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Scaling limits for the peeling process on random maps

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Abstract. We study the scaling limit of the volume and perimeter of the discovered regions in the Markovian explorations known as peeling processes for infinite random planar maps such as the uniform infinite planar triangulation (UIPT) or quadrangulation (UIPQ). In particular, our results apply to the metric exploration or peeling by layers algorithm, where the discovered regions are (almost) completed balls, or hulls, centered at the root vertex. The scaling limits of the perimeter and volume of hulls can be expressed in terms of the hull process of the Brownian plane studied in our previous work. Other applications include the metric exploration of the dual graph of our infinite random lattices, and first-passage percolation with exponential edge weights on the dual graph, also known as the Eden model or uniform peeling.

Résumé. Nous étudions la limite d'échelle du processus des volumes et des périmètres des régions explorées par un algorithme « d'épluchage » sur les cartes infinies aléatoires telles que l'UIPT (la triangulation infinie uniforme du plan) ou son analogue quadrangulaire l'UIPQ. Nos résultats s'appliquent en particulier à l'exploration des boules (pour la distance de graphe) complétées et centrées à la racine de la carte. Dans ce cas, la limite d'échelle coïncide avec le processus du périmètre et du volume des boules complétées dans le plan brownien. Parmi les autres applications, mentionnons l'exploration des boules complétées sur la carte duale et la percolation de premier passage avec poids exponentiels sur la carte duale. Ce dernier modèle, équivalent au modèle d'Eden sur la carte initiale, correspond à l'algorithme d'épluchage uniforme.

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1. Introduction

The spatial Markov property of random planar maps is one of the most important properties of these random lattices. Roughly speaking, this property says that, after a region of the map has been explored, the law of the remaining part only depends on the perimeter of the discovered region. The spatial Markov property was first used in the physics literature, without a precise justification: Watabiki [31] introduced the so-called "peeling process," which is a growth process discovering the random lattice step by step. A rigorous version of the peeling process and its Markovian properties was given by Angel [3] in the case of the Uniform Infinite Planar Triangulation (UIPT), which had been defined by Angel and Schramm [6] as the local limit of uniformly distributed plane triangulations with a fixed size. The peeling process has been used since to derive information about the metric properties of the UIPT [3], about percolation [3,4,26] and simple random walk [7] on the UIPT and its generalizations, and more recently about the conformal structure [15] of random planar maps. It also plays a crucial role in the construction of "hyperbolic" random triangulations [5,14].

In the present paper, we derive scaling limits for the perimeter and the volume of the discovered region in a peeling process of the UIPT. Our methods also apply to the Uniform Infinite Planar Quadrangulation (UIPQ), which was constructed independently by Krikun [21] and by Chassaing and Durhuus [13] (the equivalence between these two constructions was obtained by Ménard [25]). By considering the special case of the peeling by layers, we get scaling limits for the volume and the boundary length of the hull of radius r centered at the root of the UIPT, or of the UIPQ (the hull of radius r is obtained by "filling in the finite holes" in the ball of radius r). The limiting processes that arise in these scaling limits coincide with those that appeared in our previous work [17] dealing with the hull process of the Brownian plane. This is not surprising since the Brownian plane is conjectured to be the universal scaling limit of many infinite random lattices such as the UIPT, and it is known that this conjecture holds in the special case of the UIPQ [18]. We also apply our results to both the dual graph distance and the first-passage percolation distance corresponding to exponential edge weights on the dual graph of the UIPT (this first-passage percolation model is also known as the Eden model). In particular, we show that the volume and perimeter of the hulls with respect to each of these two metrics have the same scaling limits as those corresponding to the graph distance, up to explicit deterministic multiplicative factors.

For the sake of clarity, the following results are stated and proved in the case of the UIPT corresponding to type II triangulations in the terminology of Angel and Schramm [6]. In type II triangulations, loops are not allowed but there may be multiple edges. Section 6 explains the changes that are needed for the extension of our results to other random lattices such as the UIPT for type I triangulations or the UIPQ. In these extensions, scaling limits remain the same, but different constants are involved. In the case of type II triangulations, the three basic constants that arise in our results are

$$\mathsf{p}_{\triangle^2} = \left(\frac{2}{3}\right)^{2/3}, \qquad \mathsf{v}_{\triangle^2} = \left(\frac{2}{3}\right)^{7/3} \quad \text{and} \quad \mathsf{h}_{\triangle^2} = 12^{-1/3}.$$

Here the subscript $\triangle^2$ emphasizes the fact that these constants are relevant to the case of type II triangulations.

So, except in Section 6, all triangulations in this article are type II triangulations. The corresponding UIPT is denoted by T_∞. This is an infinite random triangulation of the plane given with a distinguished oriented edge whose tail vertex is called the origin (or root vertex) of the map. If $\mathbf{t}$ is a rooted finite triangulation with a simple boundary $\partial\mathbf{t}$, we denote the number of inner vertices of $\mathbf{t}$ by $|\mathbf{t}|$ and the boundary length of $\mathbf{t}$ by $|\partial\mathbf{t}|$. Furthermore, we say that $\mathbf{t}$ is a subtriangulation of T_∞ and write $\mathbf{t} \subset T_\infty$, if T_∞ is obtained from $\mathbf{t}$ by gluing an infinite triangulation with a simple boundary along the boundary of $\mathbf{t}$ (of course we also require that the root of T_∞ coincides with the root of $\mathbf{t}$ after this gluing operation). If $\mathbf{t} \subset T_\infty$ and e is an edge of $\partial\mathbf{t}$, the triangulation obtained by the peeling of e is the triangulation $\mathbf{t}$ to which we add the face incident to e that was not already in $\mathbf{t}$, as well as the finite region that the union of $\mathbf{t}$ and this added face may enclose (recall that the UIPT has only one end [6]). An exploration process $(\mathsf{T}_i)_{i\geq 0}$ is a sequence of subtriangulations of the UIPT with a simple boundary such that T_0 consists only of the root edge (viewed as a trivial triangulation) and for every $i \geq 0$ the map T_{i+1} is obtained from T_i by peeling one edge of its boundary. If the choice of this edge is independent of $T_\infty \setminus \mathsf{T}_i$, the exploration is said to be *Markovian* and we call it a *peeling process*. Different peeling processes correspond to different ways of choosing the edge to be peeled at every step. See Section 3.1 for a more rigorous presentation.

Our first theorem complements results due to Angel [3] by describing the scaling limit of the perimeter and volume of the discovered region in a peeling process. We let $(S_t)_{t\geq 0}$ denote the stable Lévy process with index $3/2$ and only negative jumps, which starts from 0 and is normalized so that its Lévy measure is $3/(4\sqrt{\pi})|x|^{-5/2}\mathbf{1}_{x<0}$, or equivalently $\mathbb{E}[\exp(\lambda S_t)] = \exp(t\lambda^{3/2})$ for any $\lambda, t \geq 0$. The process $(S_t)_{t\geq 0}$ conditioned to stay nonnegative is then denoted by $(S_t^+)_{t\geq 0}$ (see [8, Chapter VII] for a rigorous definition of $(S_t^+)_{t\geq 0}$). We also let $\xi_1, \xi_2, \ldots$ be a sequence of independent real random variables with density

$$\frac{1}{\sqrt{2\pi x^5}} e^{-1/(2x)} \mathbf{1}_{\{x>0\}}.$$

We assume that this sequence is independent of the process $(S_t^+)_{t\geq 0}$ and, for every $t \geq 0$, we set $Z_t = \sum_{t_i \leq t} \xi_i \cdot (\Delta S_{t_i}^+)^2$ where $t_1, t_2, \ldots$ is a measurable enumeration of the jumps of S^+.

Theorem 1 (Scaling limit for general peelings). *For any peeling process $(\mathsf{T}_n)_{n\geq 0}$ of the UIPT, we have the following convergence in distribution in the sense of Skorokhod*

$$\left(\frac{|\partial \mathsf{T}_{[nt]}|}{\mathsf{p}_{\triangle^2} \cdot n^{2/3}}, \frac{|\mathsf{T}_{[nt]}|}{\mathsf{v}_{\triangle^2} \cdot n^{4/3}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(S_t^+, Z_t\right)_{t\geq 0}.$$

The proof of Theorem 1 relies on the explicit expression of the transition probabilities of the peeling process. It follows from this explicit expression that the process of perimeters $(|\partial \mathsf{T}_n|)_{n\geq 0}$ is a h-transform of a random walk with independent increments in the domain of attraction of a spectrally negative stable distribution with index $3/2$ (Proposition 6). This h-transform is interpreted as conditioning the random walk to stay above level 2, and in the scaling limit this leads to the process $(S_t^+)_{t\geq 0}$. The common distribution of the variables ξ_i is the scaling limit of the volume of a Boltzmann triangulation (see Section 2.1) conditioned to have a large boundary size. The appearance of this distribution is explained by the fact that the "holes" created by the peeling process are filled in by finite triangulations distributed according to Boltzmann weights (this is called the free distribution in [6, Definition 2.3]). As a corollary of Theorem 1, we prove that any peeling process of the UIPT will eventually discover the whole triangulation, i.e., $\bigcup \mathsf{T}_n = T_\infty$, no matter what peeling algorithm is used (of course as long as the exploration is Markovian), see Corollary 7. We note that Theorem 1 can be applied to various peeling processes that have been considered in earlier works: peeling along percolation interfaces [3,4], peeling along simple random walk [7], peeling along a Brownian or a SLE_6 exploration of the Riemann surface associated with the UIPT [15], etc. In the present work, we apply Theorem 1 to three specific peeling algorithms, each of which is related to a "metric" exploration of the UIPT. The first one is the peeling by layers, which essentially grows balls for the graph distance on the UIPT, the second one is the peeling by layers in the dual map of the UIPT and the last one is the uniform peeling, which is related to first-passage percolation with exponential edge weights on the dual map of the UIPT.

Scaling limits for the hulls

For every integer $r\geq 1$, the ball $B_r(T_\infty)$ is defined as the union of all faces of T_∞ whose boundary contains at least one vertex at graph distance smaller than or equal to $r-1$ from the origin (when $r=0$ we agree that $B_0(T_\infty)$ is the trivial triangulation consisting only of the root edge). The hull $B_r^\bullet(T_\infty)$ is then obtained by adding to the ball $B_r(T_\infty)$ the bounded components of the complement of this ball (see Figure 1). Note that $B_r^\bullet(T_\infty)$ is a finite triangulation with a simple boundary. One can define a particular peeling process $(\mathsf{T}_i)_{i\geq 0}$ (called the peeling by layers) such that, for every $n\geq 0$, there exists a random integer H_n such that $B^\bullet_{H_n}(T_\infty)\subset \mathsf{T}_n\subset B^\bullet_{H_n+1}(T_\infty)$. Scaling limits for the volume and the boundary length of the hulls can then be derived by applying Theorem 1 to this particular peeling algorithm. A crucial step in this derivation is to get information about the asymptotic behavior of H_n when $n\to\infty$ (Proposition 10). Before stating our limit theorem for hulls, we need to introduce some notation.

For every real $u\geq 0$, set $\psi(u)=u^{3/2}$. The continuous-state branching process with branching mechanism ψ is the Feller Markov process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}_+$, whose semigroup is characterized as follows: for every $x,t\geq 0$ and every $\lambda>0$,

$$E\left[e^{-\lambda X_t}\mid X_0=x\right]=\exp\left(-x\left(\lambda^{-1/2}+t/2\right)^{-2}\right).$$

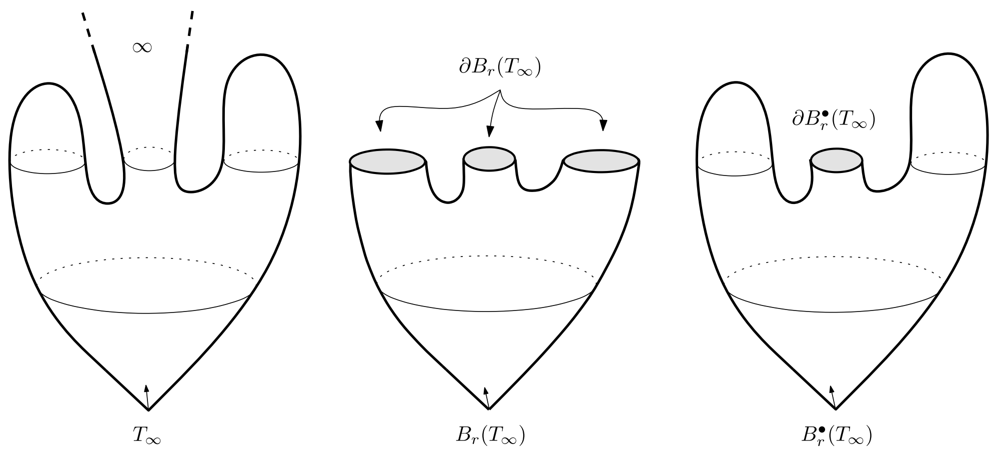

Fig. 1. From left to right, the "cactus" representation of the UIPT, the ball $B_r(T_\infty)$, whose boundary may have several components, and the hull $B_r^\bullet(T_\infty)$, whose boundary is a simple cycle.

Note that X gets absorbed at 0 in finite time. It is easy to construct a process $(\mathcal{L}_t)_{t\geq 0}$ with càdlàg paths such that the time-reversed process $(\mathcal{L}_{(-t)-})_{t\leq 0}$ (indexed by negative times) is distributed as X "started from $+\infty$ at time $-\infty$" and conditioned to hit zero at time 0 (see [17, Section 2.1] for a detailed presentation of the process $\mathcal{L}$). We consider the sequence $(\xi_i)_{i\geq 1}$ introduced before Theorem 1, and we assume that this sequence is independent of $\mathcal{L}$. We then set, for every $t \geq 0$

$$\mathcal{M}_t = \sum_{s_i \leq t} \xi_i \cdot (\Delta \mathcal{L}_{s_i})^2,$$

where $s_1, s_2, \dots$ is a measurable enumeration of the jumps of $\mathcal{L}$.

Theorem 2 (Scaling limit of the hull process). *We have the following convergence in distribution in the sense of Skorokhod*,

$$\big(n^{-2}\big|\partial B^\bullet_{[nt]}(T_\infty)\big|, n^{-4}\big|B^\bullet_{[nt]}(T_\infty)\big|\big)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} (\mathsf{p}_{\triangle^2} \cdot \mathcal{L}_{t/\mathsf{h}_{\triangle^2}}, \mathsf{v}_{\triangle^2} \cdot \mathcal{M}_{t/\mathsf{h}_{\triangle^2}})_{t\geq 0}.$$

A scaling argument shows that the limiting process has the same distribution as

$$\left(\frac{\mathsf{p}_{\triangle^2}}{(\mathsf{h}_{\triangle^2})^2}\mathcal{L}_t, \frac{\mathsf{v}_{\triangle^2}}{(\mathsf{h}_{\triangle^2})^4}\mathcal{M}_t\right)_{t\geq 0}$$

but the form given in Theorem 2 helps to understand the connection with Theorem 1.

We note that the convergence in distribution of the variables $r^{-2}|\partial B^\bullet_r(T_\infty)|$ as $r \to \infty$ had already been obtained by Krikun [22, Theorem 1.4] via a different approach. The limiting process in Theorem 2 appeared in the companion paper [17] as the process describing the evolution of the boundary length and the volume of hulls in the Brownian plane (in the setting of the Brownian plane, the length of the boundary has to be defined in a generalized sense). The paper [17] contains detailed information about distributional properties of this limiting process (see Proposition 1.2 and Theorem 1.4 in [17]). In particular, for every fixed $s > 0$, the joint distribution of the pair $(\mathcal{L}_s, \mathcal{M}_s)$ is known explicitly. Here we mention only the Laplace transform of the marginal laws:

$$\mathbb{E}\big[e^{-\lambda \mathcal{L}_s}\big] = \left(1 + \frac{\lambda s^2}{4}\right)^{-3/2},$$

$$\mathbb{E}\big[e^{-\lambda \mathcal{M}_s}\big] = 3^{3/2}\cosh\left(\frac{(2\lambda)^{1/4}s}{\sqrt{8/3}}\right)\left(\cosh^2\left(\frac{(2\lambda)^{1/4}s}{\sqrt{8/3}}\right) + 2\right)^{-3/2}.$$

Note in particular that $\mathcal{L}_r$ follows a Gamma distribution with parameter $3/2$.

Metric exploration of the dual map

Consider now the dual map T^*_∞ of the UIPT, whose vertices are the faces of the UIPT, and where two vertices are connected by an edge if the corresponding faces of the UIPT share a common edge. The origin of T^*_∞, or root face of T_∞, is the face incident to the right side of the root edge of T_∞. We equip T^*_∞ with the dual graph distance, and we let $B^{\bullet,*}_r(T_\infty)$ denote the hull of the ball of radius r in T^*_∞, i.e. the map made of all the faces of T_∞ which are at dual graph distance less than or equal to r from the root face, together with the finite regions these faces may enclose. Then the techniques developed for the proof of Theorem 2 also give the following result.

Theorem 3 (Scaling limit of the hull process on the dual map). *We have the following convergence in distribution in the sense of Skorokhod*,

$$\big(n^{-2}\big|\partial B^{\bullet,*}_{[nt]}(T_\infty)\big|, n^{-4}\big|B^{\bullet,*}_{[nt]}(T_\infty)\big|\big)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} (\mathsf{p}_{\triangle^2} \cdot \mathcal{L}_{t/\mathsf{h}^*_{\triangle^2}}, \mathsf{v}_{\triangle^2} \cdot \mathcal{M}_{t/\mathsf{h}^*_{\triangle^2}})_{t\geq 0},$$

where $\mathsf{h}^*_{\triangle^2} = \mathsf{h}_{\triangle^2} + (\mathsf{p}_{\triangle^2})^{-1}$.

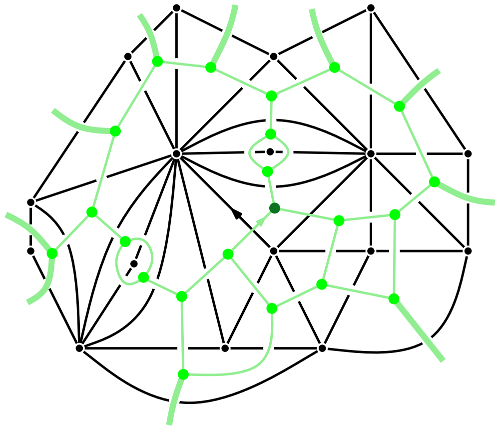

Fig. 2. Illustration of the exploration along first-passage percolation on the dual of the UIPT. We represented $\mathsf{F}_t^\bullet$ for some value of $t > 0$. By standard properties of exponential variables, the next dual edge to be explored is uniformly distributed on the boundary.

