

BERRY–ESSEEN THEOREMS UNDER WEAK DEPENDENCE

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Let $\{X_k\}_{k \geq \mathbb{Z}}$ be a stationary sequence. Given $p \in (2, 3]$ moments and a mild weak dependence condition, we show a Berry–Esseen theorem with optimal rate $n^{p/2-1}$. For $p \geq 4$, we also show a convergence rate of $n^{1/2}$ in \mathcal{L}^q -norm, where $q \geq 1$. Up to $\log n$ factors, we also obtain nonuniform rates for any $p > 2$. This leads to new optimal results for many linear and nonlinear processes from the time series literature, but also includes examples from dynamical system theory. The proofs are based on a hybrid method of characteristic functions, coupling and conditioning arguments and ideal metrics.

1. Introduction. Let $\{X_k\}_{k \in \mathbb{Z}}$ be a zero mean process having second moments $\mathbb{E}[X_k^2] < \infty$. Consider the partial sum $S_n = \sum_{k=1}^n X_k$ and its normalized variance $s_n^2 = n^{-1} \text{Var}[S_n]$. A very important issue in probability theory and statistics is whether or not the central limit theorem holds, that is, if we have

$$(1.1) \quad \lim_{n \rightarrow \infty} \left| P\left(S_n \leq x \sqrt{ns_n^2}\right) - \Phi(x) \right| = 0,$$

where $\Phi(x)$ denotes the standard normal distribution function. Going one step further, we can ask ourselves about the possible rate of convergence in (1.1), more precisely, if it holds that

$$(1.2) \quad \lim_{n \rightarrow \infty} d(P_{S_n/\sqrt{ns_n^2}}, P_Z) \tau_n < \infty \quad \text{for a sequence } \tau_n \rightarrow \infty,$$

where $d(\cdot, \cdot)$ is a probability metric, Z follows a standard normal distribution and P_X denotes the probability measure induced by the random variable X . The rate τ_n can be considered as a measure of reliability for statistical inference based on S_n , and large rates are naturally preferred. The question of rate of convergence has been addressed under numerous different setups with respect to the metric and underlying structure of the sequence $\{X_k\}_{k \in \mathbb{Z}}$ in the literature. Perhaps one of the most important metrics is the Kolmogorov (uniform) metric, given as

$$(1.3) \quad \Delta_n = \sup_{x \in \mathbb{R}} \left| P\left(S_n \leq x \sqrt{ns_n^2}\right) - \Phi(x) \right|.$$

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The latter has been studied extensively in the literature under many different notions of (weak) dependence for $\{X_k\}_{k \in \mathbb{Z}}$. One general way to measure dependence is in terms of various mixing conditions. In the case of the uniform metric, Bolthausen [6] and Rio [43] showed that it is possible to obtain the rate $\tau_n = \sqrt{n}$ in (1.3), given certain mixing assumptions and a bounded support of the underlying sequence $\{X_k\}_{k \in \mathbb{Z}}$; see also [9, 12, 25, 33], among others, for related results and extensions. Under the notion of α -mixing, Tikhomirov [45] obtained $\tau_n = n^{1/2}/(\log n)^2$, provided that $\mathbb{E}[|X_k|^3] < \infty$ and the mixing coefficient decays exponentially fast; see also [2]. Martingales constitute another important class for the study of (1.3). Some relevant contributions in this context are, for instance, Brown and Heyde [26], Bolthausen [7] and more recently Dedecker et al. [11]. In the special case of functionals of Gaussian or Poissonian sequences, deep results have been obtained by Noudin and Peccati et al.; see, for instance, [37, 38] and [39]. Another stream of significant works focuses on stationary (causal) Bernoulli-shift processes, given as

$$(1.4) \quad X_k = g_k(\varepsilon_k, \varepsilon_{k-1}, \dots) \quad \text{where } \{\varepsilon_k\}_{k \in \mathbb{Z}} \text{ is an i.i.d. sequence.}$$

The study of (1.3) given the structure in (1.4) has a long history, and dates back to Kac [30] and Postnikov [42]. Ibragimov [28] established a rate of convergence, $\tau_n = n^{1/2}/\sqrt{\log n}$, subject to an exponentially fast decaying weak dependence coefficient. Using the technique of Tikhomirov [45], Götze and Hipp obtained Edgeworth expansions for processes of type (1.4) in a series of works; cf. [19–21]; see also Heinrich [24] and Lahiri [32]. This approach, however, requires the validity of a number of technical conditions. This includes in particular a conditional Crámer-like condition subject to an exponential decay, which is somewhat difficult to verify. In contrast, it turns out that a Berry–Esseen theorem only requires a simple, yet fairly general dependence condition where no exponential decay is required. Indeed, we will see that many popular examples from the literature are within our framework. Unlike previous results in the literature, we also obtain optimal rates for $p \in (2, 3)$ given (infinite) weak dependence, which to the best of our knowledge is new (excluding special cases as linear processes). The proofs are based on an m -dependent approximation ($m \rightarrow \infty$), which is quite common in the literature. The substantial difference here is the subsequent treatment of the m -dependent sequence. To motivate one of the main ideas of the proofs, let us assume $p = 3$ for a moment. Given a weakly, m -dependent sequence $\{X_k\}_{k \in \mathbb{Z}}$, one may show via classic arguments that

$$(1.5) \quad \Delta_n \leq C\sqrt{m/n}\mathbb{E}[|X_1|^3],$$

provided that $\mathbb{E}[|X_1|^3] < \infty$ and $s_n^2 > 0$. Note, however, since X_k is weakly dependent, one finds that

$$(1.6) \quad m^{-3/2}|\mathbb{E}[S_m^3]| \leq \frac{C}{\sqrt{m}}.$$

Hence if one succeeds in replacing $\mathbb{E}[|X_1|^3]$ in (1.5) with (1.6), one obtains the optimal rate $\tau_n = \sqrt{n}$. A similar reasoning applies to $p \in (2, 3)$. Unfortunately though, setting this idea to work leads to rather intricate problems, and a technique like that of Tikhomirov [45] is not fruitful, inevitably leading to a suboptimal rate. Our approach is based on coupling and conditioning arguments and ideal (Zolotarev) metrics. Interestingly, there is a connection to more recent results of Dedecker et al. [11], who consider different (smoother) probability metrics. We will see that at least some of the problems we encounter may be redirected to these results after some preparation.

2. Main results. Throughout this paper, we will use the following notation: for a random variable X and $p \geq 1$, we denote with $\|X\|_p = \mathbb{E}[X^p]^{1/p}$ the \mathcal{L}^p norm. Let $\{\varepsilon_k\}_{k \in \mathbb{Z}}$ be a sequence of independent and identically distributed random variables with values in a measurable space \mathbb{S} . Denote the corresponding σ -algebra with $\mathcal{E}_k = \sigma(\varepsilon_j, j \leq k)$. Given a real-valued stationary sequence $\{X_k\}_{k \in \mathbb{Z}}$, we always assume that X_k is adapted to \mathcal{E}_k for each $k \in \mathbb{Z}$. Hence we implicitly assume that X_k can be written as in (1.4). For convenience, we write $X_k = g_k(\theta_k)$ with $\theta_k = (\varepsilon_k, \varepsilon_{k-1}, \dots)$. The class of processes that fits into this framework is large and contains a variety of functionals of linear and nonlinear processes including ARMA, GARCH and related processes (see, e.g., [18, 46, 48]), but also examples from dynamic system theory. Some popular examples are given below in Section 3. A nice feature of the representation given in (1.4) is that it allows us to give simple, yet very efficient and general dependence conditions. Following Wu [47], let $\{\varepsilon'_k\}_{k \in \mathbb{Z}}$ be an independent copy of $\{\varepsilon_k\}_{k \in \mathbb{Z}}$ on the same probability space, and define the “filter” $\theta_k^{(l, \prime)}$ as

$$(2.1) \quad \theta_k^{(l, \prime)} = (\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon_{k-l}, \varepsilon'_{k-l-1}, \dots).$$

We put $\theta'_k = \theta_k^{(k, \prime)} = (\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon'_0, \varepsilon_{-1}, \dots)$ and $X_k^{(l, \prime)} = g_k(\theta_k^{(l, \prime)})$, and in particular we set $X'_k = X_k^{(k, \prime)}$. As a dependence measure, we then consider the quantity $\sup_{k \in \mathbb{Z}} \|X_k - X_k^{(l, \prime)}\|_p$, $p \geq 1$. Dependence conditions of this type are quite general and easy to verify in many cases; cf. [1, 48] and the examples below. Observe that if the function $g = g_k$ does not depend on k , we obtain the simpler version

$$(2.2) \quad \sup_{k \in \mathbb{Z}} \|X_k - X_k^{(l, \prime)}\|_p = \|X_l - X'_l\|_p.$$

Note that it is actually not trivial to construct a stationary process $\{X_k\}_{k \in \mathbb{Z}}$ that can only be represented as $X_k = g_k(\theta_k)$; that is, a function g independent of k such that $X_k = g(\theta_k)$ for all $k \in \mathbb{Z}$ does not exist. We refer to Corollary 2.3 in Feldman and Rudolph [16] for such an example.

We will derive all of our results under the following assumptions.

ASSUMPTION 2.1. Let $\{X_k\}_{k \in \mathbb{Z}}$ be stationary such that for some $p \geq 2$:

- (i) $\|X_k\|_p < \infty, \mathbb{E}[X_k] = 0,$
- (ii) $\sum_{l=1}^{\infty} l^2 \sup_{k \in \mathbb{Z}} \|X_k - X_k^{(l, \cdot)}\|_p < \infty,$
- (iii) $s^2 > 0,$ where $s^2 = \sum_{k \in \mathbb{Z}} \mathbb{E}[X_0 X_k].$

In the sequel, B denotes a varying absolute constant, depending only on $p, \sum_{l=1}^{\infty} l^2 \sup_{k \in \mathbb{Z}} \|X_k - X_k^{(l, \cdot)}\|_p$ and s^2 . The following theorem is one of the main results of this paper.

THEOREM 2.2. *Grant Assumption 2.1 for some $p \in (2, 3],$ and let $s_n^2 = n^{-1} \|S_n\|_2^2.$ Then*

$$\sup_{x \in \mathbb{R}} \left| P\left(S_n / \sqrt{ns_n^2} \leq x\right) - \Phi(x) \right| \leq \frac{B}{n^{p/2-1}},$$

and hence we may select $\tau_n = n^{p/2-1}.$

Theorem 2.2 provides optimal convergence rates under mild conditions. In particular, it seems that this is the first time optimal rates are shown to hold under general infinite weak dependence conditions if $p \in (2, 3).$ Examples to demonstrate the versatility of the result are given in Section 3. In particular, we consider functions of the dynamical system $Tx = 2x \text{ mod } 1$ in Example 3.2, a problem which has been studied in the literature for decades. Combining Theorem 2.2 with results of Dedecker and Rio [13], we also obtain optimal results for the \mathcal{L}^q -norm.

THEOREM 2.3. *Grant Assumption 2.1 for some $p \geq 4,$ and let $s_n^2 = n^{-1} \|S_n\|_2^2.$ Then for any $q \geq 1$ we have*

$$\int_{\mathbb{R}} \left| P\left(S_n / \sqrt{ns_n^2} \leq x\right) - \Phi(x) \right|^q dx \leq Bn^{-q/2}.$$

Note that in the case $q = 1,$ the results of Dedecker and Rio [13] are more general. The nonuniform analogue to Theorems 2.2 and 2.3 is given below. Here, we obtain optimality up to logarithmic factors.

THEOREM 2.4. *Grant Assumption 2.1 for some $p > 2.$ Then for any $x \in \mathbb{R},$*

$$\left| P\left(S_n / \sqrt{ns_n^2} \leq x\right) - \Phi(x) \right| \leq n^{-(p \wedge 3)/2+1} \frac{B(\log n)^{p/2}}{1 + |x|^p},$$

where $a \wedge b = \min\{a, b\}.$

As a particular application of Theorem 2.4, consider $f(|S_n|/\sqrt{ns_n^2})$ where the function $f(\cdot)$ satisfies

$$(2.3) \quad f(0) = 0 \quad \text{and} \quad \int_0^{\infty} \frac{|f'(x)|}{1 + |x|^p} dx < \infty$$

for some $p > 0$, and the derivative $f'(x)$ exists for $x \in (0, \infty)$. If $\|S_n\|_p < \infty$, property (2.3) implies the identity

$$\mathbb{E}\left[f\left(|S_n|/\sqrt{ns_n^2}\right)\right] = \int_0^\infty f'(x)P\left(|S_n|/\sqrt{ns_n^2} \geq x\right) dx,$$

and we thus obtain the following corollary.

COROLLARY 2.5. *Grant Assumption 2.1 for some $p > 2$. If (2.3) holds, then*

$$\left|\mathbb{E}\left[f\left(|S_n|/\sqrt{ns_n^2}\right)\right] - \int_{\mathbb{R}} f(|x|) d\Phi(x)\right| \leq Bn^{-(p\wedge 3)/2+1}(\log n)^{p/2}.$$

As a special case, consider $f(|x|) = |x|^q, q > 0$. We may then use Corollary 2.5 to obtain rates of convergence for moments.

COROLLARY 2.6. *Grant Assumption 2.1 for some $p > 2$. Then for any $0 < q < p$, we have*

$$\left\|\left\|S_n/\sqrt{ns_n^2}\right\|_q^q - \int_{\mathbb{R}} |x|^q d\Phi(x)\right| \leq Bn^{-(p\wedge 3)/2+1}(\log n)^{p/2}.$$

In the special case of i.i.d. sequences and $0 < p < 4$, sharp results in this context have been obtained in Hall [23]. It seems that related results for dependent sequences are unknown.

3. Applications and examples. All examples considered here are time-homogenous Bernoulli-shift processes; that is, $g = g_k$ does not depend on k , and hence equality (2.2) holds.

EXAMPLE 3.1 (Functions of linear process). Let $\mathbb{S} = \mathbb{R}$, and suppose that the sequence $\{\alpha_i\}_{i \in \mathbb{N}}$ satisfies $\sum_{i=0}^\infty \alpha_i^2 < \infty$. If $\|\varepsilon_k\|_2 < \infty$, then one may show that the linear process

$$Y_k = \sum_{i=0}^\infty \alpha_i \varepsilon_{k-i} \quad \text{exists and is stationary.}$$

Let f be a measurable function such that $\mathbb{E}[X_k] = 0$, where $X_k = f(Y_k)$. If f is Hölder continuous with regularity $0 < \beta \leq 1$, that is, $|f(x) - f(y)| \leq c|x - y|^\beta$, then for any $p \geq 1$

$$\|X_k - X'_k\|_p \leq c\alpha_k^\beta \|\varepsilon_0\|_p.$$

Hence if $\sum_{i=0}^\infty i^2 |\alpha_i|^\beta < \infty$ and $s^2 > 0$, then Assumption 2.1 holds.

