PROPAGATION OF CHAOS FOR THE LANDAU EQUATION WITH MODERATELY SOFT POTENTIALS

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We consider the 3D Landau equation for moderately soft potentials [$\gamma \in (-2, 0)$ with the usual notation] as well as a stochastic system of N particles approximating it. We first establish some strong/weak stability estimates for the Landau equation, which are fully satisfactory only when $\gamma \in [-1, 0)$. We next prove, under some appropriate conditions on the initial data, the so-called propagation of molecular chaos, that is, that the empirical measure of the particle system converges to the unique solution of the Landau equation. The main difficulty is the presence of a singularity in the equation. When $\gamma \in (-1, 0)$, the strong-weak uniqueness estimate allows us to use a coupling argument and to obtain a rate of convergence. When $\gamma \in (-2, -1]$, we use the classical martingale method introduced by McKean. To control the singularity, we have to take advantage of the regularity provided by the entropy dissipation. Unfortunately, this dissipation is too weak for some (very rare) aligned configurations. We thus introduce a perturbed system with an additional noise, show the propagation of chaos for this system and finally prove that the additional noise is almost never used in the limit $N \to \infty$.

CONTENTS

1. Introduction and main results.

1.1. *The Landau equation.* The 3D homogeneous Landau equation for moderately soft potentials reads

$$(1.1) \qquad \partial_t f_t(v) = \frac{1}{2} \operatorname{div}_v \left(\int_{\mathbb{R}^3} a(v - v_*) \big[f_t(v_*) \nabla f_t(v) - f_t(v) \nabla f_t(v_*) \big] dv_* \right),$$

where the initial distribution $f_0 : \mathbb{R}^3 \to \mathbb{R}$ is given. The unknown $f_t : \mathbb{R}^3 \mapsto \mathbb{R}$ stands for the velocity-distribution in a plasma. The matrix $a : \mathbb{R}^3 \mapsto \mathbb{M}_{3\times3}(\mathbb{R})$ is symmetric nonnegative and given by

$$a(v) = |v|^{2+\gamma} \left(I - \frac{v \otimes v}{|v|^2} \right)$$

for some $\gamma \in (-2, 0)$. We will also use the notation

$$b(v) = \operatorname{div} a(v) = -2|v|^\gamma v.$$

This equation, with $\gamma = -3$, replaces the Boltzmann equation when particles are subjected to a Coulomb interaction. It was derived by Landau in 1936. Physically, only the case $\gamma = -3$ is really interesting: it is explained in [2] that the case

$\gamma = -3$ is the only one that you can obtain from a particle system in a suitable *weak coupling* limit, even if the interaction potential has a finite range (a fact that was already discovered by Bogolyubov). Concerning this Coulomb case $\gamma = -3$, the strongest existence result was recently obtained by Desvillettes [7], which improves on that of Villani [38]. See also Arsen'ev–Peskov [1] for a local existence result, [13] for a local uniqueness result and Gualdani–Guillen [19] for a detailed study of the radially symmetric case.

When $\gamma \in (-3, 1]$, the Landau equation can be seen as an approximation of the corresponding Boltzmann equation in the asymptotics of *grazing collisions*. The reference [38] that was already mentioned above rigorously proves the convergence of the Boltzmann equation to the Landau equation, together with the existence of solutions to the Landau equation in the whole range. For moderately soft potentials, that is, when $\gamma \in (-2, 0)$, a global well-posedness result was obtained in [14]. For hard potentials, that is, when $\gamma \in (0, 1]$, well-posedness, regularity and large-time behavior have been studied by Desvillettes–Villani [8, 9]. A probabilistic interpretation was introduced by Funaki in [17]. We also refer to the book of Villani [39] for a long review on kinetic models, including this one.

Let us finally mention a few important properties: the Landau equation preserves mass, momentum, kinetic energy and dissipates entropy.

1.2. *Some notation.* We denote by $\mathcal{P}(\mathbb{R}^3)$, the set of probability measures on $\mathbb{R}^3$. When $f \in \mathcal{P}(\mathbb{R}^3)$ has a density, we also denote by $f \in L^1(\mathbb{R}^3)$ this density. For $q > 0$, $\mathcal{P}_q(\mathbb{R}^3)$ stands for the set of all $f \in \mathcal{P}(\mathbb{R}^3)$ such that $m_q(f) = \int_{\mathbb{R}^3} |v|^q f(dv) < \infty$.

For $f \in \mathcal{P}(\mathbb{R}^3)$, we introduce the notation

$$b(f, v) := \int_{\mathbb{R}^3} b(v - v_*) f(dv_*), \qquad a(f, v) := \int_{\mathbb{R}^3} a(v - v_*) f(dv_*),$$

(1.2)
$$\sigma(f, v) := \big(a(f, v)\big)^{1/2}.$$

Observe that $a(f, v)$ is symmetric nonnegative [since $a(v)$ is so for all $v \in \mathbb{R}^3$], so that it indeed admits a unique nonnegative symmetric square root. The Landau equation (1.1) can be rewritten as

$$(1.3) \qquad \partial_t f_t(v) = \tfrac{1}{2} \operatorname{div}_v \big(a(f_t, v)\nabla_v f_t(v) - b(f_t, v) f_t(v)\big).$$

We will denote by H the entropy functional: for $f \in \mathcal{P}_q(\mathbb{R}^3)$, for some $q > 0$ we define $H(f) = \int_{\mathbb{R}^3} f(v) \log f(v) \, dv \in (-\infty, +\infty]$ if f has a density and $H(f) = \infty$ otherwise. Thanks to the moment assumption, this entropy is always well defined; see, for instance, [21], Lemma 3.1.

We shall also use some similar notation for probability distributions of systems of N particles. Precisely, for $N \geq 1$, $\mathcal{P}((\mathbb{R}^3)^N)$ stands for the set of probability measures on $(\mathbb{R}^3)^N$. When $F^N \in \mathcal{P}((\mathbb{R}^3)^N)$ has a density, we also denote by

$F^N \in L^1((\mathbb{R}^3)^N)$ this density. We write $\mathcal{P}_{\text{sym}}((\mathbb{R}^3)^N)$ for the set of all exchange-able elements of $\mathcal{P}((\mathbb{R}^3)^N)$. For $F^N \in \mathcal{P}_{\text{sym}}((\mathbb{R}^3)^N)$, we introduce $m_q(F^N) :=$ $\int_{(\mathbb{R}^3)^N} |v_1|^q F^N(dv_1, \ldots, dv_N)$ and define $\mathcal{P}_q((\mathbb{R}^3)^N) = \{F^N \in \mathcal{P}_{\text{sym}}((\mathbb{R}^3)^N) :$ $m_q(F^N) < \infty\}$. Finally, we introduce the entropy of $F^N \in \mathcal{P}_q((\mathbb{R}^3)^N)$ for some $q > 0$ by setting

$$H(F^N) := \begin{cases} N^{-1} \int_{(\mathbb{R}^3)^N} F^N(v_1, \ldots, v_N) \log F^N(v_1, \ldots, v_N) \, dv_1 \cdots dv_N, \\ \qquad \text{if } F^N \text{ has a density,} \\ +\infty, \qquad \text{otherwise.} \end{cases}$$

We will use the MKW (Monge–Kantorovich–Wasserstein) distance; see Villani [40] for many details: for $f, g \in \mathcal{P}_2(\mathbb{R}^3)$,

$$W_2(f, g) = \inf\left\{\left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 R(dv, dw)\right)^{1/2} : R \in \Pi(f, g)\right\},$$

where $\Pi(f, g) = \{R \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3) : R \text{ has marginals } f \text{ and } g\}$.

We now recall what is usually called *propagation of molecular chaos*.

DEFINITION 1.1. Let $\mathcal{Y}$ be a random variable taking values in a Polish space E with law g and, for each $N \geq 2$, a family $(\mathcal{Y}_1^N, \ldots, \mathcal{Y}_N^N)$ of exchangeable E-valued random variables. The sequence $(\mathcal{Y}_1^N, \ldots, \mathcal{Y}_N^N)$ is said to be $\mathcal{Y}$-chaotic (or g-chaotic) if one of the three equivalent conditions is satisfied:

  (i) $(\mathcal{Y}_1^N, \mathcal{Y}_2^N)$ goes in law to $g \otimes g$ as $N \to \infty$;
  (ii) for all $j \geq 1$, $(\mathcal{Y}_1^N, \ldots, \mathcal{Y}_j^N)$ goes in law to $g^{\otimes j}$ as $N \to \infty$;
  (iii) $N^{-1} \sum_1^N \delta_{\mathcal{Y}_i^N}$ goes in probability to g as $N \to \infty$.

We refer for instance to Sznitman [36] for the equivalence of the three conditions and to [21] for quantitative versions of this equivalence. Propagation of chaos holds for a particle system [towards the solution $(f_t)_{t \geq 0}$ of its limit equation] if, starting with f_0-chaotic initial data, the particles are f_t-chaotic for all times t. And *trajectorial* propagation of chaos holds when the trajectories of the particles are Y-chaotic, for a suitable process Y associated to the limit equation.

We will use the generic notation C for all positive constant appearing in the sequel. When needed, we will indicate in subscript the quantities on which it depends.

1.3. *Well-posedness and strong/weak stability.* We first recall that for $\alpha \in (-3, 0)$ and $f \in \mathcal{P}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with $p > 3/(3 + \alpha)$, it holds that

$$(1.4) \qquad \sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\alpha f(v_*) \, dv_* \leq 1 + C_{\alpha, p} \|f\|_{L^p}.$$

This is easily checked: write that $\int_{\mathbb{R}^3} |v - v_*|^\alpha f(v_*) \, dv_* \leq 1 + \int_{v_* \in B(v,1)} |v - v_*|^\alpha f(v_*) \, dv_*$ and use the Hölder inequality.

We next define the (classical) notion of weak solutions we use.

DEFINITION 1.2.   Let $\gamma \in (-2, 0)$. We say that f is a weak solution to (1.1) if it satisfies:

(i) $f \in L^{\infty}_{\text{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$,
(ii) if $\gamma \in (-2, -1)$, $f \in L^1_{\text{loc}}([0, \infty), L^p(\mathbb{R}^3))$ for some $p > 3/(4 + \gamma)$,
(iii) for all $\varphi \in C^2_b(\mathbb{R}^3)$, all $t \geq 0$,

(1.5)
$$\int_{\mathbb{R}^3} \varphi(v) f_t(dv) = \int_{\mathbb{R}^3} \varphi(v) f_0(dv)$$
$$+ \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L\varphi(v, v_*) f_s(dv) f_s(dv_*) \, ds,$$

where

$$L\varphi(v, v_*) := \frac{1}{2} \sum_{k,l=1}^{3} a_{kl}(v - v_*) \partial^2_{kl} \varphi(v) + \sum_{k=1}^{3} b_k(v - v_*) \partial_k \varphi(v).$$

Remark that every term is well defined in (1.5) under our assumptions on f and φ, since

(1.6) $$\left| L\varphi(v, v_*) \right| \leq C_{\varphi} \left(|v - v_*|^{\gamma+1} + |v - v_*|^{\gamma+2} \right).$$

If $\gamma \in [-1, 0)$, we have $|L\varphi(v, v_*)| \leq C_{\varphi}(1 + |v|^2 + |v_*|^2)$ so that condition (i) is sufficient, while if $\gamma \in (-2, -1)$, we have $|L\varphi(v, v_*)| \leq C_{\varphi}(1 + |v|^2 + |v_*|^2 + |v - v_*|^{\gamma+1})$, so that conditions (i) and (ii) are enough: use (1.4) with $\alpha = \gamma + 1$.

For $\gamma \in (-2, 0)$, we set

(1.7)
$$q(\gamma) := \frac{\gamma^2}{2 + \gamma}, \qquad p_1(\gamma) := \frac{3}{3 + \gamma} \quad \text{and, for } q > q(\gamma),$$
$$p_2(\gamma, q) := \frac{3q - 3\gamma}{q - 3\gamma}.$$

It can be checked that $1 < p_1(\gamma) < p_2(\gamma, q) < 3$. Let us recall the well-posedness result of [14].

THEOREM 1.3 (Corollary 1.4 in [14]).   Let $\gamma \in (-2, 0)$ and $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfy $H(f_0) < \infty$ and $m_q(f_0) < \infty$ for some $q > q(\gamma)$. Then (1.1) has a unique weak solution $f \in L^{\infty}_{\text{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\text{loc}}([0, \infty), L^p(\mathbb{R}^3))$ for some $p > p_1(\gamma)$. Moreover, this unique solution satisfies $m_2(f_t) = m_2(f_0)$ and $H(f_t) \leq H(f_0)$ for all $t \geq 0$ and f belongs to $L^1_{\text{loc}}([0, \infty), L^p(\mathbb{R}^3))$ for all $p \in (p_1(\gamma), p_2(\gamma, q))$.

We first state some weak/strong stability estimates and improve the above uniqueness result.

THEOREM 1.4. *Let $\gamma \in (-2, 0)$ and $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfy also $H(f_0) < \infty$ and $m_q(f_0) < \infty$ for some $q > q(\gamma)$. Let f be the unique weak solution of* (1.1) *built in Theorem* 1.3.

(i) *Assume that $\gamma \in (-1, 0]$. For any other weak solution $g \in L^\infty_{\mathrm{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ to* (1.1) *starting from $g_0 \in \mathcal{P}_2(\mathbb{R}^3)$, any $p \in (p_1(\gamma), p_2(\gamma, q))$, any $t \geq 0$,*

$$W_2^2(f_t, g_t) \leq W_2^2(f_0, g_0) \exp\left(C_{\gamma, p} \int_0^t \left(1 + \|f_s\|_{L^p}\right) ds\right).$$

We thus have uniqueness for (1.1) *(with initial condition f_0) in the class of all measure solutions in $L^\infty_{\mathrm{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$.*

(ii) *Assume that $\gamma \in (-2, -1]$. For any $p \in (p_1(\gamma), p_2(\gamma, q))$, any*

$$(1.8) \qquad\qquad r > \frac{2p - 3}{(3 + \gamma)(p - 1) - 1}$$

and any other weak solution $g \in L^\infty_{\mathrm{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0, \infty), L^r(\mathbb{R}^3))$ to (1.1),

$$W_2^2(f_t, g_t) \leq W_2^2(f_0, g_0) \exp\left(C_{\gamma, p, r} \int_0^t \left(1 + \|f_s\|_{L^p} + \|g_s\|_{L^r}\right) ds\right).$$

In particular, for any $r > [3(q + \gamma)]/[5q + 2\gamma q + 3\gamma]$, we have uniqueness for (1.1) *(with initial condition f_0) in the class $L^\infty_{\mathrm{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0, \infty), L^r(\mathbb{R}^3))$ [it suffices to let $p \uparrow p_2(\gamma, q)$ in* (1.8)].

When $\gamma \in (-1, 0)$, we thus prove the uniqueness in the class of all measure solutions in $L^\infty_{\mathrm{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$. This is quite satisfactory and interesting for the well-posedness theory, but there is another important consequence: we will be able to apply (up to some fluctuations) the stability result to the empirical measure of an associated particle system.

When $\gamma \in (-2, -1]$, the strong/weak estimate is of course less satisfactory, since we do not manage to completely get rid of the regularity assumptions on g. The uniqueness we deduce is slightly better than that stated in [14], Corollary 1.4, but the stability result cannot be applied to the empirical measure of a particle system.

1.4. *Entropy dissipation and a priori bounds.* The fact that *smooth* solutions to the Landau equation (1.1) belong to $L^1_{\mathrm{loc}}([0, \infty), L^p(\mathbb{R}^3))$ is a consequence of the entropy dissipation. We sketch here the argument for the sake of completeness and also because we will use, in the proof of Theorem 1.8, a similar strategy to obtain some regularity estimate on the particle system. Precisely, the entropy dissipation for a solution f_t of the Landau equation (1.1) reads

$$\frac{d}{dt} H(f_t) = - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(\nabla \log f_t(v) - \nabla \log f_t(v_*)\right)^* a(v - v_*)$$
$$\times \left(\nabla \log f_t(v) - \nabla \log f_t(v_*)\right) f_t(dv) f_t(dv_*)$$

(where the superscript $*$ stands for the transposition), which is certainly nonnegative thanks to the nonnegativity of a. Using the drift and diffusion introduced in (1.2), the dissipation of entropy may be rewritten as

$$\frac{d}{dt} H(f_t) = -\int_{\mathbb{R}^3} \nabla \log f_t(v) a(f_t, v) \nabla \log f_t(v) f_t(dv) - \int_{\mathbb{R}^3} \operatorname{div} b(f_t, v) f_t(dv).$$

Then since the entropy of f_t is decreasing and its second moment constant, Lemma 2.1 (taken from [8], Proposition 4) shows that the first term in the right-hand side controls a weighted Fisher information $I_\gamma(f_t) := \int_{\mathbb{R}^3} |\nabla \log f_t|^2 (1 + |v|)^\gamma f_t(dv)$. This in turn allows us to control the L^p-norm of f_t (for some values of p), thanks to Lemma C.3, provided we have sufficiently many moments. Finally, a L^p-norm of f_t with p large enough is sufficient to bound $\| \operatorname{div} b(f_t) \|_\infty$. All in all, if f_0 has a finite entropy and moment of order q, the entropy dissipation leads, for any $p \in (p_1(\gamma), p_2(\gamma, q))$, to

$$\int_0^T \| f_t \|_{L^p} \, dt \le C_{\gamma, p, q, T}.$$

Let us mention that in his recent paper [7], Desvillettes introduces a very tricky method and manages to derive a similar estimate when $\gamma \in [-3, -2]$.

1.5. *The particle system.* We now consider the following toy model: we have N particles characterized by their velocities $\mathcal{V}_i^N$, solving the following system of $\mathbb{R}^3$-valued SDEs

(1.9)
$$\forall i = 1, \dots, N \qquad \mathcal{V}_i^N(t) = \mathcal{V}_i^N(0) + \int_0^t b(\tilde{\mu}_s^N, \mathcal{V}_i^N(s)) \, ds$$
$$+ \int_0^t \sigma(\tilde{\mu}_s^N, \mathcal{V}_i^N(s)) \, d\mathcal{B}_i(s),$$

with the notation (1.2). Here, $(\mathcal{B}_i(t))_{i=1,\dots,N,t \ge 0}$ is an independent family of $3D$ standard Brownian motions independent of $(\mathcal{V}_i^N(0))_{i=1,\dots,N}$ and finally, for some $\eta_N \in (0, 1)$,

$$\tilde{\mu}_t^N = \mu_t^N \star \phi_{\eta_N} \qquad \text{where } \mu_t^N = \frac{1}{N} \sum_1^N \delta_{\mathcal{V}_i^N(t)} \quad \text{and} \quad \phi_\eta(x) = \frac{3}{4\pi \eta^3} \mathbb{1}_{\{|x| < \eta\}}.$$

We could probably also study the same system without the smoothing by convolution with ϕ_{η_N}. But without this smoothing, the particle system is not clearly well-posed. Since the paper is already technical enough, we decided not to study this (possibly difficult) issue. However, this is not really a limitation, *since in all the results below, we allow η_N to tend to 0 as fast as one wants.*

PROPOSITION 1.5. *For any $N \ge 1$, any initial condition $(\mathcal{V}_i^N(0))_{i=1,\dots,N}$, any $\eta_N \in (0, 1)$, (1.9) has a pathwise unique strong solution $(\mathcal{V}_i^N(t))_{i=1,\dots,N,t \ge 0}$.*

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The main topic of this paper is to show that, provided $\eta_N \to 0$ and under suitable conditions on $(\mathcal{V}_i^N(0))_{i=1,...,N}$, the empirical measure μ_t^N converges, as $N \to \infty$, to the weak solution f_t of (1.1) built in Theorem 1.3. When $\gamma = 0$, the coefficients a, b are smooth and such a convergence has been proved by Fontbona–Guérin–Méléard [11], Carrapatoso [4]; see also [12]. In a work in preparation [3], Bolley–Guillin–Fournier obtain some results when $\gamma \in (0, 1)$. In [4] and [3], a slightly more natural particle system, which a.s. preserves momentum and kinetic energy, is considered. We have not been able to study this system with the present technique, although the difference of its behavior seems very light.

When $\gamma < 0$, the situation is rather more complicated, due to the singularity of the coefficients a, b. To our knowledge, there are no available results in that context, except the one of Miot–Pulvirenti–Saffirio [25], where a partial result has been obtained when $\gamma = -3$: they prove the convergence to the Landau hierarchy, which unfortunately does not allow one to conclude. Here, we propose two methods.

When $\gamma \in (-1, 0)$, we handle a direct computation, mimicking the strong/weak stability study, and we get a result with a rate of convergence. We have to use a blob approximation of the empirical measure, in the spirit of [20].

When $\gamma \in (-2, 0)$, we use that the dissipation of entropy of the particle system provides some regularity enough to control the singularity. By this way, we obtain a convergence result, without rate, by using a tightness/uniqueness principle. We follow here [16], where we studied a similar problem for the 2D Navier–Stokes equation. However, a major difficulty appears: the entropy dissipation is actually not sufficient, due to the lack of ellipticity of the matrix a. The diffusion coefficients in (1.9) may degenerate for almost aligned configurations. We thus have to show such configurations almost never appear.

1.6. *A convergence result with rate for $\gamma \in (-1, 0)$.* In that case, the singularity is not too large, and we can use our strong/weak uniqueness principle to study the propagation of chaos.

THEOREM 1.6. *Assume that $\gamma \in (-1, 0)$ and let $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ for some $q \geq 8$ be such that $H(f_0) < \infty$. Consider f the unique weak solution to (1.1) defined in Theorem 1.3. For each $N \geq 2$, consider an exchangeable family $(\mathcal{V}_i^N(0))_{i=1,...,N}$ with $\sup_{N \geq 2} \mathbb{E}[|\mathcal{V}_1^N(0)|^4] < \infty$ and the corresponding unique solution $(\mathcal{V}_i^N(t))_{i=1,...,N, t \geq 0}$ to (1.9) with some $\eta_N \in (0, N^{-1/3})$. Denote by $\mu_t^N = N^{-1} \sum_1^N \delta_{\mathcal{V}_i^N(t)}$ the associated empirical measure. Then for $\alpha = (1 - 6/q)(2 + 2\gamma)/3$, for all $T > 0$,*

$$\sup_{[0,T]} \mathbb{E}[W_2^2(\mu_t^N, f_t)] \leq C_{T,q}(N^{-\alpha} + N^{-1/2} + \mathbb{E}[W_2^2(\mu_0^N, f_0)]).$$

In particular, the propagation of molecular chaos holds true.

Recall (see [15]) that the best general rate we can hope for the expected squared 2-Wasserstein distance between an empirical measure of some i.i.d. $\mathbb{R}^3$-valued random variables and their common law is in $N^{-1/2}$. Hence, for $\gamma \in (-1/4, 0)$, and when the particles are initially i.i.d. and f_0-distributed, the rate of convergence is optimal as soon as f_0 has a finite moment of sufficiently high order. Of course, it is likely that $N^{-1/2}$ is the true rate for all values of γ, while our rate deteriorates considerably when approaching $\gamma = -1$. However, we are quite satisfied, since to our knowledge, there are very few quantitative results of propagation of chaos for systems with a singular interaction.

1.7. *Trajectories.* Our second method will prove a slightly stronger result (although without rate): the convergence at the level of trajectories. We thus need to introduce the stochastic paths associated to the Landau equation. These paths will furthermore be used to prove the strong/weak estimates.

Given a Brownian motion $(\mathcal{B}(t))_{t \geq 0}$, independent of an initial condition $\mathcal{V}(0)$ with law f_0, we are interested in a continuous adapted $\mathbb{R}^3$-valued process $(\mathcal{V}(t))_{t \geq 0}$ solution to

$$(1.10) \qquad \mathcal{V}(t) = \mathcal{V}(0) + \int_0^t b(f_s, \mathcal{V}(s)) \, ds + \int_0^t \sigma(f_s, \mathcal{V}(s)) \, d\mathcal{B}(s),$$

where $f_t \in \mathcal{P}(\mathbb{R}^3)$ is the law of $\mathcal{V}(t)$ and using notation (1.2). By *adapted*, we mean adapted to the filtration $\mathcal{F}_t = \sigma(\mathcal{V}(0), (\mathcal{B}(s))_{s \in [0,t]})$.

The process $(\mathcal{V}(t))_{t \geq 0}$ represents the time evolution of the velocity of a *typical* particle in a plasma whose velocity distribution solves the Landau equation. Such a probabilistic interpretation (in the case of the Boltzmann equation) was initiated by Tanaka [37]. See Funaki [17] for the case of the Landau equation. The following results of existence and uniqueness are proved in [14], but with another formulation involving a white noise. We will shortly prove them again, since we need to extend them to another nonlinear SDE that we will introduce later.

PROPOSITION 1.7. *Let $\gamma \in (-2, 0)$ and $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfy also $H(f_0) < \infty$ and $m_q(f_0) < \infty$ for some $q > q(\gamma)$; recall (1.7).*

(i) *There exists a pathwise unique continuous adapted solution $(\mathcal{V}(t))_{t \geq 0}$ to (1.10) such that $f = (f_t)_{t \geq 0} \in L_{\text{loc}}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L_{\text{loc}}^1([0, \infty), L^p(\mathbb{R}^3))$ for some $p \in (p_1(\gamma), p_2(\gamma, q))$.*

(ii) *Furthermore, f is the weak solution to the Landau equation (1.1) given by Theorem 1.3.*

1.8. *A convergence result without rate.* We will assume the following hypothesis on the initial conditions of (1.9) and (1.1):

$$
(1.11) \quad \begin{cases}
\text{(i) } f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap \mathcal{P}_q(\mathbb{R}^3) \text{ for some } q > q(\gamma) \text{ and } H(f_0) < \infty; \\
\text{(ii) the sequence } \big(\mathcal{V}_1^N(0), \ldots, \mathcal{V}_N^N(0)\big) \\
\qquad \text{with law } F_0^N \text{ is exchangeable and } f_0\text{-chaotic}; \\
\text{(iii) } \sup_{N \geq 2} \mathbb{E}\big[|\mathcal{V}_1^N(0)|^2 + |\mathcal{V}_1^N(0)|^q\big] < \infty \text{ and } \sup_{N \geq 2} H(F_0^N) < \infty.
\end{cases}
$$

All these conditions hold true if f_0 satisfies point (i) and if $(\mathcal{V}_1^N(0), \ldots, \mathcal{V}_N^N(0))$ are i.i.d. and f_0-distributed.

THEOREM 1.8. *Assume that $\gamma \in (-2, 0)$. Consider $f_0 \in \mathcal{P}(\mathbb{R}^3)$ and, for each $N \geq 2$, a family $(\mathcal{V}_i^N(0))_{i=1,\ldots,N}$ of $\mathbb{R}^3$-valued random variables. Assume (1.11). For each $N \geq 2$, consider the unique solution $(\mathcal{V}_i^N(t))_{i=1,\ldots,N,t\geq 0}$ to (1.9) with some $\eta_N \in (0, 1)$. Let also $(f_t)_{t\geq 0}$ be the unique weak solution to (1.1) given by Theorem 1.3 and $(\mathcal{V}(t))_{t\geq 0}$ the unique solution to (1.10) (see Proposition 1.7). Then, as soon as $\lim_{N\to\infty} \eta_N = 0$, the sequence $((\mathcal{V}_i^N(t))_{t\geq 0})_{i=1,\ldots,N}$ is $(\mathcal{V}(t))_{t\geq 0}$-chaotic. In particular, if we set*

$$
\mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{\mathcal{V}_i^N(t)},
$$

then $(\mu_t^N)_{t\geq 0}$ goes in probability, in $C([0, \infty), \mathcal{P}(\mathbb{R}^3))$, to $(f_t)_{t\geq 0}$.

1.9. *Comments.* Propagation of chaos was initiated by Kac [23] as a step to the derivation of the Boltzmann equation. Since then, many models have been studied. For nonsingular interactions, things are more or less well understood and there has even been recently some important progress to get uniform in time propagation of chaos (see Mischler–Mouhot [26]), to which we refer for many references including the important works of Sznitman [35, 36], Méléard [24]; see also Mischler–Mouhot–Wennberg [27] and [21]. As already mentioned, most of the results concerning the Landau equation [4, 11, 12] concern the case where $\gamma = 0$ (or, at least, where the singularity is removed).

The case of a singular interaction is much more complicated and there are only very few works. Osada [28, 29] has obtained some remarkable results concerning the convergence of the vortex model to the Navier–Stokes equation (in dimension 2, with a divergence-free interaction in $1/|x|$), improved recently by the authors and Mischler [16]. In dimension one, Cepa and Lepingle have also studied the (very singular) Dyson model [5]. We shall also mention the case of a deterministic particle system (with position and velocity) in singular interaction studied by the second author and Jabin [20]. Here, the first quantitative method is in some sense inspired from [20]. The second one (without rate) relies on the entropy dissipation technique introduced in [16].

Finally, let us mention again the work in preparation [3], which treats a similar problem when $\gamma \in [0, 1)$, with more satisfactory results. Some of the techniques used here are common with [3]. In particular, the introductory part of Section 3 is reproduced from it.

1.10. *Plan of the paper.* In the next section, we prove several regularity estimates concerning the coefficients a and b, we check Proposition 1.5 (well-posedness of the particle system) and Proposition 1.7 (well-posedness of the nonlinear SDE). Section 3 is devoted to the proof of Theorem 1.4 (strong/weak stability estimates). We prove the (uniform in N) propagation of moments for the particle system in Section 4. The proof of Theorem 1.6 [propagation of chaos with rate when $\gamma \in (-1, 0)$] is given in Section 5. We next study precisely, in Section 6, the ellipticity of $a(\mu, v)$ when μ is an empirical measure. We give the proof of Theorem 1.8 (propagation of chaos without rate when $\gamma \in (-2, 0)$) in Section 7. In Appendix A, we extend a coupling result of [21]. In Appendix B, we generalize slightly a result of Figalli [10] on the equivalence between PDEs and SDEs, that we use several times. Finally, Appendix C about entropy and weighted Fisher information lies at the end of the paper.

1.11. *Final notation.* We recall that $a(v) = |v|^\gamma (|v|^2 I - v \otimes v)$ and that $b(v) = -2|v|^\gamma v$. We introduce $\sigma(v) = (a(v))^{1/2} = |v|^{\gamma/2-1}(|v|^2 I - v \otimes v)$.

For $\eta \in (0, 1)$, we recall that $\phi_\eta(x) = (3/(4\pi\eta^3))\mathbb{1}_{\{|x|<\eta\}}$. We introduce $a_\eta = a \star \phi_\eta$, $b_\eta = b \star \phi_\eta$ and $\sigma_\eta = a_\eta^{1/2}$, and we define, for $f \in \mathcal{P}(\mathbb{R}^3)$ and $v \in \mathbb{R}^3$,

$$a_\eta(f, v) = \int_{\mathbb{R}^3} a_\eta(v - w) f(dw), \qquad b_\eta(f, v) = \int_{\mathbb{R}^3} b_\eta(v - w) f(dw) \quad \text{and}$$

$$\sigma_\eta(f, v) = (a_\eta(f, v))^{1/2}.$$

Remark that it is very similar to the formula (1.2) which corresponds to the case $\eta = 0$ with the convention that $a_0 = a$ and $b_0 = b$.

For $f \in \mathcal{P}(\mathbb{R}^3)$, we put $f^\eta = f \star \phi_\eta$. We observe that $a(f^\eta, v) = a_\eta(f, v)$, $b(f^\eta, v) = b_\eta(f, v)$ and $\sigma(f^\eta, v) = \sigma_\eta(f, v)$.

We will use the standard notation $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ for $x, y \in \mathbb{R}$.

Finally, for M, N two 3×3 matrices, we set $\langle\langle M, N \rangle\rangle = \operatorname{Tr} MN^*$ and $\|M\|^2 = \operatorname{Tr} MM^*$.

2. First regularity estimates and well-posedness of the nonlinear SDE.

2.1. *Ellipticity.* We recall the ellipticity estimate of Desvillettes–Villani [8], Proposition 4. It is stated when $\gamma > 0$ but the proof only uses that $\gamma + 2 > 0$. Section 6 will be devoted to the proof of a more general result.

LEMMA 2.1. *Let $\gamma \in (-2, 0)$. For all $f \in \mathcal{P}_2(\mathbb{R}^3)$ satisfying $H(f) < +\infty$, there exists a constant $\kappa_0 > 0$ depending only on γ, $H(f)$ and $m_2(f)$ such that for all $v \in \mathbb{R}^3$,*

$$\inf_{|\xi|=1} \xi^* a(f, v)\xi \geq \kappa_0 (1 + |v|)^\gamma.$$

2.2. *Rough regularity estimates and well-posedness of the particle system.* We will frequently use the following lemma stated in [12], Lemma 11 (with $C = 1$ but with another norm).

LEMMA 2.2. *There is a constant $C > 0$ such that for any pair of nonnegative symmetric 3×3 matrices A, B,*

$$\|A^{1/2} - B^{1/2}\| \leq C\|A - B\|^{1/2} \quad and$$

$$\|A^{1/2} - B^{1/2}\| \leq C(\|A^{-1}\| \wedge \|B^{-1}\|)^{1/2}\|A - B\|.$$

We next collect some rough regularity estimates on a, b and σ.

LEMMA 2.3. *For any $\gamma \in (-2, 0)$, there is a constant C such that for all $v, w \in \mathbb{R}^3$,*

$$\big|b(v) - b(w)\big| \leq C|v - w|(|v|^\gamma + |w|^\gamma),$$

$$\|a(v) - a(w)\| \leq C|v - w|(|v|^{1+\gamma} + |w|^{1+\gamma}),$$

$$\|\sigma(v) - \sigma(w)\| \leq C|v - w|(|v|^{\gamma/2} + |w|^{\gamma/2}).$$

PROOF. The inequality concerning b is proved in [14], Remark 2.2. Since by definition $\sigma(v) = |v|^{\gamma/2-1}(|v|^2 I - v \otimes v)$, we see that σ is a C^1 function outside the origin and that $\|D\sigma(v)\| \leq C|v|^{\gamma/2}$. To go from v to w, it is possible to find a path $S : [0, 1] \to \mathbb{R}^3$, of length smaller than $\pi|v - w|/2$ and that always remains in the crown $\{z \in \mathbb{R}^3 : \min\{|v|, |w|\} \leq |z| \leq \max\{|v|, |w|\}\}$. For instance, some circular arc will do the job. Then the claimed inequality follows from

$$\|\sigma(v) - \sigma(w)\| \leq \int_0^1 \|D\sigma(S(t))\|\,|S'(t)|\,dt \leq C|v - w| \max\{|v|^{\gamma/2}, |w|^{\gamma/2}\}.$$

The inequality concerning a is proved similarly, using that $\|Da(v)\| \leq C|v|^{1+\gamma}$. This proof may also be adapted to b, using that $\|Db(v)\| \leq C|v|^\gamma$. $\quad\square$

We next study a_η, b_η and σ_η.

LEMMA 2.4. *Let $\gamma \in (-2, 0)$.*

(i) *For each $\eta \in (0, 1)$, there is C_η such that for any $y, z \in \mathbb{R}^3$, $\|(a_\eta(y))^{-1}\| \leq C_\eta(1 + |y|)^{|\gamma|}$ and*

$$|b_\eta(y) - b_\eta(z)| \leq C_\eta|y - z|,$$

$$\|\sigma_\eta(y) - \sigma_\eta(z)\| + \|a_\eta(y) - a_\eta(z)\| \leq C_\eta(1 + |y| + |z|)|y - z|.$$

(ii) *There is $C > 0$ such that for all $\eta \in (0, 1)$, all $v \in \mathbb{R}^3$,*

$$\|a_\eta(v) - a(v)\| \leq C\eta^2(\eta + |v|)^\gamma \quad and$$

$$|b_\eta(v) - b(v)| \leq \begin{cases} C\eta(\eta + |v|)^\gamma, & \text{if } \gamma \in (-1, 0), \\ C\min\{\eta, |v|\}|v|^\gamma, & \text{if } \gamma \in (-2, -1]. \end{cases}$$

(iii) *There is $C > 0$ such that for all $\eta \in (0, 1)$, all $f \in \mathcal{P}(\mathbb{R}^3)$, all $v \in \mathbb{R}^3$,*

$$\|a_\eta(f, v) - a(f, v)\| \leq C\eta^{2+\gamma} \quad and, \text{ if } \gamma \in (-1, 0)$$

$$|b_\eta(f, v) - b(f, v)| \leq C\eta^{1+\gamma}.$$

PROOF. We first check point (i). First, it holds that $a_\eta(y) = a(\phi_\eta, y)$ when using notation (1.2). But ϕ_η obviously belongs to $\mathcal{P}_2(\mathbb{R}^3)$ and has a finite entropy. Hence, Lemma 2.1 tells us that for all $\xi \in \mathbb{R}^3$, all $y \in \mathbb{R}^3$, $\xi^* a_\eta(y)\xi \geq c_\eta(1 + |y|)^\gamma|\xi|^2$, which implies that $\|(a_\eta(y))^{-1}\| \leq C_\eta(1 + |y|)^{|\gamma|}$.