First-passage percolation

Consider again the dual map T_∞^* of the UIPT. We assign independently to each edge of the dual map an exponential weight with parameter 1. For every $t \geq 0$, we write F_t for the union of all faces that may be reached from the root face by a (dual) path whose total weight is at most t. As usual, $\mathsf{F}_t^\bullet$ stands for the hull of F_t, which is obtained by filling in the finite holes of F_t inside T_∞, see Figure 2. Then $\mathsf{F}_t^\bullet$ is a triangulation with a simple boundary. If $0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots$ are the jump times of the process $t \mapsto \mathsf{F}_t^\bullet$, it is not hard to verify that the sequence $(\mathsf{F}_{\tau_n}^\bullet)_{n\geq 0}$ is a *uniform* peeling process, meaning that at each step the edge to be peeled off is chosen uniformly at random among all edges of the boundary. See Proposition 15 for a precise statement. Then Theorem 1 leads to the following result:

Theorem 4 (Scaling limits for first passage percolation). *We have the following convergence in distribution for the Skorokhod topology*

$$\left(n^{-2}\left|\partial \mathsf{F}_{[nt]}^\bullet\right|, n^{-4}\left|\mathsf{F}_{[nt]}^\bullet\right|\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} (\mathsf{p}_{\triangle^2} \cdot \mathcal{L}_{\mathsf{p}_{\triangle^2} t}, \mathsf{v}_{\triangle^2} \cdot \mathcal{M}_{\mathsf{p}_{\triangle^2} t})_{t\geq 0}.$$

Set $c_1 = \mathsf{h}_{\triangle^2}^*/\mathsf{h}_{\triangle^2} = 4$ and $c_2 = (\mathsf{p}_{\triangle^2}\mathsf{h}_{\triangle^2})^{-1} = 3$. If we compare Theorem 2, Theorem 3 and Theorem 4, we see that the scaling limits of the volume and the perimeter are the same for $B_r^\bullet(T_\infty)$, for $B_{c_1\cdot r}^{\bullet,*}(T_\infty)$ and for $\mathsf{F}_{c_2\cdot r}^\bullet$. This is consistent with the conjecture saying that balls for the dual graph distance or for first-passage percolation distance grow like deterministic balls, up to a constant multiplicative factor (this property is not expected to hold for deterministic lattices such as $\mathbb{Z}^2$, but in some sense the UIPT is more isotropic). Informally, writing $\mathrm{d_{gr}}$ for the graph distance (on the UIPT), $\mathrm{d_{gr}^*}$ for the dual graph distance and $\mathrm{d_{fpp}}$ for the first-passage percolation distance, our results suggest that in large scales,

$$\mathrm{d}_{\mathrm{gr}}^*(\cdot,\cdot) \approx c_1 \cdot \mathrm{d_{gr}}(\cdot,\cdot), \qquad \mathrm{d_{fpp}}(\cdot,\cdot) \approx c_2 \cdot \mathrm{d_{gr}}(\cdot,\cdot).$$

Note that $\mathrm{d_{gr}}$ is a metric on the UIPT, whereas $\mathrm{d_{fpp}}$ or $\mathrm{d_{gr}^*}$ are metrics on the dual graph. Still it is easy to restate the previous display in the form of a precise conjecture (see Section 5.3). This conjecture is consistent with the recent calculations of Ambjørn and Budd [1] for two and three-point functions in first-passage percolation on random triangulations, and is the subject of the forthcoming work [16].

We finally note that our uniform peeling process can be viewed as a variant of the classical Eden model on the (dual graph of the) UIPT. The same variant has been considered by Miller and Sheffield [27] and served as a motivation for the construction of Quantum Loewner Evolutions. In fact the process QLE$(\frac{8}{3}, 0)$ that is constructed in [27] is a continuum analog of the Eden model on the UIPT. See Section 2.2 in [27] for more details.

The organization of the paper follows the preceding presentation. In Section 2, we recall some enumeration results for triangulations that play an important role in the paper, and we also give a result connecting the UIPT with Boltzmann triangulations, which is of independent interest (Theorem 5). This result shows that the distributions of the ball of radius r in the UIPT and in a Boltzmann triangulation are linked by an absolute continuity relation involving a martingale, which has an explicit expression in terms of the sizes of the cycles bounding the connected components of the complement of the ball.

2. Preliminaries

Throughout this work, we consider only *rooted* planar maps, and we often omit the word rooted. We view planar maps as graphs drawn on the sphere, with the usual identification modulo orientation-preserving homeomorphisms. Recall that, except in Section 6 below, we restrict our attention to type II triangulations, meaning that there are no loops, but multiple edges are allowed. We define a triangulation with a boundary as a rooted planar map without loops, with a distinguished face (the external face) bounded by a simple cycle (called the boundary), such that all faces except possibly the distinguished one are triangles. If τ is a triangulation with a boundary, we denote its boundary by $\partial\tau$. Vertices of τ not on the boundary are called inner vertices. The size $|\tau|$ of τ is defined as the number of inner vertices of τ. The length $|\partial\tau|$ of $\partial\tau$ (or perimeter of τ) is the number of edges, or equivalently the number of vertices, in $\partial\tau$. Note that $|\partial\tau| \geq 2$ since loops are not allowed.

2.1. *Enumeration*

We gather here several results about the asymptotic enumeration of planar triangulations, see [3,6] and the references therein. For every $n \geq 0$ and $p \geq 2$, we let $\mathcal{T}_{n,p}$ denote the set of all (type II) triangulations of size n with a simple boundary of length p, that are rooted at an edge of the boundary oriented so that the external face lies on the right of the root edge. We have

$$\#\mathcal{T}_{n,p} = \frac{2^{n+1}(2p-3)!(2p+3n-4)!}{(p-2)!^2 n!(2p+2n-2)!} \underset{n\to\infty}{\sim} C(p)\left(\frac{27}{2}\right)^n n^{-5/2}, \tag{1}$$

where

$$C(p) = \frac{4}{3^{7/2}\sqrt{\pi}} \frac{(2p-3)!}{(p-2)!^2}\left(\frac{9}{4}\right)^p \underset{p\to\infty}{\sim} \frac{1}{54\pi\sqrt{3}} 9^p \sqrt{p}. \tag{2}$$

The exact formula for $\#\mathcal{T}_{n,p}$ in (1) gives $\#\mathcal{T}_{n,p} = 1$ for $n = 0$ and $p = 2$. This formula is valid provided we make the special convention that the rooted planar map consisting of a single (oriented) edge between two vertices is viewed as a triangulation with a simple boundary of length 2: This will be called the trivial triangulation. It will be used in the sequel as the starting point of the peeling process, and also sometimes to "fill in" holes of size two arising in this process.

The exponent $5/2$ in (1) is typical of the enumeration of planar maps and shows that

$$Z(p) := \sum_{n=0}^{\infty} \left(\frac{2}{27}\right)^n \#\mathcal{T}_{n,p} < \infty.$$

The numbers $Z(p)$ can be computed exactly (see [3, Proposition 1.7]): for every $p \geq 2$,

$$Z(p) = \frac{(2p-4)!}{(p-2)!p!}\left(\frac{9}{4}\right)^{p-1}. \tag{3}$$

Triangulations in $\mathcal{T}_{n,p}$, for some $n \geq 0$, are also called triangulations of the p-gon. By definition, the (critical) Boltzmann distribution on triangulations of the p-gon is the probability measure on $\bigcup_{n\geq 0} \mathcal{T}_{n,p}$ that assigns mass

$(2/27)^n Z(p)^{-1}$ to each triangulation of $\mathcal{T}_{n,p}$. This is also called the free distribution in [6]. It follows from (3) that for every $x \in [0, 1/9]$,

$$\sum_{p=1}^{\infty} Z(p+1)x^p = \frac{1}{2} + \frac{(1-9x)^{3/2}-1}{27x}.$$

From (3) and the last display, we get that

$$Z(p+1) \underset{p\to\infty}{\sim} \mathrm{t}_{\triangle^2} \cdot 9^p p^{-5/2}, \quad \text{where } \mathrm{t}_{\triangle^2} = \frac{1}{4\sqrt{\pi}}, \tag{4}$$

$$\sum_{p=1}^{\infty} Z(p+1)9^{-p} = \frac{1}{6}, \tag{5}$$

$$\sum_{p=1}^{\infty} pZ(p+1)9^{-p} = \frac{1}{3}. \tag{6}$$

Finally, we note that there is a bijection between rooted triangulations of the 2-gon having n inner vertices and rooted plane triangulations having $n+2$ vertices: Just glue together the two boundary edges of a triangulation of the 2-gon to get a triangulation of the sphere. The Boltzmann distribution on rooted triangulations of the 2-gon thus induces a probability measure on the space of all triangulations of the sphere (including the trivial one). A random triangulation distributed according to this probability measure is called a Boltzmann triangulation of the sphere. Equivalently, the law of a Boltzmann triangulation of the sphere assigns a mass $(2/27)^{n-2}Z(2)^{-1}$ to every triangulation of the sphere with n vertices (including the trivial triangulation for which $n = 2$).

2.2. *Boltzmann triangulations and the UIPT*

In this section, we describe a relation between Boltzmann triangulations of the sphere and the UIPT. This relation is not really needed in what follows but it helps to understand the importance of Boltzmann triangulations in the subsequent developments.

Let T_{Bol} be a Boltzmann triangulation of the sphere. As in the introduction above, for every integer $r \geq 1$, let $B_r(T_{\mathrm{Bol}})$ denotes the ball of radius r in T_{Bol}. So $B_r(T_{\mathrm{Bol}})$ is the rooted planar map obtained by keeping only those faces of T_{Bol} that are incident to at least one vertex at distance at most $r-1$ from the root vertex. We view $B_r(T_{\mathrm{Bol}})$ as a random variable with values in the space of all (type II) triangulations with holes. Here, a triangulation with holes is a planar map without loops, with a finite number of distinguished faces called the holes, such that all faces except possibly the holes are triangles, the boundary of every hole is a simple cycle, whose length is called the size of the hole, and two distinct holes cannot share a common edge (the triangulations with a simple boundary that we considered above are just triangulations with a single hole). In the case of $B_r(T_{\mathrm{Bol}})$, holes obviously correspond to the connected components of the complement of the ball, in a way analogous to the middle part of Figure 1. We write $\ell_1(r), \ell_2(r), \dots, \ell_{n_r}(r)$ for the sizes of the holes of $B_r(T_{\mathrm{Bol}})$ enumerated in nonincreasing order. We also write $\mathcal{F}_r$ for the σ-field generated by $B_r(T_{\mathrm{Bol}})$ and we let $\mathcal{F}_0$ be the trivial σ-field. Recall our notation $B_r(T_\infty)$ for the ball of radius r in the UIPT, which is also viewed as a random triangulation with holes.

Theorem 5. *Let $f(n) = \frac{n}{2} \cdot (n-1) \cdot (2n-3)$ for every integer $n \geq 3$ and $f(2) = 9$. The random process $(\mathbf{M}_r)_{r\geq 0}$ defined by*

$$\mathbf{M}_r := \sum_{i=1}^{n_r} f\big(\ell_i(r)\big), \quad \textit{for } r \geq 1,$$

and $\mathbf{M}_0 = 1$, is a martingale with respect to the filtration $(\mathcal{F}_r)_{r\geq 0}$. Moreover, if F is any nonnegative measurable function on the space of triangulations with holes, we have, for every $r \geq 1$,

$$\mathbb{E}\big[F\big(B_r(T_\infty)\big)\big] = \mathbb{E}\big[\mathbf{M}_r F\big(B_r(T_{\mathrm{Bol}})\big)\big]. \tag{7}$$

The second part of the theorem shows that the law of a ball in the UIPT can be obtained by biasing the law of the corresponding ball in a Boltzmann triangulation using the martingale $\mathbf{M}_r$. This is an analog of a classical result for Galton–Watson trees: In order to get the first k generations of a Galton–Watson tree conditioned on non-extinction, one biases the law of the first k generations of an unconditioned Galton–Watson tree using a martingale which is simply the size of generation k of the tree (see e.g. [24, Chapter 12]). In a sense, the UIPT can thus be viewed as a Boltzmann triangulation conditioned to be infinite. This is related to the discussion in Section 6 of [6], which associates with a Boltzmann triangulation a multitype Galton–Watson tree describing the structure of balls, in such a way that the tree associated with the UIPT is just the same Galton–Watson tree conditioned on non-extinction.

Proof. It suffices to prove the second part of the theorem. Indeed, if (7) holds, we immediately get, for every $1 \le k \le \ell$, and every function F,

$$\mathbb{E}\big[\mathbf{M}_\ell F\big(B_k(T_{\mathrm{Bol}})\big)\big] = \mathbb{E}\big[\mathbf{M}_k F\big(B_k(T_{\mathrm{Bol}})\big)\big],$$

and it follows that $\mathbb{E}[\mathbf{M}_\ell \mid \mathcal{F}_k] = \mathbf{M}_k$.

In order to verify the second assertion of the theorem, we will provide explicit formulas for the probability that the ball of radius r in T_{Bol}, resp. in T_∞, is equal to a given triangulation with holes. Let $\mathbf{t}$ be a fixed triangulation with holes. Note that $\mathbb{P}(B_r(T_{\mathrm{Bol}}) = \mathbf{t}) > 0$ if and only if all vertices belonging to the boundaries of the holes of $\mathbf{t}$ are at distance r from the root vertex, and all faces of $\mathbf{t}$ other than the holes are incident to (at least) one vertex at distance at most $r-1$ from the root vertex. Furthermore, the preceding conditions are also necessary for $\mathbb{P}(B_r(T_\infty) = \mathbf{t})$ to be positive.

Write n for the total number of vertices of $\mathbf{t}$, $m \ge 0$ for the number of holes of $\mathbf{t}$ and $p_1, \dots, p_m$ for the respective sizes of the holes of $\mathbf{t}$ – the holes are enumerated in some deterministic manner given $\mathbf{t}$. Then, for every integer $q \ge n$, the number of triangulations with q vertices whose ball of radius r coincides with $\mathbf{t}$ is equal to

$$\sum_{n_1+\cdots+n_m=q-n} \left(\prod_{j=1}^m \#\mathcal{T}_{n_j,p_j}\right),$$

where the sum is over all choices of the nonnegative integers $n_1, \dots, n_m$ such that $n_1 + \cdots + n_m = q - n$, with the additional constraint that $n_i > 0$ if $p_i = 2$. The reason for this last constraint if the fact that a hole of size 2 cannot be filled by the trivial triangulation, because this would mean that we glue the two edges of the boundary. Note that when there is no hole ($m = 0$) the quantity in the last display should be interpreted as equal to 1 if $q = n$ and to 0 otherwise. The total Boltzmann weight of those triangulations whose ball of radius r coincides with $\mathbf{t}$ is then

$$\sum_{q=n}^{\infty} \left(\frac{2}{27}\right)^{q-2} Z(2)^{-1} \sum_{n_1+\cdots+n_m=q-n} \left(\prod_{j=1}^m \#\mathcal{T}_{n_j,p_j}\right),$$

where we impose the same constraint as before on the integers $n_1, \dots, n_m$ in the sum. We set $Z'(p) = Z(p)$ if $p > 2$ and $Z'(2) = Z(2) - 1$. The quantity in the last display equals

$$\left(\frac{2}{27}\right)^{n-2} Z(2)^{-1} \sum_{n_1=\mathbf{1}_{\{p_1=2\}}}^{\infty} \cdots \sum_{n_m=\mathbf{1}_{\{p_m=2\}}}^{\infty} \prod_{j=1}^m \left(\left(\frac{2}{27}\right)^{n_j} \#\mathcal{T}_{n_j,p_j}\right) = \left(\frac{2}{27}\right)^{n-2} Z(2)^{-1} \prod_{j=1}^m Z'(p_j),$$

and so we have proved that

$$\mathbb{P}\big(B_r(T_{\mathrm{Bol}}) = \mathbf{t}\big) = \left(\frac{2}{27}\right)^{n-2} Z(2)^{-1} \prod_{j=1}^m Z'(p_j). \tag{8}$$

Next consider the UIPT T_∞. We can similarly compute $\mathbb{P}(B_r(T_\infty) = \mathbf{t})$, using the fact that T_∞ is the local limit of triangulations with a large size. If, for every integer $q \ge 3$, $T_{(q)}$ denotes a uniformly distributed plane triangulation with q vertices, we have

$$\mathbb{P}\big(B_r(T_\infty) = \mathbf{t}\big) = \lim_{q\to\infty} \mathbb{P}\big(B_r(T_{(q)}) = \mathbf{t}\big).$$

Recalling that the number of rooted plane triangulations (of type II) with q vertices is $\#\mathcal{T}_{q-2,2}$ the same counting argument as above gives for $q \geq n$,

$$\mathbb{P}\big(B_r(T_{(q)}) = \mathbf{t}\big) = (\#\mathcal{T}_{q-2,2})^{-1} \sum_{n_1+\cdots+n_m=q-n} \left(\prod_{j=1}^{m} \#\mathcal{T}_{n_j,p_j}\right),$$

where the sum is again over nonnegative integers $n_1, \dots, n_m$ such that $n_1 + \cdots + n_m = q - n$, with the same additional constraint that $n_i > 0$ if $p_i = 2$. From the asymptotics in (1), it is an easy matter to verify that, for any $\varepsilon > 0$, we can choose K sufficiently large so that the asymptotic contribution of terms corresponding to choices of $n_1, \dots, n_m$ where $n_i \geq K$ for two distinct values of $i \in \{1, \dots, m\}$ is bounded above by ε (compare with [6, Lemma 2.5]). Thanks to this observation, we get from the asymptotics (1) that

$$\mathbb{P}\big(B_r(T_\infty) = \mathbf{t}\big) = \left(\frac{2}{27}\right)^{n-2} C(2)^{-1} \sum_{j=1}^{m} C(p_j) \sum_{n_1,\dots,n_{j-1},n_{j+1},\dots,n_m} \left(\prod_{\substack{i=1\\ i\neq j}}^{m} \left(\frac{2}{27}\right)^{n_i} \#\mathcal{T}_{n_i,p_i}\right),$$

where the second sum is over all choices of $n_1, \dots, n_{j-1}, n_{j+1}, \dots, n_m \geq 0$ such that $n_i > 0$ if $p_i = 2$. It follows that

$$\mathbb{P}\big(B_r(T_\infty) = \mathbf{t}\big) = \left(\frac{2}{27}\right)^{n-2} C(2)^{-1} \sum_{j=1}^{m} C(p_j) \left(\prod_{\substack{i=1\\ i\neq j}}^{m} Z'(p_i)\right). \tag{9}$$

Comparing (9) with (8), we get

$$\mathbb{P}\big(B_r(T_\infty) = \mathbf{t}\big) = \left(\frac{Z(2)}{C(2)} \sum_{j=1}^{m} \frac{C(p_j)}{Z'(p_j)}\right) \mathbb{P}\big(B_r(T_{\mathrm{Bol}}) = \mathbf{t}\big).$$

Note that, for every integer $p \geq 2$,

$$\frac{Z(2)}{C(2)} \frac{C(p)}{Z'(p)} = f(p),$$

and so we have obtained $\mathbb{P}(B_r(T_\infty) = \mathbf{t}) = g(\mathbf{t})\mathbb{P}(B_r(T_{\mathrm{Bol}}) = \mathbf{t})$, where $g(\mathbf{t}) := \sum_{j=1}^{m} f(p_j)$. Formula (7) now follows since $\mathbf{M}_r = g(B_r(T_{\mathrm{Bol}}))$ by definition. $\square$

Remark. *Formula* (9) *is obviously related to Proposition* 4.10 *in* [6]. *We did not use directly that result because it is apparently restricted to type III triangulations* (*the formula of Proposition* 4.10 *in* [6] *does not seem to take into account the possibility of holes of size* 2).

3. Asymptotics for a general peeling process

3.1. *Peeling*

The peeling process is an algorithmic procedure that "discovers" the UIPT step by step. We give a brief presentation of this algorithm and refer to [2–4,7] for details.

Formally, the algorithm produces a nested sequence of rooted triangulations with a simple boundary $\mathsf{T}_0 \subset \mathsf{T}_1 \subset \cdots \subset \mathsf{T}_n \subset \cdots \subset T_\infty$, such that, for every $i \geq 0$, conditionally on T_i, the remaining part $T_\infty \setminus \mathsf{T}_i$ has the same distribution as a UIPT of the $|\partial \mathsf{T}_i|$-gon (see [3, Section 1.2.2] for the definition of the UIPT of the p-gon).

