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RANDOM REVERSIBLE MARKOV MATRICES WITH TUNABLE EXTREMAL EIGENVALUES

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Random sampling of large Markov matrices with a tunable spectral gap, a nonuniform stationary distribution and a nondegenerate limiting empirical spectral distribution (ESD) is useful. Fix $c > 0$ and $p > 0$. Let A_n be the adjacency matrix of a random graph following $\mathrm{G}(n, p/n)$, known as the Erdős–Rényi distribution. Add c/n to each entry of A_n and then normalize its rows. It is shown that the resulting Markov matrix has the desired properties. Its ESD weakly converges in probability to a symmetric nondegenerate distribution, and its extremal eigenvalues, other than 1, fall in $[-1/\sqrt{1+c/k}, -b] \cup [b, 1/\sqrt{1+c/k}]$ for any $0 < b < 1/\sqrt{1+c}$, where $k = \lfloor p \rfloor + 1$. Thus, for $p \in (0, 1)$, the spectral gap tends to $1 - 1/\sqrt{1+c}$.

1. Introduction. The spectral properties of random Markov matrices have received increasing attention over the years [4–6, 10, 24, 28]. In applications, it is useful to randomly sample a large Markov matrix, such that the mixing rate of the associated Markov chain is controllable. The chain can be used, for example, to evaluate the performance of a data analytic procedure under various strengths of statistical dependency within data [25]. By the well-known connection between mixing rate and eigenvalues of Markov matrix [12, 26], the issue may be cast as how to sample large Markov matrices with a specified spectral gap. This note addresses the issue for reversible Markov matrices.

Denote by $\mathcal{M}_n$ the set of $n \times n$ matrices with all entries being nonnegative. For $M \in \mathcal{M}_n$, if its eigenvalues are $\lambda_1(M), \dots, \lambda_n(M)$, counting multiplicity, then its spectral radius is $\varrho(M) = \max|\lambda_i(M)|$ and its empirical spectral distribution (ESD) is

$$\mu_M = n^{-1} \sum_{i=1}^{n} \delta_{\lambda_i(M)},$$

where δ_x is the probability measure concentrated at x. By the Perron–Frobenius theorem ([16], page 534), $\varrho(M)$ is an eigenvalue of M. If $M 1_n = 1_n$, where 1_n is the column vector of n 1's, then M is called a Markov matrix and $\varrho(M) = 1$. Letting $\lambda_n(M) = \varrho(M)$, $\lambda_\star(M) = \max_{i<n} |\lambda_i(M)|$ and $1 - \lambda_\star(M)$ are known as the second largest absolute eigenvalue and the spectral gap of M, respectively.

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For $X \in \mathcal{M}_n$, if all the entries of $a := X1_n$ are positive, then its row-normalized version refers to the Markov matrix $M = D_a^{-1}X$, where D_a denotes the diagonal matrix whose diagonal equals a. If X is symmetric, then the Markov chain with transition matrix M and initial distribution $\pi = a/1_n' a$ is stationary and has the same distribution as its time reversal, and for this reason M is called reversible relative to π. Moreover, all $\lambda_i(M)$ are real as M is similar to $D_a^{-1/2} M D_a^{-1/2}$, where $D_a^{1/2}$ denotes any symmetric matrix whose square equals D_a.

Let $X_n \in \mathcal{M}_n$ be symmetric random matrices with positive entries almost surely (a.s.). Let M_n be its row-normalized version. Suppose the diagonal and upper diagonal entries of X_n are i.i.d. $\sim \nu_n$. If $\nu_n = \nu$ for all n, then by [4], provided that the 4th moment of ν is finite, $\lambda_\star(M_n) \to 0$ a.s. as $n \to \infty$. On the other hand, if ν is in the domain of attraction of a stable law of index in $(0, 2)$, then by [5], $\lambda_\star(M_n) \to 1$ a.s. In either case, the spectral gap of M_n cannot be tuned. The results suggest that, in order for the spectral gap or, equivalently, $\lambda_\star(M_n)$ to be tunable, the (marginal) distribution of the entries of X_n needs to change according to n.

Indeed, there are simple solutions along this line. Given $n \ge 5$, randomly pick four different numbers k, l, s and t from $1, \dots, n$. Let $A_n = (\varepsilon_{ij}) \in \mathcal{M}_n$ with $\varepsilon_{ij} = \mathbf{1}\{\{i, j\} = \{k, l\} \text{ or } \{s, t\}\}$, where $\mathbf{1}\{\cdot\}$ is the indicator function; A_n is the adjacency matrix of a graph on n vertices with only two edges. Given $c > 0$, let M_n be the row-normalized version of $cJ_n + A_n$, with $J_n \in \mathcal{M}_n$ a matrix of $1/n$'s. As $\det(z - M_n) = [z + 1/(1 + c)]^2 z^{n-5}[z - (1 - 4/n)/(1 + c)][z - 1/(1 + c)](z - 1)$, $\lambda_\star(M_n) = 1/(1 + c)$, so it can be set at any value in $(0, 1)$.

The main problem with the example is that M_n has few features. It is nearly the transition matrix of a chain of i.i.d. random variables uniformly taking n values. The lack of features is also reflected in the ESD of M_n, which converges to δ_0 as $n \to \infty$. Despite this, the example shows that it is possible to tune the spectral gap by using sparse random graphs. In general, let A_n be the adjacency matrix of a random graph. Define the row-normalized version of $cJ_n + A_n$ as

$$M_n = D_n^{-1}(cJ_n + A_n) \qquad \text{with } D_n = D_{c1_n + A_n 1_n}. \tag{1}$$

Although A_n can be highly reducible, M_n is always irreducible and aperiodic and so $\lambda_\star(M_n) < 1$. Since all the eigenvalues of M_n are real, we always assume that they are sorted as

$$-1 < \lambda_1(M_n) \le \cdots \le \lambda_{n-1}(M_n) < \lambda_n(M_n) = 1.$$

Then $\lambda_\star(M_n) = \max(|\lambda_1(M_n)|, |\lambda_{n-1}(M_n)|)$. We simply call M_n reversible, as there is only one stationary distribution associated with it. The closely related matrix $I_n - D_n^{-1/2}(cJ_n + A_n)D_n^{-1/2}$ is known as a normalized Laplacian regularized by c. The effects of c on spectral clustering and concentration of the ESD have been studied in statistical machine learning [19, 21].

