Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 1, 493–502
DOI: 10.1214/15-AIHP733

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Nonexistence of Lyapunov exponents for matrix cocycles

Xueting Tian

School of Mathematical Science, Fudan University, Shanghai 200433, People's Republic of China. E-mail: xuetingtian@fudan.edu.cn

Received 25 July 2015; revised 16 November 2015; accepted 20 November 2015

Abstract. It follows from Oseledec Multiplicative Ergodic Theorem (or Kingman's Sub-additive Ergodic Theorem) that the Lyapunov-irregular set of points for which the Oseledec averages of a given continuous cocycle diverge has zero measure with respect to any invariant probability measure. In strong contrast, for any dynamical system $f : X \to X$ with exponential specification property and a Hölder continuous matrix cocycle $A : X \to \mathrm{GL}(m, \mathbb{R})$, we show here that if there exist ergodic measures with different Lyapunov spectrum, then the Lyapunov-irregular set of A is residual (i.e., containing a dense G_δ set). Here we point out that exponential specification is introduced and plays critical role, and it is still unknown whether specification is enough. The above result can be used not only for all mixing hyperbolic systems but also for some non-hyperbolic systems.

Résumé. Le théorème ergodique multiplicatif d'Oseledets (ou le théorème ergodique sous-additif de Kingman) implique que l'ensemble Lyapounov-irrégulier (les points pour lesquels la moyenne d'Oseledets d'un cocycle continu donné diverge) est de mesure nulle pour toute mesure de probabilité invariante. Par contraste avec ce fait, nous montrons que pour tout système dynamique $f : X \to X$ satisfaisant la spécification exponentielle, et pour tout cocycle de matrices $A : X \to \mathrm{GL}(m, \mathbb{R})$ Hölder continu, s'il existe des mesures ergodiques avec des spectres de Lyapounov distincts, alors l'ensemble Lyapounov-irrégulier de A est résiduel (i.e., il contient un G_δ-dense). Nous mettons donc en évidence le rôle critique de la spécification exponentielle. Il n'est pas connu si cette propriété est suffisante. Notre résultat s'applique à tous les systèmes hyperboliques mélangeants et à certains systèmes non-hyperboliques.

MSC: 37H15; 37D20; 37D25; 37C50

Keywords: Oseledec Multiplicative Ergodic Theorem; Kingman's Sub-additive Ergodic Theorem; Lyapunov exponents and cocycles; (Exponential) specification property; Uniformly hyperbolic systems

1. Introduction

In page 264 of his book [15], Ricardo Mañé wrote: "In general, (Lyapunov) regular points are very few from the topological point of view – they form a set of first category." One could try to formalize the enigmatic statement in different ways. For example, Theorem 3.14 of [1] by Abdenur, Bonatti and Crovisier is one such kind formalization, where Mañé's statement is made precise by interpreting "in general" as "for C^1-generic diffeomorphisms." In this paper, we aim to give another formalization of Mañé's statement, considering Hölder continuous cocycle over dynamics with exponential specification (particularly considering derivative cocycle over hyperbolic dynamics), see Theorems 1.4 and 1.7 below. Before stating our main results, let us introduce some basic notions and facts.

1.1. *Lyapunov exponents*

Let f be an invertible map of a compact metric space X and let $A : X \to \mathrm{GL}(m, \mathbb{R})$ be a continuous matrix function. One main object of interest is the asymptotic behavior of the products of A along the orbits of the transformation f, called cocycle induced from A: for $n > 0$

$$A(x,n) := A\big(f^{n-1}(x)\big) \cdots A\big(f(x)\big)A(x),$$

and

$$A(x,-n) := A\big(f^{-n}(x)\big)^{-1} \cdots A\big(f^{-2}(x)\big)^{-1} A\big(f^{-1}(x)\big)^{-1} = A\big(f^{-n}x, n\big)^{-1}.$$

Definition 1.1. *We say $x \in X$ to be (forward)* Lyapunov-regular *for A, if there exist numbers $\chi_1 < \chi_2 < \cdots < \chi_l$ ($l \le m$), and an A-invariant decomposition of $\mathbb{R}^m$*

$$\mathbb{R}^m_x = G_1(x) \oplus G_2(x) \oplus \cdots \oplus G_l(x)$$

such that for any $i = 1, \dots, l$ and any $0 \neq v \in G_i(x)$ one has

$$\lim_{n\to+\infty} \frac{1}{n} \log\big\|A(x,n)v\big\| = \chi_i.$$

Otherwise, x is called to be Lyapunov-irregular *for A. Let* $\mathrm{LI}(A, f)$ *denote the space of all Lyapunov-irregular points for A.*

Remark 1.2. *There are many definitions for Lyapunov-regularity, e.g., Barreira and Pesin's book* [4] *and Mañé's book* [15]. *Definition* 1.1 *of Lyapunov-regular point is similar as the one in* [15], *which was originally defined for derivative cocycle. For the definition of Barreira and Pesin, see* [4] *for more details.*

If x is Lyapunov-regular for cocycle A, it is easy to see that the limit $\lambda(A,x,v) := \lim_{n\to+\infty} \frac{1}{n}\log\|A(x,n)v\|$ exists for all vector $v \in \mathbb{R}^m \setminus \{0\}$.

