The Annals of Applied Probability
2016, Vol. 26, No. 3, 1807–1836
DOI: 10.1214/15-AAP1134

HACK'S LAW IN A DRAINAGE NETWORK MODEL: A BROWNIAN WEB APPROACH

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Hack [Studies of longitudinal stream profiles in Virginia and Maryland (1957). Report], while studying the drainage system in the Shenandoah valley and the adjacent mountains of Virginia, observed a power law relation $l \sim a^{0.6}$ between the length l of a stream from its source to a divide and the area a of the basin that collects the precipitation contributing to the stream as tributaries. We study the tributary structure of Howard's drainage network model of headward growth and branching studied by Gangopadhyay, Roy and Sarkar [*Ann. Appl. Probab.* **14** (2004) 1242–1266]. We show that the exponent of Hack's law is $2/3$ for Howard's model. Our study is based on a scaling of the process whereby the limit of the watershed area of a stream is area of a Brownian excursion process. To obtain this, we define a dual of the model and show that under diffusive scaling, both the original network and its dual converge jointly to the standard Brownian web and its dual.

1. Introduction. River basin geomorphology is a very old subject of study initiated by Horton [14]. Hack [13], studying the drainage system in the Shenandoah valley and the adjacent mountains of Virginia, observed a power law relation

$$l \sim a^{0.6} \tag{1}$$

between the length l of a stream from its source to a divide and the area of the basin a that collects the precipitation contributing to the stream as tributaries. Hack also corroborated this power law through the data gathered by Langbein [17] of nearly 400 different streams in the northeastern United States. This empirical relation (1) is widely accepted nowadays albeit with a different exponent (see Gray [12], Muller [22]) and is called Hack's law. Mandelbrot [21] mentions Hack's law to strengthen his contention that "*if all rivers as well as their basins are mutually similar, the fractal length-area argument predicts* (*river's length*)$^{1/D}$ *is proportional to* (*basin's area*)$^{1/2}$" where $D > 1$ is the fractal dimension of the river. In this connection, it is worth remarking that the Hurst exponent in fractional Brownian motion and in time series analysis arose from the study of the Nile basin by Hurst [15] where he proposed the relation $l_\perp = l_\parallel^{0.9}$ as that governing the width, $l_\perp$, and the length, $l_\parallel$, of the smallest rectangular region containing the drainage system.

Received February 2015; revised July 2015.
MSC2010 subject classifications. 60D05.
Key words and phrases. Brownian excursion, Brownian meander, Brownian web, Hack's law.

Various statistical models of drainage networks have been proposed (see Rodriguez-Iturbe and Rinaldo [26] for a detailed survey). In this paper, we study the tributary structure of a two-dimensional drainage network called the Howard's model of headward growth and branching (see Rodriguez-Iturbe and Rinaldo [26]). Our study is based on a scaling of the process and we obtain the watershed area of a stream as the area of a Brownian excursion process. This gives a statistical explanation of Hack's law and justifies the remark of Giacometti et al. [11]: "*From the results, we suggest that a statistical framework referring to the scaling invariance of the entire basin structure should be used in the interpretation of Hack's law*."

We first present an informal description of the model: suppose that the vertices of the d-dimensional lattice $\mathbb{Z}^d$ are open or closed with probability p $(0 < p < 1)$ and $1 - p$, respectively, independently of all other vertices. Each open vertex $\mathbf{u} \in \mathbb{Z}^d$ represents a water source and connects to a unique open vertex $\mathbf{v} \in \mathbb{Z}^d$. These edges represent the channels through which water can flow. The connecting vertex $\mathbf{v}$ is chosen so that the dth coordinate of $\mathbf{v}$ is one more than that of $\mathbf{u}$ and $\mathbf{v}$ has the minimum L_1 distance from $\mathbf{u}$. In case of nonuniqueness of such a vertex, we choose one of the closest open vertices with equal probability, independently of everything else.

Let V denote the set of *open* vertices and $h(\mathbf{u})$ denote the uniquely chosen vertex to which $\mathbf{u}$ connects, as described above. Set $\langle \mathbf{u}, h(\mathbf{u})\rangle$ as the edge (channel) connecting $\mathbf{u}$ and $h(\mathbf{u})$. From the construction, it follows that the random graph, $\mathcal{G} = (V, E)$ with edge set $E := \{\langle \mathbf{u}, h(\mathbf{u})\rangle : \mathbf{u} \in V\}$, does not contain any circuit. This model has been studied by Gangopadhyay, Roy and Sarkar [10] and the following results were obtained.

THEOREM 1.1. *Let* $0 < p < 1$.

(i) *For* $d = 2$ *and* $d = 3$, $\mathcal{G}$ *consists of one single tree almost surely, and for* $d \geq 4$, $\mathcal{G}$ *is a forest consisting of infinitely many disjoint trees almost surely.*

(ii) *For any* $d \geq 2$, *the graph* $\mathcal{G}$ *contains no bi-infinite path almost surely.*

In this paper, we consider only $d = 2$. Before proceeding further, we present a formal description for $d = 2$ which will be used later. Fix $0 < p < 1$ and let $\{B_{\mathbf{u}} : \mathbf{u} = (\mathbf{u}(1), \mathbf{u}(2)) \in \mathbb{Z}^2\}$ be an i.i.d. collection of Bernoulli random variables with success probability p. Set $V = \{\mathbf{u} \in \mathbb{Z}^2 : B_{\mathbf{u}} = 1\}$. Let $\{U_{\mathbf{u}} : \mathbf{u} \in \mathbb{Z}^2\}$ be another i.i.d. collection of random variables, independent of the collection of random variables $\{B_{\mathbf{u}} : \mathbf{u} \in \mathbb{Z}^2\}$, taking values in the set $\{1, -1\}$, with $\mathbb{P}(U_{\mathbf{u}} = 1) = \mathbb{P}(U_{\mathbf{u}} = -1) = 1/2$. For a vertex $(x, t) \in \mathbb{Z}^2$, we consider $k_0 = \min\{|k| : k \in \mathbb{Z}, B_{(x+k,t+1)} = 1\}$. Clearly, k_0 is almost surely finite. Now, we define

$$h(x,t) := \begin{cases} (x + k_0, t + 1) \in V, & \text{if } (x - k_0, t + 1) \notin V, \\ (x - k_0, t + 1) \in V, & \text{if } (x + k_0, t + 1) \notin V, \\ (x + U_{(x,t)} k_0, t + 1) \in V, & \text{if } (x \pm k_0, t + 1) \in V. \end{cases}$$

For any $k \geq 0$, let

$$h^{k+1}(x,t) := h\big(h^k(x,t)\big) \qquad \text{with } h^0(x,t) := (x,t),$$

$$C_k(x,t) := \begin{cases} \{(y,t-k) \in V : h^k(y,t-k) = (x,t)\}, & \text{if } (x,t) \in V, \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$C(x,t) := \bigcup_{k \geq 0} C_k(x,t).$$

Here, $h^k(x,t)$ represents the "kth generation progeny" of (x,t), the sets $C_k(x,t)$ and $C(x,t)$ denote, respectively, the set of kth generation ancestors and the set of all ancestors of (x,t); $C(x,t) = \varnothing$ if $(x,t) \notin V$. In the terminology of drainage network, $C(x,t)$ represents the region of precipitation, the water from which is channelled through the open point (x,t) (see Figure 1). From Theorem 1.1(ii), we have that $C(x,t)$ is finite almost surely.

Now, we define

$$L(x,t) := \inf\{k \geq 0 : C_k(x,t) = \varnothing\},$$

as the "length of the channel," which as earlier is finite almost surely. We observe that for any $(x,t) \in \mathbb{Z} \times \mathbb{Z}$, $L(x,t) \geq 0$ and the distribution of $L(x,t)$ does not depend upon (x,t). Our first result is about the length of the channel. We remark here that Newman, Ravishankar and Sun [23] has a similar result in a set-up which allows crossing of paths.

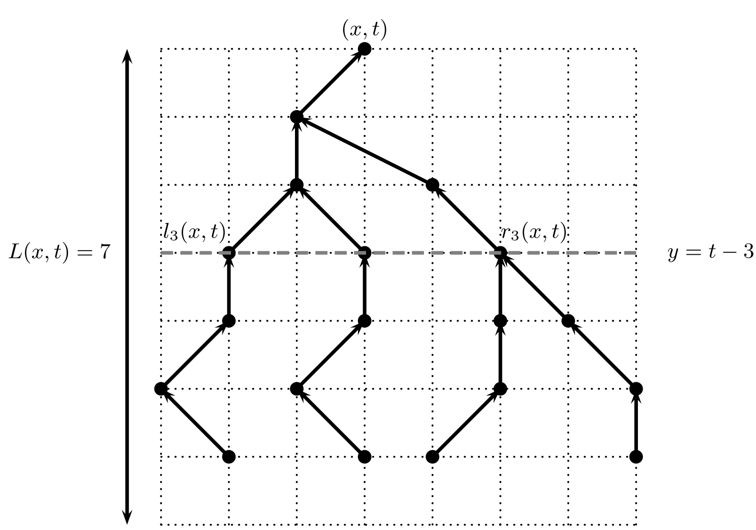

FIG. 1. *The bold vertices on the line $y = t - 3$ constitute the set $C_3(x,t)$ and all the bold vertices together constitute the cluster $C(x,t)$.*

THEOREM 1.2. *We have*

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}\big(L(0,0) > n\big) = \frac{1}{\gamma_0\sqrt{\pi}},$$

where $\gamma_0^2 := \gamma_0^2(p) = \frac{(1-p)(2-2p+p^2)}{p^2(2-p)^2}$.

Next, we define

$$r_k(x,t) := \begin{cases} \max\{u : (u, t-k) \in C_k(x,t)\}, & \text{if } 0 \le k < L(x,t), (x,t) \in V, \\ 0, & \text{otherwise,} \end{cases}$$

$$l_k(x,t) := \begin{cases} \min\{u : (u, t-k) \in C_k(x,t)\}, & \text{if } 0 \le k < L(x,t), (x,t) \in V, \\ 0, & \text{otherwise,} \end{cases}$$

$$D_k(x,t) := r_k(x,t) - l_k(x,t).$$

The quantity $D_k(x,t)$ denotes the *width* of the set of all kth generation ancestors of (x,t). We define the *width process* $D_n^{(x,t)}(s)$ and the *cluster process* $K_n^{(x,t)}(s)$ for $s \ge 0$ as follows: for $k = 0, 1, \dots$ and $k/n \le s \le (k+1)/n$,

$$(2) \qquad \begin{aligned} D_n^{(x,t)}(s) &:= \frac{D_k(x,t)}{\gamma_0\sqrt{n}} + \frac{(ns - [ns])}{\gamma_0\sqrt{n}}\big(D_{k+1}(x,t) - D_k(x,t)\big), \\ K_n^{(x,t)}(s) &:= \frac{\#C_k(x,t)}{\gamma_0\sqrt{n}} + \frac{(ns - [ns])}{\gamma_0\sqrt{n}}\big(\#C_{k+1}(x,t) - \#C_k(x,t)\big), \end{aligned}$$

where $\gamma_0 > 0$ is as in the statement of Theorem 1.2. In other words, $D_n^{(x,t)}(s)$ [resp., $K_n^{(x,t)}(s)$] is defined $D_k(x,t)/(\gamma_0\sqrt{n})$ [resp., $\#C_k(x,t)/(\gamma_0\sqrt{n})$] at time points $s = k/n$ and, at other time points defined by linear interpolation. The distributions of both $D_n^{(x,t)}$ and $K_n^{(x,t)}$ are independent of (x,t).

To describe our results, we need to introduce two processes, Brownian meander and Brownian excursion, studied by Durrett, Iglehart and Miller [7]. Let $\{W(s) : s \ge 0\}$ be a standard Brownian motion with $W(0) = 0$. Let $\tau_1 := \sup\{s \le 1 : W(s) = 0\}$ and $\tau_2 := \inf\{s \ge 1 : W(s) = 0\}$. Note that $\tau_1 < 1$ and $\tau_2 > 1$ almost surely. The standard Brownian meander, $W^+(s)$, and the standard Brownian excursion, $W_0^+(s)$, are given by

$$(3) \qquad W^+(s) := \frac{|W(\tau_1 + s(1-\tau_1))|}{\sqrt{1-\tau_1}}, \qquad s \in [0,1],$$

$$(4) \qquad W_0^+(s) := \frac{|W(\tau_1 + s(\tau_2-\tau_1))|}{\sqrt{\tau_2-\tau_1}}, \qquad s \in [0,1].$$

Both of these processes are a continuous nonhomogeneous Markov process (see Durrett and Iglehart [6] and references therein). Further, $W^+(0) = 0$ and, for $x \ge 0$, $\mathbb{P}(W^+(1) \le x) = 1 - \exp(-x^2/2)$, that is, $W^+(1)$ follows a Rayleigh distribution.