The close relation between random matrices and random graphs is well known; see [5, 7, 13, 15, 17, 18, 20, 22, 23, 27–29] and references therein. In [22], it is shown that if A_n is the adjacency matrix of a random graph following the uniform distribution $\mathrm{G}_{n,d}$ on the set of regular graphs on n vertices with fixed degree $d \geq 2$, then as $n \to \infty$, μ_{A_n} weakly converges a.s. with limiting density $f(x) = d(4d-4-x^2)_+^{1/2}/[2\pi(d^2-x^2)]$, where $a_+ := \max(a, 0)$. By Weyl's inequality, $\lambda_i(D_n^{-1}A_n) \leq \lambda_i(M_n) \leq \lambda_{i+1}(D_n^{-1}A_n)$ for $i < n$ [cf. (2)]. Consequently, M_n and $D_n^{-1}A_n = (c+d)^{-1}A_n$ have the same limiting ESD density $(c+d)f((c+d)x)$, whose support is the interval between $\pm 2\sqrt{d-1}/(c+d)$. Thus $\lambda_\star(M_n)$ is asymptotically lower bounded by $2\sqrt{d-1}/(c+d)$. On the other hand, by the above Weyl's inequality and the fact that $\varrho(A_n)$ is less than the maximum row sum of A_n ([16], pages 345–347), $\lambda_\star(M_n) \leq \varrho(A_n)/(c+d) \leq d/(c+d)$. In particular, when $d = 2$, in which case the graph consists of disjoint cycles, $\lambda_\star(M_n) \to 2/(c+2)$ a.s. Also, under various distributions on regular multigraphs of fixed degree d that allow multiple edges and, in some cases, self-loops, for any fixed l, $|\lambda_l(A_n)|$ and $\lambda_{n-l}(A_n)$ converge to $2\sqrt{d-1}$ in probability, yielding $\lambda_\star(M_n) \to 2\sqrt{d-1}/(c+d)$ [14]. However, when $d > 2$, $\lambda_\star(M_n)$ cannot be arbitrarily tuned as it is asymptotically upper bounded by $2\sqrt{d-1}/d < 1$. Perhaps important, under any distribution on regular (multi)graphs, since M_n is doubly Markov, that is, M_n' is Markov as well, the stationary distribution associated with M_n is uniform. If one wishes to sample a large Markov matrix with a nonuniform stationary distribution, then a different random graph needs to be exploited. We also mention that for a uniformly sampled doubly Markov matrix, which is irreversible a.s., its limiting ESD is degenerate [24].

We shall consider the row-normalized version M_n of $cJ_n + A_n$ with A_n the adjacency matrix of a random graph following $\mathrm{G}(n, p/n)$, the distribution on graphs on n vertices such that each pair of vertices is connected by an edge with probability p/n, independently from the other pairs ([2], VII). It is easy to see that for large n, the stationary distribution associated with M_n is nonuniform with high probability. We shall fix $c > 0$ and $p > 0$ when deriving the asymptotic spectral properties of M_n. It is known that for both $\mathrm{G}_{n,d}$ and $\mathrm{G}(n, p/n)$, if $d \to \infty$ and $p \to \infty$ as $n \to \infty$, then the ESD of suitably scaled and centered A_n tends to the semi-circle law [13, 28]. It is also known that when $p > 1$ is fixed, the adjacency matrix of the giant component of a $\mathrm{G}(n, p/n)$-distributed graph has a spectral gap asymptotically equal to 0 [23]. However, these results provide no indication on the spectral properties of M_n.

For the rest of the note, denote

$$\tau_c = 1/\sqrt{1+c}, \qquad c \geq 0.$$

One of the main results of the note is the following.

THEOREM 1. *Fix $c > 0$ and $p > 0$. For $n > p$, let A_n be the adjacency matrix of a random graph following $\mathrm{G}(n, p/n)$. Let $k = \lfloor p \rfloor + 1$. Fix $l \geq 1$ and $0 < b < \tau_c$. Then $\mathrm{P}\{b \leq \lambda_{n-l}(M_n) \leq \tau_{c/k}$ and $-\tau_{c/k} \leq \lambda_l(M_n) \leq -b\} \to 1$ as $n \to \infty$.*

Thus, roughly speaking, $\lambda_\star(M_n)$ asymptotically lies between τ_c and $\tau_{c/k}$. In particular, if $p \in (0,1)$, then $\lambda_\star(M_n) \to \tau_c$ in probability. To prove Theorem 1, in Section 2, we show that $\lambda_\star(M_n)$ is asymptotically dominated by $\tau_{c/k}$. Then, in Section 3, we show that μ_{M_n} weakly converges in probability to a symmetric non-degenerate distribution and characterize the moments of the limiting distribution in terms of a random walk on a Galton–Watson tree. The proof uses the local convergence of random graphs [5, 8]. In Section 4, we show that the essential supremum of the limiting distribution is τ_c, which together with the result in Section 2 proves Theorem 1. In this section, we also report some numerical results which suggest that bounds for $\lambda_\star(M_n)$ are not tight, especially the upper bound when p is large. Finally, in Section 5, we provide a more explicit formula for the moments of the limit of μ_{M_n}, using the standard moment method. Some of the results in previous sections can also be established by the method [11].

1.1. *Notation.* Following [2], a (labeled) graph G has no multiple edges or self-loops, and all its edges are undirected. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. Each $e \in E(G)$ is an unordered pair $\{u, v\}$, with $u \neq v \in V(G)$; u is called an endpoint of e, denoted $u \in e$. When direction has to be taken into account, denote by (u, v) the directed edge starting at u and ending at v. The adjacency matrix of G is $A_G = (\varepsilon_{uv})_{u,v\in G}$ with $\varepsilon_{uv} = \mathbf{1}\{\{u, v\} \in E(G)\}$. Denote by $|A|$ the cardinality of a set A. Denote $|G| = |V(G)|$ and $e(G) = |E(G)|$, and refer to them as the order and size of G, respectively. For brevity, denote $u \in G$ if $u \in V(G)$. Denote by $d(u, G) := |\{e \in E(G) : u \in e\}|$ the degree of $u \in G$. If G' is another graph, denote by $G \cup G'$ the graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$, and denote $G \sim G'$ if the two graphs are isomorphic ([2], page 3). If $v \in G$ and $v' \in G'$, and if there is a graph isomorphism $\sigma : G \to G'$, such that $\sigma(v) = v'$, then (G, v) and (G', v') are called isomorphic rooted graphs (rooted with v and v', resp.). For a finite set I, denote by 1_I the column vector of 1's indexed by I. For $k \geq 1$, a path on I of length k is a sequence $\mathbf{v} = (v_1, \ldots, v_{k+1})$ with $v_i \in I$ and $v_i \neq v_{i+1}$ for $i \leq k$; note the requirement that adjacent v_i's be different. If $v_{k+1} = v_1$, then $\mathbf{v}$ is said to be closed.

For properties of $\mathrm{G}(n, a)$, see [2, 3]. For $a \in [0, 1]$, denote by $\mathrm{Bern}(a)$ the Bernoulli distribution with mass a on 1. Denote by $\mathrm{Po}(p)$ the Poisson distribution with mean $p \geq 0$. The essential supremum of a measure ν on $\mathbb{R}$ is $\operatorname{ess\,sup} \nu = \sup\{x : \nu(x, \infty) > 0\}$. For $M \in \mathcal{M}_n$ and $k \geq 0$, denote by $\beta_k(M)$ the kth moment of μ_M, which equals $(1/n)\operatorname{tr}(M^k)$ ([1], equation (1.3.2)).

2. Upper bound of spectral radius. Fix $c > 0$ and $p > 0$. For $n > p$, let $A_n = A_G$ with $G \sim \mathrm{G}(n, p/n)$. Define M_n and D_n by (1). The spectrum of M_n is identical to that of $D_n^{-1/2}(cJ_n + A_n)D_n^{-1/2}$. Since $D_n^{-1/2} J_n D_n^{-1/2}$ is of rank one with the only nonzero eigenvalue being positive, by Weyl's inequality ([16], Corollary 4.3.3)

$$\lambda_i(D_n^{-1}A_n) \leq \lambda_i(M_n) \leq \lambda_{i+1}(D_n^{-1}A_n), \qquad 1 \leq i < n. \tag{2}$$

Consequently, to prove the bound involving $\tau_{c/k}$ in Theorem 1, that is, given $l \ge 1$, $\mathrm{P}\{\lambda_{n-l}(M_n) \le \tau_{c/k}$ and $\lambda_l(M_n) \ge -\tau_{c/k}\} \to 1$, it suffices to prove the following.