Oseledec Multiplicative Ergodic Theorem ([4, Theorem 3.4.4] (or see [17])). *Let f be an invertible ergodic measure-preserving transformation of a Lebesgue probability measure space (X, μ). Let A be a measurable cocycle whose generator satisfies* $\log\|A^{\pm}(x)\| \in L^1(X,\mu)$. *Then there exist numbers*

$$\chi_1 < \chi_2 < \cdots < \chi_l,$$

an f-invariant set $\mathcal{R}^\mu$ with $\mu(\mathcal{R}^\mu) = 1$, and an A-invariant decomposition of $\mathbb{R}^m$ for $x \in \mathcal{R}^\mu$,

$$\mathbb{R}^m_x = E_{\chi_1}(x) \oplus E_{\chi_2}(x) \oplus \cdots \oplus E_{\chi_l}(x)$$

with $\dim E_{\chi_i}(x) = m_i$, *such that for any $i = 1, \dots, l$ and any $0 \neq v \in E_{\chi_i}(x)$ one has*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\big\|A(x,n)v\big\| = \chi_i$$

and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\det A(x,n) = \sum_{i=1}^{l} m_i \chi_i.$$

Definition 1.3. *The numbers $\chi_1, \chi_2, \dots, \chi_l$ are called the* Lyapunov exponents *of measure μ for cocycle A and the dimension m_i of the space $E_{\chi_i}(x)$ is called the* multiplicity *of the exponent χ_i. The collection of pairs*

$$\mathrm{Sp}(\mu, A) = \big\{(\chi_i, m_i) : 1 \le i \le l\big\}$$

is the Lyapunov spectrum *of measure μ. $\mathcal{R}^\mu$ is called the* Oseledec basin *of μ and the decomposition $\mathbb{R}^m = E_{\chi_1} \oplus E_{\chi_2} \oplus \cdots \oplus E_{\chi_l}$ is called the* Oseledec splitting *of μ.*

Note that for any ergodic measure μ, all the points in the set $\mathcal{R}^\mu$ are Lyapunov-regular. By Oseledec Multiplicative Ergodic Theorem and Ergodic Decomposition Theorem, the set

$$\Delta := \bigcup_{\mu \in \mathcal{M}_f^e(X)} \mathcal{R}^\mu$$

is a Borel set with total measure, that is, Δ has full measure for any invariant Borel probability measure, where $\mathcal{M}_f^e(X)$ denotes the space of all ergodic measures. In other words, the Lyapunov-irregular set is always of zero measure for any invariant probability measure. This does not mean that the set of Lyapunov-irregular points, where the Lyapunov exponents do not exist, is empty, even if it is completely negligible from the point of view of measure theory. One such interesting result is from [8] that for any uniquely ergodic system, there exists some matrix cocycle whose Lyapunov-irregular set can be "large" as a set of second Baire category (also see [12,19] for similar discussion).

1.2. *Results*

Recall that Y is called residual in X, if Y contains a dense G_δ subset of X. The notion of residual set is usually used to describe a set being "large" in the topological sense.

Theorem 1.4. *Let $f : X \to X$ be a homeomorphism of a compact metric space X with exponential specification. Let $A : X \to \mathrm{GL}(m, \mathbb{R})$ be a Hölder continuous matrix function. Then either all ergodic measures have same Lyapunov spectrum or the Lyapunov-irregular set $\mathrm{LI}(A, f)$ is residual in X.*

Remark 1.5. *In* [8, *Theorem* 4] *Furman proved that some smooth cocycles over irrational rotations* (*which were previously studied by Herman*) *have a residual set of Lyapunov-irregular points.*

Remark 1.6. *If $m = 1$, the Lyapunov exponent can be written as Birkhoff ergodic average*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi\big(f^j(x)\big),$$

where $\phi(x) = \log \|A(x)\|$ is a continuous function. If all ergodic measures have same Lyapunov spectrum, then by Ergodic Decomposition Theorem so do all invariant measures and thus by compactness of the weak topology, the limit $\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$ should exist at every point $x \in X$ and equal to the given spectrum. Moreover, the case of $m = 1$ is in fact to study Birkhoff ergodic average and it has been studied for systems with specification or its variants by many authors, see* [2,3,13,14,16] *and reference therein.*

As a particular case of Theorem 1.4 we have a consequence for the derivative cocycle of hyperbolic systems. Let $\mathrm{LI}(f) := \mathrm{LI}(Df, f)$. It is called Lyapunov-irregular set of system f.

Theorem 1.7. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold M and $X \subseteq M$ be a topologically mixing locally maximal hyperbolic invariant subset. Then either all ergodic measures supported on X have same Lyapunov spectrum or the Lyapunov-irregular set $\mathrm{LI}(f)$ is residual in X.*

We point out that for a diffeomorphism, exponential specification does not imply hyperbolicity, see Example 2.4.