EXAMPLE 3.2 [Sums of the form $\sum f(t2^k)$]. Consider the measure preserving transformation $Tx = 2x \text{ mod } 1$ on the probability space $([0, 1], \mathcal{B}, \lambda)$, with Borel σ -algebra \mathcal{B} and Lebesgue measure λ . Let $U_0 \sim \text{Uniform}[0, 1]$. Then $TU_0 = \sum_{j=0}^{\infty} 2^{-j-1} \zeta_j$, where ζ_j are Bernoulli random variables. The flow $T^k U_0$ can then be written as $T^k U_0 = \sum_{j=0}^{\infty} 2^{-j-1} \zeta_{j+k}$; see [28]. The study about the behavior of $S_n = \sum_{k=1}^n f(T^k U_0)$ for appropriate functions f has a very long history and dates back to Kac [30]. Since then, numerous contributions have been made; see, for instance, [4, 5, 13, 14, 27, 28, 31, 33, 35, 36, 40, 42], to name a few. Here, we consider the following class of functions. Let f be a function defined on the unit interval $[0, 1]$, such that

$$(3.1) \quad \int_0^1 f(t) dt = 0, \quad \int_0^1 |f(t)|^p dt < \infty \quad \text{and} \\ \int_0^1 t^{-1} |\log(t)|^2 w_p(f, t) dt < \infty,$$

where $w_p(f, t)$ denotes the $\mathcal{L}^p([0, 1], \lambda)$ modulus of continuity of $f \in \mathcal{L}^p([0, 1], \lambda)$. This setup is a little more general than in [28]. For $x \in \mathbb{R}^+$, let $\bar{f}(x) = f(x - \lfloor x \rfloor)$; that is, \bar{f} is the one-periodic extension to the positive real line. One then often finds the equivalent formulation $S_n = \sum_{k=1}^n \bar{f}(2^k U_0)$ in the literature. Consider now the partial sum $S_n = \sum_{k=1}^n \bar{f}(2^k U_0)$. Ibragimov [28] showed that

$$(3.2) \quad \sup_{x \in \mathbb{R}} \left| P\left(S_n / \sqrt{ns_n^2} \leq x\right) - \Phi(x) \right| \leq C \left(\frac{\log n}{n}\right)^{p/2-1}.$$

By alternative methods, Le Borgne and Pène [33], extending Rio [43], managed to remove the logarithmic factor if $\|f\|_{\infty} < \infty$. A priori, the sequence $\{T^k U_0\}_{k \in \mathbb{Z}}$ does not directly fit into our framework, which, however, can be achieved by a simple time flip. Define the function $T_n(i) = n - i + 1$ for $i \in \{n, n - 1, \dots\}$, and let $\varepsilon_k = \zeta_{T_n(k)}$. Then we may write

$$X_k = f(T^k U_0) = f\left(\sum_{j=0}^{\infty} \varepsilon_{k-j} 2^{-j-1}\right), \quad k \in \{1, \dots, n\}.$$

Note that we have to perform this time flip for every $n \in \mathbb{N}$, which, however, has no impact on the applicability of our results. Using the same arguments as in [28], we find that (3.1) implies that for $p \in (2, 3]$

$$\sum_{k=1}^{\infty} k^2 \|X_k - X'_k\|_p < \infty.$$

If $s^2 > 0$, we see that Assumption 2.1 holds. In particular, an application of Theorem 2.2 gives the rate $\tau_n = n^{p/2-1}$, thereby removing the unnecessary $\log n$ factor in (3.2) for the whole range $p \in (2, 3]$.

EXAMPLE 3.3 (*m*-dependent processes). Consider the zero mean *m*-dependent process $Y_k = f(\zeta_k, \dots, \zeta_{k-m+1})$, where $m \in \mathbb{N}$ and f is a measurable function and $\{\zeta_k\}_{k \in \mathbb{Z}}$ is i.i.d. and takes values in \mathbb{S} . m may depend on n such that $n/m \rightarrow \infty$, but we demand in addition that

$$(3.3) \quad \liminf_{n \rightarrow \infty} \text{Var} \left[\sum_{k=1}^n Y_k \right] / (nm) > 0.$$

In this context, it is useful to work with the transformed block-variables

$$X_k = \frac{1}{m} \sum_{l=0}^{m-1} Y_{mk-l}, \quad k \in \mathbb{Z},$$

and write $X_k = g(\varepsilon_k, \varepsilon_{k-1})$ where $\varepsilon_k = (\zeta_{km}, \dots, \zeta_{(k-1)m+1})^\top \in \mathbb{S}^m$; hence $\{X_k\}_{k \in \mathbb{Z}}$ is a two-dependent sequence. This representation ensures that Assumption 2.1(i) and (ii) hold for $\{X_k\}_{k \in \mathbb{Z}}$, independently of the value of m . The drawback of this block-structure is that we loose a factor m , since we have

$$\frac{1}{\sqrt{nm}} S_n = \frac{1}{\sqrt{nm}} \sum_{k=1}^n Y_k = \frac{1}{\sqrt{n/m}} \sum_{k=1}^{n/m} X_k,$$

where we assume that $n/m \in \mathbb{N}$ for simplicity. However, this loss is known in the literature: Theorem 2.2 now yields the commonly observed rate $\tau_n = (n/m)^{p/2-1}$ in the context of *m*-dependent sequences satisfying (3.3); see, for instance, Theorem 2.6 in [9]. In the latter, the rate $\tau_n = (n/m)^{p/2-1}$ is not immediately obvious, but follows from elementary computations using (3.3).

EXAMPLE 3.4 (Iterated random function). Iterated random functions (cf. [15]) are an important class of processes. Many nonlinear models like ARCH, bilinear and threshold autoregressive models fit into this framework. Let $\mathbb{S} = \mathbb{R}$ and $\{X_k\}_{k \in \mathbb{Z}}$ be defined via the recursion

$$X_k = G(X_{k-1}, \varepsilon_k),$$

commonly referred to as *iterated random functions*; see, for instance, [15]. Let

$$(3.4) \quad L_\varepsilon = \sup_{x \neq y} \frac{|G(x, \varepsilon) - G(y, \varepsilon)|}{|x - y|}$$

be the Lipschitz coefficient. If $\|L_\varepsilon\|_p < 1$ and $\|G(x_0, \varepsilon)\|_p < \infty$ for some x_0 , then X_k can be represented as $X_k = g(\varepsilon_k, \varepsilon_{k-1}, \dots)$ for some measurable function g . In addition, we have

$$(3.5) \quad \|X_k - X'_k\|_p \leq C\rho^{-k} \quad \text{where } 0 < \rho < 1;$$

see [49]. Hence if $\mathbb{E}[X_k] = 0$ and $s^2 > 0$, Assumption 2.1 holds. As an example, consider the *stochastic recursion*

$$X_{k+1} = a_{k+1}X_k + b_{k+1}, \quad k \in \mathbb{Z},$$

where $\{a_k, b_k\}_{k \in \mathbb{Z}}$ is an i.i.d. sequence. Let $\varepsilon_k = (a_k, b_k)$. If we have, for some $p \geq 2$,

$$(3.6) \quad \|a_k\|_p < 1 \quad \text{and} \quad \|b_k\|_p < \infty, \quad \mathbb{E}[b_k] = 0,$$

then $\|L_\varepsilon\|_p \leq \|a_k\|_p < 1$, and Assumption 2.1 holds if $s^2 > 0$. In particular, if a_k, b_k are independent, then one readily verifies that

$$s^2 = \frac{\|b_0\|_2^2}{1 - \|a_0\|_2^2} \left(1 + \frac{2\mathbb{E}[a_0]}{1 - \mathbb{E}[a_0]} \right),$$

which is strictly positive since $|\mathbb{E}[a_0]| < 1$ by Jensen’s inequality. Hence if (3.6) holds for $p > 2$, then Assumption 2.1 holds for p . Analogue conditions can be derived for higher order recursions.

EXAMPLE 3.5 [GARCH(p, q) sequences]. Let $\mathbb{S} = \mathbb{R}$. Another very prominent stochastic recursion is the GARCH(p, q) sequence, given through the relations

$$\begin{aligned} X_k &= \varepsilon_k L_k \quad \text{where } \{\varepsilon_k\}_{k \in \mathbb{Z}} \text{ is a zero mean i.i.d. sequence and} \\ L_k^2 &= \mu + \alpha_1 L_{k-1}^2 + \dots + \alpha_p L_{k-p}^2 + \beta_1 X_{k-1}^2 + \dots + \beta_q X_{k-q}^2, \end{aligned}$$

with $\mu, \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \in \mathbb{R}$. We assume that $\|\varepsilon_k\|_p < \infty$ for some $p \geq 2$. An important quantity is

$$\gamma_C = \sum_{i=1}^r \|\alpha_i + \beta_i \varepsilon_i^2\|_2 \quad \text{with } r = \max\{p, q\},$$

where we replace possible undefined α_i, β_i with zero. If $\gamma_C < 1$, then $\{X_k\}_{k \in \mathbb{Z}}$ is stationary; cf. [8]. In particular, it was shown in [3] that $\{X_k\}_{k \in \mathbb{Z}}$ may be represented as

$$X_k = \sqrt{\mu} \varepsilon_k \left(1 + \sum_{n=1}^{\infty} \sum_{1 \leq l_1, \dots, l_n \leq r} \prod_{i=1}^n (\alpha_{l_i} + \beta_{l_i} \varepsilon_{j-l_1-\dots-l_i}^2) \right)^{1/2}.$$

Using this representation and the fact that $|x - y|^p \leq |x^2 - y^2|^{p/2}$ for $x, y \geq 0$, $p \geq 1$, one can follow the proof of Theorem 4.2 in [1] to show that

$$\|X_k - X'_k\|_p \leq C \rho^k \quad \text{where } 0 < \rho < 1.$$

Since $\mathbb{E}[X_k] = \mathbb{E}[\varepsilon_k] = 0$, Assumption 2.1 holds if $s^2 > 0$. We remark that previous results on Δ_n , in the case of GARCH(p, q) sequences, either require heavy additional assumptions or have suboptimal rates; cf. [27].

EXAMPLE 3.6 (Volterra processes). In the study of nonlinear processes, Volterra processes are of fundamental importance. Following Berkes et al. [4], we consider

$$X_k = \sum_{i=1}^{\infty} \sum_{0 \leq j_1 < \dots < j_i} a_k(j_1, \dots, j_i) \varepsilon_{k-j_1} \cdots \varepsilon_{k-j_i},$$

where $\mathbb{S} = \mathbb{R}$ and $\|\varepsilon_k\|_p < \infty$ for $p \geq 2$, and a_k are called the k th Volterra kernel. Let

$$A_{k,i} = \sum_{k \in \{j_1, \dots, j_i\}, 0 \leq j_1 < \dots < j_i} |a_k(j_1, \dots, j_i)|.$$

Then there exists a constant C such that

$$\|X_k - X'_k\|_p \leq C \sum_{i=1}^{\infty} \|\varepsilon_0\|_p^i A_{k,i}.$$

Thus if $\sum_{k,i=1}^{\infty} k^2 A_{k,i} < \infty$ and $s^2 > 0$, then Assumption 2.1 holds.

4. Proofs. The main approach consists of an m -dependent approximation where $m \rightarrow \infty$, followed by characteristic functions and Esseen’s inequality. However, here the trouble starts, since we cannot factor the characteristic function as in the classic proof, due to the m -dependence. Tikhomirov [45] uses a chaining-type argument, which is also fruitful for Edgeworth expansions; cf. [19]. However, since this approach inevitably leads to a loss in the rate, this is not an option for Berry–Esseen-type results. In order to circumvent this problem, we first work under an appropriately chosen conditional probability measure $P_{\mathbb{F}_m}$. Unfortunately though, this leads to rather intricate problems, since all involved quantities of interest are then random. We first consider the case of a weakly m -dependent sequence $\{X_k\}_{k \in \mathbb{Z}}$, where $m \rightarrow \infty$ as n increases. Note that this is different from Example 3.3. For the general case, we then construct a suitable m -dependent approximating sequence such that the error of approximation is negligible, which is carried out in Section 4.2. The overall proof of Theorem 2.2 is lengthy. Important technical auxiliary results are therefore established separately in Section 4.5. Minor additionally required results are collected in Section 4.6. The proofs of Theorems 2.3 and 2.4 are given in Sections 4.3 and 4.4. To simplify the notation in the proofs, we restrict ourselves to the case of homogeneous Bernoulli shifts, that is, where $X_k = g(\varepsilon_k, \varepsilon_{k-1}, \dots)$, and the function g does not depend on k . This requires substantially fewer indices and notation throughout the proofs, and, in particular, (2.2) holds. The more general nonhomogenous (but still stationary) case follows from straightforward (notational) adaptations. This is because the key ingredient we require for the proof is the Bernoulli-shift structure (1.4) in connection with the summability condition, Assumption 2.1(ii). Whether or not g depends on k is of no relevance in this context.

4.1. *m-dependencies.* In order to deal with m -dependent sequences, we require some additional notation and definitions. Throughout the remainder of this section, we let

$$X_k = f_m(\varepsilon_k, \dots, \varepsilon_{k-m+1}) \quad \text{for } m \in \mathbb{N}, k \in \mathbb{Z},$$

and measurable functions $f_m : \mathbb{S}^m \mapsto \mathbb{R}$, where $m = m_n \rightarrow \infty$ as n increases. We work under the following conditions:

ASSUMPTION 4.1. Let $\{X_k\}_{k \geq \mathbb{Z}}$ be such that for some $p \geq 2$, uniformly in m :

- (i) $\|X_k\|_p < \infty, \mathbb{E}[X_k] = 0,$
- (ii) $\sum_{k=1}^{\infty} k^2 \|X_k - X'_k\|_p < \infty,$
- (iii) $s_m^2 > 0,$

where $s_m^2 = \sum_{k \in \mathbb{Z}} \mathbb{E}[X_0 X_k] = \sum_{k=-m}^m \mathbb{E}[X_0 X_k].$

Observe that this setup is fundamentally different from that considered in Example 3.3. In particular, here we have that $\text{Var}[S_n] \sim n.$ Define the following σ -algebra:

$$(4.1) \quad \mathbb{F}_m = \sigma(\varepsilon_{-m+1}, \dots, \varepsilon_0, \varepsilon'_1, \dots, \varepsilon'_m, \varepsilon_{m+1}, \dots, \varepsilon_{2m}, \varepsilon'_{2m+1}, \dots),$$

where we recall that $\{\varepsilon_k\}_{k \in \mathbb{Z}}$ and $\{\varepsilon'_k\}_{k \in \mathbb{Z}}$ are mutually independent, identically distributed random sequences. We write $P_{\mathbb{F}_m}(\cdot)$ for the conditional law and $\mathbb{E}_{\mathbb{F}_m}[\cdot]$ (or $\mathbb{E}_{\mathcal{H}}[\cdot]$) for the conditional expectation with respect to \mathbb{F}_m (or some other filtration \mathcal{H}). We introduce

$$S_{|m}^{(1)} = \sum_{k=1}^n X_k - \mathbb{E}[X_k | \mathbb{F}_m] \quad \text{and} \quad S_{|m}^{(2)} = \sum_{k=1}^n \mathbb{E}[X_k | \mathbb{F}_m],$$

hence

$$S_n = \sum_{k=1}^n X_k = S_{|m}^{(1)} + S_{|m}^{(2)}.$$

To avoid any notational problems, we put $X_k = 0$ for $k \notin \{1, \dots, n\}.$ Let $n = 2(N - 1)m + m',$ where N, m are chosen such that $c_0 m \leq m' \leq m$ and $c_0 > 0$ is an absolute constant, independent of $m, n.$ For $1 \leq j \leq N,$ we construct the block random variables

$$U_j = \sum_{k=(2j-2)m+1}^{(2j-1)m} X_k - \mathbb{E}[X_k | \mathbb{F}_m] \quad \text{and} \quad R_j = \sum_{k=(2j-1)m+1}^{2jm} X_k - \mathbb{E}[X_k | \mathbb{F}_m],$$

and put $Y_j^{(1)} = U_j + R_j,$ hence $S_{|m}^{(1)} = \sum_{j=1}^N Y_j^{(1)}.$ Note that by construction of the blocks, $Y_j^{(1)}, j = 1, \dots, N$ are independent random variables under the

conditional probability measure $P_{\mathbb{F}_m}(\cdot)$, and are identically distributed at least for $j = 1, \dots, N - 1$ under P . We also put $Y_1^{(2)} = \sum_{k=1}^m \mathbb{E}[X_k | \mathbb{F}_m]$ and $Y_j^{(2)} = \sum_{k=(j-1)m+1}^{(j+1)m} \mathbb{E}[X_k | \mathbb{F}_m]$ for $j = 2, \dots, N$. Note that $Y_j^{(2)}$, $j = 1, \dots, N$ is a sequence of independent random variables. The following partial and conditional variances are relevant for the proofs:

$$\begin{aligned} \sigma_{j|m}^2 &= \frac{1}{2m} \mathbb{E}_{\mathbb{F}_m} [(Y_j^{(1)})^2] \quad \text{and} \quad \sigma_j^2 = \mathbb{E}[\sigma_{j|m}^2], \\ \sigma_{|m}^2 &= \frac{1}{n} \mathbb{E}[(S_{|m}^{(1)})^2 | \mathbb{F}_m] = \frac{1}{N + m'/2m} \sum_{j=1}^N \sigma_{j|m}^2, \\ \bar{\sigma}_m^2 &= \mathbb{E}[\sigma_{|m}^2] = \frac{1}{N + m'/2m} \sum_{j=1}^N \sigma_j^2, \\ \hat{\sigma}_m^2 &= \frac{1}{2m} \sum_{k=1}^m \sum_{l=1}^m \mathbb{E}[X_k X_l]. \end{aligned}$$

As we shall see below, these quantities are all closely connected. Note that $\sigma_i^2 = \sigma_j^2$ for $1 \leq i, j \leq N - 1$, but $\sigma_1^2 \neq \sigma_N^2$ in general. Moreover, we have the equation

$$(4.2) \quad 2m\hat{\sigma}_m^2 = ms_m^2 - \sum_{k \in \mathbb{Z}} m \wedge |k| \mathbb{E}[X_0 X_k].$$

The above relation is important, since Lemma 4.6 yields that under Assumption 4.1 we have $2\hat{\sigma}_m^2 = s_m^2 + \mathcal{O}(m^{-1})$. Moreover, Lemma 4.7 gives $\sigma_j^2 = \hat{\sigma}_m^2 + \mathcal{O}(m^{-1})$ for $1 \leq j \leq N - 1$. We conclude that

$$(4.3) \quad \sigma_j^2 = s_m^2/2 + \mathcal{O}(m^{-1}) > 0 \quad \text{for sufficiently large } m.$$

The same is true for σ_N^2 , since $m' \geq c_0 m$. Summarizing, we see that we do not have any degeneracy problems for the partial variances σ_j^2 , $1 \leq j \leq N$ under Assumption 4.1. For the second part $S_{|m}^{(2)}$, we introduce $\bar{s}_m^2 = n^{-1} \|S_{|m}^{(2)}\|_2^2$. One then readily derives via conditioning arguments that

$$(4.4) \quad s_{nm}^2 \stackrel{\text{def}}{=} n^{-1} \|S_n\|_2^2 = n^{-1} \|S_{|m}^{(1)}\|_2^2 + n^{-1} \|S_{|m}^{(2)}\|_2^2 = \bar{\sigma}_m^2 + \bar{s}_m^2.$$

We are now ready to give the main result of this section.