We next use Lemma 2.3 to get

$$|b_\eta(y) - b_\eta(z)| \leq C\eta^{-3}\int_{|u|<\eta} |y - z|(|y + u|^\gamma + |z + u|^\gamma)\,du \leq C_\eta|y - z|,$$

as well as

$$\|a_\eta(y) - a_\eta(z)\| \leq C\eta^{-3}\int_{|u|<\eta} |y - z|(|y + u|^{\gamma+1} + |z + u|^{\gamma+1})\,du,$$

which is easily bounded by $C_\eta|y - z|$ if $\gamma \in (-2, -1)$ and by $C_\eta|y - z|(1 + |y|^{\gamma+1} + |z|^{\gamma+1})$ if $\gamma \in [-1, 0)$. By using the second estimate of Lemma 2.2 and the estimate on $\|(a_\eta(y))^{-1}\|$, we conclude that $\|(a_\eta(y))^{1/2} - (a_\eta(z))^{1/2}\| \leq C_\eta h(|y|, |z|)|y - z|$, where $h(|y|, |z|) = \min\{(1 + |y|)^{|\gamma|/2}, (1 + |z|)^{|\gamma|/2}\}$ if $\gamma \in (-2, -1)$ and $h(|y|, |z|) = \min\{(1 + |y|)^{|\gamma|/2}, (1 + |z|)^{|\gamma|/2}\}(1 + |y| + |z|)^{1+\gamma}$ if $\gamma \in [-1, 0)$. In any case, $h(|y|, |z|) \leq C(1 + |y| + |z|)$, which ends the proof of point (i).

Concerning point (ii), we first study b and separate two cases. If $|v| \leq 2\eta$, then we have $|b(v)| = 2|v|^{1+\gamma}$ and $|b_\eta(v)| \leq C|v|^{1+\gamma} + C\eta^{1+\gamma}$. Hence, $|b(v) - b_\eta(v)| \leq C|v|^{1+\gamma} + C\eta^{1+\gamma}$, which is smaller than $C\eta^{1+\gamma} \leq C\eta(\eta + |v|)^\gamma$ if $\gamma \in (-1, 0]$ and than $C|v|^{1+\gamma} \leq \min\{\eta, |v|\}|v|^\gamma$ if $\gamma \in (-2, -1]$. If now $|v| > 2\eta$, we use that $|b(y) - b(z)| \leq C|y - z|(|y|^\gamma + |z|^\gamma)$ by Lemma 2.3, so that $|b(v) - b_\eta(v)| \leq C\eta\int_{\mathbb{R}^3}(|v|^\gamma + |v - u|^\gamma)\phi_\eta(u)\,du \leq C\eta|v|^\gamma$, which is bounded by $C\eta(\eta + |v|)^\gamma$ if $\gamma \in (-1, 0]$ and by $C\min\{\eta, |v|\}|v|^\gamma$ if $\gamma \in (-2, -1]$.

We now study $\|a(v) - a_\eta(v)\|$. If $|v| \leq 2\eta$, we immediately get that $\|a(v) - a_\eta(v)\| \leq \|a(v)\| + \|a_\eta(v)\| \leq C\eta^{2+\gamma} \leq C\eta^2(\eta + |v|)^\gamma$. If now $|v| \geq 2\eta$, we use a Taylor expansion: write, for $|u| < \eta$, that $a(v - u) = a(v) - u\nabla a(v) + \zeta(u, v)$, where $\|\zeta(u, v)\| \leq |u|^2 \sup_{B(v,\eta)} \|D^2 a\| \leq C\eta^2 |v|^\gamma$. Consequently, since ϕ_η is symmetric,

$$\|a(v) - a_\eta(v)\| = \left\| \int_{|u| \leq \eta} (a(v) - a(v - u))\phi_\eta(u)\, du \right\|$$

$$= \left\| \int_{|u| \leq \eta} \zeta(u, v)\phi_\eta(u)\, du \right\| \leq C\eta^2 |v|^\gamma,$$

which is controlled by $C\eta^2(\eta + |v|)^\gamma$ as desired.

We finally have to prove (iii). If $\gamma \in (-1, 0)$, we have $|b_\eta(x) - b(x)| \leq C\eta^{1+\gamma}$ by (ii), whence

$$|b_\eta(f, v) - b(f, v)| = \left| \int_{\mathbb{R}^3} [b_\eta(v - w) - b(v - w)] f(dw) \right| \leq C\eta^{1+\gamma}.$$

Now for any value of $\gamma \in (-2, 0)$, we see from point (ii) that $\|a_\eta(x) - a(x)\| \leq C\eta^{2+\gamma}$, so that we obtain $\|a_\eta(f, v) - a(f, v)\| \leq C\eta^{2+\gamma}$ by a simple integration. $\square$

At this point, we can prove the strong well-posedness of the particle system.

PROOF OF PROPOSITION 1.5.   Let

$$b_i(v_1, \ldots, v_N) := b\left(\frac{1}{N} \sum_{j=1}^N \delta_{v_j} \star \phi_{\eta_N}, v_i \right) = \frac{1}{N} \sum_{j=1}^N b_{\eta_N}(v_i - v_j),$$

$$\sigma_i(v_1, \ldots, v_N) := \left[a\left(\frac{1}{N} \sum_{j=1}^N \delta_{v_j} \star \phi_{\eta_N}, v_i \right) \right]^{1/2} = \left[\frac{1}{N} \sum_{j=1}^N a_{\eta_N}(v_i - v_j) \right]^{1/2}.$$

These are the coefficients of the system of SDEs (1.9). We claim that these coefficients have at most linear growth and are locally Lipschitz continuous, from which strong existence and uniqueness classically follow. First, b_i is globally Lipschitz continuous by Lemma 2.4(i). Next, it follows from Lemma 2.4(i) that σ_i is locally Lipschitz continuous. Finally, recalling that $\|A\| = \operatorname{Tr} AA^*$ and that $\operatorname{Tr}(a(v)) = 2|v|^{\gamma+2}$, we find that

$$\|\sigma_i(v_1, \ldots, v_N)\|^2 = \frac{2}{N} \sum_{j=1}^N \int_{\mathbb{R}^3} |v_i - v_j - u|^{\gamma+2} \phi_{\eta_N}(u)\, du$$

$$\leq \frac{2}{N} \sum_{j=1}^N (|v_i| + |v_j| + \eta_N)^{\gamma+2}.$$

Since $\gamma + 2 \in (0, 2]$, we conclude that σ_i has at most linear growth. $\square$

2.3. *Regularity estimates for the nonlinear SDE.* We now give some growth and regularity estimates on the fields $a(f)$, $\sigma(f)$ and $b(f)$ created by some probability f, that will allow us to study the well-posedness of the linear version of the SDE (1.10), as well as that of a perturbed version of it.

LEMMA 2.5. *Let $\gamma \in (-2, 0)$ and $f \in \mathcal{P}_2(\mathbb{R}^3)$. We have the following estimates for all $v, w \in \mathbb{R}^3$:*

(i) *In any case, $\|\sigma(f, v)\|^2 \leq 4 + 4m_2(f) + 4|v|^{\gamma+2}$.*

(ii) *If $\gamma \in [-1, 0)$, then $|b(f, v)| \leq 2 + 2|v| + 2m_1(f)$.*

(iii) *If $\gamma \in (-2, -1)$ and $f \in L^p(\mathbb{R}^3)$ with $p > 3/(4 + \gamma)$, then $|b(f, v)| \leq 2 + C_p \|f\|_{L^p}$.*

(iv) *If $f \in L^p(\mathbb{R}^3)$ with $p > p_1(\gamma)$, then $|b(f, v) - b(f, w)| \leq C_p(1 + \|f\|_{L^p})|v - w|$.*

(v) *If $f \in L^p(\mathbb{R}^3)$ with $p > p_1(\gamma)$, then $\|\sigma(f, v) - \sigma(f, w)\|^2 \leq C_p(1 + \|f\|_{L^p})|v - w|^2$.*

(vi) *If $\gamma \in [-1, 0)$, then $\|a(f, v) - a(f, w)\| \leq C(1 + |v| + |w| + m_1(f))|v - w|$.*

(vii) *If $\gamma \in (-2, -1)$ and $f \in L^p(\mathbb{R}^3)$ with $p > 3/(4 + \gamma)$, then $\|a(f, v) - a(f, w)\| \leq C_p(1 + \|f\|_{L^p})|v - w|$.*

PROOF. Recalling that $\operatorname{Tr} a(v) = 2|v|^{\gamma+2}$,

$$\|\sigma(f, v)\|^2 = \operatorname{Tr} a(f, v) = 2 \int_{\mathbb{R}^3} |v - w|^{\gamma+2} f(dw).$$

Point (i) follows, since $|v - w|^{\gamma+2} \leq 2|v|^{\gamma+2} + 2|w|^{\gamma+2} \leq 2 + 2|w|^2 + 2|v|^{\gamma+2}$.

Next, $|b(f, v)| \leq 2 \int_{\mathbb{R}^3} |v - w|^{\gamma+1} f(dw)$. Point (ii) follows, since for $\gamma \in [-1, 0)$, $|v - w|^{\gamma+1} \leq 1 + |v| + |w|$. Point (iii) also follows, using (1.4) and that $p > 3/(4 + \gamma)$.

Point (iv) is obtained by integration of the Lipschitz estimate on b given in Lemma 2.3 and by using (1.4) [recall that $p > p_1(\gamma) = 3/(3 + \gamma)$]:

$$|b(f, v) - b(f, w)| \leq C|v - w| \int_{\mathbb{R}^3} (|v - x|^\gamma + |w - x|^\gamma) f(dx)$$

$$\leq C|v - w|(1 + C_p \|f\|_{L^p}).$$

For (v), we use a classical result (see, e.g., Stroock–Varadhan [34], Theorem 5.2.3), which asserts that there is $C > 0$ such that for all $A : \mathbb{R}^3 \mapsto S_3^+$ (the set of symmetric nonnegative 3×3-matrices with real entries), $\|D(A^{1/2})\|_\infty \leq C\|D^2 A\|_\infty^{1/2}$. Here, we apply it to $A(v) = a(f, v)$. First,

$$\|D^2 A(v)\| = \left\| \int_{\mathbb{R}^3} D^2 a(v - x) f(dx) \right\| \leq C \int_{\mathbb{R}^3} |v - x|^\gamma f(dx)$$

$$\leq C(1 + C_p \|f\|_{L^p}),$$

where we have used that $\|D^2 a(v)\| \le |v|^\gamma$ and (1.4) [since $p > 3/(3 + \gamma)$]. Consequently,

$$\|\sigma(f, v) - \sigma(f, w)\|^2 = \|(A(v))^{1/2} - (A(w))^{1/2}\|^2 \le \|D(A^{1/2})\|_\infty^2 |v - w|^2$$

is smaller than $C(1 + C_p \|f\|_{L^p})|v - w|^2$.

Finally, integrating the Lipschitz estimate on a given in Lemma 2.3, we find the inequality $\|a(f, v) - a(f, w)\| \le C|v - w| \int_{\mathbb{R}^3} (|v - x|^{1+\gamma} + |w - x|^{1+\gamma}) f(dx)$. Using the same arguments as in (ii) and (iii), points (vi) and (vii) immediately follow. $\quad\square$

2.4. *Well-posedness of the nonlinear SDE.* We now have all the ingredients to give the following.

PROOF OF THEOREM 1.7. Recall that $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ is assumed to satisfy $m_q(f_0) < \infty$ for some $q > q(\gamma)$ and $H(f_0) < \infty$. Let $f \in L_{\text{loc}}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L_{\text{loc}}^1([0, \infty), L^p(\mathbb{R}^3))$ [for all $p \in (p_1(\gamma), p_2(\gamma, q))$] be the unique weak solution to (1.1); see Theorem 1.3. Let also $\mathcal{V}(0)$ be f_0-distributed and consider a $3D$ Brownian motion $\mathcal{B}$ independent of $\mathcal{V}(0)$.

Step 1: *The linear SDE.* Recalling that f is given, we prove here that there is strong existence and uniqueness for the linear SDE

$$(2.1) \qquad \mathcal{V}(t) = \mathcal{V}(0) + \int_0^t b(f_s, \mathcal{V}(s)) \, ds + \int_0^t \sigma(f_s, \mathcal{V}(s)) \, d\mathcal{B}(s).$$

Fix $p \in (p_1(\gamma), p_2(\gamma, q))$ and recall that $f \in L_{\text{loc}}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L_{\text{loc}}^1([0, \infty), L^p(\mathbb{R}^3))$. By Lemma 2.5(iv)–(v), the coefficients of this SDE are Lipschitz continuous in v, with a Lipschitz constant (locally) integrable in time for b and (locally) square integrable in time for σ. The conclusion follows from classical arguments; see, for instance, [30], Theorem 3.17.

Step 2: *Uniqueness for the linear PDE.* Recalling that f is fixed, we consider the Kolmogorov equation associated to (2.1):

$$(2.2) \qquad \partial_t g_t(v) = \tfrac{1}{2} \operatorname{div}_v \big(a(f_t, v) \nabla g_t(v) - b(f_t, v) g_t(v) \big).$$

As for the Landau equation, $g \in L_{\text{loc}}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ is a weak solution if for all $\varphi \in C_b^2(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \varphi(v) g_t(dv) = \int_{\mathbb{R}^3} \varphi(v) g_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L\varphi(v, v_*) g_s(dv) f_s(dv_*) \, ds.$$

By Lemma 2.5(i)–(ii)–(iii), we know that $\|\sigma(f_t, v)\|^2 \le C(1 + |v|^2)$ and $|b(f_t, v)| \le C(1 + \|f_t\|_{L^p})(1 + |v|)$, with $\|f_t\|_{L^p} \in L_{\text{loc}}^1([0, \infty))$. Hence, Proposition B.1, which slightly extends [10], Theorem 2.6, tells us that any weak solution $(g_t)_{t \ge 0}$ to (2.1) can be represented as the family of time marginals of a solution to

(2.1). Consequently, step 1 implies the uniqueness of the solution to (2.1), for any given $g_0 \in \mathcal{P}_2(\mathbb{R}^3)$.

Step 3: *Strong existence.* From step 1, we know that there exists a solution $\mathcal{V}$ to (2.1). Put $g_t = \mathcal{L}(\mathcal{V}(t))$. A direct application of the Itô formula shows that g is a weak solution to (2.2), with $g_0 = f_0$. But f, being a weak solution to the Landau equation (1.1) [see also (1.3)], is also a weak solution to (2.2). By step 2, we deduce that $g = f$ as desired.

Step 4: *Strong uniqueness.* Consider another solution $\mathcal{W}$ [with $\mathcal{L}(\mathcal{W}(t)) = h_t$] to the nonlinear SDE (1.10) (with the same initial condition and the same Brownian motion). Assume also that $h \in L^\infty_{\text{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\text{loc}}([0, \infty), L^p(\mathbb{R}^3))$ for some $p \in (p_1(\gamma), p_2(\gamma, q))$. A direct application of the Itô formula shows that h is a weak solution to (1.1), whence $f = h$ by Theorem 1.3. Consequently, $\mathcal{W}$ also solves the linear SDE (2.1), so that $\mathcal{V} = \mathcal{W}$ by step 2. $\quad\square$

3. Fine regularity estimates and strong/weak stability principles. The goal of this section is to prove Theorem 1.4, that is the strong/weak stability principles. These principles will be proved using a coupling argument between two solutions of the nonlinear SDE (1.10). Unfortunately, we cannot use the same Brownian motion for both solutions: this is due to the fact that we really need a fine estimate and that the best coupling between two $3D$ Gaussian distribution $\mathcal{N}(0, \Sigma_1)$ and $\mathcal{N}(0, \Sigma_2)$ *does not* consist in setting $X_1 = \Sigma_1^{1/2} Y$ and $X_2 = \Sigma_2^{1/2} Y$ for the same Y with law $\mathcal{N}(0, I)$. Actually, as shown in Givens–Shortt [18], the optimal coupling is obtained when setting $X_1 = \Sigma_1^{1/2} Y$ and $X_2 = \Sigma_2^{1/2} U(\Sigma_1, \Sigma_2) Y$, where the orthogonal matrix $U(\Sigma_1, \Sigma_2)$ is given by

$$(3.1) \qquad U(\Sigma_1, \Sigma_2) = \Sigma_2^{-1/2} \Sigma_1^{-1/2} \big(\Sigma_1^{1/2} \Sigma_2 \Sigma_1^{1/2} \big)^{1/2}.$$

More precisely, we will use the following lemma.

LEMMA 3.1. *For any probability measure m on a measurable space F, any pair of measurable families of 3×3 matrices $(\sigma_1(x))_{x \in F}$ and $(\sigma_2(x))_{x \in F}$, introducing the matrices $\Sigma_1 = \int_F \sigma_1(x) \sigma_1^*(x) m(dx)$ and $\Sigma_2 = \int_F \sigma_2(x) \sigma_2^*(x) m(dx)$, it holds that*

$$\big\| \Sigma_1^{1/2} - \Sigma_2^{1/2} U(\Sigma_1, \Sigma_2) \big\|^2 \leq \int_F \big\| \sigma_1(x) - \sigma_2(x) \big\|^2 m(dx).$$

PROOF. Developing both terms and using the definition (3.1) of U, we realize that the issue is to prove that

$$\text{Tr}\bigg(\int_F \sigma_1(x) \sigma_2^*(x) m(dx) \bigg) \leq \text{Tr}\big((\Sigma_1^{1/2} \Sigma_2 \Sigma_1^{1/2})^{1/2} \big).$$

By Givens and Shortt [18], proof of Proposition 7, we have $\text{Tr}(M) \leq \text{Tr}((\Sigma_1^{1/2} \times \Sigma_2 \Sigma_1^{1/2})^{1/2})$ for all 3×3 matrix M such that

$$\begin{pmatrix} \Sigma_1 & M \\ M^* & \Sigma_2 \end{pmatrix}$$

is nonnegative. But with $M = \int_F \sigma_1(x)\sigma_2^*(x)m(dx)$,

$$\begin{pmatrix} \Sigma_1 & M \\ M^* & \Sigma_2 \end{pmatrix} = \int_F \begin{pmatrix} \sigma_1(x) \\ \sigma_2(x) \end{pmatrix} \begin{pmatrix} \sigma_1(x) \\ \sigma_2(x) \end{pmatrix}^* m(dx)$$

is clearly nonnegative. The conclusion follows. $\square$

3.1. *Main ideas of the proof of Theorem* 1.4.   Unfortunately, the rigorous proof is very technical. Let us explain here the main ideas without justification. Let thus f and g be two weak solutions to (1.1). We associate (with some work, using some regularization) to these solutions $(\mathcal{V}(t))_{t \geq 0}$ [with $\mathcal{L}(\mathcal{V}(t)) = f_t$] and $(\mathcal{Z}(t))_{t \geq 0}$ [with $\mathcal{L}(\mathcal{Z}(t)) = g_t$] solving

$$d\mathcal{V}(t) = a\big(f_t, \mathcal{V}(t)\big)U(t)\,d\mathcal{B}(t) + b\big(f_t, \mathcal{V}(t)\big)\,dt \quad \text{and}$$

$$d\mathcal{Z}(t) = a\big(g_t, \mathcal{Z}(t)\big)\,d\mathcal{B}(t) + b\big(g_t, \mathcal{Z}(t)\big)\,dt,$$

where $U(t) = U(a(g_t, \mathcal{Z}(t)), a(f_t, \mathcal{V}(t)))$. Recalling that U takes values in the set of orthogonal matrices, we realize that $U(t)\,d\mathcal{B}(t)$ is a Brownian motion. Hence, $\mathcal{V}$ and $\mathcal{Z}$ are coupled solutions, with different initial values, to the nonlinear SDE (1.10). We will of course bound $W_2^2(f_t, g_t)$ by $\mathbb{E}[|\mathcal{V}(t) - \mathcal{Z}(t)|^2]$. Using the Itô formula, we directly find that

$$(3.2) \qquad \frac{d}{dt}\mathbb{E}\big[|\mathcal{Z}(t) - \mathcal{V}(t)|^2\big] = \Gamma(R_t),$$

where $R_t = \mathcal{L}(\mathcal{Z}(t), \mathcal{V}(t))$ and where, for R a probability on $\mathbb{R}^3 \times \mathbb{R}^3$, with marginals μ and ν,

$$\Gamma(R) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \big(2(z - v) \cdot \big(b(\mu, z) - b(\nu, v)\big)$$

$$+ \big\|\sigma(\mu, z) - \sigma(\nu, v)U\big(a(\mu, z), a(\nu, v)\big)\big\|^2\big)R(dz, dv).$$

Using Lemma 3.1 and that R has marginals μ and ν, we easily bound

$$\Gamma(R) \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Delta(z, z_*, v, v_*)R(dz_*, dv_*)R(dz, dv),$$

where $\Delta(z, z_*, v, v_*) := 2(z - v) \cdot (b(z - z_*) - b(v - v_*)) + \|\sigma(z - z_*) - \sigma(v - v_*)\|^2$. A simple computation (see Lemma 3.3 below) shows that $\Delta = \Delta_1 + \Delta_2$, where Δ_1 is antisymmetric and disappears when integrating, and where Δ_2 can be controlled explicitly. We have verified that

$$\Gamma(R) \leq 8 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 \frac{(|z - z_*| \wedge |v - v_*|)^{1+\gamma}}{|z - z_*| \vee |v - v_*|} R(dz_*, dv_*)R(dz, dv).$$

Assume first that $\gamma \in (-1, 0)$. We get

$$\Gamma(R) \leq 8 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 |z - z_*|^\gamma R(dz_*, dv_*)R(dz, dv)$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} |z - v|^2 |z - z_*|^\gamma \mu(dz_*)R(dz, dv).$$

Using finally (1.4) with some $p > p_1(\gamma)$,

$$\Gamma(R) \leq C_p\big(1 + \|\mu\|_{L^p}\big) \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 R(dz, dv).$$

Coming back to (3.2), we end with $(d/dt)\mathbb{E}[|\mathcal{Z}(t) - \mathcal{V}(t)|^2] \leq C_p(1 + \|f_t\|_{L^p}) \times \mathbb{E}[|\mathcal{Z}(t) - \mathcal{V}(t)|^2]$, whence, using the Grönwall lemma,

$$W_2^2(f_t, g_t) \leq W_2^2(f_0, g_0) \exp\bigg(C_p \int_0^t \big(1 + \|f_s\|_{L^p}\big) ds \bigg)$$

as desired. When now $\gamma \in (-2, -1]$, a slightly more complicated study shows that

$$\Gamma(R) \leq C_{p,r}\big(1 + \|\mu\|_{L^p} + \|\nu\|_{L^r}\big) \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 R(dz, dv),$$

for any $p > p_1(\gamma)$ and any $r > (2p - 3)/[(3 + \gamma)(p - 1) - 1]$.

3.2. *Fine regularity estimates.* The subsection is devoted to the proof of the following estimate.

PROPOSITION 3.2. *For $f, g \in \mathcal{P}_2(\mathbb{R}^3)$, $R \in \Pi(g, f)$ and $0 \leq \eta \leq \varepsilon < 1$, we set*

$$\Gamma_{\eta,\varepsilon}(R) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \big(2(z - v) \cdot \big(b(g^\eta, z) - b(f, v)\big)$$

$$+ \big\|\sigma(g^\eta, z) - \sigma(f, v)U\big(a(g^\varepsilon, z), a(f^\varepsilon, v)\big)\big\|^2\big) R(dz, dv).$$

Assume that $f \in L^p(\mathbb{R}^3)$ for some $p > p_1(\gamma)$.

(i) *If $\gamma \in (-1, 0)$, then*

$$\Gamma_{\eta,\varepsilon}(R) \leq C\varepsilon^{2+2\gamma} + C_p\big(1 + \|f^\varepsilon\|_{L^p}\big) \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 R(dz, dv).$$

(ii) *If $\gamma \in (-2, -1]$, we assume also that $g \in L^r$ for some $r > (2p - 3)/[(3 + \gamma)(p - 1) - 1]$. There is $\delta > 0$ (depending only on r, p, γ) such that*

$\Gamma_{\eta,\varepsilon}(R)$

$$\leq C_{r,p}\big(1 + \|f + g\|_{L^r}\big)\big(1 + m_1(f + g)\big)\varepsilon^\delta + C_{r,p}\big(1 + \|f^\varepsilon\|_{L^p} + \|g^\varepsilon\|_{L^r}\big)$$

$$\times \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 R(dz, dv).$$

We start with a simple computation.

LEMMA 3.3. *Assume that $\gamma \in (-2, 0)$. Then for any v, v_*, w, w_* in $\mathbb{R}^3$,*

$$\Delta(v, v_*, w, w_*) := 2(v - w) \cdot \big(b(v - v_*) - b(w - w_*)\big)$$

$$+ \big\|\sigma(v - v_*) - \sigma(w - w_*)\big\|^2$$

$$= \Delta_1(v, v_*, w, w_*) + \Delta_2(v, v_*, w, w_*),$$

where

$$\Delta_1(v, v_*, w, w_*) = \big((v-w) + (v_* - w_*)\big) \cdot \big(b(v-v_*) - b(w-w_*)\big)$$

is antisymmetric [i.e., $\Delta_1(v, v_*, w, w_*) = -\Delta_1(v_*, v, w_*, w)$] and

$$\Delta_2(v, v_*, w, w_*) \le 4\big(|v-w|^2 + |v_* - w_*|^2\big) K\big(|v-v_*|, |w-w_*|\big),$$

where

$$(3.3) \qquad\qquad K(x, y) = \frac{(x \wedge y)^{1+\gamma}}{x \vee y}.$$

Let us recall that for M, N two 3×3 matrices, $\langle\langle M, N \rangle\rangle = \operatorname{Tr} MN^*$ and $\|M\|^2 = \operatorname{Tr} MM^*$.

PROOF OF LEMMA 3.3. Defining $\Delta_2 = \Delta - \Delta_1$, we find, with the notation $X = v - v_*$ and $Y = w - w_*$,

$$\Delta_2 = (X-Y) \cdot \big(b(X) - b(Y)\big) + \big\|\sigma(X) - \sigma(Y)\big\|^2.$$

Recalling that $b(X) = -2X|X|^\gamma$, using that $\|\sigma(X)\|^2 = 2|X|^{2+\gamma}$ and that

$$\langle\langle\sigma(X), \sigma(Y)\rangle\rangle = |X|^{1+\gamma/2}|Y|^{1+\gamma/2}\left(1 + \frac{(X \cdot Y)^2}{|X|^2|Y|^2}\right) \ge 2|X|^{\gamma/2}|Y|^{\gamma/2}(X \cdot Y)$$

(we used that $1 + x^2 \ge 2x$ for the last inequality), one easily checks that

$$\Delta_2 \le 2(X \cdot Y)\big(|X|^{\gamma/2} - |Y|^{\gamma/2}\big)^2 \le 2|X\|Y|\big(|X|^{\gamma/2} - |Y|^{\gamma/2}\big)^2$$

witch equals $2(|X\|Y|)^{1+\gamma}\big(|X|^{|\gamma|/2} - |Y|^{|\gamma|/2}\big)^2$. Next, remark that for $\alpha \in [0, 1]$ and $x, y \ge 0$,

$$|x^\alpha - y^\alpha| = (x \vee y)^\alpha\left(1 - \left[\frac{x \wedge y}{x \vee y}\right]^\alpha\right) \le (x \vee y)^\alpha\left(1 - \frac{x \wedge y}{x \vee y}\right) = |x - y|(x \vee y)^{\alpha-1},$$

whence

$$\Delta_2 \le 2|X-Y|^2(|X\|Y|)^{1+\gamma}(|X| \vee |Y|)^{|\gamma|-2}$$
$$= 2|X-Y|^2(|X| \wedge |Y|)^{1+\gamma}(|X| \vee |Y|)^{-1}.$$

One easily completes the proof, using that $|X-Y|^2 \le 2(|v-w|^2 + |v_* - w_*|^2)$. □

We next prove an intermediate result.

LEMMA 3.4. *Let $\gamma \in (-2, 0)$. Consider $f, g \in \mathcal{P}_2(\mathbb{R}^3)$ and $R \in \Pi(g, f)$. For any $\varepsilon \in (0, 1)$,*

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \left(2(z - v) \cdot \left(b(g^\varepsilon, z) - b(f^\varepsilon, v)\right)\right.$$

$$+ \left.\left\| \sigma(g^\varepsilon, z) - \sigma(f^\varepsilon, v) U\left(a(g^\varepsilon, z), a(f^\varepsilon, v)\right) \right\|^2\right) R(dz, dv)$$

$$\leq 8 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 K\left(|z - z_*|, |v - v_*|\right)$$

$$\times R^\varepsilon(dz_*, dv_*) R(dz, dv),$$

where K was introduced in (3.3) and where $R^\varepsilon = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} \delta_{(u+z, u+v)} \phi_\varepsilon(u) \, du \times R(dz, dv)$ belongs to $\Pi(g^\varepsilon, f^\varepsilon)$.

In other words, $R^\varepsilon(A) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{1}_A(u + z, u + v) \phi_\varepsilon(u) R(dz, dv) \, du$ for all $A \in \mathcal{B}(\mathbb{R}^3 \times \mathbb{R}^3)$.

PROOF OF LEMMA 3.4. Let us denote by I the left-hand side of the desired inequality. We claim that

$$(z - v) \cdot \left(b(g^\varepsilon, z) - b(f^\varepsilon, v)\right)$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}^3} (z - v) \cdot \left(b(z - z_*) - b(v - v_*)\right) R^\varepsilon(dz_*, dv_*),$$

$$\left\| \sigma(g^\varepsilon, z) - \sigma(f^\varepsilon, v) U\left(a(g^\varepsilon, z), a(f^\varepsilon, v)\right) \right\|^2$$

$$\leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left\| \sigma(z - z_*) - \sigma(v - v_*) \right\|^2 R^\varepsilon(dz_*, dv_*).$$

The first point is obvious: it follows from the fact that the marginals of R^ε are g^ε and f^ε. The second point relies on Lemma 3.1 applied with (here $z, v \in \mathbb{R}^3$ are fixed) $F = \mathbb{R}^3 \times \mathbb{R}^3$, $m = R^\varepsilon(dz_*, dv_*)$, $\sigma_1(z_*, v_*) = \sigma(z - z_*)$ and $\sigma_2(z_*, v_*) = \sigma(v - v_*)$. Indeed, since $\sigma(x) = (a(x))^{1/2}$, we have

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \sigma(z - z_*) \sigma^*(z - z_*) R^\varepsilon(dz_*, dv_*) = a(g^\varepsilon, z),$$

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \sigma(v - v_*) \sigma^*(v - v_*) R^\varepsilon(dz_*, dv_*) = a(f^\varepsilon, v).$$

Recalling the notation of Lemma 3.3, we thus deduce that

$$I \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Delta(z, z_*, v, v_*) R^\varepsilon(dz_*, dv_*) R(dz, dv).$$

By Lemma 3.3, we know that $\Delta = \Delta_1 + \Delta_2$, whence, with obvious notation, $I \leq I_1 + I_2$. Using the substitution $u \mapsto -u$ and that $\Delta_1(z, z_* - u, v, v_* - u) =$

$$-\Delta_1(z_*, z + u, v_*, v + u),$$

$$I_1 = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} \Delta_1(z, z_* + u, v, v_* + u) \phi_\varepsilon(u) \, du \, R(dz_*, dv_*) R(dz, dv)$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} \Delta_1(z, z_* - u, v, v_* - u) \phi_\varepsilon(u) \, du \, R(dz_*, dv_*) R(dz, dv)$$

$$= -\int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} \Delta_1(z_*, z + u, v_*, v + u) \phi_\varepsilon(u) \, du \, R(dz_*, dv_*) R(dz, dv)$$

$$= -I_1,$$

so that $I_1 = 0$. This computation is licit: recalling the expression of Δ_1 and that $|b(x)| \le 2|x|^{\gamma+1}$,

$$\kappa_\varepsilon(z, z_*, v, v_*)$$

$$:= \int_{\mathbb{R}^3} |\Delta_1(z, z_* + u, v, v_* + u)| \phi_\varepsilon(u) \, du$$

$$\le 2(|z| + |v| + |z_*| + |v_*|)$$

$$\times \int_{\mathbb{R}^3} (|z - z_* - u|^{1+\gamma} + |v - v_* - u|^{1+\gamma}) \phi_\varepsilon(u) \, du$$

$$\le C_\varepsilon (1 + |z| + |v| + |z_*| + |v_*|)^2.$$

For the last inequality, separate the cases $\gamma \in [-1, 0)$ and $\gamma \in (-2, -1)$. This last expression is integrable against $R(dz_*, dv_*)R(dz, dv)$ because f and g belong to $\mathcal{P}_2(\mathbb{R}^3)$ by assumption.

Finally, using again the substitution $u \mapsto -u$, it is easily checked that

$$I_2 = 4 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} (|z - v|^2 + |z_* - v_*|^2)$$

$$\times K(|z - z_* - u|, |v - v_* - u|) \phi_\varepsilon(u) \, du \, R(dz_*, dv_*) R(dz, dv)$$

$$= 8 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} |z - v|^2$$

$$\times K(|z - z_* - u|, |v - v_* - u|) \phi_\varepsilon(u) \, du \, R(dz_*, dv_*) R(dz, dv),$$

which is nothing but the right-hand side of the target inequality.   □

We can finally give the following.

PROOF OF PROPOSITION 3.2.   Let thus $\gamma \in (-2, 0)$, $f, g \in \mathcal{P}_2(\mathbb{R}^3)$, $R \in \Pi(g, f)$ and $0 \le \eta < \varepsilon < 1$ be fixed. Let also $p > p_1(\gamma)$ be fixed and assume that $f \in L^p(\mathbb{R}^3)$. If $\gamma \in (-2, -1]$, we assume moreover that $g \in L^r(\mathbb{R}^3)$ for some fixed $r > (2p - 3)/[(3 + \gamma)(p - 1) - 1]$ and that $r < p$ without loss of generality.

We start with $\Gamma_{\eta,\varepsilon}(R) = I_0 + I_1 + I_2 + I_3$, where

$$I_0 := \int_{\mathbb{R}^3 \times \mathbb{R}^3} [2(z-v) \cdot (b(g^\varepsilon, z) - b(f^\varepsilon, v))$$
$$+ \|\sigma(g^\varepsilon, z) - \sigma(f^\varepsilon, v)U(a(g^\varepsilon, z), a(f^\varepsilon, v))\|^2] R(dz, dv),$$

$$I_1 := 2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} (z-v) \cdot (b(g^\eta, z) - b(g^\varepsilon, z) - b(f, v) + b(f^\varepsilon, v)) R(dz, dv),$$

$$I_2 := \int_{\mathbb{R}^3 \times \mathbb{R}^3} \|(\sigma(g^\varepsilon, z) - \sigma(g^\eta, z))$$
$$- (\sigma(f^\varepsilon, v) - \sigma(f, v))U(a(g^\varepsilon, z), a(f^\varepsilon, v))\|^2 R(dz, dv),$$

$$I_3 := 2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mathrm{Tr}[(\sigma(g^\varepsilon, z) - \sigma(f^\varepsilon, v)U(a(g^\varepsilon, z), a(f^\varepsilon, v)))$$
$$\times ((\sigma(g^\eta, z)$$
$$- \sigma(g^\varepsilon, z)) - (\sigma(f, v) - \sigma(f^\varepsilon, v))U(a(g^\varepsilon, z), a(f^\varepsilon, v)))^*] R(dz, dv).$$

Lemma 3.4 tells us that

$$I_0 \le 8 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z-v|^2 K(|z-z_*|, |v-v_*|) R^\varepsilon(dz_*, dv_*) R(dz, dv),$$

with $R^\varepsilon \in \Pi(g^\varepsilon, f^\varepsilon)$ defined by $R^\varepsilon = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} \delta_{(u+z, u+v)} \phi_\varepsilon(u) R(dz, dv)$.

We now prove (i). We thus assume that $\gamma \in (-1, 0)$. We start with I_0. Since $\gamma \in (-1, 0)$, it holds that $K(x, y) \le \min\{x^\gamma, y^\gamma\} \le y^\gamma$, whence

$$I_0 \le 8 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z-v|^2 |v-v_*|^\gamma R^\varepsilon(dz_*, dv_*) R(dz, dv).$$

Using that R^ε has g^ε for second marginal, we see that $\sup_v \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v-v_*|^\gamma \times R^\varepsilon(dz_*, dv_*) = \sup_v \int_{\mathbb{R}^3} |v-v_*|^\gamma f^\varepsilon(dv_*) \le 1 + C_p \|f^\varepsilon\|_{L^p}$ by (1.4), recall that $p > p_1(\gamma)$. Consequently,

$$I_0 \le C_p(1 + \|f^\varepsilon\|_{L^p}) \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z-v|^2 R(dz, dv).$$

Lemma 2.4(iii) tells us that $|b(f, v) - b(f^\varepsilon, v)| \le C\varepsilon^{1+\gamma}$ and implies, since $\eta \in [0, \varepsilon)$, that $|b(g^\eta, z) - b(g^\varepsilon, z)| \le C\varepsilon^{1+\gamma}$. Using that $x \cdot y \le |x|^2 + |y|^2$, we thus find

$$I_1 \le C \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\varepsilon^{2+2\gamma} + |z-v|^2) R(dz, dv)$$

$$= C\varepsilon^{2+2\gamma} + C \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z-v|^2 R(dz, dv).$$

Using that $\|a - b\|^2 \le 2(\|a\|^2 + \|b\|^2)$, that U is an orthogonal matrix and then Lemma 2.2, we see that

$$I_2 \le 2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\|\sigma(g^\varepsilon, z) - \sigma(g^\eta, z)\|^2 + \|\sigma(f^\varepsilon, v) - \sigma(f, v)\|^2) R(dz, dv)$$

$$\le C \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\|a(g^\varepsilon, z) - a(g^\eta, z)\| + \|a(f^\varepsilon, v) - a(f, v)\|) R(dz, dv).$$

Lemma 2.4(iii) thus implies that $I_2 \le C\varepsilon^{2+\gamma}$, which is smaller than $C\varepsilon^{2+2\gamma}$.