Remark 1.8. *From* [20] (*or* [10]) *we know that for a $C^{1+\alpha}$ diffeomorphism, the Lyapunov exponents of a hyperbolic ergodic measure can be approximated by ones of periodic measures. Note that every ergodic measure supported a hyperbolic set is hyperbolic. Thus for the statements of Theorem* 1.7, *"all ergodic measures supported on X have same Lyapunov spectrum" can be replaced by "all periodic measures supported on X have same Lyapunov spectrum."*

Recently we also consider the topological entropy of Lyapunov-irregular set for cocycles over hyperbolic systems, which may carry full entropy. In this process, another concept called exponential shadowing is introduced and plays important role, see [18] for precise details.

2. Specification and Lyapunov metric

2.1. *Specification and exponential specification*

Now we introduce (*exponential*) *specification* property. Let f be a continuous map of a compact metric space X.

Definition 2.1. *f is called to have specification property, if the following holds: for any $\delta > 0$ there exists an integer $N = N(\delta) > 0$ such that for any $k \geq 1$, any points $x_1, x_2, \dots, x_k \in X$, any integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_{j+1} - b_j \geq N$ $(1 \leq j \leq k-1)$, there exists a point $y \in X$ such that*

$$d\big(f^i(y), f^i(x_j)\big) < \delta, \quad a_j \leq i \leq b_j, 1 \leq j \leq k.$$

Remark that the specification property introduced by Bowen [5] (or see [7,11]) required that the shadowing point y is periodic. That is, for any $p \geq b_k - a_1 + N$, the chosen point y in above definition further satisfies $f^p(y) = y$. We call this to be *Bowen's specification property*.

Definition 2.2. *f is called to have exponential specification property with exponent $\lambda > 0$ (only dependent on the system f itself), if specification property holds and the inequality in specification can be shadowed exponentially, i.e.,*

$$d\big(f^i(x_j), f^i(y)\big) < \delta e^{-\lambda \min\{i-a_j, b_j-i\}}, \quad a_j \leq i \leq b_j, 1 \leq j \leq k.$$

If further the tracing point y is periodic with $f^p(y) = y$, then we say f has Bowen's exponential specification property.

For convenience, we say the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n y$ are exponentially δ close with exponent λ, meaning that

$$d\big(f^i(x), f^i(y)\big) < \delta e^{-\lambda \min\{i, n-i\}}, \quad 0 \leq i \leq n.$$

Remark 2.3. *It is not difficult to see that*

$$\begin{aligned} &(\textit{Bowen's})\ \textit{specification} + \textit{Local product structure} \\ &\quad \Rightarrow \quad (\textit{Bowen's})\ \textit{exponential specification}. \end{aligned}$$

Recall that every locally maximal hyperbolic set has local product structure and every topologically mixing locally maximal hyperbolic set has Bowen's specification property [5] (*or see* [7,11]). *So every topologically mixing locally maximal hyperbolic set has exponential specification property. As a particular case, every transitive Anosov diffeomorphism has exponential specification property, since it is known that every transitive Anosov diffeomorphism is topologically mixing. If a homeomorphism f is topologically conjugated to a homeomorphism g satisfying (Bowen's) exponential specification property with some exponent $\beta > 0$, and the inverse conjugation is γ-Hölder continuous, then it is not difficult to see that f has (Bowen's) exponential specification property with exponent $\beta\gamma > 0$.*

Example 2.4. *Some non-hyperbolic systems with exponential specification*:

(1) *From* [9] *there is some non-hyperbolic diffeomorphism f with $C^{1+\mathrm{Lip}}$ smoothness such that f is conjugated to a transitive Anosov diffeomorphism, the conjugation and its inverse is Hölder continuous. By Remark* 2.3 *this example satisfies Bowen's exponential specification property.*
(2) *The time-1 map of a geodesic flow of compact connected negative curvature manifolds is a partially (non-hyperbolic) hyperbolic dynamical system. Its exponential specification property can be deduced from the local product structure and specification property of the flow which is naturally hyperbolic, see* [6]. *Here we remark that the shadowing point may be not periodic with respect to the time-1 map because the shadowing of flow has a small time reparameterization.*

2.2. *Lyapunov exponents and Lyapunov metric*

In this section let us recall some Pesin-theoretic techniques, which are mainly from [10] (also see [4]).

Suppose $f : X \to X$ to be an invertible map on a compact metric space X and $A : X \to \mathrm{GL}(m, \mathbb{R})$ to be a continuous matrix function. For an ergodic measure μ, let $\chi_1 < \chi_2 < \cdots < \chi_l$ be the Lyapunov exponents of μ, $\mathcal{R}^\mu$ be the Oseledec basin of μ and the decomposition $\mathbb{R}^m = E_{\chi_1} \oplus E_{\chi_2} \oplus \cdots \oplus E_{\chi_l}$ be the Oseledec splitting of μ. We denote the standard scalar product in $\mathbb{R}^m$ by $\langle \cdot, \cdot \rangle$. For a fixed $\varepsilon > 0$ and a point $x \in \mathcal{R}^\mu$, the *ε-Lyapunov scalar product* (*or metric*) $\langle \cdot, \cdot \rangle_{x,\varepsilon}$ in $\mathbb{R}^m$ is defined as follows.