THEOREM 4.2. *Grant Assumption 4.1, and let $p \in (2, 3]$. Assume in addition that $N = N_n = n^\lambda$ for $0 < \lambda \leq p/(2p + 2)$. Then*

$$\sup_{x \in \mathbb{R}} |P(S_n/\sqrt{n} \leq x) - \Phi(x/s_{nm})| \leq c(\lambda, p)n^{-p/2+1},$$

where $c(\lambda, p) > 0$ depends on $\lambda, p, \sum_{k=1}^\infty k^2 \|X_k - X'_k\|_p$ and $\inf_m s_m^2 > 0$.

The proof of Theorem 4.2 is based on the following decomposition. Let Z_1, Z_2 be independent unit Gaussian random variables. Then

$$\begin{aligned} & \sup_{x \in \mathbb{R}} |P(S_n/\sqrt{n} \leq x) - \Phi(x/s_{nm})| \\ &= \sup_{x \in \mathbb{R}} |P(S_m^{(1)} \leq x\sqrt{n} - S_m^{(2)}) - P(Z_1\bar{\sigma}_m \leq x - Z_2\bar{\zeta}_m)| \\ &\leq \mathbf{A} + \mathbf{B} + \mathbf{C}, \end{aligned}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined as

$$\begin{aligned} \mathbf{A} &= \sup_{x \in \mathbb{R}} |\mathbb{E}[P_{\mathbb{F}_m}(S_m^{(1)}/\sqrt{n} \leq x - S_m^{(2)}/\sqrt{n}) - P_{\mathbb{F}_m}(Z_1\sigma_m \leq x - S_m^{(2)}/\sqrt{n})]|, \\ \mathbf{B} &= \sup_{x \in \mathbb{R}} |\mathbb{E}[P_{\mathbb{F}_m}(Z_1\sigma_m \leq x - S_m^{(2)}/\sqrt{n}) - P_{\mathbb{F}_m}(Z_1\bar{\sigma}_m \leq x - S_m^{(2)}/\sqrt{n})]|, \\ \mathbf{C} &= \sup_{x \in \mathbb{R}} |P(S_m^{(2)}/\sqrt{n} \leq x - Z_1\bar{\sigma}_m) - P(Z_2\bar{\zeta}_m \leq x - Z_1\bar{\sigma}_m)|. \end{aligned}$$

We will treat the three parts separately, and show that $\mathbf{A}, \mathbf{B}, \mathbf{C} \leq \frac{c(\lambda, p)}{3} n^{-p/2+1}$, which proves Theorem 4.2. As a brief overview, the proof consists of the following steps:

- (a) apply Esseen’s smoothing inequality, and factor the resulting characteristic function into a (conditional) product of characteristic functions $\varphi_j(x)$ under the conditional probability measure $P_{\mathbb{F}_m}$;
- (b) use ideal metrics to control the distance between $\varphi_j(x)$ and corresponding Gaussian versions under $P_{\mathbb{F}_m}$;
- (c) based on Renyi’s representation, control the (conditional) characteristic functions $\varphi_j(x)$ under P ;
- (d) replace conditional variances under the overall probability measure P .

One of the main difficulties arises from working under the conditional measure $P_{\mathbb{F}_m}$. For the proof, we require some additional notation. In analogy to the filter $\theta_k^{(l, \cdot)}$, we denote with $\theta_k^{(l, *)}$,

$$(4.5) \quad \theta_k^{(l, *)} = (\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon'_{k-l}, \varepsilon'_{k-l-1}, \varepsilon'_{k-l-2}, \dots).$$

We put $\theta_k^* = \theta_k^{(k, *)} = (\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon'_0, \varepsilon'_{-1}, \varepsilon'_{-2}, \dots)$ and $X_k^{(l, *)} = g(\theta_k^{(l, *)})$, and in particular, we have $X_k^* = X_k^{(k, *)}$. Similarly, let $\{\varepsilon''_k\}_{k \in \mathbb{Z}}$ be independent copies of $\{\varepsilon_k\}_{k \in \mathbb{Z}}$ and $\{\varepsilon'_k\}_{k \in \mathbb{Z}}$. For $l \leq k$, we then introduce the quantities $X_k^{(l, \prime)}, X_k^{(l, **)}, X''_k, X_k^{(**)}$ in analogy to $X_k^{(l, \cdot)}, X_k^{(l, *)}, X'_k, X_k^*$. This means that we replace every ε'_k with ε''_k at all corresponding places. For $k \geq 0$, we also introduce the σ -algebras

$$(4.6) \quad \begin{aligned} \mathcal{E}'_k &= \sigma(\varepsilon_j, j \leq k \text{ and } j \neq 0, \varepsilon'_0) \quad \text{and} \\ \mathcal{E}^*_k &= \sigma(\varepsilon_j, 1 \leq j \leq k \text{ and } \varepsilon'_i, i \leq 0). \end{aligned}$$

Similarly, we introduce filtrations \mathcal{E}_k'' and \mathcal{E}_k^{**} .

Throughout the proofs, we make the following conventions:

(1) We do not distinguish between N and $N + m'/2m$ since the difference $m'/2m$ is not of any particular relevance for the proofs. We use N for both expressions.

(2) The abbreviations *I, II, III, ...*, for expressions (possible with some additional indices) vary from proof to proof.

(3) We use $\lesssim, \gtrsim, (\sim)$ to denote (two-sided) inequalities involving a multiplicative constant.

(4) If there is no confusion, we put $Y_j = (2m)^{-1/2}Y_j^{(1)}$ for $j = 1, \dots, N$ to lighten the notation, particularly in part **A**.

(5) We write [as in (4.4)] $\stackrel{\text{def}}{=}$ if we make definitions on the fly.

4.1.1. *Part A.* The proof of part **A** is divided into four major steps. Some more technical arguments are deferred to Sections 4.5.2 and 4.5.1.

PROOF. For $L > 0$, put $\mathcal{B}_L = \{L^{-1} \sum_{j=1}^L \sigma_{j|m}^2 \geq s_m^2/4\}$, and denote with \mathcal{B}_L^c its complement. Since $S_{|m}^{(2)} \in \mathbb{F}_m$, we obtain that

$$\begin{aligned} \mathbf{A} &= \sup_{x \in \mathbb{R}} |\mathbb{E}[P_{|\mathbb{F}_m}(S_{|m}^{(1)}/\sqrt{n} \leq x - S_{|m}^{(2)}/\sqrt{n}) - P_{|\mathbb{F}_m}(Z_1\sigma_{|m} \leq x - S_{|m}^{(2)}/\sqrt{n})]| \\ &\leq \mathbb{E}\left[\sup_{y \in \mathbb{R}} |P_{|\mathbb{F}_m}(S_{|m}^{(1)}/\sqrt{n} \leq y) - P_{|\mathbb{F}_m}(Z_1\sigma_{|m} \leq y)| \mathbb{1}(\mathcal{B}_N)\right] + 2P(\mathcal{B}_N^c). \end{aligned}$$

Corollary 4.8 yields that $P(\mathcal{B}_N^c) \lesssim n^{-p/2}N \lesssim n^{-p/2+1}$ since $N \leq n$, and it thus suffices to treat

$$(4.7) \quad \Delta_{|m} \stackrel{\text{def}}{=} \sup_{y \in \mathbb{R}} |P_{|\mathbb{F}_m}(S_{|m}^{(1)}/\sqrt{n} \leq y) - P_{|\mathbb{F}_m}(Z_1\sigma_{|m} \leq y)| \mathbb{1}(\mathcal{B}_N).$$

Step 1: Berry–Esseen inequality. Denote with $\Delta_{|m}^T$ the smoothed version of $\Delta_{|m}$ (cf. [17]) as in the classical approach. Since $\sigma_{|m}^2 \geq s_m^2/4 > 0$ on the set \mathcal{B}_N by construction, the smoothing inequality (cf. [17], Lemma 1, XVI.3) is applicable, and it thus suffices to treat $\Delta_{|m}^T$. Let $\varphi_j(x) = \mathbb{E}[e^{ixY_j} | \mathbb{F}_m]$, and put $T = n^{p/2-1}c_T$, where $c_T > 0$ will be specified later. Due to the independence of $\{Y_j\}_{1 \leq j \leq N}$ under $P_{|\mathbb{F}_m}$ and since $\mathbb{1}(\mathcal{B}_N) \leq 1$, it follows that

$$(4.8) \quad \mathbb{E}[|\Delta_{|m}^T|] \leq \int_{-T}^T \mathbb{E}\left[\left|\prod_{j=1}^N \varphi_j(\xi/\sqrt{N}) - \prod_{j=1}^N e^{-\sigma_{j|m}^2 \xi^2/2N}\right|\right] / |\xi| d\xi.$$

Put $t = \xi/\sqrt{N}$. Then $\prod_{j=1}^N a_j - \prod_{j=1}^N b_j = \sum_{i=1}^N (\prod_{j=1}^{i-1} b_j)(a_i - b_i)(\prod_{j=i+1}^N a_j)$, where we use the convention that $\prod_{j=1}^{i-2}(\cdot) = \prod_{j=i+2}^N(\cdot) = 1$ if $i - 2 < 1$ or

$i + 2 > N$. Hence we have

$$\begin{aligned} & \prod_{j=1}^N \varphi_j(t) - \prod_{j=1}^N e^{-\sigma_{j|m}^2 t^2/2} \\ &= \sum_{i=1}^N \left(\prod_{j=1}^{i-1} \varphi_j(t) \right) (\varphi_i(t) - e^{-\sigma_{i|m}^2 t^2/2}) \left(\prod_{j=i+1}^N e^{-\sigma_{j|m}^2 t^2/2} \right). \end{aligned}$$

Note that both $\{\varphi_j(t)\}_{1 \leq j \leq N}$ and $\{e^{-\sigma_{j|m}^2 t^2/2}\}_{1 \leq j \leq N}$ are two-dependent sequences. Since $|\varphi_j(t)|, e^{-\sigma_{j|m}^2 t^2/2} \leq 1$, it then follows by the triangle inequality, stationarity and “leave one out” that

$$\begin{aligned} & \left\| \prod_{j=1}^N \varphi_j(t) - \prod_{j=1}^N e^{-\sigma_{j|m}^2 t^2/2} \right\|_1 \\ & \leq \sum_{i=1}^N \left\| \prod_{j=1}^{i-1} e^{-\sigma_{j|m}^2 t^2/2} \right\|_1 \left\| \varphi_i(t) - e^{-\sigma_{i|m}^2 t^2/2} \right\|_1 \left\| \prod_{j=i+2}^N |\varphi_j(t)| \right\|_1 \\ & \leq N \left\| \varphi_1(t) - e^{-\sigma_{1|m}^2 t^2/2} \right\|_1 \left\| \prod_{j=N/2}^{N-1} |\varphi_j(t)| \right\|_1 \\ & \quad + N \left\| \prod_{j=1}^{N/2-3} e^{-\sigma_{j|m}^2 t^2/2} \right\|_1 \left\| \varphi_1(t) - e^{-\sigma_{1|m}^2 t^2/2} \right\|_1 \\ & \quad + \left\| \prod_{j=1}^{N/2-3} e^{-\sigma_{j|m}^2 t^2/2} \right\|_1 \left\| \varphi_N(t) - e^{-\sigma_{N|m}^2 t^2/2} \right\|_1 \\ & = I_N(\xi) + II_N(\xi) + III_N(\xi). \end{aligned}$$

We proceed by obtaining upper bounds for $I_N(\xi), II_N(\xi)$ and $III_N(\xi)$.

Step 2: Bounding $\|\varphi_i(t) - e^{-\sigma_{i|m}^2 t^2/2}\|_1, i \in \{1, N\}$. Let $Z_i, i \in \{1, N\}$ be two zero mean standard Gaussian random variables. Then

$$\begin{aligned} \|\varphi_i(t) - e^{-\sigma_{i|m}^2 t^2/2}\|_1 & \leq \|\mathbb{E}_{\mathbb{F}_m} [\cos(tY_i) - \cos(t\sigma_{i|m} Z_i)]\|_1 \\ & \quad + \|\mathbb{E}_{\mathbb{F}_m} [\sin(tY_i) - \sin(t\sigma_{i|m} Z_i)]\|_1. \end{aligned}$$

Due to the very nice analytical properties of $\sin(y), \cos(y)$, one may reformulate the above in terms of ideal-metrics; cf. [50] and Section 4.5.2. This indeed leads to the desired bound

$$(4.9) \quad \|\varphi_i(t) - e^{-\sigma_{i|m}^2 t^2/2}\|_1 \lesssim |t|^p m^{-p/2+1}.$$

The precise derivation is carried out in Section 4.5.2 via Lemmas 4.9 and 4.10, and Corollary 4.11. Whether $i = 1$ or $i = N$ makes no difference.