Finally, to bound I_3, we start from the inequality

$$\text{Tr}(AB) \le \|A\| \|B\| \le \tfrac{1}{2}(\varepsilon^{-\gamma} \|A\|^2 + \varepsilon^\gamma \|B\|^2),$$

from which

$$I_3 \le \varepsilon^{-\gamma} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \|\sigma(g^\varepsilon, z) - \sigma(f^\varepsilon, v) U(a(g^\varepsilon, z), a(f^\varepsilon, v))\|^2 R(dz, dv) + \varepsilon^\gamma I_2.$$

First, we already know that $I_2 \le C\varepsilon^{2+\gamma}$. Next, using the same argument as in the very beginning of the proof of Lemma 3.4, since R^ε has g^ε and f^ε for marginals, we deduce that

$$I_3 \le C\varepsilon^{2+2\gamma}$$
$$+ \varepsilon^{-\gamma} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \|\sigma(z - z_*) - \sigma(v - v_*)\|^2 R^\varepsilon(dz_*, dv_*) R(dz, dv).$$

We then use Lemma 2.3 to write

$$I_3 \le C\varepsilon^{2+2\gamma} + C\varepsilon^{-\gamma} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (|z - v|^2 + |z_* - v_*|^2)$$
$$\times (|z - z_*|^\gamma + |v - v_*|^\gamma) R^\varepsilon(dz_*, dv_*) R(dz, dv).$$

Recalling the definition of R^ε and using some symmetry arguments, we find that

$$I_3 \le C\varepsilon^{2+2\gamma}$$
$$+ C\varepsilon^{-\gamma} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} |z - v|^2 (|z - z_* + u|^\gamma + |v - v_* + u|^\gamma)$$
$$\times \phi_\varepsilon(u) \, du \, R(dz_*, dv_*) R(dz, dv).$$

But $\sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} |x - u|^\gamma \phi_\varepsilon(u) \, du \le C\varepsilon^\gamma$, so that finally,

$$I_3 \le C\varepsilon^{2+2\gamma} + C \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 R(dz, dv).$$

We next prove (ii) and thus assume that $\gamma \in (-2, -1]$. We need to improve the estimates of (i), but the additional integrability allows us to do that. We first observe that our conditions $p > p_1(\gamma)$ and $r > (2p - 3)/[(3 + \gamma)(p - 1) - 1]$ imply that $p > 3/2$, that $0 \le p|1 + \gamma|/(2p - 3) < 1$ and that $p|1 + \gamma|/(2p - 3) < 4 + \gamma - 3/r$.

Hence, we can find $\delta \in (0, 1)$ such that $\delta < 4 + \gamma - 3/r$ and $\delta > p|1 + \gamma|/(2p - 3)$. We fix such a δ for the whole proof.

We first treat I_0. By the Young inequality,

$$K(x, y) = \frac{[\min\{x, y\}]^{1+\gamma}}{\max\{x, y\}} \leq \frac{x^{1+\gamma}}{y} + y^\gamma \leq C_\delta(x^{1+\gamma-\delta} + y^{-a}) + y^\gamma,$$

where $a = (\delta - 1 - \gamma)/\delta$. But $a > |\gamma|$ (because $\delta < 1$), whence finally

$$K(x, y) \leq C_\delta(1 + x^{1+\gamma-\delta} + y^{-(\delta-1-\gamma)/\delta}).$$

Consequently,

$$I_0 \leq C_\delta \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 (1 + |z - z_*|^{1+\gamma-\delta} + |v - v_*|^{-(\delta-1-\gamma)/\delta})$$
$$\times R^\varepsilon(dz_*, dv_*) R(dz, dv).$$

Using that the marginals of R^ε are g^ε and f^ε, as well as (1.4), we conclude that

$$I_0 \leq C_{r,p,\delta}(1 + \|g^\varepsilon\|_{L^r} + \|f^\varepsilon\|_{L^p}) \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 R(dz, dv),$$

provided that $r > 3/[4 + \gamma - \delta]$, i.e. $\delta < 4 + \gamma - 3/r$, and $p > 3/[3 - (\delta - 1 - \gamma)/\delta]$, i.e. $\delta > p|1 + \gamma|/(2p - 3)$. Both hold true.

We next treat I_1. Lemma 2.4(ii) implies that

$$|b(x) - b_\varepsilon(x)| \leq C \min\{\varepsilon, |x|\}|x|^\gamma \leq C\varepsilon^\delta |x|^{1+\gamma-\delta}.$$

As a consequence,

$$|b(f, v) - b(f^\varepsilon, v)| = |b(f, v) - b_\varepsilon(f, v)| \leq C\varepsilon^\delta \int_{\mathbb{R}^3} |v - v_*|^{1+\gamma-\delta} f(dv_*)$$

$$\leq C_{r,\delta}(1 + \|f\|_{L^r})\varepsilon^\delta$$

by (1.4) because $r > 3/(4 + \gamma - \delta)$. Using a similar computation for g, we easily find

$$I_1 \leq C_{r,\delta}(1 + \|f\|_{L^r} + \|g\|_{L^r})(m_1(f) + m_1(g))\varepsilon^\delta.$$

As in point (i),

$$I_2 \leq C \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\|a(g^\varepsilon, z) - a(g^\eta, z)\| + \|a(f^\varepsilon, v) - a(f, v)\|) R(dz, dv).$$

By Lemma 2.4(ii), we have $\|a(x) - a_\varepsilon(x)\| \leq C\varepsilon^2(\varepsilon + |x|)^\gamma \leq C\varepsilon^{1+\delta}|x|^{1+\gamma-\delta}$, so that, with the same arguments as when bounding I_1,

$$\|a(f^\varepsilon, v) - a(f, v)\| \leq C\varepsilon^{1+\delta} \int_{\mathbb{R}^3} |v - v_*|^{1+\gamma-\delta} f(dv_*) \leq C_{r,\delta}(1 + \|f\|_{L^r})\varepsilon^{1+\delta},$$

and the same holds for g. Consequently,

$$I_2 \leq C_{r,\delta}(1 + \|f\|_{L^r} + \|g\|_{L^r})\varepsilon^{1+\delta}.$$

To bound I_3, we start as in (i): for any $\alpha > 0$,

$$I_3 \leq \varepsilon^{\alpha} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \| \sigma(g^{\varepsilon}, z) - \sigma(f^{\varepsilon}, v) U(a(g^{\varepsilon}, z), a(f^{\varepsilon}, v)) \|^2 R(dz, dv) + \varepsilon^{-\alpha} I_2.$$

Exactly as in (i), we deduce that

$$I_3 \leq \varepsilon^{-\alpha} I_2 + C \varepsilon^{\alpha} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} |z - v|^2 (|z - z_* + u|^{\gamma} + |v - v_* + u|^{\gamma})$$
$$\times \phi_{\varepsilon}(u) \, du \, R(dz_*, dv_*) R(dz, dv).$$

But for all $x \in \mathbb{R}^3$, $\int_{\mathbb{R}^3} |x + u|^{\gamma} \phi_{\varepsilon}(u) \, du \leq C(\varepsilon + |x|)^{\gamma} \leq C \varepsilon^{-1+\delta} |x|^{1+\gamma-\delta}$, whence

$$I_3 \leq C \varepsilon^{-\alpha} I_2$$
$$+ C_{r,\delta} \varepsilon^{\alpha-1+\delta} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} |z - v|^2 (|z - z_*|^{1+\gamma-\delta} + |v - v_*|^{1+\gamma-\delta})$$
$$\times R(dz_*, dv_*) R(dz, dv).$$

With the same arguments as in the bound of I_1, we conclude, using that $R \in \Pi(g, f)$, that

$$I_3 \leq C \varepsilon^{-\alpha} I_2 + C_{r,\delta} \varepsilon^{\alpha-1+\delta} (1 + \|f\|_{L^r} + \|g\|_{L^r}) \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 R(dz, dv)$$

$$\leq C_{r,\delta} (1 + \|f\|_{L^r} + \|g\|_{L^r}) \left(\varepsilon^{1+\delta-\alpha} + \varepsilon^{\alpha-1+\delta} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 R(dz, dv) \right).$$

Choosing $\alpha = 1 - \delta$, we end with

$$I_3 \leq C_{r,\delta} (1 + \|f\|_{L^r} + \|g\|_{L^r}) \left(\varepsilon^{2\delta} + \int_{\mathbb{R}^3 \times \mathbb{R}^3} |z - v|^2 R(dz, dv) \right).$$

This completes the proof. $\square$

3.3. *Strong/weak stability principles.* The rest of the section is devoted to the following.

PROOF OF THEOREM 1.4. Let $\gamma \in (-2, 0)$ and $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfy also $H(f_0) < \infty$ and $m_q(f_0) < \infty$ for some $q > q(\gamma)$. We consider the unique weak solution f to (1.1) starting from f_0 built in Theorem 1.3, as well as another weak solution $g \in L^{\infty}_{\text{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ starting from $g_0 \in \mathcal{P}_2(\mathbb{R}^3)$. We fix $p \in (p_1(\gamma), p_2(\gamma, q))$. We know that f belongs to $L^{\infty}_{\text{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\text{loc}}([0, \infty), L^p(\mathbb{R}^3))$. If finally $\gamma \in (-2, -1]$, we assume that $g \in L^1_{\text{loc}}([0, \infty), L^r(\mathbb{R}^3))$ for some $r > (2p - 3)/[(3 + \gamma)(p - 1) - 1]$. We assume without loss of generality that $r < p$.

Step 1. By Lemma 2.5(i)–(ii)–(iii), we see that $\sigma(g_t, z)$ and $b(g_t, z)$ satisfy the assumptions of Proposition B.1. Indeed, $\|\sigma(g_t, z)\|^2 \leq C(1 + |z|^2)$ and, if

$\gamma \in [-1, 0)$, $|b(g_t, z)| \leq C(1 + |z|)$. If now $\gamma \in (-2, -1)$, since $r > 3/(4 + \gamma)$, $b(g_t, z) \leq C(1 + \|f_t\|_{L^r})$, which belongs to $L^1_{\mathrm{loc}}([0, \infty))$ by assumption. Consequently, we can find, on some probability space, a Brownian motion $(\mathcal{B}(t))_{t \geq 0}$, independent of a g_0-distributed random variable $\mathcal{Z}(0)$ such that there is a solution $(\mathcal{Z}(t))_{t \geq 0}$ to

$$\mathcal{Z}(t) = \mathcal{Z}(0) + \int_0^t b(g_s, \mathcal{Z}(s)) \, ds + \int_0^t \sigma(g_s, \mathcal{Z}(s)) \, d\mathcal{B}(s)$$

satisfying $\mathcal{L}(\mathcal{Z}(t)) = g_t$ for all $t \geq 0$.

Step 2. Let $\mathcal{V}(0)$ with law f_0 be such that $\mathbb{E}[|\mathcal{V}(0) - \mathcal{Z}(0)|^2] = W_2^2(f_0, g_0)$. Recall that the function U, taking value in the set of orthogonal matrices, was defined in (3.1). We claim that for any $\varepsilon \in (0, 1)$, there is a strong solution $(\mathcal{V}_\varepsilon(t))_{t \geq 0}$ to

$$(3.4) \quad \begin{aligned} \mathcal{V}_\varepsilon(t) = \mathcal{V}(0) + &\int_0^t b(f_s, \mathcal{V}_\varepsilon(s)) \, ds \\ + &\int_0^t \sigma(f_s, \mathcal{V}_\varepsilon(s)) U(a(g_s^\varepsilon, \mathcal{Z}(s)), a(f_s^\varepsilon, \mathcal{V}_\varepsilon(s))) \, d\mathcal{B}_s, \end{aligned}$$

where $f_s^\varepsilon := f_s \star \phi_\varepsilon$ and that furthermore, $\mathcal{L}(\mathcal{V}_\varepsilon(t)) = f_t$ for all $t \geq 0$.

Recalling that $(\mathcal{Z}(t))_{t \geq 0}$ has already been built in step 1, the strong uniqueness for (3.4) classically follows the following facts that we will check a few lines below: $v \mapsto b(f_s, v)$ is Lipschitz (with a Lipschitz constant locally integrable in time), $v \mapsto \sigma(f_s, v)$ is Lipschitz (with a Lipschitz constant locally square-integrable in time) and finally, $v \mapsto U(a(g_s^\varepsilon, \mathcal{Z}(s)), a(f_s^\varepsilon, v))$ is uniformly bounded and locally Lipschitz.

Next, recalling that $U(\Sigma_1, \Sigma_2)$ is orthogonal for all symmetric positive matrices Σ_1 and Σ_2, we deduce that $\beta(t) = \int_0^t U(a(f_s, \mathcal{V}_\varepsilon(s)), a(g_s, \mathcal{Z}(s))) \, d\mathcal{B}(s)$ is also a $3D$ Brownian motion: it suffices to note that β is a $3D$ martingale and that its quadratic variation matrix is $I_3 t$. Hence, $\mathcal{V}_\varepsilon$ also solves the linear SDE (2.1), with initial condition $\mathcal{V}(0)$ and Brownian motion $(\beta(t))_{t \geq 0}$, associated with the weak solution f to (1.1). Consequently, one can check as in the proof of Theorem 1.7, step 2, that $\mathcal{L}(\mathcal{V}_\varepsilon(t)) = f_t$ for all $t \geq 0$.

As already seen in step 1 of the proof of Theorem 1.7, using the fact that $f \in L^\infty_{\mathrm{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0, \infty), L^p(\mathbb{R}^3))$ and Lemma 2.5(iv)–(v), we see that $b(f_s, v)$ is Lipschitz continuous in v, with a Lipschitz constant (locally) integrable in time and that $\sigma(f_s, v)$ is Lipschitz continuous in v, with a Lipschitz constant (locally) square integrable in time as desired. It only remains to check that $v \mapsto U(a(g_s^\varepsilon, \mathcal{Z}(s)), a(f_s^\varepsilon, v))$ is locally Lipschitz continuous. First, we obviously have, by convexity and invariance by translation of the entropy, $H(f_s^\varepsilon) = H(f_s \star \phi_\varepsilon) \leq H(\phi_\varepsilon) \leq C_\varepsilon$, while $m_2(f_s^\varepsilon) \leq 2m_2(f_s) + 2\varepsilon^2$. The same arguments apply for g_s^ε, so that, since the second moment of f and g are (locally) bounded in time by assumption, we get that

$$\sup_{s \in [0, T]} \left[H(f_s^\varepsilon) + H(g_s^\varepsilon) + m_2(f_s^\varepsilon) + m_2(g_s^\varepsilon) \right] \leq C_{\varepsilon, T}.$$

Consequently, we can apply Proposition 2.1 and deduce that, for $s \in [0, T]$ and $v \in \mathbb{R}^3$,

$$(3.5) \qquad \left\| \left(a(f_s^\varepsilon, v) \right)^{-1} \right\| + \left\| \left(a(g_s^\varepsilon, v) \right)^{-1} \right\| \leq C_{\varepsilon, T} \left(1 + |v| \right)^{|\gamma|}.$$

By Lemma 2.5(i), we also see that

$$(3.6) \qquad \left\| a(f_s^\varepsilon, v) \right\| + \left\| a(g_s^\varepsilon, v) \right\| \leq C \left(1 + |v| \right)^2.$$

Using next Lemma 2.5(vi)–(vii) and that $\|f_t^\varepsilon\|_{L^p} \leq \|\phi_\varepsilon\|_{L^p} \leq C_\varepsilon$,

$$(3.7) \qquad \left\| a(f_s^\varepsilon, v) - a(f_s^\varepsilon, v') \right\| \leq C_\varepsilon \left(1 + |v| + |v'| \right) |v - v'|.$$

From the definition (3.1) and Lemma 2.2, it is tedious but straightforward to see that there is $i > 0$ and $C > 0$ such that for all symmetric positive matrices $\Sigma_1, \Sigma_2, \Sigma_2'$,

$$\left\| U(\Sigma_1, \Sigma_2) - U(\Sigma_1, \Sigma_2') \right\|$$
$$\leq C \left(\left\| \Sigma_1^{-1/2} \right\| + \left\| \Sigma_1^{1/2} \right\| + \left\| \Sigma_2^{-1/2} \right\| + \left\| \Sigma_2^{1/2} \right\| + \left\| (\Sigma_2')^{-1/2} \right\| \right.$$
$$\left. + \left\| (\Sigma_2')^{1/2} \right\| \right)^i \left\| \Sigma_2 - \Sigma_2' \right\|.$$

Using this inequality with $\Sigma_1 = a(g_s^\varepsilon, \mathcal{Z}(s))$, $\Sigma_2 = a(f_s^\varepsilon, v)$ and $\Sigma_2' = a(f_s^\varepsilon, v')$, using also (3.5), (3.6) and (3.7), we conclude that indeed,

$$\left\| U\left(a(g_s^\varepsilon, \mathcal{Z}(s)), a(f_s^\varepsilon, v) \right) - U\left(a(g_s^\varepsilon, \mathcal{Z}(s)), a(f_s^\varepsilon, v') \right) \right\|$$
$$\leq C_{\varepsilon, T} \left(1 + |v| + |v'| + |\mathcal{Z}(s)| \right)^j |v - v'|$$

for some $j > 0$. This completes the step.

Step 3. Recall that $\mathcal{L}(\mathcal{Z}(t)) = g_t$ and $\mathcal{L}(\mathcal{V}_\varepsilon(t)) = f_t$ [for any value of $\varepsilon \in (0, 1)$]. We will bound $W_2^2(f_t, g_t)$ by $\mathbb{E}[|\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t)|^2]$. Applying the Itô formula, we directly find that

$$\frac{d}{dt} \mathbb{E} \left[|\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t)|^2 \right]$$
$$= \mathbb{E} \left[2 \left(\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t) \right) \cdot \left(b(g_t, \mathcal{Z}(t)) - b(f_t, \mathcal{V}_\varepsilon(t)) \right) \right.$$
$$\left. + \left\| \sigma(g_t, \mathcal{Z}(t)) - \sigma(f_t, \mathcal{V}_\varepsilon(t)) U\left(a(g_t^\varepsilon, \mathcal{Z}(t)), a(f_t^\varepsilon, \mathcal{V}_\varepsilon(t)) \right) \right\|^2 \right].$$

Setting $R_{\varepsilon, t} = \mathcal{L}(\mathcal{Z}(t), \mathcal{V}_\varepsilon(t))$, which belongs to $\Pi(g, f)$, we realize that with the notation of Lemma 3.2,

$$\frac{d}{dt} \mathbb{E} \left[|\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t)|^2 \right] = \Gamma_{0, \varepsilon}(R_{\varepsilon, t}).$$

Step 4: Proof of point (i). We thus assume that $\gamma \in (-1, 0)$. Applying Proposition 3.2(i), we conclude that

$$\frac{d}{dt} \mathbb{E} \left[|\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t)|^2 \right] \leq C \varepsilon^{2 + 2\gamma} + C_p \left(1 + \|f_t^\varepsilon\|_{L^p} \right) \mathbb{E} \left[|\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t)|^2 \right].$$

But $\mathbb{E}[|\mathcal{Z}(0) - \mathcal{V}(0)|^2] = W_2^2(f_0, g_0)$, so that, by the Grönwall lemma,

$$\mathbb{E}[|\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t)|^2] \le (W_2^2(f_0, g_0) + C\varepsilon^{2+2\gamma}) \exp\left(C_p \int_0^t (1 + \|f_s^\varepsilon\|_{L^p}) \, ds\right).$$

Since finally $\mathcal{L}(\mathcal{V}_\varepsilon(t)) = f_t$ and $\mathcal{L}(\mathcal{Z}(t)) = g_t$, it holds that $W_2^2(f_t, g_t) \le \mathbb{E}[|\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t)|^2]$. To complete the proof, it suffices to let ε tend to 0, noting that $\|f_t^\varepsilon\|_{L^p} \le \|f_t\|_{L^p}$ for all $\varepsilon \in (0, 1)$.

Step 5: *Proof of point* (ii). We assume here that $\gamma \in (-2, -1)$. Applying Proposition 3.2(ii), we conclude that there is $\delta > 0$ such that

$$\frac{d}{dt}\mathbb{E}[|\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t)|^2] \le C_{r,p}(1 + \|f_t^\varepsilon\|_{L^p} + \|g_t^\varepsilon\|_{L^r})\mathbb{E}[|\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t)|^2]$$

$$+ C_{r,p}(1 + \|f_t\|_{L^r} + \|g_t\|_{L^r})(1 + m_1(f_t) + m_1(g_t))\varepsilon^\delta.$$

Using that $\mathbb{E}[|\mathcal{Z}(0) - \mathcal{V}(0)|^2] = W_2^2(f_0, g_0)$, that $(1 + \|f_t\|_{L^r} + \|g_t\|_{L^r})(1 + m_1(f_t) + m_1(g_t))$ is (locally) time-integrable because $f, g \in L_{\text{loc}}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L_{\text{loc}}^1([0, \infty), L^r(\mathbb{R}^3))$ by assumption, the Grönwall lemma, that $W_2^2(f_t, g_t) \le \mathbb{E}[|\mathcal{V}_\varepsilon(t) - \mathcal{Z}(t)|^2]$ for any $\varepsilon \in (0, 1)$ and letting ε tend to 0, we find that

$$W_2^2(f_t, g_t) \le W_2^2(f_0, g_0) \exp\left(C_{r,p} \int_0^t (1 + \|f_s\|_{L^p} + \|g_s\|_{L^r}) \, ds\right).$$

The proof is complete. □

4. Moment estimates for the particle system.

The goal of this section is to check the following propagation of moments. When $\gamma \in [-1, 0)$, we could handle a simpler proof, but the case $\gamma \in (-2, -1)$ really requires a tedious computation.

PROPOSITION 4.1. *Let* $\gamma \in (-2, 0)$. *Let* $N \ge 2$ *and* $\eta_N \in (0, 1)$. *Assume that the initial condition* $(\mathcal{V}_i^N(0))_{i=1,\dots,N}$ *is exchangeable and that* $\mathbb{E}[|\mathcal{V}_1^N(0)|^r] < \infty$ *for some* $r \ge 2$. *Consider the unique solution* $(\mathcal{V}_i^N)_{i=1,\dots,N}$ *to* (1.9). *For all* $T > 0$, *there is a finite constant* $C_{T,r}$ (*not depending on* N) *such that*

$$\sup_{[0,T]} \mathbb{E}[|\mathcal{V}_1^N(t)|^r] \le C_{T,r}\mathbb{E}[|\mathcal{V}_1^N(0)|^r].$$

PROOF. By the Itô formula, setting $\varphi(x) = |x|^r$,

$$\mathbb{E}[|\mathcal{V}_1^N(t)|^r] = \mathbb{E}[|\mathcal{V}_1^N(0)|^r] + r \int_0^t \mathbb{E}[|\mathcal{V}_1^N(s)|^{r-2}\mathcal{V}_1^N(s) \cdot b(\tilde{\mu}_s^N, \mathcal{V}_1^N(s))] \, ds$$

$$+ \frac{1}{2} \int_0^t \mathbb{E}[\text{Tr}(\nabla^2\varphi(\mathcal{V}_1^N(s))a(\tilde{\mu}_s^N, \mathcal{V}_1^N(s)))] \, ds$$

$$=: \mathbb{E}[|\mathcal{V}_1^N(0)|^r] + I_t^N + J_t^N.$$

Using that $b(\tilde{\mu}_s^N, \mathcal{V}_1^N(s)) = N^{-1}\sum_{j=1}^N (b \star \phi_{\eta_N})(\mathcal{V}_1^N(s) - \mathcal{V}_j^N(s))$, that $(b \star \phi_{\eta_N})(0) = 0$ and exchangeability, we find

$$I_t^N = \frac{r(N-1)}{N}\int_0^t \mathbb{E}[|\mathcal{V}_1^N(s)|^{r-2}\mathcal{V}_1^N(s)\cdot(b\star\phi_{\eta_N})(\mathcal{V}_1^N(s)-\mathcal{V}_2^N(s))]\,ds.$$

Using again exchangeability [and that $(b\star\phi_{\eta_N})(-x) = -(b\star\phi_{\eta_N})(x)$], we can symmetrize the expression, which gives

$$I_t^N = \frac{r(N-1)}{2N}\int_0^t \mathbb{E}[(b\star\phi_{\eta_N})(\mathcal{V}_1^N(s)-\mathcal{V}_2^N(s))$$
$$\times(\mathcal{V}_1^N(s)|\mathcal{V}_1^N(s)|^{r-2} - \mathcal{V}_2^N(s)|\mathcal{V}_2^N(s)|^{r-2})]\,ds.$$

Finally, we observe that since $b(x) = -2|x|^\gamma x$ and since $\eta_N \in (0,1)$, $|(b\star\phi_{\eta_N})(x)| \leq C(1+|x|^{\gamma+1})$ for all $x\in\mathbb{R}^3$. Furthermore, $|x|x|^{r-2} - y|y|^{r-2}| \leq C_r|x-y|(1+|x|^{r-2}+|y|^{r-2})$. All this implies that

$$I_t^N \leq C_r \int_0^t \mathbb{E}[(1+|\mathcal{V}_1^N(s)-\mathcal{V}_2^N(s)|^{\gamma+1})|\mathcal{V}_1^N(s)-\mathcal{V}_2^N(s)|$$
$$\times(1+|\mathcal{V}_1^N(s)|^{r-2}+|\mathcal{V}_2^N(s)|^{r-2})]\,ds.$$

Since $\gamma \in (-2,0)$, we easily check that $(1+|v-w|^{\gamma+1})|v-w|(1+|v|^{r-2}+|w|^{r-2}) \leq C_r(1+|v|^r+|w|^r)$, whence finally

$$I_t^N \leq C_r \int_0^t \mathbb{E}[1+|\mathcal{V}_1^N(s)|^r+|\mathcal{V}_2^N(s)|^r]\,ds \leq C_r \int_0^t \mathbb{E}[1+|\mathcal{V}_1^N(s)|^r]\,ds$$

by exchangeability.

Using next the easy estimate $\|a(\tilde{\mu}_s^N, \mathcal{V}_1^N(s))\| = \|N^{-1}\sum_{j=1}^N (a\star\phi_{\eta_N})(\mathcal{V}_1^N(s) - \mathcal{V}_j^N(s))\| \leq CN^{-1}\sum_{j=1}^N (1+|\mathcal{V}_1^N(s)-\mathcal{V}_j^N(s)|^{\gamma+2})$ [because $\eta_N \in (0,1)$], that $|D^2\varphi(x)| \leq C_r|x|^{r-2}$, and exchangeability,

$$J_t^N \leq C_r \int_0^t \mathbb{E}\left[|\mathcal{V}_1^N(s)|^{r-2}\left(1+\frac{1}{N}\sum_{j\neq 1}|\mathcal{V}_1^N(s)-\mathcal{V}_j^N(s)|^{\gamma+2}\right)\right]\,ds$$

$$\leq C_r \int_0^t \mathbb{E}[|\mathcal{V}_1^N(s)|^{r-2}(1+|\mathcal{V}_1^N(s)|^{\gamma+2}+|\mathcal{V}_2^N(s)|^{\gamma+2})]\,ds.$$

Using the Hölder inequality and exchangeability again, we easily conclude that

$$J_t^N \leq C_r \int_0^t \mathbb{E}[1+|\mathcal{V}_1^N(s)|^{r+\gamma}]\,ds \leq C_r \int_0^t \mathbb{E}[1+|\mathcal{V}_1^N(s)|^r]\,ds.$$

We have checked that $\mathbb{E}[|\mathcal{V}_1^N(t)|^r] \leq \mathbb{E}[|\mathcal{V}_1^N(0)|^r] + C_r \int_0^t \mathbb{E}[1+|\mathcal{V}_1^N(s)|^r]\,ds$. The conclusion follows from the Grönwall lemma. $\quad\square$

5. Chaos with rate. The goal of this section is to give the proof of Theorem 1.6. We first introduce a suitable coupling between our particle system $(\mathcal{V}_i^N)_{i=1,\ldots,N}$ and an i.i.d. family $(\mathcal{W}_i^N)_{i=1,\ldots,N}$ of solutions to the nonlinear SDE (1.10). We next prove that we can control the L^2-norm of a blob approximation of the empirical measure of the family $(\mathcal{W}_i^N(t))_{i=1,\ldots,N}$. This allows us to apply our strong/weak stability principle to estimate the mean squared distance between $\mathcal{V}_1^N(t)$ and $\mathcal{W}_1^N(t)$.

In the whole section, we assume that $\gamma \in (-1, 0)$, that $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfies $H(f_0) < \infty$ and $m_q(f_0) < \infty$ for some $q \geq 8$. Observe that $8 > q(\gamma)$, recall (1.7), because $\gamma \in (-1, 0)$. Hence, we can apply Theorem 1.3 and we denote by f the unique weak solution to the Landau equation (1.1), which furthermore belongs to $L_{\mathrm{loc}}^1([0, \infty), L^p(\mathbb{R}^3))$ for any $p \in [1, p_2(\gamma, q))$. Actually, we will only use this estimate with $p = 2$, which is indeed smaller than $p_2(\gamma, q)$, since $q \geq 8$ and $\gamma \in (-1, 0)$.

We fix $N \geq 2$, an independent family of $3D$ Brownian motions $(\mathcal{B}_i)_{i=1,\ldots,N}$, as well as an exchangeable family $(\mathcal{V}_i^N(0))_{i=1,\ldots,N}$, with $\mathbb{E}[|\mathcal{V}_1^N(0)|^4]$ bounded (uniformly in N). We consider $\eta_N \in (0, N^{-1/3})$ as in the statement and the unique solution $(\mathcal{V}_i^N(t))_{i=1,\ldots,N,t\geq0}$ to (1.9) with this η_N. Finally, we fix $\delta \in (0, 1)$ (close to 0) and $\varepsilon_N = N^{-(1-\delta)/3}$. Observe that $\eta_N < \varepsilon_N$.

5.1. *The coupling.* We recall that for $\varepsilon > 0$ and $u \in \mathbb{R}^3$, $\phi_\varepsilon(u) = (4\pi\varepsilon^3/3)^{-1} \times \mathbb{1}_{\{|u|<\varepsilon\}}$, that $\mu_t^N = N^{-1}\sum_{i=1}^N \delta_{\mathcal{V}_i^N(t)}$, that $\tilde{\mu}_t^N = \mu_t^N \star \phi_{\eta_N}$ and we introduce $\bar{\mu}_t^N = \mu_t^N \star \phi_{\varepsilon_N}$. For $i = 1, \ldots, N$, by definition

$$(5.1) \quad \mathcal{V}_i^N(t) = \mathcal{V}_i^N(0) + \int_0^t b(\tilde{\mu}_s^N, \mathcal{V}_i^N(s)) \, ds + \int_0^t \sigma(\tilde{\mu}_s^N, \mathcal{V}_i^N(s)) \, d\mathcal{B}_i(s).$$

By Proposition A.1, we can introduce an i.i.d. family $(\mathcal{W}_i^N(0))_{i=1,\ldots,N}$ of f_0-distributed random variables such that, denoting by F_0^N the law of $(\mathcal{V}_i^N(0))_{i=1,\ldots,N}$, the following properties hold true:

$$(5.2) \qquad \mathbb{E}\left[\sum_1^N |\mathcal{V}_i^N(0) - \mathcal{W}_i^N(0)|^2\right] = W_2^2(F_0^N, f_0^{\otimes N}),$$

$$(5.3) \qquad \text{the family } \{(\mathcal{V}_i^N(0), \mathcal{W}_i^N(0)), i = 1, \ldots, N\} \text{ is exchangeable,}$$

$$\text{a.s.,} \qquad W_2^2\left(N^{-1}\sum_1^N \delta_{\mathcal{V}_i^N(0)}, N^{-1}\sum_1^N \delta_{\mathcal{W}_i^N(0)}\right)$$

$$(5.4)$$

$$= N^{-1}\sum_1^N |\mathcal{V}_i^N(0) - \mathcal{W}_i^N(0)|^2.$$

We finally introduce the system of SDEs with unknown $(\mathcal{W}_i^N(t))_{t \geq 0, i=1,\ldots,N}$:

$$(5.5) \quad \mathcal{W}_i^N(t) = \mathcal{W}_i^N(0) + \int_0^t b(f_s, \mathcal{W}_i^N(s))\, ds$$

$$(5.6) \qquad\qquad + \int_0^t \sigma(f_s, \mathcal{W}_i^N(s)) U\big(a(\bar{\mu}_s^N, \mathcal{V}_i^N(s)), a(\bar{v}_s^N, \mathcal{W}_i^N(s))\big)\, d\mathcal{B}_i(s)$$

with the notation $v_t^N = N^{-1} \sum_{i=1}^N \delta_{\mathcal{W}_i^N(t)}$ and $\bar{v}_t^N = v_t^N \star \phi_{\varepsilon_N}$.

LEMMA 5.1. *The system* (5.5) *has a unique strong solution* $(\mathcal{W}_i^N(t))_{t \geq 0, i=1,\ldots,N}$. *Furthermore, the family* $((\mathcal{W}_i^N(t))_{t \geq 0})_{i=1,\ldots,N}$ *is independent and, for each* $i = 1, \ldots, N$, *the process* $(\mathcal{W}_i^N(t))_{t \geq 0}$ *has the same law as the unique solution* $(\mathcal{V}(t))_{t \geq 0}$ *to the nonlinear SDE* (1.10).

PROOF. The existence and uniqueness can be checked as in step 2 of the proof of Theorem 1.4: the only additional difficulty is to show that the map $(w_1, \ldots, w_N) \mapsto U(a(\bar{\mu}_s^N, \mathcal{V}_i^N(s)), a(N^{-1} \sum_{k=1}^N \delta_{w_k} \star \phi_{\varepsilon_N}, w_i))$ is locally Lipschitz continuous, which is not very hard using that $\bar{\mu}_s^N$ and $N^{-1} \sum_{k=1}^N \delta_{w_k} \star \phi_{\varepsilon_N}$ are bounded probability density functions (recall that $\varepsilon_N > 0$ is fixed here).

Since for each $i = 1, \ldots, N$, the matrix $U_i^N(s) = U(a(\bar{\mu}_s^N, \mathcal{V}_i^N(s)), a(\bar{v}_s^N, \mathcal{W}_i^N(s)))$ is orthogonal, it follows that the family $\beta_i^N(t) = \int_0^t U_i^N(s)\, d\mathcal{B}_i(s)$ consists of N independent $3D$ Brownian motion. To get convinced, it suffices to observe that these are continuous martingales and to compute the quadratic variation matrix. We thus can write

$$\mathcal{W}_i^N(t) = \mathcal{W}_i^N(0) + \int_0^t b(f_s, \mathcal{W}_i^N(s))\, ds + \int_0^t \sigma(f_s, \mathcal{W}_i^N(s))\, d\beta_i^N(s),$$

and the family $((\mathcal{W}_i^N(t))_{t \geq 0})_{i=1,\ldots,N}$ consists of N i.i.d. solutions to the nonlinear SDE (2.1). □

REMARK 5.2. The family $\{(\mathcal{V}_i^N(t), \mathcal{W}_i^N(t))_{t \geq 0}, i = 1, \ldots, N\}$ is exchangeable. This follows from (5.3) and from the symmetry and well-posedness of the systems (5.1) and (5.5).

5.2. *Preliminaries.* We start with two easy lemmas.

LEMMA 5.3. *Here, we only use that* $f \in L^1_{\text{loc}}([0, \infty), L^2(\mathbb{R}^3))$ *and that* $m_2(f_t) = m_2(f_0)$ *for all* $t \geq 0$:

(i) *There is a constant* $c_0 > 0$ *such that for all* $t \geq 0$, $\|f_t\|_{L^2} \geq c_0$.

(ii) *For any* $T \geq 0$, *we can find* $0 = t_0^N < t_1^N < \cdots < t_{K_N}^N \leq T \leq t_{K_N+1}^N$, *with* $K_N \leq 2TN^{1/3}$, *such that* $\sup_{\ell=0,\ldots,K_N}(t_{\ell+1}^N - t_\ell^N) \leq N^{-1/3}$ *and such that, setting*

$$h^N(t) = \sum_{\ell=1}^{K_N+1} \|f(t_\ell^N)\|_{L^2} \mathbb{1}_{\{t \in (t_{\ell-1}^N, t_\ell^N]\}},$$

it holds that $\int_0^T h^N(t)\,dt \leq 2\int_0^T \|f(t)\|_{L^2}\,dt$.

