\mathbb{E}f(t \wedge \tau_R, X_{t \wedge \tau_R}) = f(0, x) + \mathbb{E}\left(\int_0^{t \wedge \tau_R} (\partial_s f + \mathcal{L}_s^{\sigma, b} f)(s, X_s) ds\right).$$

Notice that

$$\begin{aligned} & (\partial_s f + \mathcal{L}_s^{\sigma, b} f)(s, x) \\ &= \alpha(1 + |x|^2)^\alpha e^{-\lambda s} f(s, x) \left(-\frac{\lambda}{\alpha} + \frac{2\langle b, x \rangle + \|\sigma\|^2}{1 + |x|^2}\right) \\ (3.5) \quad & + 2 \sum_{ijk} \sigma_{ik} \sigma_{jk} \left[\alpha(1 + |x|^2)^{\alpha-2} e^{-\lambda s} + \frac{\alpha - 1}{(1 + |x|^2)^2}\right] x_i x_j \\ & \leq \alpha(1 + |x|^2)^\alpha e^{-\lambda s} f(s, x) \left(-\frac{\lambda}{\alpha} + \frac{2\langle b, x \rangle + 2(\alpha + 1)(1 + |x|^2)^\alpha \|\sigma\|^2}{1 + |x|^2}\right) \\ & \stackrel{(1.5)}{\leq} \alpha(1 + |x|^2)^\alpha e^{-\lambda s} f(s, x) \left(-\frac{\lambda}{\alpha} + 2C_{\alpha+1}\right). \end{aligned}$$

Hence, if $\lambda \geq 2\alpha C_{\alpha+1}$, then

$$\mathbb{E}f(t \wedge \tau_R, X_{t \wedge \tau_R}) \leq f(0, x).$$

By letting $R \rightarrow \infty$, one sees that $\tau_\infty = \infty$, that is, no explosion, and (3.1) holds. □

The following global Krylov estimate is an easy consequence of Lemmas 2.4 and 3.1.

LEMMA 3.2. *Under (H1), (1.5) and (1.7), for any $q > d + 1$ and $T > 0$, there exist constants $C, \gamma > 0$ such that for all nonnegative $f \in L^q([0, T] \times \mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned} & \mathbb{E}\left(\int_0^T f(t, X_t(x)) dt\right) \\ & \leq C(1_{\alpha>0} e^{(1+|x|^2)^\alpha} + 1_{\alpha=0}(1 + |x|^2)^\gamma) \left(\int_0^T \int_{\mathbb{R}^d} f^q(t, y) dy dt\right)^{1/q}. \end{aligned}$$

PROOF. In Lemma 2.4, let us take

$$m(t) := \int_0^t \sigma(s, X_s) dW_s, \quad V(t) := \int_0^t b(s, X_s) ds,$$

so that

$$a(t) = \frac{d\langle m \rangle_t}{2 dt} = \frac{1}{2}(\sigma\sigma^*)(t, X_t).$$

By Hölder’s inequality and Lemma 2.4, we have

$$\begin{aligned}
 \mathbb{E}\left(\int_0^T f(t, X_t(x)) dt\right) &= \mathbb{E}\int_0^T f(t, X_t(x))(\det a(t))^{1/q}(\det a(t))^{-1/q} dt \\
 &\leq \left(\mathbb{E}\int_0^T f^{q/(d+1)}(t, X_t(x))(\det a(t))^{1/(d+1)} dt\right)^{(d+1)/q} \\
 (3.6) \quad &\quad \times \left(\mathbb{E}\int_0^T (\det a(t))^{-1/(q-d-1)} dt\right)^{(q-d-1)/q} \\
 &\leq C_{T,d}(\mathbb{V}^2 + \mathbb{A})^{d/(2q)}\left(\int_0^T \int_{\mathbb{R}^d} f^q(t, x) dx dt\right)^{1/q} \\
 &\quad \times \left(\mathbb{E}\int_0^T (\det a(t))^{-1/(q-d-1)} dt\right)^{(q-d-1)/q},
 \end{aligned}$$

where \mathbb{A} and \mathbb{V} are defined by (2.17). By (1.7), (3.1) and Young’s inequality, we have

$$\begin{aligned}
 \mathbb{A} + \mathbb{V}^2 &\leq C\mathbb{E}\int_0^T (|b(t, X_t)|^2 + \|\sigma(t, X_t)\|^2) dt \\
 &\leq C\mathbb{E}\int_0^T (1_{\alpha>0}\exp\{2C_2(1 + |X_t|^2)^{\alpha'}\} + 1_{\alpha=0}C_2^2(1 + |X_t|^2)^{2\gamma_2}) dt \\
 (3.7) \quad &\leq C\int_0^T (1_{\alpha>0}\mathbb{E}\exp\{e^{-\lambda t}(1 + |X_t|^2)^\alpha\} + 1_{\alpha=0}\mathbb{E}(1 + |X_t|^2)^{2\gamma_2}) dt \\
 &\leq C(1_{\alpha>0}e^{(1+|x|^2)^\alpha} + 1_{\alpha=0}(1 + |x|^2)^{2\gamma_2}).
 \end{aligned}$$

Similarly, by (1.4), we have for some $\gamma_3 > 0$,

$$(3.8) \quad \mathbb{E}\int_0^T (\det a(t))^{-1/(q-d-1)} dt \leq C(1_{\alpha>0}e^{(1+|x|^2)^\alpha} + 1_{\alpha=0}(1 + |x|^2)^{\gamma_3}).$$

Substituting (3.7) and (3.8) into (3.6), we obtain the desired estimate. \square

Taking into account Lemma 2.3, we can prove the following global-exponential moment estimate of Krylov’s type, which will play a crucial role in the proof of Theorem 1.2. Since f is allowed to be singular in a ball and of linear growth at infinity, we need to separately consider the interior and exterior parts of a ball by using Lemmas 2.3, 3.1 and 3.2.