Assuming that we are given the UIPT T_∞, the sequence $\mathsf{T}_0, \mathsf{T}_1, \dots$ is constructed inductively as follows. First T_0 is the trivial triangulation. Then, for every $n \geq 0$, conditionally on T_n we pick an edge e_n on $\partial \mathsf{T}_n$, *either deterministically (i.e. as a deterministic function of* T_n*) or via a randomized algorithm* that may involve only random quantities

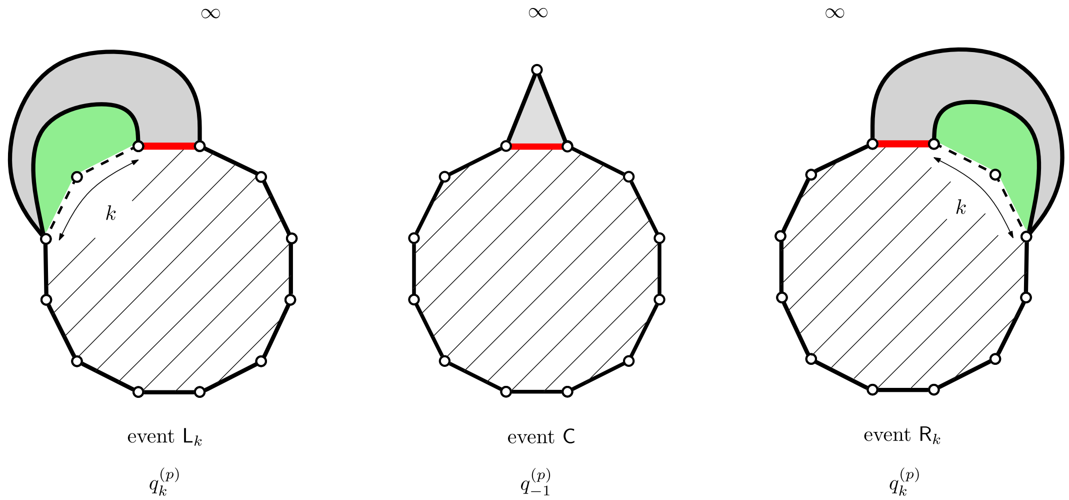

Fig. 3. Illustration of cases C, L_k, and R_k.

independent of T_∞. The triangulation T_{n+1} is then obtained by adding to T_n the triangle incident to e_n which was not contained in T_n (this is called the revealed triangle) and the bounded region that may be enclosed in the union of T_n and the revealed triangle. We sometimes say that T_{n+1} is obtained from T_n by peeling the edge e_n. Notice that, at the first step, there is only one (oriented) edge in the boundary of T_0, but we can choose to reveal the triangle on the right or on the left of this oriented edge.

The point is the fact that the distribution of the whole sequence $\mathsf{T}_0, \mathsf{T}_1, \dots$ can be described in a simple way and provides a construction of T_∞ (although this is not obvious, we shall see later that T_∞ is the limit of the finite triangulations T_n). Remarkably, the description of the law of $\mathsf{T}_0, \mathsf{T}_1, \dots$ is essentially the same independently of the (deterministic or randomized) algorithm that we use to choose the peeled edge at step n.

In order to describe the conditional law of T_{n+1} given T_n and the peeled edge e_n, we need to distinguish several cases. Suppose that at step $n \geq 0$ the triangulation T_n has a boundary of length p. The revealed triangle at time n may be of several different types (see Figure 3):

1. *Type* C: The revealed triangle has a vertex in the "unknown region." This occurs with probability

$$\mathbb{P}\big(\mathsf{C} \mid |\partial \mathsf{T}_n| = p\big) = q_{-1}^{(p)} = \frac{2}{27} \frac{C(p+1)}{C(p)}. \tag{10}$$

2. *Types* L_k *and* R_k: The three vertices of the revealed triangle lie on the boundary of T_n. This triangle thus "swallows" a piece of the boundary of $\partial \mathsf{T}_n$ of length $k \in \{1, \dots, p-2\}$. These events are denoted by R_k or L_k, depending on whether the edge of the revealed triangle that comes after the peeled edge in clockwise order is incident or not to the infinite part of the triangulation (see Figure 3). These events have a probability equal to

$$\mathbb{P}\big(\mathsf{L}_k \mid |\partial \mathsf{T}_n| = p\big) = \mathbb{P}\big(\mathsf{R}_k \mid |\partial \mathsf{T}_n| = p\big) := q_k^{(p)} = Z(k+1) \frac{C(p-k)}{C(p)}. \tag{11}$$

In cases R_k and L_k, we also need to specify the distribution of the triangulation with a boundary of length $k+1$ that is enclosed in the union of T_k and the revealed triangle. If by convention we root this triangulation at the unique edge of its boundary incident to the revealed triangle, we specify its distribution by saying that it is a Boltzmann triangulation of the $(k+1)$-gon. Note that when $k = 1$, there is a positive probability that this Boltzmann triangulation is the trivial one, and this simply means that the enclosed region is empty, or equivalently that the revealed triangle has two edges on the boundary of T_n.

The preceding considerations completely describe the distribution of the sequence $\mathsf{T}_0, \mathsf{T}_1, \dots$ – modulo of course the deterministic or randomized algorithm that is used at every step to select the peeled edge. The choices of types

C, L_k, and R_k, and of the Boltzmann triangulations that are used (whenever needed) to "fill in the holes" are made independently at every step with the probabilities given above.

At this point, we note that the geometry of the random triangulations T_n depends on the peeling algorithm used to choose the peeled edge at every step. On the other hand, it should be clear from the previous description that the law of the process $(|\mathsf{T}_n|, |\partial\mathsf{T}_n|)_{n\geq 0}$ does not depend on this algorithm. In the present section, we will be interested only in this process, and for this reason we do not need to specify the peeling algorithm. Later, in Sections 3 and 4, we will consider particular choices of the peeling algorithm, which are useful to investigate various properties of the UIPT.

To simplify notation, we set, for every $n \geq 0$,

$$P_n = |\partial\mathsf{T}_n| \quad \text{and} \quad V_n = |\mathsf{T}_n|.$$

In the remaining part of this section, we will prove Theorem 1 describing the scaling limit of the process $(P_n, V_n)_{n\geq 0}$ (see [3] and [7, Theorem 5] in the quadrangular case for related statements). We will also establish a few consequences of Theorem 1, which are of independent interest.

3.2. *The scaling limit of perimeters*

The description of the previous section shows that both processes $(P_n)_{n\geq 0}$ and $(P_n, V_n)_{n\geq 0}$ are Markov chains. The Markov chain $(P_n)_{n\geq 0}$ starts from $P_0 = 2$ and takes values in $\{2, 3, \ldots\}$. Its transition probabilities are given by

$$\mathbb{E}\big[f(P_{n+1}) \mid P_n\big] = f(P_n + 1) \cdot q_{-1}^{(P_n)} + 2\sum_{k=1}^{p-2} f(P_n - k) \cdot q_k^{(P_n)}. \tag{12}$$

Using (2), we may set $q_{-1} = \lim_{p\to\infty} q_{-1}^{(p)} = \frac{2}{3}$ and similarly $q_k = \lim_{p\to\infty} q_k^{(p)} = Z(k+1)9^{-k}$ for every $k \geq 1$. From (5) and (6), it is an easy matter to verify that

$$q_{-1} + 2\sum_{k\geq 1} q_k = 1 \quad \text{and} \quad q_{-1} - 2\sum_{k\geq 1} kq_k = 0,$$

so that the probability measure ν on $\mathbb{Z}$ given by $\nu(1) = q_{-1}$ and $\nu(-k) = 2q_k$ for every $k \geq 1$ is centered (note that ν is supported on $\{\ldots, -3, -2, -1, 1\}$). In fact, the weights q_i describe the law of the one-step peeling in the half-plane version of the UIPT, see [2,4].

We write $(W_n)_{n\geq 0}$ for a random walk with values in $\mathbb{Z}$, started from $W_0 = 2$ and with jump distribution ν. Notice that the jumps of W are bounded above by 1. Furthermore, using (4) we have for every $n \geq 0$,

$$\nu(-k) = 2q_k \underset{k\to\infty}{\sim} 2\mathsf{t}_{\triangle^2} k^{-5/2}. \tag{13}$$

It follows that ν is in the domain of attraction of a spectrally negative stable law of index $3/2$. This implies the convergence in distribution in the Skorokhod sense,

$$\left(\frac{W_{[nt]}}{\mathsf{p}_{\triangle^2} \cdot n^{2/3}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} (S_t)_{t\geq 0}, \tag{14}$$

where

$$\mathsf{p}_{\triangle^2} = \left(\frac{8\mathsf{t}_{\triangle^2}\sqrt{\pi}}{3}\right)^{2/3} = (2/3)^{2/3}, \tag{15}$$

and S is the stable Lévy process with index $3/2$ and no positive jumps, whose distribution is determined by the Laplace transform $\mathbb{E}[\exp(\lambda S_t)] = \exp(t\lambda^{3/2})$ for every $t, \lambda \geq 0$. Note that the Lévy measure of S is $\frac{3}{4\sqrt{\pi}}|x|^{-5/2}\mathbf{1}_{\{x<0\}}\mathrm{d}x$.

Our first objective is to get a scaling limit analogous to (14) for $(P_n)_{n\geq 0}$. To this end, recall from [8, Section VII.3] that one can define a process $(S_t^+)_{t\geq 0}$ with càdlàg sample paths, which is distributed as $(S_t)_{t\geq 0}$ "conditioned to stay positive forever." The scaling limit in the following result was suggested in [3] before Lemma 3.1. To simplify notation we write $[\![k,\infty[\![= \{k, k+1, k+2, \ldots\}$ and $]\!]-\infty, k]\!] = \{\ldots, k-2, k-1, k\}$ for every integer $k \in \mathbb{Z}$.

Proposition 6.

(i) *The Markov chain* $(P_n)_{n\geq 0}$ *is distributed as the random walk* $(W_n)_{n\geq 0}$ *conditioned not to hit* $]\!]-\infty,1]\!]$. *Equivalently*, $(P_n)_{n\geq 0}$ *is distributed as the* h*-transform of the random walk* $(W_n)_{n\geq 0}$ *killed upon hitting* $]\!]-\infty,1]\!]$, *where the function* h *defined on* $\mathbb{Z}$ *by*

$$h(p) := \begin{cases} 9^{-p}C(p) & \text{if } p\geq 2, \\ 0 & \text{if } p\leq 1, \end{cases} \tag{16}$$

is, *up to multiplication by a positive constant*, *the unique nontrivial nonnegative function that is* ν*-harmonic on* $[\![2,\infty[\![$ *and vanishes on* $]\!]-\infty,1]\!]$.

(ii) *The following convergence in distribution holds in the Skorokhod sense*,

$$\left(\frac{P_{[nt]}}{\mathsf{p}_{\triangle^2}\cdot n^{2/3}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(S_t^+\right)_{t\geq 0}, \tag{17}$$

where we recall that $\mathsf{p}_{\triangle^2} = (2/3)^{2/3}$.

Proof. (i) Let h be defined by (16). From the explicit formulas (10) and (11), one immediately gets that, for every $p\geq 2$ and every $k\in\{-1,1,2,\dots,p-2\}$,

$$q_k^{(p)} = \frac{h(p-k)}{h(p)}q_k. \tag{18}$$

It then follows from (12) and the definition of ν that, for every $p\geq 2$ and $k\in\{-p+2,-p+3,\dots,-1,1\}$,

$$\mathbb{P}(P_{n+1}=p+k \mid P_n=p) = \frac{h(p+k)}{h(p)}\nu(k) = \frac{h(p+k)}{h(p)}\mathbb{P}(W_{n+1}=p+k \mid W_n=p). \tag{19}$$

By summing over k, we get, for every $p\geq 2$,

$$\sum_{k\in\mathbb{Z}}\frac{h(p+k)}{h(p)}\nu(k) = 1$$

so that h is ν-harmonic on $[\![2,\infty[\![$. Note that the uniqueness (up to a multiplicative constant) of a positive function that is ν-harmonic on $[\![2,\infty[\![$ and vanishes on $]\!]-\infty,1]\!]$ is easy, since, for every $p\geq 2$, the value of this function at $p+1$ is determined from its values for $2\leq i\leq p$. Furthermore, formula (19) precisely says that $(P_n)_{n\geq 0}$ is distributed as the h-transform of the random walk $(W_n)_{n\geq 0}$ killed upon hitting $]\!]-\infty,1]\!]$. The fact that this h-transform can be interpreted as the random walk W conditioned to stay in $[\![2,\infty[\![$ is classical, see e.g. [9].

(ii) This follows from the invariance principle proved in [12]. □

From (2), we have

$$h(p) \underset{p\to\infty}{\sim} \frac{1}{54\pi\sqrt{3}}\sqrt{p}. \tag{20}$$

Still from (2), we can write, for $p\geq 2$,

$$h(p) = \frac{1}{3^{7/2}4\sqrt{\pi}}\frac{(2p-3)\times(2p-5)\times\cdots\times 3\times 1}{(2p-4)\times(2p-6)\times\cdots\times 4\times 2},$$

so that $h(p+1)/h(p) = (2p-1)/(2p-2)$, proving that h is monotone increasing on $[\![2,\infty[\![$. Then, for every $j\geq 1$, and every p with $p\geq j+2$,

$$q_j^{(p)} = \frac{h(p-j)}{h(p)}q_j \leq q_j \tag{21}$$

and similarly, for every $p \geq 2$,

$$q_{-1}^{(p)} = \frac{h(p+1)}{h(p)} q_{-1} \geq q_{-1}. \tag{22}$$

These bounds will be useful later.

3.3. *A few applications*

Let us give a few applications of Proposition 6. First, it is easy to recover from this proposition the known fact (see [3, Claim 3.3]) that the Markov chain $(P_n)_{n\geq 0}$ is transient,

$$P_n \xrightarrow[n\to\infty]{\text{a.s.}} +\infty. \tag{23}$$

To see this, let $p \geq 2$ and write $\mathbb{P}_p$ for a probability measure under which the random walk W with jump distribution ν starts from p. For every $y \in \mathbb{Z}$, set $T_y = \min\{n \geq 0 : W_n = y\}$. Note that $T_y < \infty$ a.s. because the random walk W is recurrent. Similarly, suppose that $\widetilde{T}_y$ is distributed under $\mathbb{P}_p$ as the hitting time of y for a Markov chain with the same transition kernel as $(P_n)_{n\geq 0}$ but started from p. Then, standard properties of h-transforms give for every $p, y \in [\![2, \infty[\![$,

$$\mathbb{P}_p(\widetilde{T}_y < \infty) = \frac{h(y)}{h(p)} \mathbb{P}_p(W_k \geq 2, \forall k \leq T_y).$$

Since h is monotone increasing on $[\![2, \infty[\![$, the right-hand side is smaller than 1 when $p > y$, giving the desired transience.

The following corollary was conjectured in [7, Section 5.1].

Corollary 7. *Any peeling $(\mathsf{T}_n)_{n\geq 0}$ of the UIPT will eventually discover T_∞ entirely, that is*

$$\bigcup_{n\geq 0} \mathsf{T}_n = T_\infty, \quad a.s.$$

Proof. It is enough to prove that, if $n_0 \geq 1$ is fixed, then a.s. every vertex of $\partial \mathsf{T}_{n_0}$ belongs to the interior of T_{n_1} for some $n_1 > n_0$ sufficiently large. Indeed, if this property holds, an inductive argument shows that the minimal distance between a vertex outside T_n and the root tends to infinity as $n \to \infty$, which gives the desired result.

So let us fix n_0 and a vertex v of $\partial \mathsf{T}_{n_0}$, and argue conditionally on T_{n_0} and v. We note that, for every $n \geq n_0$, conditionally on the event that v is still on the boundary of T_n, the probability that v will be "surrounded" by the revealed triangle at step $n+1$, and therefore will belong to the interior of T_{n+1}, is at least

$$\sum_{k=[P_n/2]+1}^{P_n-2} q_k^{(P_n)}$$

with the convention that the sum is 0 if $[P_n/2]+1 > P_n - 2$. If P_n is large enough, the latter quantity is bounded below by

$$\sum_{k=[P_n/2]+1}^{[3P_n/4]} q_k^{(P_n)} = \sum_{k=[P_n/2]+1}^{[3P_n/4]} \frac{h(P_n-k)}{h(P_n)} q_k \geq c P_n^{-3/2},$$

where c is a positive constant and we used (4) and (20) in the last inequality. Recalling that $P_n \to \infty$ a.s., we see that the proof will be complete if we can verify that the series

$$\sum_{n=1}^{\infty} P_n^{-3/2}$$

diverges a.s.

To this end, we argue by contradiction and assume that we can find two constants $M<\infty$ and $\varepsilon>0$ such that the probability of the event

$$\left\{\sum_{n=1}^{\infty} P_n^{-3/2} \le M\right\}$$

is greater than ε. On this event, for any $t>1$ and any $n\ge 1$, we have

$$\int_1^t \mathrm{d}u \left(\frac{P_{[nu]}}{n^{2/3}}\right)^{-3/2} \le \frac{1}{n}\sum_{i=n}^{[nt]}\left(\frac{P_i}{n^{2/3}}\right)^{-3/2} = \sum_{i=n}^{[nt]} P_i^{-3/2} \le M.$$

Using the convergence of Proposition 6(ii), we obtain that, for every $t>1$, the probability of the event $\{\int_1^t \mathrm{d}u (S_u^+)^{-3/2} \le (\mathsf{p}_{\triangle^2})^{-3/2} M\}$ is greater than ε. Letting $t\to\infty$ we get that

$$\mathbb{P}\left(\int_1^{\infty} \frac{\mathrm{d}u}{(S_u^+)^{3/2}} \le (\mathsf{p}_{\triangle^2})^{-3/2} M\right) \ge \varepsilon.$$

This is a contradiction because

$$\int_1^{\infty} \frac{\mathrm{d}u}{(S_u^+)^{3/2}} = \infty \quad \text{a.s.}$$

as can be seen by an application of Jeulin's lemma [20, Proposition 4 c)], noting that we have $(S_u^+)^{-3/2} \overset{(d)}{=} u^{-1}(S_1^+)^{-3/2}$ by scaling and that the law of S_1^+ is diffuse, for instance by [8, Corollary VII.16]. $\square$

The next lemma will be an important tool in the proof of Theorems 2 and 4.