We also need some random variables obtained as functionals of these two processes. In particular, let

$$I_0^+ := \int_0^1 W_0^+(t)\,dt \quad \text{and} \quad M_0^+ := \max\{W_0^+(t) : t \in [0,1]\}.$$

Louchard and Janson [20] showed that, as $x \to \infty$, the distribution function and the density are, respectively, given by

$$\mathbb{P}(I_0^+ > x) \sim \frac{6\sqrt{6}}{\sqrt{\pi}} x \exp(-6x^2) \quad \text{and} \quad f_{I_0^+}(x) \sim \frac{72\sqrt{6}}{\sqrt{\pi}} x^2 \exp(-6x^2).$$

The random variable M_0^+ is continuous, having a strictly positive density on $(0,\infty)$ (see Durrett and Iglehart [6]) and for $x > 0$,

$$\mathbb{P}(M_0^+ \le x) = 1 + 2\sum_{k=1}^{\infty} \exp(-(2kx)^2/2)[1-(2kx)^2] \qquad \text{with } \mathbb{E}(M_0^+) = \sqrt{\pi/2}.$$

For $f \in C[0,\infty)$, let $f|_{[0,1]}$ denotes the restriction of f over $[0,1]$. Our next result is about the weak convergence of the width process $D_n^{(0,0)}|_{[0,1]}$ and the cluster process $K_n^{(0,0)}|_{[0,1]}$ under diffusive scaling. Here and subsequently, as is commonly used in statistics, we use the notation $X|Y$ to denote the conditional random variable X given Y.

THEOREM 1.3. *As $n \to \infty$, we have*:

(i) $D_n^{(0,0)}|_{[0,1]} | \mathbf{1}_{\{L(0,0)>n\}} \Rightarrow \sqrt{2}W^+$,

(ii) $\sup\{|pD_n^{(0,0)}(s) - K_n^{(0,0)}(s)| : s \in [0,1]\} | \mathbf{1}_{\{L(0,0)>n\}} \xrightarrow{P} 0$.

The following corollary is an immediate consequence of Theorem 1.3.

COROLLARY 1.3.1. *For $u > 0$, as $n \to \infty$ we have*:

(i) $\sqrt{n}\mathbb{P}(\#C_n(0,0) > \sqrt{n}\gamma_0 u) \to \frac{1}{\gamma_0\sqrt{\pi}} \exp(-u^2/4p^2)$,

(ii) $\mathbb{P}(\sum_{k=0}^n \#C_k(0,0) > n^{3/2}\gamma_0 u | L(0,0) > n) \to \mathbb{P}(p\sqrt{2}I^+ > u)$.

Before we proceed to state Theorem 1.4, we recall some results regarding random vectors whose distribution functions have regularly varying tails (see Resnick [25], page 172). A random vector Z on $(0,\infty)^d$ with a distribution function F has a regularly varying tail if, as $n \to \infty$, there exists a sequence $b_n \to \infty$ such that $n\mathbb{P}\{Z/b_n \in \cdot\} \xrightarrow{v} \nu(\cdot)$ for some $\nu \in M_+$ where $M_+ := \{\mu : \mu \text{ is a non-negative Radon measure on } (0,\infty)^d\}$. Here, $\xrightarrow{v}$ denotes vague convergence. It is in this context that Theorem 1.4 obtains a regularly varying tail for the distribution of $(L(x,t), (\#C(x,t))^{2/3})$; which justifies that the exponent of Hack's law is $2/3$ for Howard's model. In addition, we obtain a scaling law, with a Hack exponent of $1/2$, for the length of the stream, vis-à-vis the maximum width of the region of precipitation, that is,

$$D_{\max}(0,0) := \max\{D_k(0,0) : 0 \le k < L(0,0)\}. \tag{5}$$

It should be noted that Leopold and Langbein [18] obtained an exponent of 0.64 through computer simulations.

THEOREM 1.4. *Let* $\mathbf{E} := [0,\infty) \times [0,\infty) \setminus \{(0,0)\}$. *There exist measures* μ *and* ν *on the Borel* σ*-algebra on* $\mathbf{E}$ *such that for any Borel set* $B \subseteq \mathbf{E}$ *we have*

$$\sqrt{n}\mathbb{P}\left[\frac{(L(0,0), (\#C(0,0))^{2/3})}{n} \in B\right] \to \mu(B), \tag{6}$$

$$\sqrt{n}\mathbb{P}\left[\frac{(L(0,0), (D_{\max}(0,0))^{1/2})}{n} \in B\right] \to \nu(B), \tag{7}$$

with μ *and* ν *being given by*

$$\mu(B) = \iint_B \frac{3\sqrt{v}}{4\sqrt{2\pi}\gamma_0^2 p t^3} f_{I_0^+}\left(\frac{v^{3/2}}{\gamma_0 p \sqrt{2t^3}}\right) dv\, dt,$$

$$\nu(B) = \iint_B \frac{v}{\sqrt{2\pi}\gamma_0^2 p t^2} f_{M_0^+}\left(\frac{v^2}{\gamma_0 p \sqrt{2t}}\right) dv\, dt$$

where $f_{I_0^+}$ *and* $f_{M_0^+}$ *denote the density functions of* I_0^+ *and* M_0^+, *respectively. Moreover, for* $\lambda, \tau > 0$, *we have*

$$\begin{aligned}&\sqrt{n}\mathbb{P}\left[\frac{(L(0,0), (\#C(0,0))^{\alpha})}{n} \in (\tau,\infty) \times (\lambda,\infty)\right]\\ &\quad = \begin{cases} 0, & \text{if } \alpha < \dfrac{2}{3}, \\ \dfrac{1}{\sqrt{\pi\tau\gamma_0^2}}, & \text{if } \alpha > \dfrac{2}{3}\end{cases}\end{aligned} \tag{8}$$

and

$$\begin{aligned}&\sqrt{n}\mathbb{P}\left[\frac{(L(0,0), (D_{\max}(0,0))^{\alpha})}{n} \in (\tau,\infty) \times (\lambda,\infty)\right]\\ &\quad = \begin{cases} 0, & \text{if } \alpha < \dfrac{1}{2}, \\ \dfrac{1}{\sqrt{\pi\tau\gamma_0^2}}, & \text{if } \alpha > \dfrac{1}{2}.\end{cases}\end{aligned} \tag{9}$$

The estimates of the densities $f_{I_0^+}$ and $f_{M_0^+}$ imply that μ and ν are finite measures on $\mathbf{E}$. An immediate consequence of the above theorem is the following.

COROLLARY 1.4.1. *As $n \to \infty$ for $u > 0$, we have*:

(i) $\sqrt{n}\mathbb{P}(\#C(0,0) > \sqrt{2n^3}\gamma_0 pu) \to \frac{1}{2\sqrt{\pi}\gamma_0}\int_0^\infty t^{-3/2}\overline{F}_{I_0^+}(ut^{-3/2})\,dt$,

(ii) $\sqrt{n}\mathbb{P}(D_{\max}(0,0) > \sqrt{2n}\gamma_0 pu) \to \frac{1}{2\sqrt{\pi}\gamma_0}\int_0^\infty t^{-3/2}\overline{F}_{M_0^+}(ut^{-1/2})\,dt$,

where $F_{I_0^+}$ and $F_{M_0^+}$ are the distribution functions of I_0^+ and M_0^+, respectively, and $\overline{F}_{I_0^+} := 1 - F_{I_0^+}$, $\overline{F}_{M_0^+} := 1 - F_{M_0^+}$.

The proofs of the above theorems are based on a scaling of the process. In the next section, we define a dual graph and show that as processes, under a suitable scaling, the original and the dual processes converge jointly to the Brownian web and its dual in distribution (the double Brownian web). This invariance principle is used in Sections 3 and 4 to prove the theorems. In this connection, it is worth noting that in Proposition 2.7, we have provided an alternate characterization of the dual of Brownian web which is of independent interest. This characterization is suitable for proving the joint convergence of coalescing noncrossing path family and its dual to the double Brownian web and has been used in Theorem 2.9 to achieve the required convergence.

We should mention here that the Brownian web appears as a universal scaling limit for various network models (see Fontes et al. [9], Ferrari, Fontes and Wu [8], Coletti, Fontes and Dias [5]). It is reasonable to expect that with suitable modifications our method will give similar results in other network models. Our results will hold for any network model which admits a dual and satisfies (i) conditions listed in Remark 2.1, (ii) the scaled model and its dual converges weakly to the double Brownian web (see Section 2) and (iii) a certain sequence of counting random variables are uniformly integrable (see Lemma 3.3). In this sense, our result can be considered as a universality class result.

2. Dual process and the double Brownian web.

2.1. *Dual process*. For the graph $\mathcal{G}$, we now describe a dual process such that the set of ancestors $C(x,t)$ (defined in the previous section) of a vertex $(x,t) \in V$ is bounded by two dual paths. The dependency inherent in the graph $\mathcal{G}$ implies that, although the cluster is bounded by two dual paths, these paths are *not given by independent random walks*. The dual vertices are precisely the mid-points between two consecutive open vertices on each horizontal line $\{y = n\}, n \in \mathbb{Z}$ with each dual vertex having a unique offspring dual vertex in the negative direction of the y-axis. Before giving a formal definition, we direct the attention of the reader to Figure 2.

For $\mathbf{u} \in \mathbb{Z}^2$, we define

$$J_{\mathbf{u}}^+ := \inf\{k : k \geq 1, (\mathbf{u}(1) + k, \mathbf{u}(2)) \in V\},$$
$$J_{\mathbf{u}}^- := \inf\{k : k \geq 1, (\mathbf{u}(1) - k, \mathbf{u}(2)) \in V\}. \tag{10}$$

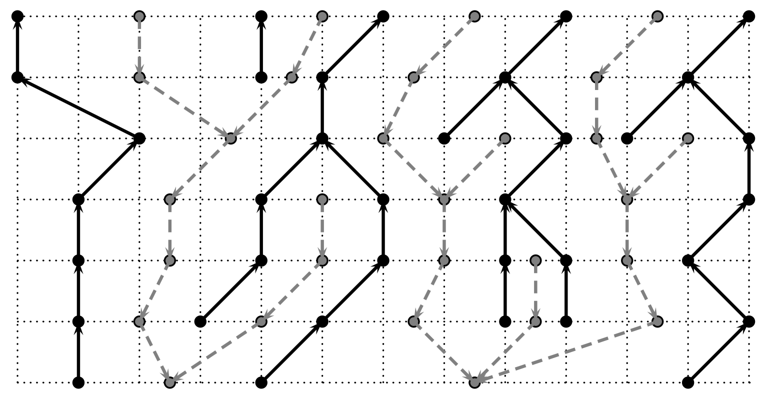

FIG. 2. *The black points are open vertices, the gray points are the vertices of the dual process and the gray (dashed) paths are the dual paths.*

Next, we define $r(\mathbf{u}) := (\mathbf{u}(1) + J^+_{\mathbf{u}}, \mathbf{u}(2))$ and $l(\mathbf{u}) := (\mathbf{u}(1) - J^-_{\mathbf{u}}, \mathbf{u}(2))$, as the first open point to the right (*open right neighbour*) and the first open point to the left (*open left neighbour*) of $\mathbf{u}$, respectively. For $(x,t) \in V$, let $\hat{r}(x,t) := (x + J^+_{(x,t)}/2, t)$ and $\hat{l}(x,t) := (x - J^-_{(x,t)}/2, t)$ denote, respectively, the right dual neighbour and the left dual neighbour of (x,t) in the dual vertex set. Finally, the dual vertex set is given by

$$\widehat{V} := \{\hat{r}(x,t), \hat{l}(x,t) : (x,t) \in V\}.$$

For a vertex $(u,s) \in \widehat{V}$, let $(v, s-1) \in \widehat{V}$ be such that the straight line segment joining (u,s) and $(v,s-1)$ does not cross any edge in $\mathcal{G}$. The dual edges are edges joining all such (u,s) and $(v,s-1)$. Formally, for $(u,s) \in \widehat{V}$, we define

$$\begin{aligned} a^l(u,s) &:= \sup\{z : (z, s-1) \in V, h(z, s-1)(1) < u\}, \\ a^r(u,s) &:= \inf\{z : (z, s-1) \in V, h(z, s-1)(1) > u\} \end{aligned} \tag{11}$$

and set $\hat{h}(u,s) := ((a^l(u,s) + a^r(u,s))/2, s-1)$. Note that $(a^r(u,s), s-1)$ and $(a^l(u,s), s-1)$ are the nearest vertices in V to the right and left, respectively, of the dual vertex $\hat{h}(u,s)$. Finally, the edge set of the dual graph $\widehat{\mathcal{G}} := (\widehat{V}, \widehat{E})$ is given by

$$\widehat{E} := \{\langle (u,s), \hat{h}(u,s)\rangle : (u,s) \in \widehat{V}\}.$$

REMARK 2.1. Note that the vertex set of the dual graph is a subset of $\frac{1}{2}\mathbb{Z} \times \mathbb{Z}$. Before we proceed, we list some properties of the graph $\mathcal{G}$ and its dual $\widehat{\mathcal{G}}$.

(1) $\mathcal{G}$ uniquely specifies the dual graph $\widehat{\mathcal{G}}$ and the dual edges do not intersect the original edges. The construction ensures that $\widehat{\mathcal{G}}$ does not contain any circuit.

(2) For $(x,t) \in V$, the cluster $C(x,t)$ is enclosed within the dual paths starting from $\hat{r}(x,t)$ and $\hat{l}(x,t)$. The boundedness of $C(x,t)$ for every $(x,t) \in V$ implies that these two dual paths coalesce, thus $\widehat{\mathcal{G}}$ is a single tree.

(3) Since paths starting from any two open vertices in the original graph coalesce and the dual edges do not cross the original edges, there is no bi-infinite path in $\widehat{\mathcal{G}}$.