PROPOSITION 2. *Let $p > 0$ and $k = \lfloor p \rfloor + 1$. Then $\mathrm{P}\{\varrho(D_n^{-1} A_n) \le \tau_{c/k}\} \to 1$ as $n \to \infty$.*

For graph G, denote

$$K_G = K_G(c) = D^{-1}_{c1_{V(G)}+A_G 1_{V(G)}} A_G \tag{3}$$

and analogously $K_n = D_n^{-1} A_n$. Put $q = \varrho(K_G)$. By the Perron–Frobenius theorem ([16], page 534) $q = \lambda_{|G|}(K_G)$ and if $|G| > 1$ and G is connected, then $q > 0$ and there is a vector $f = (f(u))_{u\in G}$ with all $f_u > 0$, such that

$$K_G f = q f. \tag{4}$$

Denote by $N(u)$ the neighborhood of u in G, that is, the set of $v \in G$ with $\{u, v\} \in E(G)$.

LEMMA 3. *Let G be connected with $|G| > 1$ and $C \ne \varnothing$ be a subset of $V(G)$. Denote by $h(u)$ the distance of $u \in G$ to C. Define $\omega(u) = f(u)q^{h(u)}$. For $i = 0, \pm 1$, define $N_i(u) = \{v \in N(u) : h(v) = h(u) + i\}$ and $d_i(u) = |N_i(u)|$. Then*

$$\begin{aligned} q^{-1} \sum d_0(u)\omega(u) + q^{-2} \sum d_{-1}(u)\omega(u) \\ = \sum [d_0(u) + d_{-1}(u) + c]\omega(u). \end{aligned} \tag{5}$$

The result actually holds for any function h on $V(G)$ with the property that $h(v) - h(u) \in \{0, \pm 1\}$ for any $u \in G$ and $v \in N(u)$. It is also easy to generate the result to integer-valued functions on $V(G)$. However, so far only h defined in Lemma 3 has proved to be useful.

PROOF OF LEMMA 3. Since $N_0(u)$, $N_-(u)$ and $N_{+1}(u)$ partition $N(u)$, (4) can be written as

$$\sum_{v\in N_{-1}(u)} f(v) + \sum_{v\in N_0(u)} f(v) + \sum_{v\in N_{+1}(u)} f(v) = q[c + d(u, G)] f(u).$$

Multiplying both sides by $q^{h(u)-1}$ yields

$$\sum_{v\in N_{-1}(u)} \omega(v) + q^{-1} \sum_{v\in N_0(u)} \omega(v) + q^{-2} \sum_{v\in N_{+1}(u)} \omega(v) = [c + d(u, G)]\omega(u).$$

Take sum over u. Since $v \in N_{-1}(u) \iff u \in N_{+1}(v)$ and $v \in N_0(u) \iff u \in N_0(v)$,

$$\sum_u \sum_{v\in N_{-1}(u)} \omega(v) = \sum_v \sum_{u\in N_{+1}(u)} \omega(v) = \sum_v d_{+1}(v)\omega(v)$$

and likewise,

$$\sum_u \sum_{v \in N_0(u)} = \sum_v d_0(v)\omega(v), \qquad \sum_u \sum_{v \in N_{+1}(u)} = \sum_v d_{-1}(v)\omega(v).$$

Combining the equations and noticing $d(u, G) = d_0(u) + d_{-1}(u) + d_{+1}(u)$, (5) then follows. $\square$

LEMMA 4. *Let G be a connected graph. If G is a tree or a unicyclic graph, then*

$$\varrho(K_G) < \tau_c. \tag{6}$$

Furthermore, if G is a unicyclic graph, then

$$\varrho(K_G) \ge (1 + c/2)^{-1} \qquad \text{with “=”} \iff G \text{ is a cycle}. \tag{7}$$

PROOF. First, let G be a tree. If $|G| = 1$, then $K_G = 0$ and (6) is trivial. Let $|G| \ge 2$. Pick an arbitrary vertex $\theta \in G$ and let $C = \{\theta\}$. It is easy to see that for any $u \in G$, $d_0(u) = 0$ and $d_{-1}(u) = \mathbf{1}\{u \ne \theta\}$. Then (6) follows from (5), which now takes the form

$$q^{-2} \sum_{u \ne \theta} \omega(u) = (1 + c) \sum_{u \ne \theta} \omega(u) + c\omega(\theta). \tag{8}$$

Next, let G be unicyclic. Let C be the cycle subgraph of G. Then $|C| \ge 3$. The subgraph of G obtained by removing the edges in C consists of $|C|$ isolated trees, each containing exactly one vertex in C. It can be seen that $d_0(u) = 2\mathbf{1}\{u \in C\}$ and $d_{-1}(u) = \mathbf{1}\{u \notin C\}$. Then by (5),

$$(2/q) \sum_{u \in C} \omega(u) + q^{-2} \sum_{u \notin C} \omega(u) = (c + 2) \sum_{u \in C} \omega(u) + (c + 1) \sum_{u \notin C} \omega(u).$$

If G is a cycle, then $C = G$ and the equation yields $q = 1/(1 + c/2)$. If G is not a cycle, then $\sum_{u \notin C} \omega(u) > 0$. If $q \le 1/(1 + c/2)$, then from $2/q \ge c + 2$ and $\sum_{u \in C} \omega(u) > 0$, it follows that $c + 1 \ge q^{-2}$, or $q \ge \tau_c > 1/(1 + c/2)$, which is a contradiction. Thus, $q > 1/(1 + c/2)$. But then $2/q < c + 2$, implying $q^{-2} > c + 1$, or $q < \tau_c$. $\square$

It may be worth noting that if G is a tree, then $|G| \to \infty$ does not guarantee that $\varrho(K_G) \to \tau_c$. For example, suppose $d(v, G) < 1 + c$ for all $v \in G$. Put $d_0 = \lceil c \rceil$. Then $d(v, G) \le d_0$. Let f be as in (4) and $\theta = \arg\max f(v)$. Then for $k \ge 1$, $\sum_{h(u)=k} \omega(u) \le d_0^k q^k f(\theta) < [d_0/(1 + c)]^k \omega(\theta)$, giving $\sum_{u \ne \theta} \omega(u) \le b\omega(\theta)$ with $b = \sum_k [d_0/(1 + c)]^k < \infty$. Then by (8), $q \not\to \tau_c$.

PROOF OF PROPOSITION 2. By definition, $K_n = D_n^{-1} A_n = K_G$ with $G \sim \mathrm{G}(n, p/n)$. First, suppose $0 < p < 1$. Write the connected components of G as

$G_1, \dots, G_s$. Then K_G can be partitioned as

$$K_G = \begin{pmatrix} K_{G_1} & & \\ & \ddots & \\ & & K_{G_s} \end{pmatrix}.$$

The eigenvalues of K_G therefore are exactly those of K_{G_i}, counting multiplicity. Since $0 < p < 1$, P{all G_i are trees or unicyclic graphs} $\to 1$ as $n \to \infty$ ([3], Corollary 5.8). This combined with Lemma 4 yields $\mathrm{P}\{\varrho(K_n) \le \tau_c\} \to 1$.