LEMMA 3.3. *For given $q > d + 1$, let $f \in L^q_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ be a nonnegative measurable function. Let α be as in (1.5). Suppose that for some $R_0, C_0 > 0$ and $\alpha' \in [0, \alpha)$,*

$$(3.9) \quad f(t, x)1_{\{|x|>R_0\}} \leq C_0[1_{\alpha>0}(1 + |x|^2)^{\alpha'} + 1_{\alpha=0}\log(1 + |x|^2)].$$

Under (H1), (1.5) and (1.7), for any $T > 0$, there are $C, \gamma > 0$ such that for all $x \in \mathbb{R}^d$,

$$(3.10) \quad \mathbb{E} \exp \left\{ \int_0^T f(t, X_t(x)) dt \right\} \leq C(1_{\alpha>0} e^{(1+|x|^2)^\alpha} + 1_{\alpha=0}(1 + |x|^2)^\gamma).$$

PROOF. Set

$$f_{R_0}(t, x) := f(t, x)1_{|x| \leq R_0}, \quad \bar{f}_{R_0}(t, x) := f(t, x)1_{|x| > R_0}.$$

By Hölder’s inequality, we have

$$(3.11) \quad \begin{aligned} \left(\mathbb{E} \exp \left\{ \int_0^T f(t, X_t(x)) dt \right\} \right)^2 &\leq \mathbb{E} \exp \left\{ 2 \int_0^T \bar{f}_{R_0}(t, X_t(x)) dt \right\} \\ &\times \mathbb{E} \exp \left\{ 2 \int_0^T f_{R_0}(t, X_t(x)) dt \right\} \\ &=: I_1(T, x) \times I_2(T, x). \end{aligned}$$

For $I_1(T, x)$, by (3.9), Jensen’s inequality and Lemma 3.1, we have

$$(3.12) \quad \begin{aligned} I_1(T, x) &\leq \mathbb{E} \exp \left\{ 2C_0 \int_0^T [1_{\alpha>0}(1 + |X_t(x)|^2)^{\alpha'} + 1_{\alpha=0} \log(1 + |X_t(x)|^2)] dt \right\} \\ &\leq \frac{1}{T} \int_0^T \mathbb{E} \exp \{ 2C_0 T [1_{\alpha>0}(1 + |X_t(x)|^2)^{\alpha'} \\ &\quad + 1_{\alpha=0} \log(1 + |X_t(x)|^2)] \} dt \\ &\leq \frac{C}{T} \int_0^T (1_{\alpha>0} \mathbb{E} \exp \{ e^{-\lambda t} (1 + |X_t(x)|^2)^\alpha \} \\ &\quad + 1_{\alpha=0} \mathbb{E}(1 + |X_t(x)|^2)^{2C_0 T}) dt \\ &\leq C(1_{\alpha>0} \exp \{ (1 + |x|^2)^\alpha \} + 1_{\alpha=0}(1 + |x|^2)^{2C_0 T}), \end{aligned}$$

where λ is the same as in (3.1), and the third inequality is due to Young’s inequality. For $I_2(T, x)$, for any $\varepsilon \in (0, 1)$ and $\theta > 1$, by Young’s inequality we have

$$(3.13) \quad I_2(T, x) \leq e^{C\varepsilon} \mathbb{E} \exp \left\{ \varepsilon \int_0^T f_{R_0}(t, X_t(x))^\theta dt \right\}.$$

Let us choose $\theta > 1$ so that $\frac{\theta}{\theta-1} > d + 1$. Then by Lemma 3.2, we have

$$\varepsilon \sup_{|x| \leq R_0} \mathbb{E} \left(\int_0^T f_{R_0}(t, X_t(x))^\theta dt \right) \leq \varepsilon C_{R_0} \left(\int_0^T \int_{|y| < R_0} f(t, y)^\theta dy dt \right)^{\theta/q} =: c_\varepsilon.$$

Since $(t, X_t(x))$ is a time-homogenous Markov process in $\mathbb{R}_+ \times \mathbb{R}^d$, if we choose ε being small enough so that $c_\varepsilon < 1$, then by (2.15), we obtain

$$\mathbb{E} \exp \left\{ \varepsilon \int_0^T f_{R_0}(t, X_t(x))^\theta dt \right\} \leq 1 + \frac{\varepsilon}{1 - c_\varepsilon} \mathbb{E} \left(\int_0^T f_{R_0}(t, X_t(x))^\theta dt \right),$$

which, together with (3.11), (3.12), (3.13) and Lemma 3.2, yields the desired estimate. \square

We also need the following local-exponential moment estimate of Krylov’s type, which is a consequence of Lemma 2.3 and [37], Theorem 2.1, (see also [34], Theorem 4.1). In particular, the integrability index q in the following lemma can be smaller than the one in Lemma 3.3.