Definition 2.5. *For $u \in E_{\chi_i}(x)$, $v \in E_{\chi_j}(x)$, $i \neq j$ we define $\langle \cdot, \cdot \rangle_{x,\varepsilon} = 0$. For $i = 1, \dots, l$ and $u, v \in u \in E_{\chi_i}(x)$, we define*

$$\langle \cdot, \cdot \rangle_{x,\varepsilon} = m \sum_{n \in \mathbb{Z}} \bigl\langle A(x,n)u, A(x,n)v \bigr\rangle \exp\bigl(-2\chi_i n - \varepsilon |n|\bigr).$$

Note that the series in Definition 2.5 converges exponentially for any $x \in \mathcal{R}^\mu$. The constant m in front of the conventional formula is introduced for more convenient comparison with the standard scalar product. Usually, ε will be fixed and we will denote $\langle \cdot, \cdot \rangle_{x,\varepsilon}$ simply by $\langle \cdot, \cdot \rangle_x$ and call it the *Lyapunov scalar product*. The norm generated by this scalar product is called the *Lyapunov norm* and is denoted by $\| \cdot \|_{x,\varepsilon}$ or $\| \cdot \|_x$.

Let us recall some important properties of the Lyapunov scalar product and norm. For any $x \in \mathcal{R}^\mu$ and any $u \in E_{\chi_i}(x)$

$$\exp\bigl(n\chi_i - \varepsilon|n|\bigr)\|u\|_{x,\varepsilon} \le \bigl\|A(x,n)u\bigr\|_{f^n x,\varepsilon} \le \exp\bigl(n\chi_i + \varepsilon|n|\bigr)\|u\|_{x,\varepsilon} \quad \forall n \in \mathbb{Z}, \tag{1}$$

$$\exp\bigl(n\chi - \varepsilon|n|\bigr) \le \bigl\|A(x,n)u\bigr\|_{f^n x \leftarrow x} \le \exp\bigl(n\chi + \varepsilon|n|\bigr) \quad \forall n \in \mathbb{Z}, \tag{2}$$

where $\chi = \chi_l$ and $\| \cdot \|_{f^n x \leftarrow x}$ is the operator norm with respect to the Lyapunov norms. It is defined for any matrix B and any points $x, y \in \mathcal{R}^\mu$ as follows:

$$\|B\|_{y \leftarrow x} = \sup\bigl\{\|Bu\|_{y,\varepsilon} \cdot \|u\|_{x,\varepsilon}^{-1} : 0 \neq u \in \mathbb{R}^m\bigr\}.$$

It should be emphasized that, for any given $\varepsilon > 0$, Lyapunov scalar product and Lyapunov norm are defined only for $x \in \mathcal{R}^\mu$. They depend only measurably on the point even if the cocycle is Hölder. Therefore, comparison with the standard norm becomes important. The uniform lower bound follow easily from the definition:

$$\|u\|_{x,\varepsilon} \ge \|u\|.$$

The upper bound is not uniform, but it changes slowly along the orbits of each $x \in \mathcal{R}^\mu$: there exists a measurable function $K_\varepsilon(x)$ defined on the set $\mathcal{R}^\mu$ such that

$$\|u\| \le \|u\|_{x,\varepsilon} \le K_\varepsilon(x)\|u\| \quad \forall x \in \mathcal{R}^\mu, \forall u \in \mathbb{R}^m, \tag{3}$$

$$K_\varepsilon(x)e^{-\varepsilon n} \le K_\varepsilon\bigl(f^n x\bigr) \le K_\varepsilon(x)e^{\varepsilon n} \quad \forall x \in \mathcal{R}^\mu, \forall n \in \mathbb{Z}. \tag{4}$$

For any matrix B and any $x, y \in \mathcal{R}^\mu$ inequalities (3) and (4) yield

$$K_\varepsilon(x)^{-1}\|B\| \le \|B\|_{y \leftarrow x} \le K_\varepsilon(y)\|B\|. \tag{5}$$

When ε is fixed it is usually omitted and write $K(x) = K_\varepsilon(x)$. For any $l > 1$ we also define the following subsets of $\mathcal{R}^\mu$

$$\mathcal{R}^\mu_{\varepsilon,l} = \bigl\{x \in \mathcal{R}^\mu : K_\varepsilon(x) \le l\bigr\}. \tag{6}$$

Note that

$$\lim_{l \to \infty} \mu\bigl(\mathcal{R}^\mu_{\varepsilon,l}\bigr) \to 1.$$

Without loss of generality, we can assume that the set $\mathcal{R}^{\mu}_{\varepsilon,l}$ is compact and that Lyapunov splitting and Lyapunov scalar product are continuous on $\mathcal{R}^{\mu}_{\varepsilon,l}$. Indeed, by Luzin's theorem we can always find a subset of $\mathcal{R}^{\mu}_{\varepsilon,l}$ satisfying these properties with arbitrarily small loss of measure (for standard Pesin sets these properties are automatically satisfied).