To continue, note that given $0 < p_0 < p_1 < 1$, as $n \to \infty$,

$$\inf_{p_0 \le p \le p_1} \mathrm{P}_p\{\varrho(K_n) \le \tau_c\} \to 1, \tag{9}$$

where P_p denotes probability under $\mathrm{G}(n, p/n)$. Indeed, from the proof of Theorem 5.7 and Corollary 5.8 in [3], as $n \to \infty$, $\inf_{p_0 \le p \le p_1}$ P{every component of G is a tree or a unicyclic graph} $\to 1$. Then (9) follows from the same argument for the already-proved case $0 < p < 1$.

Now let $p \ge 1$. Then $k := \lfloor p \rfloor + 1 > 1$. For $n > p$, let $T_{1,n}, \dots, T_{k,n}$ be i.i.d. $\sim A_G$ with $G \sim \mathrm{G}(n, p'_n/n)$, where

$$p'_n = \frac{p}{k - (k-1)p/n}.$$

Since $p'_n \in (0, n)$, G is well defined. Let $T_n = (t_{ij}) = \sum_{s=1}^k T_{s,n}$. Since t_{ij}, $i < j$, are i.i.d., for any $B = (b_{ij}) \in \{0, 1\}^{n \times n}$ with $b_{ij} = b_{ji}$ and $b_{ii} = 0$,

$$\begin{aligned} \mathrm{P}\{T_n = B | T_n \in \{0,1\}^{n \times n}\} &= \frac{\mathrm{P}\{t_{ij} = b_{ij}, i < j\}}{\mathrm{P}\{t_{ij} \in \{0,1\}, i < j\}} \\ &= \prod_{i<j} \frac{\mathrm{P}\{t_{ij} = b_{ij}\}}{\mathrm{P}\{t_{ij} \in \{0,1\}\}}. \end{aligned}$$

Since $\mathrm{P}\{t_{ij} = 0\} = (1 - p'_n/n)^k$ and $\mathrm{P}\{t_{ij} = 1\} = k(p'_n/n)(1 - p'_n/n)^{k-1}$, direct calculation shows that conditional on it being in $\{0,1\}^{n \times n}$, T_n has the same distribution as A_n. For $i < j$, $\mathrm{P}\{t_{ij} \in \{0,1\}\} \ge 1 - [k(k-1)/2](p'_n/n)^2$. On the other hand, $p'_n \to p/k$ as $n \to \infty$. Then for n large enough, $\mathrm{P}\{T_n \in \{0,1\}^n\} > \exp(-p^2)$, so letting $\Delta_n = D_{c1_n + T_n 1_n}$ and $C = \exp(p^2)$, for any x,

$$\begin{aligned} \mathrm{P}\{\varrho(K_n) > x\} &= \mathrm{P}\{\varrho(\Delta_n^{-1} T_n) > x | T_n \in \{0,1\}^{n \times n}\} \\ &\le C\mathrm{P}\{\varrho(\Delta_n^{-1/2} T_n \Delta_n^{-1/2}) > x\}. \end{aligned} \tag{10}$$

Put $\Delta_{s,n} = D_{c1_n/k + T_{s,n}1_n}$ and $B_{s,n} = \Delta_{s,n}^{-1/2} T_{s,n} \Delta_{s,n}^{-1/2}$. Then $\Delta_n = \Delta_{1,n} + \cdots + \Delta_{k,n}$ and

$$\Delta_n^{-1/2} T_n \Delta_n^{-1/2} = \sum_s \Delta_n^{-1/2} \Delta_{s,n}^{1/2} B_{s,n} \Delta_{s,n}^{1/2} \Delta_n^{-1/2}.$$

Fix an arbitrary $a \in (p/k, 1)$. For n large enough, $p'_n \in [p/k, a]$. Then by (9), the probability of the event that $\varrho(B_{s,n}) \le \tau_{c/k}$ for all $1 \le s \le k$ tends to 1. On this event, for any $u \in \mathbb{R}^n$,

$$\begin{aligned}
|u' \Delta_n^{-1/2} T_n \Delta_n^{-1/2} u| &\le \sum_s |u' \Delta_n^{-1/2} \Delta_{s,n}^{1/2} B_{s,n} \Delta_{s,n}^{1/2} \Delta_n^{-1/2} u| \\
&\le \sum_s \varrho(B_{s,n}) |\Delta_{s,n}^{1/2} \Delta_n^{-1/2} u|^2 \\
&\le \tau_{c/k} \sum_s |\Delta_{s,n}^{1/2} \Delta_n^{-1/2} u|^2 = \tau_{c/k} |u|^2.
\end{aligned}$$

It follows that $\mathrm{P}\{\varrho(\Delta_n^{-1/2} T_n \Delta_n^{-1/2}) \le \tau_{c/k}\} \to 1$, so by (10), $\mathrm{P}\{\varrho(D_n^{-1} A_n) \le \tau_{c/k}\} \to 1$. $\square$

3. Convergence of ESD. Let A_n, M_n, D_n and $K_n = D_n^{-1} A_n$ be as in previous sections. We shall show that μ_{M_n} weakly converges as $n \to \infty$. From Weyl's inequality (2), μ_{M_n} weakly converges in probability (resp., a.s.) $\Longleftrightarrow$ μ_{K_n} does so in probability (resp., a.s.) and, provided the convergence holds, the two ESDs have the same limit. Therefore, we shall focus on μ_{K_n} instead. The approach we shall take is the local convergence of random graphs; see [8] and references therein, and [5] for extension to the ESD of random matrices whose entries belong to the domain of attraction of stable laws.

Let G be a graph and $v_0 \in G$. Fix $c \ge 0$. Consider the following random walk on G starting from v_0 at step 0. If $d(v_0, G) \ge 1$, then if at step $k \ge 0$ the random walk is at v with $d(v, G) = d(\ge 1)$, then at step $k+1$, it either moves to a neighbor of v with probability $1/(c+d)$, or is killed with probability $c/(c+d)$. If $d(v_0, G) = 0$, then the random walk is killed at step 1, regardless of the value of c. Let

$$r_k(G, v_0, c) = \mathrm{P}\{\text{the random walk is alive and at } v_0 \text{ at step } k\}.$$

Let $\varnothing$ be an arbitrary element. Denote by $[(G, v_0)]$ the class of graphs rooted with $\varnothing$ that are isomorphic to (G, v_0). Then $r_k(G, v_0, c)$ depends on (G, v_0) only through $[(G, v_0)]$.

Recall that in order for $K_n = D_n^{-1} A_n$ to be always well defined, c has to be strictly positive. In the following, we redefine D_n such that its ith diagonal element is 1 if the entire ith row of A_n is 0. With this definition, c can be 0.

THEOREM 5. *Let $c \ge 0$. As $n \to \infty$, μ_{K_n} weakly converges in probability. The weak convergence is a.s. if n is replaced with any subsequence n_j with $\sum n_j^{-1} < \infty$. The limiting distribution is symmetric and nondegenerate, and for $k \ge 1$, its kth moment is $\beta_k = \mathrm{E} r_k(T, \varnothing, c)$, where T is a random Galton–Watson tree rooted with $\varnothing$ and with* $\mathrm{Po}(p)$ *offspring distribution.*

Note that for any tree T, if k is odd, then $r_k(T, \varnothing, c) = 0$ and hence $\beta_k = 0$. This immediately leads to the symmetry of the limiting distribution.