LEMMA 3.4. *For $R \geq 1$, let τ_R be defined by (3.3). Under (H1) and (1.7), for any $q > \frac{d}{2} + 1$ and $T > 0$, there exists a constant $C_R > 0$ such that for all $f \in L^q_{\text{loc}}(\mathbb{R}^{d+1})$ and $|x| < R$,*

$$(3.14) \quad \mathbb{E} \exp \left\{ \int_0^{T \wedge \tau_R} f(t, X_t(x)) dt \right\} < \infty.$$

PROOF. Let χ_R be a smooth cutoff function with $\chi_R(x) = 1$ for $|x| \leq R$ and $\chi_R = 0$ for $|x| \geq R + 1$, and set

$$b_R(t, x) = b(t, x)\chi_R(x), \quad \sigma_R(t, x) := \sigma(t, \chi_R(x)x).$$

By (H1) and (1.7), it is easy to see that for some $C_R > 0$ and $\alpha \in (0, 1)$,

$$\begin{aligned} |b_R(t, x)| &\leq C_R, & C_R^{-1}|\xi| &\leq |\sigma_R(t, x)\xi| \leq C_R|\xi|, \\ |\sigma_R(t, x) - \sigma_R(t, y)| &\leq C_R|x - y|^\alpha. \end{aligned}$$

Let $X_t^R(x)$ solve SDE (1.1) with (b_R, σ_R) in place of (b, σ) . By the local uniqueness, one has

$$X_t^R(x) = X_t(x), \quad t < \tau_R.$$

Hence, letting $\theta > 1$ so that $\frac{q}{\theta} > \frac{d}{2} + 1$, by [37], Theorem 2.1, we have for all $|x| < R$,

$$\begin{aligned} \mathbb{E} \left(\int_0^{T \wedge \tau_R} |f(t, X_t(x))|^\theta dt \right) &= \mathbb{E} \left(\int_0^T |f(t, X_t^R(x))|^\theta 1_{t < \tau_R} dt \right) \\ &\leq \mathbb{E} \left(\int_0^T |f(t, X_t^R(x))|^\theta 1_{|X_t^R(x)| < R} dt \right) \\ &\leq C_R \left(\int_0^T \int_{B_R} |f(t, y)|^q dy dt \right)^{\theta/q}. \end{aligned}$$

Thus, using the same technique as in the proof of Lemma 3.3 and by (2.15), we get (3.14). \square

The following lemma will be used in the proof of irreducibility.

LEMMA 3.5. For given $x_0, y_0 \in \mathbb{R}^d$ and $m \geq 1$, let Y_t solve the following SDE:

$$(3.15) \quad dY_t = -m(Y_t - y_0) dt + b(t, Y_t) dt + \sigma(t, Y_t) dW_t, \quad Y_0 = x_0.$$

Under (H1) and (H2), for any $T > 0$, there exist constants $C_0, C_1 > 0$ such that for all $t \in [0, T]$ and $m \geq 1$,

$$(3.16) \quad \mathbb{E}|Y_t - y_0|^2 \leq C_0 e^{-mt} |x_0 - y_0|^2 + \frac{C_1}{\sqrt{m}}$$

and

$$(3.17) \quad \mathbb{E}\left(\sup_{t \in [0, T]} |Y_t|^2\right) < \infty.$$

PROOF. Let $\tilde{b}(t, x) := -m(x - y_0) + b(t, x)$ and f be as in (3.4). As in the calculations of (3.5), we have

$$\begin{aligned} &(\partial_s f + \mathcal{L}_s^{\sigma, \tilde{b}} f)(s, x) \\ &\leq \alpha(1 + |x|^2)^\alpha e^{-\lambda s} f(s, x) \left(-\frac{\lambda}{\alpha} + \frac{2\langle \tilde{b}, x \rangle + 2(\alpha + 1)(1 + |x|^2)^\alpha \|\sigma\|^2}{1 + |x|^2}\right) \\ &\stackrel{(1.5)}{\leq} \alpha(1 + |x|^2)^\alpha e^{-\lambda s} f(s, x) \left(-\frac{\lambda}{\alpha} + 2C_{\alpha+1} + \frac{2m(|x| \cdot |y_0| - |x|^2)}{1 + |x|^2}\right). \end{aligned}$$

If $|x| \leq |y_0|$, then

$$\begin{aligned} &(\partial_s f + \mathcal{L}_s^{\sigma, \tilde{b}} f)(s, x) \\ &\leq \alpha(1 + |y_0|^2)^\alpha e^{-\lambda s} \exp(e^{-\lambda s} (1 + |y_0|^2)^\alpha) \{2C_{\alpha+1} + 2m|y_0|^2\}. \end{aligned}$$

If $|x| > |y_0|$ and choose $\lambda > 2\alpha C_{\alpha+1}$, then

$$(\partial_s f + \mathcal{L}_s^{\sigma, \tilde{b}} f)(s, x) \leq 0.$$

Hence,

$$(3.18) \quad \mathbb{E} \exp\{e^{-\lambda t} (1 + |Y_t|^2)^\alpha\} = \mathbb{E} f(t, Y_t) \leq f(0, x_0) + C(y_0)(1 + m)t.$$