3. Norm estimate of cocycles and generic property of Lyapunov-irregularity

3.1. *Estimate of the norm of Hölder cocycles*

The maximal (or largest) Lyapunov exponent (or simply, MLE) of $A : X \to \mathrm{GL}(m, \mathbb{R})$ at one point $x \in X$ is defined as the limit

$$\chi_{\max}(A, x) := \lim_{n\to\infty} \frac{1}{n} \log\big\| A(x, n)\big\|,$$

if it exists. In this case x is called to be (forward) *Max-Lyapunov-regular*. Otherwise, x is *Max-Lyapunov-irregular*. By Kingman's Sub-additive Ergodic Theorem, for any ergodic measure μ and μ a.e. point x, MLE always exists and is constant, denoted by $\chi_{\max}(A, \mu)$. From Oseledec Multiplicative Ergodic Theorem (as stated above), it is easy to see that $\chi_{\max}(A, \mu) = \chi_l$ where $\chi_1 < \chi_2 < \cdots < \chi_l$ are the Lyapunov exponents of μ. Let $\mathrm{MLI}(A, f)$ denote the set of all Max-Lyapunov-irregular points. Then it is of zero measure for any ergodic measure and by Ergodic Decomposition Theorem so does it for all invariant measures.

Now let us recall a general estimate of the norm of A along any orbit segment close to one orbit of $x \in \mathcal{R}^{\mu}$ [10].

Lemma 3.1 ([10, Lemma 3.1]). *Let A be an α-Hölder cocycle $(\alpha > 0)$ over a continuous map f of a compact metric space X and let μ be an ergodic measure for f with the maximal Lyapunov exponent $\chi_{\max}(A, \mu) = \chi$. Then for any positive λ and ε satisfying $\lambda > \varepsilon/\alpha$ there exists $c > 0$ such that for any $n \in \mathbb{N}$, any point $x \in \mathcal{R}^{\mu}$ with both x and $f^n x$ in $\mathcal{R}^{\mu}_{\varepsilon,l}$, and any point $y \in X$ such that the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n(y)$ are exponentially δ close with exponent λ for some $\delta > 0$ we have*

$$\big\|A(y, n)\big\|_{f^n x\leftarrow x} \le e^{cl\delta^\alpha} e^{n(\chi+\varepsilon)} \le e^{2n\varepsilon+cl\delta^\alpha} \big\|A(x, n)\big\|_{f^n x\leftarrow x} \tag{7}$$

and

$$\big\|A(y, n)\big\| \le l e^{cl\delta^\alpha} e^{n(\chi+\varepsilon)} \le l^2 e^{2n\varepsilon+cl\delta^\alpha} \big\|A(x, n)\big\|. \tag{8}$$

The constant c depends only on the cocycle A and on the number $(\alpha\lambda - \varepsilon)$.

Lemma 3.2. *Let A be an α-Hölder cocycle $(\alpha > 0)$ over a continuous map f of a compact metric space X and let μ be an ergodic measure for f with the maximal Lyapunov exponent $\chi_{\max}(A, \mu) = \chi$. Then for any positive λ and ε satisfying $\lambda > \varepsilon/\alpha$ there exists $\delta > 0$ such that for any $n \in \mathbb{N}$, any point $x \in \mathcal{R}^{\mu}$ with both x and $f^n x$ in $\mathcal{R}^{\mu}_{\varepsilon,l}$, and any point $y \in X$, if the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n(y)$ are exponentially δ close with exponent λ, we have*

$$\big\|A(y, n)\big\| \le l e^l e^{n(\chi+\varepsilon)} \le l^2 e^l e^{2n\varepsilon} \big\|A(x, n)\big\|. \tag{9}$$

Proof. For Lemma 3.1, let $\delta > 0$ small enough such that

$$c\delta^\alpha < 1.$$

Then the estimate (9) is obvious from Lemma 3.1. □

Another lemma is to estimate the growth of vectors in a certain cone $K \subseteq \mathbb{R}^m$ invariant under $A(x, n)$ [10]. Let $\chi_1 < \chi_2 < \cdots < \chi_l$ be the Lyapunov exponents of μ. Let x be a point in $\mathcal{R}^{\mu}_{\varepsilon,l}$ and $y \in X$ be a point such that the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n y$ are exponentially δ close with exponent λ. We denote $x_i = f^i x$ and

$y_i = f^i y$, $i = 0, 1, \ldots, n$. For each i we have orthogonal splitting $\mathbb{R}^m = E_i \oplus F_i$ with respect to the Lyapunov norm, where E_i is the Lyapunov space at x_i corresponding to the maximal Lyapunov exponent $\chi = \chi_l$ and F_i is the direct sum of all other Lyapunov spaces at x_i corresponding to the Lyapunov exponents less than χ. For any vector $u \in \mathbb{R}^m$ we denote by $u = u' + u^{\perp}$ the corresponding splitting with $u' \in E_i$ and $u^{\perp} \in F_i$; the choice of i will be clear from the context. To simplify notation, we write $\| \cdot \|_i$ for the Lyapunov norm at x_i. For each $i = 0, 1, \ldots, n$ we consider cones

$$K_i = \{u \in \mathbb{R}^m : \|u^{\perp}\|_i \le \|u'\|_i\} \quad \text{and} \quad K_i^{\eta} = \{u \in \mathbb{R}^m : \|u^{\perp}\|_i \le (1-\eta)\|u'\|_i\}$$