Lemma 8. *There exist two constants $0<c_1<c_2<\infty$ such that, for all $n\ge 1$, we have*

$$c_1 n^{-2/3} \le \mathbb{E}\left[\frac{1}{P_n}\right] \le c_2 n^{-2/3}.$$

Proof. The lower bound is easy since Proposition 6(ii) gives

$$\mathbb{E}\left[\frac{n^{2/3}}{P_n}\right] \ge \mathbb{E}\left[\frac{n^{2/3}}{P_n}\wedge 1\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\frac{1}{\mathsf{p}_{\triangle^2} S_1^+}\wedge 1\right] > 0.$$

To prove the upper bound, we first fix $k\ge 2$ and $n\ge 1$, and we evaluate $\mathbb{P}(P_n=k)$. By Proposition 6(i) and properties of h-transforms, we have

$$\mathbb{P}(P_n=k) = \frac{h(k)}{h(2)}\cdot \mathbb{P}\big(\{W_i\ge 2, \forall i\le n\}\cap\{W_n=k\}\big).$$

We set $\widetilde{W}_i = W_n - W_{n-i}$ for $0\le i\le n$ and note that we can also define $\widetilde{W}_i$ for $i>n$ in such a way that $(\widetilde{W}_i)_{i\ge 0}$ is a random walk with the same jump distribution as W and $\widetilde{W}_0=0$. We have then

$$\mathbb{P}\big(\{W_i\ge 2, \forall 0\le i\le n\}\cap\{W_n=k\}\big) = \mathbb{P}\big(\{\widetilde{W}_n=k-2\}\cap\{\widetilde{W}_i\le k-2, \forall i\le n\}\big) = \frac{\mathbb{P}(\widetilde{T}_{k-1}=n+1)}{q_{-1}},$$

where we have set $\widetilde{T}_{k-1}=\min\{i\ge 0: \widetilde{W}_i=k-1\}$. Note that $\widetilde{W}$ has positive jumps only of size 1. We can thus use Kemperman's formula (see e.g. [28, p. 122]) to get

$$\mathbb{P}(\widetilde{T}_{k-1}=n+1) = \frac{k-1}{n+1}\mathbb{P}(\widetilde{W}_{n+1}=k-1).$$

From the last three displays, we have

$$\mathbb{P}(P_n = k) = \frac{3}{2}\frac{h(k)}{h(2)}\frac{k-1}{n+1}\mathbb{P}(\widetilde{W}_{n+1} = k-1).$$

Using the local limit theorem for random walk in the domain of attraction of a stable distribution (see e.g. [19, Theorem 4.2.1]), we can find a constant c'' such that

$$\mathbb{P}(\widetilde{W}_n = k) \le c'' n^{-2/3}, \tag{24}$$

for every $n \ge 1$ and $k \in \mathbb{Z}$. Then, for every $n \ge 1$,

$$\begin{aligned}\mathbb{E}\left[\frac{1}{P_n}\right] &= \mathbb{E}\left[\frac{1}{P_n}\mathbf{1}_{\{P_n > n^{2/3}\}}\right] + \mathbb{E}\left[\frac{1}{P_n}\mathbf{1}_{\{P_n \le n^{2/3}\}}\right] \\ &\le n^{-2/3} + \sum_{k=1}^{[n^{2/3}]} \frac{3}{2}\frac{h(k)}{h(2)}\frac{k-1}{n+1}\frac{1}{k}\mathbb{P}(\widetilde{W}_{n+1} = k-1) \\ &\le n^{-2/3} + \frac{3c''}{2h(2)} n^{-5/3} \sum_{k=1}^{[n^{2/3}]} h(k).\end{aligned}$$

The upper bound of the lemma follows using (20). □

3.4. *The scaling limit of volumes*

Our goal is now to study the scaling limit of the process $(V_n)_{n\ge 0}$. We start with a result similar to [3, Proposition 6.4] about the distribution of the size of a Boltzmann triangulation with a large perimeter. For every $p \ge 2$, we let $T^{(p)}$ denote a random triangulation of the p-gon with Boltzmann distribution.

Proposition 9. *Set* $\mathsf{b}_{\triangle^2} = \frac{2}{3}$.

1. *We have* $\mathbb{E}[|T^{(p)}|] \sim \mathsf{b}_{\triangle^2} \cdot p^2$ *as* $p \to \infty$.
2. *The following convergence in distribution holds*:

$$p^{-2}\big|T^{(p)}\big| \xrightarrow[p\to\infty]{(d)} \mathsf{b}_{\triangle^2} \cdot \xi,$$

where ξ *is a random variable with density* $\frac{e^{-1/2x}}{x^{5/2}\sqrt{2\pi}}$ *on* $\mathbb{R}_+$.

Remark. *We have* $\mathbb{E}[\xi] = 1$ *and the size-biased version of the distribution of* ξ (*with density* $\frac{e^{-1/2x}}{x^{3/2}\sqrt{2\pi}}$ *on* $\mathbb{R}_+$) *is the* $1/2$-*stable distribution with Laplace transform* $e^{-\sqrt{2\lambda}}$. *Consequently, for* $\lambda > 0$, *we have*

$$\mathbb{E}\big[e^{-\lambda\xi}\big] = (1 + \sqrt{2\lambda})e^{-\sqrt{2\lambda}}.$$

Proof of Proposition 9. The first assertion follows from the formula $\mathbb{E}[|T^{(p)}|] = \frac{1}{3}(p-1)(2p-3)$ for $p \ge 2$ which is easily derived from the exact formula for the generating function of the sequence $(\#\mathcal{T}_{n,p})_{n\ge 0}$ found in [6, Proposition 2.4]. See also [29, Proposition 3.4].

For the second assertion, we proceed as in [3, Proposition 6.4]. From the explicit expressions (1) and (3), an asymptotic expansion using Stirling's formula shows that, for every fixed $x > 0$, we have

$$p^2\mathbb{P}\big(\big|T^{(p)}\big| = \big[p^2x\big]\big) = p^2\frac{(2/27)^{[p^2x]}\#\mathcal{T}_{[p^2x],p}}{Z(p)} \xrightarrow[p\to\infty]{} \frac{2e^{-1/(3x)}}{3x^{5/2}\sqrt{3\pi}},$$

and the convergence holds uniformly when x varies over a compact subset of $\mathbb{R}_+$. Since the right-hand side of the last display is the density of the variable $2\xi/3$, the desired result follows. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. We will verify that

$$\left(\frac{P_{[nt]}}{\mathsf{p}_{\triangle^2}\cdot n^{2/3}},\frac{V_{[nt]}}{\mathsf{v}_{\triangle^2}\cdot n^{4/3}}\right)_{0\le t\le 1}\xrightarrow[n\to\infty]{(d)}(S_t^+,Z_t)_{0\le t\le 1}. \tag{25}$$

The statement of Theorem 1 follows, noting that there is no loss of generality in restricting the time interval to [0, 1]. The constant $\mathsf{v}_{\triangle^2}$ will appear below as

$$\mathsf{v}_{\triangle^2}=(\mathsf{p}_{\triangle^2})^2\mathsf{b}_{\triangle^2}. \tag{26}$$

The convergence of the first component in (25) is given by Proposition 6. We will thus study the conditional distribution of the second component given the first one, and Proposition 9 will be our main tool. We first note that, for every $n\ge 1$, we can write

$$V_n=|\mathsf{T}_n|=V_n^*+\widetilde{V}_n,$$

where V_n^* denotes the number of inner vertices of T_n that belong to $\partial\mathsf{T}_i$ for some $i\le n-1$, and $\widetilde{V}_n$ is thus the total number of inner vertices in the Boltzmann triangulations that were used to fill in the holes in the case of occurrence of events L_k or R_k at some step $i\le n$ of the peeling process. Since $\#(\partial\mathsf{T}_i\setminus\partial\mathsf{T}_{i-1})\le 1$ for $1\le i\le n$, it is clear that $V_n^*\le n+2$ for every $n\ge 0$. It follows that (25) is equivalent to the same statement where $V_{[nt]}$ is replaced by $\widetilde{V}_{[nt]}$.

Next we can write, for every $k\in\{1,\dots,n\}$,

$$\widetilde{V}_k=\sum_{i=1}^k\mathbf{1}_{\{P_i<P_{i-1}\}}U_i, \tag{27}$$

where, conditionally on $(P_0,P_1,\dots,P_n)$, the random variables U_i (for i such that $P_i<P_{i-1}$) are independent, and U_i is distributed as $|T^{(P_{i-1}-P_i+1)}|$, with the notation of Proposition 9.

Fix $\varepsilon>0$ and set, for every $k\in\{1,\dots,n\}$,

$$\widetilde{V}_k^{\le\varepsilon}=\sum_{i=1}^k\mathbf{1}_{\{0<P_{i-1}-P_i\le\varepsilon n^{2/3}\}}U_i,\qquad \widetilde{V}_k^{>\varepsilon}=\sum_{i=1}^k\mathbf{1}_{\{P_{i-1}-P_i>\varepsilon n^{2/3}\}}U_i. \tag{28}$$

We first observe that $n^{-4/3}\mathbb{E}[\widetilde{V}_n^{\le\varepsilon}]$ is small uniformly in n when ε is small. Indeed, it follows from Proposition 9 that there is a constant C such that $\mathbb{E}[|T^{(p)}|]\le Cp^2$ for every $p\ge 2$, which gives

$$\mathbb{E}\big[\widetilde{V}_n^{\le\varepsilon}\big]\le C\sum_{i=1}^n\mathbb{E}\big[(P_{i-1}-P_i+1)^2\mathbf{1}_{\{0<P_{i-1}-P_i\le\varepsilon n^{2/3}\}}\big].$$

On the other hand, from the bound (21) and (4), it is straightforward to verify that, for every $i\ge 1$ and every $p\ge 2$,

$$\mathbb{E}\big[(P_{i-1}-P_i+1)^2\mathbf{1}_{\{0<P_{i-1}-P_i\le\varepsilon n^{2/3}\}}\mid P_{i-1}=p\big]\le C'\sum_{j=1}^{[\varepsilon n^{2/3}]}(j+1)^2j^{-5/2}\le C''\sqrt{\varepsilon}n^{1/3},$$

with some constants C' and C'' independent of n and ε. By combining the last two displays, we obtain, for every $n\ge 1$,

$$n^{-4/3}\mathbb{E}\big[\widetilde{V}_n^{\le\varepsilon}\big]\le CC''\sqrt{\varepsilon}. \tag{29}$$

Let us turn to $\widetilde{V}_n^{>\varepsilon}$. We write $s_1, s_2, \dots$ for the jump times of S^+ before time 1 listed in decreasing order of their absolute values. For every $n \ge 1$, let $\ell_1^{(n)}, \dots, \ell_{k_n}^{(n)}$ be all integers $i \in \{1, \dots, n\}$ such that $P_{i-1} - P_i > 0$, listed in decreasing order of the quantities $P_{i-1} - P_i$ (and in the usual order of $\mathbb{N}$ for indices such that $P_{i-1} - P_i$ is equal to a given value). For definiteness, we also set $\ell_i^{(n)} = 1$ if $i > k_n$. It follows from (17) that, for every integer $K \ge 1$,

$$\begin{aligned}&\big(n^{-1}\ell_1^{(n)}, \dots, n^{-1}\ell_K^{(n)}, n^{-2/3}\big(P^{(n)}_{\ell_1^{(n)}} - P^{(n)}_{\ell_1^{(n)}-1}\big), \dots, n^{-2/3}\big(P_{\ell_K^{(n)}} - P_{\ell_K^{(n)}-1}\big)\big)\\ &\quad\xrightarrow[n\to\infty]{(d)} \big(s_1, \dots, s_K, \mathsf{p}_{\triangle^2}\Delta S^+_{s_1}, \dots, \mathsf{p}_{\triangle^2}\Delta S^+_{s_K}\big),\end{aligned} \tag{30}$$

and this convergence in distribution holds jointly with (17). Furthermore, using the conditional distribution of the variables U_i given $(P_0, \dots, P_n)$ and Proposition 9, we also get, for every integer $K \ge 1$,

$$\left(\frac{U^{(n)}_{\ell_1^{(n)}}}{(P_{\ell_1^{(n)}} - P_{\ell_1^{(n)}-1})^2}, \dots, \frac{U^{(n)}_{\ell_K^{(n)}}}{(P_{\ell_K^{(n)}} - P_{\ell_K^{(n)}-1})^2}\right) \xrightarrow[n\to\infty]{(d)} \big(\mathsf{b}_{\triangle^2}\xi_1, \dots, \mathsf{b}_{\triangle^2}\xi_K\big), \tag{31}$$

where $\xi_1, \xi_2, \dots$ are independent copies of the variable ξ of Proposition 9. This convergence holds jointly with (17) and (30), provided that we assume that the sequence $\xi_1, \xi_2, \dots$ is independent of S^+. Now note that we can choose K sufficiently large so that the probability that $|\Delta S^+_{s_K}| < \varepsilon/(2\mathsf{p}_{\triangle^2})$ is arbitrarily close to 1. Recalling the definition of $\widetilde{V}_n^{>\varepsilon}$, we can combine (30) and (31) in order to get the convergence

$$\big(n^{-2/3}P_{[nt]}, n^{-4/3}\widetilde{V}^{>\varepsilon}_{[nt]}\big)_{0\le t\le 1} \xrightarrow[n\to\infty]{(d)} \big(\mathsf{p}_{\triangle^2}S^+_t, (\mathsf{p}_{\triangle^2})^2\mathsf{b}_{\triangle^2}Z^\varepsilon_t\big)_{0\le t\le 1}, \tag{32}$$

where the process $(Z_t^\varepsilon)_{0\le t\le 1}$ is defined by

$$Z_t^\varepsilon = \sum_{i=1}^\infty \mathbf{1}_{\{s_i \le t, |\Delta S^+_{s_i}| > \varepsilon/\mathsf{p}_{\triangle^2}\}}\big(\Delta S^+_{s_i}\big)^2 \xi_i.$$

In agreement with the notation of the introduction, set, for every $0 \le t \le 1$,

$$Z_t = \sum_{i=1}^\infty \mathbf{1}_{\{s_i \le t\}}\big(\Delta S^+_{s_i}\big)^2 \xi_i.$$

Then, it is easy to verify that, for every $\delta > 0$,

$$\mathbb{P}\Big(\sup_{0\le t\le 1}\big|Z_t - Z_t^\varepsilon\big| > \delta\Big) \xrightarrow[\varepsilon\to 0]{} 0.$$

Furthermore, (29) also gives

$$\sup_{n\ge 1}\mathbb{P}\Big(\sup_{0\le t\le 1}\big|n^{-4/3}\widetilde{V}_{[nt]} - n^{-4/3}\widetilde{V}^{\ge\varepsilon}_{[nt]}\big| > \delta\Big) \xrightarrow[\varepsilon\to 0]{} 0.$$

The convergence (25), with V replaced by $\widetilde{V}$, follows from (32) and the preceding considerations. This completes the proof. □

4. Distances in the peeling process

4.1. *Peeling by layers*

In this section, we focus on a particular peeling algorithm, which we call the peeling by layers. As previously, we start from the trivial triangulation that consists only of the root edge. At the first step, we discover the triangle on the left

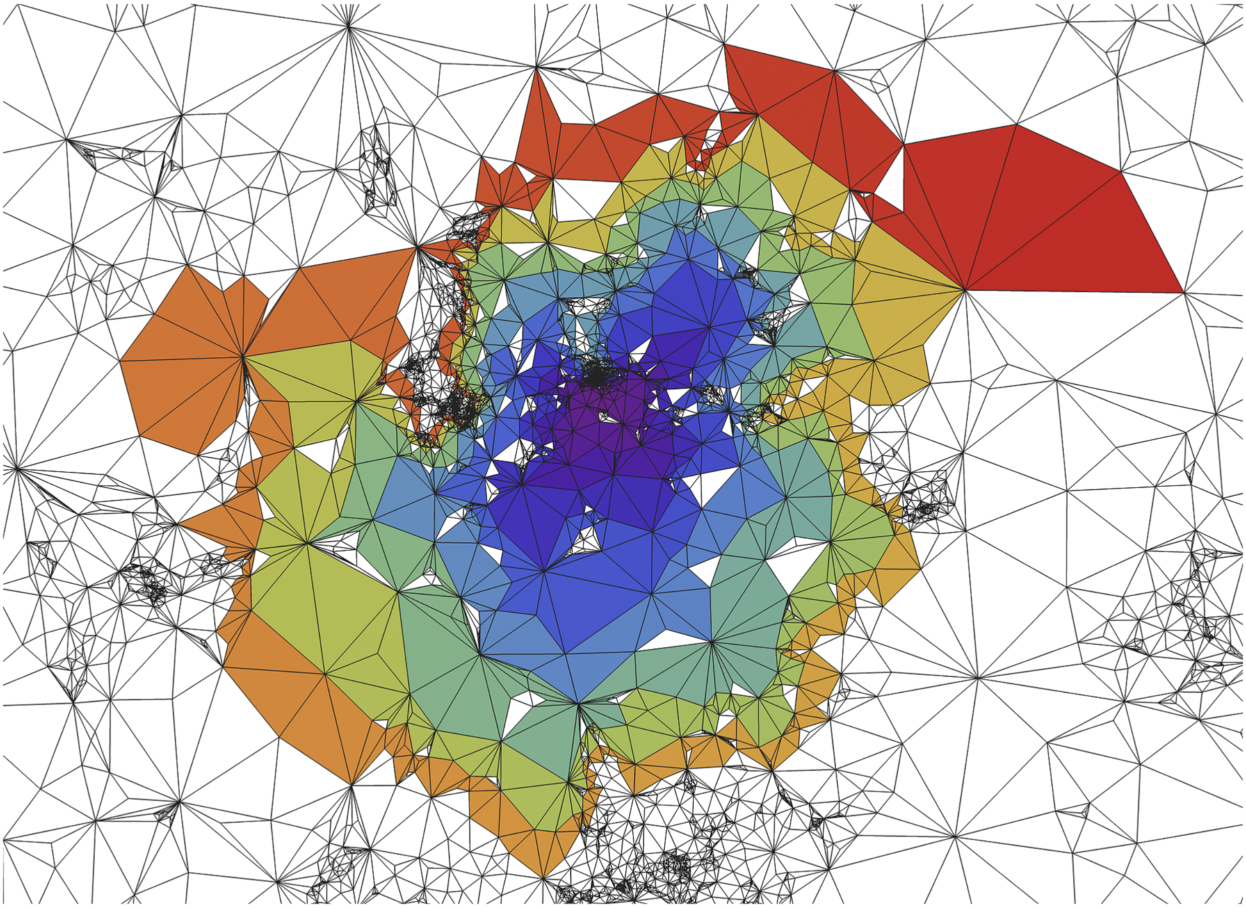

Fig. 4. The peeling by layers algorithm in a random triangulation drawn in the plane via Tutte's barycentric embedding. The successive layers are represented with different colors. Courtesy of Timothy Budd. See https://www.youtube.com/watch?v=afR9yo1P9vE for the associated movie.

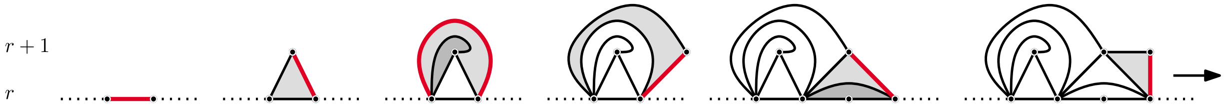

Fig. 5. Illustration of the peeling by layers. When $B_r^\bullet(T_\infty)$ has been discovered, we turn around the boundary $\partial B_r^\bullet(T_\infty)$ from left to right in order to reveal the next layer and obtain $B_{r+1}^\bullet(T_\infty)$.

side of the root edge to get T_1. To get T_2, we then discover the triangle on the right side of the root edge. Then we continue by induction in the following way. We note that the triangle revealed at step n has either one or two edges in the boundary of T_n. If it has one edge in the boundary, we discover at step $n+1$ the triangle incident to this edge which is not already in T_n. If it has two edges in the boundary, we do the same for the right-most among these two edges (this makes sense because in that case the boundary of T_n must contain at least 3 edges). See Figures 4 and 5 for an example.