On the other hand, by Itô's formula, we have for all $t \in [0, T]$

$$\begin{aligned} \frac{d\mathbb{E}|Y_t - y_0|^2}{dt} &= -2m\mathbb{E}|Y_t - y_0|^2 + 2\mathbb{E}\langle Y_t - y_0, b(t, Y_t) \rangle + \mathbb{E}\|\sigma(t, Y_t)\|^2 \\ &\stackrel{(1.5)}{\leq} -2m\mathbb{E}|Y_t - y_0|^2 + 2C_{1/2}(1 + \mathbb{E}|Y_t|^2) + 2|y_0|\mathbb{E}|b(t, Y_t)| \end{aligned}$$

$$\begin{aligned} &\stackrel{(1.7)}{\leq} 2(C_{1/2} - m)\mathbb{E}|Y_t - y_0|^2 + C + (\mathbb{E} \exp\{C(1 + |Y_t|^2)^{\alpha'}\})^{1/2} \\ &\stackrel{(3.18)}{\leq} 2(C_{1/2} - m)\mathbb{E}|Y_t - y_0|^2 + C\sqrt{m}, \end{aligned}$$

where $C = C(T, x_0, y_0)$ is independent of m . By Gronwall’s inequality, we have

$$\mathbb{E}|Y_t - y_0|^2 \leq e^{2(C_2 - m)t} |x_0 - y_0|^2 + C\sqrt{m}e^{2(C_2 - m)t} \int_0^t e^{2(m - C_2)s} ds,$$

which then gives (3.16). As for (3.17), it follows by (3.15), (1.7) and (3.18). \square

We are now in a position to give:

PROOF OF THEOREM 1.2. For any $p \geq 2$ and $T > 0$, by (1.7) and Burkholder’s inequality, we have for all $0 \leq s \leq t \leq T$,

$$\begin{aligned} &\mathbb{E}|X_t(x) - X_s(x)|^p \\ &\leq C\mathbb{E}\left(\int_s^t |b(r, X_r(x))| dr\right)^p + C\mathbb{E}\left|\int_s^t \sigma(r, X_r(x)) dW_r\right|^p \\ &\leq C\mathbb{E}\left(\int_s^t |b(r, X_r(x))|^2 dr\right)^{p/2} + C\mathbb{E}\left(\int_s^t \|\sigma(r, X_r(x))\|^2 dr\right)^{p/2} \\ (3.19) \quad &\leq C(t - s)^{p/2 - 1} \mathbb{E} \int_s^t (|b(r, X_r(x))|^p + \|\sigma(r, X_r(x))\|^p) dr \\ &\leq C(t - s)^{p/2 - 1} \mathbb{E} \int_s^t (1_{\alpha > 0} \exp\{C(1 + |X_r(x)|^2)^{\alpha'}\} \\ &\quad + 1_{\alpha = 0}(1 + |X_r(x)|^2)^{p\gamma/2}) dr \\ &\leq C(t - s)^{p/2} (1_{\alpha > 0} \exp\{(1 + |x|^2)^\alpha\} + 1_{\alpha = 0}(1 + |x|^2)^{p\gamma/2}), \end{aligned}$$

where the last step is due to $\alpha' \in [0, \alpha)$, Young’s inequality and Lemma 3.1.

Next, set $Z_t := X_t(x) - X_t(y)$. For any $p \geq 1$, by Itô’s formula we have

$$\begin{aligned} |Z_t|^{2p} &= |x - y|^{2p} + 2p \int_0^t |Z_s|^{2(p-1)} \langle Z_s, [\sigma(s, X_s(x)) - \sigma(s, X_s(y))] dW_s \rangle \\ &\quad + 2p \int_0^t |Z_s|^{2(p-1)} \langle Z_s, [b(s, X_s(x)) - b(s, X_s(y))] \rangle ds \\ (3.20) \quad &+ 2p \int_0^t |Z_s|^{2(p-1)} \|\sigma(s, X_s(x)) - \sigma(s, X_s(y))\|^2 ds \\ &\quad + 2p(p - 1) \int_0^t |Z_s|^{2(p-2)} |[\sigma(s, X_s(x)) - \sigma(s, X_s(y))]^* Z_s|^2 ds \\ &=: |x - y|^{2p} + \int_0^t |Z_s|^{2p} (\xi(s) dW_s + \eta(s) ds), \end{aligned}$$

where

$$\xi(s) := \frac{2p[\sigma(s, X_s(x)) - \sigma(s, X_s(y))]^* Z_s}{|Z_s|^2}$$

and

$$\begin{aligned} \eta(s) := & \frac{2p\langle Z_s, [b(s, X_s(x)) - b(s, X_s(y))] \rangle}{|Z_s|^2} \\ & + \frac{2p\|\sigma(s, X_s(x)) - \sigma(s, X_s(y))\|^2}{|Z_s|^2} \\ & + \frac{2p(p-1)[\sigma(s, X_s(x)) - \sigma(s, X_s(y))]^* Z_s|^2}{|Z_s|^4}. \end{aligned}$$

Here, we use the convention $\frac{0}{0} = 0$. By Doléans–Dade’s exponential formula, we have

$$|Z_t|^{2p} = |x - y|^{2p} \exp \left\{ \int_0^t \xi(s) dW_s - \frac{1}{2} \int_0^t |\xi(s)|^2 ds + \int_0^t \eta(s) ds \right\}.$$

For $R > |x| \vee |y|$, define a stopping time

$$\tau_R := \inf\{t \geq 0 : |X_t(x)| \vee |X_t(y)| \geq R\}.$$

By (2.13), we have for $s < \tau_R$,

$$|\xi(s)| \leq 2^{d+1} p(\mathcal{M}_{2R} |\nabla \sigma(t, \cdot)|(X_s(x)) + \mathcal{M}_{2R} |\nabla \sigma(t, \cdot)|(X_s(y))).$$