Step 3: Bounding $\|\prod_{j=N/2}^{N-1} |\varphi_j(t)|\|_1$: in order to bound $\|\prod_{j=N/2}^{N-1} |\varphi_j(t)|\|_1$, we require good enough estimates for $|\varphi_j(t)|$ where $0 \leq t < 1$. As already mentioned, we cannot directly follow the classical approach. Instead, we use a refined version based on a conditioning argument. To this end, let us first deal with $\varphi_j(t)$. Put

$$(4.10) \quad \mathcal{G}_j^{(l)} = \mathcal{E}_{(2j-2)m+l} \quad \text{for } j, l \geq 1.$$

We first consider the case $j = 1$. Introduce

$$(4.11) \quad \begin{aligned} IV_1^{(l)}(m) &= \sum_{k=l+1}^m (X_k - \mathbb{E}_{\mathbb{F}_m}[X_k]) + R_1 \quad \text{and} \\ V_1^{(l)}(m) &= IV_1^{(l)}(m) - \mathbb{E}_{\mathcal{G}_1^{(l)}}[IV_1^{(l)}(m)]. \end{aligned}$$

Then since $\mathbb{F}_m \cap \sigma(-\varepsilon_{-m+1}, \dots, \varepsilon_0) \subseteq \mathcal{G}_1^{(l)}$, we have

$$(4.12) \quad |\varphi_1(t)| \leq \mathbb{E}_{\mathbb{F}_m} [|\mathbb{E}_{\mathcal{G}_1^{(l)}}[e^{it(2m)^{-1/2}V_1^{(l)}(m)}]|].$$

Clearly, this is also valid for $\varphi_j(t)$, $j = 2, \dots, N$, with corresponding $\mathcal{G}_j^{(l)}$ and $IV_j^{(l)}(m)$, $V_j^{(l)}(m)$, defined analogously to (4.11). Let

$$(4.13) \quad \varphi_j^{(l)}(x) = \mathbb{E}_{\mathcal{G}_j^{(l)}}[e^{ix(m-l)^{-1/2}V_j^{(l)}(m)}],$$

and $\mathcal{J} = \{j : N/2 \leq j \leq N - 1 \text{ and } 2 \text{ divides } j\}$, and hence \mathcal{J} denotes the set of all even numbers between $N/2$ and $N - 1$. Then

$$\prod_{j=N/2}^{N-1} |\varphi_j(t)| \leq \prod_{j \in \mathcal{J}} \mathbb{E}_{\mathbb{F}_m} [|\mathbb{E}_{\mathcal{G}_j^{(l)}}[e^{it(2m)^{-1/2}V_j^{(l)}(m)}]|] = \prod_{j \in \mathcal{J}} \mathbb{E}_{\mathbb{F}_m} [|\varphi_j^{(l)}(x)|],$$

where $x = t\sqrt{(m-l)/2m}$. Note that $\{V_j^{(l)}(m)\}_{j \in \mathcal{J}}$ is a sequence of i.i.d. random variables, particularly with respect to $P_{\mathbb{F}_m}$. Hence by independence and Jensen's inequality, it follows from the above that

$$(4.14) \quad \begin{aligned} \left\| \prod_{j=N/2}^{N-1} |\varphi_j(t)| \right\|_1 &\leq \prod_{j \in \mathcal{J}} \|\mathbb{E}_{\mathbb{F}_m}[|\varphi_j^{(l)}(x)|]\|_1 \\ &\leq \prod_{j \in \mathcal{J}} \|\varphi_j^{(l)}(x)\|_1 = \left\| \prod_{j \in \mathcal{J}} |\varphi_j^{(l)}(x)| \right\|_1. \end{aligned}$$

We thus see that it suffices to deal with $\varphi_j^{(l)}(x)$. The classical argument uses the estimate

$$\varphi(\xi/\sqrt{\sigma^2 n}) \leq e^{-5\xi^2/18n} \quad \text{for } \xi^2/n \leq c, c > 0$$

for the characteristic function φ . Since in our case φ_j is random, we cannot use this estimate. Instead, we will use Lemma 4.5, which provides a similar result. In order to apply it, set $J = |\mathcal{J}| \geq N/8$,

$$(4.15) \quad H_j = \frac{1}{\sqrt{m-l}} V_j^{(l)}(m) \quad \text{and} \quad \mathcal{H}_j = \mathcal{G}_j^{(l)}.$$

For the applicability of Lemma 4.5, we need to verify that:

- (i) $\mathbb{E}_{\mathcal{H}_j}[H_j] = 0$;
- (ii) there exists a $u^- > 0$ such that $P(\mathbb{E}_{|\mathcal{H}_j}[H_j^2] \leq u^-) < 1/7$, uniformly for $j \in \mathcal{J}$;
- (iii) $\|H_j\|_p \leq c_1$ uniformly for $j \in \mathcal{J}$ and some $c_1 < \infty$.

Now (i) is true by construction. Claim (ii) is dealt with via Lemma 4.14, which yields that

$$(4.16) \quad P(\mathbb{E}_{|\mathcal{H}_j}[H_j^2] \leq \hat{\sigma}_{m-l}^2) \lesssim \frac{1}{\sqrt{m-l}}.$$

Since $\hat{\sigma}_{m-l}^2 \geq s_m^2/4$ for large enough $m-l$ (say $m-l \geq K_0 > 0$) by Lemma 4.6, we may set $0 < u^- = s_m^2/8 \leq \hat{\sigma}_{m-l}^2/2$. For showing (iii), it suffices to treat the case $j = 1$. Note that (for $k \leq m$)

$$(4.17) \quad \mathbb{E}_{\mathcal{G}_1^{(l)}}[X_k] = \mathbb{E}_{\mathcal{E}_l}[X_k - X_k^{(k-l,*)}] \quad \text{and} \quad \mathbb{E}_{\mathbb{F}_m}[X_k] = \mathbb{E}_{\mathbb{F}_m}[X_k - X_k^*].$$

By stationarity and the triangle and Jensen inequalities, we then have that

$$(4.18) \quad \begin{aligned} \sqrt{m-l} \|H_j\|_p &\leq \left\| \sum_{k=l+1}^m X_k \right\|_p + \left\| \sum_{k=l+1}^m \mathbb{E}_{\mathcal{E}_l}[X_k - X_k^{(k-l,*)}] \right\|_p \\ &\quad + 2 \left\| \sum_{k=l+1}^m \mathbb{E}_{\mathbb{F}_m}[X_k - X_k^*] \right\|_p + 2 \|R_1\|_p. \end{aligned}$$

Using Jensen’s inequality and arguing similar to Lemma 4.13, it follows that

$$\begin{aligned} \left\| \sum_{k=l+1}^m \mathbb{E}_{\mathcal{G}_l}[X_k - X_k^{(k-l,*)}] \right\|_p &\leq \sum_{k=l+1}^m \|X_k - X_k^{(k-l,*)}\|_p \\ &\leq \sum_{k=1}^{\infty} k \|X_k - X_k'\|_p < \infty. \end{aligned}$$

Similarly, using also Lemma 4.13 to control $\|R_1\|_p$, we obtain that

$$(4.19) \quad \left\| \sum_{k=l+1}^m \mathbb{E}_{\mathbb{F}_m}[X_k] \right\|_p + \|R_1\|_p < \infty.$$

By Lemma 4.12, we have $\|\sum_{k=l+1}^m X_k\|_p \lesssim \sqrt{m-l}$, and hence (iii) follows. We can thus apply Lemma 4.5 with $u^- = s_m^2/8$ and $J = |\mathcal{J}| \geq N/8$, which yields

$$(4.20) \quad \left\| \prod_{j \in \mathcal{J}} |\varphi_j^{(l)}(x)| \right\|_1 \lesssim e^{-c_{\varphi,1}x^2N/16} + e^{-\sqrt{N/32}\log 8/7} \quad \text{for } x^2 < c_{\varphi,2},$$

where $x = t\sqrt{(m-l)/2m}$. It is important to emphasize that both $c_{\varphi,1}, c_{\varphi,2}$ do not depend on l, m and are strictly positive. Moreover, we find from (4.16) that l can be chosen freely, as long as $m-l$ is larger than K_0 , which will be important in the next step.

Step 4: Bounding and integrating $I_N(\xi), II_N(\xi), III_N(\xi)$.

We first treat $I_N(\xi)$. Recall that $t = \xi/\sqrt{N}$, hence

$$|t|^p m^{-p/2+1} \lesssim |\xi|^p n^{-p/2+1} N^{-1}.$$

By (4.9), (4.14) and (4.20), it then follows for $\xi^2(m-l) < c_{\varphi,2}n$ that

$$(4.21) \quad I_N(\xi) \lesssim |\xi|^p n^{-p/2+1} (e^{-c_{\varphi,1}\xi^2(m-l)/16m} + e^{-\sqrt{N/32}\log 8/7}).$$

To make use of this bound, we need to appropriately select $l = l(\xi)$. Recall that $N = n^\lambda, 0 < \lambda \leq p/(2p+2)$ by assumption. Choosing

$$l(\xi) = \mathbb{1}(\xi^2 < n^\lambda c_{\varphi,2}) + \left(m - \frac{c_{\varphi,2}n}{2\xi^2} \vee K_0\right) \mathbb{1}(\xi^2 \geq n^\lambda c_{\varphi,2})$$

and $c_T^2 < c_{\varphi,2}/K_0$, we obtain from the above that

$$(4.22) \quad \int_{-T}^T I_N(\xi)/\xi \, d\xi \lesssim n^{-p/2+1}.$$

In order to treat $II_N(\xi)$, let $N' = N/2 - 3$, and $\mathcal{B}_{N'} = \{\sum_{j=1}^{N'} \sigma_{j|m}^2 \geq s_m^2/4\}$. Denote with $\mathcal{B}_{N'}^c$ its complement. Then by Corollary 4.8 (straightforward adaption is necessary) and (4.9), it follows that

$$(4.23) \quad \begin{aligned} II_N(\xi) \mathbb{1}(|\xi| \leq N) &\leq N \|\varphi_1(t) - e^{-\sigma_{1|m}^2 \xi^2/2}\|_1 \left\| \prod_{j=1}^{N'} e^{-\sigma_{j|m}^2 \xi^2/2} \mathbb{1}(\mathcal{B}_{N'}) \right\|_1 \\ &\quad + N \|\varphi_1(t) - e^{-\sigma_{1|m}^2 \xi^2/2}\|_1 P(\mathcal{B}_{N'}^c) \\ &\lesssim |\xi|^p n^{-p/2+1} e^{-s_m^2 \xi^2/16} + |\xi|^p n^{-p/2+1} N n^{-p/2}. \end{aligned}$$

Similarly, using $\|\varphi_1(t) - e^{-\sigma_{1|m}^2 \xi^2/2}\|_1 \leq 2$ one obtains

$$(4.24) \quad II_N(\xi) \mathbb{1}(|\xi| > N) \lesssim |\xi|^p n^{-p/2+1} e^{-s_m^2 \xi^2 N/16} + n^{-p/2} N^2.$$

Hence employing (4.23) and (4.24) yields

$$\begin{aligned}
 \int_{-T}^T II_N(\xi)/\xi \, d\xi &\lesssim n^{-p/2+1} \int_{|\xi| \leq N} |\xi|^{p-1} (e^{-s_m^2 \xi^2/16} + n^{-p/2} N) \, d\xi \\
 &\quad + \int_{N < |\xi| \leq T} (n^{-p/2+1} |\xi|^{p-1} e^{-s_m^2 \xi^2/16} + n^{-p/2} N^2 \xi^{-1}) \, d\xi \\
 (4.25) \qquad &\lesssim n^{-p/2+1} + n^{-p/2+1} n^{-p/2} N^{p+1} + n^{-p/2} N^2 \log T \\
 &\lesssim n^{-p/2+1},
 \end{aligned}$$

since $N = n^\lambda$, $0 < \lambda \leq p/(2p + 2)$ by assumption. Similarly, one obtains the same bound for $III_N(\xi)$. This completes the proof of part **A**. \square

4.1.2. *Part B.*

PROOF. Let

$$\Delta^{(2)}(x) \stackrel{\text{def}}{=} \mathbb{E}[P_{|\mathbb{F}_m}(Z_1 \sigma_m \leq x - S_m^{(2)}/\sqrt{n}) - P_{|\mathbb{F}_m}(Z_1 \bar{\sigma}_m \leq x - S_m^{(2)}/\sqrt{n})].$$

Recall that $\mathcal{B}_N = \{N^{-1} \sum_{j=1}^N \sigma_{j|m}^2 \geq s_m^2/4\}$ and $P(\mathcal{B}_N^c) \lesssim n^{-p/2+1}$ by Corollary 4.8. Using properties of the Gaussian distribution, it follows that

$$\begin{aligned}
 \mathbf{B} &\leq \sup_{x \in \mathbb{R}} |\mathbb{E}[\Delta^{(2)}(x) \mathbb{1}(\mathcal{B}_N)]| + \sup_{x \in \mathbb{R}} |\mathbb{E}[\Delta^{(2)}(x) \mathbb{1}(\mathcal{B}_N^c)]| \\
 &\lesssim \mathbb{E}[|1/\sigma_m - 1/\bar{\sigma}_m| \mathbb{1}(\mathcal{B}_N)] + n^{-p/2+1}.
 \end{aligned}$$

Using $(a - b)(a + b) = a^2 - b^2$, Hölders inequality and Lemma 4.7, we obtain that

$$\mathbb{E}[|1/\sigma_m - 1/\bar{\sigma}_m| \mathbb{1}(\mathcal{B}_N)] \lesssim \|\sigma_m^2 - \bar{\sigma}_m^2\|_{p/2} \lesssim n^{-p/2+1}.$$

Hence we conclude that $\mathbf{B} \lesssim n^{-p/2+1}$. \square

4.1.3. *Part C.*

PROOF. Due to the independence of Z_1, Z_2 , we may rewrite **C** as

$$\mathbf{C} = \sup_{x \in \mathbb{R}} |\Phi((x - S_m^{(2)}/\sqrt{n})/\bar{\sigma}_m) - \Phi((x - Z_2 \bar{s}_m)/\bar{\sigma}_m)|,$$

where $\Phi(\cdot)$ denotes the c.d.f. of a standard normal distribution. This induces a “natural” smoothing. The claim now follows by repeating the same arguments as in part **A**. Note however, that the present situation is much easier to handle, due to the already smoothed version, and since $Y_k^{(2)}, k = 1, \dots, N$ is a sequence of independent random variables. Alternatively, one may also directly appeal to the results in [11]. \square

4.2. *Proof of Theorem 2.2.* The proof of Theorem 2.2 mainly consists of constructing a good m -dependent approximation and then verifying the conditions of Theorem 4.2. To this end, set $m = cn^{3/4}$ for some $c > 0$, and note that $1/4 < p/(2p + 2)$ for $p \in (2, 3]$. Let $\mathcal{E}_k^m = \sigma(\varepsilon_j, k - m + 1 \leq j \leq k)$, and define the approximating sequence as

$$(4.26) \quad \begin{aligned} X_k^{(\leq m)} &= \mathbb{E}[X_k | \mathcal{E}_k^m] \quad \text{and} \\ X_k^{(> m)} &= X_k - X_k^{(\leq m)} = X_k - \mathbb{E}[X_k | \mathcal{E}_k^m]. \end{aligned}$$

We also introduce the corresponding partial sums as

$$(4.27) \quad S_n^{(\leq m)} = \sum_{k=1}^n X_k^{(\leq m)}, \quad S_n^{(> m)} = \sum_{k=1}^n X_k^{(> m)}.$$

Further, let $s_n^2 = n^{-1} \|S_n\|_2^2$ and $s_{nm}^2 = n^{-1} \|S_n^{(\leq m)}\|_2^2 = \bar{\sigma}_m^2 + \bar{\zeta}_m^2$. We require the following auxiliary result (Lemma 5.1 in [27]).