This algorithm is particularly well suited to the study of distances from the root vertex, for the following reason. One easily proves by induction that, for every $n \geq 1$, one and only one of the two following possibilities occurs. Either all vertices of $\partial\mathsf{T}_n$ are at the same distance h from the root vertex. Or there is an integer $h \geq 0$ such that $\partial\mathsf{T}_n$ contains both vertices at distance h and at distance $h+1$ from the root vertex. In the latter case, vertices at distance h form a connected subset of $\partial\mathsf{T}_n$, and the edge that will be "peeled off" at step $n+1$ is the only edge of the boundary whose left end is at distance $h+1$ and whose right end is at distance h. In both cases we write $H_n = h$, so that the boundary

$\partial\mathsf{T}_n$ does contain vertices at distance H_n and may also contain vertices at distance H_n+1. We also set $H_0=0$ by convention.

Since the peeling algorithm discovers the whole triangulation T_∞ (Corollary 7), it is clear that H_n tends to ∞ as $n\to\infty$. Also obviously $0\le H_{n+1}-H_n\le 1$ for every $n\ge 1$, hence we may set $\sigma_r:=\min\{n\ge 0: H_n=r\}$ for every integer $r\ge 1$. A simple argument shows that for $n=\sigma_r$, all vertices of $\partial\mathsf{T}_n$ are at distance r from the root vertex (this however does not characterize σ_r since there may exist other times $n>\sigma_r$ with the same property). Furthermore, any vertex lying outside T_{σ_r} must be at distance at least $r+1$ from the root vertex, and any triangle of T_{σ_r} that is incident to an edge of the boundary contains a vertex at distance $r-1$ from the root vertex (indeed this triangle has been discovered by the peeling algorithm at a time where the boundary still contained vertices at distance $r-1$, and the corresponding peeled edge had to connect a vertex at distance r to a vertex at distance $r-1$). It follows from the previous considerations that we have $\mathsf{T}_{\sigma_r}=B^\bullet_r(T_\infty)$ for every $r\ge 1$. Furthermore, for every $n\ge 1$ such that $H_n>0$, we have $\sigma_{H_n}\le n<\sigma_{H_n+1}$ and therefore

$$B^\bullet_{H_n}(T_\infty)\subset\mathsf{T}_n\subset B^\bullet_{H_n+1}(T_\infty). \tag{33}$$

This also holds for n such that $H_n=0$, provided we define $B^\bullet_0(T_\infty)$ as the trivial triangulation consisting only of the root edge.

An important consequence is the following fact, which needs not be true for a general peeling algorithm. If $\mathcal{F}_n$ stands for the σ-field generated by $\mathsf{T}_0,\mathsf{T}_1,\dots,\mathsf{T}_n$, then the graph distances of vertices of T_n from the root vertex are measurable with respect to $\mathcal{F}_n$. This is clear since (33) shows that a geodesic from any vertex of T_n to the root visits only vertices of T_n.

At an intuitive level, the peeling algorithm "turns" around the boundary of the hull of balls of the UIPT in clockwise order and discovers T_∞ layer after layer. When turning around $\partial B^\bullet_r(T_\infty)$, the peeling process creates new vertices at distance $r+1$ from the root vertex in a way similar to a front propagation. See Figure 5.

To simplify notation, we write $B^\bullet_r$ and $\partial B^\bullet_r$ instead of $B^\bullet_r(T_\infty)$ and $\partial B^\bullet_r(T_\infty)$ in this section. As (33) suggests, the proof of Theorem 2 will rely on the convergence in distribution of a rescaled version of the process H_n. Let us sketch some ideas of the proof of the latter convergence. Between times σ_r and σ_{r+1}, the peeling process needs to turn around $\partial B^\bullet_r$, which roughly takes a time linear in $|\partial B^\bullet_r|$ (see Proposition 11 below for a precise statement). We thus expect that, for some positive constant a,

$$\sigma_{r+1}-\sigma_r\approx\frac{1}{a}\left|\partial B^\bullet_r\right|=\frac{1}{a}P_{\sigma_r} \tag{34}$$

and therefore

$$\sigma_r\approx\frac{1}{a}\sum_{i=1}^{r-1}P_{\sigma_i}.$$

A formal inversion now gives for k large,

$$H_k=\sup\{r\ge 0:\sigma_r\le k\}\approx a\sum_{i=1}^{k}\frac{1}{P_i},$$

and the limit behavior of the right-hand side can be derived from the fact that $(n^{-2/3}P_{[nt]})_{t\ge 0}$ converges in distribution to $(\mathsf{p}_{\triangle^2}S^+_t)_{t\ge 0}$ (Proposition 6). The following proposition shows that the previous heuristic considerations are indeed correct with the value of a given by $\mathsf{a}_{\triangle^2}=1/3$ (note that $\mathsf{h}_{\triangle^2}$ in Proposition 10 below is then equal to $\mathsf{a}_{\triangle^2}/\mathsf{p}_{\triangle^2}$, and see also Proposition 11).

Proposition 10 (Distances in the peeling by layers). *We have the following convergence in distribution for the Skorokhod topology*

$$\left(\frac{P_{[nt]}}{\mathsf{p}_{\triangle^2}\cdot n^{2/3}},\frac{V_{[nt]}}{\mathsf{v}_{\triangle^2}\cdot n^{4/3}},\frac{H_{[nt]}}{\mathsf{h}_{\triangle^2}\cdot n^{1/3}}\right)_{t\ge 0}\xrightarrow[n\to\infty]{(d)}\left(S^+_t,Z_t,\int_0^t\frac{\mathrm{d}u}{S^+_u}\right)_{t\ge 0},$$

where $\mathsf{h}_{\triangle^2}=12^{-1/3}$.

Noting that $|B_r^\bullet| = V_{\sigma_r}$ and $|\partial B_r^\bullet| = P_{\sigma_r}$, we will derive Theorem 2 from the last proposition via a time change argument in Section 4.4. This derivation involves time-changing the limiting processes S_t^+ and Z_t by the inverse of the increasing process $\int_0^t \frac{\mathrm{d}u}{S_u^+}$, which is clearly related to the Lamperti transformation connecting continuous-state branching processes to spectrally positive Lévy processes. In the next section, we state and prove Proposition 11, which is the key ingredient of the proof of Proposition 10. The latter proof will be given in Section 4.3.

4.2. *Turning around layers*

We write $\mathcal{L}$ for the set of all edges of T_∞ that are part of $\partial B_r^\bullet$ for some integer $r \geq 1$. Note that all these edges belong to $\partial \mathsf{T}_n$ for some $n \geq 1$ (because we know that $B_r^\bullet = \mathsf{T}_{\sigma_r}$ for every $r \geq 1$), but the converse is not true. For every $n \geq 0$, we write A_n for the number of edges of $\mathcal{L}$ belonging to $\mathsf{T}_n \setminus \partial \mathsf{T}_n$.

Clearly, $(A_n)_{n\geq 0}$ is an increasing process. Also, recalling our notation $\mathcal{F}_n$ for the σ-field generated by $\mathsf{T}_0, \mathsf{T}_1, \dots, \mathsf{T}_n$, the random variable A_n is measurable with respect to $\mathcal{F}_n$. The point is that, on one hand, the hulls $B_1^\bullet, \dots, B_{H_n}^\bullet$ are measurable functions of T_n, and, on the other hand, edges of $\mathsf{T}_n \setminus \partial \mathsf{T}_n$ which may be in $\mathcal{L}$ (i.e. which link two vertices at the same distance from the root) are at distance at most H_n from the root (here it is important that we considered only edges of $\mathsf{T}_n \setminus \partial \mathsf{T}_n$ in the definition of A_n, since the σ-field $\mathcal{F}_n$ does not give enough information to decide whether an edge of $\partial \mathsf{T}_n$ linking two vertices at distance $H_n + 1$ from the root belongs to $\mathcal{L}$ or not).

Proposition 11. *We have*

$$\frac{A_n}{n} \xrightarrow[n\to\infty]{(P)} \frac{1}{3} =: \mathsf{a}_{\triangle^2}.$$

Proof. We use the notation $\Delta A_n = A_{n+1} - A_n$ for every $n \geq 0$. We note that the inner edges of the Boltzmann triangulations that are used to fill in the holes created by the peeling algorithm cannot be in $\mathcal{L}$, and it follows that we have

$$0 \leq \Delta A_n \leq (\Delta P_n)_- + 1 \tag{35}$$

for every $n \geq 0$, the additional term 1 coming from the fact that the edge that is peeled at time n could actually be in $\mathcal{L}$ (this happens only at times of the form $n = \sigma_r$). In particular $\mathbb{E}[\Delta A_n] < \infty$ and $\mathbb{E}[A_n] < \infty$. We then set, for every $i \geq 0$,

$$\eta_i = \mathbb{E}[\Delta A_i \mid \mathcal{F}_i],$$

so that $M_n := A_n - \sum_{i=0}^{n-1} \eta_i$ is a martingale with respect to the filtration $(\mathcal{F}_n)$.

We first prove that $M_n/n \to 0$ in probability. To this end, we use bounds on the second moment of ΔM_n. Recall our bound $\Delta A_n \leq (\Delta P_n)_- + 1$, and note that, for every $k \geq 1$ and every $p \geq 2$, (13) and (21) give

$$\mathbb{P}(\Delta P_n = -k \mid P_n = p) = \frac{h(p-k)}{h(p)} \mathbb{P}(\Delta W_n = -k) \leq C k^{-5/2},$$

for some constant $C > 0$ independent of p and k. It follows that

$$\mathbb{E}\big[(\Delta A_n)^2 \mid P_n = p\big] \leq 1 + C \sum_{k=1}^{p-2} (k+1)^2 k^{-5/2} = O(\sqrt{p}).$$

Since $P_n \leq n + 2$, we deduce from the last display that

$$\mathbb{E}\big[(\Delta M_n)^2\big] = \mathbb{E}\big[(\Delta A_n - \eta_n)^2\big] \leq 2\big(\mathbb{E}\big[(\Delta A_n)^2\big] + \mathbb{E}\big[\mathbb{E}[\Delta A_n \mid \mathcal{F}_n]^2\big]\big) \leq 4\mathbb{E}\big[(\Delta A_n)^2\big] = O(\sqrt{n}).$$

Since the martingale M has orthogonal increments, we get $\mathbb{E}[M_n^2] = O(n^{3/2})$ and it follows that $M_n/n \to 0$ in L^2.

To complete the proof of Proposition 11, it is then enough to verify that

$$\frac{1}{n}\sum_{i=0}^{n-1}\eta_i \xrightarrow[n\to\infty]{(P)} \frac{1}{3}. \tag{36}$$

The idea of the proof is as follows. For most times n, the boundary $\partial\mathsf{T}_n$ has both a “large” number of vertices at distance H_n and a “large” number of vertices at distance H_n+1 from the root. Then, except on a set of small probability, the only events leading to a nonzero value of ΔA_n are events of type R_k for which

$$\Delta A_n = -\Delta P_n = k. \tag{37}$$

The conditional expectation of ΔA_n is thus computed using the probabilities of the events R_k.

To make the preceding argument rigorous, we introduce some notation. For every integer $n \geq 0$, write U_n for the number of vertices in $\partial\mathsf{T}_n$ that are at distance H_n from the root vertex. Note that the function $n \mapsto U_n$ is nonincreasing on every interval $[\sigma_r, \sigma_{r+1}[$ where H_n is equal to r. We also set $G_n = P_n - U_n$, which represents the number of vertices in $\partial\mathsf{T}_n$ that are at distance H_n+1 from the root vertex.

Lemma 12. *For every integer $L \geq 1$, we have*

$$\frac{1}{n}\sum_{i=0}^{n}\mathbf{1}_{\{U_i\leq L \text{ or } G_i\leq L\}} \xrightarrow[n\to\infty]{(P)} 0.$$

Let us postpone the proof of this lemma. To complete the proof of (36), we first use the bound (21) to deduce from the inequality $\Delta A_n \leq |\Delta P_n| + 1$ that, for every $n \geq 0$,

$$\eta_n = \mathbb{E}[\Delta A_n \mid \mathcal{F}_n] \leq \mathbb{E}[|\Delta P_n| \mid \mathcal{F}_n] + 1 \leq C_1, \tag{38}$$

for some finite constant C_1. Furthermore, using (21) again, we have also, for every integer $L \geq 1$,

$$\mathbb{E}[\Delta A_n \mathbf{1}_{\{|\Delta P_n|\geq L\}} \mid \mathcal{F}_n] \leq \mathbb{E}\big[\big(|\Delta P_n|+1\big)\mathbf{1}_{\{|\Delta P_n|\geq L\}} \mid \mathcal{F}_n\big] \leq c_{(L)}, \tag{39}$$

where the constants $c_{(L)}$ are such that $c_{(L)} \to 0$ as $L \to \infty$. Then, on the event $\{U_n \geq L, G_n \geq L\}$, the condition $|\Delta P_n| < L$ ensures that the only transitions of the peeling algorithm at step $n+1$ leading to a positive value of ΔA_n are of type R_k for some k, and in that case $\Delta A_n = -\Delta P_n = k$. It follows that, still on the event $\{U_n \geq L, G_n \geq L\}$,

$$\mathbb{E}[\Delta A_n \mathbf{1}_{\{|\Delta P_n|<L\}} \mid \mathcal{F}_n] = \sum_{k=1}^{L-1} k q_k^{(P_n)} \leq \sum_{k=1}^{\infty} k q_k = \frac{1}{3}. \tag{40}$$

Note that we have $P_n \geq 2L$ on the event $\{U_n \geq L, G_n \geq L\}$. Since $q_k^{(p)}$ converges to q_k as $p \to \infty$, the preceding considerations and (39) entail that, for every $\varepsilon > 0$, we can fix $L_0 > 0$ so that, for every $L \geq L_0$ and every n, we have, on the event $\{U_n \geq L, G_n \geq L\}$,

$$\frac{1}{3} - \varepsilon \leq \mathbb{E}[\Delta A_n \mid \mathcal{F}_n] \leq \frac{1}{3} + \varepsilon. \tag{41}$$

Finally, we have, using (38),

$$\left|\frac{1}{n}\sum_{i=0}^{n-1}\eta_i - \frac{1}{n}\sum_{i=0}^{n-1}\mathbf{1}_{\{U_i\geq L, G_i\geq L\}}\mathbb{E}[\Delta A_i \mid \mathcal{F}_i]\right| \leq \frac{C_1}{n}\sum_{i=0}^{n-1}\mathbf{1}_{\{U_i\leq L \text{ or } G_i\leq L\}},$$

and we can now combine (41) and Lemma 12 to get our claim (36). This completes the proof of Proposition 11, but we still have to prove Lemma 12. □

Proof of Lemma 12. We start with some preliminary observations. From the definition of the peeling by layers, one easily checks that the triple $(P_n, G_n, H_n)_{n\geq 0}$ is a Markov chain with respect to the filtration $(\mathcal{F}_n)$, taking values in $\{(p,\ell,h)\in\mathbb{Z}^3 : p\geq 2, 0\leq \ell\leq p-1, h\geq 0\}$, and whose transition kernel Q is specified as follows:

$$\begin{aligned}
&Q\big((p,\ell,h),(p+1,\ell+1,h)\big)=q_{-1}^{(p)},\\
&Q\big((p,\ell,h),(p-k,\ell-k,h)\big)=q_{k}^{(p)} \quad \text{for } 1\leq k\leq \ell-1,\\
&Q\big((p,\ell,h),(p-k,\ell,h)\big)=q_{k}^{(p)} \quad \text{for } 1\leq k\leq p-\ell-1,\\
&Q\big((p,\ell,h),(p-k,0,h)\big)=q_{k}^{(p)} \quad \text{for } \ell\leq k\leq p-2,\\
&Q\big((p,\ell,h),(p-k,0,h+1)\big)=q_{k}^{(p)} \quad \text{for } p-\ell\leq k\leq p-2.
\end{aligned} \tag{42}$$

The Markov chain $(P_n, G_n, H_n)_{n\geq 0}$ starts from the initial value $(2,1,0)$.

Obviously, the triple $(P_n, U_n, H_n)_{n\geq 0}$ is also a Markov chain, now with values in $\{(p,\ell,h)\in\mathbb{Z}^3 : p\geq 2, 1\leq \ell\leq p, h\geq 0\}$, and its transition kernel Q' is expressed by the formula analogous to (42), where only the first and the last two lines are different and replaced by

$$\begin{aligned}
&Q'\big((p,\ell,h),(p+1,\ell,h)\big)=q_{-1}^{(p)},\\
&Q'\big((p,\ell,h),(p-k,p-k,h+1)\big)=q_{k}^{(p)} \quad \text{for } \ell\leq k\leq p-2,\\
&Q'\big((p,\ell,h),(p-k,p-k,h)\big)=q_{k}^{(p)} \quad \text{for } p-\ell\leq k\leq p-2.
\end{aligned} \tag{43}$$

We now fix $k\in\{0,1,\dots,L\}$. We will prove that

$$\frac{1}{n}\sum_{i=0}^{n}\mathbb{P}(G_i=k)\underset{n\to\infty}{\longrightarrow}0. \tag{44}$$

Let us explain why the lemma follows from (44). If $k'\in\{1,\dots,L\}$, a simple argument using the Markov chain (P_n, U_n, H_n) shows that, for every $i\geq 1$,

$$\mathbb{P}(G_{i+1}=0\mid\mathcal{F}_i)\geq q_{k'}^{(P_i)}\mathbf{1}_{\{U_i=k'\}}\mathbf{1}_{\{P_i\geq k'+2\}}$$

and therefore

$$\mathbb{P}(G_{i+1}=0)\geq \beta\mathbb{P}\big(U_i=k', P_i\geq k'+2\big),$$

with a constant $\beta>0$ depending on k'. If we assume that (44) holds for $k=0$, the latter bound (together with the transience of the Markov chain (P_n)) implies that

$$\frac{1}{n}\sum_{i=0}^{n}\mathbb{P}\big(U_i=k'\big)\underset{n\to\infty}{\longrightarrow}0. \tag{45}$$

Clearly the lemma follows from (44) and (45).

Let us prove (44). Let $N\geq 1$, and write $T_1^N, T_2^N,\dots$ for the successive passage times of the Markov chain (P_n, G_n, H_n) in the set $\{(p,\ell,h): p\geq N, \ell=k\}$. We claim that there exist two positive constants c and α (which depend on k but not on N) such that, for every sufficiently large N and for every integer $i\geq 1$,

$$\mathbb{P}\big[T_{i+1}^N-T_i^N\geq\alpha N\mid\mathcal{F}_{T_i^N}\big]\geq c. \tag{46}$$

If the claim holds, simple arguments show that we have a.s.

$$\liminf_{j\to\infty}\frac{T_j^N}{j}\geq\alpha c N$$

and it follows that, a.s.,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \mathbf{1}_{\{P_i \geq N, G_i = k\}} \leq \frac{1}{\alpha c N}.$$

We can remove $P_i \geq N$ in the indicator function since the Markov chain $(P_n)_{n\geq 0}$ is transient. This gives (44) since N can be taken arbitrarily large.

Let us verify the claim. Applying the strong Markov property at time T_i^N leads to a Markov chain $(\widetilde{P}_n, \widetilde{G}_n, \widetilde{H}_n)$ with transition kernel Q but now started from some triple (p_0, ℓ_0, h_0) such that $p_0 \geq N$ and $\ell_0 = k$. We also set $\widetilde{U}_n = \widetilde{P}_n - \widetilde{G}_n$. The bound (46) reduces to finding two positive constants α and c such that, for every sufficiently large N,

$$\mathbb{P}(\tau_k \geq \alpha N) \geq c, \tag{47}$$

where $\tau_k = \min\{j \geq 1 : \widetilde{G}_j = k\}$. We set $\widetilde{T} := \inf\{n \geq 0 : \widetilde{P}_n = \widetilde{U}_n\}$, and observe that we have either $\widetilde{H}_{\widetilde{T}} = h_0 + 1$ or $\widetilde{H}_{\widetilde{T}} = h_0$.