We now obtain a Markov process from the dual paths. Fix $(u,s) \in \widehat{V}$ and for $k \geq 1$, set $\hat{h}^k(u,s) := \hat{h}(\hat{h}^{k-1}(u,s))$ where $\hat{h}^0(u,s) := (u,s)$. Letting $\widehat{X}_k^{(u,s)}$ denote the first coordinate of $\hat{h}^k(u,s)$, it may be observed that $\widehat{X}_{k+1}^{(u,s)}$ is a function of $\widehat{X}_k^{(u,s)}$ and the collection of random variables $\{(B_{\mathbf{u}}, U_{\mathbf{u}}) : \mathbf{u}(2) = s-k-1 \in \mathbb{Z}\}$. Thus, by the random mapping representation (see, e.g., Levin, Peres, and Wilmer [19]) we have the following.

PROPOSITION 2.2. *For $(u,s) \in \widehat{V}$, the process $\{\widehat{X}_k^{(u,s)} : k \geq 0\}$ is a time homogeneous Markov process.*

Before we proceed, we make the following observations about the transition probabilities of the Markov process. Let G be a geometric random variable taking values in $\{1,2,\ldots\}$, that is, $\mathbb{P}(G=l) = p(1-p)^{l-1}$ for $l \geq 1$. For any $\mathbf{u} \in \mathbb{Z} \times \mathbb{Z}$, the random variables $J_{\mathbf{u}}^+$ and $J_{\mathbf{u}}^-$ are i.i.d. copies of the geometric random variable G independent of $B_{\mathbf{u}}$. Further, if $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{Z}^2$ are such that $\mathbf{u}_1(1) \geq \mathbf{u}_2(1) - 1$ and $\mathbf{u}_1(2) = \mathbf{u}_2(2)$, the random variables $J_{\mathbf{u}_1}^+$ and $J_{\mathbf{u}_2}^-$ are also independent. Now, for $u \notin \mathbb{Z}$ and, $v \in \mathbb{Z}/2$, we have

$$\begin{aligned}\mathbb{P}\big(\widehat{X}_1^{(u,s)} - \widehat{X}_0^{(u,s)} = v \mid \widehat{X}_0^{(u,s)} = u\big) &= \mathbb{P}\big(J_{(u-1/2,s-1)}^+ - J_{(u+1/2,s-1)}^- = 2v\big) \\ &= \mathbb{P}(G_1 - G_2 = 2v),\end{aligned} \tag{12}$$

where G_1 and G_2 are i.i.d. copies of G, defined above. If $u \in \mathbb{Z}$ and $v \in \mathbb{Z}/2$, we have, using notations from above

$$\begin{aligned}&\mathbb{P}\big(\widehat{X}_1^{(u,s)} - \widehat{X}_0^{(u,s)} = v \mid \widehat{X}_0^{(u,s)} = u\big) \\ &\quad = (1-p)\mathbb{P}(G_1 - G_2 = 2v) + p\mathbb{P}(G = 2v)/2 + p\mathbb{P}(G = -2v)/2,\end{aligned} \tag{13}$$

where G_1 and G_2 are as above. It is therefore obvious that the transition probabilities of $\widehat{X}_k^{(u,s)}$ depend on whether the present state is an integer or not.

From equations (12) and (13), we state the following.

PROPOSITION 2.3. *For any $(u,s) \in \widehat{V}$, $\{\widehat{X}_k^{(u,s)} : k \geq 0\}$ is an L^2-martingale with respect to the filtration $\mathcal{F}_k := \sigma(\{B_{\mathbf{u}}, U_{\mathbf{u}} : \mathbf{u} \in \mathbb{Z}^2, \mathbf{u}(2) \geq s-k\})$.*

2.2. *Dual Brownian web.* In this section, we briefly describe the dual Brownian web $\widehat{\mathcal{W}}$ associated with $\mathcal{W}$ and present an alternate characterization of the dual Brownian web $\widehat{\mathcal{W}}$.

The Brownian web (studied extensively by Arratia [1, 2], Tóth and Werner [30], Fontes et al. [9]) may be viewed as a collection of one-dimensional coalescing Brownian motions starting from every point in the space time plane $\mathbb{R}^2$. We recall relevant details from Fontes et al. [9].

Let $\mathbb{R}^2_c$ denote the completion of the space time plane $\mathbb{R}^2$ with respect to the metric

$$\rho\big((x_1, t_1), (x_2, t_2)\big) := \big|\tanh(t_1) - \tanh(t_2)\big| \vee \left|\frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|}\right|.$$

As a topological space $\mathbb{R}^2_c$ can be identified with the continuous image of $[-\infty, \infty]^2$ under a map that identifies the line $[-\infty, \infty] \times \{\infty\}$ with the point $(*, \infty)$, and the line $[-\infty, \infty] \times \{-\infty\}$ with the point $(*, -\infty)$. A path π in $\mathbb{R}^2_c$ with starting time $\sigma_\pi \in [-\infty, \infty]$ is a mapping $\pi : [\sigma_\pi, \infty] \to [-\infty, \infty] \cup \{*\}$ such that $\pi(\infty) = *$ and, when $\sigma_\pi = -\infty$, $\pi(-\infty) = *$. Also $t \mapsto (\pi(t), t)$ is a continuous map from $[\sigma_\pi, \infty]$ to $(\mathbb{R}^2_c, \rho)$. We then define Π to be the space of all paths in $\mathbb{R}^2_c$ with all possible starting times in $[-\infty, \infty]$. The following metric, for $\pi_1, \pi_2 \in \Pi$

$$d_\Pi(\pi_1, \pi_2) := \max\Bigg\{\big|\tanh(\sigma_{\pi_1}) - \tanh(\sigma_{\pi_2})\big|, \sup_{t \geq \sigma_{\pi_1} \wedge \sigma_{\pi_2}} \left|\frac{\tanh(\pi_1(t \vee \sigma_{\pi_1}))}{1 + |t|} - \frac{\tanh(\pi_2(t \vee \sigma_{\pi_2}))}{1 + |t|}\right|\Bigg\}$$

makes Π a complete, separable metric space.

REMARK 2.4. Convergence in this metric can be described as locally uniform convergence of paths as well as convergence of starting times. Therefore, for any $\varepsilon > 0$ and $m > 0$, we can choose $\varepsilon_1 (= f(\varepsilon, m)) > 0$ such that for $\pi_1, \pi_2 \in \Pi$ with $\{(\pi_i(t), t) : t \in [\sigma_{\pi_i}, m]\} \subseteq [-m, m] \times [-m, m]$ for $i = 1, 2$, $d_\Pi(\pi_1, \pi_2) < \varepsilon_1$ implies that $\|(\pi_1(\sigma_{\pi_1}), \sigma_{\pi_1}) - (\pi_2(\sigma_{\pi_2}), \sigma_{\pi_2})\|_2 < \varepsilon$ and $\sup\{|\pi_1(t) - \pi_2(t)| : t \in [\max\{\sigma_{\pi_1}, \sigma_{\pi_2}\}, m]\} < \varepsilon$. We will use this later several times.

Let $\mathcal{H}$ be the space of compact subsets of (Π, d_Π) equipped with the Hausdorff metric $d_{\mathcal{H}}$. The Brownian web $\mathcal{W}$ is a random variable taking values in the complete separable metric space $(\mathcal{H}, d_{\mathcal{H}})$.

Before introducing the dual Brownian web, we require a similar metric space on the collection of backward paths. As in the definition of Π, let $\widehat{\Pi}$ be the collection of all paths $\hat{\pi}$ with starting time $\sigma_{\hat{\pi}} \in [-\infty, \infty]$ such that $\hat{\pi} : [-\infty, \sigma_{\hat{\pi}}] \to [-\infty, \infty] \cup \{*\}$ with $\hat{\pi}(-\infty) = *$ and, when $\sigma_{\hat{\pi}} = +\infty$, $\hat{\pi}(\infty) = *$. As earlier

$t \mapsto (\hat{\pi}(t), t)$ is a continuous map from $[-\infty, \sigma_{\hat{\pi}}]$ to $(\mathbb{R}_c^2, \rho)$. We equip $\widehat{\Pi}$ with the metric

$$d_{\widehat{\Pi}}(\hat{\pi}_1, \hat{\pi}_2) := \max\Biggl\{\bigl|\tanh(\sigma_{\hat{\pi}_1}) - \tanh(\sigma_{\hat{\pi}_2})\bigr|, \sup_{t \le \sigma_{\hat{\pi}_1} \vee \sigma_{\hat{\pi}_2}} \biggl|\frac{\tanh(\hat{\pi}_1(t \wedge \sigma_{\hat{\pi}_1}))}{1+|t|} - \frac{\tanh(\hat{\pi}_2(t \wedge \sigma_{\hat{\pi}_2}))}{1+|t|}\biggr|\Biggr\}$$

making $(\widehat{\Pi}, d_{\widehat{\Pi}})$ a complete, separable metric space. The complete separable metric space of compact sets of paths of $\widehat{\Pi}$ is denoted by $(\widehat{\mathcal{H}}, d_{\widehat{\mathcal{H}}})$, where $d_{\widehat{\mathcal{H}}}$ is the Hausdorff metric on $\widehat{\mathcal{H}}$, and let $\mathcal{B}_{\widehat{\mathcal{H}}}$ be the corresponding Borel σ field.

2.3. *Properties of* $(\mathcal{W}, \widehat{\mathcal{W}})$. The Brownian web and its dual $(\mathcal{W}, \widehat{\mathcal{W}})$ is a $(\mathcal{H} \times \widehat{\mathcal{H}}, \mathcal{B}_{\mathcal{H}} \times \mathcal{B}_{\widehat{\mathcal{H}}})$ valued random variable such that $\mathcal{W}$ and $\widehat{\mathcal{W}}$ uniquely determine each other almost surely with $\widehat{\mathcal{W}}$ being equally distributed as $-\mathcal{W}$, the Brownian web rotated 180° about the origin. The interaction between the paths in $\mathcal{W}$ and $\widehat{\mathcal{W}}$ is that of Skorohod reflection (see Soucaliuc, Tóth and Werner [28]).

We introduce some notation to study the sets $\{\pi(t+s) : \pi \in \mathcal{W}, \sigma_\pi \le t\}$ and $\{\hat{\pi}(t-s) : \hat{\pi} \in \widehat{\mathcal{W}}, \sigma_{\hat{\pi}} \ge t\}$. For a $(\mathcal{H}, B_{\mathcal{H}})$ valued random variable K and $t \in \mathbb{R}$, let $K^{t-} := \{\pi : \pi \in K \text{ and } \sigma_\pi \le t\}$. Similarly, for a $(\widehat{\mathcal{H}}, B_{\widehat{\mathcal{H}}})$ valued random variable $\widehat{K}$ and $t \in \mathbb{R}$, let $\widehat{K}^{t+} := \{\hat{\pi} : \hat{\pi} \in \widehat{K} \text{ and } \sigma_{\hat{\pi}} \ge t\}$. For $t_1, t_2 \in \mathbb{R}$, $t_2 > t_1$ and a $(\mathcal{H}, B_{\mathcal{H}})$ valued random variable K, define

$$\begin{aligned} &\mathcal{M}_K(t_1, t_2) := \bigl\{\pi(t_2) : \pi \in K^{t_1-}, \pi(t_2) \in [0, 1]\bigr\}; \\ &\xi_K(t_1, t_2) := \#\mathcal{M}_K(t_1, t_2), \end{aligned} \tag{14}$$

that is, $\xi_K(t_1, t_2)$ denotes the number of distinct points in $[0, 1] \times t_2$ which are on some path in K^{t_1-}. We note that for $t > 0$, $\mathcal{M}_{\mathcal{W}}(t_0, t_0 + t) = \mathcal{N}_{\mathcal{W}}(t_0, t; 0, 1)$ as defined in Sun and Swart [29]. It is known that for all $t > 0$ the random variable $\xi_{\mathcal{W}}(t_0, t_0 + t)$ is finite almost surely (see (E_1) in Theorem 1.3 in Sun and Swart [29]) with

$$\mathbb{E}\bigl(\xi_{\mathcal{W}}(t_0, t_0 + t)\bigr) = \frac{1}{\sqrt{\pi t}}. \tag{15}$$

Moreover, from the known properties of $(\mathcal{W}, \widehat{\mathcal{W}})$ the proof of the following proposition is straightforward (for details, see Roy, Saha and Sarkar [27]).

PROPOSITION 2.5. *For any* $t_0 < t_1$, *almost surely we have*:

(i) $\mathcal{M}_{\mathcal{W}}(t_0, t_1) \cap \mathbb{Q} = \varnothing$;
(ii) *each point in* $\mathcal{M}_{\mathcal{W}}(t_0, t_1)$ *is of type* $(1, 1)$;
(iii) *for each* $x \in \mathcal{M}_{\mathcal{W}}(t_0, t_1)$, *there exists* $\pi_1, \pi_2 \in \mathcal{W}$ *with* $\sigma_{\pi_1} < t_0$, $\sigma_{\pi_2} > t_0$ *and* $\pi_1(t_1) = \pi_2(t_1) = x$;

(iv) *for each* $x \in \mathcal{M}_{\mathcal{W}}(t_0, t_1)$, *there exist exactly two paths* $\hat{\pi}_r^{(x,t_1)}$ *and* $\hat{\pi}_l^{(x,t_1)}$ *in* $\widehat{\mathcal{W}}$ *starting from* (x, t_1) *with* $\hat{\pi}_r^{(x,t_1)}(t) > \hat{\pi}_l^{(x,t_1)}(t)$ *for all* $[t_0, t_1)$.