with $\eta > 0$. Note that for $u \in K_i$,

$$\|u\|_i \ge \|u'\|_i \ge \frac{1}{\sqrt{2}}\|u\|_i. \tag{10}$$

If all Lyapunov exponent of A with respect to μ are equal to χ (that is, $l = 1$), one has $F_i = \{0\}$, $K_i^{\eta} = K_i = \mathbb{R}^m$, in this case let

$$\varepsilon_0(\mu) = \lambda\alpha. \tag{11}$$

If not all Lyapunov exponent of A with respect to μ are equal to χ (that is, $l > 1$), let $\sigma < \chi$ be the second largest Lyapunov exponent of A with respect to μ, that is, $\sigma = \chi_{l-1}$. In this case set

$$\varepsilon_0(\mu) = \min\{\lambda\alpha, (\chi - \sigma)/2\}. \tag{12}$$

For $0 < \varepsilon < \varepsilon_0(\mu)$, from [10] we know the following.

Lemma 3.3 ([10, Lemma 3.3]). *In the notation above, for any set $\mathcal{R}^{\mu}_{\varepsilon,l}$, there exist $\eta, \delta > 0$ such that if $x, f^n x \in \mathcal{R}^{\mu}_{\varepsilon,l}$ and the orbit segments $x, fx, \ldots, f^n x$ and $y, fy, \ldots, f^n(y)$ are exponentially δ close with exponent λ, then for every $i = 0, 1, \ldots, n-1$ we have $A(y_i)(K_i) \subseteq K_i^{\eta}$ and $\|(A(y_i)u)'\|_{i+1} \ge e^{\chi - 2\varepsilon}\|u'\|_i$ for any $u \in K_i$.*

3.2. *Residual property of maximal Lyapunov-irregularity*

Now let us show a residual result for $\mathrm{MLI}(A, f)$.

Theorem 3.4. *Let $f : X \to X$ be a continuous map of a compact metric space X with exponential specification. Let $A : X \to \mathrm{GL}(m, \mathbb{R})$ be a α-Hölder continuous function for some $\alpha > 0$. Suppose that*

$$\inf_{\mu \in \mathcal{M}^e_f(X)} \chi_{\max}(A, \mu) < \sup_{\mu \in \mathcal{M}^e_f(X)} \chi_{\max}(A, \mu).$$

Then the Max-Lyapunov-irregular set $\mathrm{MLI}(A, f)$ *is residual in X.*

In other words, either all ergodic measures have same maximal Lyapunov exponent or Max-Lyapunov-irregular set $\mathrm{MLI}(A, f)$ is residual in X.

Proof of Theorem 3.4. Take two ergodic measures ν and ω such that

$$\chi_{\max}(A, \nu) > \chi_{\max}(A, \omega).$$

If let $a = \chi_{\max}(A, \nu)$ and $b = \chi_{\max}(A, \omega)$, we can choose $\tau > 0$ such that $a - 2\tau > b + 2\tau$.

Let $C = \max_{x \in X}\{\|A(x)\|, \|A^{-1}(x)\|\}$ and let λ be the positive number in the definition of exponential specification. Take $\varepsilon \in (0, \min\{\frac{1}{2}\tau, \varepsilon_0(\nu), \varepsilon_0(\omega)\})$ satisfying $\lambda > \varepsilon/\alpha$, where $\varepsilon_0(\mu)$ is the number w.r.t. measure μ defined in (11) or (12). For the measures ν and ω, take l large enough such that

$$\nu(\mathcal{R}^{\nu}_{\varepsilon,l}) > 0, \qquad \omega(\mathcal{R}^{\omega}_{\varepsilon,l}) > 0.$$

Define

$$O_n := \left\{ p \;\middle|\; \exists n_1, n_2 > n \text{ s.t. } \frac{1}{n_1}\log\|A(p,n_1)\| > a-\tau \text{ and } \frac{1}{n_2}\log\|A(p,n_2)\| < b+\tau \right\}.$$

By continuity of $A(x,n)$, O_n is open. It is straightforward to check that

$$\bigcap_{n\ge 1} O_n \subseteq \mathrm{MLI}(A,f).$$

So we only need to prove that for any $n \ge 1$, O_n is dense in X. Fix $x_0 \in X$, $n \ge 1$ and $t > 0$, we will show $O_n \cap B(x_0,t) \ne \varnothing$.

More precisely, firstly take $\eta > 0$, $\delta > 0$ small enough such that Lemma 3.3 applies to both measures ν and ω. Secondly we reduce δ so that $\delta \in (0,t)$ and that Lemma 3.2 applies to both measures ν and ω. For δ, λ, by assumption there is some integer $N > 0$ such that the exponential specification of Definition 2.2 holds.