Since $\mathcal{M}_{2R} |\nabla \sigma(t, \cdot)|(x) \in L^q_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ with $q > d + 2$, by Lemma 3.4, we have for any $\kappa > 0$,

$$\mathbb{E} \exp \left\{ \kappa \int_0^{T \wedge \tau_R} |\xi(s)|^2 ds \right\} < \infty.$$

Hence, for any $\kappa > 0$, by Novikov’s criterion,

$$t \mapsto \exp \left\{ \kappa \int_0^{t \wedge \tau_R} \xi(s) dW_s - \frac{\kappa^2}{2} \int_0^{t \wedge \tau_R} |\xi(s)|^2 ds \right\} =: \mathcal{E}_\kappa(t)$$

is a martingale. Thus, by Hölder’s inequality we have

$$\begin{aligned} & \mathbb{E}|Z_{t \wedge \tau_R}|^{2p} \\ (3.21) \quad & \leq |x - y|^{2p} (\mathbb{E} \mathcal{E}_2(t))^{1/2} \left(\mathbb{E} \exp \left\{ \int_0^{t \wedge \tau_R} (|\xi(s)|^2 + 2\eta(s)) ds \right\} \right)^{1/2} \\ & = |x - y|^{2p} \left(\mathbb{E} \exp \left\{ \int_0^{t \wedge \tau_R} (|\xi(s)|^2 + 2\eta(s)) ds \right\} \right)^{1/2}. \end{aligned}$$

On the other hand, in view of

$$\begin{aligned} |Z_s|^2(|\xi(s)|^2 + 2\eta(s)) &\leq 8p^2\|\sigma(s, X_s(x)) - \sigma(s, X_s(y))\|^2 \\ &\quad + 4p\langle Z_s, [b(s, X_s(x)) - b(s, X_s(y))] \rangle \\ &\stackrel{(1.6)}{\leq} 4p|Z_s|^2(F_{2p}(s, X_s(x)) + F_{2p}(s, X_s(y))), \end{aligned}$$

by (3.21), (1.8) and Lemma 3.3, as well as Lemma 3.1 and Fatou’s lemma, we further have

$$\begin{aligned} \mathbb{E}|Z_t|^{2p} \\ (3.22) \quad &\leq |x - y|^{2p} \left(\mathbb{E} \exp \left\{ 4p \int_0^t (F_{2p}(s, X_s(x)) + F_{2p}(s, X_s(y))) \, ds \right\} \right)^{1/2} \\ &\leq C|x - y|^{2p} \{g(x)g(y)\}^{1/4} \leq C|x - y|^{2p} (g(x)^{1/2} + g(y)^{1/2}), \end{aligned}$$

where $g(x) := 1_{\alpha>0}e^{(1+|x|^2)^\alpha} + 1_{\alpha=0}(1 + |x|^2)^\gamma$, which, together with (3.19) and Kolmogorov’s continuity criterion, yields that $X_t(x)$ admits a bi-continuous version, and for any $T, R > 0$ and $p \geq 1$,

$$(3.23) \quad \mathbb{E} \left(\sup_{t \in [0, T], |x| \leq R} |X_t(x)|^p \right) < +\infty.$$

(A) It follows by (3.22) and Lemma 2.1 with $q = \infty, U$ being any ball.

(B) Following the above proof, for any $T > 0$, by Hölder’s inequality and Doob’s maximal inequality, we have

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in [0, T \wedge \tau_R]} |Z_t|^{2p} \right) \\ &\leq |x - y|^{2p} \left(\mathbb{E} \sup_{t \in [0, T]} \mathcal{E}_1^2(t) \right)^{1/2} \left(\mathbb{E} \exp \left\{ \sup_{t \in [0, T \wedge \tau_R]} 2 \int_0^t \eta(s) \, ds \right\} \right)^{1/2} \\ &\leq 2|x - y|^{2p} (\mathbb{E} \mathcal{E}_1^2(T))^{1/2} \left(\mathbb{E} \exp \left\{ \sup_{t \in [0, T \wedge \tau_R]} 2 \int_0^t \eta(s) \, ds \right\} \right)^{1/2} \\ &\leq 2|x - y|^{2p} (\mathbb{E} \mathcal{E}_4(T))^{1/4} \left(\mathbb{E} \exp \left\{ 6 \int_0^T |\xi(s)|^2 \, ds \right\} \right)^{1/4} \\ &\quad \times \left(\mathbb{E} \exp \left\{ \sup_{t \in [0, T \wedge \tau_R]} 2 \int_0^t \eta(s) \, ds \right\} \right)^{1/2}. \end{aligned}$$

By the additional assumption (1.10), as in the above proof, we get

$$\mathbb{E} \left(\sup_{t \in [0, T]} |Z_t|^{2p} \right) \leq C|x - y|^{2p} \{g(x)g(y)\}^{1/4} \leq C|x - y|^{2p} (g(x)^{1/2} + g(y)^{1/2}),$$

which, together with Lemma 2.1 with $q, r = \infty$ and U being any ball, implies (1.11).