There are several ways to construct $\widehat{\mathcal{W}}$ from $\mathcal{W}$. In this paper, we follow the *wedge characterization* provided by Sun and Swart [29]. For $\pi^r, \pi^l \in \mathcal{W}$ with coalescing time t^{π^r,π^l} and $\pi^r(\max\{\sigma_{\pi^r}, \sigma_{\pi^l}\}) > \pi^l(\max\{\sigma_{\pi^r}, \sigma_{\pi^l}\})$, the wedge with right boundary π^r and left boundary π^l, is an open set in $\mathbb{R}^2$ given by

$$\begin{aligned} A &= A(\pi^r, \pi^l) \\ &:= \{(y, s) : \max\{\sigma_{\pi^l}, \sigma_{\pi^r}\} < s < t^{\pi^r,\pi^l}, \pi^l(s) < y < \pi^r(s)\}. \end{aligned} \tag{16}$$

A path $\hat{\pi} \in \widehat{\Pi}$, is said to *enter the wedge* A *from outside* if there exist t_1 and t_2 with $\sigma_{\hat{\pi}} > t_1 > t_2$ such that $(\hat{\pi}(t_1), t_1) \notin \overline{A}$ and $(\hat{\pi}(t_2), t_2) \in A$.

From Theorem 1.9 in Sun and Swart [29], it follows that the dual Brownian web $\widehat{\mathcal{W}}$ associated with the Brownian web $\mathcal{W}$ satisfies the following wedge characterization.

THEOREM 2.6. *Let* $(\mathcal{W}, \widehat{\mathcal{W}})$ *be a Brownian web and its dual. Then almost surely*

$$\widehat{\mathcal{W}} = \{\hat{\pi} : \hat{\pi} \in \widehat{\Pi} \text{ and does not enter any wedge in } \mathcal{W} \text{ from outside}\}.$$

Because of Theorem 2.6, for a $(\mathcal{H} \times \widehat{\mathcal{H}}, \mathcal{B}_{\mathcal{H}} \times \mathcal{B}_{\widehat{\mathcal{H}}})$ valued random variable $(\mathcal{W}, \mathcal{Z})$ to show that $\mathcal{Z} = \widehat{\mathcal{W}}$, it suffices to check that $\mathcal{Z}$ satisfies the wedge condition. Here we present an alternate condition which is easier to check.

PROPOSITION 2.7. *Let* $(\mathcal{W}, \mathcal{Z})$ *be a* $(\mathcal{H} \times \widehat{\mathcal{H}}, \mathcal{B}_{\mathcal{H}} \times \mathcal{B}_{\widehat{\mathcal{H}}})$ *valued random variable such that*:

(1) *for any deterministic* $(x, t) \in \mathbb{R}^2$, *there exists a path* $\hat{\pi}^{(x,t)} \in \mathcal{Z}$ *starting at* (x, t) *and going backward in time almost surely*;

(2) *paths in* $\mathcal{Z}$ *do not cross paths in* $\mathcal{W}$ *almost surely*, *that is*, *there does not exist any* $\pi \in \mathcal{W}, \hat{\pi} \in \mathcal{Z}$ *and* $t_1, t_2 \in (\sigma_\pi, \sigma_{\hat{\pi}})$ *such that* $(\hat{\pi}(t_1) - \pi(t_1))(\hat{\pi}(t_2) - \pi(t_2)) < 0$ *almost surely*;

(3) *paths in* $\mathcal{Z}$ *and paths in* $\mathcal{W}$ *do not coincide over any time interval almost surely*, *that is*, *for any* $\pi \in \mathcal{W}$ *and* $\hat{\pi} \in \mathcal{Z}$ *and for no pair of points* $t_1 < t_2$ *with* $\sigma_\pi \le t_1 < t_2 \le \sigma_{\hat{\pi}}$ *we have* $\hat{\pi}(t) = \pi(t)$ *for all* $t \in [t_1, t_2]$ *almost surely.*

Then $\mathcal{Z} = \widehat{\mathcal{W}}$ *almost surely.*

PROOF. From conditions (2) and (3), we have that $\hat{\pi} \in \mathcal{Z}$ does not enter any wedge in $\mathcal{W}$ from outside. Hence, $\mathcal{Z} \subseteq \widehat{\mathcal{W}}$. The argument for $\widehat{\mathcal{W}} \subseteq \mathcal{Z}$ follows from the *fish-trap* technique introduced in the proof of Lemma 4.7 of Sun and Swart [29]. It shows that $\widehat{\mathcal{W}} \subseteq \widetilde{\mathcal{Z}}$ almost surely for any $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ valued random variable

$\widetilde{\mathcal{Z}}$ satisfying (i) paths in $\widetilde{\mathcal{Z}}$ do not cross paths is $\mathcal{W}$ and (ii) for any deterministic countable dense set, there exist paths in $\widetilde{\mathcal{Z}}$ starting from every point of that dense set (for details, see Roy, Saha and Sarkar [27]). $\square$

2.4. *Convergence to the double Brownian web.* For any $(x,t) \in V$, the path $\pi^{(x,t)}$ in the random graph $\mathcal{G}$ is obtained as the piecewise linear function $\pi^{(x,t)}: [t,\infty) \to \mathbb{R}$ with $\pi^{(x,t)}(t+k) = h^k(x,t)(1)$ for every $k \geq 0$ and $\pi^{(x,t)}$ being linear in the interval $[t+k, t+k+1]$. Similarly, for $(x,t) \in \widehat{V}$, the dual path $\hat{\pi}^{(x,t)}$ is the piecewise linear function $\hat{\pi}^{(x,t)}: (-\infty, t] \to \mathbb{R}$ with $\hat{\pi}^{(x,t)}(t-k) = \hat{h}^k(x,t)(1)$ for every $k \geq 0$ and $\hat{\pi}^{(x,t)}$ being linear in the interval $[t-k-1, t-k]$. Let $\mathcal{X} := \{\pi^{(x,t)} : (x,t) \in V\}$ and $\widehat{\mathcal{X}} := \{\hat{\pi}^{(x,t)} : (x,t) \in \widehat{V}\}$ be the collection of all possible paths and dual paths admitted by $\mathcal{G}$ and $\widehat{\mathcal{G}}$.

For a given $\gamma > 0$ and a path π with starting time σ_π, the scaled path $\pi_n(\gamma): [\sigma_\pi/n, \infty] \to [-\infty, \infty]$ is given by $\pi_n(\gamma)(t) = \pi(nt)/(\sqrt{n}\gamma)$ for each $n \geq 1$. Thus, the starting time of the scaled path $\pi_n(\gamma)$ is $\sigma_{\pi_n(\gamma)} = \sigma_\pi/n$. Similarly, for the backward path $\hat{\pi}$, the scaled version is $\hat{\pi}_n(\gamma): [-\infty, \sigma_{\hat{\pi}}/n] \to [-\infty, \infty]$ given by $\hat{\pi}_n(\gamma)(t) = \hat{\pi}(nt)/(\sqrt{n}\gamma)$ for each $n \geq 1$. For each $n \geq 1$, let $\mathcal{X}_n = \mathcal{X}_n(\gamma) := \{\pi_n^{(x,t)}(\gamma) : (x,t) \in V\}$ and $\widehat{\mathcal{X}}_n = \widehat{\mathcal{X}}_n(\gamma) := \{\hat{\pi}_n^{(x,t)}(\gamma) : (x,t) \in \widehat{V}\}$ be the collections of all the nth order diffusively scaled paths and dual paths, respectively.

The closure $\overline{\mathcal{X}}_n(\gamma)$ of $\mathcal{X}_n(\gamma)$ in (Π, d_Π) and the closure $\overline{\widehat{\mathcal{X}}}_n(\gamma)$ of $\widehat{\mathcal{X}}_n(\gamma)$ in $(\widehat{\Pi}, d_{\widehat{\Pi}})$ are $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ and $(\widehat{\mathcal{H}}, \mathcal{B}_{\widehat{\mathcal{H}}})$ valued random variables, respectively. Coletti, Fontes and Dias [5] showed the following.

THEOREM 2.8. *For $\gamma_0 := \gamma_0(p)$ as in Theorem 1.2, as $n \to \infty$, $\overline{\mathcal{X}}_n(\gamma_0)$ converges weakly to the standard Brownian web $\mathcal{W}$.*

Our main result is the joint invariance principle for $\{(\overline{\mathcal{X}}_n(\gamma_0), \overline{\widehat{\mathcal{X}}}_n(\gamma_0)) : n \geq 1\}$ considered as $(\mathcal{H} \times \widehat{\mathcal{H}}, \mathcal{B}_{\mathcal{H}} \times \mathcal{B}_{\widehat{\mathcal{H}}})$ valued random variables.

THEOREM 2.9. *$\{(\overline{\mathcal{X}}_n(\gamma_0), \overline{\widehat{\mathcal{X}}}_n(\gamma_0)) : n \geq 0\}$ converges weakly to $(\mathcal{W}, \widehat{\mathcal{W}})$ as $n \to \infty$.*

We require the following propositions to prove Theorem 2.9. We say that $\{\widehat{W}^{(x,t)}(u) : u \leq t\}$ is a Brownian motion going *back in time* if $\widehat{W}^{(x,t)}(t-s) := W(t+s), s \geq 0$ where $\{W(u) : u \geq t\}$ is a Brownian motion with $W(t) = x$.

PROPOSITION 2.10. *For any deterministic point $(x,t) \in \mathbb{R}^2$, there exists a sequence of paths $\hat{\theta}_n^{(x,t)} \in \widehat{\mathcal{X}}_n(\gamma_0)$ which converges in distribution to $\widehat{W}^{(x,t)}$.*

PROOF. For any $(x,t) \in \mathbb{R}^2$ fix $t_n = \lfloor nt \rfloor$ and $x_n = \max\{\lfloor \sqrt{n}\gamma_0 x \rfloor + j : j \leq 0, (\lfloor \sqrt{n}\gamma_0 x \rfloor + j, t_n) \in \widehat{V}\}$. Let $\hat{\theta}_n^{(x,t)} \in \widehat{\mathcal{X}}_n(\gamma_0)$ be the scaling of the path $\hat{\pi}^{(x_n,t_n)} \in \widehat{\mathcal{X}}$.

Since $\mathcal{G}$ is invariant under translation by lattice points and $\widehat{\mathcal{G}}$ is uniquely determined by $\mathcal{G}$, the conditional distribution of $\{(x_n, t_n) + \hat{h}^j(0,0) : j \geq 0\}$ given $(0,0) \in \widehat{V}$ is the same as that of $\{\hat{h}^j(x_n, t_n) : j \geq 0\}$. We observe that $(x_n/(\sqrt{n}\gamma_0), t_n/n) \to (x,t)$ as $n \to \infty$ almost surely. Hence, it suffices to prove that the scaled dual path starting from $(0,0)$ given $(0,0) \in \widehat{V}$ converges in distribution to $\widehat{W}^{(0,0)}$.

From Proposition 2.3, we see that $\widehat{X}_j^{(0,0)} = \hat{h}^j(0,0)(1)$ is an L^2 martingale with respect to the filtration $\sigma(\{B_{(z,s)}, U_{(z,s)} : z \in \mathbb{Z}, s \geq -k\})$. Let

$$\eta_n(u) := s_n^{-1}\big[\widehat{X}_j^{(0,0)} + \big(\widehat{X}_{j+1}^{(0,0)} - \widehat{X}_j^{(0,0)}\big)\big(us_n^2 - s_j^2\big)/\big(s_{j+1}^2 - s_j^2\big)\big]$$

for $u \in [0,\infty)$ and $s_j^2 \leq us_n^2 < s_{j+1}^2$, where $s_n^2 = \sum_{j=1}^n \mathbb{E}((\widehat{X}_j^{(0,0)} - \widehat{X}_{j-1}^{(0,0)})^2)$. We know η_n converges in distribution to a standard Brownian motion (see Theorem 3, [4]). Since $s_n^2/(n\gamma_0^2) \to 1$, it can be seen that $\sup_{u\in[0,M]} |\eta_n(u) - \hat{\theta}_n^{(0,0)}(-u)| \to 0$ in probability for any $M > 0$. So by Slutsky's theorem, we conclude that $\hat{\theta}_n^{(0,0)}$ converges in distribution to a standard Brownian motion going backward in time. $\square$

The next result helps in estimating the probability that a direct path and a dual path stay close to each other for some time period. Given $m \in \mathbb{N}$ and $\varepsilon, \delta > 0$, we define the event

$$\begin{aligned}
B_n^\varepsilon &= B_n^\varepsilon(\delta, m) \\
&:= \big\{\text{there exist } \pi_1^n, \pi_2^n, \pi_3^n \in \mathcal{X}_n \text{ such that } \sigma_{\pi_1^n}, \sigma_{\pi_2^n} \leq 0, \\
&\qquad \sigma_{\pi_3^n} \leq \lfloor n\delta \rfloor/n, \pi_1^n(0) \in [-m,m], \big|\pi_1^n(0) - \pi_2^n(0)\big| < \varepsilon, \text{ with} \\
&\qquad \pi_1^n\big(\lfloor n\delta\rfloor/n\big) \neq \pi_2^n\big(\lfloor n\delta\rfloor/n\big) \text{ and } \big|\pi_1^n\big(\lfloor n\delta\rfloor/n\big) - \pi_3^n\big(\lfloor n\delta\rfloor/n\big)\big| < \varepsilon, \text{ with} \\
&\qquad \pi_1^n\big(2\lfloor n\delta\rfloor/n\big) \neq \pi_3^n\big(2\lfloor n\delta\rfloor/n\big)\big\}.
\end{aligned}$$

LEMMA 2.11. *For any $m \in \mathbb{N}$ and $\varepsilon, \delta > 0$, we have*

$$\mathbb{P}\big(B_n^\varepsilon(\delta, m)\big) \leq C_1(\delta, m)\varepsilon,$$

where $C_1(\delta, m)$ is a positive constant, depending only on δ and m.