PROOF. For point (i), we recall that $\int_{\mathbb{R}^3} f_t(dv) = \int_{\mathbb{R}^3} f_0(dv) = 1$ and $m_2(f_t) = m_2(f_0)$. We put $M = \sqrt{2m_2(f_0)}$ and we infer from the Cauchy–Schwarz inequality that $\|f_t\|_{L^2}^2 \geq \|f_t \mathbb{1}_{B(0,M)}\|_{L^2}^2 \geq (3/(4\pi M^3))(f_t(B(0,M)))^2$. Here, $f_t(B(0,M)) = \int_{B(0,M)} f_t(dv)$. Next, $f_t(B(0,M)) = 1 - f_t(B(0,M)^c) \geq 1 - m_2(f_t)/M^2 = 1/2$. We conclude with $c_0 = (3/(16\pi M^3))^{1/2}$.

Point (ii) is not difficult: consider $a_\ell = \ell N^{-1/3}/2$ for $\ell = 0, \ldots, K_N + 1$, with $K_N = \lfloor 2TN^{1/3}\rfloor$. Put $t_0^N = 0$ and, for each $i = 1, \ldots, K_N + 1$, consider $t_\ell^N \in (a_{\ell-1}, a_\ell]$ such that $\|f(t_\ell^N)\|_{L^2} \leq 2N^{1/3}\int_{a_{\ell-1}}^{a_\ell} \|f(t)\|_{L^2}$ (here, $2N^{1/3}$ is the length of $(a_{\ell-1}, a_\ell]$). One easily checks that all the conditions are satisfied. $\square$

The second lemma states a few easy and standard properties of the solution to the nonlinear SDE (which actually hold true for any SDE of which the coefficients have at most linear growth).

LEMMA 5.4. *Recall that $\gamma \in (-1,0)$, and that $m_q(f_0) < \infty$ for some $q > 6$. Consider the unique solution $\mathcal{V}$ to the nonlinear SDE* (1.10); *see Proposition* 1.7. *For all $T > 0$,*

$$\mathbb{E}\left[\sup_{t \in [0,T]} |\mathcal{V}(t)|^q + \left(\sup_{0 \leq s < t \leq T} \frac{|\mathcal{V}(t) - \mathcal{V}(s)|}{|t-s|^{1/3}}\right)^q\right] \leq C_{T,q}.$$

PROOF. Since $\gamma \in (-1,0)$ and since $m_2(f_t) = m_2(f_0)$ for all $t \geq 0$, we know from Lemma 2.5(i)–(ii) that $b(f_t, \cdot)$ and $\sigma(f_t, \cdot)$ have at most linear growth, uniformly in $t \geq 0$. Since $\mathbb{E}[|\mathcal{V}(0)|^q] = m_q(f_0) < \infty$, standard computations involving the Burkholder–Davis–Gundy inequality show that for all $T > 0$, $\mathbb{E}[\sup_{t \in [0,T]} |\mathcal{V}(t)|^q] \leq C_{T,q}$. Standard computations again show that for all $0 \leq s < t \leq T$, $\mathbb{E}[|\mathcal{V}(t) - \mathcal{V}(s)|^q] \leq C_{T,q}|t-s|^{q/2}$. By the Kolmogorov criterion (see Revuz–Yor [31], Theorem 2.1, page 26), we conclude that for any $\alpha \in (0, 1/2 - 1/q)$,

$$\mathbb{E}\left[\left(\sup_{0 \leq s < t \leq T} \frac{|\mathcal{V}(t) - \mathcal{V}(s)|}{|t-s|^{\alpha}}\right)^q\right] \leq C_{T,q}.$$

The choice $\alpha = 1/3$ is licit since $q > 6$. $\square$

5.3. *On the L^2 norm of the blob limit empirical measure.* The following proposition is an important step. Similar considerations were used in [20].

PROPOSITION 5.5. *Recall that $\gamma \in (-1,0)$, that f is the unique weak solution to* (1.1) *starting from $f_0 \in \mathcal{P}(\mathbb{R}^3)$ with a finite entropy and a finite moment of order $q > 6$. Fix $T > 0$ and consider h^N built in Lemma* 5.3. *Consider the solution*

$(\mathcal{W}_i^N)_{i=1,\ldots,N}$ to (5.5) and recall that $v_t^N = N^{-1} \sum_1^N \delta_{\mathcal{W}_i^N(t)}$ and $\bar{v}_t^N = v_t^N \star \phi_{\varepsilon_N}$ with $\varepsilon_N = N^{-(1-\delta)/3}$. We have

$$\Pr(\forall t \in [0,T], \|\bar{v}_t^N\|_{L^2} \le 173 h^N(t)) \ge 1 - C_{T,q,\delta} N^{1-\delta q/3}.$$

Of course, 173 is not at all the optimal constant.

PROOF OF PROPOSITION 5.5.   It is quite complicated, so we break it into several steps.

Step 1. We introduce the event $\Omega_{T,N}^1$ on which

$$\forall i = 1, \ldots, N$$

$$\sup_{[0,T+1]} |\mathcal{W}_i^N(t)| + \sup_{0 \le s < t \le T+1} (t-s)^{-1/3} |\mathcal{W}_i^N(t) - \mathcal{W}_i^N(s)| \le N^{\delta/3}$$

and we prove that

$$\Pr((\Omega_{T,N}^1)^c) \le C_{T,q} N^{1-\delta q/3}.$$

Since each $\mathcal{W}_i^N$ has the same law as the solution $\mathcal{V}$ to (1.10),

$$\Pr((\Omega_{T,N}^1)^c) \le N \Pr\left(\sup_{[0,T+1]} |\mathcal{V}(t)| + \sup_{0 \le s < t \le T+1} (t-s)^{-1/3} |\mathcal{V}(t) - \mathcal{V}(s)| \ge N^{\delta/3}\right).$$

Thus, Lemma 5.4 and the Markov inequality give us $\Pr((\Omega_{T,N}^1)^c) \le C_{T,q} N \cdot N^{-\delta q/3}$.

Step 2. We consider the natural partition $\mathcal{P}_N$ of $\mathbb{R}^3$ in cubes with edge length ε_N and call $\mathcal{P}_N^\delta$ the subset of its elements that intercept $B(0, N^{\delta/3})$. Observe that $\#(\mathcal{P}_N^\delta) \le (2(N^{\delta/3} + 1)\varepsilon_N^{-1})^3 \le 64 N^\delta \varepsilon_N^{-3} = 64N$. We claim that for any $(x_1, \ldots, x_N) \in B(0, N^{\delta/3})^N$ and any $(y_1, \ldots, y_N) \in B(0, N^{\delta/3})^N$ such that $\sup_{i=1,\ldots,N} |x_i - y_i| \le \varepsilon_N$,

$$\left\| \left(N^{-1} \sum_1^N \delta_{y_i} \right) \star \phi_{\varepsilon_N} \right\|_{L^2}^2 \le \frac{3731}{N^2 \varepsilon_N^3} \sum_{D \in \mathcal{P}_N^\delta} (\#\{i : x_i \in D\})^2.$$

We start with

$$\left(N^{-1} \sum_1^N \delta_{y_i} \right) \star \phi_{\varepsilon_N}(v) = \frac{3}{4\pi N \varepsilon_N^3} \#\{i : y_i \in B(v, \varepsilon_N)\}$$

$$\le \frac{3}{4\pi N \varepsilon_N^3} \#\{i : x_i \in B(v, 2\varepsilon_N)\},$$

whence

$$\left(N^{-1} \sum_1^N \delta_{y_i} \right) \star \phi_{\varepsilon_N}(v) \le \frac{3}{4\pi N \varepsilon_N^3} \sum_{D \in \mathcal{P}_N^\delta} \#\{i : x_i \in D\} \mathbb{1}_{\{D \cap B(v, 2\varepsilon_N) \ne \varnothing\}}.$$

Consequently, setting $A = \|(N^{-1} \sum_1^N \delta_{y_i}) \star \phi_{\varepsilon_N}\|_{L^2}^2$,

$$A \le \frac{9}{16\pi^2 N^2 \varepsilon_N^6} \sum_{D,D' \in \mathcal{P}_N^\delta} \#\{i : x_i \in D\} \#\{i : x_i \in D'\}$$

$$\times \int_{\mathbb{R}^3} \mathbb{1}_{\{D \cap B(v, 2\varepsilon_N) \neq \varnothing, D' \cap B(v, 2\varepsilon_N) \neq \varnothing\}} \, dv.$$

Using that $2xy \le x^2 + y^2$ and a symmetry argument, we find that

$$A \le \frac{9}{16\pi^2 N^2 \varepsilon_N^6} \sum_{D \in \mathcal{P}_N^\delta} \left(\#\{i : x_i \in D\}\right)^2 \int_{\mathbb{R}^3} \mathbb{1}_{\{D \cap B(v, 2\varepsilon_N) \neq \varnothing\}}$$

$$\times \sum_{D' \in \mathcal{P}_N^\delta} \mathbb{1}_{\{D' \cap B(v, 2\varepsilon_N) \neq \varnothing\}} \, dv.$$

But for each $v \in \mathbb{R}^3$, $\sum_{D' \in \mathcal{P}_N^\delta} \mathbb{1}_{\{D' \cap B(v, 2\varepsilon_N) \neq \varnothing\}} = \#\{D' \in \mathcal{P}_N^\delta : D' \cap B(v, 2\varepsilon_N) \neq \varnothing\} \le 5^3$ and for each $D \in \mathcal{P}_N^\delta$, we easily check that $\int_{\mathbb{R}^3} \mathbb{1}_{\{D \cap B(v, 2\varepsilon_N) \neq \varnothing\}} \, dv \le 4\pi(5\varepsilon_N)^3/3$. Finally, we have verified that

$$A \le \frac{9}{16\pi^2 N^2 \varepsilon_N^6} \times \frac{5^3 4\pi (5\varepsilon_N)^3}{3} \sum_{D \in \mathcal{P}_N^\delta} \left(\#\{i : x_i \in D\}\right)^2$$

$$\le \frac{3731}{N^2 \varepsilon_N^3} \sum_{D \in \mathcal{P}_N^\delta} \left(\#\{i : x_i \in D\}\right)^2.$$

Step 3. We now fix $t \in [0, T + 1]$, we consider the event

$$\Omega_{t,N}^2 = \left\{ N^{-2} \varepsilon_N^{-3} \sum_{D \in \mathcal{P}_N^\delta} \left(\#\{i : \mathcal{W}_i^N(t) \in D\}\right)^2 \le 8\|f_t\|_{L^2}^2 \right\}$$

and we prove that there are some positive constants C and c (depending only on δ and c_0, recall that $\|f_t\|_{L^2} \ge c_0$ by Lemma 5.3) such that

$$\Pr\left((\Omega_{t,N}^2)^c\right) \le C \exp(-cN^{\delta/2}).$$

To this end, we introduce, for $D \in \mathcal{P}_N^\delta$, $Z_D = \#\{i : \mathcal{W}_i^N(t) \in D\}$. It follows a Binomial$(N, f_t(D))$-distribution. It thus holds that

$$(5.7) \qquad \Pr(Z_D \ge x) \le \exp(-x/8) \qquad \text{for all } x \ge 2Nf_t(D).$$

Indeed, $\mathbb{E}[\exp(Z_D)] = \exp(N \log(1 + f_t(D)(e - 1))) \le \exp(N(e - 1)f_t(D))$, whence $\Pr(Z_D \ge x) \le \exp(-x + N(e - 1)f_t(D)) \le \exp(-x + (e - 1)x/2)$ if $x \ge 2Nf_t(D)$. And $1 - (e - 1)/2 > 1/8$ holds true. We next observe that on the one hand, by the Cauchy–Schwarz inequality,

$$\|f_t\|_{L^2}^2 \ge \sum_{D \in \mathcal{P}_N^\delta} \int_D f_t^2(v) \, dv \ge \varepsilon_N^{-3} \sum_{D \in \mathcal{P}_N^\delta} (f_t(D))^2,$$

and on the other hand, since $\#(\mathcal{P}_N^\delta) \leq 64 N^\delta \varepsilon_N^{-3}$,

$$\|f_t\|_{L^2}^2 \geq 64^{-1} N^{-\delta} \varepsilon_N^3 \sum_{D \in \mathcal{P}_N^\delta} \|f_t\|_{L^2}^2.$$

All in all,

$$8\|f_t\|_{L^2}^2 \geq \sum_{D \in \mathcal{P}_N^\delta} \left(4\varepsilon_N^{-3}(f_t(D))^2 + 16^{-1} N^{-\delta} \varepsilon_N^3 \|f_t\|_{L^2}^2\right).$$

Consequently, on the event $(\Omega_{t,N}^2)^c$, there is at least one $D \in \mathcal{P}_N^\delta$ for which there holds $Z_D^2 \geq N^2 \varepsilon_N^3 [4\varepsilon_N^{-3}(f_t(D))^2 + 16^{-1} N^{-\delta} \varepsilon_N^3 \|f_t\|_{L^2}^2]$, whence

$$\Pr((\Omega_{t,N}^2)^c) \leq \sum_{D \in \mathcal{P}_N^\delta} \Pr\left(Z_D \geq N\varepsilon_N^{3/2}[4\varepsilon_N^{-3} f_t(D) + 16^{-1} N^{-\delta} \varepsilon_N^3 \|f_t\|_{L^2}^2]^{1/2}\right).$$

But $x_N := N\varepsilon_N^{3/2}[4\varepsilon_N^{-3}(f_t(D))^2 + 16^{-1} N^{-\delta} \varepsilon_N^3 \|f_t\|_{L^2}^2]^{1/2} \geq N\varepsilon_N^{3/2} \times (4\varepsilon_N^{-3}(f_t(D))^2)^{1/2}$ which equals $2Nf_t(D)$, so that we can apply (5.7). Since we also have the inequality $x_N \geq N\varepsilon_N^{3/2} 4^{-1} N^{-\delta/2} \varepsilon_N^{3/2} \|f_t\|_{L^2} = N^{\delta/2} \|f_t\|_{L^2}/4$, we conclude that

$$\Pr((\Omega_{t,N}^2)^c) \leq \sum_{D \in \mathcal{P}_N^\delta} \exp(-N^{\delta/2} \|f_t\|_{L^2}/32) \leq 64N \exp(-c_0 N^{\delta/2}/32).$$

We used that $\#(\mathcal{P}_N^\delta) \leq 64N$, that $N\varepsilon_N^3 = N^\delta$ and that $\|f_t\|_{L^2} \geq c_0$. This completes the step.

Step 4. We finally introduce the event

$$\Omega_{T,N} = \Omega_{T,N}^1 \cap \left(\bigcap_{\ell=1}^{K_N+1} \Omega_{t_\ell^N,N}^2\right),$$

where $0 = t_0^N < t_1^N < \cdots < t_{K_N}^N < T < t_{K_N+1}^N$, with $K_N \leq 2TN^{1/3}$ and $\sup_{\ell=0,\ldots,K_N}(t_{\ell+1}^N - t_\ell^N) \leq N^{-1/3}$, were defined in Lemma 5.3. Recall also that we introduced $h^N(t) = \sum_{\ell=1}^{K_N+1} \|f(t_\ell^N)\|_{L^2} \mathbb{1}_{\{t \in (t_{\ell-1}^N, t_\ell^N]\}}$.

We first claim that, $\Pr(\Omega_{T,N}) \geq 1 - C_{T,q,\delta} N^{1-q\delta/3}$. This follows from the fact that $K_N \leq 2TN^{1/3}$ and from steps 1 and 3: we have $\Pr(\Omega_{T,N}^c) \leq \Pr((\Omega_{T,N}^1)^c) + \sum_{\ell=1}^{K_N+1} \Pr((\Omega_{t_\ell^N,N}^2)^c) \leq C_{T,q} N^{1-q\delta/3} + C(K_N + 1) \exp(-cN^{\delta/2}) \leq C_{T,q,\delta} \times N^{1-q\delta/3}$. The claim follows.

It only remains to prove that indeed, $\|\bar{v}_t^N\|_{L^2} \leq 173 h^N(t)$ for all $t \in [0, T]$ on $\Omega_{T,N}$. Recall that $\bar{v}_t^N = (N^{-1} \sum_1^N \delta_{\mathcal{W}_i^N(t)}) \star \phi_{\varepsilon_N}$. On $\Omega_{T,N}$, we know that:

• $\mathcal{W}_i^N(t)$ belongs to $B(0, N^{\delta/3})$ for all $i = 1, \ldots, N$ and all $t \in [0, T+1]$ (thanks to $\Omega_{T,N}^1$);

- for all $\ell = 1, \ldots, K_N + 1$, all $t \in [t_{\ell-1}^N, t_\ell^N]$, all $i = 1, \ldots, N$, $|\mathcal{W}_i^N(t) - \mathcal{W}_i^N(t_\ell^N)| \leq N^{\delta/3}|t_\ell^N - t_{\ell-1}^N| \leq N^{\delta/3 - 1/3} = \varepsilon_N$ (due to $\Omega_{T,N}^1$ again);

- for all $\ell = 1, \ldots, K_N + 1$, $N^{-2}\varepsilon_N^{-3} \sum_{D \in \mathcal{P}_N^\delta} (\#\{i : \mathcal{W}_i^N(t_\ell^N) \in D\})^2 \leq 8\|f_{t_\ell^N}\|^2$ (due to $\Omega_{t_\ell^N, N}^2$).

Using step 2, we conclude that indeed, on $\Omega_{T,N}$, for all $t \in [0, T]$, defining ℓ as the index such that $t \in (t_{\ell-1}^N, t_\ell^N]$,

$$\|\bar{v}_t^N\|_{L^2}^2 \leq 3731 N^{-2}\varepsilon_N^{-3} \sum_{D \in \mathcal{P}_N^\delta} (\#\{i : \mathcal{W}_i(t_\ell^N) \in D\})^2$$

$$\leq 3731 \times 8\|f_{t_\ell^N}\|^2$$

$$= 29{,}848(h^N(t))^2.$$

This completes the proof, since $\sqrt{29{,}848} < 173$. $\quad\square$

5.4. *Computation of the mean squared error.*   Here is the main computation of the section.

PROPOSITION 5.6.   *Recall that $\gamma \in (-1, 0)$, that f is the unique weak solution to* (1.1) *starting from $f_0 \in \mathcal{P}(\mathbb{R}^3)$ with a finite entropy and with $m_8(f_0) < \infty$. Recall that $\eta_N \in (0, N^{-1/3})$, that $\varepsilon_N = N^{-(1-\delta)/3}$ and that $(\mathcal{V}_i^N)_{i=1,\ldots,N}$ and $(\mathcal{W}_i^N)_{i=1,\ldots,N}$ are the solutions to* (5.1) *and* (5.5). *Recall that $\mu_t^N = N^{-1}\sum_1^N \delta_{\mathcal{V}_i^N(t)}$, $v_t^N = N^{-1}\sum_1^N \delta_{\mathcal{W}_i^N(t)}$, $\tilde{\mu}_t^N = \mu_t^N \star \phi_{\eta_N}$, $\bar{\mu}_t^N = \mu_t^N \star \phi_{\varepsilon_N}$ and $\bar{v}_t^N = v_t^N \star \phi_{\varepsilon_N}$. Fix $T > 0$, recall that h^N was defined in Lemma 5.3 and consider the stopping time*

$$\tau_N = \inf\{t \geq 0 : \|\bar{v}_t^N\|_{L^2} \geq 173 h^N(t)\}.$$

There is a constant $C_{T,\delta} > 0$ such that for all $t \in [0, T]$,

$$\mathbb{E}[|\mathcal{V}_1^N(t \wedge \tau_N) - \mathcal{W}_1^N(t \wedge \tau_N)|^2]$$

$$\leq C_{T,\delta}(N^{-(1-\delta)(2+2\gamma)/3} + N^{-1/2} + \mathbb{E}[|\mathcal{V}_1^N(0) - \mathcal{W}_1^N(0)|^2]).$$

It seems that we could remove the term $N^{-1/2}$ with some work, but this would not improve the final result of Theorem 1.6.

PROOF OF PROPOSITION 5.6.   We put $u_t = \mathbb{E}[|\mathcal{V}_1^N(t \wedge \tau_N) - \mathcal{W}_1^N(t \wedge \tau_N)|^2]$. To lighten notation, we also set $U_i^N(s) = U(a(\bar{\mu}_s^N, \mathcal{V}_i^N(s)), a(\bar{v}_s^N, \mathcal{W}_i^N(s)))$. By Lemma 5.4, we know that $\sup_{[0,T]} m_8(f_t) < \infty$.

Step 1. A direct application of the Itô formula gives

$$u_t = \mathbb{E}\Big[\big|\mathcal{V}_1^N(0) - \mathcal{W}_1^N(0)\big|^2$$

$$+ \int_0^{t \wedge \tau_N} \big(2(\mathcal{V}_1^N(s) - \mathcal{W}_1^N(s)) \cdot (b(\tilde{\mu}_s^N, \mathcal{V}_1^N(s)) - b(f_s, \mathcal{W}_1^N(s)))$$

$$+ \big\|\sigma(\tilde{\mu}_s^N, \mathcal{V}_1^N(s)) - \sigma(f_s, \mathcal{W}_1^N(s))U_1^N(s)\big\|^2\big) \, ds\Big]$$

$$= \mathbb{E}[|\mathcal{V}_1^N(0) - \mathcal{W}_1^N(0)|^2] + \mathbb{E}\Big[\int_0^{t \wedge \tau_N} (I_s + J_s + K_s) \, ds\Big],$$

where

$$I_s := 2(\mathcal{V}_1^N(s) - \mathcal{W}_1^N(s)) \cdot (b(\tilde{\mu}_s^N, \mathcal{V}_1^N(s)) - b(v_s^N, \mathcal{W}_1^N(s)))$$

$$+ \big\|\sigma(\tilde{\mu}_s^N, \mathcal{V}_1^N(s)) - \sigma(v_s^N, \mathcal{W}_1^N(s))U_1^N(s)\big\|^2,$$

$$J_s := 2(\mathcal{V}_1^N(s) - \mathcal{W}_1^N(s)) \cdot (b(v_s^N, \mathcal{W}_1^N(s)) - b(f_s, \mathcal{W}_1^N(s))),$$

$$K_s := \big\|\sigma(\tilde{\mu}_s^N, \mathcal{V}_1^N(s)) - \sigma(f_s, \mathcal{W}_1^N(s))U_1^N(s)\big\|^2$$

$$- \big\|\sigma(\tilde{\mu}_s^N, \mathcal{V}_1^N(s)) - \sigma(v_s^N, \mathcal{W}_1^N(s))U_1^N(s)\big\|^2.$$

Let us notice at once that

$$J_s \leq \big|\mathcal{V}_1^N(s) - \mathcal{W}_1^N(s)\big|^2 + \big|b(v_s^N, \mathcal{W}_1^N(s)) - b(f_s, \mathcal{W}_1^N(s))\big|^2$$

$$=: \big|\mathcal{V}_1^N(s) - \mathcal{W}_1^N(s)\big|^2 + J_s^1$$

and, using that $\|A - B\|^2 - \|A - B'\|^2 \leq \|B - B'\|^2 + 2\|A - B'\|\|B' - B\|$ and that $U_1^N(s)$ is an orthogonal matrix,

$$K_s \leq K_s^1 + \sqrt{M_s K_s^1},$$

where

$$K_s^1 := \big\|\sigma(f_s, \mathcal{W}_1^N(s)) - \sigma(v_s^N, \mathcal{W}_1^N(s))\big\|^2,$$

$$M_s := \big\|\sigma(\tilde{\mu}_s^N, \mathcal{V}_1^N(s)) - \sigma(v_s^N, \mathcal{W}_1^N(s))U_1^N(s)\big\|^2.$$

Step 2. Using exchangeability, we realize that

$$\mathbb{E}\Big[\int_0^{t \wedge \tau_N} I_s \, ds\Big]$$

$$= \mathbb{E}\Big[\int_0^{t \wedge \tau_N} \frac{1}{N} \sum_{i=1}^N (2(\mathcal{V}_i^N(s) - \mathcal{W}_i^N(s))(b(\tilde{\mu}_s^N, \mathcal{V}_i^N(s)) - b(v_s^N, \mathcal{W}_i^N(s)))$$

$$+ \big\|\sigma(\tilde{\mu}_s^N, \mathcal{V}_i^N(s)) - \sigma(v_s^N, \mathcal{W}_i^N(s))$$

$$\times U\big(a(\bar{\mu}_s^N, \mathcal{V}_i^N(s)), a(\bar{v}_s^N, \mathcal{W}_i^N(s))\big)\big\|^2\big) \, ds\Big].$$

Setting $R_t^N = N^{-1} \sum_1^N \delta_{(\mathcal{V}_i^N(s), \mathcal{W}_i^N(s))}$, which belongs to $\Pi(\mu_t^N, \nu_t^N)$, observing that $\tilde{\mu}_s^N = (\mu_s^N)^{\eta_N}$, $\bar{\mu}_s^N = (\mu_s^N)^{\varepsilon_N}$ and $\bar{\nu}_s^N = (\nu_s^N)^{\varepsilon_N}$, and recalling the notation of Proposition 3.2, this directly gives

$$\mathbb{E}\left[\int_0^{t \wedge \tau_N} I_s \, ds\right] = \mathbb{E}\left[\int_0^{t \wedge \tau_N} \Gamma_{\eta_N, \varepsilon_N}(R_s^N) \, ds\right].$$

We thus use Proposition 3.2(i) [with $p = 2$ which is indeed greater than $p_1(\gamma)$] and obtain, since $\|\bar{\nu}_s^N\|_{L^2} \leq 173 h^N(s)$ for all $s \leq \tau_N$,

$$\mathbb{E}\left[\int_0^{t \wedge \tau_N} I_s \, ds\right]$$
$$\leq Ct\varepsilon_N^{2+2\gamma} + C\mathbb{E}\left[\int_0^{t \wedge \tau_N} (1 + h_N(s)) \frac{1}{N} \sum_{i=1}^N |\mathcal{V}_i^N(s) - \mathcal{W}_i^N(s)|^2 \, ds\right].$$

Using exchangeability again, we end with

$$\mathbb{E}\left[\int_0^{t \wedge \tau_N} I_s \, ds\right] \leq Ct\varepsilon_N^{2+2\gamma} + C\int_0^t (1 + h_N(s)) u_s \, ds.$$

Step 3. We now check that $\mathbb{E}[J_s^1 + K_s^1] \leq C_T N^{-1}$ for all $s \in [0, T]$. This will imply that

$$\mathbb{E}\left[\int_0^{t \wedge \tau_N} (J_s + K_s^1) \, ds\right] \leq C_T N^{-1} + \int_0^t u_s \, ds.$$

Recall that $m_2(f_t) = m_2(f_0)$ and $H(f_t) \leq H(f_0)$ for all $t \geq 0$ (see Theorem 1.3), so that Lemma 2.1 ensures us that $\|(a(f_t, v))^{-1}\| \leq C(1 + |v|)^{|\gamma|}$ for all $t \geq 0$ and all $v \in \mathbb{R}^3$. Consequently, using Lemma 2.2, we deduce that $K_s^1 \leq C(1 + |\mathcal{W}_1^N(s)|)^{|\gamma|} \|a(\nu_s^N, \mathcal{W}_1^N(s)) - a(f_s, \mathcal{W}_1^N(s))\|^2$. By the Cauchy–Schwarz inequality, and since $\mathcal{L}(\mathcal{W}_1^N(s)) = f_s$,

$$\mathbb{E}[K_s^1] \leq C(1 + m_{2|\gamma|}(f_s))^{1/2} \mathbb{E}[\|a(\nu_s^N, \mathcal{W}_1^N(s)) - a(f_s, \mathcal{W}_1^N(s))\|^4]^{1/2}$$
$$\leq C_T \mathbb{E}[\|a(\nu_s^N, \mathcal{W}_1^N(s)) - a(f_s, \mathcal{W}_1^N(s))\|^4]^{1/2}.$$

We also have $J_s^1 = |b(\nu_s^N, \mathcal{W}_1^N(s)) - b(f_s, \mathcal{W}_1^N(s))|^2$, so that, using the Cauchy–Schwarz inequality again,

$$\mathbb{E}[J_s^1] \leq C\mathbb{E}[\|b(\nu_s^N, \mathcal{W}_1^N(s)) - b(f_s, \mathcal{W}_1^N(s))\|^4]^{1/2}.$$

To conclude the step, it thus suffices to prove that for $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$ with at most quadratic growth (which is the case of all the entries of a and b), with the notation $\varphi(\mu, x) = \int_{\mathbb{R}^3} \varphi(x - y)\mu(dy)$,

$$\mathbb{E}[|\varphi(\nu_s^N, \mathcal{W}_1^N(s)) - \varphi(f_s, \mathcal{W}_1^N(s))|^4] \leq C_{T,\varphi} N^{-2}.$$

To this end, let us recall that there is a constant $C > 0$ such that for any sequence of i.i.d. real-valued random variables $(X_n)_{n \geq 1}$ (see Rosenthal [33] for a much more general inequality),

$$(5.8) \qquad \mathbb{E}\left[\left| N^{-1} \sum_1^N (X_i - \mathbb{E}[X_1]) \right|^4\right] \leq C(1 + \mathbb{E}[X_1^4]) N^{-2}.$$

But the family $(\mathcal{W}_i^N(s))_{i=1,\ldots,N}$ is i.i.d. with common law f_s. Let us denote by $\mathbb{E}_1$ the expectation concerning only $\mathcal{W}_1^N$, and by $\mathbb{E}_{2,N}$ the expectation concerning only $\mathcal{W}_2^N, \ldots, \mathcal{W}_N^N$. We have $\varphi(\mathcal{W}_1^N(s), v_s^N) = N^{-1} \sum_{i=1}^N \varphi(\mathcal{W}_1^N(s) - \mathcal{W}_i^N(s))$ and it holds by independence that $\varphi(\mathcal{W}_1^N(s), f_s) = (N - 1)^{-1} \sum_{i=2}^N \mathbb{E}_{2,N} \times [\varphi(\mathcal{W}_1^N(s) - \mathcal{W}_i^N(s))]$. As a consequence, since $(x + y)^4 \leq 8x^4 + 8y^4$,

$$\mathbb{E}[|\varphi(v_s^N, \mathcal{W}_1^N(s)) - \varphi(f_s, \mathcal{W}_1^N(s))|^4]$$

$$\leq 8\mathbb{E}\left[\left| \left(\frac{1}{N} - \frac{1}{N-1}\right) \sum_1^N \varphi(\mathcal{W}_1^N(s) - \mathcal{W}_i^N(s)) \right|^4\right]$$

$$+ 8\mathbb{E}_1\left[\mathbb{E}_{2,N}\left[\left| \frac{1}{N-1} \sum_2^N \varphi(\mathcal{W}_1^N(s) - \mathcal{W}_i^N(s)) \right.\right.\right.$$

$$\left.\left.\left. - \mathbb{E}_{2,N}\left[\frac{1}{N-1} \sum_2^N \varphi(\mathcal{W}_1^N(s) - \mathcal{W}_i^N(s))\right]\right|^4\right]\right]$$

$$=: 8A_N + 8B_N.$$

Using that $|\varphi(x)| \leq C_\varphi(1 + |x|^2)$, we easily get $A_N \leq C_\varphi(1 + m_8(f_s))N^{-4} \leq C_{\varphi,T} N^{-4}$. Since the random variables $\varphi(\mathcal{W}_1^N(s) - \mathcal{W}_i^N(s))$ are i.i.d. under $\mathbb{E}_{2,N}$, we may apply (5.8), which gives

$$B_N \leq CN^{-2}\mathbb{E}_1[1 + \mathbb{E}_{2,N}(\varphi^4(\mathcal{W}_1^N(s) - \mathcal{W}_i^N(s)))].$$

Using again that $|\varphi(x)| \leq C_\varphi(1 + |x|^2)$, we deduce that $B_N \leq C_\varphi(1 + m_8(f_s)) \times N^{-2} \leq C_{\varphi,T} N^{-2}$.

Step 4. We finally show that $\sup_{[0,T]} \mathbb{E}[M_s] \leq C_T$. This will imply, by step 3 and the Cauchy–Schwarz inequality, that $\mathbb{E}[\sqrt{K_s^1 M_s}] \leq C_T N^{-1/2}$, whence

$$\mathbb{E}\left[\int_0^{t \wedge \tau_N} \sqrt{K_s^1 M_s}\, ds\right] \leq C_T N^{-1/2}.$$

Thanks to Lemma 2.5(i), we have

$$M_s \leq C(1 + m_2(\tilde{\mu}_s^N) + m_2(v_s^N) + |\mathcal{V}_1^N(s)|^{2+\gamma} + |\mathcal{W}_1^N(s)|^{2+\gamma}).$$

Using exchangeability, that $\eta_N \leq 1$ and that $0 \leq \gamma + 2 \leq 2$, we see that

$$\mathbb{E}[M_s] \leq C(1 + \mathbb{E}[|\mathcal{V}_1^N(s)|^2 + |\mathcal{W}_1^N(s)|^2]).$$

We conclude the step using that $\mathbb{E}[|\mathcal{W}_1^N(s)|^2] = m_2(f_s) = m_2(f_0)$ and that, by Proposition 4.1, $\sup_{[0,T]} \mathbb{E}[|\mathcal{V}_1^N(s)|^2]$ is bounded (uniformly in N).

Step 5. We now gather everything:

$$u_t \leq u_0 + C \int_0^t (1 + h_s^N) u_s \, ds + C_T (\varepsilon_N^{2\gamma+2} + N^{-1/2}).$$

Using the Grönwall lemma, we get

$$\sup_{[0,T]} u_t \leq C_T (u_0 + \varepsilon_N^{2\gamma+2} + N^{-1/2}) \exp\Big(C \int_0^T h^N(s) \, ds \Big)$$

$$\leq C_T (u_0 + \varepsilon_N^{2\gamma+2} + N^{-1/2}),$$

since $\int_0^T h^N(s) \, ds \leq 2 \int_0^T \|f(s)\|_{L^2} \, ds$ by Lemma 5.3 and since $f \in L_{\mathrm{loc}}^1([0, \infty), L^2(\mathbb{R}^3))$. This completes the proof. $\square$

5.5. *Conclusion.* We finally can give the following.

PROOF OF THEOREM 1.6. Recall that $q \geq 8$, fix $\delta = 6/q$ and recall that $\tau_N = \inf\{t \geq 0 : \|\bar{v}_t^N\|_{L^2} \geq 173 h^N(t)\}$. By Proposition 5.5, we know that $\Pr(\tau_N \leq T) \leq C_{T,q,\delta} N^{1-q\delta/3} = C_{T,q} N^{-1}$. We then write, for $t \in [0, T]$,

$$\mathbb{E}[|\mathcal{V}_1^N(t) - \mathcal{W}_1^N(t)|^2]$$

$$\leq \mathbb{E}[|\mathcal{V}_1^N(t \wedge \tau_N) - \mathcal{W}_1^N(t \wedge \tau_N)|^2] + \mathbb{E}[|\mathcal{V}_1^N(t) - \mathcal{W}_1^N(t)|^2 \mathbb{1}_{\{\tau_N \leq T\}}]$$

$$\leq C_{T,\delta}(\mathbb{E}[|\mathcal{V}_1^N(0) - \mathcal{W}_1^N(0)|^2] + N^{-(1-\delta)(2\gamma+2)/3} + N^{-1/2})$$

$$+ C\mathbb{E}[|\mathcal{V}_1^N(t)|^4 + |\mathcal{W}_1^N(t)|^4]^{1/2} (\Pr(\tau_N \leq T))^{1/2}$$

by Proposition 5.6 and the Cauchy–Schwarz inequality. By assumption, $\mathbb{E}[|\mathcal{V}_1^N(0)|^4]$ is bounded (uniformly in N), so that Proposition 4.1 implies that $\mathbb{E}[|\mathcal{V}_1^N(t)|^4] \leq C_T$. Next, Lemma 5.4 gives us $\mathbb{E}[|\mathcal{W}_1^N(t)|^4] \leq C_T$. As a conclusion,

$$\sup_{[0,T]} \mathbb{E}[|\mathcal{V}_1^N(t) - \mathcal{W}_1^N(t)|^2]$$

$$\leq C_{T,\delta}(\mathbb{E}[|\mathcal{V}_1^N(0) - \mathcal{W}_1^N(0)|^2] + N^{-(1-\delta)(2\gamma+2)/3} + N^{-1/2}),$$

whence, by exchangeability,

$$\sup_{[0,T]} \mathbb{E}[W_2^2(\mu_t^N, v_t^N)]$$

$$\leq \sup_{[0,T]} \mathbb{E}\left[\frac{1}{N} \sum_1^N |\mathcal{V}_i^N(t) - \mathcal{W}_i^N(t)|^2 \right]$$

$$\leq C_{T,\delta}\left(\mathbb{E}\left[\frac{1}{N}\sum_{1}^{N}|\mathcal{V}_i^N(0)-\mathcal{W}_i^N(0)|^2\right]+N^{-(1-\delta)(2\gamma+2)/3}+N^{-1/2}\right)$$

$$\leq C_{T,\delta}\left(\mathbb{E}[W_2^2(\mu_0^N,v_0^N)]+N^{-(1-\delta)(2\gamma+2)/3}+N^{-1/2}\right),$$

where we used (5.4) for the last inequality. But for each $t \in [0,T]$, v_t^N is, by Lemma 5.1, the empirical measure of N i.i.d. f_t-distributed random variables. We thus infer from [15], Theorem 1 (with $d = 3$, $p = 2$ and $q = 5$), that $\mathbb{E}[W_2^2(v_t^N, f_t)] \leq C(m_5(f_t))^{2/5}N^{-1/2} \leq C_T N^{-1/2}$ for any $t \in [0,T]$. As a conclusion,

$$\sup_{[0,T]}\mathbb{E}[W_2^2(\mu_t^N, f_t)] \leq 2\sup_{[0,T]}\mathbb{E}[W_2^2(\mu_t^N, v_t^N)+W_2^2(v_t^N, f_t)]$$

$$\leq 2C_{T,\delta}\left(\mathbb{E}[W_2^2(\mu_0^N,v_0^N)]+N^{-(1-\delta)(2\gamma+2)/3}+N^{-1/2}\right).$$

Observing that $\mathbb{E}[W_2^2(\mu_0^N,v_0^N)] \leq 2\mathbb{E}[W_2^2(\mu_0^N,f_0)]+2\mathbb{E}[W_2^2(v_0^N,f_0)] \leq 2\mathbb{E}[W_2^2(\mu_0^N,f_0)]+CN^{-1/2}$ completes the proof. $\square$

6. More ellipticity estimates. We now turn to the proof of our second result: the propagation of chaos without rate when $\gamma \in (-2,0)$.