LEMMA 4.3. *For every $\delta > 0$, every $m, n \geq 1$ and every $x \in \mathbb{R}$, the following estimate holds:*

$$\begin{aligned} &|P(S_n/\sqrt{n} \leq xs_n) - \Phi(x)| \\ &\leq A_0(x, \delta) + A_1(m, n, \delta) \\ &\quad + \max\{A_2(m, n, x, \delta) + A_3(m, n, \delta), A_4(m, n, x, \delta) + A_5(m, n, x, \delta)\}, \end{aligned}$$

where:

$$\begin{aligned} A_0(x, \delta) &= |\Phi(x) - \Phi(x + \delta)|; \\ A_1(m, n, \delta) &= P(|S_n - S_n^{(\leq m)}| \geq \delta s_n \sqrt{n}); \\ A_2(m, n, x, \delta) &= |P(S_n^{(\leq m)} \leq (x + \delta)s_n \sqrt{n}) - \Phi((x + \delta)s_n/s_{nm})|; \\ A_3(m, n, x, \delta) &= |\Phi((x + \delta)s_n/s_{nm}) - \Phi(x + \delta)|; \\ A_4(m, n, x, \delta) &= A_2(m, n, x, -\delta) \quad \text{and} \quad A_5(m, n, x, \delta) = A_3(m, n, x, -\delta). \end{aligned}$$

PROOF OF THEOREM 2.2. As a preparatory result, note that

$$(4.28) \quad ns_n^2 = ns^2 + \sum_{k \in \mathbb{Z}} (n \wedge |k|) \mathbb{E}[X_0 X_k].$$

Using the same arguments as in Lemma 4.6, it follows that $ns_n = ns^2 + \mathcal{O}(1) > 0$. By the properties of Gaussian distribution,

$$\sup_{x \in \mathbb{R}} |\Phi(x/\sqrt{s^2}) - \Phi(x/\sqrt{s_n})| \lesssim n^{-1},$$

and we may thus safely interchange s_n^2 and s^2 . We first deal with $A_1(m, n, \delta)$. For $j \in \mathbb{Z}$, denote with $\mathcal{P}_j(X_k^{(>m)})$ the projection operator

$$(4.29) \quad \mathcal{P}_j(X_k^{(>m)}) = \mathbb{E}[X_k^{(>m)} | \mathcal{E}_j] - \mathbb{E}[X_k^{(>m)} | \mathcal{E}_{j-1}].$$

Proceeding as in the proof of Lemma 3.1 in [29], it follows that for $k \geq 0$,

$$(4.30) \quad \|\mathcal{P}_0(X_k^{(>m)})\|_p \leq 2 \min \left\{ \|X_k - X'_k\|_p, \sum_{l=m}^{\infty} \|X_l - X'_l\|_p \right\}.$$

An application of Theorem 1 in [48] now yields that

$$(4.31) \quad n^{-1/2} \|S_n^{(>m)}\|_p \leq c(p) \sum_{k=1}^{\infty} \|\mathcal{P}_0(X_k^{(>m)})\|_p$$

for some absolute constant $c(p)$ that only depends on p . By (4.30), it follows that the above is of magnitude

$$(4.32) \quad \sum_{k=L}^{\infty} L^{-2} k^2 \|X_k - X'_k\|_p + L \sum_{k=m}^{\infty} m^{-2} k^2 \|X_k - X'_k\|_p \lesssim L^{-2} + Lm^{-2}.$$

Setting $L = m^{2/3}$, we obtain the bound $\mathcal{O}(m^{-4/3}) = \mathcal{O}(n^{-1})$. We thus conclude from the Markov inequality that

$$P(|S_n - S_n^{(\leq m)}| \geq \delta s_n \sqrt{n}) = P(|S_n^{(>m)}| \geq \delta s_n \sqrt{n}) \lesssim (\delta n)^{-p},$$

hence

$$(4.33) \quad A_1(m, n, \delta) \lesssim (\delta n)^{-p}.$$

Note that a much sharper bound can be obtained via moderate deviation arguments (cf. [22]), but the current one is sufficient for our needs, and its deviation requires fewer computations. Next, we deal with $A_2(m, n, x, \delta)$. The aim is to apply Theorem 4.2 to obtain the result. In order to do so, we need to verify Assumption 4.1(i)–(iii) for $X_k^{(\leq m)}$.

Case (i): Note first that $\mathbb{E}[X_k^{(\leq m)}] = \mathbb{E}[X_k] = 0$. Moreover, Jensen’s inequality gives

$$\|X_k^{(\leq m)}\|_p = \|\mathbb{E}[X_k | \mathcal{E}_k^m]\|_p \leq \|X_k\|_p < \infty.$$

Hence Assumption 4.1(i) is valid.

Case (ii): Note that we may assume $k \leq m$, since otherwise $(X_k^{(\leq m)})' - X_k^{(\leq m)} = 0$, and Assumption 4.1(ii) is trivially true. Put

$$\mathcal{E}_k^{(m, \cdot)} = \sigma(\varepsilon_j, k - m + 1 \leq j \leq k, j \neq 0, \varepsilon'_0).$$

Since $\mathbb{E}[X_k | \mathcal{E}_k^m]' = \mathbb{E}[X'_k | \mathcal{E}_k^{(m,\cdot)}]$, it follows that

$$\begin{aligned}
 (X_k^{(\leq m)})' - X_k^{(\leq m)} &= \mathbb{E}_{\mathcal{E}_k^{(m,\cdot)}}[X'_k] - \mathbb{E}_{\mathcal{E}_k^m}[X_k] \\
 (4.34) \qquad \qquad \qquad &= \mathbb{E}_{\mathcal{E}_k^{(m,\cdot)}}[X'_k - X_k] + \mathbb{E}_{\mathcal{E}_k^{(m,\cdot)}}[X_k] - \mathbb{E}_{\mathcal{E}_k^m}[X_k]
 \end{aligned}$$

$$\begin{aligned}
 (4.35) \qquad \qquad \qquad &= \mathbb{E}_{\mathcal{E}_k^{(m,\cdot)}}[X'_k - X_k] + \mathbb{E}_{\mathcal{E}_k^m}[X'_k] - \mathbb{E}_{\mathcal{E}_k^m}[X_k] \\
 &= \mathbb{E}_{\mathcal{E}_k^{(m,\cdot)}}[X'_k - X_k] + \mathbb{E}_{\mathcal{E}_k^m}[X'_k - X_k].
 \end{aligned}$$

Hence by Jensen’s inequality $\|(X_k^{(\leq m)})' - X_k^{(\leq m)}\|_p \leq 2\|X_k - X'_k\|_p$, which gives the claim.

Case (iii): We have $X_k^{(\leq m)} = \mathbb{E}[X_k^{(m,*)} | \mathcal{E}_k]$. Then

$$\begin{aligned}
 (4.36) \qquad \|X_k^{(> m)}\|_p &= \|\mathbb{E}[X_k - X_k^{(m,*)} | \mathcal{E}_k]\|_p \leq \|X_k - X_k^{(m,*)}\|_p \\
 &\leq m^{-2} \sum_{l=m}^{\infty} l^2 \|X_l - X'_l\|_p \lesssim m^{-2}.
 \end{aligned}$$

By the Cauchy–Schwarz, triangle and Jensen inequalities, we have

$$\begin{aligned}
 &|\mathbb{E}[X_k X_0] - \mathbb{E}[X_k^{(\leq m)} X_0^{(\leq m)}]| \\
 &\leq \|X_0\|_2 \|X_k^{(> m)}\|_2 + \|X_k\|_2 \|X_0^{(> m)}\|_2 + \|X_0^{(> m)}\|_2 \|X_k^{(> m)}\|_2.
 \end{aligned}$$

By (4.36), this is of the magnitude $\mathcal{O}(m^{-2})$. We thus conclude that

$$(4.37) \qquad \left| \sum_{k=0}^m \mathbb{E}[X_k X_0] - \sum_{k=0}^m \mathbb{E}[X_k^{(\leq m)} X_0^{(\leq m)}] \right| \lesssim m^{-1}.$$

On the other hand, we have

$$\left| \sum_{k>m} \mathbb{E}[X_k X_0] \right| \leq \sum_{k>m} \|X_0\|_2 \|X_k^* - X_k\|_2 \leq \frac{1}{m} \sum_{k>m} k^2 \|X_k - X'_k\|_2 \|X_0\|_2 \lesssim \frac{1}{m}.$$

This yields

$$(4.38) \qquad \left| \sum_{k \in \mathbb{Z}} \mathbb{E}[X_k X_0] - s^2 \right| \lesssim \frac{1}{m},$$

which gives (iii) for large enough m . Since $cn^{3/4}$, we see that we may apply Theorem 4.2 which yields

$$(4.39) \qquad \sup_{x \in \mathbb{R}} A_2(m, n, x, \delta) \lesssim n^{-p/2+1}.$$

Next, we deal with $A_3(m, n, x, \delta)$. Properties of the Gaussian distribution function give

$$\sup_{x \in \mathbb{R}} A_3(m, n, x, \delta) \lesssim \delta + |s_n^2 - s_{nm}^2|.$$

However, by the Cauchy–Schwarz inequality and (4.32), it follows that

$$(4.40) \quad |s_n^2 - s_{nm}^2| \leq n^{-1} \|S_n^{(>m)}\|_2 \|S_n + S_n^{(\leq m)}\|_2 \lesssim m^{-4/3} \lesssim n^{-1},$$

and we thus conclude that

$$(4.41) \quad \sup_{x \in \mathbb{R}} A_3(m, n, x, \delta) \lesssim \delta + n^{-1}.$$

Finally, setting $\delta = n^{-1/2}$, standard arguments involving the Gaussian distribution function yield that

$$(4.42) \quad \sup_{x \in \mathbb{R}} A_0(x, \delta) \lesssim \delta = n^{-1/2}.$$

Piecing together (4.33), (4.39), (4.41) and (4.42), Lemma 4.3 yields

$$(4.43) \quad \sup_{x \in \mathbb{R}} |P(S_n/\sqrt{n} \leq x) - \Phi(x/s_n)| \lesssim n^{-p/2+1},$$

which completes the proof. \square

4.3. Proof of Theorem 2.3. Recall that

$$\Delta_n(x) = \left| P\left(S_n \leq x\sqrt{ns_n^2}\right) - \Phi(x) \right| \quad \text{and} \quad \Delta_n = \sup_{x \in \mathbb{R}} \Delta_n(x).$$

We first consider the case $q > 1$. Using Theorem 2.2, we have

$$(4.44) \quad \int_{\mathbb{R}} |\Delta_n(x)|^q dx \leq \Delta_n^{q-1} \int_{\mathbb{R}} |\Delta_n(x)| dx \lesssim n^{-(q-1)/2} \int_{\mathbb{R}} |\Delta_n(x)| dx.$$

In order to bound $\int_{\mathbb{R}} |\Delta_n(x)| dx$, we apply [13], Theorem 3.2, which will give us the bound

$$(4.45) \quad \int_{\mathbb{R}} |\Delta_n(x)| dx \lesssim \frac{1}{\sqrt{n}}.$$

To this end, we need to verify that

$$(4.46) \quad \begin{aligned} & \sum_{k>0} \left(\|X_0^2 \vee 1(\mathbb{E}[X_k^2 - \mathbb{E}[X_k^2]|\mathcal{E}_0])\|_1 \right. \\ & \left. + \frac{1}{k} \sum_{i=1}^k \|X_{-i} X_0 \mathbb{E}[X_k^2 - \mathbb{E}[X_k^2]|\mathcal{E}_0]\|_1 \right) < \infty \quad \text{and} \\ & \sum_{k>0} \frac{1}{k} \sum_{i=\lfloor k/2 \rfloor}^k \| |X_0| \vee 1 \mathbb{E}[X_i X_k^2 - \mathbb{E}[X_i X_k^2]|\mathcal{E}_0] \|_1 < \infty. \end{aligned}$$

Applying the Hölder, Jensen and triangle inequalities, we get

$$\begin{aligned} \|X_0^2 \vee 1(\mathbb{E}[X_k^2 - \mathbb{E}[X_k^2]|\mathcal{E}_0])\|_1 & \leq \|X_0^2 \vee 1\|_2 \|X_k - X_k^*\|_4 \|X_k + X_k^*\|_4 \\ & \lesssim k \|X_k - X_k'\|_4. \end{aligned}$$

Similarly, with $\mathcal{E}_{-i} = \sigma(\varepsilon_k, k \leq -i)$, we obtain that

$$\begin{aligned} \|X_{-i} X_0 \mathbb{E}[X_k^2 - \mathbb{E}[X_k^2] | \mathcal{E}_0]\|_1 &\lesssim \|X_{-i}\|_4 \|\mathbb{E}[X_0 | \mathcal{E}_{-i}]\|_4 k \|X_k - X'_k\|_4 \\ &\lesssim \|X_i - X_i^*\|_4 k \|X_k - X'_k\|_4 \\ &\lesssim i \|X_i - X'_i\|_4 k \|X_k - X'_k\|_4. \end{aligned}$$

In the same manner, we get that

$$\begin{aligned} \| |X_0| \vee 1 \mathbb{E}[X_i X_k^2 - \mathbb{E}[X_i X_k^2] | \mathcal{E}_0]\|_1 &\lesssim \|X_i - X_i^*\|_4 + \|X_k - X_k^*\|_4 \\ &\lesssim i \|X_i - X'_i\|_4 + k \|X_k - X'_k\|_4. \end{aligned}$$

Combining all three bounds, the validity of (4.46) follows, and hence (4.45). For (4.44), we thus obtain

$$\int_{\mathbb{R}} |\Delta_n(x)|^q dx \lesssim n^{-(q-1)/2-1/2} \lesssim n^{-q/2},$$

which completes the proof for $q > 1$. For $q = 1$, we may directly refer to [13], Theorem 3.2, using the above bounds.

4.4. *Proof of Theorem 2.4.* For the proof, we require the following result; cf. [41], Lemma 5.4.

LEMMA 4.4. *Let Y be a real-valued random variable. Put*

$$\tilde{\Delta} = \sup_{x \in \mathbb{R}} |P(Y \leq x) - \Phi(x)|,$$

and assume that $\|Y\|_q < \infty$ for $q > 0$ and $0 \leq \tilde{\Delta} \leq e^{-1/2}$. Then

$$|P(Y \leq x) - \Phi(x)| \leq \frac{c(q)\tilde{\Delta}(\log 1/\tilde{\Delta})^{q/2} + \lambda_q}{1 + |x|^q}$$

for all x , where $c(q)$ is a positive constant depending only on q , and

$$\lambda_q = \left| \int_{\mathbb{R}} |x|^q d\Phi(x) - \mathbb{E}[|Y|^q] \right|.$$

Consider first the case where $|x| \leq c_0 \sqrt{\log n}$, for $c_0 > 0$ large enough (see below). Then by the Markov inequality and Lemma 4.12, it follows that

$$(4.47) \quad \left| P\left(S_n \mathbb{1}(|S_n| \leq n) \leq x \sqrt{ns_n^2}\right) - P\left(S_n \leq x \sqrt{ns_n^2}\right) \right| \lesssim n^{-p/2}.$$

Combining Theorem 2.2 with (4.47) and Lemma 4.4, we see that it suffices to consider λ_p with $Y = S_n \mathbb{1}(|S_n| \leq n)$. Using again Theorem 2.2 together with (4.47), standard tail bounds for the Gaussian distribution and elementary computations give

$$(4.48) \quad \lambda_p \lesssim n^{-(p \wedge 3)/2+1} (\log n)^{p/2} + \int_{c_0 \sqrt{\log n}}^n x^{p-1} P(|S_n| \geq x \sqrt{ns_n^2}) dx.$$

According to a Fuk–Nagaev-type inequality for dependent sequences in [34], Theorem 2, if it holds that

$$(4.49) \quad \sum_{k=1}^{\infty} (k^{p/2-1} \|X_k - X'_k\|_p^p)^{1/(p+1)} < \infty,$$

then for large enough $c_0 > 0$ and $x \geq c_0\sqrt{\log n}$ we get

$$(4.50) \quad P(|S_n| \geq x\sqrt{ns_n^2}) \lesssim n^{-p/2+1}x^{-p},$$

and hence,

$$(4.51) \quad \int_{c_0\sqrt{\log n}}^n x^{p-1} P(|S_n| \geq x\sqrt{ns_n^2}) dx \lesssim n^{-p/2+1}(\log n)^{p/2} \log n.$$

However, setting $a_k = k^{-1/2-1/(3(p+1))}$, an application of the Cauchy–Schwarz inequality yields

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} (k^{p/2-1} \|X_k - X'_k\|_p^p)^{1/(p+1)} \right)^2 \\ & \leq \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} a_k^{-2} k^{(p-2)/(p+1)} \|X_k - X'_k\|_p^{2p/(p+1)} \\ & \lesssim \sum_{k=1}^{\infty} k^2 \|X_k - X'_k\|_p < \infty \end{aligned}$$

by Assumption 2.1. Hence (4.49) holds, and thus (4.50) and (4.51). To complete the proof, it remains to treat the case $|x| > c_0\sqrt{\log n}$. But in this case, we may directly appeal to (4.50) which gives the result.

4.5. Proof of main lemmas.

4.5.1. *Bounding conditional characteristic functions and variances.* Suppose we have a sequence of random variables $\{H_j\}_{1 \leq j \leq J}$ and a sequence of filtrations $\{\mathcal{H}_j\}_{1 \leq j \leq J}$, such that both $\{\mathbb{E}_{\mathcal{H}_j}[H_j^2]\}_{1 \leq j \leq J}$ and $\{\mathbb{E}_{\mathcal{H}_j}[|H_j|^p]\}_{1 \leq j \leq J}$ are independent sequences. Note that this does not necessarily mean that $\{H_j\}_{1 \leq j \leq J}$ is independent, and indeed this is not the case when we apply Lemma 4.5 in step 4 of the proof of part A. Introduce the conditional characteristic function

$$(4.52) \quad \varphi_j^{\mathcal{H}}(x) = \mathbb{E}[\exp(ixH_j)|\mathcal{H}_j].$$

Given the above conditions, we have the following result.