PROOF OF THEOREM 5. Put $K_n = (x_{ij})$. Denote by $C_{k,n}$ the set of closed paths of length k on $\{1, \dots, n\}$. For $\mathbf{i} \in C_{k,n}$, denote $x(\mathbf{i}) = x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{k-1} i_k} x_{i_k i_1}$. Then for $k \ge 1$ and $s = 1, \dots, n$, the sth diagonal entry of K_n^k is

$$\big(K_n^k\big)_{ss} = \sum_{\mathbf{i} \in C_{k,n}, i_1 = i_k = s} x(\mathbf{i}).$$

Since $x(\mathbf{i})$ is the probability that the random walk is alive after traversing the closed path $\mathbf{i}$,

$$\big(K_n^k\big)_{ss} = r_k(G, s, c), \qquad s = 1, \dots, n.$$

For a random walk on G that starts from s, if it returns to s at step k, then the vertices it visits by then each has at most distance $k - 1$ from s, and so the neighbors of each such vertex has at most distance k from s. Denote by $G_{k,s}$ the subgraph of G whose vertex set consists of vertices with distance from s no greater than k and whose edge set consists of edges in G connecting these vertices. Then $r_k(G, s, c) = r_k(G_{k,s}, s, c)$. It is well known that, given s, as $n \to \infty$, G rooted with s converges locally to T in distribution. This means that for any k, $G_{k,s}$ rooted with s converges in distribution to T_k, the subtree of T consisting of $\varnothing$ and its first k generations of descendants; see, for example, [8]. As a result, $r_k(G_{k,1}, 1, c) \to r_k(T_k, \varnothing, c) = r_k(T, \varnothing, c)$ in distribution. By the above displays, $\beta_k(K_n) = n^{-1} \sum_{s=1}^n r_k(G, s, c)$. Then by exchangeability and dominated convergence, $\mathrm{E}\beta_k(K_n) = \mathrm{E}r_k(G, 1, c) = \mathrm{E}r_k(G_{k,1}, 1, c) \to \mathrm{E}r_k(T, \varnothing, c)$.

We need to show that $\beta_k(K_n) \to \beta_k$ in probability as $n \to \infty$, and a.s. if n is replaced with $n_j \to \infty$ such that $\sum n_j^{-1} < \infty$. Put $\xi_s = r_k(G, s, c)$. By exchangeability,

$$\begin{aligned} \mathrm{Var}\big[\beta_k(K_n)\big] &= n^{-1} \mathrm{Var}(\xi_1) + 2\big(1 - n^{-1}\big) \mathrm{Cov}(\xi_1, \xi_2) \\ &\le n^{-1} + 2\big|\mathrm{Cov}(\xi_1, \xi_2)\big|. \end{aligned} \tag{11}$$

Let $S_1 = \mathbf{1}\{\text{distance between 1 and 2 in } G \text{ is} > 2k\}$. Then

$$\begin{aligned} \mathrm{P}\{S_1 = 0\} &\le \sum_{l=0}^{2k-1} \mathrm{P}\big\{\exists i_1, \dots, i_{2k-1} \text{ s.t. } \{i_t, i_{t+1}\} \in E(G), \\ &\qquad 0 \le t < 2k, \text{ with } i_0 = 1, i_{2k} = 2\big\} \\ &\le \sum_{l=0}^{2k-1} n^{2k-1} (p/n)^{2k} = O_k(1/n), \end{aligned}$$

where $O_k(\cdot)$ denotes that the implicit constant depends only on k in addition to the fixed p and c. Note that when $S_1 = 1$, $G_{k,1}$ and $G_{k,2}$ are disjoint. Let $S_2 =$

$\mathbf{1}\{|G_{k,s}| \le n/2, s = 1, 2\}$. Denote by $\mathrm{dis}(u, v) =$ distance between u and v in G. By $|G_{0,s}| = |\{u : \mathrm{dis}(s, u) = 0\}| = 1$,

$$\mathrm{E}\big|\{u : \mathrm{dis}(s, u) = k\}\big| \le \mathrm{E}\Bigg[\sum_{v:\mathrm{dis}(s,v)=k-1} \sum_{u=1}^{n} \mathbf{1}\big\{\{u, v\} \in E(G)\big\}\Bigg]$$
$$\le p\mathrm{E}\big|\{u : \mathrm{dis}(s, u) = k - 1\}\big|,$$

and induction, $\mathrm{E}|G_{k,s}| \le 1 + p + \cdots + p^k = O_k(1)$. Then by Markov inequality, $\mathrm{P}\{S_2 = 0\} = O_k(1/n)$. Let $S = S_1S_2$. Then $\mathrm{P}\{S = 0\} = O_k(1/n)$. Conditioning on $S = 1$, $[(G_{k,1}, 1)]$ and $[(G_{k,2}, 2)]$ are i.i.d. $\sim [(G_{k,1}, 1)]$ conditioning on $|G_{k,1}| \le n/2$. Since for $s = 1, 2$, $\xi_s = r_k(G_{k,s}, s, c)$ only depends on $[(G_{k,s}, s)]$, $\mathrm{Cov}(\xi_1, \xi_2|S = 1) = 0$. By exchangeability $\mathrm{E}(\xi_1|S) = \mathrm{E}(\xi_2|S)$, denoted by h_S. Then

$$\begin{aligned}\mathrm{Cov}(\xi_1, \xi_2) &= \mathrm{E}\big[\mathrm{Cov}(\xi_1, \xi_2|S)\big] + \mathrm{Cov}\big(\mathrm{E}(\xi_1|S), \mathrm{E}(\xi_2|S)\big)\\ &= \mathrm{Cov}(\xi_1, \xi_2|S = 0)\mathrm{P}\{S = 0\} + \mathrm{Var}(h_S)\\ &= O_k(1/n) + (h_1 - h_0)^2\mathrm{P}\{S = 0\}\mathrm{P}\{S = 1\} = O_k(1/n),\end{aligned}$$

so by (11), $\mathrm{Var}[\beta_k(K_n)] = O_k(n^{-1})$. This implies that $\beta_k(K_n) - \mathrm{E}[\beta_k(K_n)] \to 0$ in probability, so $\beta_k(K_n) \to \beta_k$ in probability. Moreover, for $n_j \to \infty$ with $\sum n_j^{-1} < \infty$, by the Borel–Cantelli lemma, $\beta_k(K_{n_j}) - \mathrm{E}[\beta_k(K_{n_j})] \to 0$ a.s., giving $\beta_k(K_{n_j}) \to \beta_k$ a.s.

Since the entries of K_n are nonnegative with row sums no greater than 1, $\varrho(D_n^{-1}A_n) \le 1$ and hence μ_{K_n} is supported in $[-1, 1]$. Meanwhile, by the Weierstrass theorem, polynomials are dense in $C([-1, 1])$. Then by the convergence in probability of $\beta_k(K_n)$ and standard results on weak convergence ([9], Section 8.4), μ_{K_n} weakly converges in probability to a probability distribution with support in $[-1, 1]$ and moments β_k. Finally, for any $n_j \to \infty$ with $\sum n_j^{-1} < \infty$, the a.s. weak convergence of the ESD of K_{n_j} follows from the a.s. convergence of $\beta_k(K_{n_j})$. $\square$

4. Essential supremum of the limit of ESD.

Let μ_∞ be the limiting distribution of μ_{K_n}, where again $K_n = D_n^{-1}A_n$. The main result of this section is the following.