By looking at the transition kernel Q and using the bounds (21) and (22), we see that we can couple the Markov chain $(\widetilde{P}_n, \widetilde{G}_n, \widetilde{H}_n)$ with a random walk (Y_n) started from $\ell_0 = k$, whose jump distribution μ is given by $\mu(1) = q_{-1}$, $\mu(-j) = q_j$ for every $j \geq 1$, and $\mu(0) = 1 - \mu(1) - \sum_{j\geq 1} \mu(-j)$, in such a way that

$$\widetilde{G}_n \geq Y_n, \quad \text{for every } 0 \leq n < \widetilde{T},$$

and on the event where $Y_1 = k + 1$ and $\min_{j\geq 1} Y_j = k + 1$ we have $\widetilde{H}_{\widetilde{T}} = h_0 + 1$ (the point is that on the latter event, the transition corresponding to the last line of (43) will not occur, at any time n such that $0 \leq n < \widetilde{T}$). Since the random walk Y has a positive drift to ∞, the latter event occurs with probability $c_0 > 0$. We have thus obtained that

$$\mathbb{P}\big(\{\widetilde{G}_n \geq k + 1, \text{ for every } 1 \leq n < \widetilde{T}\} \cap \{\widetilde{H}_{\widetilde{T}} = h_0 + 1\}\big) \geq c_0. \tag{48}$$

Next we observe that there is a positive constant c_1 such that, for every $\varepsilon > 0$, we have, for all sufficiently large N,

$$\mathbb{P}\big(\big\{\widetilde{T} \leq c_1(N - k)\big\} \cap \{H_{\widetilde{T}} = h_0 + 1\}\big) < \varepsilon. \tag{49}$$

To get this bound, we now consider the transition kernel Q': We use (21) to observe that we can couple $(\widetilde{P}_n, \widetilde{U}_n, \widetilde{H}_n)$ with a random walk Y' started from $N - k$, with only nonpositive jumps distributed according to $\mu'(-k) = q_k$ for every $k \geq 1$ (and of course $\mu'(0) = 1 - \sum_{k\geq 1} \mu'(-k)$), in such a way that

$$\widetilde{U}_n \geq Y'_n, \quad \text{for every } 0 \leq n < \widetilde{T},$$

and $Y'_{\widetilde{T}} \leq 0$ on the event $\{\widetilde{H}_{\widetilde{T}} = h_0 + 1\}$. In particular on the event $\{\widetilde{H}_{\widetilde{T}} = h_0 + 1\}$ the hitting time of the negative half-line by Y' must be smaller than or equal to $\widetilde{T}$. Since μ' has a finite first moment, the law of large numbers gives a constant c_1 such that (49) holds.

By combining (48) and (49), and recalling the definition of τ_k, we get

$$\begin{aligned}
&\mathbb{P}\big(\tau_k \geq c_1(N - k)\big) \\
&\quad \geq \mathbb{P}\big(\{\widetilde{G}_n \geq k + 1, \text{ for every } 1 \leq n < \widetilde{T}\} \cap \{H_{\widetilde{T}} = h_0 + 1\}\big) - \mathbb{P}\big(\big\{\widetilde{T} \leq c_1(N - k)\big\} \cap \{H_{\widetilde{T}} = h_0 + 1\}\big) \\
&\quad \geq c_0 - \varepsilon.
\end{aligned}$$

Our claim (47) now follows since we can choose $\varepsilon < c_0$. □

4.3. *Distances in the peeling by layers*

We need another lemma before we proceed to the proof of Proposition 10.

Lemma 13. *There exists a constant C such that $\mathbb{E}[H_n] \leq C n^{1/3}$, for every $n \geq 1$.*

Proof. It will be convenient to introduce a process H'_n which coincides with H_n at times of the form σ_r, $r \geq 1$, but which "interpolates" H_n on every interval $[\sigma_r, \sigma_{r+1}]$. To be specific, we recall the notation introduced in the proof of Lemma 12, and we set for every $n \geq 0$,

$$H'_n = H_n + \frac{G_n}{P_n}.$$

From the form of the transition kernel of the Markov chain (P_n, G_n, H_n) (see the proof of Lemma 12), we get, for every triple (p, ℓ, h) such that $\mathbb{P}(P_n = p, G_n = \ell, H_n = h) > 0$,

$$\begin{aligned}\mathbb{E}\big[\big|\Delta H'_n\big| \big| P_n = p, G_n = \ell, H_n = h\big] &= q_{-1}^{(p)}\left|\frac{\ell+1}{p+1} - \frac{\ell}{p}\right| + \sum_{k=1}^{p-2} q_k^{(p)}\left|\frac{(\ell-k)\vee 0}{p-k} - \frac{\ell}{p}\right| \\ &\quad + \sum_{k=1}^{p-\ell-1} q_k^{(p)}\left|\frac{\ell}{p-k} - \frac{\ell}{p}\right| + \sum_{k=p-\ell}^{p-2} q_k^{(p)}\left(1 - \frac{\ell}{p}\right).\end{aligned}$$

Then it is not hard to verify that each term in the right-hand side is bounded above by c/p, with some constant c independent of (p, ℓ, h). Indeed, writing c for a constant that may vary from line to line, and using (21), we have

$$q_{-1}^{(p)}\left|\frac{\ell+1}{p+1} - \frac{\ell}{p}\right| \leq \frac{1}{p+1},$$

and similarly,

$$\begin{aligned}&\sum_{k=1}^{\ell} q_k^{(p)}\left|\frac{\ell-k}{p-k} - \frac{\ell}{p}\right| = \sum_{k=1}^{\ell} q_k^{(p)}\frac{k(p-\ell)}{p(p-k)} \leq \frac{1}{p}\sum_{k=1}^{\infty} kq_k = \frac{c}{p}, \\ &\sum_{k=\ell+1}^{p-2} q_k^{(p)}\frac{\ell}{p} \leq \frac{1}{p}\sum_{k=\ell+1}^{\infty} kq_k \leq \frac{c}{p}, \\ &\sum_{k=1}^{p-\ell-1} q_k^{(p)}\left|\frac{\ell}{p-k} - \frac{\ell}{p}\right| = \sum_{k=1}^{p-\ell-1} q_k^{(p)}\frac{\ell k}{p(p-k)} \leq \sum_{k=1}^{p-\ell-1} q_k^{(p)}\frac{k}{p} \leq \frac{c}{p}, \\ &\sum_{k=p-\ell}^{p-2} q_k^{(p)}\left(1 - \frac{\ell}{p}\right) \leq \left(1 - \frac{\ell}{p}\right)\sum_{k=p-\ell}^{\infty} q_k \leq \left(1 - \frac{\ell}{p}\right) \times c(p-\ell)^{-3/2} = \frac{c}{p}(p-\ell)^{-1/2}.\end{aligned}$$

We conclude that there exists a constant C' such that $\mathbb{E}[\Delta H'_n \mid \mathcal{F}_n] \leq C'/P_n$. By Lemma 8, we have then $\mathbb{E}[\Delta H'_n] \leq C''n^{-2/3}$ with some other constant C''. It follows that $\mathbb{E}[H'_n] \leq C'''n^{1/3}$, giving the bound of the lemma since $H_n \leq H'_n$. □

Proof of Proposition 10. It follows from Theorem 1 and Proposition 11, together with monotonicity arguments for the last component, that we have the joint convergence in distribution

$$\big(n^{-2/3}P_{[nt]}, n^{-4/3}V_{[nt]}, n^{-1}A_{[nt]}\big)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \big(\mathsf{p}_{\triangle^2}S_t^+, \mathsf{v}_{\triangle^2}Z_t, \mathsf{a}_{\triangle^2}t\big)_{t\geq 0} \tag{50}$$

in the Skorokhod sense. We now need to deal with the convergence of the (rescaled) process H. We first note that by construction we have $A_{\sigma_{r+1}} - A_{\sigma_r} = P_{\sigma_r}$ for every $r \geq 1$. More precisely, for every $r \geq 1$ and every n with $\sigma_r \leq n < \sigma_{r+1}$, we have

$$\begin{aligned}&A_{\sigma_{r+1}} - A_n = U_n \leq P_n, \\ &A_n - A_{\sigma_r} = P_{\sigma_r} - U_n \leq P_{\sigma_r}.\end{aligned}$$

It easily follows that, for every $0 \le n_1 \le n_2$, we have

$$\frac{A_{n_2} - A_{n_1}}{\max_{n_1 \le i \le n_2} P_i} \le H_{n_2} - H_{n_1} + 1, \tag{51}$$

and

$$H_{n_2} - H_{n_1} \le \frac{A_{n_2} - A_{n_1}}{\min_{n_1 \le i \le n_2} P_i} + 1. \tag{52}$$

Fix $0 < s < t$. By (50),

$$n^{-2/3} \min_{[ns] \le k \le [nt]} P_k \xrightarrow[n\to\infty]{(d)} \mathsf{p}_{\triangle^2} \inf_{s \le u \le t} S_u^+,$$

and the limit is a (strictly) positive random variable. Using also Proposition 11, we then deduce from the bound (52) that the sequence $n^{-1/3}(H_{[nt]} - H_{[ns]})$ is tight. Hence we can assume that along a suitable subsequence, for every integer $k \ge 0$, for every $1 \le i \le 2^k$, we have the convergence in distribution

$$n^{-1/3}(H_{[n(s+i2^{-k}(t-s))]} - H_{[n(s+(i-1)2^{-k}(t-s))]}) \xrightarrow[n\to\infty]{(d)} \Lambda_{k,i}^{(s,t)}, \tag{53}$$

where $\Lambda_{k,i}^{(s,t)}$ is a nonnegative random variable. Moreover, we can assume that the convergences (53) hold jointly, and jointly with (50). It then follows from the bounds (51) and (52) that, for every k and i,

$$\frac{\mathsf{a}_{\triangle^2}}{\mathsf{p}_{\triangle^2}} \frac{2^{-k}(t-s)}{\sup_{s+(i-1)2^{-k}(t-s) \le u \le s+i2^{-k}(t-s)} S_u^+} \le \Lambda_{k,i}^{(s,t)} \le \frac{\mathsf{a}_{\triangle^2}}{\mathsf{p}_{\triangle^2}} \frac{2^{-k}(t-s)}{\inf_{s+(i-1)2^{-k}(t-s) \le u \le s+i2^{-k}(t-s)} S_u^+}.$$

Note that $\mathsf{a}_{\triangle^2}/\mathsf{p}_{\triangle^2} = 12^{-1/3} =: \mathsf{h}_{\triangle^2}$. By summing over i, we get

$$\mathsf{h}_{\triangle^2} \sum_{i=1}^{2^k} \frac{2^{-k}(t-s)}{\sup_{s+(i-1)2^{-k}(t-s) \le u \le s+i2^{-k}(t-s)} S_u^+} \le \Lambda_{0,1}^{(s,t)} \le \mathsf{h}_{\triangle^2} \sum_{i=1}^{2^k} \frac{2^{-k}(t-s)}{\inf_{s+(i-1)2^{-k}(t-s) \le u \le s+i2^{-k}(t-s)} S_u^+}.$$

When $k \to \infty$, both the right-hand side and the left-hand-side of the previous display converge a.s. to

$$\mathsf{h}_{\triangle^2} \int_s^t \frac{\mathrm{d}u}{S_u^+}.$$

This argument (and the fact that the limit does not depend on the chosen subsequence) thus gives

$$n^{-1/3}(H_{[nt]} - H_{[ns]}) \xrightarrow[n\to\infty]{(d)} \mathsf{h}_{\triangle^2} \int_s^t \frac{\mathrm{d}u}{S_u^+}, \tag{54}$$

and this convergence holds jointly with (50).

At this point, we use Lemma 13, which tells us that $\mathbb{E}[n^{-1/3} H_{[ns]}]$ can be made arbitrarily small, uniformly in n, by choosing s small. Also Lemma 13, (54) and Fatou's lemma imply that

$$\mathbb{E}\left[\int_s^t \frac{\mathrm{d}u}{S_u^+}\right]$$

is bounded above independently of $s \in (0, t]$, and therefore $\int_0^t \frac{\mathrm{d}u}{S_u^+} < \infty$ a.s. (we could have obtained this more directly). Letting $s \to 0$, we deduce from the previous considerations that

$$n^{-1/3} H_{[nt]} \xrightarrow[n\to\infty]{(d)} \mathsf{h}_{\triangle^2} \int_0^t \frac{\mathrm{d}u}{S_u^+}, \tag{55}$$

jointly with (50). The statement of Proposition 10 now follows from monotonicity arguments using the fact that the limit in (55) is continuous in t. $\square$

4.4. *From Proposition 10 to Theorem 2*

In this section, we deduce Theorem 2 from Proposition 10 via a time change argument. We start with some preliminary observations.

We fix $x > 0$ and write $(\Gamma_t^x)_{t\geq 0}$ for the stable Lévy process with index $3/2$ and no negative jumps started from x, whose distribution is characterized by the formula

$$\mathbb{E}\big[\exp\big(-\lambda\big(\Gamma_t^x - x\big)\big)\big] = \exp\big(\lambda t^{3/2}\big), \quad \lambda, t \geq 0.$$

Equivalently, $\Gamma_t^x = x - S_t$ where S_t is as in the introduction. Set $\gamma_x := \inf\{t \geq 0 : \Gamma_t^x = 0\}$. Then $\gamma_x < \infty$ a.s., and a classical time-reversal theorem (see e.g. [8, Theorem VII.18]) states that the law of $(\Gamma^x_{(\gamma_x - t)-})_{0\leq t\leq \gamma_x}$ (with $\Gamma^x_{0-} = x$) coincides with the law of $(S_t^+)_{0\leq t\leq \rho_x}$, where $\rho_x := \sup\{t \geq 0 : S_t^+ = x\}$.

On the other hand, consider the process $\mathcal{L}$ of Section 1. If $\lambda_x := \sup\{t \geq 0 : \mathcal{L}_t \leq x\}$, then $\lambda_x < \infty$ a.s. and setting $X_t^x = \mathcal{L}_{(\lambda_x - t)-}$ for $0 \leq t \leq \lambda_x$ (with $\mathcal{L}_{0-} = 0$), the process $(X_t^x)_{0\leq t\leq \lambda_x}$ is distributed as the continuous-state branching process with branching mechanism $\psi(u) = u^{3/2}$ started from x and stopped when it hits 0. See [17, Section 2.1] for more details.

The classical Lamperti transformation asserts that, if we set

$$\tau_t^x := \inf\left\{s \geq 0 : \int_0^s \frac{\mathrm{d}u}{\Gamma_u^x} \geq t\right\}$$

for $0 \leq t \leq R_x := \int_0^{\gamma_x} \frac{\mathrm{d}u}{\Gamma_u^x}$, the time-changed process $(\Gamma^x_{\tau_t^x})_{0\leq t\leq R_x}$ has the same distribution as $(X_t^x)_{0\leq t\leq \lambda_x}$. We can then combine the Lamperti transformation with the preceding observations to obtain that, if

$$\eta_t := \inf\left\{s \geq 0 : \int_0^s \frac{\mathrm{d}u}{S_u^+} \geq t\right\},$$

for every $t \geq 0$, the process

$$\left(S^+_{\eta_t}, 0 \leq t \leq \int_0^{\rho_x} \frac{\mathrm{d}u}{S_u^+}\right)$$

has the same distribution as $(\mathcal{L}_t)_{0\leq t\leq \lambda_x}$. Since this holds for every $x > 0$, we conclude that the processes $(S^+_{\eta_t})_{t\geq 0}$ and $(\mathcal{L}_t)_{t\geq 0}$ have the same distribution. It easily follows that we have also

$$\big(S^+_{\eta_t}, Z_{\eta_t}\big)_{t\geq 0} \overset{(d)}{=} (\mathcal{L}_t, \mathcal{M}_t)_{t\geq 0}, \tag{56}$$

with the notation of Section 1.

Let us turn to the proof of Theorem 2. We recall that, for every integer $r \geq 1$, we have $|\partial B_r^\bullet| = P_{\sigma_r}$ and $|B_r^\bullet| = V_{\sigma_r}$, with $\sigma_r = \min\{n : H_n \geq r\}$. We use the convergence in distribution of Proposition 10 and the Skorokhod representation theorem to find, for every $n \geq 1$, a triple $(P^{(n)}, V^{(n)}, H^{(n)})$ having the same distribution as (P, V, H), in such a way that we now have the almost sure convergence

$$\left(\frac{P^{(n)}_{[nt]}}{\mathsf{p}_{\triangle^2} \cdot n^{2/3}}, \frac{V^{(n)}_{[nt]}}{\mathsf{v}_{\triangle^2} \cdot n^{4/3}}, \frac{H^{(n)}_{[nt]}}{\mathsf{h}_{\triangle^2} \cdot n^{1/3}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{\text{a.s.}} \left(S_t^+, Z_t, \int_0^t \frac{\mathrm{d}u}{S_u^+}\right)_{t\geq 0}, \tag{57}$$

for the Skorokhod topology. For every $n \geq 1$, and every $r \geq 1$, set

$$\sigma_r^{(n)} = \min\big\{k : H_k^{(n)} \geq r\big\}.$$

Then it easily follows from (57) that

$$\left(\frac{1}{n}\sigma^{(n)}_{[n^{1/3}t]}\right)_{t\geq 0}\xrightarrow[n\to\infty]{\text{a.s.}}(\eta_{t/\mathsf{h}_{\triangle^2}})_{t\geq 0},$$

uniformly on every compact time set. By combining the latter convergence with (57) we arrive at the a.s. convergence in the Skorokhod sense,

$$\left(n^{-2/3}P^{(n)}_{\sigma^{(n)}_{[n^{1/3}t]}}, n^{-4/3}V^{(n)}_{\sigma^{(n)}_{[n^{1/3}t]}}\right)_{t\geq 0}\xrightarrow[n\to\infty]{\text{a.s.}}\left(\mathsf{p}_{\triangle^2}S^{+}_{\eta_{t/\mathsf{h}_{\triangle^2}}}, \mathsf{v}_{\triangle^2}Z_{\eta_{t/\mathsf{h}_{\triangle^2}}}\right)_{t\geq 0}.$$

Recalling the identity in distribution (56), we get the convergence in distribution of Theorem 2 since

$$\left(P^{(n)}_{\sigma^{(n)}_r}, V^{(n)}_{\sigma^{(n)}_r}\right)_{r\geq 0}\overset{(d)}{=}(P_{\sigma_r}, V_{\sigma_r})_{r\geq 0}=\left(\left|\partial B^{\bullet}_r\right|, \left|B^{\bullet}_r\right|\right)_{r\geq 0}.$$

This completes the proof.

5. Application to other distances

In this section, we apply our techniques to study other distances on the UIPT (in fact on the dual graph of the UIPT) in order to get similar results for the scaling limits of the associated hull processes. Specifically, we will consider the dual graph distance and the first-passage percolation distance with exponential edge weights on the dual graph.

5.1. *The dual graph distance*

We consider the dual map of the UIPT, whose vertices are in one-to-one correspondence with the faces of the UIPT, and each edge e of the UIPT corresponds to an edge of the dual map between the two faces incident to e. This dual map is denoted by T^*_∞. By convention, the root vertex of T^*_∞ or root face is the face incident to the right-hand side of the root edge of the UIPT. We denote the graph distance on T^*_∞ or dual graph distance by $\mathrm{d}^*_{\mathrm{gr}}$. For every integer $r\geq 0$, we let $B^{\bullet,*}_r(T_\infty)$ denote the hull of the ball of radius r for $\mathrm{d}^*_{\mathrm{gr}}$. This is the union of all faces of T_∞ that are at dual graph distance smaller than or equal to r from the root face, together with the finite regions these faces may enclose.