To control the singularity of the coefficients, we will need some regularity of the law of the particle system. Such regularity will be obtained thanks to the diffusion. We thus will need to show that the diffusion coefficients $a(\mu^N, v)$ are sufficiently elliptic. Lemma 2.1, which proves some ellipticity of $a(f, v)$, requires the finiteness of the entropy of f, and thus cannot apply to empirical measures. We will rather use the following lemmas. They all rely on a geometric condition on triplet of points, saying roughly that they are far enough from being aligned.

DEFINITION 6.1. Let $\delta > 0$. We say that a triplet of points (x_1, x_2, x_3) satisfy the δ-nonalignment condition if

$$(6.1) \quad |x_2 - x_1| \geq 6\sqrt{\delta}, \qquad |p_{(x_2-x_1)^\perp}(x_3-x_1)| \geq 24\delta+2\sqrt{\delta}|x_3-x_1|,$$

where $p_{(x_2-x_1)^\perp}$ is the projection onto the plane orthogonal to $x_2 - x_1$.

This condition is not invariant by permutation of the three points. Also, a triplet satisfying the δ-nonalignment condition also satisfy the δ'-nonalignment condition for all $\delta' \in [0,\delta]$.

LEMMA 6.2. Let $\gamma \in (-2,0)$, $\delta \in (0,1)$ and $R > 1$. There exists a constant $\kappa > 0$ depending only on γ, δ, R such that, for any triplet of points (x_1, x_2, x_3) belonging to $B(0, R)$ and satisfying the δ-nonalignment condition, for any $f \in \mathcal{P}(\mathbb{R}^3)$,

$$\inf_{|\xi|=1}\xi^*a(f,v)\xi \geq \kappa(1+|v|)^\gamma \inf_{k=1,2,3}f(B(x_k,\delta)).$$

The proof of this lemma is strongly inspired by that of Desvillettes–Villani [8], Proposition 4 (which is recalled in Lemma 2.1). In fact, [8], Proposition 4, may be seen as a consequence of Lemma 6.2, thanks to the following lemma.

LEMMA 6.3. *Let $\ell \in \mathbb{N}^*$, $H_0 > 0$ and $\mathcal{E}_0 > 0$ be fixed. There exist some constants $\delta \in (0, 1)$, $R > 0$ and $\kappa > 0$ such that for any $f \in \mathcal{P}(\mathbb{R}^3)$ with $H(f) \leq H_0$ and $m_2(f) \leq \mathcal{E}_0$, there are x_1, x_2, x_3 belonging to $B(0, R)$, satisfying the $(\ell\delta)$-nonalignment condition and such that*

$$\inf_{k=1,2,3} f\big(B(x_k, \delta)\big) \geq \kappa.$$

We will actually need the following time-dependent version of Lemma 6.3.

LEMMA 6.4. *Let $\gamma \in (-2, 0)$ and $q > q(\gamma)$. Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfy also $H(f_0) < \infty$ and $m_q(f_0) < \infty$. Let $T > 0$, let $p \in (p_1(\gamma), p_2(\gamma, q))$, and let $f \in L^\infty_{\text{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\text{loc}}([0, \infty), L^p(\mathbb{R}^3))$ be the solution of (1.1) given by Theorem 1.3. There exist four constants $\delta_0 \in (0, 1)$, $R_0 > 0$, $\kappa_0 > 0$ and $\tau_0 > 0$ such that for any $t \in [0, T]$, we can find three points x^t_1, x^t_2, x^t_3 in $B(0, R_0)$ satisfying the $(4\delta_0)$-nonalignment condition and such that*

$$\inf_{s \in [t, t+\tau_0]} \inf_{k=1,2,3} f_s\big(B(x^t_k, \delta_0)\big) \geq \kappa_0.$$

We now prove all these lemmas.

PROOF OF LEMMA 6.2. We fix $(x_1, x_2, x_3) \in B(0, R)^3$ satisfying the δ-nonalignment condition, we fix $v \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$ such that $|\xi| = 1$, and we divide the proof into two steps.

Step 1. *A geometric claim.* We introduce the cone C centered at v, with axis ξ and angle $\arcsin[\delta/(2 + R + |v|)]$ that can also be defined by

$$C = \left\{ v_* \in \mathbb{R}^3 : \frac{|p_{\xi^\perp}(v - v_*)|}{|v - v_*|} \leq \frac{\delta}{2 + R + |v|} \right\}.$$

We claim that the cone C cannot intersect the three balls $B^{2\delta}_k := B(x_k, 2\delta)$.

We thus assume that C intersects the balls $B^{2\delta}_1$ and $B^{2\delta}_2$ and show it does not intersect $B^{2\delta}_3$. We first check that ξ and $\xi_0 := (x_2 - x_1)/|x_2 - x_1|$ are almost aligned, in the sense that

(6.2) $$|p_{\xi^\perp}(\xi_0)| \leq \sqrt{\delta} \quad \text{and} \quad |p_{\xi^\perp_0}(\xi)| \leq \sqrt{\delta}.$$

We may find $w_1, w_2 \in C$ such that $|w_1 - x_1| \leq 2\delta$ and $|w_2 - x_2| \leq 2\delta$. This implies that $|w_i| \leq R + 2\delta \leq R + 2$ for $i = 1, 2$. Starting from $x_2 - x_1 = (x_2 - w_2) + (w_2 - v) + (v - w_1) + (w_1 - x_1)$, we write

$$|p_{\xi^\perp}(x_2 - x_1)| \leq 2\delta + |p_{\xi^\perp}(v - w_2)| + |p_{\xi^\perp}(v - w_1)| + 2\delta$$

$$\leq 4\delta + \frac{\delta}{2 + R + |v|}(|v - w_2| + |v - w_1|) \leq 6\delta.$$

We conclude that $|p_{\xi^\perp}(\xi_0)| \leq 6\delta/|x_2 - x_1| \leq \sqrt{\delta}$ thanks to the first assumption in (6.1). Since finally ξ_0 and ξ are unit vectors, $|p_{\xi^\perp}(\xi_0)| \leq \sqrt{\delta}$ means that $|\xi \cdot \xi_0|^2 \geq 1 - \delta$, whence $|p_{\xi_0^\perp}(\xi)| \leq \sqrt{\delta}$.

Next, for any $w \in C \setminus \{v\}$, we show that there is a point w^* on the line (v, w) (whence $w^* \in C$) that is very close to x_1, in the sense that $|w^* - x_1| \leq 6\delta$. Let thus

$$w^* = v + \lambda^*(w - v) \qquad \text{with } \lambda^* = \frac{(w_1 - v) \cdot \xi}{(w - v) \cdot \xi},$$

which satisfies $w^* \cdot \xi = w_1 \cdot \xi$, whence $|w_1 - w^*| = |p_{\xi^\perp}(w_1 - w^*)|$. Then

$$|w_1 - w^*| \leq |p_{\xi^\perp}(w_1 - v)| + |p_{\xi^\perp}(v - w^*)|$$

$$\leq \frac{\delta}{2 + R + |v|}(|w_1 - v| + |w^* - v|)$$

$$\leq \frac{\delta}{2 + R + |v|}(2|w_1 - v| + |w^* - w_1|).$$

We conclude that [recall that $\delta \in (0, 1)$, that $R > 1$ and that $|w_1| \leq R + 2$]

$$|w^* - w_1| \leq \frac{2|w_1 - v|\delta}{2 + R + |v| - \delta} \leq 4\delta,$$

whence $|w^* - x_1| \leq 6\delta$ as desired.

This allows us to conclude that C is included in a narrow cone centered at x_1 and directed by ξ_0. More precisely, for any $w \in C$,

$$(6.3) \qquad |p_{\xi_0^\perp}(w - x_1)| \leq 18\delta + 2\sqrt{\delta}|w - x_1|.$$

Indeed, consider w^* as previously, recall that $|w^* - x_1| \leq 6\delta$ and write

$$|p_{\xi_0^\perp}(w - x_1)| \leq 6\delta + |p_{\xi_0^\perp}(w - w^*)|$$

$$\leq 6\delta + |p_{\xi_0^\perp}(p_{\xi^\perp}(w - w^*))| + |(w - w^*) \cdot \xi||p_{\xi_0^\perp}(\xi)|,$$

because $x = p_{\xi^\perp}(x) + (x \cdot \xi)\xi$ for all x (since ξ is unitary). But $|p_{\xi_0^\perp}(\xi)| \leq \sqrt{\delta}$ by (6.2). Also, since $w - w^* = (1 - \lambda^*)(w - v)$ and since w belongs to C, we have $|p_{\xi^\perp}(w - w^*)|/|w - w^*| \leq \delta/(2 + R + |v|)$. All in all,

$$|p_{\xi_0^\perp}(w - x_1)| \leq 6\delta + \frac{\delta}{2 + R + |v|}|w - w^*| + \sqrt{\delta}|w - w^*| \leq 6\delta + 2\sqrt{\delta}|w - w^*|.$$

Using finally that $|w - w^*| \leq |w - x_1| + |x_1 - w^*| \leq |w - x_1| + 6\delta$, the conclusion follows.

Assume finally that there is $w_3 \in B_3^{2\delta} \cap C$. Then, using (6.3) with $w = w_3$, one finds that

$$|p_{\xi_0^\perp}(x_3 - x_1)| < |p_{\xi_0^\perp}(w_3 - x_1)| + 2\delta \leq 20\delta + 2\sqrt{\delta}|w_3 - x_1|$$

$$\leq 24\delta + 2\sqrt{\delta}|x_3 - x_1|.$$

But $\xi_0 = (x_2 - x_1)/|x_2 - x_1|$ so that $p_{\xi_0^\perp} = p_{(x_2-x_1)^\perp}$: this contradicts (6.1).

Step 2. *Conclusion.* We choose $k \in \{1, 2, 3\}$ such that C does not intersect $B_k^{2\delta}$ and we write, recalling that $a(f, v) = \int_{\mathbb{R}^3} a(v - v_*) f(dv_*)$ and the expression of a,

$$\xi^* a(f, v)\xi = \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2}\left(1 - \frac{((v - v_*) \cdot \xi)^2}{|v - v_*|^2}\right) f(dv_*)$$

$$= \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2}\frac{|p_{\xi^\perp}(v - v_*)|^2}{|v - v_*|^2} f(dv_*).$$

Since now C does not intersect $B(x_k, \delta) \subset B_k^{2\delta}$, we find that

$$\xi^* a(f, v)\xi \geq \left(\frac{\delta}{2 + R + |v|}\right)^2 \int_{B(x_k,\delta)} |v - v_*|^{\gamma+2} f(dv_*)$$

$$\geq \left(\frac{\delta}{2 + R + |v|}\right)^2 f\big(B(x_k, \delta)\big) \inf_{v_* \in B(x_k,\delta)} |v - v_*|^{\gamma+2}.$$

Since moreover $v \notin B_k^{2\delta}$ (because $v \in C$), one easily bounds the infimum by

$$\inf_{v_* \in B(x_k,\delta)} |v - v_*|^{\gamma+2} \geq \max\{|v| - R - 1, \delta\}^{\gamma+2} \geq \left(\frac{(1 + |v|)\delta}{R + 3}\right)^{\gamma+2}.$$

The last inequality is easily checked separating the cases $|v| \leq R + 2$ and $|v| > R + 2$. Finally, we have checked that

$$\xi^* a(f, v)\xi \geq \frac{\delta^2}{(2 + R + |v|)^2}\left(\frac{(1 + |v|)\delta}{R + 3}\right)^{\gamma+2} f\big(B(x_k, \delta)\big)$$

$$\geq \kappa \big(1 + |v|\big)^\gamma f\big(B(x_k, \delta)\big)$$

with $\kappa := [\delta/(R + 3)]^{4+\gamma}$. This concludes the proof. $\square$

PROOF OF LEMMA 6.3.   Let thus $f \in \mathcal{P}(\mathbb{R}^3)$ with $m_2(f) \leq \mathcal{E}_0$, $H(f) \leq H_0$ and let $\ell \in \mathbb{N}^*$. We set $R := 1 + \sqrt{2\mathcal{E}_0}$ and observe that $f(B(0, R)) \geq 1 - m_2(f)/R^2 \geq 1/2$.

By Lemma C.1(ii), there is a universal constant $C > 0$ such that for any Borelian $A \subset \mathbb{R}^3$ (here $|\cdot|$ stands for the Lebesgue measure)

$$|A| \leq \exp\big(-4(C + H_0 + \mathcal{E}_0)\big) \qquad \text{implies } f(A) \leq 1/4.$$

We now fix

$$\delta = \min\left\{\frac{\exp(-4(C + H_0 + \mathcal{E}_0))}{2\pi R\ell^2(24 + 4R)^2}, \left(\frac{\exp(-4(C + H_0 + \mathcal{E}_0))}{1000\ell^{3/2}}\right)^{2/3}\right\}.$$

For any couple of points y_1, y_2 in $B(0, R)$, we introduce the zone

$$D_{y_1,y_2} := \big\{y \in B(0, R) : \big|p_{(y_2-y_1)^\perp}(y - y_1)\big| \leq 24\ell\delta + 2\sqrt{\ell\delta}|y - y_1|\big\}.$$

Using the rough upper bound $|y - y_1| \leq 2R$, we see that D_{y_1, y_2} is included in a truncated cylinder with length $2R$ and radius $24\ell\delta + 4R\sqrt{\ell\delta}$, so that $|D_{y_1, y_2}| \leq 2\pi R(24\ell\delta + 4R\sqrt{\ell\delta})^2 \leq 2\pi R\ell^2(24 + 4R)^2\delta$. By definition of δ, we deduce that $|D_{y_1, y_2}| \leq \exp(-4(C + H_0 + \mathcal{E}_0))$, whence $f(D_{y_1, y_2}) \leq 1/4$. Similarly, for any $x \in \mathbb{R}^3$, $|B(x, 6\sqrt{\ell\delta})| \leq 1000\ell^{3/2}\delta^{3/2} \leq \exp(-4(C + H_0 + \mathcal{E}_0))$, so that $f(B(x, 6\sqrt{\ell\delta})) \leq 1/4$.

Since we can cover $B(0, R)$ with $(10R\delta^{-1})^3$ balls of radius δ [and centered in $B(0, R)$] and since $f(B(0, R)) \geq 1/2$, we can find $x_1 \in B(0, R)$ such that

$$f(B(x_1, \delta)) \geq \frac{1}{2}\left(\frac{\delta}{10R}\right)^3.$$

But we know that $f(B(x_1, 6\sqrt{\ell\delta})) \leq 1/4$, so that $f(B(0, R) \setminus B(x_1, 6\sqrt{\ell\delta})) \geq 1/4$. Of course, $B(0, R) \setminus B(x_1, 6\sqrt{\ell\delta})$ can also be covered by $(10R\delta^{-1})^3$ balls of radius δ [and centered in $B(0, R) \setminus B(x_1, 6\sqrt{\ell\delta})$]. We deduce that there is $x_2 \in B(0, R) \setminus B(x_1, 6\sqrt{\ell\delta})$ such that

$$f(B(x_2, \delta)) \geq \frac{1}{4}\left(\frac{\delta}{10R}\right)^3.$$

We obviously have $|x_1 - x_2| \geq 6\sqrt{\ell\delta}$. We finally recall that $f(D_{x_1, x_2}) \leq 1/4$, whence $f(B(0, R) \setminus D_{x_1, x_2}) \geq 1/4$. We thus we can find, as usual, $x_3 \in B(0, R) \setminus D_{x_1, x_2}$ such that

$$f(B(x_3, \delta)) \geq \frac{1}{4}\left(\frac{\delta}{10R}\right)^3.$$

By definition of D_{x_1, x_2}, and since $|x_1 - x_2| \geq 6\sqrt{\ell\delta}$, the triplet (x_1, x_2, x_3) satisfies the $(\ell\delta)$-nonalignment condition. This completes the proof, with $\kappa := (\delta/(10R))^3/4$. $\square$

PROOF OF LEMMA 6.4.   Recalling that $H(f_t) \leq H(f_0)$ and $m_2(f_t) = m_2(f_0)$ for all $t \geq 0$, we can apply Lemma 6.3, with $\ell = 8$, with the same constants for all times: there are $\delta \in (0, 1)$, $R_0 > 1$ and $\kappa > 0$ such that for all $t \in [0, T]$, there are three points (x_1^t, x_2^t, x_3^t) in $B(0, R_0)$ satisfying the (8δ)-nonalignment condition and such that $f_t(B(x_k^t, \delta)) \geq \kappa$ for $k = 1, 2, 3$.

Next, we choose a smooth function $h : \mathbb{R}^3 \mapsto [0, 1]$ such that $\mathbb{1}_{\{|x| \leq 1\}} \leq h \leq \mathbb{1}_{\{|x| \leq 2\}}$. For $k = 1, 2, 3$, we define, for $0 \leq t \leq s$,

$$w_k^t(s) := \int_{\mathbb{R}^3} h\left(\frac{v - x_k^t}{\delta}\right) f_s(dv).$$

By construction, we have $w_k^t(t) \geq f_t(B(x_k^t, \delta)) \geq \kappa$. Using the weak formulation (1.5) and the bound (1.4) with the smooth function $\varphi = h(\delta^{-1}(\cdot - x_k^t))$, we get

[recall that $p \in (p_1(\gamma), p_2(\gamma, q))$ is fixed and consider $p' \in (p_1(\gamma), p)$], for $0 \leq t \leq s \leq T + 1$,

$$\left| w_k^t(s) - w_k^t(t) \right| \leq C_\delta \int_t^s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(|v - v_*|^{\gamma+1} + |v - v_*|^{\gamma+2} \right) f_u(dv) f_u(dv_*) \, du$$

$$\leq C_\delta \int_t^s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(1 + |v|^2 + |v_*|^2 + |v - v_*|^\gamma \right) f_u(dv) f_u(dv_*) \, du$$

$$\leq C_\delta \int_t^s \left(1 + m_2(f_u) + \| f_u \|_{L^{p'}} \right) du.$$

The last inequality uses (1.6). Since f_u is a probability density function and since $p > p'$, we have $\| f_u \|_{L^{p'}} \leq \| f_u \|_{L^p}^a$ with $a = [p(p' - 1)]/[p'(p - 1)] \in (0, 1)$. Using the Hölder inequality and that $1 + m_2(f_u) + \| f_u \|_{L^p} \in L^1([0, T + 1])$, we deduce that, still for $0 \leq t \leq s \leq T + 1$,

$$\left| w_k^t(s) - w_k^t(t) \right| \leq C_{\delta, T} (s - t)^{1-a}.$$

From this and since $w_k^t(t) \geq \kappa$ for all $t \in [0, T]$, it is clear that there exists τ_0 such that, setting $\kappa_0 = \kappa/2$,

$$\forall t \in [0, T], \forall s \in [t, t + \tau_0] \qquad w_k^t(s) \geq \kappa_0.$$

The bound is valid for $k = 1, 2, 3$. By definition of w_k^t, this implies that

$$\inf_{t \in [0, T]} \inf_{s \in [t, t+\tau_0]} \inf_{k=1,2,3} f_s \left(B(x_k^t, 2\delta) \right) \geq \kappa_0.$$

Setting $\delta_0 = 2\delta$ completes the proof: since $8\delta = 4\delta_0$, the points x_k^t satisfy the $(4\delta_0)$-nonalignment condition. $\quad \square$

7. Chaos without rate. The goal of this section is to prove Theorem 1.8. When applying the methods of [16], the main difficulty is that the matrix a is not uniformly elliptic. Ellipticity is important, since it provides the regularity that allows us to show that the singularity of the coefficients is not too much visited. In the particle system (1.9), there is a lack of diffusion in some directions when the particles are almost aligned. We thus will proceed as follows. We will first introduce a perturbed particle system, by adding some diffusion when particles are in a bad (almost aligned) configuration. We then will prove propagation of chaos for this perturbed system. Finally, we will show that the artificial noise is used only with a very small probability (as $N \to \infty$).

In the rest of that section, we consider $\gamma \in (-2, 0)$ and $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfying $H(f_0) < \infty$ and $m_q(f_0) < \infty$ for some $q > q(\gamma)$. We consider the unique solution f belonging to $L^\infty_{\text{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\text{loc}}([0, \infty), L^p(\mathbb{R}^3))$ for all $p \in (p_1(\gamma), p_2(\gamma, q))$ to (1.1) given by Theorem 1.3. We recall that it satisfies $H(f_t) \leq H(f_0) < \infty$ and $m_2(f_t) = m_2(f_0) < \infty$. We also fix an arbitrary final time T. We finally consider, for each $N \geq 2$, an exchangeable initial condition $(\mathcal{V}_1^N(0), \ldots, \mathcal{V}_N^N(0))$ satisfying the set of conditions (1.11).

7.1. *Definition of a perturbed system.* We first introduce some notation.

NOTATION 7.1.   (i) Recall the constants $R_0 > 0$, $\delta_0 > 0$, $\kappa_0 > 0$ and $\tau_0 > 0$ introduced in Lemma 6.4. We also denote by $n_0 = \lfloor T/\tau_0 \rfloor$, so that

$$[0, T] \subset \bigcup_{l=0}^{n_0} \lfloor l\tau_0, (l+1)\tau_0 \rfloor.$$

Lemma 6.4 implies that for each $l = 0, \ldots, n$, we can find three points x_1^l, x_2^l, x_3^l in $B(0, R_0)$ satisfying the $(4\delta_0)$-nonalignment condition (6.1) and such that

$$(7.1) \qquad \inf_{l=0,\ldots,n_0} \inf_{t \in [l\tau_0, (l+1)\tau_0]} \inf_{k=1,2,3} f_t\big(B(x_k^l, \delta_0)\big) \geq \kappa_0.$$

(ii) Let $h : \mathbb{R}^3 \mapsto [0, 1]$ and $\chi : [0, \infty) \mapsto [0, 1]$ be smooth and satisfy $\mathbb{1}_{\{|v| \leq 1\}} \leq h(v) \leq \mathbb{1}_{\{|v| \leq 2\}}$ and $\mathbb{1}_{\{r \leq 1\}} \leq \chi(r) \leq \mathbb{1}_{\{r \leq 2\}}$. For $l = 0, \ldots, n_0$ and $g \in \mathcal{P}(\mathbb{R}^3)$, we put

$$c_l(g) := \sum_{k=1}^{3} \chi\left(\frac{4}{\kappa_0} \int_{\mathbb{R}^3} h\left(\frac{v - x_k^l}{2\delta_0}\right) g(dv)\right) \in [0, 3].$$

We also shorten the notation of the coefficients of the particle system.

NOTATION 7.2.   For $\mathbf{v}^N = (v_1, \ldots, v_N) \in (\mathbb{R}^3)^N$, we introduce

$$b_i^N(\mathbf{v}^N) = b\left(\frac{1}{N} \sum_1^N \delta_{v_j} \star \phi_{\eta_N}, v_i\right) \quad \text{and} \quad a_i^N(\mathbf{v}^N) = a\left(\frac{1}{N} \sum_1^N \delta_{v_j} \star \phi_{\eta_N}, v_i\right),$$

and $\sigma_i^N(\mathbf{v}^N) = (a_i^N(\mathbf{v}^N))^{1/2}$. For $l = 1, \ldots, n_0$, we set

$$c_l^N(\mathbf{v}^N) = c_l\left(\frac{1}{N} \sum_1^N \delta_{v_j}\right).$$

We will use the following properties.

PROPOSITION 7.3.   *Let* $\gamma \in (-2, 0)$. *Let* $g \in \mathcal{P}(\mathbb{R}^3)$, $\mathbf{v}^N = (v_1, \ldots, v_N) \in (\mathbb{R}^3)^N$, *and* $l \in \{0, \ldots, n_0\}$:

(i)   *We have* $c_l(g) = 0$ *if* $\inf_{k=1,2,3} g(B(x_k^l, 2\delta_0)) \geq \kappa_0/2$.
(ii) *We have* $c_l(g) \geq 1$ *as soon as* $\inf_{k=1,2,3} g(B(x_k^l, 4\delta_0)) \leq \kappa_0/4$.
(iii) *There is a constant* $C > 0$ *such that*

$$|\nabla_{v_i} c_l^N(\mathbf{v}^N)| \leq \frac{C}{N\kappa_0\delta_0}.$$

(iv) *There is $\kappa_1 > 0$ such that for all $\xi \in \mathbb{R}^3$,*

$$\xi^* a_i^N(\mathbf{v}^N)\xi + |\xi|^2 c_l^N(\mathbf{v}^N) \geq \kappa_1 (1 + |v_i|)^\gamma |\xi|^2.$$

(v) $c_l(f_t) = 0$ *for all* $t \in [l\tau_0, (l+1)\tau_0]$.
(vi) *There is a constant $C > 0$ such that for all $g_1, g_2 \in \mathcal{P}(\mathbb{R}^3)$,*

$$\left| c_l(g_1) - c_l(g_2) \right| \leq \frac{C}{\kappa_0 \delta_0} W_1(g_1, g_2) \leq \frac{C}{\kappa_0 \delta_0} W_2(g_1, g_2).$$

We need the following easy remark.

LEMMA 7.4.   *For all $\delta > 0$, there is $c_\delta > 0$ such that for all $x \in \mathbb{R}^3$, all $\mu \in \mathcal{P}(\mathbb{R}^3)$, all $\eta \in (0, 1)$,*

$$(\mu \star \phi_\eta)\big(B(x, \delta)\big) \geq c_\delta \mu\big(B(x, \delta)\big).$$

PROOF.   It of course suffices to treat the case where $x = 0$. Let thus $\delta > 0$ be fixed. For $z \in B(0, \delta)$, we introduce $C_z := \{u \in B(0, \delta/2) : u \cdot z \leq -|u| \|z\|/2\}$. Clearly, the Lebesgue measure p_δ of C_z does not depend on z and is positive. We next claim that for any $u \in C_z$ and $\eta \in (0, 1)$, $|z + \eta u| < \delta$. Indeed, this is obvious if $|z| \leq \delta/2$, while one easily checks that $|z + \eta u|^2 \leq |z|^2 < \delta^2$ when $|z| \in [\delta/2, \delta)$. Consequently,

$$(\mu \star \phi_\eta)\big(B(0, \delta)\big) = \frac{3}{4\pi} \int_{\mathbb{R}^3} \mu(dz) \int_{B(0,1)} du \mathbb{1}_{\{|z+\eta u|<\delta\}}$$

$$\geq \frac{3}{4\pi} \int_{B(0,\delta)} \mu(dz) \int_{C_z} du,$$

which is greater than $(3p_\delta/(4\pi))\mu(B(0, \delta))$ as desired.   □

PROOF OF PROPOSITION 7.3.   Point (i) is immediate: for example, $g(B(x_k^l, 2\delta_0)) \geq \kappa_0/2$ implies that $(4/\kappa_0) \int_{\mathbb{R}^3} h((v - x_k^l)/(2\delta_0))g(dv) \geq 2$, from which we deduce that $\chi((4/\kappa_0) \int_{\mathbb{R}^3} h((v - x_k^l)/(2\delta_0))g(dv)) = 0$. Point (ii) is also obvious: if there is k such that $g(B(x_k^l, 4\delta_0)) \leq \kappa_0/4$, then $(4/\kappa_0) \int_{\mathbb{R}^3} h((v - x_k^l)/(2\delta_0))g(dv) \leq 1$, hence $\chi((4/\kappa_0) \int_{\mathbb{R}^3} h((v - x_k^l)/(2\delta_0))g(dv)) = 1$. Point (iii) follows from the fact that h and χ are smooth (with bounded derivatives), since

$$c_l^N(\mathbf{v}^N) = \sum_{k=1}^3 \chi\left(\frac{4}{N\kappa_0} \sum_{i=1}^N h\left(\frac{v_i^N - x_k^l}{\delta_0} \right) \right).$$

We next check (iv), recalling that $a_i^N(\mathbf{v}^N) = a(\mu^N \star \phi_{\eta_N}, v_i)$ and $c_l^N(\mathbf{v}^N) = c_l(\mu^N)$, with $\mu^N = N^{-1} \sum_1^N \delta_{v_i}$. Assume first that $\inf_{k=1,2,3} \mu^N(B(x_k^l, 4\delta_0)) \geq \kappa_0/4$. Then by Lemma 7.4, we have $\inf_{k=1,2,3}(\mu^N \star \phi_{\eta_N})(B(x_k^l, 4\delta_0)) \geq c_{\delta_0}\kappa_0/4$.

Since now the triplet (x_1^l, x_2^l, x_3^l) satisfy the $(4\delta_0)$-nonalignment condition, we can apply Lemma 6.2 and obtain

$$\xi^* a_i^N(\mathbf{v}^N)\xi \geq \kappa(1 + |v_i|)^\gamma |\xi|^2 \inf_{k=1,2,3} (\mu^N \star \phi_{\eta_N})(B(x_k^l, 4\delta_0))$$

$$\geq \frac{\kappa c_{\delta_0}\kappa_0}{4}(1 + |v_i|)^\gamma |\xi|^2.$$

Assume next that $\inf_{k=1,2,3} \mu^N(B(x_k^l, 4\delta_0)) \leq \kappa_0/4$. Then $c_l^N(\mathbf{v}^N) \geq 1$ by point (ii), so that

$$|\xi|^2 c_l^N(\mathbf{v}^N) \geq |\xi|^2 \geq (1 + |v_i|)^\gamma |\xi|^2$$

since $\gamma \leq 0$. We conclude with the choice $\kappa_1 := (\kappa c_{\delta_0}\kappa_0/4) \wedge 1$. Point (v) is a direct consequence of (7.1) and of point (i). Finally, to prove point (vi), we consider $g_1, g_2 \in \mathcal{P}(\mathbb{R}^3)$ and $R \in \Pi(g_1, g_2)$ such that $\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v_1 - v_2| R(dv_1, dv_2) = W_1(g_1, g_2)$. Then we write, using that χ and h are globally Lipschitz-continuous and bounded,

$$|c_l(g_1) - c_l(g_2)|$$

$$= \left| \chi\left(\frac{4}{\kappa_0}\int_{\mathbb{R}^3} h\left(\frac{v - x_k^l}{2\delta_0}\right)g_1(dv)\right) - \chi\left(\frac{4}{\kappa_0}\int_{\mathbb{R}^3} h\left(\frac{v - x_k^l}{2\delta_0}\right)g_2(dv)\right) \right|$$

$$\leq \frac{C}{\kappa_0}\left| \int_{\mathbb{R}^3} h\left(\frac{v - x_k^l}{2\delta_0}\right)g_1(dv) - \int_{\mathbb{R}^3} h\left(\frac{v - x_k^l}{2\delta_0}\right)g_2(dv) \right|$$

$$= \frac{C}{\kappa_0}\left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left[h\left(\frac{v_1 - x_k^l}{2\delta_0}\right) - h\left(\frac{v_2 - x_k^l}{2\delta_0}\right)\right] R(dv_1, dv_2) \right|$$

$$\leq \frac{C}{\kappa_0 \delta_0} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v_1 - v_2| R(dv_1, dv_2),$$

from which the conclusion follows. □

We now introduce our perturbed particle system $\mathcal{U}^N = (\mathcal{U}_1^N, \ldots, \mathcal{U}_N^N)$, defined on the time interval $[0, T]$, as the solution to

$$\mathcal{U}_i^N(t) = \mathcal{V}_i^N(0) + \int_0^t b_i^N(\mathcal{U}_s^N)\,ds + \int_0^t \sigma_i^N(\mathcal{U}_s^N)\,d\mathcal{B}^i(s)$$

(7.2)

$$+ \int_0^t c_{\lfloor s/\tau_0 \rfloor}^N(\mathcal{U}_s^N)\,d\mathcal{W}_i(s) + \int_0^t c_{\lfloor s/\tau_0 \rfloor}^N(\mathcal{U}_s^N)\nabla_{v_i}c_{\lfloor s/\tau_0 \rfloor}^N(\mathcal{U}_s^N)\,ds,$$

for all $i = 1, \ldots, N$. Here, $((\mathcal{W}_i(t))_{t \in [0,T]})_{i=1,\ldots,N}$ is an independent family of $3D$ standard Brownian motions independent of $(\mathcal{V}_i^N(0), (\mathcal{B}_i(t))_{t \in [0,T]})_{i=1,\ldots,N}$.

PROPOSITION 7.5. *Recall that $\gamma \in (-2, 0)$, that $\eta_N \in (0, 1)$ is fixed and that the initial condition $(\mathcal{V}_i^N(0))_{i=1,\ldots,N}$ is exchangeable. There is strong existence*

and uniqueness for (7.2). *Moreover, if* $\mathbb{E}[|\mathcal{V}_1^N(0)|^r] < \infty$ *for some* $r \geq 2$, *then*

$$\sup_{N \geq 2} \sup_{[0,T]} \mathbb{E}[|\mathcal{U}_1^N(t)|^r] < C_{T,r},$$

for some constant $C_{T,r}$ *not depending on* N.

PROOF. The strong existence and uniqueness is clear, since all the coefficients are locally Lipschitz continuous and have at most linear growth: this has already been seen in the proof of Proposition 1.5 for b_i^N and σ_i^N, and $c_{\lfloor s/\tau_0 \rfloor}^N$ and $c_{\lfloor s/\tau_0 \rfloor}^N \nabla_{v_i} c_{\lfloor s/\tau_0 \rfloor}^N$ are obviously bounded and smooth in x (recall that here, N is fixed). Concerning the moment estimates, it suffices to handle the same proof as that of Proposition 4.1. The additional terms cause no difficulty (and are much easier to treat) because $c_{\lfloor s/\tau_0 \rfloor}^N$ is uniformly bounded (by 3) and $c_{\lfloor s/\tau_0 \rfloor}^N \nabla_{v_i} c_{\lfloor s/\tau_0 \rfloor}^N$ is uniformly bounded [by $3C/(N\kappa_0\delta_0) \leq C$] thanks to Proposition 7.3(iii). $\quad\square$

We will first study propagation of chaos for the perturbed particle system. We thus introduce the corresponding nonlinear process $(\mathcal{U}(t))_{t \in [0,T]}$, solution to

(7.3)
$$\mathcal{U}(t) = \mathcal{V}(0) + \int_0^t b(g_s, \mathcal{U}(s))\,ds + \int_0^t \sigma(g_s, \mathcal{U}(s))\,d\mathcal{B}(s)$$
$$+ \int_0^t c_{\lfloor s/\tau_0 \rfloor}(g_s)\,d\mathcal{W}(s),$$

where $\mathcal{V}(0)$ is f_0-distributed, independent of the (independent) $3D$-Brownian motions $(\mathcal{B}(t))_{t \in [0,T]}$ and $(\mathcal{W}(t))_{t \in [0,T]}$, and where $g_t \in \mathcal{P}(\mathbb{R}^3)$ is the law of $\mathcal{U}(t)$. Observe that for $(\mathcal{U}(t))_{t \in [0,T]}$ a solution to (7.3), its family $(g_t)_{t \in [0,T]}$ of time marginals is a weak solution to

(7.4) $\quad \partial_t g_t(v) = \frac{1}{2} \operatorname{div}_v \big(a(g_t, v)\nabla g_t(v) - b(g_t, v)g_t(v) + c_{\lfloor t/\tau_0 \rfloor}^2(g_t)\nabla g_t(v)\big).$

PROPOSITION 7.6. *Recall that* $\gamma \in (-2, 0)$ *and that* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ *satisfies* $H(f_0) < \infty$ *and* $m_q(f_0) < \infty$ *for some* $q > q(\gamma)$:

(i) *There exists a unique weak solution* $(g_t)_{t \in [0,T]}$ *to* (7.4) *such that* $g_0 = f_0$ *and such that* $(g_t)_{t \in [0,T]} \in L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3)) \cap L^1([0,T], L^p(\mathbb{R}^3))$ *for some* $p \in (p_1(\gamma), p_2(\gamma, q))$. *Furthermore, it holds that* $(g_t)_{t \in [0,T]} = (f_t)_{t \in [0,T]}$, *where* $(f_t)_{t \geq 0}$ *is the unique weak solution to the Landau equation* (1.1) *built in Theorem* 1.3.