LEMMA 4.5. *Let $p > 2$, and assume that:*

- (i) $\mathbb{E}_{\mathcal{H}_j}[H_j] = 0$ uniformly for $j = 1, \dots, J$,

(ii) *there exists a $u^- > 0$ such that $P(\mathbb{E}_{\mathcal{H}_j}[H_j^2] \leq u^-) < 1/7$ uniformly for $j = 1, \dots, J$,*

(iii) $\mathbb{E}[|H_j|^p] \leq c_1 < \infty$ uniformly for $j = 1, \dots, J$.

Then there exist constants $c_{\varphi,1}, c_{\varphi,2} > 0$, only depending on u^-, c_1 and p , such that

$$\mathbb{E} \left[\prod_{j=1}^J |\varphi_j^{\mathcal{H}}(x)| \right] \lesssim e^{-c_{\varphi,1}x^2J} + e^{-\sqrt{J/4} \log 8/7} \quad \text{for } x^2 \leq c_{\varphi,2}.$$

PROOF. Let

$$I(s, x) = \mathbb{E}_{\mathcal{H}_j} [H_j^2 ((\cos(sxH_j) - \cos(0)) + i(\sin(sxH_j) - \sin(0)))].$$

Using a Taylor expansion and writing $e^{ix} = \cos(x) + i \sin(x)$, we obtain that

$$\mathbb{E}_{\mathcal{H}_j} [e^{ixH_j}] = 1 - \mathbb{E}_{\mathcal{H}_j} [H_j^2]x^2/2 + x^2/2 \int_0^1 (1-s)I(s, x) ds.$$

Using the Lipschitz property of $\cos(y)$ and $\sin(y)$, it follows that

$$(4.53) \quad |I(s, x)| \leq 2\mathbb{E}_{\mathcal{H}_j} [H_j^2|xh| + 2H_j^2\mathbb{1}(|H_j| \geq h)], \quad h > 0.$$

For $h > 0$ we have from the Markov inequality

$$\begin{aligned} \mathbb{E}_{\mathcal{H}_j} [H_j^2\mathbb{1}(|H_j| \geq h)] &\leq 2 \int_h^\infty x P_{\mathcal{H}_j}(|H_j| \geq x) dx + h^2 P_{\mathcal{H}_j}(|H_j| \geq h) \\ &\leq 2h^{-p+2} \int_0^\infty x^{p-1} P_{\mathcal{H}_j}(|H_j| \geq x) dx + h^2 P_{\mathcal{H}_j}(|H_j| \geq h) \\ &\leq \frac{2+p}{p} h^{-p+2} \mathbb{E}_{\mathcal{H}_j} [|H_j|^p] < h^{-p+2} \mathbb{E}_{\mathcal{H}_j} [|H_j|^p]. \end{aligned}$$

We thus conclude from (4.53) that

$$|I(s, x)| \leq 2\mathbb{E}_{\mathcal{H}_j} [H_j^2]|xh| + 4h^{-p+2}\mathbb{E}_{\mathcal{H}_j} [|H_j|^p].$$

This gives us

$$(4.54) \quad \begin{aligned} &|\mathbb{E}_{\mathcal{H}_j} [e^{ixH_j}] - 1 + \mathbb{E}_{\mathcal{H}_j} [H_j^2]x^2/2| \\ &\leq \mathbb{E}_{\mathcal{H}_j} [H_j^2]h|x|^3 + 2h^{-p+2}x^2\mathbb{E}_{\mathcal{H}_j} [|H_j|^p]. \end{aligned}$$

Let $\mathcal{I} = \{1, \dots, J\}$, and put $\sigma_j^{\mathcal{H}} = \mathbb{E}[H_j^2|\mathcal{H}_j]$ and $\rho_j^{\mathcal{H}} = \mathbb{E}[|H_j|^p|\mathcal{H}_j]$. Consider

$$\rho_{1,J}^{\mathcal{H}} \geq \rho_{2,J}^{\mathcal{H}} \geq \dots \geq \rho_{J,J}^{\mathcal{H}},$$

where $\rho_{j,J}^{\mathcal{H}}$ denotes the j th largest random variable for $1 \leq j \leq J$. Let $E_j, j = 1, \dots, J$ denote i.i.d. unit exponential random variables, and denote with $E_{j,J}$ the j th largest. Further, denote with $F_{\rho_j}(\cdot)$ the c.d.f. of $\rho_j^{\mathcal{H}}, j = 1, \dots, J$, and with

$F_\rho(\cdot) = \min_{1 \leq j \leq J} F_{\rho_j}(\cdot)$. Using the transformation $-\log(1 - F_{\rho_j}(\cdot))$, we thus obtain

$$(4.55) \quad \begin{aligned} & \{\rho_j^{\mathcal{H}} \leq x_j : 1 \leq j \leq J\} \\ & \stackrel{d}{=} \{E_j \leq -\log(1 - F_{\rho_j}(x_j)) : 1 \leq j \leq J\}, \quad x_j \in \mathbb{R}, \end{aligned}$$

which is the well-known Renyi representation; cf. [10, 44]. In particular, by the construction of $F_\rho(\cdot)$ it follows that

$$P(\rho_{J/2,J}^{\mathcal{H}} \leq u^+) \geq P(E_{J/2,J} \leq -\log(1 - F_\rho(u^+)))$$

for $0 \leq u^+ < \infty$. Let $u_{\mathcal{H}}^+ = -\log(1 - F_\rho(u^+))$ and $u_{\mathcal{H}}^+(J) = \sqrt{J/2}(u_{\mathcal{H}}^+ - \log 2)$. We wish to find a u^+ such that $u_{\mathcal{H}}^+ > \log 2$. This is implied by $F_\rho(u^+) > 1/2$. We will now construct such an u^+ . Since

$$c_1 \geq \mathbb{E}[|\rho_j^{\mathcal{H}}|] = \int_0^\infty (1 - F_{\rho_j}(x)) dx \geq c_2 P(\rho_j^{\mathcal{H}} \geq c_2) \quad \text{for } c_2 > 0,$$

it follows that $c_1/c_2 \geq 1 - P(\rho_j^{\mathcal{H}} < c_2)$. Hence choosing $u^+ = c_2 = 4c_1$, we obtain $F_{\rho_j}(u^+) \geq 3/4$ and hence $F_\rho(u^+) \geq 3/4$, which leads to $u_{\mathcal{H}}^+ \geq \sqrt{J/2} \log 2$. Thus by known properties of exponential order statistics (cf. [10, 17]), we have

$$\begin{aligned} P(E_{J/2,J} \leq u_{\mathcal{H}}^+) &= P(\sqrt{J/2}(E_{J/2,J} - \log 2) \leq \sqrt{J/2}(u_{\mathcal{H}}^+ - \log 2)) \\ &= 1 - P(\sqrt{J/2}(E_{J/2,J} - \log 2) > u_{\mathcal{H}}^+(J)) \\ &\geq 1 - \mathbb{E}[e^{\sqrt{J/2}(E_{J/2,J} - \log 2)}] e^{-u_{\mathcal{H}}^+(J)} \geq 1 - \mathcal{O}(e^{-\sqrt{J/2} \log 2}), \end{aligned}$$

for sufficiently large J . We thus conclude that

$$(4.56) \quad P(\rho_{J/2,J}^{\mathcal{H}} \leq u^+) \geq 1 - \mathcal{O}(e^{-\sqrt{J/2} \log 2}).$$

Let us denote this set with $\mathcal{A}^+ = \{\rho_{J/2,J}^{\mathcal{H}} \leq u^+\}$, and put $\mathcal{I}_{\mathcal{A}^+}^+ = \{1 \leq j \leq J : \rho_j^{\mathcal{H}} \leq u^+\}$. Note that the index set $\mathcal{I}_{\mathcal{A}^+}^+$ has at least cardinality $J/2$ given event \mathcal{A}^+ . For the sake of simplicity, let us assume that $|\mathcal{I}_{\mathcal{A}^+}^+| = J/2$, which, as is clear from the arguments below, has no impact on our results. Let us introduce

$$\sigma_{1,J/2}^{\mathcal{H},\diamond} \geq \sigma_{2,J/2}^{\mathcal{H},\diamond} \geq \dots \geq \sigma_{J/2,J/2}^{\mathcal{H},\diamond}$$

the order statistics of $\sigma_j^{\mathcal{H}}$ within the index set $\mathcal{I}_{\mathcal{A}^+}^+$. This means that $\sigma_{J/2,J/2}^{\mathcal{H},\diamond}$ is not necessarily the smallest value of $\sigma_j^{\mathcal{H}}$, $1 \leq j \leq J$. More generally, it holds that

$$(4.57) \quad \sigma_{j,J/2}^{\mathcal{H},\diamond} \geq \sigma_{J/2+j,J}^{\mathcal{H}}, \quad j \in \{1, \dots, J/2\}.$$

Now, similar to as before, let $F_{\sigma_j}(\cdot)$ be the c.d.f. of $\sigma_j^{\mathcal{H}}$, $j = 1, \dots, J$, and put $F_\sigma(\cdot) = \max_{1 \leq j \leq J} F_{\sigma_j}(\cdot)$, $u_{\mathcal{H}}^- = -\log(1 - F_\sigma(u^-))$ and $u_{\mathcal{H}}^-(J) = \sqrt{J/4}(\log 4/$

$3 - u_{\mathcal{H}}^-)$ for some $0 \leq u^- \leq u^+$. We search for a $u^- > 0$ such that $u_{\mathcal{H}}^- < \log 7/6$, which is true if $\max_{1 \leq j \leq J} F_{\sigma_j}(u^-) < 1/7$. However, this is precisely what we demanded in the assumptions. Then proceeding as before, we have

$$\begin{aligned} P(E_{3J/4,J} \geq u_{\mathcal{H}}^-) &= P(\sqrt{J/4}(E_{3J/4,J} - \log 4/3) \geq \sqrt{J/4}(u_{\mathcal{H}}^- - \log 4/3)) \\ &= 1 - P(\sqrt{J/4}(\log 4/3 - E_{3J/4,J}) > u_{\mathcal{H}}^-(J)) \\ &\geq 1 - \mathbb{E}[e^{\sqrt{J/4}(\log 4/3 - E_{3J/4,J})}]e^{-u_{\mathcal{H}}^-(J)} \\ &\geq 1 - \mathcal{O}(e^{-\sqrt{J/4}\log 8/7}). \end{aligned}$$

We thus conclude from (4.57) and the construction of $F_{\sigma}(\cdot)$ that

$$\begin{aligned} (4.58) \quad P(\sigma_{J/4,J/2}^{\mathcal{H},\diamond} \geq u^-) &\geq P(\sigma_{3J/4,J}^{\mathcal{H}} \geq u^-) \geq P(E_{3J/4,J} \geq u_{\mathcal{H}}^-) \\ &\geq 1 - \mathcal{O}(e^{-\sqrt{J/4}\log 8/7}). \end{aligned}$$

Put $\mathcal{I}_{\mathcal{A}} = \{j \in \mathcal{I}_{\mathcal{A}}^+ : \sigma_j^{\mathcal{H},\diamond} \geq u^-\}$. Combining (4.56) and (4.58) we obtain

$$(4.59) \quad P(\{\sigma_{J/4,J/2}^{\mathcal{H},\diamond} \geq u^-\} \cap \{\rho_{J/2,J}^{\mathcal{H}} \leq u^+\}) \geq 1 - \mathcal{O}(e^{-\sqrt{J/4}\log 8/7}).$$

We denote this set with $\mathcal{A} = \{\sigma_{J/4,J/2}^{\mathcal{H},\diamond} \geq u^-\} \cap \{\rho_{J/2,J}^{\mathcal{H}} \leq u^+\}$. Also note that by the (conditional) Lyapunov inequality, we have

$$(4.60) \quad \rho_{j,J}^{\mathcal{H}} \geq (\sigma_{j,J}^{\mathcal{H}})^{p/2}.$$

Note that $|\mathcal{I}_{\mathcal{A}}| \geq J/4$ on the event \mathcal{A} , and, by the above, we get

$$(4.61) \quad \rho_j^{\mathcal{H}} \leq u^+ \quad \text{and} \quad u^- \leq \sigma_j^{\mathcal{H}} \leq (u^+)^{p/2} \quad \text{for } j \in \mathcal{I}_{\mathcal{A}}.$$

Using (4.54), this implies that for every $j \in \mathcal{I}_{\mathcal{A}}$, we have

$$|\mathbb{E}_{|\mathcal{H}_j}[e^{ixH_j}] - 1 + \mathbb{E}_{|\mathcal{H}_j}[H_j^2]x^2/2| \leq (u^+)^{p/2}h|x|^3 + 2h^{-p+2}u^+x^2.$$

Hence, if $(u^+)^{p/2}x^2/2 < 1$ and $h = x^{-1/(p-1)}$, we conclude from the above and the triangle inequality that for $j \in \mathcal{I}_{\mathcal{A}}$,

$$(4.62) \quad |\varphi_j^{\mathcal{H}}(x)| < 1 - u^-x^2/2 + (2u^+ + (u^+)^{p/2})|x|^{2+\delta(p)} \quad \text{for } (u^+)^{p/2}x^2/2 < 1,$$

where $\delta(p) = (p - 2)/(p - 1) > 0$. Since $0 < u^-, u^+ < \infty$ and $\delta(p) > 0$, there exist absolute constants $0 < c_{\varphi,1}, c_{\varphi,2}$, chosen sufficiently small, such that

$$(4.63) \quad u(x) \stackrel{\text{def}}{=} u^-x^2/2 - (2u^+ + (u^+)^{p/2})|x|^{2+\delta(p)} \geq 8c_{\varphi,1}x^2 \quad \text{for } x^2 \leq c_{\varphi,2}.$$

Next, observe that

$$\begin{aligned}
 \mathbb{E} \left[\left| \prod_{j=1}^J \varphi_j^{\mathcal{H}}(x) \right| \right] &= \mathbb{E} \left[\left| \prod_{j=1}^J \varphi_j^{\mathcal{H}}(x) \right| (\mathbb{1}(\mathcal{A}) + \mathbb{1}(\mathcal{A}^c)) \right] \\
 (4.64) \qquad &\leq P(\mathcal{A}^c) + \mathbb{E} \left[\left| \prod_{j \in \mathcal{I}} \varphi_j^{\mathcal{H}}(x) \right| \mathbb{1}(\mathcal{A}) \right] \\
 &\leq P(\mathcal{A}^c) + \mathbb{E} \left[\left| \prod_{j \in \mathcal{I}_{\mathcal{A}}} \varphi_j^{\mathcal{H}}(x) \right| \mathbb{1}(\mathcal{A}) \right].
 \end{aligned}$$

Moreover, using (4.62) and (4.63) and since $|\mathcal{I}_{\mathcal{A}}| = J/4$ on \mathcal{A} , it follows that

$$\begin{aligned}
 \mathbb{E} \left[\left| \prod_{j \in \mathcal{I}_{\mathcal{A}}} \varphi_j^{\mathcal{H}}(x) \right| \mathbb{1}(\mathcal{A}) \right] &\leq \mathbb{E} \left[\left| \prod_{j \in \mathcal{I}_{\mathcal{A}}} (1 - u(x)) \right| \mathbb{1}(\mathcal{A}) \right] \\
 &\leq \mathbb{E} \left[\prod_{j \in \mathcal{I}_{\mathcal{A}}} e^{-u(x)} \mathbb{1}(\mathcal{A}) \right] \leq e^{-u(x)J/8} \leq e^{-c_{\varphi,1}Jx^2}.
 \end{aligned}$$

Hence we conclude from the above and (4.59) that

$$\mathbb{E} \left[\left| \prod_{j \in \mathcal{I}} \varphi_j^{\mathcal{H}}(x) \right| \right] \lesssim e^{-c_{\varphi,1}x^2J} + e^{-\sqrt{J/4} \log 8/7} \quad \text{for } x^2 \leq c_{\varphi,2},$$

which yields the claim. \square

LEMMA 4.6. *Grant Assumption 4.1. Then $\sum_{k=1}^{\infty} k|\mathbb{E}[X_0X_k]| < \infty$ and $\widehat{\sigma}_m^2 = s_m^2/2 + \mathcal{O}(m^{-1})$. Moreover, we have $\widehat{\sigma}_l^2 = s_m^2/2 + \mathcal{O}(1)$ as $l \rightarrow m$.*

PROOF. Since $\mathbb{E}[X_k|\mathcal{E}_0] = \mathbb{E}[X_k - X_k^*|\mathcal{E}_0]$, the Cauchy-Schwarz and Jensen inequalities imply

$$\begin{aligned}
 \sum_{k=0}^{\infty} |\mathbb{E}[X_0X_k]| &\leq \|X_0\|_2 \sum_{k=0}^{\infty} \|\mathbb{E}[X_k|\mathcal{E}_0]\|_2 \leq \|X_0\|_2 \sum_{k=0}^{\infty} \|X_k - X_k^*\|_2 \\
 &\leq \|X_0\|_2 \sum_{k=1}^{\infty} k^2 \|X_k - X_k'\|_2 < \infty.
 \end{aligned}$$

The decomposition $\widehat{\sigma}_m^2 = s_m^2/2 + \mathcal{O}(m^{-1})$ now follows from (4.2). Claim $\widehat{\sigma}_l^2 = s_m^2/2 + \mathcal{O}(1)$ as $l \rightarrow m$ readily follows from the previous computations. \square

LEMMA 4.7. *Grant Assumption 4.1. Then:*

- (i) $\|\sigma_{j|m}^2 - \sigma_j^2\|_{p/2} \lesssim \|\sigma_{j|m}^2 - \widehat{\sigma}_m^2\|_{p/2} + m^{-1} \lesssim m^{-1}$ for $1 \leq j \leq N$,
- (ii) $\sigma_j^2 = \widehat{\sigma}_m^2 + \mathcal{O}(m^{-1})$ for $1 \leq j \leq N$,
- (iii) $\|\sigma_{|m}^2 - \overline{\sigma}_m^2\|_{p/2} \lesssim n^{-1}N^{2/p}$.