PROOF. Let D_n^ε be the unscaled version of the event B_n^ε, that is,

$$\begin{aligned}
D_n^\varepsilon := \big\{&\text{there exist } (x,0), (y,0), \big(z, \lfloor n\delta\rfloor\big) \in V \text{ such that} \\
&x \in [-m\sqrt{n}\gamma_0, m\sqrt{n}\gamma_0], |x - y| < \sqrt{n}\varepsilon\gamma_0 \text{ and } h^{\lfloor n\delta\rfloor}(x,0) \neq h^{\lfloor n\delta\rfloor}(y,0), \\
&\big|h^{\lfloor n\delta\rfloor}(x,0)(1) - z\big| < \sqrt{n}\varepsilon\gamma_0, h^{2\lfloor n\delta\rfloor}(x,0) \neq h^{\lfloor n\delta\rfloor}\big(z, \lfloor n\delta\rfloor\big)\big\}.
\end{aligned}$$

On the event D_n^ε there exists $l \in [-m\sqrt{n}\gamma_0, m\sqrt{n}\gamma_0] \cap \mathbb{Z}$ such that the unscaled paths starting from $(l,0)$ and $(l+1,0)$ (as in Figure 3) do not meet in time $\lfloor n\delta\rfloor$—an event which occurs with probability at most $C_2/\sqrt{n\delta}$ for some constant $C_2 > 0$

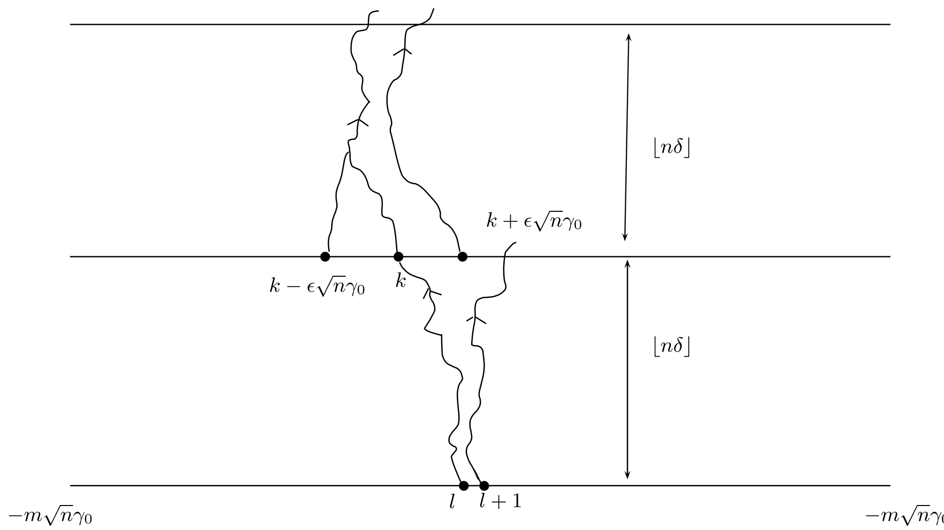

FIG. 3. *The vertices* $(l, 0)$ *and* $(l + 1, 0)$ *and the corresponding vertex* $(k, \lfloor n\delta \rfloor)$ *as required in the proof of Lemma* 2.11.

(see Theorem 4 of Coletti, Fontes and Dias [5]). Supposing $h^{\lfloor n\delta \rfloor}(l, 0)(1) = k$, two unscaled paths, one starting from a vertex $\lfloor \sqrt{n}\varepsilon\gamma_0 \rfloor$ distance to the left of k and the other starting from a vertex $\lfloor \sqrt{n}\varepsilon\gamma_0 \rfloor$ distance to the right of k, do not meet in time $\lfloor n\delta \rfloor$ has a probability at most $C_2 2\sqrt{n}\varepsilon\gamma_0/\sqrt{n\delta}$ for all $k \in \mathbb{Z}$. Thus, summing over all possibilities of l and k and using Markov property we have

$$\begin{aligned}
\mathbb{P}(D_n^\varepsilon) &\le \mathbb{P}\Biggl(\bigcup_{l=-2m\sqrt{n}\gamma_0}^{2m\sqrt{n}\gamma_0} \bigcup_{k\in\mathbb{Z}} \{h^{\lfloor n\delta \rfloor}(l, 0)(1) = k \neq h^{\lfloor n\delta \rfloor}(l + 1, 0)(1) \text{ and} \\
&\qquad h^{\lfloor n\delta \rfloor}(k - \lfloor \sqrt{n}\varepsilon\gamma_0 \rfloor, \lfloor n\delta \rfloor) \neq h^{\lfloor n\delta \rfloor}(k + \lfloor \sqrt{n}\varepsilon\gamma_0 \rfloor, \lfloor n\delta \rfloor)\}\Biggr) \\
&\le \sum_{l=-2m\sqrt{n}\gamma_0}^{2m\sqrt{n}\gamma_0} \frac{2C_2\sqrt{n}\varepsilon\gamma_0}{\sqrt{n\delta}} \sum_{k\in\mathbb{Z}} \mathbb{P}\{h^{\lfloor n\delta \rfloor}(l, 0)(1) = k \neq h^{\lfloor n\delta \rfloor}(l + 1, 0)(1)\} \\
&\le \sum_{l=-2m\sqrt{n}\gamma_0}^{2m\sqrt{n}\gamma_0} \frac{2C_2\sqrt{n}\varepsilon\gamma_0}{\sqrt{n\delta}} \mathbb{P}\{h^{\lfloor n\delta \rfloor}(l, 0)(1) \neq h^{\lfloor n\delta \rfloor}(l + 1, 0)(1)\} \\
&\le \sum_{l=-2m\sqrt{n}\gamma_0}^{2m\sqrt{n}\gamma_0} \frac{2C_2\sqrt{n}\varepsilon\gamma_0}{\sqrt{n\delta}} \frac{C_2}{\sqrt{n\delta}} \\
&\le C_1(\delta, m)\varepsilon. \qquad \square
\end{aligned}$$

PROOF OF THEOREM 2.9. Since $\widehat{\mathcal{X}}$ consists of noncrossing paths only, Proposition 2.10 implies the tightness of the family $\{\overline{\widehat{\mathcal{X}}}_n : n \geq 1\}$ (see Proposition B.2 in the Appendix of Fontes et al. [9]). The joint family $\{(\overline{\mathcal{X}}_n, \overline{\widehat{\mathcal{X}}}_n) : n \geq 1\}$ is tight since each of the two marginal families is tight. To prove Theorem 2.9, it suffices to show that for any subsequential limit $(\mathcal{W}, \mathcal{Z})$ of $\{(\overline{\mathcal{X}}_n, \overline{\widehat{\mathcal{X}}}_n) : n \geq 1\}$, the random variable $\mathcal{Z}$ satisfies the conditions given in Proposition 2.7.

Consider a convergent subsequence of $\{(\overline{\mathcal{X}}_n, \overline{\widehat{\mathcal{X}}}_n) : n \geq 1\}$ such that $(\mathcal{W}, \mathcal{Z})$ is its weak limit and by Skorohod's representation theorem, we may assume that the convergence happens almost surely. For ease of notation, we denote the convergent subsequence by itself.

From Proposition 2.10, it follows that for any deterministic $(x, t) \in \mathbb{R}^2$ there exists a path $\hat{\pi} \in \mathcal{Z}$ starting at (x, t) going backward in time almost surely.

Since $(\overline{\mathcal{X}}_n, \overline{\widehat{\mathcal{X}}}_n)$ converges to $(\mathcal{W}, \mathcal{Z})$ almost surely, if a dual path in $\mathcal{Z}$ crosses a path in $\mathcal{W}$, there exists a dual path in $\widehat{\mathcal{X}}_n$ which crosses a path in $\mathcal{X}_n$, for some $n \geq 1$, yielding a contradiction. Hence, the paths in $\mathcal{Z}$ do not cross paths in $\mathcal{W}$ almost surely (for details, see Roy, Saha and Sarkar [27]).

Now, to prove that condition (3) in Proposition 2.7 is satisfied, we define the following event: for $\delta > 0$ and positive integer $m \geq 1$, let

$$\begin{aligned} A(\delta, m) := \bigl\{ &\text{there exist paths } \pi \in \mathcal{W} \text{ and } \hat{\pi} \in \mathcal{Z} \text{ with } \sigma_\pi, \sigma_{\hat{\pi}} \in (-m, m), \\ &\text{and there exists } t_0 \text{ such that } \sigma_\pi < t_0 < t_0 + \delta < \sigma_{\hat{\pi}}, \\ &\text{and } -m < \pi(t) = \hat{\pi}(t) < m \text{ for all } t \in [t_0, t_0 + \delta] \bigr\}. \end{aligned}$$

It is enough to show that for any fixed $\delta > 0$ and for $m \geq 1$, we have $\mathbb{P}(A(\delta, m)) = 0$.

We present here the idea of the proof; more details are available in Roy, Saha and Sarkar [27]. Fix $\varepsilon > 0$. Since we are in a setup where the scaled paths converge almost surely, for all large n there exist $\pi_1^n \in \mathcal{X}_n$ and $\hat{\pi}^n \in \widehat{\mathcal{X}}_n$ within ε distance of π and $\hat{\pi}$, respectively. Using the fact that a dual vertex lies in the middle of two open vertices and the forward paths cannot cross the dual paths, it follows that for all large n there exist $\pi_2^n, \pi_3^n \in \widehat{\mathcal{X}}_n$ such that:

(a) $\max\{|\pi_1^n(\sigma_{\pi_2^n}) - \pi_2^n(\sigma_{\pi_2^n})|, |\pi_1^n(\sigma_{\pi_3^n}) - \pi_3^n(\sigma_{\pi_3^n})|\} < 4\varepsilon$;
(b) $\pi_1^n(\sigma_{\pi_2^n} + \delta/3) \neq \pi_2^n(\sigma_{\pi_2^n} + \delta/3)$ and $\pi_1^n(\sigma_{\pi_3^n} + \delta/3) \neq \pi_3^n(\sigma_{\pi_3^n} + \delta/3)$.

This gives us that $A(\delta, m) \subseteq \liminf_{n\to\infty} \bigcup_{j=1}^{\lfloor 6m/\delta \rfloor} B_n^{4\varepsilon}(\delta/3, 2m; j)$.

Here, $B_n^{4\varepsilon}(\delta/3, 2m; j)$ is a translation of the event $B_n^{4\varepsilon}(\delta/3, 2m)$, considered in Lemma 2.11; translated such that the starting time of the paths π_n^1 and π_n^2 are shifted by $-m + j\lfloor n\delta/3 \rfloor / n$ (see Figure 4).

By translation invariance of our model and Lemma 2.11, for all $n \geq 1$ we have $\mathbb{P}(B_n^{4\varepsilon}(\delta/3, 2m; j)) \leq 4C_1(\delta/3, 2m)\varepsilon$. This completes the proof. □

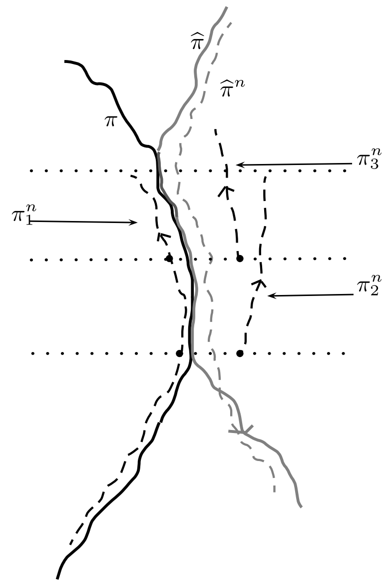

FIG. 4. *The event $A(\delta, m)$. The bold paths are from $(\mathcal{W}, \widehat{\mathcal{W}})$ and the approximating dashed paths are from $(\mathcal{X}_n, \widehat{\mathcal{X}}_n)$.*

3. Proof of Theorem 1.2. Let $\xi := \xi_{\mathcal{W}}(0, 1)$ and $\xi_n := \xi_{\overline{\mathcal{X}}_n}(0, 1)$ be as defined in (14). The proof of Theorem 1.2 follows from the following proposition.

PROPOSITION 3.1. *$\mathbb{E}[\xi_n] \to \mathbb{E}[\xi]$ as $n \to \infty$.*

We first complete the proof of Theorem 1.2 assuming Proposition 3.1.