(ii) *There is a pathwise unique continuous adapted solution* $(\mathcal{U}(t))_{t \in [0,T]}$ *to* (7.3) *such that, setting* $g_t = \mathcal{L}(\mathcal{U}(t))$, $(g_t)_{t \in [0,T]} \in L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3)) \cap L^1([0,T], L^p(\mathbb{R}^3))$ *for some* $p \in (p_1(\gamma), p_2(\gamma, q))$. *Furthermore,* $(g_t)_{t \in [0,T]} = (f_t)_{t \in [0,T]}$.

(iii) *Finally* $(\mathcal{U}(t))_{t \in [0,T]}$ *equals* $(\mathcal{V}(t))_{t \in [0,T]}$, *the unique solution to* (1.10) (*see Theorem* 1.7).

PROOF.   Consider the unique solution $(\mathcal{V}(t))_{t\geq 0}$ to the nonlinear SDE (1.10), the unique weak solution $f \in L^\infty_{\mathrm{loc}}([0,\infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0,\infty), L^p(\mathbb{R}^3))$ to (1.1), and recall that $f_t = \mathcal{L}(\mathcal{V}(t))$. Then we know from Proposition 7.3(v) that $c_{\lfloor t/\tau_0 \rfloor}(f_t) = 0$ for all $t \in [0,T]$. Consequently, $(\mathcal{V}(t))_{t\geq 0}$ also solves (7.3) and $(f_t)_{t\in[0,T]}$ also solves (7.4) [compare with (1.3)]. Next, we claim that once the uniqueness of the weak solution to (7.4) is established, the pathwise uniqueness of the solution $(\mathcal{U}(t))_{t\in[0,T]}$ to (7.3) can be checked exactly as in the proof of Theorem 1.7. We thus only have to prove the uniqueness part in point (i).

Let thus $p \in (p_1(\gamma), p_2(\gamma, q))$ be fixed, and consider two solutions $(g_t)_{t\in[0,T]}$, $(k_t)_{t\in[0,T]}$ to (7.4), both lying in $L^\infty_{\mathrm{loc}}([0,\infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0,\infty), L^p(\mathbb{R}^3))$. Exactly as in the proof of Theorem 1.4—steps 1–2, we can build, for each $\varepsilon > 0$, two processes $(\mathcal{Z}(t))_{t\in[0,T]}$ and $(\mathcal{V}_\varepsilon(t))_{t\in[0,T]}$ with respective families of time marginals $(g_t)_{t\in[0,T]}$ and $(k_t)_{t\in[0,T]}$, solving

$$\mathcal{Z}(t) = \mathcal{V}(0) + \int_0^t b(g_s, \mathcal{Z}(s))\,ds + \int_0^t \sigma(g_s, \mathcal{Z}(s))\,d\mathcal{B}(s)$$
$$+ \int_0^t c_{\lfloor s/\tau_0 \rfloor}(g_s)\,d\mathcal{W}(s),$$

$$\mathcal{V}_\varepsilon(t) = \mathcal{V}(0) + \int_0^t b(k_s, \mathcal{V}_\varepsilon(s))\,ds$$
$$+ \int_0^t \sigma(k_s, \mathcal{V}_\varepsilon(s))U(a(g_s^\varepsilon, \mathcal{Z}(s)), a(k_s^\varepsilon, \mathcal{V}_\varepsilon(s)))\,d\mathcal{B}_s$$
$$+ \int_0^t c_{\lfloor s/\tau_0 \rfloor}(k_s)\,d\mathcal{W}(s).$$

We can then reproduce exactly the same computations as in Theorem 1.4—steps 3–4–5 to prove uniqueness in the class $L^\infty_{\mathrm{loc}}([0,\infty), \mathcal{P}_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0,\infty), L^p(\mathbb{R}^3))$ (some strong/weak stability estimates could also be checked here). When computing $(d/dt)\mathbb{E}[|\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t)|^2]$, there is the additional term $3(c_{\lfloor t/\tau_0 \rfloor}(g_t) - c_{\lfloor t/\tau_0 \rfloor}(k_t))^2$. But this is controlled, using Proposition 7.3, by $C W_2^2(g_t, k_t)$ which is itself bounded by $C\mathbb{E}[|\mathcal{Z}(t) - \mathcal{V}_\varepsilon(t)|^2]$. This term thus causes no difficulty. $\quad\square$

7.2. *Regularity estimate for the perturbed particle system.*   We now need to introduce the entropy and weighted Fisher information. These functionals are studied in detail in the Appendix.

NOTATION 7.7.   For $\mathcal{V}^N$ a random variable with law $F \in \mathcal{P}((\mathbb{R}^3)^N)$ with a density, we define

$$H(\mathcal{V}^N) = H(F) := \frac{1}{N} \int_{(\mathbb{R}^3)^N} F(\mathbf{v}^N) \log(F(\mathbf{v}^N))\,d\mathbf{v}^N,$$

$$I_\gamma(\mathcal{V}^N) = I_\gamma(F) := \frac{1}{N} \int_{(\mathbb{R}^3)^N} \frac{|\nabla_\gamma F(\mathbf{v}^N)|^2}{F(\mathbf{v}^N)}\,d\mathbf{v}^N,$$

where the differential operator ∇_γ is the weighted gradient

$$\nabla_\gamma F := ((1 + |v_1|^2)^{\gamma/4}\nabla_{v_1} F, \ldots, (1 + |v_N|^2)^{\gamma/4}\nabla_{v_n} F).$$

If F has no density, we simply set $H(\mathcal{V}^N) = H(F) = \infty$ and $I_\gamma(\mathcal{V}^N) = I_\gamma(F) = \infty$.

The following lemma uses strongly the perturbation.

PROPOSITION 7.8.   *For each $N \geq 2$, we consider the solution $(\mathcal{U}_1^N(t), \ldots, \mathcal{U}_N^N(t))_{t\in[0,T]}$ to (7.2). For each $t \in [0, T]$, we denote by G_t^N the law of $\mathcal{U}^N(t) = (\mathcal{U}_1^N(t), \ldots, \mathcal{U}_N^N(t))$. Recall that by (1.11), $\sup_{N\geq 2} H(G_0^N) < \infty$ and $\sup_{N\geq 2} \mathbb{E}[|\mathcal{U}_1^N(0)|^q] < \infty$ for some $q > q(\gamma)$. It holds that*

$$\sup_{N\geq 2}\sup_{t\in[0,T]} H(G_t^N) < \infty \quad and \quad \sup_{N\geq 2}\int_0^T I_\gamma(G_t^N)\,dt < \infty.$$

PROOF.   The first step is to derive the master (or Kolmogorov) equation for the time marginals G_t^N. Applying the Itô formula to compute the expectation of $\varphi(\mathcal{U}_t^N)$, we deduce that for all $\varphi \in C_b^2((\mathbb{R}^3)^N)$,

$$\int_{(\mathbb{R}^3)^N} \varphi(\mathbf{v}^N) G_t^N(d\mathbf{v}^N)$$

$$= \int_{(\mathbb{R}^3)^N} \varphi(\mathbf{v}^N) G_0^N(d\mathbf{v}^N)$$

$$+ \int_0^t \int_{(\mathbb{R}^3)^N} \sum_{i=1}^N \nabla_{v_i}\varphi(\mathbf{v}^N)$$

$$\times [b_i^N(\mathbf{v}^N) + c_{\lfloor s/\tau_0\rfloor}^N(\mathbf{v}^N)\nabla_{v_i} c_{\lfloor s/\tau_0\rfloor}^N(\mathbf{v}^N)] G_s^N(d\mathbf{v}^N)\,ds$$

$$+ \frac{1}{2}\int_0^t \int_{(\mathbb{R}^3)^N} \sum_{i=1}^N \sum_{k,l=1}^3 \partial_{v_{ik}v_{il}}\varphi(\mathbf{v}^N)((a_i^N(\mathbf{v}^N))_{kl}$$

$$+ (c_{\lfloor s/\tau_0\rfloor}^N(\mathbf{v}^N))^2\delta_{kl}) G_s^N(d\mathbf{v}^N)\,ds.$$

Recall now that $b_k = \sum_{l=1}^3 \partial_l a_{lk}$. Hence, we see that G^N is a weak solution to

$$\partial_t G_t^N(\mathbf{v}^N) = \frac{1}{2}\sum_{i=1}^N \sum_{k,l=1}^3 \partial_{v_{ik}}[\{(a_i^N(\mathbf{v}^N))_{kl} + (c_{\lfloor t/\tau_0\rfloor}^N(\mathbf{v}^N))^2\delta_{kl})\}\partial_{v_{il}} G_t^N(\mathbf{v}^N)]$$

$$- \frac{1}{2}\sum_{i=1}^N \sum_{k=1}^3 \partial_{v_{ik}}[(b_i^N(\mathbf{v}^N))_k G_t^N(\mathbf{v}^N)],$$

or equivalently to

$$\partial_t G_t^N = \frac{1}{2} \sum_{i=1}^{N} \mathrm{div}_{v_i}\big[\{a_i^N + (c_{\lfloor t/\tau_0 \rfloor}^N)^2 I\}\nabla_{v_i} G_t^N\big] - \frac{1}{2}\sum_{i=1}^N \mathrm{div}_{v_i}[b_i^N G_t^N].$$

We thus have, performing some integrations by parts,

$$\frac{d}{dt} H(G_t^N) = \frac{1}{N}\int_{(\mathbb{R}^3)^N} (1 + \log G_t^N(\mathbf{v}^N))\partial_t G_t^N(\mathbf{v}^N)\,d\mathbf{v}^N$$

$$= -\frac{1}{2N}\sum_{i=1}^N \int_{(\mathbb{R}^3)^N} \nabla_{v_i} \log G_t^N$$

$$\times [a_i^N(\mathbf{v}^N) + (c_{\lfloor t/\tau_0 \rfloor}^N(\mathbf{v}^N))^2 I]\nabla_{v_i}\log G_t^N(\mathbf{v}^N) G_t^N(d\mathbf{v}^N)$$

$$-\frac{1}{2N}\sum_{i=1}^N \int_{(\mathbb{R}^3)^N} \mathrm{div}\,b_i^N(\mathbf{v}^N) G_t^N(d\mathbf{v}^N).$$

We next use Proposition 7.3(iv) for the diffusion term and, for the drift term, that $\mathrm{div}\,b(v) = -(6 + 2\gamma)|v|^\gamma$, whence $-\mathrm{div}(b \star \phi_{\eta_N})(v) = (6 + 2\gamma)\int_{\mathbb{R}^3} |v - z|^\gamma \phi_{\eta_N}(z)\,dz \le C|v|^\gamma$, and exchangeability. We find that

$$\frac{d}{dt} H(G_t^N) \le -\frac{\kappa_1}{2N}\sum_{i=1}^N \int_{(\mathbb{R}^3)^N} \frac{(1 + |v_i|)^\gamma |\nabla_{v_i} G_t^N(\mathbf{v}^N)|^2}{G_t^N(\mathbf{v}^N)}\,d\mathbf{v}^N$$

$$+ C\int_{(\mathbb{R}^3)^N} |v_1 - v_2|^\gamma G_t^N(d\mathbf{v}^N)$$

$$\le -2^{\gamma/2 - 1}\kappa_1 I_\gamma(G_t^N) + C\mathbb{E}[|\mathcal{U}_1^N(t) - \mathcal{U}_2^N(t)|^\gamma].$$

We used that $(1 + x)^\gamma \ge 2^{\gamma/2}(1 + x^2)^{\gamma/2}$. Denote by $G_{t,2}^N$ the law of $(\mathcal{U}_1^N(t), \mathcal{U}_2^N(t))$. We now apply Lemma C.4 with $\kappa = -\gamma$. Since $q > q(\gamma) = \gamma^2/(2 + \gamma)$, we have $\alpha := \max\{2, q\} > |\gamma| \max\{1, \kappa/(2 - \kappa)\}$, whence, with $r := \alpha/(\alpha - \gamma)$,

$$\frac{d}{dt} H(G_t^N) \le -2^{\gamma/2 - 1}\kappa_1 I_\gamma(G_t^N) + C_{\gamma,q}(1 + I_\gamma(G_{t,2}^N)^r (1 + m_\alpha(G_{2,t}^N)^{1-r}))$$

$$\le -2^{\gamma/2 - 1}\kappa_1 I_\gamma(G_t^N) + C_{T,\gamma,q}(1 + I_\gamma(G_t^N)^r).$$

We finally used the moment bound proved in Proposition 7.5 [and the moment assumption in (1.11)] and Lemma C.2(i), from which $I_\gamma(G_{t,2}^N) \le I_\gamma(G_t^N)$. Since $r \in (0, 1)$, we easily conclude, using the Young inequality, that

$$\frac{d}{dt} H(G_t^N) \le -2^{\gamma/2 - 2}\kappa_1 I_\gamma(G_t^N) + C_{T,\gamma,q}.$$

Integrating this inequality and using again (1.11), we get, for any $t \in [0, T]$,

$$H(G_t^N) + \frac{\kappa_1}{8}\int_0^t I_\gamma(G_s^N)\,ds \le H(F_0^N) + C_{T,\gamma,q}t \le C_{T,\gamma,q}.$$

We immediately deduce that $H(F_t^N)$ is bounded uniformly in N and $t \in [0, T]$. Finally, it follows from the fact that $\sup_{N \geq 2} \sup_{[0,T]} \mathbb{E}[|\mathcal{U}_1^N(t)|^2] < \infty$ [by (1.11) and Proposition 7.5] and Lemma C.1(i) that $\inf_{N \geq 2} H(F_T^N) > -\infty$. Consequently, $\int_0^T I_\gamma(F_s^N) \, ds$ is also uniformly bounded. This completes the proof. $\square$

7.3. *Some more estimates.* We will of course need, in several steps, to control the singularity of b. Also, to verify that the limit points of the empirical measure of the particle system belong, in some sense, to $L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$, we will need to control $\mathbb{E}[\sup_{[0,T]} |\mathcal{U}_1^N(s)|^2]$, with the supremum inside the expectation. All this is more or less obvious when $\gamma \in (-1, 0)$, but requires a little work when $\gamma \in (-2, -1]$, based on the regularity estimate checked in the previous subsection.

LEMMA 7.9. *For each $N \geq 2$, we consider the solution $(\mathcal{U}_1^N(t), \ldots, \mathcal{U}_N^N(t))_{t \in [0,T]}$ to (7.2). Recall that by (1.11), $\sup_{N \geq 2} \mathbb{E}[|\mathcal{U}_1^N(0)|^q] < \infty$ for some $q > q(\gamma)$. It holds that*

   (i) $\sup_{N \geq 2} \mathbb{E}[\sup_{t \in [0,T]} |\mathcal{U}_1^N(s)|^2] < \infty$;
   (ii) $\sup_{N \geq 2} \int_0^T \mathbb{E}[|\mathcal{U}_1^N(s) - \mathcal{U}_2^N(s)|^\gamma] \, ds < \infty$.

Here, we prove point (i) using point (ii), but it seems that a refinement of the proof of Proposition 4.1 could also work.

PROOF OF LEMMA 7.9. We put $\alpha = 2 \vee q$ and recall that $\sup_{N \geq 2} \sup_{t \in [0,T]} \mathbb{E}[|\mathcal{U}_1^N(s)|^\alpha] < \infty$ by Proposition 7.5. We first prove (ii). Denote by $G_{s,2}^N$ the two-marginal of G_s^N, which is the law of $(\mathcal{U}_1^N(s), \mathcal{U}_2^N(s))$. Observe that $\alpha > |\gamma| \max\{1, |\gamma|/(2 + \gamma)\}$ [because $\alpha \geq 2$ and $\alpha \geq q > q(\gamma) = \gamma^2/(2 + \gamma)$]. By Lemma C.4, we know that, with $r = \alpha/(\alpha - \gamma)$, $\mathbb{E}[|\mathcal{U}_1^N(s) - \mathcal{U}_2^N(s)|^\gamma] \leq C_{\kappa,\alpha}(1 + (I_\gamma(G_{s,2}^N))^r(1 + m_\alpha(G_{s,2}^N))^{1-r}) \leq C_{\kappa,\alpha,T}(1 + I_\gamma(G_{s,2}^N))$ because $r \in (0, 1)$ and $m_\alpha(G_{s,2}^N) = \mathbb{E}[|\mathcal{U}_1^N(s)|^\alpha]$ is controlled uniformly in $N \geq 2$ and $s \in [0, T]$. Finally, we know from Lemma C.2(i) that $I_\gamma(G_{s,2}^N) \leq I_\gamma(G_s^N)$, whence $\mathbb{E}[|\mathcal{U}_1^N(s) - \mathcal{U}_2^N(s)|^\gamma] \leq C_{\kappa,\alpha}(1 + I_\gamma(G_s^N))$. Integrating in time and using Proposition 7.8 completes the proof of (ii).

To prove (i), we start from $\sup_{[0,T]} |\mathcal{U}_1^N(t)|^2 \leq C[|\mathcal{U}_1^N(0)|^2 + I^N + J^N + K^N + L^N]$, where

$$I^N := \sup_{[0,T]} \left(\int_0^t b_1^N(\mathcal{U}^N(s)) \, ds \right)^2,$$

$$J^N := \sup_{[0,T]} \left(\int_0^t \sigma_1^N(\mathcal{U}^N(s)) \, d\mathcal{B}_1(s) \right)^2,$$

$$K^N := \sup_{[0,T]} \left(\int_0^t c_{\lfloor s/\tau_0 \rfloor}^N(\mathcal{U}^N(s)) \, d\mathcal{W}_1(s) \right)^2,$$

$$L^N = \sup_{[0,T]} \left(\int_0^t c_{\lfloor s/\tau_0 \rfloor}^N(\boldsymbol{\mathcal{U}}^N(s)) \nabla_{v_1} c_{\lfloor s/\tau_0 \rfloor}^N(\boldsymbol{\mathcal{U}}^N(s)) \, ds \right)^2.$$

First, $\sup_N \mathbb{E}[|\mathcal{U}_1^N(0)|^2] < \infty$ by assumption (1.11) and $\mathbb{E}[K^N + L^N]$ is uniformly bounded because c_l^N and $\nabla_{v_1} c_l^N$ are uniformly bounded; see Proposition 7.3. By Doob's inequality,

$$\mathbb{E}[J^N] \leq C \int_0^T \mathbb{E}[\|\sigma_1^N(\boldsymbol{\mathcal{U}}^N(s))\|^2] \, ds$$

$$\leq C \int_0^T \mathbb{E}\left[\frac{1}{N} \sum_{j=1}^N \|(a \star \phi_{\eta_N}(\mathcal{U}_1^N - \mathcal{U}_j^N(s)))\| \right] ds.$$

Using exchangeability and that $\|(a \star \phi_{\eta_N})(v)\| \leq C(1 + |v|^{\gamma+2}) \leq C(1 + |v|^2)$,

$$\mathbb{E}[J^N] \leq C \int_0^T \mathbb{E}[1 + |\mathcal{U}_1^N(s) - \mathcal{U}_2^N(s)|^2] \, ds \leq C \int_0^T \mathbb{E}[1 + |\mathcal{U}_1^N(s)|^2] \, ds,$$

which is also uniformly bounded. Finally, Hölder's inequality, exchangeability and the inequality $|(b \star \phi_{\eta_N})(v)| \leq C(1 + |v|^{\gamma+1})$ lead us to

$$\mathbb{E}[I^N] \leq C_T \int_0^T \mathbb{E}[|(b \star \phi_{\eta_N})(\mathcal{U}_1^N(s) - \mathcal{U}_2^N(s))|^2] \, ds$$

$$\leq C_T \int_0^T \mathbb{E}[1 + |\mathcal{U}_1^N(s) - \mathcal{U}_2^N(s)|^{2\gamma+2}] \, ds.$$

If $\gamma \in [-1, 0)$, we bound $\mathbb{E}[|\mathcal{U}_1^N(s) - \mathcal{U}_2^N(s)|^{2\gamma+2}]$ by $CE[1 + |\mathcal{U}_1^N(s)|^2 + |\mathcal{U}_2^N(s)|^2] \leq C(1 + \mathbb{E}[|\mathcal{U}_1^N(s)|^2])$ and immediately deduce that $\mathbb{E}[I^N]$ is uniformly bounded. If now $\gamma \in (-2, -1)$, then it holds true that $\gamma < 2\gamma + 2 < 0$, so that also we conclude, using point (ii), that $\mathbb{E}[I^N]$ is uniformly bounded. $\square$

7.4. *Tightness for the perturbed particle system.* We can now prove the tightness of our perturbed particle system.

PROPOSITION 7.10. *We still assume* (1.11) *and consider, for each* $N \geq 2$*, the unique solution* $(\mathcal{U}_1^N(t), \ldots, \mathcal{U}_N^N(t))_{t \in [0,T]}$ *to* (7.2). *We also set*

$$\mathcal{Q}^N := \frac{1}{N} \sum_{i=1}^N \delta_{(\mathcal{U}_i^N(t))_{t \in [0,T]}}.$$

(i) *The family* $\{\mathcal{L}((\mathcal{U}_1^N(t))_{t \in [0,T]}), N \geq 2\}$ *is tight in* $\mathcal{P}(C([0, T], \mathbb{R}^3))$.
(ii) *The family* $\{\mathcal{L}(\mathcal{Q}^N), N \geq 2\}$ *is tight in* $\mathcal{P}(\mathcal{P}(\mathbb{R} \times C([0, T], \mathbb{R}^3)))$.

PROOF.   As is well known, point (ii) is implied by point (i) and the exchangeability of the system; see Sznitman [36], I-Proposition 2.2. To prove point (i), we use the shortened notation $b_i^N(s) = b_i^N(\mathcal{U}_s^N)$, $\sigma_i^N(s) = \sigma_i^N(\mathcal{U}_s^N)$ and $c^N(s) = c_{\lfloor s/\tau_0 \rfloor}^N(\mathcal{U}_s^N)$, we write

$$\mathcal{U}_1^N(t) = \mathcal{V}_1^N(0) + \int_0^t b_1^N(s)\,ds + \int_0^t \sigma_1^N(s)\,d\mathcal{B}_1(s) + \int_0^t c^N(s)\,d\mathcal{W}_1(s)$$

$$+ \int_0^t c^N(s)\nabla_{v_1}c^N(s)\,ds$$

$$= \mathcal{V}_1^N(0) + I^N(t) + J^N(t) + K^N(t) + L^N(t)$$

and prove separately that each term is tight.

First, $\{\mathcal{V}_1^N(0)\}_{N\geq 2}$ is tight by (1.11)(iii). Next, the families $\{(K^N(t))_{t\in[0,T]}\}_{N\geq 2}$ and $\{(L^N(t))_{t\in[0,T]}\}_{N\geq 2}$ are obviously tight (use the Kolmogorov criterion for K^N), because c^N and $\nabla_{v_1}c^N(s)$ are uniformly bounded, see Proposition 7.3(iii).

To prove that $\{(J^N(t))_{t\in[0,T]}\}_{N\geq 2}$ is tight, we use the Kolmogorov criterion (see, e.g., Stroock–Varadhan [34], Corollary 2.1.4): it suffices to show that there are some constants $p > 0$, $\beta > 1$ and C such that for all $N \geq 2$, all $0 \leq s \leq t \leq T$,

$$\mathbb{E}[|J^N(t) - J^N(s)|^p] \leq C|t - s|^\beta.$$

We consider $p = 4/(2 + \gamma) > 2$. By the Burkholder–Davis–Gundy inequality and since $[\sigma_1^N(s)]^2 = a_1^N(s)$, we have, for $0 \leq s \leq t \leq T$,

$$\mathbb{E}[|J^N(t) - J^N(s)|^p] \leq C\mathbb{E}\left[\left(\int_s^t \|a_1^N(u)\|\,du\right)^{p/2}\right]$$

$$\leq C\mathbb{E}\left[\left(\int_s^t \frac{1}{N}\sum_{j\neq 1}\|(a \star \phi_{\eta_N})(\mathcal{U}_1^N(u) - \mathcal{U}_j^N(u))\|\,du\right)^{p/2}\right].$$

Recalling next that $\|(a \star \phi_{\eta_N})(v)\| \leq C(1 + |v|^{\gamma+2})$ and using the Hölder inequality,

$$\mathbb{E}[|J^N(t) - J^N(s)|^p]$$

$$\leq C(t-s)^{p/2-1}\int_s^t \mathbb{E}\left[\left(1 + \frac{1}{N}\sum_{j\neq 1}|\mathcal{U}_1^N(u) - \mathcal{U}_j^N(u)|^{2+\gamma}\right)^{p/2}\right]du$$

$$\leq C(t-s)^{p/2}\sup_{[0,T]}\mathbb{E}\left[1 + \frac{1}{N}\sum_{j\neq 1}|\mathcal{U}_1^N(u) - \mathcal{U}_j^N(u)|^2\right]$$

since $p/2 > 1$ and $(2 + \gamma)p/2 = 2$. Using finally exchangeability and the moment bound proved in Lemma 7.5 [together with (1.11)], we deduce that $\mathbb{E}[|J^N(t) - J^N(s)|^p] \leq C_p(t - s)^{p/2}$. Since $p/2 > 1$, the tightness of $\{(J^N(t))_{t\geq 0}\}_{N\geq 2}$ follows.

The remaining term $I^N(t)$ is the more difficult, since it is singular [when $\gamma \in (-2, -1)$]. For some $p > 1$ to be chosen later, we write

$$|I^N(t) - I^N(s)| = \left| \int_s^t b_1^N(s)\, ds \right| \leq (t-s)^{1-1/p} \left(\int_s^t |b_1^N(s)|^p\, ds \right)^{1/p}.$$

Thus, with $\alpha = 1 - 1/p > 0$,

$$\sup_{0 \leq s < t \leq T} \frac{|I^N(t) - I^N(s)|}{|t-s|^\alpha} \leq \left(\int_0^T [b_1^N(s)]^p\, ds \right)^{1/p}.$$

Taking expectations and using the Hölder inequality, we get

$$\mathbb{E}\left[\sup_{0 \leq s < t \leq T} \frac{|I^N(t) - I^N(s)|}{|t-s|^\alpha} \right] \leq \left(\int_0^T \mathbb{E}[(b_1^N(s))^p]\, ds \right)^{1/p}.$$

Using the expression of $b_1^N(s) = N^{-1} \sum_{j=2}^N (b \star \phi_{\eta_N})(\mathcal{U}_1^N(s) - \mathcal{U}_j^N(s))$, the Hölder inequality and exchangeability, we find

$$\mathbb{E}\left[\sup_{0 \leq s < t \leq T} \frac{|I^N(t) - I^N(s)|}{|t-s|^\alpha} \right]^p \leq C \int_0^T \mathbb{E}[1 + |\mathcal{U}_1^N(s) - \mathcal{U}_2^N(s)|^{(\gamma+1)p}]\, ds.$$

If $\gamma \in (-1, 0)$, we choose $p = 2/(1+\gamma)$. By the boundedness of the moment of order two obtained in Proposition 7.5, we get, with $\alpha = (1-\gamma)/2$,

$$\mathbb{E}\left[\sup_{0 \leq s < t \leq T} \frac{|I^N(t) - I^N(s)|}{|t-s|^\alpha} \right]^p \leq C.$$

If $\gamma = -1$, we choose $p = 2$ (or any other value) and deduce that, with $\alpha = 1/2$,

$$\mathbb{E}\left[\sup_{0 \leq s < t \leq T} \frac{|I^N(t) - I^N(s)|}{|t-s|^\alpha} \right]^2 \leq C.$$

We now consider the case where $\gamma \in (-2, -1)$. Since $\gamma < 2\gamma + 2 < 0$, we deduce from Lemma 7.9(ii) that $\sup_{N \geq 2} \int_0^T \mathbb{E}[|\mathcal{U}_1^N(s) - \mathcal{U}_2^N(s)|^{2(\gamma+1)}]\, ds < \infty$. Consequently, we may choose $p = 2$ and get, with $\alpha = 1/2$,

$$\mathbb{E}\left[\sup_{0 \leq s < t \leq T} \frac{|I^N(t) - I^N(s)|}{|t-s|^\alpha} \right]^2 \leq C.$$

In any case, the tightness of the family $\{(I^N(t))_{t \in [0,T]}\}$ follows from the Ascoli theorem. $\square$

7.5. *Propagation of chaos for the perturbed system.* With the several estimates obtained in the previous subsections, we can now prove the propagation of chaos for the perturbed system. The proof uses some martingale techniques, as was initiated by Sznitman [36]. The only difficulty in the present case is to control the singularity of b (when $\gamma \in (-2, -1)$).

PROPOSITION 7.11. *Assume* (1.11). *For each* $N \geq 2$, *let* $(\mathcal{U}_1^N(t), \ldots, \mathcal{U}_N^N(t))_{t \in [0,T]}$ *be the unique solution to* (7.2). *Let* $(\mathcal{U}(t))_{t \in [0,T]}$ *be the solution to the perturbed nonlinear SDE* (7.3), *given by Proposition* 7.6. *The sequence* $((\mathcal{U}_1^N(t))_{t \in [0,T]}, \ldots, (\mathcal{U}_N^N(t))_{t \in [0,T]})$ *is* $(\mathcal{U}(t))_{t \in [0,T]}$-*chaotic.*

PROOF. We define $\mathcal{S}$ as the set of all probability measures $g \in \mathcal{P}(C([0,T], \mathbb{R}^3))$ such that g is the law of $(\mathcal{U}(t))_{t \in [0,T]}$ solution to the perturbed nonlinear SDE (7.3) associated with f_0 and satisfying, for $g_t \in \mathcal{P}(\mathbb{R}^3)$ the law of $\mathcal{U}(t)$,

$$(g_t)_{t \in [0,T]} \in L^1([0,T], L^p(\mathbb{R}^3)) \quad \text{and}$$
(7.5)
$$(g_t)_{t \in [0,T]} \in L^\infty([0,T], \mathcal{P}_2(\mathbb{R}^3))$$

for some $p > p_1(\gamma)$. The uniqueness result shown in Proposition 7.6 implies that the set $\mathcal{S}$ contains only one element.

We recall that $\mathcal{Q}^N = N^{-1} \sum_{i=1}^N \delta_{(\mathcal{U}_i^N(t))_{t \in [0,T]}}$ stands for the empirical distribution of the trajectories. By Proposition 7.10, this sequence is tight, we thus can consider a (not relabeled) subsequence of $\mathcal{Q}^N$ going in law to some $\mathcal{Q}$. We will show that $\mathcal{Q}$ almost surely belongs to $\mathcal{S}$. This will conclude the proof, since $\mathcal{S}$ contains only one element. The definition of propagation of chaos was recalled in Definition 1.1. We recall that $G_t^N = \mathcal{L}(\mathcal{U}_1^N(t), \ldots, \mathcal{U}_N^N(t))$.

Step 1. Consider the identity map $X : C([0,T], \mathbb{R}^3) \mapsto C([0,T], \mathbb{R}^3)$. Using the classical theory of martingale problems, we realize that g belongs to $\mathcal{S}$ as soon as

(a) $g \circ X_0^{-1} = f_0$;
(b) setting $g_t = g \circ X_t^{-1}$, (7.5) holds true;
(c) for all $0 < t_1 < \cdots < t_k < s < t \leq T$, all $\varphi_1, \ldots, \varphi_k \in C_b(\mathbb{R}^3)$, all $\varphi \in C_b^2(\mathbb{R}^3)$,

$$
\begin{aligned}
\mathcal{F}(g) := \int \int & g(d\beta) g(d\tilde{\beta}) \varphi_1(\beta_{t_1}) \cdots \varphi_k(\beta_{t_k}) \\
& \times \left[\varphi(\beta_t) - \varphi(\beta_s) - \int_s^t \mathbb{1}_{\{\beta_u \neq \tilde{\beta}_u\}} b(\beta_u - \tilde{\beta}_u) \cdot \nabla \varphi(\beta_u) \, du \right. \\
& - \frac{1}{2} \int_s^t \left[a(\beta_u - \tilde{\beta}_u) : \nabla^2 \varphi(\beta_u) \right] du \\
& \left. - \frac{1}{2} \int_s^t c_{\lfloor u/\tau_0 \rfloor}^2 (g_u) \Delta \varphi(\beta_u) \, du \right] = 0.
\end{aligned}
$$

Here and below, we use the notation $A : B = \sum_{k,l=1}^3 A_{kl} B_{kl}$ for two 3×3-matrices A and B.

Indeed, let $(\mathcal{U}(t))_{t \geq 0}$ be g-distributed. Then (a) implies that $\mathcal{U}(0)$ is f_0-distributed and (b) says that the requirement (7.5) is fulfilled. Finally, point (c)

tells us that for all $\varphi \in C_b^2(\mathbb{R}^2)$,

$$\varphi(\mathcal{U}(t)) - \varphi(\mathcal{U}(0)) - \int_0^t \int \mathbb{1}_{\{\mathcal{U}(s) \neq \tilde{\beta}_s\}} b(\mathcal{U}(s) - \tilde{\beta}_s) \cdot \nabla \varphi(\mathcal{U}(s)) g(d\tilde{\beta}) \, ds$$

$$- \frac{1}{2} \int_0^t \int [a(\mathcal{U}(s) - \tilde{\beta}_s) : \nabla^2 \varphi(\mathcal{U}(s))] g(d\tilde{\beta}) \, ds$$

$$- \frac{1}{2} \int_0^t c_{\lfloor s/\tau_0 \rfloor}^2 (g_s) \Delta \varphi(\mathcal{U}(s)) \, ds$$

is a martingale. Observe that $\int \mathbb{1}_{\{\mathcal{U}(s) \neq \tilde{\beta}_s\}} b(\mathcal{U}(s) - \tilde{\beta}_s) g(d\tilde{\beta}) \, ds = \int_{\mathbb{R}^3} \mathbb{1}_{\{\mathcal{U}(s) \neq x\}} \times b(\mathcal{U}(s) - x) g_s(dx) = \int_{\mathbb{R}^3} b(\mathcal{U}(s) - x) g_s(dx) = b(g_s, \mathcal{U}(s))$. We used here that g_s does not weight points, since it has a density by point (b). Similarly, $\int a(\mathcal{U}(s) - \tilde{\beta}_s) g(d\tilde{\beta}) = a(g_s, \mathcal{U}(s))$. All this classically implies the existence of two independent 3D-Brownian motions $(\mathcal{B}(t))_{t \geq 0}$ and $(\mathcal{W}(t))_{t \geq 0}$ such that

$$\mathcal{U}(t) = \mathcal{U}(0) + \int_0^t b(g_s, \mathcal{U}(s)) \, ds + \int_0^t \sigma(g_s, \mathcal{U}(s)) \, d\mathcal{B}(s)$$

$$+ \int_0^t c_{\lfloor s/\tau_0 \rfloor}(g_s) \, d\mathcal{W}(s).$$

Hence, $(\mathcal{U}(t))_{t \in [0,T]}$ solves (7.3) as desired.

We thus only have to prove that $\mathcal{Q}$ a.s. satisfies points (a), (b) and (c). For each $t \in [0, T]$, we set $\mathcal{Q}_t = \mathcal{Q} \circ X_t^{-1}$ and $\mathcal{Q}_t^N = \mathcal{Q}^N \circ X_t^{-1} = N^{-1} \sum_1^N \delta_{\mathcal{U}_i^N(t)}$.

Step 2. We know from (1.11) that the sequence $G_0^N = F_0^N$ is f_0-chaotic, which implies that $\mathcal{Q}_0^N = \mathcal{Q}^N \circ X_0^{-1}$ (this is nothing but the law of $N^{-1} \sum_1^N \delta_{\mathcal{V}_i^N(0)}$) goes weakly to f_0 in law (and thus in probability since f_0 is deterministic), whence $\mathcal{Q}_0 = f_0$ a.s. Hence, $\mathcal{Q}$ satisfies (a).

Step 3. Point (b) follows from Lemma 7.9, Proposition 7.5 and Corollary C.6. First, Lemma 7.9(i) tells us that $\sup_N \mathbb{E}[\sup_{[0,T]} |\mathcal{U}_1^N(s)|^2] < \infty$, which implies by exchangeability that $\sup_N \mathbb{E}[\sup_{[0,T]} m_2(\mathcal{Q}_t^N)] < \infty$. Since $\mathcal{Q}$ is a weak limit of $\mathcal{Q}^N$, we easily conclude that $\mathbb{E}[\sup_{[0,T]} m_2(\mathcal{Q}_t)] < \infty$, whence $(\mathcal{Q}_t)_{t \in [0,T]} \in L^\infty([0, T], \mathcal{P}_2(\mathbb{R}^3))$ a.s.