By Poincaré recurrence theorem, there exist two points $x \in \mathcal{R}^{\nu}_{\varepsilon,l}$, $z \in \mathcal{R}^{\omega}_{\varepsilon,l}$ and two increasing sequences $\{H_i\}, \{L_i\} \nearrow \infty$ such that $f^{H_i}(x) \in \mathcal{R}^{\nu}_{\varepsilon,l}$, $f^{L_i}(z) \in \mathcal{R}^{\omega}_{\varepsilon,l}$. Take $H = H_i \gg \max\{N,n\}$ such that

$$\frac{1}{\sqrt{2}l}e^{H(a-2\varepsilon)} > C^N e^{(H+N)(a-\tau)}$$

and take $L = L_j \gg H+N$ large enough such that

$$le^l e^{L(b+\varepsilon)} C^{H+2N} < e^{(b+\tau)(L+H+2N)}.$$

Now let us consider three orbit segments

$$\{x_0\}, \qquad \{x, fx, \dots, f^H x\}, \qquad \{z, fz, \dots, f^L z\}$$

(hint: $a_1 = b_1 = 0$, $a_2 = N$, $b_2 = a_2 + H$, $a_3 = b_2 + N$, $b_3 = a_3 + L$) for the exponential specification. Then there is $y_0 \in X$ such that $d(y_0,x_0) < \delta$, the orbit segments $x, fx, \dots, f^H x$ and $y, fy, \dots, f^H y$ are exponentially δ close with exponent λ where $y = f^N y_0$, and simultaneously the orbit segments $z, fz, \dots, f^L z$ and $y', fy', \dots, f^L y'$ are exponentially δ close with exponent λ where $y' = f^{H+2N} y_0$.

Firstly let us consider the orbit segments $x, fx, \dots, f^H x$ and $y, fy, \dots, f^H y$. By Lemma 3.3 (in this estimate $\chi = a$, being the maximal Lyapunov exponent of ν), for any $u \in K_0$ with $\|u\| = 1$,

$$\big\|\big(A(y,H)u\big)'\big\|_H \ge e^{H(a-2\varepsilon)}\|u'\|_0.$$

Together with (3) and (10), we have

$$\begin{aligned}\|A(y,H)\| &\ge \|A(y,H)u\| \ge \frac{1}{l}\|A(y,H)u\|_H \ge \frac{1}{l}\big\|\big(A(y,H)u\big)'\big\|_H \ge \frac{1}{l}e^{H(a-2\varepsilon)}\|u'\|_0 \\ &\ge \frac{1}{\sqrt{2}l}e^{H(a-2\varepsilon)}\|u\|_0 \ge \frac{1}{\sqrt{2}l}e^{H(a-2\varepsilon)}\|u\| = \frac{1}{\sqrt{2}l}e^{H(a-2\varepsilon)} > C^N e^{(H+N)(a-\tau)}.\end{aligned}$$

Then

$$\begin{aligned}\|A(y_0,H+N)\| &\ge m\big(A(y_0,N)\big)\cdot\|A(y,H)\| \\ &\ge C^{-N}\cdot\|A(y,H)\| > e^{(H+N)(a-\tau)}, \end{aligned} \tag{13}$$

where $m(B) = \|B^{-1}\|^{-1}$.

Secondly let us consider the orbit segments $z, fz, \dots, f^L z$ and $y', fy', \dots, f^L y'$. By the first estimate in (9) of Lemma 3.2 (in this estimate $\chi = b$, being the maximal Lyapunov exponent of ω) we have

$$\|A(y',L)\| \le le^l e^{L(b+\varepsilon)}.$$

Then

$$\begin{aligned}\|A(y_0, L+H+2N)\| &\le \|A(y', L)\| \cdot \|A(y_0, H+2N)\| \\ &\le l e^l e^{L(b+\varepsilon)} C^{H+2N} < e^{(b+\tau)(L+H+2N)}. \end{aligned} \tag{14}$$

Take $n_1 = H + N$ and $n_2 = L + H + 2N$, then (13) and (14) imply $y_0 \in O_n$. Recall $d(y_0, x_0) < \delta$ and $\delta < t$ so that $y_0 \in B(x_0, t)$. So we complete the proof. □

3.3. *Proof of Theorems 1.4 and 1.7*

For a cocycle A and an ergodic measure μ, let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m$ (counted with their multiplicities) denote the Lyapunov exponents of μ for A. Let

$$\Lambda_i^A(\mu) = \sum_{j=1}^{i} \lambda_j.$$

Then it is easy to see that: for any two ergodic measures $\mu, \nu \in \mathcal{M}_f^e(X)$,

$$\mathrm{Sp}(\mu, A) = \mathrm{Sp}(\nu, A) \quad \Leftrightarrow \quad \Lambda_i^A(\mu) = \Lambda_i^A(\nu) \quad \forall i. \tag{15}$$

Let us consider cocycle $\bigwedge^i A(x,n)$ induced by cocycle $A(x,n)$ on the i-fold exterior powers $\bigwedge^i \mathbb{R}^m$. For an ergodic measure μ, it is standard to see that for any $1 \le i \le m$

$$\chi_{\max}\Big(\bigwedge^i A, \mu\Big) = \sum_{j=1}^{i} \lambda_j = \Lambda_i^A(\mu). \tag{16}$$