THEOREM 6. *For any $p > 0$, $\operatorname{ess\,sup} \mu_\infty = \tau_c$.*

Thus, for fixed $c > 0$, as $p \to \infty$, $\operatorname{ess\,sup} \mu_\infty$ does not vanish. This may be compared to the case where the underlying random graph follows $\mathrm{G}_{n,d}$. By [22], the corresponding essential supremum is $2\sqrt{d-1}/(c+d)$ so, given $c > 0$, it tends to 0 as $d \to \infty$.

PROOF OF THEOREM 6. Given $s \ge 1$, the probability q_s that T is a tree on $\{\varnothing, v_1, \ldots, v_s\}$ with $E(T) = \{\{\varnothing, v_i\}, i = 1, \ldots, s\}$ is positive. For this T and $k =$

$2m$, it is easy to get $r_k(T, \varnothing, c) = [s/(c+s)]^m[1/(c+1)]^m$. Then by Theorem 5, $\beta_k \ge q_s[s/(c+s)]^m[1/(c+1)]^m$, yielding $\operatorname{ess\,sup} \mu_\infty \ge \sqrt{s/(c+s)}\tau_c$. Letting $s \to \infty$ then gives $\operatorname{ess\,sup} \mu_\infty \ge \tau_c$.

To show $\operatorname{ess\,sup} \mu_\infty \le \tau_c$, it suffices to consider $c > 0$. For $k = 2m$, arguing as in the proof of Theorem 5, $\beta_k = \mathrm{E} r_k(T, \varnothing, c) = \mathrm{E} r_k(T_{m+1}, \varnothing, c) = \mathrm{E}(K^k_{T_{m+1}})_{\varnothing\varnothing}$. We claim that for any finite graph G, $v \in G$, and $k \ge 1$,

$$|(K^k_G)_{vv}| \le \varrho(K_G)^k. \tag{12}$$

Together with Lemma 4, this implies $|(K^k_{T_{m+1}})_{\varnothing\varnothing}| \le \tau^k_c$. As a result $\beta_k \le \tau^k_c$, and hence $\operatorname{ess\,sup} \mu_\infty \le \tau_c$, which completes the proof.

To prove (12), suppose $V(G) = \{1, \dots, n\}$. Then $K_G = D^{-1}_a A_G$, where $a = (a_1, \dots, a_n)' = c1_n + A_G 1_n$. Let $b = (\sqrt{a_1}, \dots, \sqrt{a_n})'$. Then $K^k_G = D^{-1}_b B^k D_b$, where $B = D^{-1}_b A_G D^{-1}_b$. Since D_b is diagonal, $(K^k_G)_{ii} = (B^k)_{ii}$, $i \le n$. Since B is symmetric, so is B^k. In general, for any symmetric real-valued matrix H, since $\varrho(H) \pm H$ is nonnegative definite, $\max_i |H_{ii}| \le \varrho(H)$. Thus, $|(K^k_G)_{ii}| \le \varrho(B^k) = \varrho(B)^k = \varrho(K_G)^k$, as claimed. $\square$

We now can prove Theorem 1. Without loss of generality, let $b \in (0, \tau_c)$ be a continuity point of the distribution function of μ_∞. Then by Theorems 5–6, $\mu_{K_n}(J) \to \mu_\infty(b, \infty) > 0$ in probability for $J = (-\infty, -b)$, (b, ∞), and so for any $l \ge 1$, $\mathrm{P}\{\lambda_l(K_n) \le -b \text{ and } \lambda_{n-l+1}(K_n) \ge b\} \to 1$. Together with Proposition 2, this yields

$$\mathrm{P}\bigl\{b \le \lambda_{n-l+1}(K_n) \le \tau_{c/k} \text{ and } -\tau_{c/k} \le \lambda_l(K_n) \le -b\bigr\} \to 1.$$

Since the convergence holds for all $l \ge 1$, then by Weyl's inequality (2), the proof is complete.

We conducted a simulation study to examine the tightness of the bounds in Theorem 1. Given c, for each $p \in \{0.5, 1, 1.5, 2, 2.5, 3\}$, we used MATLAB function `eig` to calculate $\lambda_\star(M_n)$ for 200 randomly sampled M_n with $n = 4000$. In each panel of Figure 1, the boxplot of the sample values of $\lambda_\star(M_n)$ is shown as a function p. On each box, the central mark is the sample median, the edges of the box are the 1st and 3rd sample quartiles, the whiskers extend to the most extreme sample values considered by MATLAB to be nonoutliers, and the outliers are plotted individually as "x". The y-coordinate of the long horizon line extending from $p = 0.5$ to 3 equals τ_c. As Figure 1 shows, for $p = 0.5$ and most of $p \ge 1$, even when $n = 4000$, $\lambda_\star(M_n)$ is still quite below τ_c. Since $\lambda_\star(M_n)$ is asymptotically lower bounded by τ_c according to Theorem 6, this suggests that its convergence is slow. The plots also indicate that for different values of c, there are different values of p for which the convergence is fastest in terms of how fast $\lambda_\star(M_n)$ approaches or goes above τ_c and how fast its variation decreases. Among all the pairs of c and p, only $(c, p) = (0.5, 1.5)$ and $(1, 2)$ generated a significant number of $\lambda_\star(M_n)$ that

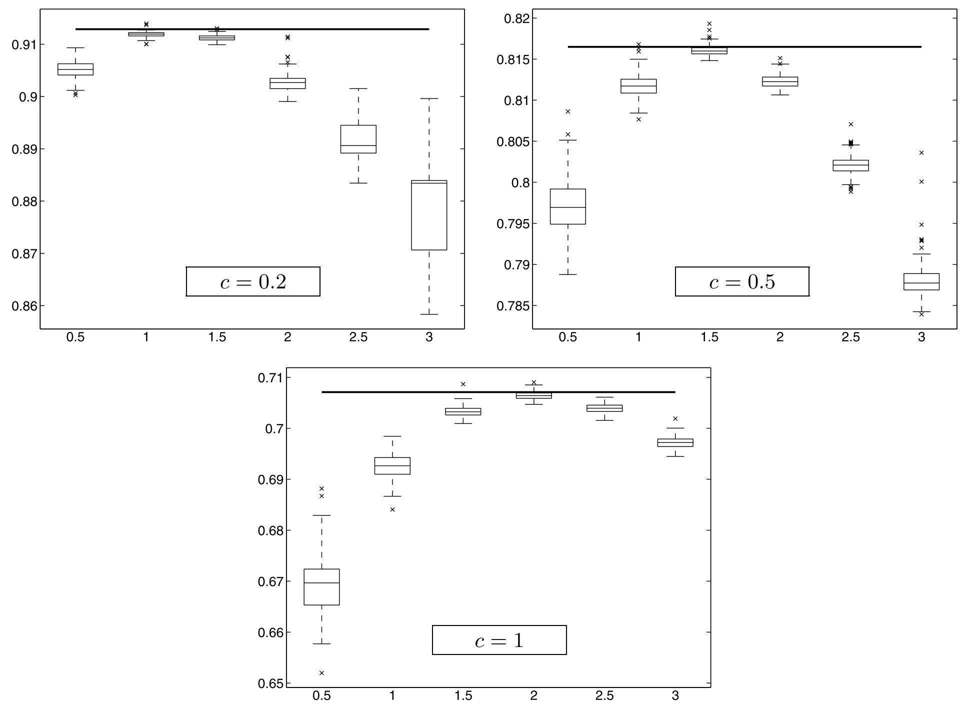

FIG. 1. *Boxplots of randomly sampled* $\lambda_\star(M_n)$.