PROOF OF THEOREM 1.2. Using the translation invariance of our model, we have

$$\sqrt{n}\gamma_0 \mathbb{P}\big(L(0,0) > n\big) = \sum_{k=0}^{\lfloor \sqrt{n}\gamma_0 \rfloor} \mathbb{E}(\mathbf{1}_{\{L(k,n)>n\}}) \times \frac{\sqrt{n}\gamma_0}{\lfloor \sqrt{n}\gamma_0 \rfloor + 1}$$
$$= \mathbb{E}(\xi_n) \times \frac{\sqrt{n}\gamma_0}{\lfloor \sqrt{n}\gamma_0 \rfloor + 1} \to \mathbb{E}(\xi) = \frac{1}{\sqrt{\pi}} \qquad \text{as } n \to \infty.$$

This proves Theorem 1.2. □

Proposition 3.1 will be proved through a sequence of lemmas.

To state the next lemma, we recall from Theorem 2.9 that $(\overline{\mathcal{X}}_n, \overline{\widehat{\mathcal{X}}}_n) \Rightarrow (\mathcal{W}, \widehat{\mathcal{W}})$ as $n \to \infty$. Using Skorohod's representation theorem, we assume that we are working on a probability space where $d_{\mathcal{H}\times\widehat{\mathcal{H}}}((\overline{\mathcal{X}}_n, \overline{\widehat{\mathcal{X}}}_n), (\mathcal{W}, \widehat{\mathcal{W}})) \to 0$ almost surely as $n \to \infty$.

LEMMA 3.2. *For $t_1 > t_0$, we have*

$$\mathbb{P}\big(\xi_{\overline{\mathcal{X}}_n}(t_0, t_1) \neq \xi_{\mathcal{W}}(t_0, t_1) \textit{ for infinitely many } n\big) = 0.$$

PROOF. We prove the lemma for $t_0 = 0$ and $t_1 = 1$, that is, for $\xi_n = \xi_{\overline{\mathcal{X}}_n}(0, 1)$ and $\xi_{\mathcal{W}}(0, 1)$, the proof for general t_0, t_1 being similar. First, we show that, for all $k \geq 0$,

$$\liminf_{n\to\infty} \mathbf{1}_{\{\xi_n \geq k\}} \geq \mathbf{1}_{\{\xi \geq k\}} \qquad \text{almost surely.} \tag{17}$$

Indeed, for $k = 0$, both $\mathbf{1}_{\{\xi_n \geq k\}}$ and $\mathbf{1}_{\{\xi \geq k\}}$ equal 1. For $k \geq 1$, (17) follows from almost sure convergence of $(\overline{\mathcal{X}}_n, \widehat{\overline{\mathcal{X}}}_n)$ to $(\mathcal{W}, \widehat{\mathcal{W}})$ and from the properties of the set $\mathcal{M}_{\mathcal{W}}(0, 1)$ as described in Proposition 2.5.

To complete the proof, we need to show that $\mathbb{P}(\limsup_{n\to\infty}\{\xi_n > \xi\}) = 0$. This is equivalent to showing that $\mathbb{P}(\Omega_0^k) = 0$ for all $k \geq 0$, where

$$\Omega_0^k := \big\{\omega : \xi_n(\omega) > \xi(\omega) = k \text{ for infinitely many } n\big\}.$$

Consider $k = 0$ first. From Proposition 2.5, it follows that on the event $\xi = 0$, almost surely we can obtain $\gamma := \gamma(\omega) > 0$ such that $\mathcal{M}_{\mathcal{W}}(0, 1) \cap (-\gamma, 1 + \gamma) = \varnothing$. From the almost sure convergence of $(\overline{\mathcal{X}}_n, \widehat{\overline{\mathcal{X}}}_n)$ to $(\mathcal{W}, \widehat{\mathcal{W}})$, we have $\mathbb{P}(\Omega_0^0) = 0$.

For $k > 0$, on the event Ω_0^k we show a forward path $\pi \in \mathcal{W}$ coincides with a dual path $\hat{\pi} \in \widehat{\mathcal{W}}$ for a positive time which leads to a contradiction. From Proposition 2.5, it follows that given $\eta > 0$, there exist $m_0 \in \mathbb{N}$ and $s_0 \in (1/m_0, 1)$ such that $\mathbb{P}(\xi_{\mathcal{W}}(1/m_0, 1) = \xi_{\mathcal{W}}(1/m_0, s_0) = \xi_{\mathcal{W}}(0, 1) = k) > 1 - \eta$, that is, the paths leading to any single point considered in $\mathcal{M}_{\mathcal{W}}(0, 1) = \mathcal{M}_{\mathcal{W}}(1/m_0, 1)$ have coalesced before time s_0. Fix $0 < \varepsilon < 1/m_0$ such that $(x - \varepsilon, x + \varepsilon) \subset (0, 1)$ for all $x \in \mathcal{M}_{\mathcal{W}}(1/m_0, 1)$ and the ε-tubes around the k paths contributing to $\mathcal{M}_{\mathcal{W}}(s_0, 1)$, viz., $\pi_1(t), \ldots, \pi_k(t)$, $t \in [s_0, 1]$, given by

$$T_\varepsilon^i := \big\{(x, t) : \pi_i(t) - \varepsilon \leq x \leq \pi_i(t) + \varepsilon, s_0 \leq t \leq 1\big\} \qquad \text{for } i = 1, \ldots, k,$$

are disjoint. Since we have almost sure convergence on the event Ω_0^k, there exists n_0 such that one of the k tubes must contain at least two paths, $\pi_1^{n_0}, \pi_2^{n_0}$ (say) of $\mathcal{X}_{n_0}$ which do not coalesce by time 1. From the construction of dual paths, it follows that there exists at least one dual path $\hat{\pi}^{n_0} \in \widehat{\overline{\mathcal{X}}}{}_{n_0}^{1+}$ lying between $\pi_1^{n_0}$ and $\pi_2^{n_0}$ for $t \in [s_0, 1]$, and hence we must have an approximating $\hat{\pi} \in \widehat{\mathcal{W}}^{1+}$ close to $\hat{\pi}^{n_0}$ for $t \in [s_0, 1]$. Since we have only finitely many disjoint k tubes, taking $\varepsilon \to 0$ and using compactness of $\widehat{\mathcal{W}}$ we obtain that there exists $\hat{\pi} \in \widehat{\mathcal{W}}$ such that $\hat{\pi}(t) = \pi_i(t)$ for $t \in [s_0, 1]$ and for some $1 \leq i \leq k$. This violates the property of Brownian web and its dual that they do not spend positive Lebesgue time together. Hence, $\mathbb{P}(\Omega_0^k) = 0$ for all $k \geq 0$ and this completes the proof of the lemma. $\square$

Lemma 3.2 immediately gives the following corollary.

COROLLARY 3.2.1. *As $n \to \infty$, ξ_n converges in distribution to ξ.*

Corollary 3.2.1 along with the following lemma completes the proof of Proposition 3.1.

LEMMA 3.3. *The family $\{\xi_n : n \in \mathbb{N}\}$ is uniformly integrable.*

PROOF. For $m \in \mathbb{N}$, let

$$K_m = [-m, m]^2 \cap \mathbb{Z}^2 \quad \text{and} \quad \Omega_m := \{(0,1), (0,-1), (1,1), (1,-1)\}^{K_m}.$$

We assign the product probability measure $\mathbb{P}'$ whose marginals for $\mathbf{u} \in K_m$ are given by

$$\mathbb{P}'\{\zeta : \zeta(\mathbf{u}) = (a,b)\} = \begin{cases} \dfrac{p}{2}, & \text{for } a = 1 \text{ and } b \in \{1,-1\}, \\ \dfrac{(1-p)}{2}, & \text{for } a = 0 \text{ and } b \in \{1,-1\}. \end{cases}$$

$\mathbb{P}'$ is the measure induced by the random variables $\{(B_{\mathbf{u}}, U_{\mathbf{u}}) : \mathbf{u} \in K_m\}$.

For $\zeta \in \Omega_m$ and for $K \subseteq K_m$, the K cylinder of ζ is given by $C(\zeta, K) := \{\zeta' : \zeta'(\mathbf{u}) = \zeta(\mathbf{u}) \text{ for all } \mathbf{u} \in K\}$. For any two events $A, B \subseteq \Omega_m$, let

$$\begin{aligned} A \square B := \{\zeta &: \text{there exists } K = K(\zeta) \subseteq K_m \text{ such that } C(\zeta, K) \subseteq A, \\ &\text{and } C(\zeta, K') \subseteq B \text{ for } K' = K_m \setminus K\} \end{aligned}$$

denote the disjoint occurrence of A and B. Note that this definition is associative, that is, for any $A, B, C \subseteq \Omega_m$ we have $(A \square B) \square C = A \square (B \square C)$.

Let

$$\begin{aligned} F_n^m := \{&\text{there exist } (u_1, n), (u_2, n) \in \widehat{V} \text{ with } 0 \le u_1 < u_2 \le \sqrt{n}\gamma_0 \text{ and} \\ &(v_1^l, l), (v_2^l, l) \in V \text{ for all } 0 \le l \le n \text{ such that} \\ &-m \le v_1^l < \hat{h}^l(u_1, n)(1) < \hat{h}^l(u_2, n)(1) < v_2^l \le m\}, \\ E_n^m(k) := \{&\text{for } 1 \le i \ne j \le k, \text{ there exists } (x_i, 0) \in V \text{ with} \\ &h^n(x_i, 0)(1) \in [0, \sqrt{n}\gamma_0], h^n(x_i, 0) \ne h^n(x_j, 0), h^l(x_i, 0)(1) \in [-m, m] \\ &\text{for all } 0 \le l \le n\}. \end{aligned}$$

We claim that for all $k \ge 2$,

$$E_n^m(3k) \subseteq \underbrace{F_n^m \square F_n^m \square \cdots \square F_n^m}_{k \text{ times}}. \tag{18}$$

We prove it for $k = 2$. For general k, the proof is similar. Let $(u_i, n) \in \widehat{V}$, $1 \le i \le 5$ and $(x_i, 0) \in V$, $1 \le i \le 6$ be as in Figure 5. The region explored to obtain the

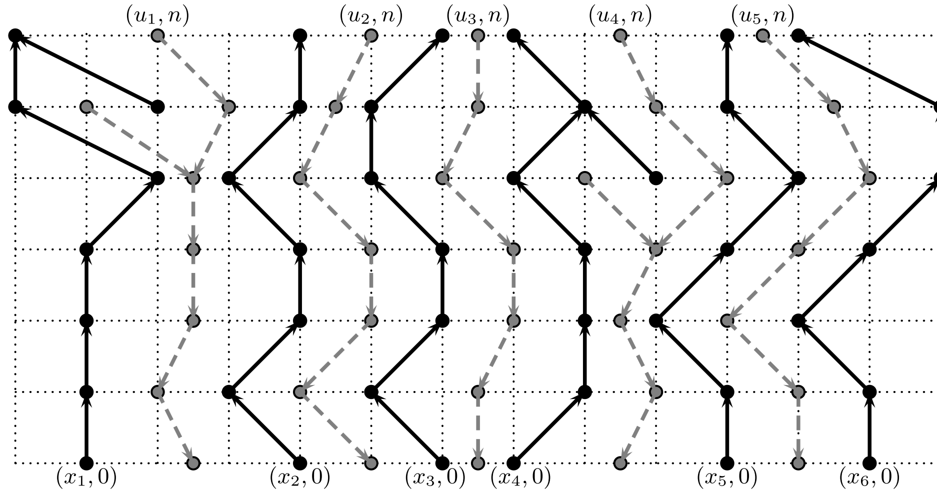

FIG. 5. *The event $E_n^m(6)$.*

vertex $\hat{h}^j(u_i, n)$ for $1 \le j \le n$ is contained in $\bigcup_{l=0}^{n-1}[h^l(x_i, 0)(1), h^l(x_{i+1}, 0)(1)] \times \{l\}$. Thus, the regions explored to obtain the dual paths starting from (u_1, n), (u_2, n) and the dual paths starting from (u_4, n), (u_5, n) are disjoint (see Figure 5). Hence, it follows that $E_n^m(6) \subseteq F_n^m \square F_n^m$.

Since the event $E_n^m(k)$ is monotonic in m, from (18) we get

$$\begin{aligned}\mathbb{P}(\xi_n \ge 3k) &= \mathbb{P}\Big(\lim_{m\to\infty} E_n^m(3k)\Big) = \lim_{m\to\infty} \mathbb{P}\big(E_n^m(3k)\big) \\ &\le \lim_{m\to\infty} \mathbb{P}\big(F_n^m \square \cdots \square F_n^m\big) = \lim_{m\to\infty} \mathbb{P}'\big(F_n^m \square \cdots \square F_n^m\big).\end{aligned}$$

Applying the BKR inequality (see Reimer [24]), we get

$$\mathbb{P}(\xi_n \ge 3k) \le \lim_{m\to\infty} \big(\mathbb{P}'(F_n^m)\big)^k = \Big(\mathbb{P}\Big(\lim_{m\to\infty} F_n^m\Big)\Big)^k = \big(\mathbb{P}(F_n)\big)^k, \tag{19}$$

where $F_n := \{$there exist $(u_1, n), (u_2, n) \in \widehat{V}$ with $0 \le u_1 < u_2 \le \sqrt{n}\gamma_0$ such that $\hat{h}^n(u_1, n) \ne \hat{h}^n(u_2, n)\}$.