The integrability condition is slightly more complicated. Since $\mathcal{Q}_t^N$ goes in law to $\mathcal{Q}_t$ for each $t \in [0, T]$, we may apply Corollary C.6 and deduce that $\mathbb{E}[I_\gamma(\mathcal{Q}_t)] \leq \liminf_N I_\gamma(G_t^N)$. By the Fatou lemma, this yields

$$\int_0^T \mathbb{E}[I_\gamma(\mathcal{Q}_s)] \, ds \leq \int_0^T \liminf_N I_\gamma(G_t^N) \, dt \leq \liminf_N \int_0^T I_\gamma(G_t^N) \, dt < \infty$$

by Proposition 7.8. Next, we know from Proposition 7.5 and from (1.11) that, for $\alpha = \max\{2, q\}$ [with $q > q(\gamma)$], $\sup_N \sup_{[0,T]} \mathbb{E}[|\mathcal{U}_1^N(s)|^\alpha] < \infty$. Using exchangeability, this gives $\sup_N \sup_{[0,T]} \mathbb{E}[m_\alpha(\mathcal{Q}_t^N)] < \infty$, from which we deduce that $\sup_{[0,T]} \mathbb{E}[m_\alpha(\mathcal{Q}_t)] < \infty$. Consider now $p \in (\max\{3/2, p_1(\gamma)\}, \max\{(6 +$

$3|\gamma|)/(2 + 3|\gamma|)$, $p_2(\gamma, q)\})$ [observe that $(6 + 3|\gamma|)/(2 + 3|\gamma|) > 3/2$ because $|\gamma| < 2$]. We always have $p \in (3/2, 3)$ [because $p_2(\gamma, q) < 3$]. Set $r = (3p - 3)/(2p)$, which belongs to $(1/2, 1)$ and $a = |\gamma|r/(1 - r) = |\gamma|(3p - 3)/(3 - p)$. Then, applying Lemma C.3, we get

$$\|\mathcal{Q}_t\|_{L^p} \leq C\big(I_\gamma(\mathcal{Q}_t)\big)^r \big(1 + m_a(\mathcal{Q}_t)\big)^{1-r}$$

whence, by the Hölder inequality,

$$\mathbb{E}\big[\|\mathcal{Q}_t\|_{L^p}\big] \leq C\mathbb{E}\big[I_\gamma(\mathcal{Q}_t)\big]^r \mathbb{E}\big[1 + m_a(\mathcal{Q}_t)\big]^{1-r}.$$

But $a \leq \alpha$. Indeed, just use that $p < [(6 + 3|\gamma|)/(2 + 3|\gamma|)] \vee p_2(\gamma, q)$. But $p \leq (6 + 3|\gamma|)/(2 + 3|\gamma|)$ implies that $a \leq 2 \leq \alpha$ and $p \leq p_2(\gamma, q) = (3q + 3|\gamma|)/(q + 3|\gamma|)$ gives us $a \leq q \leq \alpha$. Consequently, recalling that $\sup_{[0,T]} \mathbb{E}[m_\alpha(\mathcal{Q}_t)] < \infty$, we get

$$\mathbb{E}\big[\|\mathcal{Q}_t\|_{L^p}\big] \leq C\mathbb{E}\big[I_\gamma(\mathcal{Q}_t)\big]^r \leq C\big(1 + \mathbb{E}[I_\gamma(\mathcal{Q}_t)]\big).$$

Integrating in time, we deduce that $\int_0^T \mathbb{E}[\|\mathcal{Q}_t\|_{L^p}]\,dt < \infty$, whence finally, $\int_0^T \|\mathcal{Q}_t\|_{L^p}\,dt < \infty$ a.s. We have checked that $\mathcal{Q}$ satisfies point (b).

Step 4. From now on, we consider some fixed $\mathcal{F} : \mathcal{P}(C([0, T], \mathbb{R}^3)) \mapsto \mathbb{R}$ as in point (c). We will check that $\mathcal{F}(\mathcal{Q}) = 0$ a.s. and this will complete the proof.

Step 4.1. Here, we prove that for all $N \geq 2$,

$$\mathbb{E}\big[|\mathcal{F}(\mathcal{Q}^N)|\big] \leq C_\mathcal{F}(N^{-1/2} + \eta_N).$$

To this end, we recall that $\varphi \in C_b^2(\mathbb{R}^3)$ is fixed and we define, for $i = 1, \dots, N$,

$$
\begin{aligned}
O_i^N(t) :=\ &\varphi(\mathcal{U}_i^N(t)) \\
&- \frac{1}{N} \sum_{j=1}^{N} \int_0^t \mathbb{1}_{\{\mathcal{U}_i^N(s) \neq \mathcal{U}_j^N(s)\}} b(\mathcal{U}_i^N(s) - \mathcal{U}_j^N(s)) \cdot \nabla\varphi(\mathcal{U}_i^N(s))\,ds \\
&- \frac{1}{2N} \sum_{j=1}^{N} \int_0^t \big[a(\mathcal{U}_i^N(s) - \mathcal{U}_j^N(s)) : \nabla^2\varphi(\mathcal{U}_i^N(s))\big]\,ds \\
&- \frac{1}{2} \int_0^t \big[c_{\lfloor s/\tau_0 \rfloor}^N(\boldsymbol{\mathcal{U}}^N(s))\big]^2 \Delta\varphi(\mathcal{U}_i^N(s))\,ds.
\end{aligned}
$$

By definition of $\mathcal{F}$, we have

$$\mathcal{F}(\mathcal{Q}^N) = \frac{1}{N} \sum_{i=1}^{N} \varphi_1(\mathcal{U}_i^N(t_1)) \cdots \varphi_k(\mathcal{U}_i^N(t_k))\big[O_i^N(t) - O_i^N(s)\big].$$

Applying the Itô formula to compute $\varphi(\mathcal{U}_i^N(t))$, we realize that, for $i = 1, \dots, N$, $O_i^N(t) = \varphi(\mathcal{U}_i^N(0)) + M_i^N(t) + \Delta_i^N(t)$, where

$$M_i^N(t) = \int_0^t \nabla\varphi(\mathcal{U}_i^N(s)) \cdot \big[\sigma_i^N(\boldsymbol{\mathcal{U}}^N(s))\,d\mathcal{B}^i(s) + c_{\lfloor s/\tau_0 \rfloor}^N(\boldsymbol{\mathcal{U}}^N(s))\,d\mathcal{W}^i(s)\big],$$

$$\Delta_i^N(t) = \int_0^t \nabla\varphi(\mathcal{U}_i^N(s)) \cdot \left[c_{\lfloor s/\tau_0 \rfloor}^N(\boldsymbol{\mathcal{U}}^N(s)) \nabla_{v_i} c_{\lfloor s/\tau_0 \rfloor}^N(\boldsymbol{\mathcal{U}}^N(s)) \right.$$

$$+ \frac{1}{N} \sum_{j=1}^N \mathbb{1}_{\{\mathcal{U}_i^N(s) \neq \mathcal{U}_j^N(s)\}} (b \star \phi_{\eta_N} - b)(\mathcal{U}_i^N(s) - \mathcal{U}_j^N(s)) \bigg] ds$$

$$+ \int_0^t \frac{1}{2N} \sum_{j=1}^N [(a \star \phi_{\eta_N} - a)(\mathcal{U}_i^N(s) - \mathcal{U}_j^N(s)) : \nabla^2\varphi(\mathcal{U}_i^N(s))] \, ds.$$

We used here that $(b \star \phi_{\eta_N})(x) = \mathbb{1}_{\{x \neq 0\}} (b \star \phi_{\eta_N})(x)$ since $(b \star \phi_{\eta_N})(0) = 0$ by symmetry. Performing now some classical stochastic calculus, using that $0 < t_1 < \cdots < t_k < s < t$, that $\varphi_1, \ldots, \varphi_k, \nabla\varphi$ and c_l^N are uniformly bounded and that the Brownian motions $\mathcal{B}^1, \ldots, \mathcal{B}^N$ and $\mathcal{W}^1, \ldots, \mathcal{W}^N$ are independent, we easily obtain

$$\mathbb{E}\left[\left(\frac{1}{N} \sum_{i=1}^N \varphi_1(\mathcal{U}_i^N(t_1)) \cdots \varphi_k(\mathcal{U}_i^N(t_k)) [M_i^N(t) - M_i^N(s)] \right)^2 \right]$$

$$\leq \frac{C_{\mathcal{F}}}{N^2} \sum_{i=1}^N \int_s^t \mathbb{E}[1 + \|\sigma_i^N(\boldsymbol{\mathcal{U}}^N(s))\|^2] \, ds \leq \frac{C_{\mathcal{F}}}{N},$$

where we have used the fact that $\|\sigma_i^N(\boldsymbol{\mathcal{U}}^N(s))\|^2 \leq CN^{-1} \sum_{j=1}^N (1 + |\mathcal{U}_i^N(s) - \mathcal{U}_j^N(s)|^{2+\gamma})$, exchangeability, and the moment estimate of Proposition 7.5 (since $\gamma + 2 \in (0, 2]$).

Next, we use exchangeability and the boundedness of $\varphi_1, \ldots, \varphi_k, \nabla\varphi, \nabla^2\varphi$ to write

$$\mathbb{E}\left[\left| \frac{1}{N} \sum_{i=1}^N \varphi_1(\mathcal{U}_i^N(t_1)) \cdots \varphi_k(\mathcal{U}_i^N(t_k)) [\Delta_i^N(t) - \Delta_i^N(s)] \right| \right]$$

$$\leq C_{\mathcal{F}} \mathbb{E}[I^N + J^N + K^N],$$

where

$$I^N := \int_0^t |c_{\lfloor s/\tau_0 \rfloor}^N(\boldsymbol{\mathcal{U}}^N(s)) \nabla_{v_1} c_{\lfloor s/\tau_0 \rfloor}^N(\boldsymbol{\mathcal{U}}^N(s))| \, ds,$$

$$J^N := \int_0^t |(b \star \phi_{\eta_N} - b)(\mathcal{U}_1^N(s) - \mathcal{U}_2^N(s))| \, ds,$$

$$K^N := \int_0^t \|(a \star \phi_{\eta_N} - a)(\mathcal{U}_1^N(s) - \mathcal{U}_2^N(s))\| \, ds.$$

Since $c_l^N \nabla_{v_1} c_l^N$ is bounded by C/N by Proposition 7.3(iii), we immediately find that $\mathbb{E}[I^N] \leq C/N$. Next, using Lemma 2.4(ii),

$$\mathbb{E}[J^N] \leq C\eta_N \int_0^t \mathbb{E}[|\mathcal{U}_s^N(s) - \mathcal{U}_2^N(s)|^\gamma] \, ds \leq C_t \eta_N$$

by Lemma 7.9(ii). Finally, using again Lemma 2.4(ii),

$$\mathbb{E}[K^N] \le C\eta_N^2 \int_0^t \mathbb{E}[|\mathcal{U}_s^N(s) - \mathcal{U}_2^N(s)|^\gamma]\, ds \le C_t \eta_N.$$

Step 4.2. We introduce, for $\varepsilon \in (0, 1)$, a smooth and bounded approximation $b_\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $b_\varepsilon(x) = b(x)$ for $|x| \ge \varepsilon$ and $|b_\varepsilon(x)| \le |b(x)| = 2|x|^{1+\gamma}$ for all x. This smoothing is useless if $\gamma \in [-1, 0)$, but we treat all the cases similarly to avoid repetitions. We also introduce $\mathcal{F}_\varepsilon$ defined as $\mathcal{F}$ with b replaced by b_ε. The diffusion coefficient a is continuous and so remains unchanged. For each fixed $\varepsilon \in (0, 1)$, for every $M > 0$, the map $g \mapsto \mathcal{F}_\varepsilon(g)$ is continuous and bounded on the set $g \in \mathcal{P}(C([0, T], \mathbb{R}^3))$, $\int(\sup_{[0,T]} |\beta_s|^2)g(d\beta) \le M$. This is not hard to check, using that $\varphi_1, \ldots, \varphi_k, \varphi, \nabla\varphi, \nabla^2\varphi$ are continuous and bounded, that $b_\varepsilon(z)$ and $a(z)$ are continuous and bounded by $C_\varepsilon(1 + z^2)$, and finally that c_l is bounded (by 3) and Lipschitz continuous for the W_2 topology by Proposition 7.3(vi). Since $\mathcal{Q}^N$ goes in law to $\mathcal{Q}$ and since $\sup_N \mathbb{E}[\int(\sup_{[0,T]} |\beta_s|^2)\mathcal{Q}^N(d\beta)] = \sup_N \mathbb{E}[\sup_{[0,T]} |\mathcal{U}_1^N(s)|^2] < \infty$ by exchangeability and Lemma 7.9(i), we deduce that for any $\varepsilon \in (0, 1)$,

$$\mathbb{E}[|\mathcal{F}_\varepsilon(\mathcal{Q})|] = \lim_N \mathbb{E}[|\mathcal{F}_\varepsilon(\mathcal{Q}^N)|].$$

Step 4.3. We now prove that for all $N \ge 2$, all $\varepsilon \in (0, 1)$,

$$\mathbb{E}[|\mathcal{F}(\mathcal{Q}^N) - \mathcal{F}_\varepsilon(\mathcal{Q}^N)|] \le C_\mathcal{F}\varepsilon.$$

Using that all the functions (including the derivatives) involved in $\mathcal{F}$ are bounded and that $|b_\varepsilon(x) - b(x)| \le 2|x|^{1+\gamma}\mathbb{1}_{\{|x|<\varepsilon\}}$, we get

$$|\mathcal{F}(g) - \mathcal{F}_\varepsilon(g)|$$

(7.6)
$$\le C_\mathcal{F} \iint \int_0^T \mathbb{1}_{\{0<|\beta_t-\tilde\beta_t|<\varepsilon\}} |\beta_t - \tilde\beta_t|^{1+\gamma}\, dt\, g(d\tilde\beta)g(d\beta)$$

$$\le C_\mathcal{F}\varepsilon \iint \int_0^T \mathbb{1}_{\{\beta_t \ne \tilde\beta_t\}} |\beta_t - \tilde\beta_t|^\gamma\, dt\, g(d\tilde\beta)g(d\beta).$$

Thus,

$$|\mathcal{F}(\mathcal{Q}^N) - \mathcal{F}_\varepsilon(\mathcal{Q}^N)| \le C_\mathcal{F}\frac{\varepsilon}{N^2} \sum_{i \ne j} \int_0^T |\mathcal{U}_i^N(t) - \mathcal{U}_j^N(t)|^\gamma\, dt.$$

It suffices to take expectations, to use exchangeability and then Lemma 7.9(ii) to complete the step.

Step 4.4. We next check that a.s.,

$$\lim_{\varepsilon \to 0} |\mathcal{F}(\mathcal{Q}) - \mathcal{F}_\varepsilon(\mathcal{Q})| = 0.$$

Starting from (7.6), using (1.4) and that $(\mathcal{Q}_t)_{t\in[0,T]} \in L^1([0,T], L^p(\mathbb{R}^3))$ a.s. for some $p > p_1(\gamma)$ by step 3, we get

$$\left|\mathcal{F}(\mathcal{Q}) - \mathcal{F}_\varepsilon(\mathcal{Q})\right| \le C_{\mathcal{F}}\varepsilon \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x-y|^\gamma \mathcal{Q}_t(dx)\mathcal{Q}_t(dy)$$

$$\le C_{\mathcal{F}}\varepsilon \int_0^T \left(1 + \|\mathcal{Q}_t\|_{L^p}\right) dt,$$

whence the conclusion.

Step 4.5. We finally conclude: for any $\varepsilon \in (0,1)$, we write, using steps 4.1, 4.2 and 4.3,

$$\mathbb{E}\left[\left|\mathcal{F}(\mathcal{Q})\right| \wedge 1\right] \le \mathbb{E}\left[\left|\mathcal{F}_\varepsilon(\mathcal{Q})\right|\right] + \mathbb{E}\left[\left|\mathcal{F}(\mathcal{Q}) - \mathcal{F}_\varepsilon(\mathcal{Q})\right| \wedge 1\right]$$

$$= \lim_{N \to +\infty} \mathbb{E}\left[\left|\mathcal{F}_\varepsilon(\mathcal{Q}^N)\right|\right] + \mathbb{E}\left[\left|\mathcal{F}(\mathcal{Q}) - \mathcal{F}_\varepsilon(\mathcal{Q})\right| \wedge 1\right]$$

$$\le \limsup_{N \to +\infty} \mathbb{E}\left[\left|\mathcal{F}(\mathcal{Q}^N)\right|\right] + \limsup_{N \to +\infty} \mathbb{E}\left[\left|\mathcal{F}(\mathcal{Q}^N) - \mathcal{F}_\varepsilon(\mathcal{Q}^N)\right|\right]$$

$$\quad + \mathbb{E}\left[\left|\mathcal{F}(\mathcal{Q}) - \mathcal{F}_\varepsilon(\mathcal{Q})\right| \wedge 1\right]$$

$$\le C_{\mathcal{F}}\varepsilon + \mathbb{E}\left[\left|\mathcal{F}(\mathcal{Q}) - \mathcal{F}_\varepsilon(\mathcal{Q})\right| \wedge 1\right].$$

Now we let $\varepsilon \to 0$ and use that $\lim_\varepsilon \mathbb{E}[|\mathcal{F}(\mathcal{Q}) - \mathcal{F}_\varepsilon(\mathcal{Q})| \wedge 1] = 0$ thanks to step 4.4 by dominated convergence. Consequently, $\mathbb{E}[|\mathcal{F}(\mathcal{Q})| \wedge 1] = 0$, whence $\mathcal{F}(\mathcal{Q}) = 0$ a.s. as desired. $\quad\square$

7.6. *Asymptotic annihilation of the perturbation.* We finally show that the perturbed and unperturbed particle systems are asymptotically the same. This uses the propagation of chaos for the perturbed system.

PROPOSITION 7.12. *Recall that $\gamma \in (-2, 0)$, assume (1.11) and consider, for each $N \ge 2$, the solution $(\mathcal{V}^N(t))_{t\ge0}$ to (1.9) and the solution $(\mathcal{U}^N(t))_{t\in[0,T]}$ to (7.2), with the same initial conditions $\mathcal{U}_i^N(0) = \mathcal{V}_i^N(0)$, $i = 1, \ldots, N$ and same Brownian motions $\mathcal{B}^i$, $i = 1, \ldots, N$. Then*

$$\lim_N \Pr\left[(\mathcal{V}^N(t))_{t\in[0,T]} = (\mathcal{U}^N(t))_{t\in[0,T]}\right] = 1.$$

PROOF. By the strong uniqueness for the particle system (1.9) (see Proposition 1.5), we see that $(\mathcal{V}^N(t))_{t\in[0,T]} = (\mathcal{U}^N(t))_{t\in[0,T]}$ as soon as $c^N_{\lfloor t/\tau_0\rfloor}(\mathcal{U}^N(t)) = 0$ for all $t \in [0,T]$. Recalling the definition of c_l (see Notation 7.1), it thus suffices to prove that

$$(7.7) \quad \lim_N \Pr\left[\forall l = 0, \ldots, n_0, \forall t \in [l\tau_0, (l+1)\tau_0], c_l^N(\mathcal{U}^N(t)) = 0\right] = 1.$$

We denote by $d : \mathcal{P}(C([0,T], \mathbb{R}^3)) \mapsto \mathbb{R}^+$ the function defined by

$$d(g) := \inf_{l=0,\ldots,n_0} \inf_{t\in[l\tau_0,(l+1)\tau_0]} \inf_{k=1,2,3} \int_{\mathbb{R}^3} h\left(\frac{v - x_k^l}{\delta_0}\right) g_t(dv),$$

where $h : \mathbb{R}^3 \to [0,1]$ (a smooth function satisfying $\mathbb{1}_{\{|v| \leq 1\}} \leq h \leq \mathbb{1}_{\{|v| \leq 2\}}$), the x_k^l's and δ_0 have been introduced in Notation 7.1. Remark first that d is continuous (and bounded) with respect to the weak topology of measure on $\mathcal{P}(C([0,T], \mathbb{R}^3))$. Consequently, we know from Proposition 7.11 that $d(\mathcal{Q}^N)$ goes in law to $d(g)$, where $\mathcal{Q}^N = N^{-1} \sum_1^N \delta_{(\mathcal{U}_i^N(t))_{t \in [0,T]}}$ and where $g = \mathcal{L}((\mathcal{U}(t))_{t \in [0,T]})$, where $(\mathcal{U}(t))_{t \in [0,T]}$ is the unique solution to the perturbed nonlinear SDE (7.3). But we also know, from Proposition 7.6 that $g_t = f_t$ for all $t \in [0,T]$, where f is the unique weak solution to the Landau equation (1.1). We finally recall (7.1)

$$\inf_{l=0,\dots,n_0} \inf_{t \in [l\tau_0, (l+1)\tau_0]} \inf_{k=1,2,3} f_t\big(B(x_k^l, \delta_0)\big) \geq \kappa_0,$$

which implies that $d(g) \geq \kappa_0$. Consequently, it holds that

$$\lim_{N \to +\infty} \Pr\left[d(\mathcal{Q}^N) \geq \frac{\kappa_0}{2}\right] = 1,$$

whence

$$\lim_{N \to +\infty} \Pr\left[\inf_{l=0,\dots,n_0} \inf_{t \in [l\tau_0, (l+1)\tau_0]} \inf_{k=1,2,3} \mathcal{Q}_t^N\big(B(x_k^l, 2\delta_0)\big) \geq \frac{\kappa_0}{2}\right] = 1.$$

Using Proposition 7.3(i) [which implies that $c_l(\mu) = 0$ as soon as $\inf_{k=1,2,3} \mu(B(x_k^l, 2\delta_0)) \geq \kappa_0/2$] and that $c_l^N(\mathcal{U}_t^N) = c_l(\mathcal{Q}_t^N)$ by definition (see Notation 7.1 again), we conclude that indeed, (7.7) holds true. $\square$

7.7. *Conclusion.* We now have all the weapons to give the proof of our main result.

PROOF OF THEOREM 1.8. We know from Proposition 7.11 that the perturbed system $(\mathcal{U}^N(t))_{t \in [0,T]}$ is $(\mathcal{U}(t))_{t \in [0,T]}$-chaotic, where $(\mathcal{U}(t))_{t \in [0,T]}$ is the unique solution to the perturbed nonlinear SDE (7.3). But Proposition 7.6 tells us that $(\mathcal{U}(t))_{t \in [0,T]} = (\mathcal{V}(t))_{t \in [0,T]}$, where $(\mathcal{V}(t))_{t \in [0,T]}$ is the unique solution to the (nonperturbed) nonlinear SDE (7.3), while Proposition 7.12 tells us that $\lim_N \Pr((\mathcal{V}^N(t))_{t \in [0,T]} = (\mathcal{U}^N(t))_{t \in [0,T]}) = 1$. We immediately conclude that $(\mathcal{V}^N(t))_{t \in [0,T]}$ is $(\mathcal{V}(t))_{t \in [0,T]}$-chaotic. Recalling that $T > 0$, which has been fixed at the beginning of the section, can be chosen arbitrarily large, we deduce that the sequence $(\mathcal{V}^N(t))_{t \geq 0}$ is $(\mathcal{V}(t))_{t \geq 0}$-chaotic. As already mentioned (see Definition 1.1), this implies that $\mathcal{Q}^N := N^{-1} \sum_1^N \delta_{(\mathcal{V}_i^N(t))_{t \geq 0}}$ goes in probability to $\mathcal{L}((\mathcal{V}(t))_{t \geq 0})$ as $N \to \infty$ in $\mathcal{P}(C([0, \infty), \mathbb{R}^3))$. But $\mathcal{L}(\mathcal{V}(t)) = f_t$ for all $t \geq 0$. It is not hard to conclude, that for $\mu_t^N := N^{-1} \sum_1^N \delta_{\mathcal{V}_i^N(t)}$, the sequence $(\mu_t^N)_{t \geq 0}$ goes in probability to $(f_t)_{t \geq 0}$ in $C([0, \infty), \mathcal{P}(\mathbb{R}^3))$. $\square$

APPENDIX A: A COUPLING RESULT

We reformulate and extend a result found in [21], Proposition 2.4, for the distance W_1. Here, $|\cdot|$ is any fixed norm in $\mathbb{R}^d$.

PROPOSITION A.1. *Let $p > 0$ and $d \geq 1$. For $N \geq 2$, let F_N and G_N be two symmetric probability measures in $\mathcal{P}_p((\mathbb{R}^d)^N)$. There exists $(X_1, \ldots, X_N)$ with law F^N and $(Y_1, \ldots, Y_N)$ with law G^N enjoying the following properties*:

(a) *The coupling is optimal in the sense that $\mathbb{E}[\sum_1^N |X_i - Y_i|^p] = \inf\{\mathbb{E}[\sum_1^N |U_i - V_i|^p]\}$, the infimum being taken over all random vectors $(U_1, \ldots, U_N)$ and $(V_1, \ldots, V_N)$ with laws F^N and G^N.*
(b) *The family $\{(X_i, Y_i), i = 1, \ldots, N\}$ is exchangeable.*
(c) *Almost surely, $W_p^p(N^{-1} \sum_1^N \delta_{X_i}, N^{-1} \sum_1^N \delta_{Y_i}) = N^{-1} \sum_1^N |X_i - Y_i|^p$.*

PROOF. We only sketch the proof, since it is very similar to [21], Proposition 2.4.

We start with a coupling $\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_N)$, $\tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_N)$ of F^N and G^N satisfying only point (a). Such an optimal coupling is well known to exist; see, for example, Villani [40].

Next, we consider a (uniform) random $\sigma \in \mathfrak{S}_N$, the set of permutations of $\{1, \ldots, N\}$, independent of $\tilde{X}, \tilde{Y}$, and we put $\hat{X}_i = \tilde{X}_{\sigma(i)}$ and $\hat{Y}_i = \tilde{Y}_{\sigma(i)}$. It is straightforward to check that $\hat{X} = (\hat{X}_1, \ldots, \hat{X}_N)$, $\hat{Y} = (\hat{Y}_1, \ldots, \hat{Y}_N)$ is still a coupling between F^N and G^N (because these distributions are symmetric), still satisfies (a), and now satisfies (b).

We finally introduce, for $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ in $(\mathbb{R}^d)^N$,

$$S_{x,y} = \left\{ \tau \in \mathfrak{S}_N : W_p^p\left(N^{-1} \sum_1^N \delta_{x_i}, N^{-1} \sum_1^N \delta_{y_i}\right) = N^{-1} \sum_1^N |x_i - y_{\tau(i)}|^p \right\}.$$

Conditionally on $\hat{X}$ and $\hat{Y}$, we consider a random permutation τ uniformly chosen in $S_{\hat{X}, \hat{Y}}$ and we set $X_i = \hat{X}_i$ and $Y_i = \hat{Y}_{\tau(i)}$. It remains to prove that $X = (X_1, \ldots, X_N)$, $Y = (Y_1, \ldots, Y_N)$ is a coupling between F^N and G^N satisfying points (a), (b) and (c). Point (c) is satisfied because $\tau \in S_{\hat{X}, \hat{Y}}$ a.s. By definition of τ, we see that $\sum_1^N |X_i - Y_i|^p \leq \sum_1^N |\hat{X}_i - \hat{Y}_i|^p$ a.s., so that (a) is satisfied. And of course, X is F^N-distributed, since $X = \hat{X}$.

To check point (b), let $\sigma \in \mathfrak{S}_N$ be fixed. For $x \in (\mathbb{R}^d)^N$, we put $x_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(N)})$. We observe that for any $x, y \in (\mathbb{R}^d)^N$, it holds that $S_{x_\sigma, y_\sigma} = \sigma^{-1} S_{x,y} \sigma$. Thus, conditionally on $\hat{X}$ and $\hat{Y}$, $\sigma^{-1} \circ \tau \circ \sigma$ is uniformly distributed on $S_{\hat{X}_\sigma, \hat{Y}_\sigma}$. Hence, by exchangeability of $(\hat{X}, \hat{Y})$, the triple $(\hat{X}, \hat{Y}, \tau)$ has the same law as the triple $(\hat{X}_\sigma, \hat{Y}_\sigma, \sigma^{-1} \circ \tau \circ \sigma)$. Thus, $(\hat{X}, \hat{Y}_\tau) \overset{d}{=} (\hat{X}_\sigma, \hat{Y}_{\tau \circ \sigma})$. In other words, (X, Y) has the same law as (X_σ, Y_σ).

We finally check that Y is G^N-distributed. Consider a bounded measurable $\varphi :$ $(\mathbb{R}^3)^N \mapsto \mathbb{R}$ and its symmetrization $\tilde{\varphi}(y_1, \ldots, y_N) = (N!)^{-1} \sum_{\sigma \in \mathfrak{S}_N} \varphi(y_{\sigma(1)}, \ldots,$ $y_{\sigma(n)})$. Using the exchangeability of $(Y_1, \ldots, Y_N)$, we can write $\mathbb{E}[\varphi(Y_1, \ldots,$ $Y_N)] = \mathbb{E}[\tilde{\varphi}(Y_1, \ldots, Y_N)]$. But $\tilde{\varphi}$ being symmetric, we a.s. have $\tilde{\varphi}(Y_1, \ldots, Y_N) =$ $\tilde{\varphi}(\hat{Y}_1, \ldots, \hat{Y}_N)$, whence $\mathbb{E}[\varphi(Y_1, \ldots, Y_N)] = \mathbb{E}[\tilde{\varphi}(\hat{Y}_1, \ldots, \hat{Y}_N)]$. Using finally the exchangeability of $(\hat{Y}_1, \ldots, \hat{Y}_N)$, we deduce the equality $\mathbb{E}[\varphi(Y_1, \ldots, Y_N)] =$ $\mathbb{E}[\varphi(\hat{Y}_1, \ldots, \hat{Y}_N)]$. This completes the proof. $\square$

APPENDIX B: SOLUTIONS TO A SDE ASSOCIATED TO WEAK SOLUTIONS OF A PDE

Here, we extend a result of Figalli [10], Theorem 2.6.

PROPOSITION B.1. *Let $\alpha : [0, \infty) \times \mathbb{R}^3 \mapsto \mathbb{M}_{3 \times 3}(\mathbb{R})$ and $\beta : [0, \infty) \times \mathbb{R}^3 \mapsto$ $\mathbb{R}^3$ be measurable and satisfy $\|\alpha(t, x)\| \leq \rho(t)(1 + |x|^2)$ and $|\beta(t, x)| \leq \rho(t)(1 +$ $|x|)$ for some $\rho \in L^1_{\text{loc}}([0, \infty))$. Consider $(\mu_t)_{t \geq 0} \in L^\infty_{\text{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$, weak solution to*

$$\text{(B.1)} \qquad \partial_t \mu_t = - \sum_{k=1}^3 \partial_{x_k}(\beta(t)\mu_t) + \frac{1}{2} \sum_{k,l=1}^3 \partial_{x_k x_l}(\alpha_{kl}(t)\mu_t).$$

There exists, on some probability space, a μ_0-distributed random variable X_0, independent of a d-dimensional Brownian motion $(B_t)_{t \geq 0}$ and a solution $(X_t)_{t \geq 0}$ to

$$\text{(B.2)} \qquad X_t = X_0 + \int_0^t \beta(s, X_s) \, ds + \int_0^t (\alpha(s, X_s))^{1/2} \, dB_s,$$

which furthermore satisfies that $\mathcal{L}(X_t) = \mu_t$ for all $t \geq 0$.

We follow the proof of [10], which concerns the case where α and β are bounded.

PROOF OF PROPOSITION B.1.   It suffices to prove the result when $\rho \equiv 1$. Indeed, consider, in the general case, the time change $h(t) = \int_0^t (1 + \rho(s)) \, ds$, its inverse function $g(t) = h^{-1}(t)$ and set $\bar{\mu}_t := \mu_{g(t)}$. Then $(\bar{\mu}_t)_{t \geq 0}$ still belongs to $L^\infty_{\text{loc}}([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ and solves (B.1) with α and β replaced by

$$\bar{\alpha}(t, x) = (1 + \rho(g(t)))^{-1} \alpha(g(t), x) \quad \text{and} \quad \bar{\beta}(t, x) = (1 + \rho(g(t)))^{-1} \beta(g(t), x).$$

These functions satisfy $\|\bar{\alpha}(t, x)\| \leq 1 + |x|^2$ and $|\bar{\beta}(t, x)| \leq 1 + |x|$. Thus, if we have proved the result when $\rho \equiv 1$, we can find a solution $(\bar{X}_t)_{t \geq 0}$ to $\bar{X}_t = \bar{X}_0 + \int_0^t \bar{\beta}(s, \bar{X}_s) \, ds + \int_0^t (\bar{\alpha}(s, \bar{X}_s))^{1/2} \, dB_s$ and such that $\mathcal{L}(\bar{X}_t) = \bar{\mu}_t$. It is then not hard to see that $X_t := \bar{X}_{h(t)}$ satisfies $\mathcal{L}(X_t) = \mu_t$ and solves $X_t = X_0 + \int_0^t \beta(s, X_s) \, ds +$

$\int_0^t (\alpha(s, X_s))^{1/2} \, dW_s$, where $W_t = \int_0^{h(t)} (1 + \rho(g(s)))^{-1/2} \, dB_s$ is still a Brownian motion.

From now on, we thus assume that $\|\alpha(t, x)\| \leq 1 + |x|^2$ and $|\beta(t, x)| \leq 1 + |x|$ and we consider $(\mu_t)_{t \geq 0}$ as in the statement. We divide the proof in several steps.

Step 1. We introduce $\phi_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/(2t)}$ and $\mu_t^\varepsilon := \mu_t \star \phi_{\varepsilon(1+t)}$. For each $t \geq 0$, μ_t^ε is a positive smooth function. Then $(\mu_t^\varepsilon)_{t \geq 0}$ solves (B.1) with α and β replaced by

$$(\text{B.3}) \quad \alpha^\varepsilon(t) := \frac{(\alpha(t)\mu_t) \star \phi_{\varepsilon(1+t)}}{\mu_t^\varepsilon} + \varepsilon \mathrm{Id} \quad \text{and} \quad \beta^\varepsilon(t) := \frac{(\beta(t)\mu_t) \star \phi_{\varepsilon(1+t)}}{\mu_t^\varepsilon}.$$

We now check that there is a constant C [not depending on $\varepsilon \in (0, 1)$] such that

$$|\beta^\varepsilon(t, x)| \leq C\left(1 + \sqrt{m_2(\mu_t)} + |x|\right) \quad \text{and} \quad \|\alpha^\varepsilon(t, x)\| \leq C\left(1 + m_2(\mu_t) + |x|^2\right).$$

First, since $\beta(t, x) \leq 1 + |x|$,

$$|\beta^\varepsilon(t, x)| \leq \frac{\int_{\mathbb{R}^3} |\beta(t, y)| \phi_{\varepsilon(1+t)}(x - y) \mu_t(dy)}{\int_{\mathbb{R}^3} \phi_{\varepsilon(1+t)}(x - y) \mu_t(dy)}$$

$$\leq 1 + \frac{\int_{\mathbb{R}^3} |y| \phi_{\varepsilon(1+t)}(x - y) \mu_t(dy)}{\int_{\mathbb{R}^3} \phi_{\varepsilon(1+t)}(x - y) \mu_t(dy)}.$$

We next introduce $R := \sqrt{2m_2(\mu_t)}$, for which $\mu_t(B(0, R)) \geq 1/2$, and write

$$|\beta^\varepsilon(t, x)| \leq 1 + 2|x| + R + \frac{\int_{|y-x| \geq |x|+R} |y| \phi_{\varepsilon(1+t)}(x - y) \mu_t(dy)}{\int_{|y-x| \leq |x|+R} \phi_{\varepsilon(1+t)}(x - y) \mu_t(dy)}.$$

But ϕ_ε is radially symmetric and decreasing, so that

$$\int_{|y-x| \geq |x|+R} |y| \phi_{\varepsilon(1+t)}(x - y) \mu_t(dy) \leq \phi_{\varepsilon(1+t)}(|x| + R) \sqrt{m_2(\mu_t)}$$

and [observe that $B(0, R) \subset B(x, |x| + R)$]

$$\int_{|y-x| \leq |x|+R} \phi_{\varepsilon(1+t)}(x - y) \mu_t(dy) \geq \phi_{\varepsilon(1+t)}(|x| + R) \mu_t\left(B(x, |x| + R)\right)$$

$$\geq \frac{\phi_{\varepsilon(1+t)}(|x| + R)}{2}.$$

This gives

$$|\beta^\varepsilon(t, x)| \leq 1 + 2|x| + R + 2\sqrt{m_2(\mu_t)} \leq 1 + 2|x| + 4\sqrt{m_2(\mu_t)}$$

as desired. The very same arguments show that

$$\|\alpha^\varepsilon(t, x)\| \leq 1 + (2|x| + R)^2 + 2m_2(\mu_t) \leq 1 + 8|x|^2 + 6m_2(\mu_t).$$

Finally, it is easy to check that for all $\varepsilon \in (0, 1)$, all $T > 0$, all $R > 0$,

$$\sup_{t \in [0,T]} \sup_{|x| \leq R} \left(|D\alpha^\varepsilon(t, x)| + |D\beta^\varepsilon(t, x)|\right) < \infty.$$

Step 2. The coefficient $\beta^\varepsilon(t, \cdot)$ is locally Lipschitz continuous (locally uniformly in time) and has at most linear growth, while $\alpha^\varepsilon(t, \cdot)$ is locally Lipschitz continuous (locally uniformly in time) and has at most quadratic growth. Since furthermore $\alpha^\varepsilon(t, \cdot)$ is uniformly elliptic (for ε fixed), we can apply Lemma 2.2 and obtain that $(\alpha^\varepsilon(t, \cdot))^{1/2}$ is also locally Lipschitz. It is thus classical that for a given initial condition X_0^ε with law μ_0^ε and a given Brownian motion $(B_t)_{t \geq 0}$, there exists a pathwise unique solution to

$$(\text{B.4}) \qquad X_t^\varepsilon = X_0^\varepsilon + \int_0^t \beta^\varepsilon(s, X_s^\varepsilon)\,ds + \int_0^t (\alpha^\varepsilon(s, X_s^\varepsilon))^{1/2}\,dB_s,$$

which furthermore satisfies $\mathcal{L}(X_t^\varepsilon) = \mu_t^\varepsilon$, by uniqueness of the weak solution to the PDE (B.1) with α and β replaced by α^ε and β^ε.