were no less than τ_c, with the fraction of such $\lambda_\star(M_n)$ equal to 22% and 20%, respectively. When n was increased to 6000, the fraction changed to 18% and 22%, respectively. However, the differences in fraction are not statistically significant. To see if the relatively high fractions were due to fluctuations of the bulk of the eigenvalues, we counted the total number of eigenvalues with absolute values no less than τ_c. When $n = 4000$, for each pair, there were $200 \times 4000 = 8 \times 10^5$ absolute eigenvalues in total. Only 246 and 243 of them, respectively, were no less than τ_c. When $n = 6000$, the counts changed to 239 and 246, respectively. Thus, the fluctuation of the bulk had little to do with the relatively high percentages of $\lambda_\star(M_n)$ greater than τ_c.

We were unable to go beyond $n = 6000$ due to limited computing capacity. Nevertheless, the numerical results suggest that for $p > 1$, τ_c is not a tight lower bound, at least in the probabilistic sense that there is $t' > \tau_c$, such that $\mathrm{P}\{\lambda_\star(M_n) \geq t'\} \not\to 0$. The numerical results also suggest, more convincingly, that the upper bound in Theorem 1 is far from being tight, especially for large p. It would be interesting to see whether $\lambda_\star(M_n)$ has a nonrandom limit or weakly converges to a nondegenerate distribution and in either case, at what rate of convergence.

5. A formula for moments of the limit of ESD. This section gives a more explicit formula for the moments β_k of the limiting distribution μ_∞. To state the result, if $\mathbf{v}$ is a path on $I_n := \{1, \dots, n\}$, denote by $[\mathbf{v}]$ the graph whose vertex set consists of the distinct elements among v_i, and whose edge set consists of the distinct unordered pairs among $\{v_i, v_{i+1}\}$, $1 \le i \le k$. Denote by $C_{k,n}$ the set of closed paths of length k on I_n. Denote by $n(\cdot, \mathbf{v})$ the number of times an object appears in $\mathbf{v}$. Thus, for $u, u' \in I$ and $e = \{u, u'\}$, $n(u, \mathbf{v}) = \sum_{i=1}^{k+1} \mathbf{1}\{v_i = u\}$ and $n(e, \mathbf{v}) = \sum_{i=1}^{k} \mathbf{1}\{\{v_i, v_{i+1}\} = e\}$. Also, denote $n_+(u, \mathbf{v}) = \sum_{x \in I} n((u, x), \mathbf{v})$, that is, the number of directed edges in $\mathbf{v}$ starting at u. Denote by E_n the set of edges of the complete graph on I_n. Following the definition on page 17 of [1], a path $\mathbf{i} \in C_{k,n}$ is called canonical if $i_1 = 1$ and $i_j \le \max(i_1, \dots, i_{j-1}) + 1$ for $2 \le j \le k$. For such a path $\mathbf{i}$, if $|[\mathbf{i}]| = t$, then the set of distinct values of i_j is $\{1, \dots, t\}$. Let

$$\Gamma_{k,t} = \{\mathbf{i} \in C_{k,n} : \mathbf{i} \text{ is canonical, } [\mathbf{i}] \text{ is a tree of size } t\}.$$

As long as $n \ge t$, the definition is independent of n. Note that for $\mathbf{i} \in C_{k,n}$, $[\mathbf{i}]$ is a tree $\iff |[\mathbf{i}]| = e([\mathbf{i}]) + 1$, and when this is the case, each $e \in E([\mathbf{i}])$ is traversed by $\mathbf{i}$ on both directions the same number of times, and hence k is even. Therefore, $\Gamma_{k,t} = \varnothing$ if k is odd.

PROPOSITION 7. *Let $c > 0$. Then for even $k = 2m$,*

$$\beta_k = \sum_{t=2}^{m+1} p^{t-1} \sum_{\mathbf{i} \in \Gamma_{k,t}} \prod_{a=1}^{t} \mathrm{E}\big[\big(c + d\big(a, [\mathbf{i}]\big) + \xi\big)^{-n_+(a,\mathbf{i})}\big], \qquad \xi \sim \mathrm{Po}(p).$$

PROOF. Given n, write $A_n = (\varepsilon_{ij}) \in \{0, 1\}^{n \times n}$. For $e = \{i, j\} \in E_n$, denote $\varepsilon_e = \varepsilon_{ij} = \varepsilon_{ji}$. Put

$$w_i = c + \sum_{j=1}^{n} \varepsilon_{ij}, \qquad x_{ij} = \varepsilon_{ij}/w_i.$$

For $(y_{ij}) \in \mathbb{R}^{n \times n}$ and $(z_i) \in \mathbb{R}^n$, and for $\mathbf{i} \in C_{k,n}$, denote $y(\mathbf{i}) = y_{i_1 i_2} y_{i_2 i_3} \cdots y_{i_{k-1} i_k} y_{i_k i_1}$ and $z(\mathbf{i}) = z_{i_1} \cdots z_{i_k}$. Then from the proof of Theorem 5,

$$\mathrm{E}\big[\mathrm{tr}\big(K_n^k\big)\big] = \sum_{\mathbf{i} \in C_{k,n}} \mathrm{E}\big[x(\mathbf{i})\big]. \tag{13}$$

Given $\mathbf{i} \in C_{k,n}$, let $t = |[\mathbf{i}]|$ and $s = e([\mathbf{i}])$. Since $\varepsilon(\mathbf{i}) = \prod_{e \in E([\mathbf{i}])} \varepsilon_e^{n(e,\mathbf{i})}$ with all $n(e, \mathbf{i}) \ge 1$, $x(\mathbf{i}) = \varepsilon(\mathbf{i})/w(\mathbf{i}) \ne 0 \iff \varepsilon_e = 1$ for all $e \in E([\mathbf{i}]) \iff \varepsilon(\mathbf{i}) = 1$. As ε_e, $e \in E([\mathbf{i}])$, are i.i.d. $\sim$ Bern(p/n), $\mathrm{P}\{\varepsilon(\mathbf{i}) = 1\} = (p/n)^s$. For $j \le k$, $w_{i_j} \ge c + \varepsilon_{i_j i_{j+1}}$ and $\{i_j, i_{j+1}\} \in E([\mathbf{i}])$. Consequently, $\varepsilon(\mathbf{i}) = 1$ implies $w_{i_j} \ge c + 1$ for all j. As a result,

$$\mathrm{E}\big[x(\mathbf{i})\big] = \mathrm{E}\big[\mathbf{1}\{\varepsilon(\mathbf{i}) = 1\}/w(\mathbf{i})\big] \le (c+1)^{-k} \mathrm{P}\{\varepsilon(\mathbf{i}) = 1\} = (c+1)^{-k} (p/n)^s.$$