For any $(x, t) \in \mathbb{R}^2$ fix $t_n = \lfloor nt \rfloor$ and $x_n = \max\{\lfloor \sqrt{n}\gamma_0 x \rfloor + j : j \le 0, (\lfloor \sqrt{n}\gamma_0 x \rfloor + j, t_n) \in \widehat{V}\}$. Let $\hat{\theta}_n^{(x,t)} \in \widehat{\mathcal{X}}_n(\gamma_0)$ be the scaling of the path $\hat{\pi}^{(x_n, t_n)} \in \widehat{\mathcal{X}}$. Define

$$F_n' := \big\{\hat{\theta}_n^{(0,1)} \text{ and } \hat{\theta}_n^{(1,1)} \text{ do not coalesce in time } 1\big\}.$$

We observe that $F_n \subseteq F_n'$. Now $\mathbb{P}(F_n')$ converges to the probability that two independent Brownian motions starting at a distance 1 from each other do not meet by time 1. Since $\lim_{n\to\infty} \mathbb{P}(F_n') < 1$, the family $\{\xi_n : n \in \mathbb{N}\}$ is uniformly integrable. $\square$

REMARK 3.4. It is to be noted that Newman, Ravishankar and Sun [23] also used ideas of negative correlation to establish the weak convergence of $\mathcal{M}_{\overline{\mathcal{X}}_n}$ as a point process on $\mathbb{R}$ for a more general setup where paths can cross each other. In our case, the negative correlation ideas come in a much less essential manner only to establish uniform integrability as the noncrossing nature of paths enable us to obtain Corollary 3.2.1.

4. Proofs of Theorems 1.3 and 1.4. In this section, we prove Theorems 1.3 and 1.4. The main idea of the proof is that the horizontal distance between the dual paths $\hat{\pi}^{\hat{r}(x,t)}$ and $\hat{\pi}^{\hat{l}(x,t)}$ (see Figure 6) form a Brownian excursion process after scaling. The cluster $C(x,t)$ being enclosed between these two paths, its size is related to the area under the Brownian excursion.

For a formal proof, we need to introduce some notation. For $\tau > 0$, let $S^\tau, S^{\tau^+} : C[0,\infty) \to \mathbb{R}$ be defined by $S^\tau(f) := \inf\{t \geq 0 : f(t+s) \geq f(t) \text{ for all } 0 \leq s \leq \tau\}$ and $S^{\tau^+}(f) := \inf\{t \geq 0 : f(t+s) > f(t) \text{ for all } 0 < s \leq \tau\}$. Let $T^{\tau^+} : C[0,\infty) \to C[0,\infty)$ be the map given by

$$T^{\tau^+}(f)(s) := \begin{cases} f(S^{\tau^+}+s) - f(S^{\tau^+}), & \text{if } S^{\tau^+} < \infty, \\ f(s), & \text{otherwise.} \end{cases} \tag{20}$$

For a Brownian motion W with $W(0) = 0$, we define $W^\tau = T^{\tau^+}(W)$. From Bolthausen [3], we have $S^{\tau^+} = S^\tau < \infty$ almost surely under the measure induced by W on $C[0,\infty)$ and $W^1|_{[0,1]} \stackrel{d}{=} W^+$ where W^+ is the standard Brownian meander process defined in (3). From the scaling property of Brownian motion, it

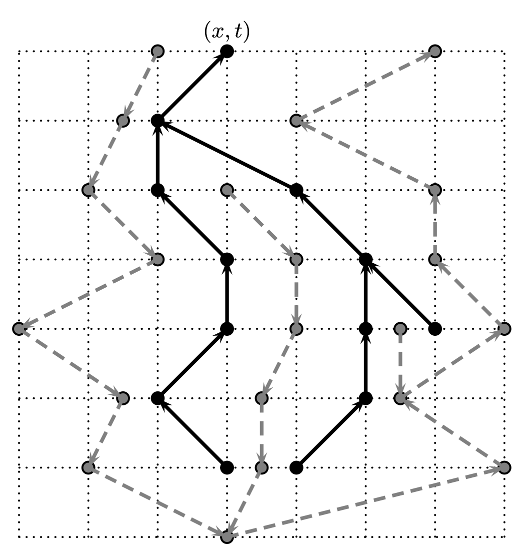

FIG. 6. *The two dual paths $\hat{\pi}^{\hat{l}(x,t)}$ and $\hat{\pi}^{\hat{r}(x,t)}$ enclose the cluster $C(x,t)$. These dual paths after scaling are each Brownian paths.*

follows that $\{W^\tau(s) : s \in [0,\tau]\} \stackrel{d}{=} \{\sqrt{\tau} W^+(s/\tau) : s \in [0,\tau]\}$. Durrett, Iglehart and Miller [7] (Theorem 2.1) proved that $W|\mathbf{1}_{\{\min_{s\in[0,1]} W(s)\geq -\varepsilon\}} \Rightarrow W^+$ as $\varepsilon \downarrow 0$. Using this result and the scaling property of W^τ, given above, straightforward calculations imply the following lemma and its corollary (for details, see Roy, Saha and Sarkar [27]).

LEMMA 4.1. *For $\tau > 0$ considering W as a standard Brownian motion on $[0,\infty)$ starting from 0, we have $W|\mathbf{1}_{\{\min_{t\in[0,\tau]} W(t)\geq -1/n\}} \Rightarrow W^\tau$ as $n \to \infty$.*

Define $\widetilde{W}^\tau$ as the process on $C[0,\infty)$ given by

$$\widetilde{W}^\tau(t) := \begin{cases} W^\tau(t), & \text{if } 0 \leq t \leq \tau, \\ W^\tau(\tau) + \widetilde{W}(t-\tau), & \text{otherwise,} \end{cases}$$

where $\widetilde{W}$ is a Brownian motion on $[0,\infty)$, independent of W^τ, with $\widetilde{W}(0) = 0$. For $f \in C[0,\infty)$, let $t_f := \inf\{s > 0 : f(s) = 0\}$ with $t_f = \infty$ if $f(s) \neq 0$ for all $s > 0$. Consider the mapping $H : C[0,\infty) \to C[0,\infty)$ given by $H(f)(t) := \mathbf{1}_{\{t\leq t_f\}} f(t)$. We define $W^{+,\tau} = H(W^\tau)$. A similar argument as that of Lemma 4.1 gives us the following corollary.

COROLLARY 4.1.1. *For $\tau > 0$, we have, $W^\tau \stackrel{d}{=} \widetilde{W}^\tau$ and $W^{+,\tau} \stackrel{d}{=} H(\widetilde{W}^\tau)$.*

Let $A \subset C[0,\infty)$ be such that

$$A := \big\{ f \in C[0,\infty) : t_f < \infty \text{ and for every } \varepsilon > 0 \text{ there exists } s \in (t_f, t_f + \varepsilon) \text{ with } f(s) < 0 \big\}. \tag{21}$$

From Corollary 4.1.1, it follows that $\mathbb{P}(W^\tau \in A) = 1$. Hence, H is continuous almost surely under the measure induced by W^τ on $C[0,\infty)$.

Next, we obtain the distribution of $\int_0^\infty W^{+,\tau}(t)\,dt$.

LEMMA 4.2. *For $\tau, \lambda > 0$, we have*

$$\mathbb{P}\left(\int_0^\infty W^{+,\tau}(t)\,dt > \lambda\right) = \frac{\sqrt{\tau}}{2} \int_\tau^\infty t^{-3/2} \overline{F}_{I_0^+}(\lambda t^{-3/2})\,dt.$$

PROOF. We give here a straightforward proof using random walk. Let $\{S_n : n \geq 0\}$ be a symmetric random walk with variance 1 starting at $S_0 = 0$. Since $\mathbb{P}(W^\tau \in A) = 1$, minor modification of the argument used to prove Lemmas 2.4 and 2.5, Bolthausen [3] shows that $H \circ T^{\tau^+}$ is almost surely continuous under the measure induced by W on $C[0,\infty)$ (for details, see Roy, Saha and Sarkar [27]). From Donsker's invariance principle and from the continuous mapping theorem, it follows that for $\lambda > 0$, a continuity point of $\int_0^\infty W^{+,\tau}(t)\,dt$, we have

$$\mathbb{P}\left(\int_0^\infty W^{+,\tau}(t)\,dt > \lambda\right) = \lim_{n\to\infty} \mathbb{P}\left(\int_0^\infty H\big(T^{\tau^+}(Y_n)\big)(t)\,dt > \lambda\right),$$

where

$$Y_n(t) := \frac{S_k}{\sqrt{n}} + \frac{(nt - [nt])}{\sqrt{n}}(S_{k+1} - S_k) \qquad \text{for } \frac{k}{n} \le t < \frac{k+1}{n}. \tag{22}$$

A similar argument as in Lemma 3.1 of Bolthausen [3] gives us that (for details, see Roy, Saha and Sarkar [27])

$$\begin{aligned}\mathbb{P}\Big(\int_0^\infty H\big(T^{\tau^+}(Y_n)\big)(t)\,dt > \lambda\Big)\\ &= \mathbb{P}\Big(\int_0^\infty H(Y_n)(t)\,dt > \lambda \Big| \min_{t\in[0,\tau]} Y_n(t) \ge 0, t_0 > n\tau\Big),\end{aligned}$$

where $t_0 := \inf\{n > 0 : S_n = 0\}$ is the first return time to 0 of the random walk. Hence for $\lambda > 0$, a continuity point of $W^{+,\tau}$, we obtain

$$\begin{aligned}\mathbb{P}\Big(\int_0^\infty W^{+,\tau}(t)\,dt > \lambda\Big)\\ &= \lim_{n\to\infty} \mathbb{P}\Big(\int_0^\infty H(Y_n)(t)\,dt > \lambda \Big| \min_{t\in[0,\tau]} Y_n(t) \ge 0, t_0 > n\tau\Big)\\ &= \lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{n^{3/2}\mathbb{P}(t_0 = \lfloor n\tau\rfloor + j)}{n(\sqrt{n}\mathbb{P}(t_0 > n\tau))}\\ &\quad \times \mathbb{P}\Big(\int_0^\infty H(Y_n)(t)\,dt > \lambda \Big| \min_{t\in[0,\tau]} Y_n(t) \ge 0, t_0 = \lfloor n\tau\rfloor + j\Big)\\ &= \lim_{n\to\infty} \frac{1}{\sqrt{n}\mathbb{P}(t_0 > n\tau)} \int_{\lfloor n\tau\rfloor/n}^\infty g_n(t) f_n(t)\,dt,\end{aligned}$$

where for $t \ge \lfloor n\tau\rfloor/n$, $f_n(t) = \mathbb{P}(\int_0^\infty H(Y_n)(u)\,du > \lambda | \min_{t\in[0,\tau]} Y_n(t) \ge 0, t_0 = \lfloor nt\rfloor + 1)$ and $g_n(t) = n^{3/2}\mathbb{P}(t_0 = \lfloor nt\rfloor + 1)$. It is known that (see Kaigh [16])

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}(t_0 > n) = \sqrt{\frac{2}{\pi}} \quad \text{and} \quad \lim_{n\to\infty} n^{3/2}\mathbb{P}(t_0 = n) = \frac{1}{\sqrt{2\pi}}.$$

Hence, from Theorem 2.6 Kaigh [16] together with the continuous mapping theorem and the scaling property of the Brownian motion we have $\mathbb{P}(\int_0^\infty W^{+,\tau}(t)\,dt > \lambda) = \frac{\sqrt{\tau}}{2}\int_\tau^\infty t^{-3/2}\overline{F}_{I_0^+}(\lambda t^{-3/2})\,dt$. Finally, I_0^+ being a continuous random variable (see Louchard and Janson [20]), it follows that the random variable $\int_0^\infty W^{+,\tau}(t)\,dt$ is continuous. This completes the proof. $\square$

4.1. *Proof of Theorem* 1.3. Recall that $\hat{r}(x,t)$ and $\hat{l}(x,t)$ denote the right and left dual neighbours, respectively, of $(x,t) \in V$. Let $\hat{D}_k(x,t) := \hat{h}^k(\hat{r}(x,t))(1) - \hat{h}^k(\hat{l}(x,t))(1)$ where $\hat{h}$ is as defined after (11). Consider the continuous function

$\hat{D}_n^{(x,t)} \in C[0,\infty)$ given by

$$\hat{D}_n^{(x,t)}(s) := \frac{\hat{D}_k(x,t)}{\gamma_0\sqrt{n}} + \frac{(ns - [ns])}{\gamma_0\sqrt{n}}\big(\hat{D}_{k+1}(x,t) - \hat{D}_k(x,t)\big) \quad \text{for } \frac{k}{n} \le s \le \frac{k+1}{n}. \tag{23}$$

Fix $\tau > 0$. For an $\mathcal{H} \times \widehat{\mathcal{H}}$ valued random variable $(K, \widehat{K})$ and for $x \in \mathcal{M}_K(0,\tau)$ let $\hat{\pi}_r^{(x,\tau)}$ be defined as

$$\hat{\pi}_r^{(x,\tau)} := \begin{cases} \hat{\pi}, & \text{if } \sigma_{\hat{\pi}} = \tau \text{ and there is no } \hat{\pi}_1 \in \widehat{K}^{\tau+} \text{ with } x < \hat{\pi}_1(\tau) < \hat{\pi}(\tau), \\ \hat{\pi}_0, & \text{otherwise,} \end{cases}$$

where $\hat{\pi}_0$ denotes the constant zero function with $\sigma_{\hat{\pi}_0} = \tau$. In other words, $\hat{\pi}_r^{(x,\tau)} \in \widehat{K}^{\tau+}$ is such that among all $\hat{\pi} \in \widehat{K}^{\tau+}$, $\hat{\pi}_r^{(x,\tau)}(\tau)$ is closest to (x,τ) on the right. Similarly, $\hat{\pi}_l^{(x,\tau)}$ is defined as the path closest to (x,τ) on the left.