Proof of Theorem 1.4. Assume that there are two ergodic measures with different Lyapunov spectrum. By (15), there is some $1 \le i \le m$ such that

$$\inf_{\mu \in \mathcal{M}_f^e(X)} \Lambda_i^A(\mu) < \sup_{\mu \in \mathcal{M}_f^e(X)} \Lambda_i^A(\mu).$$

By (16), one has

$$\inf_{\mu \in \mathcal{M}_f^e(X)} \chi_{\max}\Big(\bigwedge^i A, \mu\Big) < \sup_{\mu \in \mathcal{M}_f^e(X)} \chi_{\max}\Big(\bigwedge^i A, \mu\Big).$$

Then we can apply Theorem 3.4 to the cocycle $\bigwedge^i A(x,n)$ and obtain that the Max-Lyapunov-irregular set of $\bigwedge^i A$, $\mathrm{MLI}(\bigwedge^i A, f)$, is residual in X. Note that $\mathrm{LI}(A, f) \supseteq \mathrm{MLI}(\bigwedge^i A, f)$, since a point Lyapunov-regular for A should be also Lyapunov-regular for $\bigwedge^i A$. So $\mathrm{LI}(A, f)$ is also residual in X. Now we complete the proof. □

Proof of Theorem 1.7. From Remark 2.3 we know $f|_X$ has exponential specification. Applying Theorem 1.4 for cocycle $A(x,n) = D_x f^n$, one ends the proof. □

Acknowledgement

The author thanks the anonymous referee very much for his or her careful reading and constructive suggestions, particularly for the part of quoting Mañé's statement (in the first paragraph of this article). The research of X. Tian was supported by National Natural Science Foundation of China (grant no. 11301088).

References

[1] F. Abdenur, C. Bonatti and S. Crovisier. Nonuniform hyperbolicity of C^1-generic diffeomorphisms. *Israel J. Math.* **183** (1) (2011) 1–60. MR2811152

[2] S. Albeverio, M. Pratsiovytyi and G. Torbin. Topological and fractal properties of subsets of real numbers which are not normal. *Bull. Sci. Math.* **129** (2005) 615–630. MR2166730

[3] L. Barreira, J. Li and C. Valls. Irregular sets for ratios of Birkhoff averages are residual. *Publ. Mat.* **58** (2014) 49–62. MR3211826

[4] L. Barreira and Y. B. Pesin. *Nonuniform Hyperbolicity*. Cambridge Univ. Press, Cambridge, 2007. MR2348606

[5] R. Bowen. Periodic points and measures for Axiom A diffeomorphisms. *Trans. Amer. Math. Soc.* **154** (1971) 377–397. MR0282372

[6] R. Bowen. Periodic orbits for hyperbolic flows. *Amer. J. Math.* **94** (1972) 1–30. MR0298700

[7] M. Denker, C. Grillenberger and K. Sigmund. *Ergodic Theory on Compact Space*. *Lecture Notes in Mathematics* **527**, 1976. MR0457675

[8] A. Furman. On the multiplicative ergodic theorem for uniquely ergodic systems. *Ann. Inst. Henri Poincaré Probab. Stat.* **33** (6) (1997) 797–815. MR1484541

[9] A. Gogolev. Diffeomorphisms Hölder conjugate to Anosov diffeomorphisms. *Ergodic Theory Dynam. Systems* **30** (2) (2010) 441–456. MR2599887

[10] B. Kalinin. Livšic theorem for matrix cocycles. *Ann. of Math.* **173** (2) (2011) 1025–1042. MR2776369

[11] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. *Encyclopedia of Mathematics and Its Applications* **54**. Cambridge Univ. Press, Cambridge, 1995. MR1326374

[12] D. Lenz. Existence of non-uniform cocycles on uniquely ergodic systems. *Ann. Inst. Henri Poincaré Probab. Stat.* **40** (2) (2004) 197–206. MR2044815

[13] J. Li and M. Wu. The sets of divergence points of self-similar measures are residual. *J. Math. Anal. Appl.* **404** (2013) 429–437. MR3045184

[14] J. Li and M. Wu. Generic property of irregular sets in systems satisfying the specification property. *Discrete Contin. Dyn. Syst.* **34** (2014) 635–645. MR3094597

[15] R. Mañé. *Ergodic Theory and Differentiable Dynamics*. Springer-Verlag, Berlin, London, 1987. MR0889254

[16] L. Olsen. Extremely non-normal numbers. *Math. Proc. Cambridge Philos. Soc.* **137** (2004) 43–53. MR2075041

[17] V. I. Oseledec. Multiplicative ergodic theorem, Liapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.* **19** (1968) 197–221. Translated from Russian. MR0240280

[18] X. Tian. Lyapunov 'non-typical' points of matrix cocycles and topological entropy. Preprint. Available at arXiv:1505.04345.

[19] P. Walters. Unique ergodicity and random matrix products. In *Lyapunov Exponents* 37–55. Springer, Berlin, Heidelberg, 1986. MR0850069

[20] Z. Wang and W. Sun. Lyapunov exponents of hyperbolic measures and hyperbolic periodic orbits. *Trans. Amer. Math. Soc.* **362** (2010) 4267–4282. MR2608406