(C) For each $n \in \mathbb{N}$, let $\chi_n(x)$ be a nonnegative smooth function in \mathbb{R}^d with $\chi_n(x) = 1$ for all $x \in B_n$ and $\chi_n(x) = 0$ for all $x \notin B_{n+1}$. Let

$$b_n(t, x) := \chi_n(x)b(t, x), \quad \sigma_n(t, x) := \sigma(t, \chi_n(x)x).$$

Clearly, for any $T > 0$,

$$b_n \in L^q([0, T] \times \mathbb{R}^d), \quad \nabla \sigma_n \in L^q([0, T] \times \mathbb{R}^d),$$

and for some $K_n > 0$,

$$K_n^{-1}|\xi| \leq |\sigma_n(t, x)\xi| \leq K_n|\xi|, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \xi \in \mathbb{R}^m.$$

Let $X_t^n(x)$ be the solution of SDE (1.1) corresponding to b_n and σ_n . By [37], Theorem 1.1 or [34], for any bounded measurable function f and $t > 0$,

$$(3.24) \quad x \mapsto \mathbb{E}f(X_t^n(x)) \quad \text{is continuous.}$$

Fix $R > 0$. For $n > R$, define a stopping time

$$\tau_{n,R} := \left\{ t \geq 0 : \sup_{|x| \leq R} |X_t(x)| \geq n \right\}.$$

By Chebyshev’s inequality and (3.23), we have

$$(3.25) \quad \lim_{n \rightarrow \infty} \mathbb{P}(t > \tau_{n,R}) \leq \lim_{n \rightarrow \infty} \mathbb{E} \left(\sup_{s \in [0,t], |x| \leq R} |X_s(x)|^p \right) / n = 0.$$

Moreover, by the local uniqueness of solutions to SDE (1.1) (see [37]), we have

$$X_t(x) = X_t^n(x), \quad |x| \leq R, t \in [0, \tau_{n,R}].$$

Let f be a bounded measurable function. For any $x, y \in B_R$, we have

$$\begin{aligned} & |\mathbb{E}(f(X_t(x)) - f(X_t(y)))| \\ & \leq |\mathbb{E}(f(X_t(x)) - f(X_t(y))1_{t \leq \tau_{n,R}})| + 2\|f\|_\infty \mathbb{P}(t > \tau_{n,R}) \\ & = |\mathbb{E}(f(X_t^n(x)) - f(X_t^n(y))1_{t \leq \tau_{n,R}})| + 2\|f\|_\infty \mathbb{P}(t > \tau_{n,R}) \\ & \leq |\mathbb{E}(f(X_t^n(x)) - f(X_t^n(y)))| + 4\|f\|_\infty \mathbb{P}(t > \tau_{n,R}), \end{aligned}$$

which together with (3.24) and (3.25) yields the continuity of $x \mapsto \mathbb{E}(f(X_t(x)))$.

(D) Our proof is adapted from [28]. It suffices to prove that for any $T, a > 0$ and $x_0, y_0 \in \mathbb{R}^d$,

$$\mathbb{P}(|X_T(x_0) - y_0| \leq a) > 0.$$

In what follows, we shall fix $T, a > 0$ and $x_0, y_0 \in \mathbb{R}^d$. Let $Y_t(x_0)$ solve SDE (3.15) and for $N > 0$, set

$$\tau_N := \inf\{t : |Y_t(x_0)| \geq N\}.$$

By (3.16) and (3.17), we may choose N and m large enough so that

$$(3.26) \quad \mathbb{P}(\tau_N \leq T) + \mathbb{P}(|Y_T(x_0) - y_0| > a) < 1.$$

Define

$$U_t := -m\sigma(t, Y_t)^*[\sigma(t, Y_t)\sigma(t, Y_t)^*]^{-1}(Y_t - y_0)$$

and

$$Z_T := \exp\left(\int_0^{T \wedge \tau_N} U_s dW_s - \frac{1}{2} \int_0^{T \wedge \tau_N} |U_s|^2 ds\right).$$

Since $|U_{t \wedge \tau_N}|^2$ is bounded, $\mathbb{E}[Z_T] = 1$ by Novikov’s criteria.

By Girsanov’s theorem, $\tilde{W}_t := W_t + V_t$ is a \mathbb{Q} -Brownian motion, where

$$V_t := \int_0^{t \wedge \tau_N} U_s ds, \quad \mathbb{Q} := Z_T \mathbb{P}.$$

By (3.26) we have

$$(3.27) \quad \mathbb{Q}(\{\tau_N \leq T\} \cup \{|Y_T(x_0) - y_0| > a\}) < 1.$$

Notice that the solution Y_t of (3.15) also solves the following SDE:

$$Y_{t \wedge \tau_N} = x_0 + \int_0^{t \wedge \tau_N} b(s, Y_s) ds + \int_0^{t \wedge \tau_N} \sigma(s, Y_s) d\tilde{W}_s.$$

Set

$$\theta_N := \inf\{t : |X_t| \geq N\}.$$

Then the uniqueness in distribution for (1.1) yields that the law of $\{(X_t \times 1_{\{\theta_N \geq t\}})_{t \in [0, T]}, \theta_N\}$ under \mathbb{P} is the same as that of $\{(Y_t 1_{\{\tau_N \geq t\}})_{t \in [0, T]}, \tau_N\}$ under \mathbb{Q} . Hence,

$$\begin{aligned} \mathbb{P}(|X_T(x_0) - y_0| > a) &\leq \mathbb{P}(\{\theta_N \leq T\} \cup \{\theta_N \geq T, |X_T(x_0) - y_0| > a\}) \\ &= \mathbb{Q}(\{\tau_N \leq T\} \cup \{\tau_N \geq T, |Y_T(x_0) - y_0| > a\}) \\ &\leq \mathbb{Q}(\{\tau_N \leq T\} \cup \{|Y_T(x_0) - y_0| > a\}) < 1. \end{aligned}$$