Since $[\mathbf{i}]$ is connected, $1 \le e([\mathbf{i}]) \le k$ and $2 \le |[\mathbf{i}]| \le e([\mathbf{i}]) + 1$. For $2 \le t \le k$, the number of $\mathbf{i} \in C_{k,n}$ with $|[\mathbf{i}]| = t$ is less than $\binom{n}{t} t^k$. As a result, for $n \ge 2$,

$$\begin{aligned}\sum_{\mathbf{i}\in C_{k,n}:|[\mathbf{i}]|\le e([\mathbf{i}])} \mathrm{E}[x(\mathbf{i})] &= \sum_{s=1}^{k}\sum_{t=2}^{s} \sum_{\mathbf{i}\in C_{k,n}:|[\mathbf{i}]|=t, e([\mathbf{i}])=s} \mathrm{E}[x(\mathbf{i})] \\ &\le \sum_{s=1}^{k}\sum_{t=1}^{s} n^t t^k (c+1)^{-k} (p/n)^s \\ &\le (c+1)^{-k} \sum_{s=1}^{k} s^k p^s \sum_{t=1}^{s} n^{t-s} \le 2(c+1)^{-k} \sum_{s=1}^{k} s^k p^s.\end{aligned}$$

Since p and c are fixed, then by (13)

$$\mathrm{E}\big[\mathrm{tr}\big(K_n^k\big)\big] = \sum_{\mathbf{i}\in C_{k,n}:|[\mathbf{i}]|=e([\mathbf{i}])+1} \mathrm{E}\big[x(\mathbf{i})\big] + O_k(1) \qquad \text{as } n\to\infty. \tag{14}$$

Let $\mathbf{i}$ be a path counted on the right-hand side of (14) and $|[\mathbf{i}]| = t$. Then $2 \le t \le m+1$ and

$$x(\mathbf{i}) = \frac{\varepsilon(\mathbf{i})}{w(\mathbf{i})} = \prod_{a,b=1}^{t} \left(\frac{\varepsilon_{ab}}{w_a}\right)^{n((a,b),\mathbf{i})}.$$

Clearly, $x(\mathbf{i})$ is a deterministic function of A_n, denoted by $F(A_n)$. Arrange the elements of $V([\mathbf{i}])$ as $z_1, \dots, z_t$ in the order of initial appearance in $\mathbf{i}$ and let $\sigma(z_l) = l$. Then $\sigma : V([\mathbf{i}]) \to \{1, \dots, t\}$ is the unique bijection such that $\sigma(\mathbf{i}) := (\sigma(i_1), \dots, \sigma(i_k), \sigma(i_1)) \in \Gamma_{k,t}$. Extend σ to a permutation of $\{1, \dots, n\}$, still denoted σ. Let $S = (s_{ij}) \in \mathcal{M}_n$ with $s_{ij} = \mathbf{1}\{i = \sigma(j)\}$. From $S' A_n S = (\tilde{\varepsilon}_{ij})$ with $\tilde{\varepsilon}_{ij} = \sum_{kl} s_{ki}\varepsilon_{ij}s_{lj} = \varepsilon_{\sigma(i)\sigma(j)}$, $x(\sigma(\mathbf{i})) = \varepsilon(\sigma(\mathbf{i}))/w(\sigma(\mathbf{i})) = F(S'A_nS)$. Since $A_n \sim S'A_nS$, then $x(\mathbf{i}) \sim x(\sigma(\mathbf{i}))$, in particular, $\mathrm{E}[x(\mathbf{i})] = \mathrm{E}[x(\sigma(\mathbf{i}))]$.

It is easy to see that for each $\mathbf{i} \in \Gamma_{k,t}$, there are exactly $n!/(n-t)!$ paths counted on the right-hand side of (14) that can be mapped in the above way to $\mathbf{i}$. As a result,

$$\mathrm{E}\big[\mathrm{tr}\big(K_n^k\big)\big] = \sum_{t=2}^{m+1} \frac{n!}{(n-t)!} \sum_{\mathbf{i}\in\Gamma_{k,t}} \mathrm{E}\big[x(\mathbf{i})\big] + O_k(1), \qquad n\to\infty. \tag{15}$$

Given $t = 2, \dots, m+1$ and $\mathbf{i} \in \Gamma_{k,t}$, for $a = 1, \dots, t$, write

$$q_a = \sum_{a\in e\in E([\mathbf{i}])} \varepsilon_e, \qquad y_a = \sum_{a\in e\in E_t\setminus E([\mathbf{i}])} \varepsilon_e, \qquad S_a = \sum_{a\in e\in E_n\setminus E_t} \varepsilon_e.$$

Then $w_a = c + q_a + y_a + S_a$. Put $y = (y_1, \dots, y_t)$. Then y, $S_1, \dots, S_t$, and ε_e, $e \in E([\mathbf{i}])$, are all independent, and $x(\mathbf{i}) \ne 0 \Longleftrightarrow \varepsilon_e = 1$ for all $e \in E([\mathbf{i}])$. Since

$e([\mathbf{i}]) = t-1$, then

$$\begin{aligned}\mathrm{E}\big[x(\mathbf{i})\big] &= \mathrm{E}\big[x(\mathbf{i})\mathbf{1}\{\varepsilon_e = 1\ \forall e \in E([\mathbf{i}])\}\big] \\ &= (p/n)^{t-1}\mathrm{E}\Bigg[\prod_{a,b=1}^{t}(c+q_a+y_a+S_a)^{-n((a,b),\mathbf{i})}\Big|\varepsilon_e = 1\ \forall e \in E([\mathbf{i}])\Bigg].\end{aligned}$$

On the other hand, when $\varepsilon_e = 1$ for all $e \in E([\mathbf{i}])$, $q_a = d(a,[\mathbf{i}])$ for all $a = 1,\dots,t$. Then

$$\begin{aligned}\mathrm{E}\big[x(\mathbf{i})\big] &= (p/n)^{t-1}\mathrm{E}\Bigg[\prod_{a=1}^{t}\big(c+d(a,[\mathbf{i}])+y_a+S_a\big)^{-\sum_{b=1}^{t}n((a,b),\mathbf{i})}\Bigg] \\ &= (p/n)^{t-1}\mathrm{E}\Bigg[\prod_{a=1}^{t}\big(c+d(a,[\mathbf{i}])+y_a+S_a\big)^{-n_+(a,\mathbf{i})}\Bigg].\end{aligned}$$

Let $n \to \infty$. Since t is fixed, $(y, S_1,\dots,S_t) \to (0,\xi_1,\dots,\xi_t)$ in distribution, with ξ_i i.i.d. $\sim \mathrm{Po}(p)$. Then by (15) and dominated convergence, for $k = 2m$,

$$\mathrm{E}\big[\beta_k(K_n)\big] = n^{-1}\mathrm{E}\big[\mathrm{tr}\big(K_n^k\big)\big] \to \sum_{t=2}^{m+1} p^{t-1}\sum_{\mathbf{i}\in\Gamma_{k,t}}\mathrm{E}\Bigg[\prod_{a=1}^{t}\big(c+d(a,[\mathbf{i}])+\xi_a\big)^{-n_+(a,\mathbf{i})}\Bigg],$$

finishing the proof. $\square$

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