Similarly as in the previous section we now design a peeling algorithm which discovers these dual hulls step by step. In the first step ($n=0$) we reveal the root face. In the second step ($n=1$), we peel any edge incident to the root face. Then inductively at step $n+1$ we peel the edge of the boundary of T_n which lies immediately on the right of the last revealed triangle (but not incident to that triangle). See Figure 6 for an illustration.

As in the case of the peeling by layers for the graph distance on the primal lattice, one can prove by induction that, for every $n\geq 0$, there is an integer $h\geq 0$ such that one and only one of the following two possibilities occurs. Either all faces incident to $\partial\mathsf{T}_n$ are at the same dual graph distance h from the root face of the UIPT. Or $\partial\mathsf{T}_n$ contains both edges incident to faces at dual distance h and edges incident to faces at dual distance $h+1$ from the root face. In the last case, these edges form two connected subsets of the boundary and the edge that will be "peeled off" at step $n+1$ is the only edge incident to a face in T_n at dual distance h such that the edge immediately on its left is incident to a face of T_n at dual distance $h+1$. In both cases we write $H^*_n=h$. As in the previous sections, we let P_n and V_n stand respectively for the perimeter and for the volume of the triangulation discovered after n peeling steps.

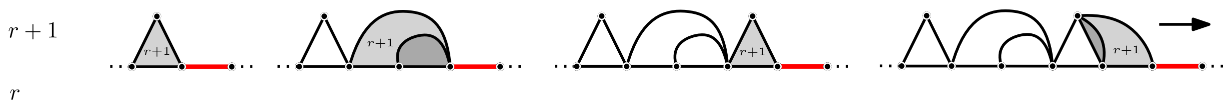

Fig. 6. Illustration of the peeling by layers on the dual map. When $B^{\bullet,*}_r(T_\infty)$ has been discovered, we turn around the boundary $\partial B^{\bullet,*}_r(T_\infty)$ from left to right in order to reveal the next layer and obtain $B^{\bullet,*}_{r+1}(T_\infty)$.

Proposition 14 (Distances in the peeling by layers on the dual map). *We have the following convergence in distribution for the Skorokhod topology*

$$\left(\frac{P_{[nt]}}{\mathsf{p}_{\triangle^2}\cdot n^{2/3}},\frac{V_{[nt]}}{\mathsf{v}_{\triangle^2}\cdot n^{4/3}},\frac{H^*_{[nt]}}{\mathsf{h}^*_{\triangle^2}\cdot n^{1/3}}\right)_{t\geq 0}\xrightarrow[n\to\infty]{(d)}\left(S_t^+,Z_t,\int_0^t\frac{\mathrm{d}u}{S_u^+}\right)_{t\geq 0},$$

where $\mathsf{h}^*_{\triangle^2}=(1+\mathsf{a}_{\triangle^2})/\mathsf{p}_{\triangle^2}=(16/3)^{1/3}$.

Theorem 3 is derived from Proposition 14 in exactly the same way as Theorem 2 is derived from Proposition 10 in Section 4.4. Let us briefly discuss the proof of Proposition 14, which follows the same lines as that of Proposition 10. The convergence of the first two components is again a consequence of Theorem 1, and we focus on the convergence of the third component. As for the peeling by layers on the primal lattice, the key idea is to consider the speed at which the peeling by layers (on the dual map) "turns" around the boundary. More precisely we denote the set of all edges of T_∞ that are part of $B_r^{\bullet,*}(T_\infty)$ for some $r\geq 0$ by $\mathcal{L}^*$, and we let A_n^* stand for the number of edges of $\mathsf{T}_n\setminus\partial\mathsf{T}_n$ that belong to $\mathcal{L}^*$. We aim at the following analog of Proposition 11:

$$\frac{A_n^*}{n}\xrightarrow[n\to\infty]{(P)}\mathsf{a}_{\triangle^2}+1=4/3. \tag{58}$$

The idea to prove this convergence is the same as before: For most times n, the boundary $\partial\mathsf{T}_n$ has both a large number of edges incident to a face of T_n at dual distance H_n^*+1 from the root face, and a large number of edges incident to a face of T_n at dual distance H_n^*. Then, except on a set of small probability, the only events leading to a nonzero value of ΔA_n^* are events of type R_k, for which

$$\Delta A_n=-\Delta P_n+1=k+1.$$

Note the additional term $+1$ in the last display (compare with (37)) coming from the fact that we peel an edge belonging to $\mathcal{L}^*$ at every step. This additional term explains why we get the limit $\mathsf{a}_{\triangle^2}+1$ in (58), instead of $\mathsf{a}_{\triangle^2}$ in Proposition 11. Apart from this difference, the technical details of the proof of (58) are very similar to those of Proposition 11. For the analog of Lemma 12, we introduce the number U_n^* of edges of $\partial\mathsf{T}_n$ that are incident to a face of T_n at dual distance H_n^* from the root face, and $G_n^*=P_n-U_n^*$. Then $(P_n,G_n^*,H_n^*)_{n\geq 0}$ is a Markov chain taking values in $\{(p,\ell,h)\in\mathbb{Z}^3:p\geq 2,0\leq\ell\leq p-1,h\geq 0\}$, whose transition kernel Q^* is specified as follows:

$$\begin{aligned}
&Q^*\big((p,\ell,h),(p+1,\ell+2,h)\big)=q_{-1}^{(p)}\quad\text{if }\ell\leq p-2,\\
&Q^*\big((p,p-1,h),(p+1,0,h+1)\big)=q_{-1}^{(p)},\\
&Q^*\big((p,\ell,h),(p-k,\ell-k+1,h)\big)=q_k^{(p)}\quad\text{for }1\leq k\leq\ell,\\
&Q^*\big((p,\ell,h),(p-k,\ell+1,h)\big)=q_k^{(p)}\quad\text{for }1\leq k\leq p-\ell-2,\\
&Q^*\big((p,\ell,h),(p-k,1,h)\big)=q_k^{(p)}\quad\text{for }\ell+1\leq k\leq p-2,\\
&Q^*\big((p,\ell,h),(p-k,0,h+1)\big)=q_k^{(p)}\quad\text{for }p-\ell-1\leq k\leq p-2.
\end{aligned}\tag{59}$$

The analog of Lemma 12 then holds with G_i and U_i replaced respectively by G_i^* and P_i^*, with a very similar proof. This provides the key technical ingredient needed to adapt the proof of Proposition 11 in order to get the convergence (58). Finally, an analog of Lemma 13 also holds with $\mathbb{E}[H_n]$ replaced by $\mathbb{E}[H_n^*]$, and Proposition 14 can then be derived from (58) in the same way as Proposition 10 was derived from Proposition 11 in Section 4.3. We leave the details to the reader.

5.2. *First-passage percolation*

We now assign independent weights exponentially distributed with parameter 1 to the edges of T_∞^*. The weight of a path in T_∞^* is just the sum of the weights of its edges. We let F_0 consist only of the root face and, for every $t>0$, we let F_t be the union of all faces of the UIPT which are connected to the root face by a dual path whose weight is less than or equal to t. We then let $\mathsf{F}^\bullet_t$ be the hull of F_t. We set $\tau_0 = 0$ and we let $0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ be the successive jump times of the process $\mathsf{F}^\bullet_t$ (a simple argument shows that $\tau_n \to \infty$ as $n \to \infty$, which will also follow from the next proposition). Note that, at each time τ_n with $n \geq 1$, a new triangle incident to the boundary of $\mathsf{F}^\bullet_{\tau_{n-1}}$ is added to $\mathsf{F}^\bullet_{\tau_{n-1}}$, together with the triangles in the "hole" that this addition may create.

By convention we let $\tilde{\mathsf{F}}_0$ be the trivial triangulation, and we set, for every $n \geq 1$

$$\tilde{\mathsf{F}}_n = \mathsf{F}^\bullet_{\tau_{n-1}}.$$

The following proposition shows that the process $(\tilde{\mathsf{F}}_n)_{n\geq 0}$ is a particular instance of a peeling process, which is called the *uniform* peeling process or Eden model on the UIPT. See also [1, Section 6].

Proposition 15. *The sequence $(\tilde{\mathsf{F}}_n)_{n\geq 0}$ has the same law as the sequence $(\mathsf{T}_n)_{n\geq 0}$ corresponding to a peeling process where at step 1 we reveal the triangle incident to the right-hand side of the root edge, and for every $n \geq 2$, conditionally on $\mathsf{T}_0, \dots, \mathsf{T}_{n-1}$, the peeled edge at step n is chosen uniformly at random among the edges of $\partial \mathsf{T}_{n-1}$. Furthermore, conditionally on the sequence $(\tilde{\mathsf{F}}_n)_{n\geq 1}$, the increments $\tau_1 - \tau_0, \tau_2 - \tau_1, \tau_3 - \tau_2, \dots$ are independent, and, for every $k \geq 1$, $\tau_k - \tau_{k-1}$ is exponentially distributed with parameter $|\partial \tilde{\mathsf{F}}_k|$.*

Remark. *Since $|\partial \tilde{\mathsf{F}}_k| \leq k + 2$, the last assertion shows that $\tau_k \uparrow \infty$ a.s. as $k \to \infty$.*

Proof of Proposition 15. Let $n \geq 1$. Consider an edge e of $\partial \tilde{\mathsf{F}}_n$. Then, e is incident to a unique face f_e of $\mathsf{F}_{\tau_{n-1}}$, and we write $\mathrm{d}_{\mathrm{fpp}}(f_e)$ for the first-passage percolation distance between f_e and the root face (in other words, this is the minimal weight of a dual path connecting the root face and f_e). We also write w_e for the weight of e, or rather of its dual edge. Since f_e is contained in $\mathsf{F}_{\tau_{n-1}}$, we have $\mathrm{d}_{\mathrm{fpp}}(f_e) \leq \tau_{n-1}$, with equality only if f_e is the triangle that was added at time τ_{n-1}. Also it is clear that

$$w_e > \tau_{n-1} - \mathrm{d}_{\mathrm{fpp}}(f_e)$$

because otherwise this would contradict the fact that the other face incident to e is not in $\mathsf{F}_{\tau_{n-1}}$.

Next the lack of memory of the exponential distribution ensures that, conditionally on the variables $(\tilde{\mathsf{F}}_0, \tilde{\mathsf{F}}_1, \dots, \tilde{\mathsf{F}}_n, \tau_1, \dots, \tau_{n-1})$, the random variables

$$w_e - \big(\tau_{n-1} - \mathrm{d}_{\mathrm{fpp}}(f_e)\big),$$

where e varies over the edges of $\partial \tilde{\mathsf{F}}_n$, are independent and exponentially distributed with parameter 1. Now observe that the next jump will occur at time

$$\tau_n = \tau_{n-1} + \min\big\{w_e - \big(\tau_{n-1} - \mathrm{d}_{\mathrm{fpp}}(f_e)\big) : e \text{ edge of } \partial \tilde{\mathsf{F}}_n\big\}.$$

It follows that, conditionally on $(\tilde{\mathsf{F}}_0, \tilde{\mathsf{F}}_1, \dots, \tilde{\mathsf{F}}_n, \tau_1, \dots, \tau_{n-1})$, the variable $\tau_n - \tau_{n-1}$ is exponential with parameter $|\partial \tilde{\mathsf{F}}_n|$, and furthermore, the new triangle added to $\tilde{\mathsf{F}}_n$ corresponds to the edge attaining the preceding minimum, which is therefore uniformly distributed over edges of $\partial \tilde{\mathsf{F}}_n$. This completes the proof. □

Proof of Theorem 4. As in the previous sections, we use the notation V_n and P_n for the volume and the perimeter of $\tilde{\mathsf{F}}_n$. We will establish the following convergence in distribution for the Skorokhod topology

$$\left(\frac{P_{[nt]}}{\mathsf{p}_{\triangle^2} \cdot n^{2/3}}, \frac{V_{[nt]}}{\mathsf{v}_{\triangle^2} \cdot n^{4/3}}, \frac{\tau_{[nt]}}{(1/\mathsf{p}_{\triangle^2}) \cdot n^{1/3}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(S^+_t, Z_t, \int_0^t \frac{\mathrm{d}u}{S^+_u}\right)_{t\geq 0}. \tag{60}$$

Theorem 4 then follows from (60) by the very same arguments we used to deduce Theorem 2 from Proposition 10 in Section 4.4.

The joint convergence of the first two components in (60) is given by Theorem 1. So we need to prove the convergence of the third component and to check that it holds jointly with the first two. As in the proof of Proposition 10, we fix $0 < s < t$ and we first consider $\tau_{[nt]} - \tau_{[ns]}$. Writing

$$\tau_{[nt]} - \tau_{[ns]} = \sum_{i=[ns]+1}^{[nt]} (\tau_i - \tau_{i-1})$$

and, using Proposition 15, we see that, conditionally on $(P_k)_{k\geq 0}$, the variable $\tau_{[nt]} - \tau_{[ns]}$ is distributed as

$$\sum_{i=[ns]+1}^{[nt]} \frac{\mathbf{e}_i}{P_i},$$

where the random variables $\mathbf{e}_1, \mathbf{e}_2, \dots$ are independent and exponentially distributed with parameter 1, and are also independent of $(P_k)_{k\geq 0}$. By the convergence of the first component in (60), we have

$$n^{-1/3} \sum_{i=[ns]+1}^{[nt]} \frac{1}{P_i} = \int_{n^{-1}([ns]+1)}^{n^{-1}([nt]+1)} \frac{\mathrm{d}u}{n^{-2/3} P_{[nu]}} \xrightarrow[n\to\infty]{(d)} \frac{1}{\mathsf{p}_{\triangle^2}} \int_s^t \frac{\mathrm{d}u}{S_u^+},$$

and, on the other hand,

$$E\left[\left(n^{-1/3} \sum_{i=[ns]+1}^{[nt]} \frac{\mathbf{e}_i}{P_i} - n^{-1/3} \sum_{i=[ns]+1}^{[nt]} \frac{1}{P_i}\right)^2 \,\middle|\, (P_k)_{k\geq 0}\right]$$
$$= n^{-2/3} \sum_{i=[ns]+1}^{[nt]} \frac{1}{(P_i)^2} = \frac{1}{n} \int_{n^{-1}([ns]+1)}^{n^{-1}([nt]+1)} \frac{\mathrm{d}u}{(n^{-2/3} P_{[nu]})^2}$$

converges to 0 in probability as $n \to \infty$. It easily follows that

$$n^{-1/3}(\tau_{[nt]} - \tau_{[ns]}) \xrightarrow[n\to\infty]{(d)} \frac{1}{\mathsf{p}_{\triangle^2}} \int_s^t \frac{\mathrm{d}u}{S_u^+}, \tag{61}$$

and the previous argument also shows that this convergence holds jointly with that of the first two components in (60). We can complete the proof by arguing in a way similar to the end of the proof of Proposition 10. It suffices to verify that

$$\sup_{n\geq 1} \mathbb{E}\big[n^{-1/3} \tau_{[ns]}\big] \xrightarrow[s\to 0]{} 0.$$

This is however very easy, since

$$\mathbb{E}[\tau_{[ns]}] = \mathbb{E}\left[\sum_{i=1}^{[ns]} \frac{1}{P_i}\right]$$

and we can use Lemma 8 to obtain that $\mathbb{E}[\tau_{[ns]}] \leq C(ns)^{1/3}$, for some constant C. $\square$

5.3. *Comparing distances*

One conjectures that balls for the dual graph distance or the first-passage percolation distance grow asymptotically like “deterministic” balls for the graph distance. More precisely, one expects that there exist two constants $c_1, c_2 > 0$

such that, for every $\varepsilon > 0$, one has

$$B_{(c_1^{-1}-\varepsilon)r} \subset B_r^* \subset B_{(c_1^{-1}+\varepsilon)r},$$
$$B_{(c_2^{-1}-\varepsilon)r} \subset \mathsf{F}_r \subset B_{(c_2^{-1}+\varepsilon)r}.$$

with high probability when r is large (here B_r^* is the ball for the dual graph distance, that is the union of all faces that are at dual graph distance less than or equal to r from the root face). The reason for this belief is the fact that the UIPT is "isotropic," in contrast with deterministic lattices such as $\mathbb{Z}^d$. Our results support the previous conjecture since the scaling limits for the perimeter and volume of hulls are the same for any the balls B_r, F_r and B_r^*, up to multiplicative constants. Note that, if the last display holds, we must have also

$$B^{\bullet,*}_{(c_1-\varepsilon)r} \subset B^\bullet_r \subset B^{\bullet,*}_{(c_1+\varepsilon)r},$$
$$\mathsf{F}^\bullet_{(c_2-\varepsilon)r} \subset B^\bullet_r \subset \mathsf{F}^\bullet_{(c_2+\varepsilon)r},$$

with high probability when r is large. By comparing the limits in distribution of $|B^\bullet_r|$ (Theorem 2), of $|B^{\bullet,*}_r|$ (Theorem 3) and of $|\mathsf{F}^\bullet_r|$ (Theorem 4), we see that if the previous conjecture holds, the constants c_1 and c_2 must be equal to

$$c_1 = \frac{\mathsf{h}^*_{\triangle^2}}{\mathsf{h}_{\triangle^2}} = \frac{1+\mathsf{a}_{\triangle^2}}{\mathsf{a}_{\triangle^2}} = 4 \quad \text{and} \quad c_2 = \frac{1}{\mathsf{p}_{\triangle^2}\mathsf{h}_{\triangle^2}} = \frac{1}{\mathsf{a}_{\triangle^2}} = 3.$$

See [1, Remark 5] for related calculations about two-point and three-point functions for first-passage percolation on type I triangulations (in that case, the analog of the constant $\mathsf{a}_{\triangle^2}$ is $\mathsf{a}_{\triangle^1} = 1/(2\sqrt{3})$, as we shall see below). The proof of the above conjecture is discussed, in the slightly different setting of type I triangulations, in the forthcoming work [16].

6. Other models

Although we chose to focus on type II triangulations, our results can be extended to other classes of infinite random planar maps. Roughly speaking, one only needs to replace the constants $\mathsf{a}_{\triangle^2}$, $\mathsf{t}_{\triangle^2}$, $\mathsf{b}_{\triangle^2}$ defined in (13), in Proposition 9 and in Proposition 11 by their appropriate values in the model in consideration. All our results should then go through with the constants $\mathsf{v}_.$, $\mathsf{p}_.$, $\mathsf{h}_.$, $\mathsf{h}^*_.$ evaluated via the same "universal" relations from the constants $\mathsf{a}_.$, $\mathsf{t}_.$, $\mathsf{b}_.$. In this section we carefully explain how to do this in two particular cases, namely type I triangulations and quadrangulations. It may well be the case that our techniques can be extended to even more general classes of random planar maps such as the (regular critical) Boltzmann triangulations considered in [11].

6.1. *Type I triangulations*

Let us consider the case of type I triangulations, where both loops and multiple edges are allowed. The construction of the UIPT in this case is not treated by Angel and Schramm [6], but the techniques of [6] can easily be extended using the corresponding enumeration results (see below). Alternatively, the construction of the type I UIPT follows as a special case of the recent results of Stephenson [30]. We denote the UIPT for type I triangulations by $T^{(1)}_\infty$. Let us list the enumeration results corresponding to those of Section 2.1. These results may be found in Krikun [23] (Krikun uses the number of edges as the size parameter and in order to apply his formulas we note that a triangulation of the p-gon with n inner vertices has $3n+2p-3$ edges).