The proof is complete. \square

4. Proof of Theorem 1.7. We first prepare the following easy lemma.

LEMMA 4.1. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function. Assume that for some $g_1 \in L_{loc}^{p_1}(\mathbb{R}^d)$, $g_2 \in L_{loc}^{p_2}(\mathbb{R}^d)$ and some $R > 0$,*

$$(4.1) \quad |f(x) - f(y)| \leq |x - y|(g_1(x) + g_1(y)) \quad \forall x, y \in B_{3R},$$

$$(4.2) \quad |f(x) - f(y)| \leq |x - y|(g_2(x) + g_2(y)) \quad \forall x, y \notin B_R.$$

Then we have for all $x, y \in \mathbb{R}^d$ with $|x - y| \leq R$,

$$|f(x) - f(y)| \leq 2^{d+1}|x - y|(g(x) + g(y)),$$

where

$$g(x) = \mathcal{M}_R g_1(x) 1_{|x| \leq 2R} + \mathcal{M}_R g_2(x) 1_{|x| > 2R},$$

and $\mathcal{M}_R g_i(x) := \sup_{s \in (0, R)} \int_{B_s} |g_i(x + z)| \, dz, i = 1, 2$.

PROOF. First of all, by the assumptions and Lemma 2.1, we have

$$|\nabla f(x)| \leq 2g_1(x), \quad |x| < 3R, \quad |\nabla f(x)| \leq 2g_2(x), \quad |x| > R.$$

By (2.13), we have for Lebesgue-almost all $x, y \in \mathbb{R}^d$ with $|x - y| < R$,

$$|f(x) - f(y)| \leq 2^d|x - y|(\mathcal{M}_R |\nabla f|(x) + \mathcal{M}_R |\nabla f|(y)),$$

which in turn implies the desired estimate by the definition of \mathcal{M}_R and redefinition of $g(x)$ on a Lebesgue zero set. \square

Below, we fix $T > 0$ and write for $p \in [1, \infty]$,

$$\mathbb{L}^p(T) := L^p([0, T] \times \mathbb{R}^d).$$

Let $\chi \in C^\infty(\mathbb{R}^d; [0, 1])$ be a cutoff function with

$$\chi(x) = 1, \quad \forall |x| \leq 1, \quad \chi(x) = 0, \quad \forall |x| > 2, \quad \|\nabla \chi\|_\infty \leq 2,$$

and for $R > 0$, we set

$$\chi_R(x) := \chi(x/R), \quad \bar{\chi}_R(x) = 1 - \chi_R(x).$$

Let R_0 be as in (H2'). Without loss of generality, we may assume $R_0 \geq 4$ so that

$$(4.3) \quad \|\nabla \chi_{R_0}\|_\infty \leq \|\nabla \chi\|_\infty / R_0 \leq 1/2.$$

We make the following decomposition for b :

$$b = b_1 + b_2, \quad b_1 := b\chi_{R_0}, \quad b_2 := b\bar{\chi}_{R_0}.$$

In view of (H1'), the function b_1 is global L^q -integrable; while (H2') implies that b_2 satisfies (H2). On the other hand, by Sobolev's embedding theorem, (H1') and (H2') also imply that for some $\alpha \in (0, 1)$ and $C > 0$,

$$\|\sigma(t, x) - \sigma(t, y)\| \leq C|x - y|^\alpha.$$

The following result is an easy combination of [21], page 120, Theorem 1, and [22], Theorem 10.3 and Lemma 10.2 (see [34], Theorem 3.5, for a detailed proof).

LEMMA 4.2. *Let $q > d + 2$. Under (H1') and (H2'), for any $\lambda > 0$, there exists a unique solution $u \in \mathbb{L}^q(T)$ with $\nabla^2 u \in \mathbb{L}^q(T)$ to the following backward PDE:*

$$(4.4) \quad \partial_t u + \mathcal{L}_t^{\sigma, b_1} u + b_1 = \lambda u, \quad u(T) = 0.$$

Moreover, there exist a $\lambda > 0$ and a positive constant $C = C(K, d, q, T, \lambda, \|b_1\|_{\mathbb{L}^q(T)})$ such that

$$(4.5) \quad \|\partial_t u\|_{\mathbb{L}^q(T)} + \|\nabla^2 u\|_{\mathbb{L}^q(T)} \leq C < \infty \quad \text{and} \quad \|u\|_{\mathbb{L}^\infty(T)} + \|\nabla u\|_{\mathbb{L}^\infty(T)} \leq \frac{1}{2}.$$

Let $u(t, x)$ be as in the above lemma. Now, we want to follow the same idea as in [37] to perform Zvonkin's transformation and transform SDE (1.1) into a new one with coefficients satisfying (H1) and (H2). However, if we argue entirely the same as usual and consider the transform

$$(t, x) \mapsto \Psi(t, x) := x + u(t, x),$$

then one finds that condition (1.8) may not be satisfied for the new coefficients (see Lemma 4.4 below). For this reason, we define

$$u_{R_0}(t, x) := u(t, x)\chi_{2R_0}(x), \quad \Phi_t(x) := x + u_{R_0}(t, x),$$

where R_0 is the same as in (H2').

We have

LEMMA 4.3. *The following statements hold:*

(i) *For each $t \in [0, T]$, the map $x \rightarrow \Phi_t(x)$ is a C^1 -diffeomorphism and*

$$\|\nabla \Phi_t\|_\infty, \|\nabla \Phi_t^{-1}\|_\infty \leq 2.$$

Moreover, $\nabla \Phi_t(x)$ and $\nabla \Phi_t^{-1}(x)$ are Hölder continuous in x uniformly in $t \in [0, T]$.