Step 3. Here, we check that the family $\{(X_t^\varepsilon)_{t \geq 0}, \varepsilon > 0\}$ is tight. Since μ_0 has only a moment of order two, we cannot directly apply the Kolmogorov criterion and have to use another approximation procedure. For $R > 0$, we consider $(X_t^{R,\varepsilon})_{t \geq 0}$ the pathwise unique solution to $X_t^{R,\varepsilon} = X_0^\varepsilon \mathbb{1}_{\{|X_0^\varepsilon| \leq R\}} + \int_0^t \beta^\varepsilon(s, X_s^{R,\varepsilon})\,ds + \int_0^t (\alpha^\varepsilon(s, X_s^{R,\varepsilon}))^{1/2}\,dB_s$.

Recall that $\sup_{s \in [0,T], \varepsilon \in (0,1)}(|\beta^\varepsilon(s, x)|^2 + \|\alpha^\varepsilon(s, x)\|) \leq C_T(1 + |x|^2)$ by step 1. It is thus completely standard to check, using the Kolmogorov criterion, that for each $R > 0$, the family $\{(X_t^{R,\varepsilon})_{t \geq 0}, \varepsilon > 0\}$ is tight: for all $A > 0$, we can find a compact subset $\mathcal{K}(R, A)$ of $C([0, \infty), \mathbb{R}^3)$ such that $\sup_{\varepsilon \in (0,1)} \Pr((X_t^{R,\varepsilon})_{t \geq 0} \notin \mathcal{K}(R, A)) \leq 1/A$.

But by pathwise uniqueness, we have that $(X_t^\varepsilon)_{t \geq 0} = (X_t^{R,\varepsilon})_{t \geq 0}$ on the event $\{|X_0^\varepsilon| \leq R\}$. And it holds that $\Pr(|X_0^\varepsilon| > R) \leq m_2(\mu_0^\varepsilon)/R^2 \leq (1 + m_2(\mu_0))/R^2$. Consequently,

$$\sup_{\varepsilon \in (0,1)} \Pr((X_t^\varepsilon)_{t \geq 0} \notin \mathcal{K}(R, A)) \leq \frac{1}{A} + \frac{1 + m_2(\mu_0)}{R^2}.$$

Choosing $\mathcal{K}_B = \mathcal{K}(\sqrt{2B(1 + m_2(\mu_0))}, 2B)$, we find that $\sup_{\varepsilon \in (0,1)} \Pr((X_t^\varepsilon)_{t \geq 0} \notin \mathcal{K}_B) \leq 1/B$, which ends the step.

Step 4. We thus can find a sequence $\varepsilon_n \searrow 0$ such that $(X_t^{\varepsilon_n})_{t \geq 0}$ goes in law (for the uniform topology on compact time intervals) to some $(X_t)_{t \geq 0}$. For all $t \geq 0$, we have $\mathcal{L}(X_t) = \mu_t$ because $\mathcal{L}(X_t^{\varepsilon_n}) = \mu_t^{\varepsilon_n} \to \mu_t$. It thus only remains to prove that $(X_t)_{t \geq 0}$ solves (B.2). By the theory of martingale problems, it suffices to prove that for any $0 < s_1 < \cdots < s_k < s < t$, any $\varphi_1, \ldots, \varphi_k \in C_b(\mathbb{R}^3)$, any $\varphi \in C_c^2(\mathbb{R}^3)$, we have $\mathbb{E}[\mathcal{F}(X)] = 0$, where $\mathcal{F} : C([0, \infty), \mathbb{R}^3) \mapsto \mathbb{R}$ is defined by

$$\mathcal{F}(\gamma) = \varphi_1(\gamma_{s_1}) \cdots \varphi_k(\gamma_{s_k}) \Bigg(\varphi(\gamma_t) - \varphi(\gamma_s)$$

$$- \int_s^t \Bigg[\nabla\varphi(\gamma_u) \cdot \beta(u, \gamma_u) + \frac{1}{2} \sum_{k,l=1}^3 \partial_{x_k x_l}\varphi(\gamma_u)\alpha_{kl}(u, \gamma_u) \Bigg] du \Bigg).$$

We also introduce some continuous functions $\tilde{\alpha} : [0, \infty) \times \mathbb{R}^3 \mapsto \mathbb{M}_{3 \times 3}(\mathbb{R})$ and $\tilde{\beta} : [0, \infty) \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ (that will be chosen later very close to α and β). We define $\tilde{\alpha}^\varepsilon$ and $\tilde{\beta}^\varepsilon$ defined exactly as in (B.3), but with $\tilde{\alpha}$ and $\tilde{\beta}$. We finally define $\tilde{\mathcal{F}}$ (resp., $\mathcal{F}_\varepsilon$, resp., $\tilde{\mathcal{F}}_\varepsilon$) as $\mathcal{F}$ but with $\tilde{\alpha}$ and $\tilde{\beta}$ (resp., α^ε and β^ε, resp., $\tilde{\alpha}^\varepsilon$ and $\tilde{\beta}^\varepsilon$) instead of α and β.

We of course start from the identity $\mathbb{E}[\mathcal{F}_{\varepsilon_n}(X^{\varepsilon_n})] = 0$, and write

$$
\begin{aligned}
\big|\mathbb{E}[\mathcal{F}(X)]\big| &\leq \big|\mathbb{E}[\mathcal{F}(X)] - \mathbb{E}[\tilde{\mathcal{F}}(X)]\big| + \big|\mathbb{E}[\tilde{\mathcal{F}}(X)] - \mathbb{E}[\tilde{\mathcal{F}}(X^{\varepsilon_n})]\big| \\
&\quad + \big|\mathbb{E}[\tilde{\mathcal{F}}(X^{\varepsilon_n})] - \mathbb{E}[\tilde{\mathcal{F}}_{\varepsilon_n}(X^{\varepsilon_n})]\big| \\
&\quad + \big|\mathbb{E}[\tilde{\mathcal{F}}_{\varepsilon_n}(X^{\varepsilon_n})] - \mathbb{E}[\mathcal{F}_{\varepsilon_n}(X^{\varepsilon_n})]\big|.
\end{aligned}
$$

First, since $\tilde{\alpha}$ and $\tilde{\beta}$ are continuous, it easy to show that $\tilde{\mathcal{F}} : C([0, \infty), \mathbb{R}^3) \mapsto \mathbb{R}$ is continuous (for the topology of uniform convergence on compact time intervals) and bounded (recall that φ is compactly supported). Consequently, $\lim_n |\mathbb{E}[\tilde{\mathcal{F}}(X)] - \mathbb{E}[\tilde{\mathcal{F}}(X^{\varepsilon_n})]| = 0$. Next, there is a constant C such that

$$
\begin{aligned}
&\big|\mathbb{E}[\mathcal{F}(X)] - \mathbb{E}[\tilde{\mathcal{F}}(X)]\big| \\
&\quad \leq C \int_0^t \mathbb{E}\big[\big\|\alpha(u, X_u) - \tilde{\alpha}(u, X_u)\big\| + \big|\beta(u, X_u) - \tilde{\beta}(u, X_u)\big|\big]\, du \\
&\quad = C \int_0^t \int_{\mathbb{R}^3} \big[\big\|\alpha(u, x) - \tilde{\alpha}(u, x)\big\| + \big|\beta(u, x) - \tilde{\beta}(u, x)\big|\big] \mu_u(dx)\, du.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\big|\mathbb{E}[\tilde{\mathcal{F}}_{\varepsilon_n}(X^{\varepsilon_n})] - \mathbb{E}[\mathcal{F}_{\varepsilon_n}(X^{\varepsilon_n})]\big| \\
&\quad \leq C \int_0^t \int_{\mathbb{R}^3} \big(\big\|\alpha^{\varepsilon_n}(u, x) - \tilde{\alpha}^{\varepsilon_n}(u, x)\big\| + \big|\beta^{\varepsilon_n}(u, x) - \tilde{\beta}^{\varepsilon_n}(u, x)\big|\big) \mu_u^{\varepsilon_n}(dx)\, du \\
&\quad \leq C \int_0^t \int_{\mathbb{R}^3} \left(\frac{[(\|\alpha(u) - \tilde{\alpha}(u)\|\mu_u + |\beta(u) - \tilde{\beta}(u)|\mu_u) \star \phi_{\varepsilon_n(1+u)}](x)}{\mu_u^{\varepsilon_n}(x)} + \varepsilon_n\right) \\
&\qquad \times \mu_u^{\varepsilon_n}(dx)\, du \\
&\quad \leq C \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \big(\big\|\alpha(u, x) - \tilde{\alpha}(u, x)\big\| + \big|\beta(u, x) - \tilde{\beta}(u, x)\big|\big) \\
&\qquad \times \phi_{\varepsilon_n(1+u)}(y - x) \mu_u(dx)\, dy\, du + C t \varepsilon_n.
\end{aligned}
$$

Using that $\phi_{\varepsilon_n(1+u)}$ has mass 1, we conclude that

$$
\begin{aligned}
&\limsup_n \big|\mathbb{E}[\tilde{\mathcal{F}}_{\varepsilon_n}(X^{\varepsilon_n})] - \mathbb{E}[\mathcal{F}_{\varepsilon_n}(X^{\varepsilon_n})]\big| \\
&\quad \leq C \int_0^t \int_{\mathbb{R}^3} \big[\big\|\alpha(u, x) - \tilde{\alpha}(u, x)\big\| + \big|\beta(u, x) - \tilde{\beta}(u, x)\big|\big] \mu_u(dx)\, du.
\end{aligned}
$$

Finally,

$$\left| \mathbb{E}[\tilde{\mathcal{F}}(X^{\varepsilon_n})] - \mathbb{E}[\tilde{\mathcal{F}}_{\varepsilon_n}(X^{\varepsilon_n})] \right|$$

$$\leq C \int_0^t \int_{\mathbb{R}^3} \left(\|\tilde{\alpha}(u,x) - \tilde{\alpha}^{\varepsilon_n}(u,x)\| + |\tilde{\beta}(u,x) - \tilde{\beta}^{\varepsilon_n}(u,x)| \right) \mu_u^{\varepsilon_n}(dx)\, du$$

$$\leq C \int_0^t \int_{\mathbb{R}^3} \left(\|\tilde{\alpha}(u,x) - \tilde{\alpha}(u,y)\| + |\tilde{\beta}(u,x) - \tilde{\beta}(u,y)| \right)$$

$$\times \phi_{\varepsilon_n(1+u)}(y-x)\, dy \mu_u(dx)\, du + C\varepsilon_n t.$$

Since $\tilde{\alpha}$ and $\tilde{\beta}$ are continuous, this clearly tends to 0 as $n \to \infty$. All in all, we have checked that

$$\left| \mathbb{E}[\mathcal{F}(X)] \right| \leq C \int_0^t \int_{\mathbb{R}^3} \left[\|\alpha(u,x) - \tilde{\alpha}(u,x)\| + |\beta(u,x) - \tilde{\beta}(u,x)| \right] \mu_u(dx)\, du.$$

But this holds true for any choice of continuous functions $\tilde{\alpha}$ and $\tilde{\beta}$. And since $\mu_s(dx)\, ds$ is a Radon measure, we can find $\tilde{\alpha}$ and $\tilde{\beta}$ continuous and arbitrarily close to α and β in $L^1([0,T] \times \mathbb{R}^3, \mu_s(dx)\, ds)$. We conclude that $|\mathbb{E}[\mathcal{F}(X)]| = 0$ as desired. $\square$

APPENDIX C: ENTROPY AND FISHER INFORMATION

In this section, we present a series of results involving the Boltzmann entropy H and some fractional (or weighted) Fisher information I^r and I_γ. They provide key estimates in order to exploit the regularity of the objects we deal with.

C.1. Notation. For $F \in \mathcal{P}((\mathbb{R}^3)^N)$ with a density (and a finite moment of positive order for the entropy), the Boltzmann entropy H, the weighted Fisher information I_γ (for $\gamma < 0$) and the fractional Fisher information I^r [for $r \in (1/2, 1)$] of F are defined as

$$H(F) := \frac{1}{N} \int_{(\mathbb{R}^3)^N} F(\mathbf{v}^N) \log F(\mathbf{v}^N)\, d\mathbf{v}^N,$$

$$I_\gamma(F) := \frac{1}{N} \int_{(\mathbb{R}^3)^N} \frac{|\nabla_\gamma F(\mathbf{v}^N)|^2}{F(\mathbf{v}^N)}\, d\mathbf{v}^N = \frac{1}{N} \int_{(\mathbb{R}^3)^N} |\nabla_\gamma \log F(\mathbf{v}^N)|^2 F(d\mathbf{v}^N),$$

$$I^r(F^N) := \frac{1}{N} \int_{(\mathbb{R}^3)^N} \frac{|\nabla F(\mathbf{v}^N)|_{2r}^{2r}}{F(\mathbf{v}^N)^{2r-1}}\, d\mathbf{v}^N = \frac{1}{N} \int_{(\mathbb{R}^3)^N} |\nabla \log F(\mathbf{v}^N)|_{2r}^{2r} F(d\mathbf{v}^N),$$

with the notation $\mathbf{v}^N = (v_1, \ldots, v_N)$, where the differential operator ∇_γ is the *weighted* gradient

$$\nabla_\gamma F(\mathbf{v}^N) := \left(\langle v_1 \rangle^{\gamma/2} \nabla_{v_1} F(\mathbf{v}^N), \ldots, \langle v_n \rangle^{\gamma/2} \nabla_{v_n} F(\mathbf{v}^N) \right),$$

where $\langle v \rangle = (1 + |v|^2)^{1/2}$ and where the norm $|\cdot|_{2r}$ is the ℓ^{2r} norm on $(\mathbb{R}^3)^N$ defined by

$$|\mathbf{v}^N|_{2r}^{2r} := \sum_{i=1}^{N} |v_i|^{2r}.$$

If $F \in \mathcal{P}((\mathbb{R}^3)^N)$ has no density, we simply put $H(F) = +\infty$. If $F \in \mathcal{P}((\mathbb{R}^3)^N)$ has no density or if its density has no gradient, we put $I_\gamma(F) = I^r(F) = +\infty$. The somewhat unusual normalization by $1/N$ is made in order that for any $f \in \mathcal{P}(\mathbb{R}^3)$,

$$H(f^{\otimes N}) = H(f), \qquad I_\gamma(f^{\otimes N}) = I_\gamma(f) \quad \text{and} \quad I^r(f^{\otimes N}) = I^r(f).$$

Recall finally that the moment of order q is defined, for any $F \in \mathcal{P}_{\mathrm{sym}}((\mathbb{R}^3)^N)$, by

$$m_q(F) := \int_{(\mathbb{R}^3)^N} |v_1|^q F(d\mathbf{v}^N).$$

C.2. First properties and estimates. We make use of the two following properties.

LEMMA C.1.

(i) *For any $q, \lambda \in (0, \infty)$, there is a constant $C_{q,\lambda} \in \mathbb{R}$ such that for any $N \geq 1$, any $F \in \mathcal{P}_q((\mathbb{R}^3)^N)$*

$$H(F) \geq -C_{q,\lambda} - \lambda m_q(F).$$

(ii) *Consider the constant $C = 1 + |C_{2,1}|$ [with $C_{2,1}$ introduced in* (i)]. *For any $f \in \mathcal{P}_2(\mathbb{R}^3)$, any measurable $A \subset \mathbb{R}^3$ with Lebesgue measure $|A| < 1$,*

$$f(A) \leq \frac{C + H(f) + m_2(f)}{-\log|A|}.$$

In particular, $|A| \leq \exp(-4(C + H(f) + m_2(f)))$ implies that $f(A) \leq 1/4$.

PROOF. The first estimate is classical. See the comments before [21], Lemma 3.1, for a proof. To prove the second one, we decompose f as

$$f = f(A)f_1 + (1 - f(A))f_2 \qquad \text{where } f_1 = \frac{\mathbb{1}_A}{f(A)}f \text{ and } f_2 = \frac{\mathbb{1}_{A^c}}{1 - f(A)}f.$$

The entropy of f may be rewritten as

$$H(f) = f(A)\log f(A) + (1 - f(A))\log(1 - f(A)) + f(A)H(f_1)$$
$$+ (1 - f(A))H(f_2).$$

Since $m_2(f_2) \leq (1 - f(A))^{-1} m_2(f)$, the application of (i) (with $q = 2$ and $\lambda = 1$) to f_2 leads us to $H(f_2) \geq -C_{2,1} - (1 - f(A))^{-1} m_2(f)$. Since $x \log x \geq -1/2$ for $x \in [0, 1]$, we find

$$H(f) \geq -1 - (1 - f(A))C_{2,1} - m_2(f) + f(A)H(f_1)$$
$$\geq -C - m_2(f) + f(A)H(f_1).$$

But Supp $f_1 \subset A$ classically implies, by the Jensen inequality, that $H(f_1) \geq -\log |A|$. Hence,

$$H(f) \geq -C - m_2(f) - f(A)\log |A|.$$

The conclusion follows. $\square$

We now state two useful properties of the fractional and weighted Fisher information.

LEMMA C.2. *Let $\gamma < 0$ and $r \in (1/2, 1)$.*

(i) *The weighted and fractional Fisher information are super-additive*: *for all $N \geq 1$, all $F \in \mathcal{P}_{\mathrm{sym}}((\mathbb{R}^3)^N)$, all $k = 1, \ldots, N$, denoting by $F_k \in \mathcal{P}_{\mathrm{sym}}((\mathbb{R}^3)^k)$ the k-marginal of F,*

$$I^r(F_k) \leq I^r(F) \quad and \quad I_\gamma(F_k) \leq I_\gamma(F).$$

(ii) *The fractional Fisher information can be controlled by weighted Fisher information*: *for all $N \geq 1$, all $F \in \mathcal{P}_{\mathrm{sym}}(\mathbb{R}^{3^N})$,*

$$I^r(F) \leq C_q \big(I_\gamma(F)\big)^r \big(1 + m_q(F)\big)^{1-r} \qquad with \ q := \frac{|\gamma| r}{1 - r}.$$

PROOF. Since F is symmetric, we can write

$$I^r(F) = \int_{(\mathbb{R}^3)^N} |\nabla_{v_1} \log F(\mathbf{v}^N)|^{2r} F(d\mathbf{v}^N) \quad and$$

$$I_\gamma(F) = \int_{(\mathbb{R}^3)^N} \langle v_1 \rangle^\gamma |\nabla_{v_1} \log F(\mathbf{v}^N)|^2 F(d\mathbf{v}^N).$$

The super-additivity of the fractional Fisher information is a consequence of the convexity of the function $\Psi_r : \mathbb{R}_*^+ \times \mathbb{R}^3 \mapsto \mathbb{R}^+$ defined by

$$\Psi_r(a, b) := \frac{|b|^{2r}}{a^{2r-1}}.$$

Computing its Hessian matrix, we find

$$\nabla^2 \Psi_r(a, b) = \frac{2r(2r-1)|b|^{2r-2}}{a^{2r+1}} \begin{pmatrix} |b|^2 & -ab^* \\ -ab & \frac{a^2}{2r-1}\big(I_3 + (2r-2)\frac{bb^*}{|b|^2}\big) \end{pmatrix}$$

$$\geq \frac{2r(2r-1)|b|^{2r-2}}{a^{2r+1}} \begin{pmatrix} |b|^2 & -ab^* \\ -ab & a^2 \frac{bb^*}{|b|^2} \end{pmatrix},$$

which is always nonnegative if $2r - 1 \geq 0$. Then, we have, for $1 \leq k \leq N$, setting $\mathbf{v}_k^N = (v_1, \ldots, v_k)$ and $\mathbf{v}_{N-k}^N = (v_{k+1}, \ldots, v_N)$, by the Jensen inequality,

$$
\frac{|\nabla_{v_1} F_k(\mathbf{v}_k^N)|^{2r}}{F_k(\mathbf{v}_k^N)^{2r-1}}
$$

$$
= \Psi_r\big(F_k(\mathbf{v}_k^N), \nabla_{v_1} F_k(\mathbf{v}_k^N)\big)
$$

$$
= \Psi_r\bigg(\int_{(\mathbb{R}^3)^{N-k}} F(\mathbf{v}_k^N, \mathbf{v}_{N-k}^N)\, d\mathbf{v}_{N-k}^N, \int_{(\mathbb{R}^3)^N} \nabla_{v_1} F(\mathbf{v}_k^N, \mathbf{v}_{N-k}^N)\, d\mathbf{v}_{N-k}^N\bigg)
$$

$$
\leq \int_{(\mathbb{R}^3)^{N-k}} \Psi_r\big(F(\mathbf{v}_k^N, \mathbf{v}_{N-k}^N), \nabla_{v_1} F(\mathbf{v}_k^N, \mathbf{v}_{N-k}^N)\big)\, d\mathbf{v}_{N-k}^N
$$

$$
= \int_{(\mathbb{R}^3)^{N-k}} \frac{|\nabla_{v_1} F(\mathbf{v}_k^N, \mathbf{v}_{N-k}^N)|^{2r}}{F(\mathbf{v}_k^N, \mathbf{v}_{N-k}^N)^{2r-1}}\, d\mathbf{v}_{N-k}^N.
$$

Integrating this inequality in $\mathbf{v}_k^N$, we obtain $I^r(F_k) \leq I^r(F)$. Choosing next $r = 1$, multiplying the inequality by $\langle v_1 \rangle^\gamma$ and integrating in $\mathbf{v}_k^N$ we obtain $I_\gamma(F_k) \leq I_\gamma(F)$.

For the second point, we use the Hölder inequality:

$$
I^r(F) = \int_{(\mathbb{R}^3)^N} \big(|\nabla_{v_1} \log F(\mathbf{v}^N)|^{2r} F^r(\mathbf{v}^N) \langle v_1 \rangle^{\gamma r}\big)\big(F^{1-r}(\mathbf{v}^N) \langle v_1 \rangle^{-\gamma r}\big)\, d\mathbf{v}^N
$$

$$
\leq [I_\gamma(F)]^r \bigg(\int_{(\mathbb{R}^3)^N} \langle v_1 \rangle^{-\gamma r/(1-r)} F(d\mathbf{v}^N)\bigg)^{1-r}.
$$

Recalling that $q = |\gamma| r/(1-r)$, the conclusion follows. $\quad\square$

We next establish some kind of Gagliardo–Nirenberg–Sobolev inequality in $\mathbb{R}^3$ involving fractional and weighted Fisher informations.

LEMMA C.3.   *Let* $\gamma < 0$. *For any* $r \in (1/2, 1)$, *for* $p := 3/(3 - 2r)$ *and* $q := |\gamma| r/(1-r)$, *for any* $f \in \mathcal{P}(\mathbb{R}^3)$,

$$
\|f\|_{L^p} \leq C_r I^r(f) \leq C_{r,\gamma} \big(I_\gamma(f)\big)^r \big(1 + m_q(f)\big)^{1-r}.
$$

PROOF.   Using the Hölder inequality, we can write, for any $p' \in (0, 2r)$

$$
\|\nabla f\|_{L^{p'}}^{p'} = \int_{\mathbb{R}^3} \bigg(\frac{|\nabla f|^{2r}}{f^{2r-1}}\bigg)^{p'/(2r)} f^{(2r-1)p'/(2r)}
$$

$$
\leq \bigg(\int_{\mathbb{R}^3} \frac{|\nabla f|^{2r}}{f^{2r-1}}\bigg)^{p'/(2r)} \bigg(\int_{\mathbb{R}^3} f^{(2r-1)p'/(2r-p')}\bigg)^{2r - p'/(2r)},
$$

whence

$$
\|\nabla f\|_{L^{p'}} \leq \big(I^r(f)\big)^{1/2r} \|f\|_{L^{(2r-1)p'/(2r-p')}}^{1-1/2r}.
$$

Together with the Gagliardo–Nirenberg–Sobolev inequality, it comes, with $1/p = 1/p' - 1/3$ (which is well defined since $p' < 2$)

$$\|f\|_{L^p} \leq C_{p'}\|\nabla f\|_{L^{p'}} \leq C_{p'}(I^r(f))^{1/2r}\|f\|_{L^{(2r-1)p'/(2r-q)}}^{1-1/2r}.$$

This inequality becomes really interesting when $p = (2r-1)p'/(2r - p')$, which leads to the choice $p' = 3/[2(2-r)]$. It that case $p = 3/(3-2r)$ and we find

$$\|f\|_{L^p} \leq C_r I^r(f).$$

Using finally Lemma C.2(ii) completes the proof. □

We deduce that pairs of particles of which the law has a finite weighted Fisher information are not too close.

LEMMA C.4. *Let* $\gamma < 0$. *Consider* $F \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)$ *and* $(\mathcal{X}_1, \mathcal{X}_2)$ *a* F-*distributed random variable. For any* $\kappa \in (0,2)$ *and any* $q > |\gamma|\max\{1, \kappa/(2-\kappa)\}$, *there exists* $C_{\kappa,q}$ *such that, with* $r := q/(q-\gamma) \in (1/2, 1)$,

$$\mathbb{E}(|\mathcal{X}_1 - \mathcal{X}_2|^{-\kappa}) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{F(dx_1, dx_2)}{|x_1 - x_2|^\kappa} \leq C_{\kappa,q}[(I_\gamma(F))^r(1 + m_q(F))^{1-r} + 1].$$

PROOF.   We introduce the unitary linear transformation $\Phi : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \mathbb{R}^3$ defined by

$$\Phi(x_1, x_2) = \tfrac{1}{\sqrt{2}}(x_1 - x_2, x_1 + x_2) =: (y_1, y_2).$$

Let $\tilde{F} := F \circ \Phi^{-1}$ and $\tilde{f}$ the first marginal of $\tilde{F}$ [$\tilde{f}$ is the law of $2^{-1/2}(\mathcal{X}_1 - \mathcal{X}_2)$]. A simple substitution shows that $I^r(\tilde{F}) \leq C_r I^r(F)$ for some constant C_r. Next, we cannot apply Lemma C.2(i) because $\tilde{F}$ is not symmetric, but we obviously have $I^r(\tilde{f}) \leq 2I^r(\tilde{F})$. We now write

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{F(x_1, x_2)}{|x_1 - x_2|^\kappa}\, dx_1\, dx_2 = 2^{\kappa/2}\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\tilde{F}(y_1, y_2)}{|y_1|^\kappa}\, dy_1\, dy_2 = 2^{\kappa/2}\int_{\mathbb{R}^3} \frac{\tilde{f}(y)}{|y|^\kappa}\, dy,$$

which is smaller than $C_{\kappa,p}(1 + \|\tilde{f}\|_{L^p})$ by (1.4), with $p := 3(q-\gamma)/(q-3\gamma) > 3/(3-\kappa)$ (thanks to our condition on q). We then use Lemma C.3 [we have $p = 3/(3-2r)$] to get

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{F(x_1, x_2)}{|x_1 - x_2|^\kappa}\, dx_1\, dx_2 \leq C_{\kappa,q}(1 + I^r(\tilde{f})) \leq C_{\kappa,q}(1 + I^r(F)).$$

Lemma C.2(ii) [observe that $q = |\gamma|r/(1-r)$] allows us to conclude. □

C.3. Many-particle weighted Fisher information. We finally need a result showing that if the particle distribution of the N-particle system has a uniformly bounded weighted Fisher information, then any limit point of the associated empirical measure has finite expected weighted Fisher information. Such a result is a consequence of some representation identities for *level-3 functionals* as first observed by Robinson–Ruelle in [32] for the entropy in a somewhat different setting. Recently in [21], this kind of representation identity has been extended to the Fisher information. The proof is mainly based on the De Finetti–Hewitt–Savage representation theorem [6, 22] (see also [21]). Unfortunately, we cannot apply directly the results of [21].

THEOREM C.5. *Let $\gamma < 0$. Consider, for each $N \geq 2$, a probability measure $F^N \in \mathcal{P}_{\mathrm{sym}}((\mathbb{R}^3)^N)$. For $j \geq 1$, denote by $F_j^N \in \mathcal{P}((\mathbb{R}^3)^j)$ the jth marginal of F^N. Assume that there exists a compatible sequence (π_j) of symmetric probability measures on $(\mathbb{R}^3)^j$ so that $F_j^N \to \pi_j$ in the weak sense of measures in $\mathcal{P}((\mathbb{R}^3)^j)$. Denoting by $\pi \in \mathcal{P}(\mathcal{P}(\mathbb{R}^3))$ the probability measure associated to the sequence (π_j) thanks to the De Finetti–Hewitt–Savage theorem, there holds*

$$\int_{\mathcal{P}(\mathbb{R}^3)} I_\gamma(f)\pi(df) = \sup_{j \geq 1} I_\gamma(\pi_j) \leq \liminf_{N \to \infty} I_\gamma(F^N).$$

The De Finetti–Hewitt–Savage theorem asserts that for a sequence (π_j) of symmetric probability measures on E^j (for some measurable space E), compatible in the sense that the k-marginal of π_j is π_k for all $1 \leq k \leq j$, there exists a unique probability measure $\pi \in \mathcal{P}(\mathcal{P}(E))$ such that $\pi_j = \int_{\mathcal{P}(E)} f^{\otimes j}\pi(df)$ for all $j \geq 1$. See, for instance, [21], Theorem 5.1.

COROLLARY C.6. *Let $\gamma < 0$. Consider, for each $N \geq 2$, a probability measure $F^N \in \mathcal{P}_{\mathrm{sym}}((\mathbb{R}^3)^N)$, and $(X_1^N, \ldots, X_N^N)$ with law F^N. Assume that $\mu_N := N^{-1} \sum_1^N \delta_{X_i^N}$ goes in law to some (possibly random) $\mu \in \mathcal{P}(\mathbb{R}^3)$. Then*

$$\mathbb{E}[I_\gamma(\mu)] \leq \liminf_{N \to \infty} I_\gamma(F^N).$$

PROOF. Denote by $\pi \in \mathcal{P}(\mathcal{P}(\mathbb{R}^3))$ the law of μ and, for $j \geq 1$, by $\pi_j = \int_{\mathcal{P}(E)} f^{\otimes j}\pi(df)$. The corollary immediately follows from Theorem C.5 once we have checked that for all $j \geq 1$, F_j^N goes weakly to π_j. But this is an easy and classical consequence of the fact that μ_N goes in law to μ; see, for example, Sznitman [36], I-Proposition 2.2 and Remark 2.3 or [21], Lemma 2.8. □

Theorem C.5 is a consequence of [21], Lemma 5.6, and of the following series of properties.

LEMMA C.7. *Let $\gamma < 0$. The weighted Fisher information satisfies the following properties*:

(i) *For any $j \geq 1$, $I_\gamma : \mathcal{P}((\mathbb{R}^3)^j) \to \mathbb{R} \cup \{+\infty\}$ is nonnegative, convex, proper and lower semi-continuous for the weak convergence.*

(ii) *For all $j \geq 1$, all $f \in \mathcal{P}(\mathbb{R}^3)$, $I_\gamma(f^{\otimes j}) = I_\gamma(f)$.*

(iii) *For all $F \in \mathbf{P}_{\mathrm{sym}}((\mathbb{R}^3)^j)$, all $\ell, n \geq 1$ with $j = \ell + n$, there holds $j I_\gamma(F) \geq \ell I_\gamma(F_\ell) + n I_\gamma(F_n)$, where $F_\ell \in \mathcal{P}((\mathbb{R}^3)^\ell)$ stands for the ℓ-marginal of F.*

(iv) *The functional $\mathcal{I}_\gamma : \mathcal{P}(\mathcal{P}(\mathbb{R}^3)) \to \mathbb{R} \cup \{\infty\}$ defined by*

$$\mathcal{I}_\gamma(\pi) := \sup_{j \geq 1} I_\gamma(\pi_j) \qquad \text{where } \pi_j := \int_{\mathcal{P}(\mathbb{R}^3)} f^{\otimes j} \pi(df) \in \mathcal{P}((\mathbb{R}^3)^j)$$

is affine in the following sense. For any $\pi \in \mathcal{P}(\mathcal{P}(\mathbb{R}^3))$ and any partition of $\mathcal{P}(\mathbb{R}^3)$ by some sets ω_i, $1 \leq i \leq M$, such that ω_i is an open set in $\mathbb{R}^3 \setminus (\omega_1 \cup \cdots \cup \omega_{i-1})$ for any $1 \leq i \leq M - 1$ and $\pi(\omega_i) > 0$ for any $1 \leq i \leq M$, defining

$$\alpha_i := \pi(\omega_i) \quad \text{and} \quad \gamma^i := \frac{1}{\alpha_i} \mathbf{1}_{\omega_i} \pi \in \mathcal{P}(\mathcal{P}(\mathbb{R}^3))$$

so that

$$\pi = \alpha_1 \gamma^1 + \cdots + \alpha_M \gamma^M \quad \text{and} \quad \alpha_1 + \cdots + \alpha_M = 1,$$

there holds

$$\mathcal{I}_\gamma(\pi) = \alpha_1 \mathcal{I}_\gamma(\gamma^1) + \cdots + \alpha_M \mathcal{I}_\gamma(\gamma^M).$$

PROOF. We only sketch the proof, which is roughly an adaptation to the weighted case of the proof of [21], Lemma 5.10.

Step 1. We first prove point (i). Let us present an alternative expression of the weighted Fisher information: for $F \in \mathcal{P}((\mathbb{R}^3)^j)$, it holds that

$$I_\gamma(G) := \frac{1}{j} \sup_{\psi \in C_c^1((\mathbb{R}^3)^j)^{3j}} \left\{ -\int_{(\mathbb{R}^3)^j} \operatorname{div}_V \psi \, F(dV) - \sum_{i=1}^j \int \frac{|\psi_i|^2}{4} \langle v_i \rangle^{-\gamma} F(dV^j) \right\}.$$

Again, the right-hand side term is well defined in $\mathbb{R}$ because the function $|\psi|^2/4 - \operatorname{div}_X \psi$ is continuous and bounded for any $\psi \in C_c^1((\mathbb{R}^3)^j)^{3j}$. We immediately deduce that I_γ is convex, lower semi-continuous and proper so that point (i) holds.

Step 2. Point (ii) is obvious from the definition of I_γ.

Step 3. We now prove (iii), reproducing the proof of the same super-additivity property established for the usual Fisher information in [21], Lemma 3.7. We define for any $i \leq j$

$$\iota_i := i I_\gamma(F_i) = \sup_{\psi \in C_c^1((\mathbb{R}^3)^i)^{3i}} \left\{ \int_{(\mathbb{R}^3)^i} \left(\nabla_V F_i \cdot \psi - F_i \sum_{j \leq i} \frac{|\psi_j|^2}{4} \langle v_j \rangle^{-\gamma} \right) \right\},$$

where the sup is taken on all $\psi = (\psi_1, \ldots, \psi_i)$, with all $\psi_\ell : (\mathbb{R}^3)^i \to \mathbb{R}^3$. We then write the previous equality for ι_j and restrict the supremum over all ψ such that for some $i \leq j$:

• the i first ψ_ℓ depend only on $(v_1, \ldots, v_i)$, with the notation $\psi^i = (\psi_1, \ldots, \psi_i)$,

• the $j - i$ last ψ_ℓ depend only on $(v_{i+1}, \ldots, v_j)$, with the notation $\psi^{j-i} = (\psi_{i+1}, \ldots, \psi_j)$.

We then have the inequality

$$
\iota_j \geq \sup_{\psi^i, \psi^{j-i}} \int_{(\mathbb{R}^3)^j} \left[\nabla_i F \cdot \psi^i + \nabla_{j-i} F \cdot \psi^{j-i} \right.
$$

$$
\left. - F \frac{1}{4} \left(\sum_{\ell \leq i} |\psi_\ell|^2 \langle v_\ell \rangle^{-\gamma} + \sum_{\ell > i} |\psi_\ell|^2 \langle v_\ell \rangle^{-\gamma} \right) \right]
$$

$$
= \sup_{\psi^i \in C_c^1((\mathbb{R}^3)^i)^{3i}} \int_{(\mathbb{R}^3)^i} \left[\nabla_i F_i \cdot \psi^i - \frac{F_i}{4} \sum_{\ell \leq i} |\psi_\ell|^2 \langle v_\ell \rangle^{-\gamma} \right]
$$

$$
+ \sup_{\psi^{j-i} \in C_c^1((\mathbb{R}^3)^{j-i})^{3(j-i)}} \int_{(\mathbb{R}^3)^{j-i}} \left[\nabla_{j-i} G_{j-i} \cdot \psi^{j-i} - \frac{F_{j-i}}{4} \sum_{\ell > i} |\psi_\ell|^2 \langle v_\ell \rangle^{-\gamma} \right]
$$

$$
= \iota_i + \iota_{j-i}.
$$

Step 3. We do not prove here the affine character of $\mathcal{I}_\gamma$ and refer to [21], Lemma 5.10, where the same property for the usual Fisher information is checked: the presence of the bounded weight does not raise any difficulty. □

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