For every $p \geq 1$ and $n \geq 0$, let $\mathcal{T}^{(1)}_{n,p}$ stand for the set of all type I triangulations with n inner vertices and a simple boundary of length p, which are rooted on an edge of the boundary in the way explained in Section 2.1. We have for $(n,p) \neq (0,1)$

$$\#\mathcal{T}^{(1)}_{n,p} = 4^{n-1}\frac{p(2p)!(2p+3n-5)!!}{(p!)^2 n!(2p+n-1)!!} \underset{n\to\infty}{\sim} C^{(1)}(p)(12\sqrt{3})^n n^{-5/2},$$

where

$$C^{(1)}(p) = \frac{3^{p-2}p(2p)!}{4\sqrt{2\pi}(p!)^2} \underset{p\to\infty}{\sim} \frac{1}{36\pi\sqrt{2}}\sqrt{p}12^p.$$

We then set $Z^{(1)}(p) = \sum_{n\geq 0} \#\mathcal{T}^{(1)}_{n,p}(12\sqrt{3})^{-n}$ and we have the formula (see [4, Section 2.2])

$$Z^{(1)}(p) = \frac{6^p(2p-5)!!}{8\sqrt{3}p!} \quad \text{if } p \geq 2, \qquad Z^{(1)}(1) = \frac{2-\sqrt{3}}{4}.$$

The generating series of $Z^1(p)$ can also be computed explicitly from [23, formula (4)] and an appropriate change of variables (we omit the details):

$$\sum_{p\geq 0} Z^{(1)}(p+1)z^p = \frac{1}{2} + \frac{(1-12z)^{3/2}-1}{24\sqrt{3}z}.$$

In particular, the analog of (4) is

$$Z^{(1)}(p+1) \underset{p\to\infty}{\sim} \frac{\sqrt{3}}{8\sqrt{\pi}}12^p p^{-5/2},$$

and similarly as in (4), we set

$$\mathsf{t}_{\triangle^1} = \frac{\sqrt{3}}{8\sqrt{\pi}}.$$

The peeling algorithm discovering $T^{(1)}_\infty$ is then described in a very similar way as in Section 3.1. The only difference is that we now need to consider the possibility of loops. With the notation of Section 3.1, and supposing that the revealed region has a boundary of size $p \geq 1$, events of type L_0 or R_0, or of type L_{p-1} or R_{p-1}, may occur (the definition of these events should be obvious from Figure 3). The respective probabilities of events C, L_k or R_k are given by formulas analogous to (10) and (11), where $2/27$ is replaced by $1/(12\sqrt{3})$, the functions C and Z are replaced respectively by $C^{(1)}$ and $Z^{(1)}$, and finally k is allowed to vary in $\{0, \ldots, p-1\}$.

An analog of Proposition 6 holds, and the constant $\mathsf{p}_{\triangle^2}$ has to be replaced by

$$\mathsf{p}_{\triangle^1} = \left(\frac{8\mathsf{t}_{\triangle^1}\sqrt{\pi}}{3}\right)^{2/3} = 3^{-1/3}.$$

Similarly, there is a version of Proposition 9 in the type I case, and the constant $\mathsf{b}_{\triangle^2}$ is replaced by

$$\mathsf{b}_{\triangle^1} = \frac{4}{3}$$

whereas the limiting distribution remains the same. Finally, the analog of Proposition 11 involves the new constant

$$\mathsf{a}_{\triangle^1} = \frac{1}{2\sqrt{3}}.$$

The proofs of Theorems 1, 2, 3 and 4 can then be adapted easily to the UIPT $T^{(1)}_\infty$. In these statements, $\mathsf{p}_{\triangle^2}$ is replaced by $\mathsf{p}_{\triangle^1}$ and the other constants $\mathsf{v}_{\triangle^2}$, $\mathsf{h}_{\triangle^2}$ and $\mathsf{h}^*_{\triangle^2}$ are replaced respectively by

$$\mathsf{v}_{\triangle^1} = (\mathsf{p}_{\triangle^1})^2\mathsf{b}_{\triangle^1} = 4\cdot 3^{-5/3}, \qquad \mathsf{h}_{\triangle^1} = \frac{\mathsf{a}_{\triangle^1}}{\mathsf{p}_{\triangle^1}} = \frac{1}{2}3^{-1/6} \quad \text{and} \quad \mathsf{h}^*_{\triangle^1} = \frac{1+\mathsf{a}_{\triangle^1}}{\mathsf{p}_{\triangle^1}}.$$

We note that $\mathsf{p}_{\triangle^1}/(\mathsf{h}_{\triangle^1})^2 = \mathsf{p}_{\triangle^2}/(\mathsf{h}_{\triangle^2})^2$, which, by Theorem 2 and its type I analog, means that the scaling limit of the perimeter of hulls is exactly the same for type I and for type II triangulations. This fact can be explained by a direct relation between the UIPTs of type I and of type II, but we omit the details.

6.2. *Quadrangulations*

Let us now consider the Uniform Infinite Planar Quadrangulation (UIPQ), which is denoted here by Q_∞. This case requires more changes in the arguments. We first note that a quadrangulation with a simple boundary necessarily has an even perimeter. For every $p \geq 1$, let $\mathcal{Q}_{n,p}$ stand for the set of all quadrangulations with a simple boundary of perimeter $2p$ and n inner vertices, which are rooted at an oriented edge of the boundary in such a way that the external face lies on the right of the root edge. For $n \geq 0$ and $p \geq 1$, we read from [10, Eq. (2.11)] that

$$\#\mathcal{Q}_{n,p} = 3^{n-1}\frac{(3p)!(3p-3+2n)!}{n!p!(2p-1)!(n+3p-1)!} \underset{n\to\infty}{\sim} C^{\square}(p)12^n n^{-5/2},$$

where

$$C^{\square}(p) = \frac{8^{p-1}(3p)!}{3\sqrt{\pi}\,p!(2p-1)!} \underset{p\to\infty}{\sim} \frac{1}{8\sqrt{3\pi}}54^p\sqrt{p}.$$

We have also, for every $p \geq 2$,

$$Z^{\square}(p) = \sum_{n\geq 0}\#\mathcal{Q}_{n,p}12^{-n} = \frac{8^p(3p-4)!}{(p-2)!(2p)!},$$

and $Z^{\square}(1) = 4/3$. Furthermore,

$$Z^{\square}(p+1) \underset{p\to\infty}{\sim} \frac{1}{\sqrt{3\pi}}54^p p^{-5/2}, \qquad \sum_{k\geq 0} Z^{\square}(k+1)54^{-k} = 3/2, \qquad \sum_{k\geq 0} kZ^{\square}(k+1)54^{-k} = 1/2. \tag{62}$$

The transitions in the peeling process of the UIPQ are more complicated than previously because of additional cases. If at step $n \geq 0$ the perimeter of the discovered quadrangulation Q_n is equal to $2m$, then the revealed quadrangle at the next step may have three different shapes (see Figure 7):

1. *Shape* C: The revealed quadrangle has two vertices in the unknown region, an event of probability

$$\mathbb{P}\big(\mathsf{C} \mid |\partial\mathsf{Q}_n| = 2m\big) = \mathbf{q}_{-2}^{(m)} = 12^{-2}\frac{C^{\square}(m+1)}{C^{\square}(m)}.$$

2. *Shapes* L_k *and* R_k*, for* $k \in \{0, 1, \dots, 2m-1\}$: The revealed quadrangle has three vertices on the boundary of Q_n. This quadrangle then "swallows" a part of the boundary of $\partial\mathsf{Q}_n$ of length k. This event is denoted by L_k or R_k according to whether the part of the boundary that is swallowed is on the right or on the left of the peeled edge. Note that the revealed face encloses a finite quadrangulation of perimeter $k+1$ if k is odd and $k+2$ if k is even. These events have probability

$$\mathbb{P}\big(\mathsf{L}_{2k} \mid |\partial\mathsf{Q}_n| = 2m\big) = \mathbb{P}\big(\mathsf{L}_{2k+1} \mid |\partial\mathsf{Q}_n| = 2m\big) = \mathbb{P}\big(\mathsf{R}_{2k} \mid |\partial\mathsf{Q}_n| = 2m\big) = \mathbb{P}\big(\mathsf{R}_{2k+1} \mid |\partial\mathsf{Q}_n| = 2m\big)$$
$$= \mathbf{q}_{2k+1}^{(m)} = \mathbf{q}_{2k}^{(m)} = \frac{Z^{\square}(k+1)}{12}\frac{C^{\square}(m-k)}{C^{\square}(m)}.$$

3. *Shapes* L_{k_1,k_2}, R_{k_1,k_2} *and* C_{k_1,k_2} *for* $k_1, k_2 \geq 1$ *odd and such that* $k_1 + k_2 < 2m$: This last case occurs when the revealed quadrangle has its four vertices on $\partial\mathsf{Q}_n$. It then encloses two finite quadrangulations of respective perimeters k_1+1 and k_2+1 either both on the left side of the peeled edge in case L_{k_1,k_2}, or one on each side of the peeled

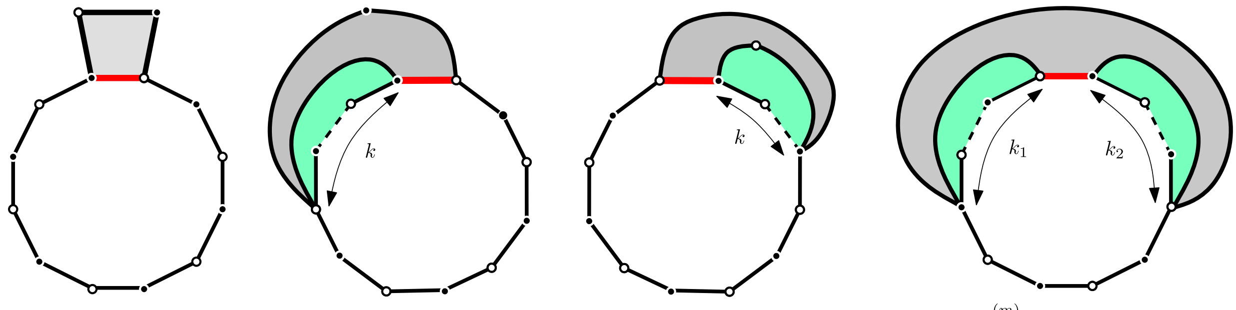

Fig. 7. A few peeling transitions in the quadrangular case.

edge in case C_{k_1,k_2}, or both on the right side of the peeled edge in case R_{k_1,k_2}. These three events have the same probability: writing $k_1 = 2j_1 + 1$ and $k_2 = 2j_2 + 1$, with $j_1 + j_2 < m - 1$,

$$\begin{aligned}\mathbb{P}\big(\mathsf{L}_{k_1,k_2} \mid |\partial \mathsf{Q}_n| = 2m\big) &= \mathbb{P}\big(\mathsf{R}_{k_1,k_2} \mid |\partial \mathsf{Q}_n| = 2m\big) = \mathbb{P}\big(\mathsf{C}_{k_1,k_2} \mid |\partial \mathsf{Q}_n| = 2m\big) \\ &= \mathbf{q}^{(m)}_{k_1,k_2} = Z^{\square}(j_1+1) Z^{\square}(j_2+1) \frac{C^{\square}(m - j_1 - j_2 - 1)}{C^{\square}(m)}.\end{aligned}$$

Furthermore, conditionally on each of the above cases, the finite quadrangulations enclosed by the revealed face are independent Boltzmann quadrangulations with the prescribed perimeters. Let $P^{\square}_n$ stand for the half-perimeter at step n in the peeling process. Then, similarly as in the triangular case, the Markov chain $(P^{\square}_n)$ is obtained by conditioning a random walk X on $\mathbb{Z}$ to stay (strictly) positive, and the increments of X are now distributed as follows:

$$\mathbb{E}\big[f(X_{n+1}) \mid X_n\big] = f(X_n+1)\cdot \mathbf{q}_{-2} + \sum_{k=0}^{\infty} f(X_n - k)\cdot \Bigg(2(\mathbf{q}_{2k} + \mathbf{q}_{2k+1}) + 3 \sum_{\substack{k_1+k_2=2k \\ k_1,k_2\geq 1 \text{ odd}}} \mathbf{q}_{k_1,k_2}\Bigg),$$

where $\mathbf{q}_j = \lim_{m\to\infty} \mathbf{q}^{(m)}_j$ and $\mathbf{q}_{k_1,k_2} = \lim_{m\to\infty} \mathbf{q}^{(m)}_{k_1,k_2}$ as in the triangular case. From the enumeration results, we get, for every $k \geq 0$,

$$\mathbb{P}(\Delta X = -k) = 2(q_{2k} + q_{2k+1}) + 3 \sum_{\substack{k_1+k_2=2k \\ k_1,k_2\geq 1 \text{ odd}}} q_{k_1,k_2} \underset{k\to\infty}{\sim} \frac{1}{2\sqrt{3\pi}} k^{-5/2}. \tag{63}$$

The results of Sections 3.2 and 3.4 can then be extended to the UIPQ Q_∞. Comparing (63) with (13), we see that the role of the constant $\mathsf{t}_{\triangle^2}$ is now played by $\mathsf{t}_{\square} = 1/(4\sqrt{3\pi})$. Then the convergence in distribution of Proposition 6 holds for $P^{\square}_n$, with the constant $\mathsf{p}_{\triangle^2}$ replaced by

$$\mathsf{p}_{\square} = \left(\frac{8\mathsf{t}_{\square}\sqrt{\pi}}{3}\right)^{2/3} = \frac{2^{2/3}}{3}.$$

An analog of Proposition 9, where we now consider a Boltzmann quadrangulation $Q^{(p)}$ of the $2p$-gon, also holds in the form

$$p^{-2}\mathbb{E}\big[|Q^{(p)}|\big] \underset{p\to\infty}{\longrightarrow} \frac{9}{2} =: \mathsf{b}_{\square}.$$

The peeling by layers requires certain modifications in the case of quadrangulations. As previously, the ball $B_r(Q_\infty)$ is the planar map obtained by keeping only those faces of Q_∞ that are incident to at least one vertex whose

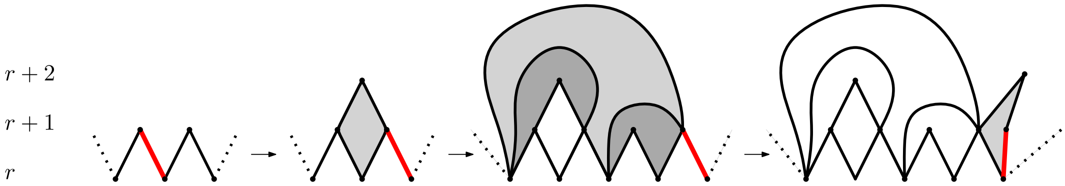

Fig. 8. Illustration of the peeling by layers in the quadrangular case: we choose an edge to start discovering the new layer and then peel from left to right all the edges that contain a vertex at distance r from the root vertex.

graph distance from the root vertex is smaller than or equal to $r-1$, and the hull $B^\bullet_r(Q_\infty)$ is obtained by filling in the finite holes of $B_r(Q_\infty)$. The boundary $\partial B^\bullet_r(Q_\infty)$ is now a simple cycle that visits alternatively vertices at distance r and $r+1$ from the root vertex. If we move around the boundary of this cycle in clockwise order, we encounter two types of (oriented) edges, edges $r+1 \to r$ connecting a vertex at distance $r+1$ to a vertex at distance r, and edges $r \to r+1$ connecting a vertex at distance r to a vertex at distance $r+1$.

To describe the peeling by layers algorithm, suppose that, at a certain step of the peeling process, the revealed region is the hull $B^\bullet_r(Q_\infty)$. Then we choose (deterministically or using some independent randomization) an edge of the boundary of type $r+1 \to r$. We reveal the face incident to this edge that is not already in $B^\bullet_r(Q_\infty)$ and as usual we fill in the holes that may have been created. At the next step, either the (new) boundary has an edge of type $r+1 \to r$ that is incident to the quadrangle revealed in the previous step, and we peel this edge, or we peel the first edge of type $r+1 \to r$ coming after the revealed quadrangle in clockwise order. We continue inductively, "moving around the boundary in clockwise order." See Figure 8 for an example. After a finite number of steps, the boundary does not contain any vertex at distance r, and it is easy to verify that the revealed region is then the hull $B^\bullet_{r+1}(Q_\infty)$, so that we can continue the construction by induction.

Proposition 11 is adapted as follows. For every $r \geq 1$, let $\mathcal{L}^\square_r$ be the set of all vertices of $\partial B^\bullet_r(Q_\infty)$ that are at distance exactly r from the root vertex. Clearly the perimeter $|\partial B^\bullet_r(Q_\infty)|$ is equal to $2\#\mathcal{L}^\square_r$. We also denote the union of all $\mathcal{L}^\square_r$ for $r \geq 1$ by $\mathcal{L}^\square$. Finally, for $n \geq 1$, we let $A^\square_n$ be the number of vertices of $\mathcal{L}^\square$ that are in the interior of the discovered region at step n. Then the analog of Proposition 11 reads

$$\frac{A_n}{n} \xrightarrow[n\to\infty]{(P)} \frac{1}{3} =: \mathsf{a}_\square.$$

The idea of the proof is the same but technicalities become somewhat more complicated (we omit the details).

Versions of Theorems 1, 2 then hold for the UIPQ Q_∞. In these statements we now interpret the size of the boundary as half its perimeter, the constant $\mathsf{p}_{\triangle^2}$ is replaced by $\mathsf{p}_\square$ and the other constants $\mathsf{v}_{\triangle^2}$ and $\mathsf{h}_{\triangle^2}$ are replaced respectively by

$$\mathsf{v}_\square = (\mathsf{p}_\square)^2 \mathsf{b}_\square = 2^{1/3} \quad \text{and} \quad \mathsf{h}_\square = \frac{\mathsf{a}_\square}{\mathsf{p}_\square} = 2^{-2/3}.$$

We observe that the convergence of volumes in the analog of Theorem 2 for the UIPQ was already obtained in [17] as a consequence of the invariance principles relating the UIPQ and the Brownian plane (see Theorems 5.1 and 1.3 in [17]). It would be significantly harder to derive the convergence of boundary lengths from the same invariance principles. On the other hand, Krikun [21] has a version of the scaling limit for boundary lengths in the case of quadrangulations, but with a different definition of hull boundaries leading to different constants.

It is also possible to adapt Theorems 3 and 4 to the setting of quadrangulations: the limiting process in the analog of Theorem 3 (where we again consider the half-perimeter rather than the perimeter) is $(\mathsf{p}_\square \cdot \mathcal{L}_{t/\mathsf{h}^*_\square}, \mathsf{v}_\square \cdot \mathcal{M}_{t/\mathsf{h}^*_\square})_{t\geq 0}$, with

$$\mathsf{h}^*_\square = \frac{1+\mathsf{a}^*_\square}{2\mathsf{p}_\square},$$

where $\mathsf{a}^*_\square = \frac{1}{2}$ is the mean number of edges "swallowed" on the right of the peeled edge in a peeling step for the half-plane UIPQ (see [4, Eq. (8)] where this quantity is denoted by $\delta^\square/2$). The extra multiplicative factor 2 in the time parameter comes from the fact that we are dealing with half-perimeters. Similarly, the limiting process in the analog of Theorem 4 is $(\mathsf{p}_\square \cdot \mathcal{L}_{2\mathsf{p}_\square t}, \mathsf{v}_\square \cdot \mathcal{M}_{2\mathsf{p}_\square t})_{t\geq 0}$.

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