(ii) *Let $q > d + 2$. We have $\partial_t \Phi_t, \nabla^2 \Phi_t, \partial_t \Phi_t^{-1}, \nabla^2 \Phi_t^{-1} \in \mathbb{L}^q(T)$ and*

$$(4.6) \quad \partial_t \Phi_t + \mathcal{L}_t^{\sigma, b_1} \Phi_t = \sigma_{ik}\sigma_{jk}\partial_i u \partial_j \chi_{2R_0} + \frac{1}{2}u\sigma_{ik}\sigma_{jk}\partial_i \partial_j \chi_{2R_0} + \lambda u_{R_0}.$$

Here and below, we use Einstein's convention for summation.

PROOF. By (4.5) and (4.3), we have

$$\frac{1}{2}|x - y| \leq |\Phi_t(x) - \Phi_t(y)| \leq \frac{3}{2}|x - y|.$$

Thus, (i) follows by (4.5) and Sobolev's embedding result (see [22], Lemma 10.2).

(ii) $\partial_t \Phi_t, \nabla^2 \Phi_t, \partial_t \Phi_t^{-1}, \nabla^2 \Phi_t^{-1} \in \mathbb{L}^q(T)$ follows by (4.5) and (i). Moreover, by elementary calculations, we have

$$(4.7) \quad \begin{aligned} \partial_t \Phi_t + \mathcal{L}_t^{\sigma, b_1} \Phi_t &= \sigma_{ik}\sigma_{jk}\partial_i u \partial_j \chi_{2R_0} + \frac{1}{2}u\sigma_{ik}\sigma_{jk}\partial_i \partial_j \chi_{2R_0} \\ &\quad + ub_1^i \partial_i \chi_{2R_0} + \lambda u_{R_0} + b_1(1 - \chi_{2R_0}). \end{aligned}$$

Notice that

$$b_1^i \partial_i \chi_{2R_0} = \chi_{R_0} b^i \partial_i \chi_{2R_0} = 0, \quad b_1(1 - \chi_{2R_0}) = b \chi_{R_0}(1 - \chi_{2R_0}) = 0.$$

Equality (4.6) follows by (4.7). \square

Using the above lemma, we may prove the following Zvonkin transformation (see [16, 37] for more details).

LEMMA 4.4. *Let h be defined by the right-hand side of (4.6). Then X_t solves SDE (1.1) if and only if $Y_t := \Phi_t(X_t)$ solves the following SDE:*

$$(4.8) \quad dY_t = \tilde{b}(t, Y_t) dt + \tilde{\sigma}(t, Y_t) dW_t,$$

where

$$(4.9) \quad \tilde{\sigma} := (\nabla \Phi \cdot \sigma) \circ \Phi^{-1}, \quad \tilde{b} := (h + b_2 \cdot \nabla \Phi) \circ \Phi^{-1}.$$

PROOF. (\Rightarrow) By (4.6) and generalized Itô's formula (see [19], page 122, Theorem 1), we have (4.8).

(\Leftarrow) By elementary calculations, it is easy to check that

$$\partial_t \Phi_t^{-1} + \mathcal{L}_t^{\tilde{\sigma}, \tilde{b}} \Phi_t^{-1} = b \circ \Phi_t^{-1}.$$

As above, using generalized Itô's formula again, we obtain that $\Phi_t^{-1}(Y_t)$ solves the SDE (1.1). \square

Now we give:

PROOF OF THEOREM 1.7. Since (A)–(D) are invariant under diffeomorphism transformation $x \mapsto \Phi_t(x)$, by Lemma 4.4, it suffices to check that $\tilde{\sigma}$ and \tilde{b} defined by (4.9) satisfy (H1)–(H2) so that we can use Theorem 1.2 to complete the proof.

First of all, (H1) is obvious by Lemma 4.3 and (H1'). For (H2), by definitions (4.9) and Lemma 4.3, it is easy to see that

$$|\tilde{b}(t, x)| \leq \|h\|_\infty + 2|b_2(t, \Phi_t^{-1}(x))| \leq C(1 + |\Phi_t^{-1}(x)|) \leq C(1 + |x|),$$

and by (2.11), for any $R > 0$, there are functions $g_R, \hat{g}_R \in L_{\text{loc}}^q(\mathbb{R}_+ \times \mathbb{R}^d)$ such that for all $x, y \in \mathbb{R}^d$ with $|x - y| \leq R$

$$\|\tilde{\sigma}(t, x) - \tilde{\sigma}(t, y)\| \leq |x - y|(g_R(t, x) + g_R(t, y)),$$

$$|\tilde{b}(t, x) - \tilde{b}(t, y)| \leq |x - y|(\hat{g}_R(t, x) + \hat{g}_R(t, y)).$$

On the other hand, by the definition of Φ , there exists a $R_1 \geq 2R_0$ large enough such that

$$\Phi_t(x) = \Phi_t^{-1}(x) = x, \quad |x| \geq R_1.$$

Hence, for $|x| \geq R_1$, we have

$$\tilde{b}(t, x) = b(t, x), \quad \tilde{\sigma}(t, x) = \sigma(t, x).$$

Thus, by (H2') and Lemma 4.1, one sees that (H2) and (1.10) hold for \tilde{b} and $\tilde{\sigma}$. The proof is complete. \square

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