PROOF. We first show (i). Without loss of generality, we may assume $j = 1$, since $m \sim m'$. To lighten the notation, we use $R_1 = R_1^{(1)}$. We will first establish that $\|\sigma_{j|m}^2 - \hat{\sigma}_m^2\|_{p/2} \lesssim m^{-1}$. We have that

$$2m(\sigma_{1|m}^2 - \hat{\sigma}_m^2) = \mathbb{E}_{\mathbb{F}_m} \left[\left(\sum_{k=1}^m (X_k^{(**)} + (X_k - X_k^{(**)}) - \mathbb{E}_{\mathbb{F}_m}[X_k]) + R_1 \right)^2 \right] - 2m\hat{\sigma}_m^2.$$

By squaring out the first expression, we obtain a sum of square terms and a sum of mixed terms. Let us first treat the mixed terms, which are

$$\begin{aligned} & 2 \sum_{k=1}^m \sum_{l=1}^m \mathbb{E}_{\mathbb{F}_m} [X_k^{(**)}(X_l - X_l^{(**)}) + X_k^{(**)}\mathbb{E}_{\mathbb{F}_m}[X_l] + \mathbb{E}_{\mathbb{F}_m}[X_k](X_l - X_l^{(**)})] \\ & + 2 \sum_{k=1}^m \mathbb{E}_{\mathbb{F}_m} [R_1 X_k^{(**)} + R_1(X_k - X_k^{(**)}) + R_1\mathbb{E}_{\mathbb{F}_m}[X_k]] \\ & = I_m + II_m + III_m + IV_m + V_m + VI_m. \end{aligned}$$

We will handle all these terms separately.

Case I_m : We have

$$\begin{aligned} I_m/2 &= \sum_{l=1}^m \sum_{k=l}^m (\dots) + \sum_{l=1}^m \sum_{k=1}^{l-1} (\dots) \\ &= \sum_{l=1}^m \sum_{k=l}^m \mathbb{E}_{\mathbb{F}_m} [(X_l - X_l^{(**)})\mathbb{E}[X_k^{(**)} | \sigma(\mathbb{F}_m, \mathcal{E}_l, \mathcal{E}_l^{(**)})]] \\ &\quad + \sum_{l=1}^m \sum_{k=1}^{l-1} \mathbb{E}_{\mathbb{F}_m} [X_k^{(**)}(X_l - X_l^{(**)})]. \end{aligned}$$

Since

$$\mathbb{E}[X_k^{(**)} | \sigma(\mathbb{F}_m, \mathcal{E}_l, \mathcal{E}_l^{(**)})] \stackrel{d}{=} \mathbb{E}[X_k | \mathcal{E}_l] = \mathbb{E}[X_k - X_k^{(k-l,*)} | \mathcal{E}_l],$$

the Cauchy–Schwarz (with respect to $\mathbb{E}_{\mathbb{F}_m}$) and Jensen inequalities thus yield

$$\begin{aligned} \|I_m\|_{p/2} &\leq 2 \sum_{l=1}^m \sum_{k=l}^m \|X_l - X_l^{(**)}\|_p \|X_k - X_k^{(k-l,*)}\|_p \\ &\quad + 2 \sum_{l=1}^m \sum_{k=1}^{l-1} \|X_k^{(**)}\|_p \|X_l - X_l^{(**)}\|_p \\ &\leq 2 \left(\sum_{l=1}^{\infty} l \|X_l - X_l'\|_p \right)^2 + 2 \sum_{l=1}^{\infty} l^2 \|X_l - X_l'\|_p \|X_1\|_p < \infty. \end{aligned}$$

Case II_m : Since $\mathbb{E}_{\mathbb{F}_m}[X_k^{(**)}] = \mathbb{E}[X_k] = 0$ ($k \leq m$) it follows that $II_m = 0$.

Case III_m : It follows via the Jensen and triangle inequalities that

$$(4.65) \quad \begin{aligned} \|\mathbb{E}[X_l|\mathbb{F}_m]\|_p &= \|\mathbb{E}[X_l - X_l^{(**)}|\mathbb{F}_m]\|_p \\ &\leq \|X_l - X_l^{(**)}\|_p \leq \sum_{j=l}^{\infty} \|X_j - X'_j\|_p. \end{aligned}$$

The Cauchy–Schwarz (with respect to $\mathbb{E}_{\mathbb{F}_m}$) and Jensen inequalities then give

$$\|III_m\|_{p/2} \leq 2 \left(\sum_{l=1}^m \sum_{j=l}^{\infty} \|X_j - X'_j\|_p \right)^2 \leq 2 \left(\sum_{l=1}^{\infty} l \|X_l - X'_l\|_p \right)^2 < \infty.$$

Case IV_m : Note that $X_k^{(**)}$ and $X_l^{(l-k,*)}$ are independent for $1 \leq k \leq m$ and $m + 1 \leq l \leq 2m$. Hence since $\mathbb{E}_{\mathbb{F}_m}[X_k^{(**)}] = 0$, we have

$$\begin{aligned} \sum_{k=1}^m \mathbb{E}_{\mathbb{F}_m}[R_1 X_k^{(**)}] &= \sum_{k=1}^m \sum_{l=m+1}^{2m} \mathbb{E}_{\mathbb{F}_m}[X_k^{(**)} X_l] \\ &= \sum_{k=1}^m \sum_{l=m+1}^{2m} \mathbb{E}_{\mathbb{F}_m}[X_k^{(**)} (X_l - X_l^{(l-k,*)} + X_l^{(l-k,*)})] \\ &= \sum_{k=1}^m \sum_{l=m+1}^{2m} \mathbb{E}_{\mathbb{F}_m}[X_k^{(**)} (X_l - X_l^{(l-k,*)})]. \end{aligned}$$

The Cauchy–Schwarz (with respect to $\mathbb{E}_{\mathbb{F}_m}$) and Jensen inequalities then yield

$$\begin{aligned} \|IV_m\|_{p/2} &\leq 2 \sum_{k=1}^m \sum_{l=m+1}^{2m} \|X_k^{(**)}\|_p \|X_l - X_l^{(l-k,*)}\|_p \\ &\leq 2 \sum_{k=1}^m \sum_{l=m+1}^{2m} \sum_{j=l-k}^{\infty} \|X_j - X'_j\|_p \|X_1\|_p \\ &\leq 2 \sum_{k=1}^m \sum_{j=m-k}^{\infty} (j - m + k) \|X_j - X'_j\|_p \|X_1\|_p \\ &\leq 2 \sum_{j=1}^{\infty} j^2 \|X_j - X'_j\|_p \|X_1\|_p < \infty. \end{aligned}$$

Case V_m : The Cauchy–Schwarz (with respect to $\mathbb{E}_{\mathbb{F}_m}$) and Jensen inequalities yield

$$\begin{aligned} \|V_m\|_{p/2} &\leq 2 \sum_{k=1}^m \|X_k - X_k^{(**)}\|_p \|R_1\|_p \\ &\leq \sum_{l=1}^{\infty} l^2 \|X_l - X'_l\|_p \|R_1\|_p < \infty, \end{aligned}$$

since $\|R_1\|_p < \infty$ by Lemma 4.13.

Case VI_m : Proceeding as above and using (4.65), we get $\|VI_m\|_{p/2} < \infty$. It thus remains to deal with the squared terms, which are

$$\begin{aligned} & \sum_{k=1}^m \sum_{l=1}^m \mathbb{E}_{\mathbb{F}_m} [X_k^{(**)} X_l^{(**)} + (X_k - X_k^{(**)})(X_l - X_l^{(**)}) \\ & \qquad \qquad \qquad + \mathbb{E}_{\mathbb{F}_m} [X_k] \mathbb{E}_{\mathbb{F}_m} [X_l] + \mathbb{E}_{\mathbb{F}_m} [R_1^2]] \\ & = 2m\widehat{\sigma}_m^2 + VII_m + VIII_m + IX_m. \end{aligned}$$

However, using the results from the previous computations and Lemma 4.13, one readily deduces that

$$(4.66) \quad \|VII_m\|_{p/2} < \infty, \quad \|VIII_m\|_{p/2} < \infty, \quad \|IX_m\|_{p/2} < \infty.$$

Piecing everything together, we have established that $\|\sigma_{j|m}^2 - \widehat{\sigma}_m^2\|_{p/2} \lesssim m^{-1}$. However, from the above arguments one readily deduces that $\sigma_j^2 = \widehat{\sigma}_m^2 + \mathcal{O}(m^{-1})$, and hence (i) and (ii) follow. We now treat (iii). Since $\{Y_j^{(1)}\}_{1 \leq j \leq N}$ is an independent sequence under $P_{\mathbb{F}_m}$, we have

$$(4.67) \quad \sigma_{|m}^2 = N^{-1} \sum_{j=1}^N \sigma_{j|m}^2.$$

Let $\mathcal{I} = \{1, 3, 5, \dots\}$ and $\mathcal{J} = \{2, 4, 6, \dots\}$ such that $\mathcal{I} \cup \mathcal{J} = \{1, 2, \dots, N\}$. Then

$$\left\| \sum_{j=1}^N \sigma_{j|m}^2 - \sigma_j^2 \right\|_{p/2} \leq \left\| \sum_{j \in \mathcal{I}} \sigma_{j|m}^2 - \sigma_j^2 \right\|_{p/2} + \left\| \sum_{j \in \mathcal{J}} \sigma_{j|m}^2 - \sigma_j^2 \right\|_{p/2}.$$

Note that $\{\sigma_{j|m}^2\}_{j \in \mathcal{I}}$ is a sequence of independent random variables, and the same is true for $\{\sigma_{j|m}^2\}_{j \in \mathcal{J}}$. Then by Lemma 4.12, it follows that

$$\left\| \sum_{j=1}^N \sigma_{j|m}^2 - \sigma_j^2 \right\|_{p/2} \lesssim N^{2/p} \|\sigma_{j|m}^2 - \sigma_j^2\|_{p/2} \quad \text{for } p \in (2, 3],$$

which by (i) is of the magnitude $\mathcal{O}(N^{2/p} m^{-1})$. Hence we conclude from (4.67) that

$$\|\sigma_{|m}^2 - \bar{\sigma}_m^2\|_{p/2} \lesssim n^{-1} N^{2/p}. \quad \square$$

COROLLARY 4.8. *Grant Assumption 4.1. Let $\mathcal{B} = \{\sigma_{|m}^2 \geq s_m^2/4\}$. Then*

$$P(\mathcal{B}^c) \lesssim n^{-p/2} N.$$

PROOF. By Markov’s inequality, Lemma 4.6 and Lemma 4.7, it follows that for large enough m

$$P(\mathcal{B}^c) \leq P(|\sigma_{|m}^2 - \bar{\sigma}_m^2| \geq s_m^2/4 - \mathcal{O}(m^{-1})) \lesssim s_m^{-p/2} n^{-p/2} N \lesssim n^{-p/2} N$$

since $s_m^2 > 0$ by Assumption 4.1(iii). \square

4.5.2. *Ideal metrics and applications.* The aim of this section is to give a proof for the inequality

$$(4.68) \quad \|\varphi_i(t) - e^{-\sigma_{i|m}^2 t^2/2}\|_1 \lesssim |t|^p m^{-p/2+1}, \quad i \in \{1, N\}$$

in Corollary 4.11. We will achieve this by employing ideal metrics. Let $s > 0$. Then we can represent s as $s = m + \alpha$, where $[s] = m$ denotes the integer part, and $0 < \alpha \leq 1$. Let \mathfrak{F}_s be the class of all real-valued functions f , such that the m th derivative exists and satisfies

$$(4.69) \quad |f^{(m)}(x) - f^{(m)}(y)| \leq |x - y|^\alpha.$$

Note that since $\cos(y), \sin(y)$ are bounded in absolute value and are Lipschitz continuous, it follows that up to some finite constant $c(\alpha) > 0$ we have $\sin(y), \cos(y) \in \mathfrak{F}_s$ for any $s > 0$. As already mentioned in step 2 of the proof of part A, we will make use of some special ideal-metrics ζ_s (Zolotarev metric). For two probability measures P, Q , the metric ζ_s is defined as

$$\zeta_s(P, Q) = \sup \left\{ \left| \int f(x)(P - Q)(dx) \right| : f \in \mathfrak{F}_s \right\}.$$

The metric $\zeta_s(P, Q)$ has the nice property of homogeneity. For random variables X, Y , induced probability measures P_{cX}, P_{cY} and constant $c > 0$, this means that $\zeta_s(P_{cX}, P_{cY}) = |c|^s \zeta_s(P_X, P_Y)$. We require some further notation. For $1 \leq j \leq N$, put

$$S_{j|m} = \frac{1}{\sqrt{2m}} \sum_{k=(2j-2)m+1}^{(2j-1)m} X_k \quad \text{and} \quad S_{j|m}^{(**)} = \frac{1}{\sqrt{2m}} \sum_{k=(2j-2)m+1}^{(2j-1)m} X_k^{((2j-2)m, **)}.$$

Note that $S_{j|m}^{(**)}$ is independent of \mathbb{F}_m , and hence $\widehat{\sigma}_m^2 = \mathbb{E}_{\mathbb{F}_m}[(S_{j|m}^{(**)})^2]$. Let $\{Z_j\}_{1 \leq j \leq N}$ be a sequence of zero mean, standard i.i.d. Gaussian random variables. In addition, let

$$\eta_{j|m}^2 = \frac{1}{\sqrt{2m}} \mathbb{E}_{\mathbb{F}_m}[(Y_j^{(1)})^2] / \widehat{\sigma}_m^2 = \sigma_{j|m}^2 / \widehat{\sigma}_m^2 \quad \text{for } 1 \leq j \leq N - 1,$$

and $\eta_{m'|m}^2 = \sigma_{m'|m}^2 / \widehat{\sigma}_{m'}^2$, for $j = N$. As first step toward (4.68), we have the following result.

LEMMA 4.9. *Grant Assumption 4.1. Then for $f(x) \in \{\cos(x), \sin(x)\}$, it holds that*

$$\|\mathbb{E}_{\mathbb{F}_m}[f(x(2m)^{-1/2}Y_j^{(1)}) - f(xS_{j|m}^{(**)}\eta_{j|m})]\|_1 \lesssim m^{-p/2+1}|x|^p \quad \text{if } p \in (2, 3],$$

where $j = 1, \dots, N$.

PROOF. To lighten the notation, we use $Y_j = (2m)^{-1/2}Y_j^{(1)}$ and $S_j = S_{j|m}^{(**)}$ in the following. Using Taylor expansion, we have

$$(4.70) \quad f(y) = f(0) + yf'(0) + y^2 f''(0)/2 + y^2 \int_0^1 (1-t)(f''(ty) - f''(0)) dt.$$

Note $\mathbb{E}_{\mathbb{F}_m}[Y_j] = 0$ and $\mathbb{E}_{\mathbb{F}_m}[S_j \eta_{j|m}] = \eta_{j|m} \mathbb{E}[S_j] = 0$. Moreover, since $\sigma_{j|m}^2 = \eta_{j|m}^2 \mathbb{E}_{\mathbb{F}_m}[S_j^2]$ by construction, we obtain from (4.70) that

$$\begin{aligned} & \mathbb{E}_{\mathbb{F}_m}[f(xY_j) - f(xS_j \eta_{j|m})] \\ &= x^2 \int_0^1 (1-t) \mathbb{E}_{\mathbb{F}_m}[Y_j^2 (f''(txY_j) - f''(0)) \\ & \quad - (S_j \eta_{j|m})^2 (f''(txS_j \eta_{j|m}) - f''(0))] / 2 dt \\ & \stackrel{\text{def}}{=} x^2 I_m(x). \end{aligned}$$

We have

$$Y_j^2 - S_j^2 \eta_{j|m}^2 = Y_j^2 - S_j^2 + S_j^2 \widehat{\sigma}_m^{-2} (\sigma_{j|m}^2 - \widehat{\sigma}_m^2),$$

where we recall that S_j and $\sigma_{j|m}$ are independent. Using the Jensen, triangle and Hölder inequalities and $f/2 \in \mathfrak{F}_p$, it follows that

$$\begin{aligned} & \|\mathbb{E}_{\mathbb{F}_m}[(Y_j^2 - (S_j \eta_{j|m})^2)(f''(txY_j) - f''(0))]\|_1 \\ & \leq 2 \|(Y_j^2 - (S_j \eta_{j|m})^2) |txY_j|^{p-2}\|_1 \\ & \leq 2 \|Y_j^2 - (S_j \eta_{j|m})^2\|_{p/2} \| |txY_j|^{p-2} \|_{p/(p-2)} \\ & \lesssim 2 \| |txY_j|^{p-2} \|_{p/(p-2)} \\ & \quad \times (\|Y_j - S_j\|_p \|Y_j + S_j\|_p + \|S_j^2\|_{p/2} \|\sigma_{j|m}^2 - \widehat{\sigma}_m^2\|_{p/2}), \end{aligned}$$

where we used that $\widehat{\sigma}_m > 0$ for large enough m . By Lemmas 4.13 and 4.7, this is of magnitude $\mathcal{O}(m^{-1/2}|tx|^{p-2})$. Hence by adding and subtracting $f''(txY_j)$ and using similar arguments as before, we obtain from the above

$$\begin{aligned} & x^2 \|I_m(x)\|_1 \\ & \lesssim m^{-1/2} |x|^p + x^2 \int_0^1 (1-t) \|\mathbb{E}_{\mathbb{F}_m}[S_j^2 \eta_{j|m}^2 (f''(txY_j) - f''(txS_j))]\|_1 dt \\ (4.71) \quad & \lesssim m^{-1/2} |x|^p + |x|^p \|S_j^2\|_{p/2} \|\eta_{j|m}^2\|_{p/2} \| |S_j - Y_j|^{p-2} \|_{p/(p-2)} \\ & \lesssim m^{-1/2} |x|^p + m^{-(p-2)/2} |x|^p \lesssim m^{-p/2+1} |x|^p, \end{aligned}$$

where we use that S_j and $\eta_{j|m} = \widehat{\sigma}_{j|m} / \widehat{\sigma}_m$ are independent. This gives the desired result. \square

As next step toward (4.68), we have the following.

LEMMA 4.10. *Grant Assumption 4.1. Then for $f(x) \in \{\cos(x), \sin(x)\}$, it holds that*

$$\|\mathbb{E}_{\mathbb{F}_m}[f(xZ_j\sigma_{j|m}) - f(xS_{j|m}^{(**)}\eta_{j|m})]\|_1 \lesssim |x|^p m^{-p/2+1} \quad \text{if } p \in (2, 3],$$

where $j = 1, \dots, N$.

PROOF. To increase the readability, we use the abbreviations $\hat{\sigma} = \hat{\sigma}_m$ and $S_j = S_{j|m}^{(**)}$ in the following. The main objective is to transfer the problem to the setup in [11] and apply the corresponding results. To this end, we first perform some necessary preparatory computations. We have that

$$(4.72) \quad \sum_{k>l} \|\mathbb{E}[X_k|\mathcal{E}_0]\|_p \leq \sum_{k>l} \|X_k - X_k^*\|_p \leq \sum_{k>l} k \|X_k - X'_k\|_p \rightarrow 0$$

as $l \rightarrow \infty$, hence it follows that

$$(4.73) \quad \sum_{k=0}^m \mathbb{E}[X_k|\mathcal{E}_0] \quad \text{converges in } \|\cdot\|_p.$$

Next, note that

$$(4.74) \quad 2m\hat{\sigma}^2 = \mathbb{E}_{\mathcal{E}_0} \left[\left(\sum_{k=1}^m X_k^* \right)^2 \right].$$

Using exactly the same arguments as in the proof of Lemma 4.7 (the present situation is much simpler), we get that

$$\left\| \mathbb{E}_{\mathcal{E}_0} \left[\left(\sum_{k=1}^m X_k \right)^2 \right] - 2m\hat{\sigma}^2 \right\|_{p/2} < \infty.$$

We thus obtain that

$$(4.75) \quad \sum_{m=1}^{\infty} m^{-3/2} \left\| \mathbb{E}_{\mathcal{E}_0} \left[\left(\sum_{k=1}^m X_k \right)^2 \right] - 2m\hat{\sigma}^2 \right\|_{p/2} \lesssim \sum_{m=1}^{\infty} m^{-3/2} < \infty.$$

We will now treat the cases $p \in (2, 3)$ and $p = 3$ separately.

Case $p \in (2, 3)$: Since $\cos(y)/2, \sin(y)/2 \in \mathfrak{F}_p$, the homogeneity of order p implies that

$$\begin{aligned} \|\mathbb{E}_{\mathbb{F}_m}[f(xS_j\eta_{j|m}) - f(xZ_j\hat{\sigma}\eta_{j|m})]\|_1 &\leq 2\|\zeta_p(P_{xS_j\eta_{j|m}|\mathbb{F}_m}, P_{xZ_j\hat{\sigma}\eta_{j|m}|\mathbb{F}_m})\|_1 \\ &= 2\||x|^p|\eta_{j|m}|^p\zeta_p(P_{S_j|\mathbb{F}_m}, P_{Z_j\hat{\sigma}|\mathbb{F}_m})\|_1. \end{aligned}$$

Since S_j, Z_j are independent of \mathbb{F}_m , we have $\zeta_p(P_{S_j|\mathbb{F}_m}, P_{Z_j\hat{\sigma}|\mathbb{F}_m}) = \zeta_p(P_{S_j}, P_{Z_j\hat{\sigma}})$. Hence

$$\|\mathbb{E}_{\mathbb{F}_m}[f(xS_j\eta_{j|m}) - f(xZ_j\hat{\sigma}\eta_{j|m})]\|_1 \leq 2\||x|^p|\eta_{j|m}|^p\|_1\zeta_p(P_{S_j}, P_{Z_j\hat{\sigma}}).$$

By Lemma 4.7, we have $\|\eta_{j|m}\|_p < \infty$. Note that $\mathbb{E}[S_j] = \mathbb{E}[Z_j] = 0$ and $\|S_j\|_2^2 = \|Z_j\hat{\sigma}\|_2^2$. Hence due to (4.73) and (4.75), we may apply Theorem 3.1(1) in [11], which gives us $\zeta_p(P_{S_j}, P_{Z_j\hat{\sigma}}) \lesssim m^{-p/2+1}$. Hence

$$\|\mathbb{E}_{\mathbb{F}_m}[f(xS_j\eta_{j|m}) - f(xZ_j\hat{\sigma}\eta_{j|m})]\|_1 \lesssim |x|^p m^{-p/2+1}.$$

Case $p = 3$: We may proceed as in the previous case, with the exception that here we need to apply Theorem 3.2 in [11]. We may do so due to (4.73) and (4.75). Hence we obtain

$$(4.76) \quad \|\mathbb{E}_{\mathbb{F}_m}[f(xS_j\eta_{j|m}) - f(xZ_j\hat{\sigma}\eta_{j|m})]\|_1 \lesssim |x|^3 m^{-1/2}. \quad \square$$

Using Lemmas 4.9 and 4.10, the triangle inequality gives the following corollary, which proves (4.68).

COROLLARY 4.11. *Grant Assumption 4.1. Then*

$$\|\varphi_i(t) - e^{-\sigma_{i|m}^2 t^2/2}\|_1 \lesssim |t|^p m^{-p/2+1}, \quad i \in \{1, N\}.$$

4.6. *Some auxiliary lemmas.* We will frequently use the following lemma, which is essentially a restatement of Theorem 1 in [48], adapted to our setting.

LEMMA 4.12. *Put $p' = \min\{p, 2\}$, $p \geq 1$. If $\sum_{l=1}^\infty \sup_{k \in \mathbb{Z}} \|X_k - X_k^{(l, \cdot)}\|_p < \infty$, then*

$$\|X_1 + \dots + X_n\|_p \lesssim n^{1/p'}.$$

For the sake of completeness, we state this result in the general, nontime-homogenous but stationary Bernoulli-shift context.

Recall that

$$Y_j = \frac{1}{\sqrt{2m}} Y_j^{(1)}, \quad Y_j^{(1)} = U_j + R_j,$$

$$S_{j|m}^{(**)} = \frac{1}{\sqrt{2m}} \sum_{k=(2j-2)m+1}^{(2j-1)m} X_k^{((2j-2)m, **)}.$$

LEMMA 4.13. *Grant Assumption 4.1. Then:*

- (i) $\|S_{j|m}^{(**)} - Y_j\|_p \lesssim m^{-1/2}(1 + \|R_j\|_p) \lesssim m^{-1/2}$ for $j = 1, \dots, N$,
- (ii) $\|Y_j\|_p < \infty$ for $j = 1, \dots, N$.

PROOF. Without loss of generality, we assume that $j = 1$ since $m \sim m'$. (i) We have the decomposition

$$\sqrt{2m} \|S_{j|m}^{(**)} - Y_j\|_p \leq \sum_{k=1}^m \|X_k - X_k^{(**)}\|_p + \sum_{k=1}^m \|\mathbb{E}_{\mathbb{F}_m}[X_k]\|_p + \|R_j\|_p.$$

We will deal with all three terms separately. The triangle inequality gives

$$\sum_{k=1}^m \|X_k - X_k^{(**)}\|_p \leq \sum_{k=1}^{\infty} k \|X_k - X'_k\|_p < \infty.$$

Next, note that $\mathbb{E}[X_k^{(**)} | \mathbb{F}_m] = \mathbb{E}[X_k] = 0$ for $1 \leq k \leq m$. Hence it follows via the Jensen and triangle inequalities that

$$\begin{aligned} \left\| \sum_{k=1}^m \mathbb{E}[X_k | \mathbb{F}_m] \right\|_p &= \left\| \sum_{k=1}^m \mathbb{E}[X_k - X_k^{(**)} | \mathbb{F}_m] \right\|_p \\ &\leq \sum_{k=1}^m \|X_k - X_k^{(**)}\|_p \leq \sum_{k=1}^{\infty} k \|X_k - X'_k\|_p < \infty. \end{aligned}$$

Similarly, since $X_k - \mathbb{E}_{\mathbb{F}_m}[X_k] \stackrel{d}{=} \mathbb{E}_{\mathbb{F}_m}[X_k^{(k-m,*)} - X_k]$ for $m + 1 \leq k \leq 2m$, we have

$$\begin{aligned} \left\| \sum_{k=m+1}^{2m} X_k - \mathbb{E}_{\mathbb{F}_m}[X_k] \right\|_p &= \left\| \sum_{k=m+1}^{2m} \mathbb{E}_{\mathbb{F}_m}[X_k^{(k-m,*)} - X_k] \right\|_p \\ &\leq \sum_{k=m+1}^{2m} \|X_k^{(k-m,*)} - X_k\|_p \leq \sum_{k=1}^{\infty} k \|X'_k - X_k\|_p < \infty. \end{aligned}$$

Combining all three bounds gives (i). This implies that for (ii), it suffices to show that $\|U_1\|_p \lesssim \sqrt{m}$. Using the above bounds and Lemma 4.12, we get

$$\|U_1\|_p \leq \left\| \sum_{k=1}^m X_k \right\|_p + \sum_{k=1}^m \|\mathbb{E}_{\mathbb{F}_m}[X_k]\|_p \lesssim \sqrt{m}. \quad \square$$

LEMMA 4.14. *Grant Assumption 4.1, and let H_j and \mathcal{H}_j , $j \in \mathcal{J}$ be as in (4.15). Then*

$$P(\mathbb{E}_{\mathcal{H}_j}[H_j^2] \leq \widehat{\sigma}_{m-l}^2) < 1/7.$$

PROOF. Since $m \sim m'$, it suffices to treat the case $j = 1$. Recall that $\mathcal{H}_1 = \mathcal{G}_1^{(l)} = \mathcal{E}_l$ and

$$\begin{aligned} \sqrt{m-l}H_1 &= \sum_{k=l+1}^m X_k - \mathbb{E}_{\mathcal{G}_1^{(l)}}[X_k] - R_1 + \mathbb{E}_{\mathcal{G}_1^{(l)}}[R_1], \\ 2(m-l)\widehat{\sigma}_{m-l}^2 &= \mathbb{E}_{\mathcal{E}_l} \left[\left(\sum_{k=l+1}^m X_k^{(k-l,*)} \right)^2 \right]. \end{aligned}$$

Let $I_k^{(l)} = \mathbb{E}_{\mathcal{E}_l}[X_k] + (R_1 - \mathbb{E}_{\mathcal{E}_l}[R_1])\mathbb{1}(k = l + 1)$. Using $a^2 - b^2 = (a - b)(a + b)$ and applying the Cauchy–Schwarz and Jensen inequalities then yields

$$\begin{aligned} & (m - l) \|\mathbb{E}_{\mathcal{H}_1}[H_1^2] - 2\hat{\sigma}_{m-l}^2\|_1 \\ &= \left\| \mathbb{E}_{\mathcal{E}_l} \left[\left(\sum_{k=l+1}^m X_k - X_k^{(k-l,*)} - I_k^{(l)} \right) \left(\sum_{k=l+1}^m X_k + X_k^{(k-l,*)} - I_k^{(l)} \right) \right] \right\|_1 \\ &\leq \left\| \sum_{k=l+1}^m X_k - X_k^{(k-l,*)} - I_k^{(l)} \right\|_2 \left\| \sum_{k=l+1}^m X_k + X_k^{(k-l,*)} - I_k^{(l)} \right\|_2 \\ &\stackrel{\text{def}}{=} I_1(l, m) I_2(l, m). \end{aligned}$$

By Lemma 4.13 and the arguments therein, it follows that $I_1(l, m) = \mathcal{O}(1)$, uniformly for $0 < l < m$. Similarly, one obtains that $(m - l)^{-1/2} I_2(l, m) = \mathcal{O}(1)$. Hence

$$(4.77) \quad \|\mathbb{E}_{\mathcal{H}_j}[H_j^2] - 2\hat{\sigma}_{m-l}^2\|_1 \lesssim \frac{1}{\sqrt{m-l}}.$$

We then have that

$$\begin{aligned} P(\mathbb{E}_{\mathcal{H}_j}[H_j^2] \leq \hat{\sigma}_{m-l}^2) &\leq P(|\mathbb{E}_{\mathcal{H}_j}[H_j^2] - 2\hat{\sigma}_{m-l}^2| > \hat{\sigma}_{m-l}^2) \\ &\leq \hat{\sigma}_{m-l}^{-2} \|\mathbb{E}_{\mathcal{H}_j}[H_j^2] - 2\hat{\sigma}_{m-l}^2\|_1 \lesssim \frac{1}{\sqrt{m-l}} \end{aligned}$$

by Markov’s inequality. Hence the claim follows if $m - l$ is large enough. Note that more detailed computations, as in Lemma 4.7, would give a more precise result. However, the current version is sufficient for our needs. \square

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