For $\hat{\pi} \in \widehat{\Pi}$ with $\sigma_{\hat{\pi}} \ge \tau$, let $g(\hat{\pi}) \in C[0,\infty)$ be given by $g(\hat{\pi})(t) := \hat{\pi}(\tau - t)$ for $t \ge 0$. Fix $f \in C_b[0,\infty)$ and define

$$\kappa_{(K,\widehat{K})}(\tau, f) := \sum_{x \in \mathcal{M}_K(0,\tau)} f\big(g\big(\hat{\pi}_r^{(x,\tau)}\big) - g\big(\hat{\pi}_l^{(x,\tau)}\big)\big).$$

Let $\kappa(\tau, f) := \kappa_{(\mathcal{W},\widehat{\mathcal{W}})}(\tau, f)$, and $\kappa_n(\tau, f) := \kappa_{(\overline{\mathcal{X}}_n, \widehat{\overline{\mathcal{X}}}_n)}(\tau, f)$. Comparing with the definitions introduced in (14), for $m_f = \sup\{|f(s)| : s \in [0,\infty)\}$ we have

$$\kappa(\tau, f) \le m_f \xi_{\mathcal{W}}(0,\tau), \kappa_n(\tau, f) \le m_f \xi_{\overline{\mathcal{X}}_n}(0,\tau) \qquad \text{for all } n \ge 1. \tag{24}$$

From Proposition 2.5, we know that for each $x \in \mathcal{M}_{\mathcal{W}}(0,\tau)$, there exist $\hat{\pi}_r^{(x,\tau)}$, $\hat{\pi}_l^{(x,\tau)} \in \widehat{\mathcal{W}}$ both starting from (x,τ) with $\hat{\pi}_r^{(x,\tau)}(0) > \hat{\pi}_l^{(x,\tau)}(0)$.

The following lemma is the main tool for establishing Theorem 1.3 and Theorem 1.4.

LEMMA 4.3. *For $\tau > 0$ and $f \in C_b[0,\infty)$, we have*

$$\lim_{n\to\infty} \mathbb{E}\big[\kappa_n(\tau, f)\big] = \mathbb{E}\big[\kappa(\tau, f)\big]. \tag{25}$$

PROOF. From (24) and Lemma 3.3, it follows that the family $\{\kappa_n(\tau, f) : n \in \mathbb{N}\}$ is uniformly integrable. Hence, it suffices to show that $\kappa_n(\tau, f)$ converges in distribution to $\kappa(\tau, f)$ as $n \to \infty$. We assume that we are working on a probability space such that $(\overline{\mathcal{X}}_n, \widehat{\overline{\mathcal{X}}}_n)$ converges to $(\mathcal{W}, \widehat{\mathcal{W}})$ almost surely in $(\mathcal{H} \times \widehat{\mathcal{H}}, d_{\mathcal{H}\times\widehat{\mathcal{H}}})$. From Lemma 3.2, we have $\lim_{n\to\infty} \xi_{\overline{\mathcal{X}}_n}(0,\tau) = \xi_{\mathcal{W}}(0,\tau)$ almost surely, and hence from (24) for $\xi_{\mathcal{W}}(0,\tau) = 0$, we have $\kappa_n(\tau, f) = \kappa(\tau, f) = 0$ for all n large. Next, we consider the case $\xi_{\mathcal{W}}(0,\tau) = k \ge 1$. Suppose $\mathcal{M}_{\mathcal{W}}(0,\tau) = \{x_1, \dots, x_k\}$. From Lemma 3.2, we have that $\mathcal{M}_{\overline{\mathcal{X}}_n}(0,\tau) = \{x_1^n, \dots, x_k^n\}$ for all large n and $\lim_{n\to\infty} x_i^n = x_i$ for all $1 \le i \le k$. Fix $T \ge 0$. To complete the proof, it is enough to show that $\sup\{|\hat{\pi}_r^{(x_i,\tau)}(\tau-s)-\hat{\pi}_r^{(x_i^n,\tau)}(\tau-s)|\vee|\hat{\pi}_l^{(x_i,\tau)}(\tau-s)-\hat{\pi}_l^{(x_i^n,\tau)}(\tau-s)|:s\in[0,\tau+T]\}\to 0$ as $n\to\infty$ for all $1\le i\le k$.

We observe that for $y_i\in(\hat{\pi}_r^{(x_i,\tau)}(0),\hat{\pi}_l^{(x_i,\tau)}(0))\cap\mathbb{Q}$ there exists $\pi^{(y_i,0)}\in\mathcal{W}$ such that $\pi^{(y_i,0)}(\tau)=x_i$. We choose $\varepsilon=\varepsilon(\omega)>0$ so that for all $1\le i\le k$:

(a) $(x_i-\varepsilon,x_i+\varepsilon)\subset(0,1)$, $(x_i-2\varepsilon,x_i+2\varepsilon)\cap\mathcal{M}_{\mathcal{W}}(0,\tau)=\{x_i\}$ and
(b) $(\hat{\pi}_r^{(x_i,\tau)}(0)-\pi^{(y_i,0)}(0))\wedge(\pi^{(y_i,0)}(0)-\hat{\pi}_l^{(x_i,\tau)}(0))>2\varepsilon$.

Let $n_0=n_0(\omega)$ be such that, for all $n\ge n_0$:

(i) $\xi_{\overline{\mathcal{X}}_n}(0,\tau)=\xi_{\mathcal{W}}(0,\tau)$ and

(ii) for all $1\le i\le k$ there exist $\hat{\pi}_i^{1,n},\hat{\pi}_i^{2,n}\in\widehat{\overline{\mathcal{X}}}_n^{\tau+}$ and $\pi_i^n\in\overline{\mathcal{X}}_n^{0-}$ such that $\sup\{|\hat{\pi}_i^{1,n}(\tau-s)-\hat{\pi}_r^{(x_i,\tau)}(\tau-s)|\vee|\hat{\pi}_i^{2,n}(\tau-s)-\hat{\pi}_l^{(x_i,\tau)}(\tau-s)|\vee|\pi_i^n(\tau-s)-\pi^{(y_i,0)}(\tau-s)|:s\in[0,\tau+T]\}<\varepsilon$.

The choice of n_0 ensures that $\mathcal{M}_{\overline{\mathcal{X}}_n}(0,\tau)\cap(x_i-\varepsilon,x_i+\varepsilon)=\{x_i^n\}$. Since there exist only two dual paths starting from (x_i,τ), because of the uniqueness of x_i^n in the interval $(x_i-\varepsilon,x_i+\varepsilon)$ and the noncrossing nature of our paths we must have $\hat{\pi}_r^{(x_i^n,\tau)}(\tau-s)=\hat{\pi}_i^{1,n}(\tau-s)$ and $\hat{\pi}_l^{(x_i^n,\tau)}(\tau-s)=\hat{\pi}_i^{2,n}(\tau-s)$ for all $s\in[0,\tau+T]$ and for all $n\ge n_0$ (for details, see Roy, Saha and Sarkar [27]). Since $T\ge 0$ is chosen arbitrarily, this completes the proof. $\square$

The next lemma calculates $\mathbb{E}[\kappa(\tau,f)]$.

LEMMA 4.4. *For $\tau>0$ and $f\in C_b[0,\infty)$, we have*

$$\mathbb{E}[\kappa(\tau,f)]=\mathbb{E}(f(\sqrt{2}W^{+,\tau}))/\sqrt{\pi\tau}.$$

PROOF. Let $I_n\subset\{0,1,\dots,n-1\}$ given by $I_n:=\{i:0\le i\le n-1,\hat{\pi}^{(i/n,\tau)},\hat{\pi}^{((i+1)/n,\tau)}\in\widehat{\mathcal{W}}$ such that $\hat{\pi}^{(i/n,\tau)}(0)<\hat{\pi}^{((i+1)/n,\tau)}(0)\}$. We define

$$\mathcal{R}_n(\tau,f)=\sum_{i\in I_n}f(g(\hat{\pi}^{((i+1)/n,\tau)}-\hat{\pi}^{(i/n,\tau)})).$$

From Proposition 2.5, we know $\mathcal{M}_{\mathcal{W}}(0,\tau)\cap\mathbb{Q}=\varnothing$. For each $x\in\mathcal{M}_{\mathcal{W}}(0,\tau)$, set $l_n^x=\lfloor nx\rfloor/n$ and $r_n^x=l_n^x+(1/n)$. Since there are exactly two dual paths $\hat{\pi}_r^{(x,\tau)}$ and $\hat{\pi}_l^{(x,\tau)}$ starting from (x,τ) with $\hat{\pi}_r^{(x,\tau)}(0)>\hat{\pi}_l^{(x,\tau)}(0)$, from Proposition 3.2(e) of Sun and Swart [29] it follows that $\{\hat{\pi}^{(l_n^x,\tau)}:n\in\mathbb{N}\}$ and $\{\hat{\pi}^{(r_n^x,\tau)}:n\in\mathbb{N}\}$ converge to $\hat{\pi}_l^{(x,\tau)}$ and $\hat{\pi}_r^{(x,\tau)}$, respectively, in $(\widehat{\Pi},d_{\widehat{\Pi}})$ as $n\to\infty$. Hence, $\mathcal{R}_n(\tau,f)\to\kappa(\tau,f)$ almost surely as $n\to\infty$. For each $i\in I_n$, there exist $y_i\in(\hat{\pi}^{(i/n,\tau)}(0),\hat{\pi}^{((i+1)/n,\tau)}(0))\cap\mathbb{Q}$ and $\pi^{(y_i,0)}\in\mathcal{W}$ such that $\pi^{(y_i,0)}(\tau)\in\mathcal{M}_{\mathcal{W}}(0,\tau)$. Hence, for $m_f=\sup\{|f(t)|:t\ge 0\}$ we have $\mathcal{R}_n(\tau,f)\le m_f\xi_{\mathcal{W}}(0,\tau)$ for all n. As $\mathbb{E}[\xi_{\mathcal{W}}(0,\tau)]<\infty$, the family $\{\mathcal{R}_n(\tau,f):n\in\mathbb{N}\}$ is uniformly integrable, and hence we have $\lim_{n\to\infty}\mathbb{E}[\mathcal{R}_n(\tau,f)]=\mathbb{E}[\kappa(\tau,f)]$. From the fact that

$g(\hat{\pi}^{((i+1)/n,\tau)}) - g(\hat{\pi}^{(i/n,\tau)}) \stackrel{d}{=} H(1/n + \sqrt{2}W)$ where W denotes the standard Brownian motion on $[0, \infty)$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}[\mathcal{R}_n(\tau, f)] \\
&\quad = \lim_{n\to\infty} n\mathbb{E}\Big[f\big(H(1/n + \sqrt{2}W)\big)\big|1/n + \min_{t\in[0,\tau]} \sqrt{2}W(t) > 0\Big] \\
&\qquad \times \mathbb{P}\Big(1/n + \min_{t\in[0,\tau]} \sqrt{2}W(t) > 0\Big) \\
&\quad = \lim_{n\to\infty} \mathbb{E}\Big[f\big(H(1/n + \sqrt{2}W)\big)\big| \min_{t\in[0,\tau]} \sqrt{2}W(t) > -1/n\Big]n \\
&\qquad \times \big(2\Phi(1/\sqrt{2\tau}n) - 1\big) \\
&\quad = \mathbb{E}\big(f\big(\sqrt{2}W^{+,\tau}\big)\big)/\sqrt{\pi\tau},
\end{aligned}$$

where the last equality follows from Lemma 4.1, Slutsky's theorem and continuous mapping theorem. This completes the proof. $\square$

Now, to complete the proof of Theorem 1.3 we need the following lemmas.

LEMMA 4.5. *For $\tau > 0$, we have $\hat{D}_n^{(0,0)}|\mathbf{1}_{\{L(0,0)>n\tau\}} \Rightarrow \sqrt{2}W^{+,\tau}$ as $n \to \infty$.*

PROOF. Using translation invariance of our model, we have

$$\mathbb{E}\big(f\big(\hat{D}_n^{(0,0)}\big)|\mathbf{1}_{\{L(0,0)>n\tau\}}\big) = \frac{\mathbb{E}[\kappa_n(\tau, f)]}{\mathbb{E}[\xi_{\overline{\mathcal{X}}_n}(0, \tau)]} \to \frac{\mathbb{E}[\kappa(\tau, f)]}{\mathbb{E}[\xi_{\mathcal{W}}(0, \tau)]} = \mathbb{E}\big(f\big(\sqrt{2}W^{+,\tau}\big)\big).$$

This holds for all $f \in C_b[0, \infty)$ which completes the proof. $\square$

LEMMA 4.6. *For $\tau > 0$, we have*:

(a) $\sup\{|\hat{D}_n^{(0,0)}(s) - D_n^{(0,0)}(s)| : s \geq 0\}|\mathbf{1}_{\{L(0,0)>n\tau\}} \xrightarrow{P} 0$ *as* $n \to \infty$,

(b) $\sup\{|K_n^{(0,0)}(s) - pD_n^{(0,0)}(s)| : s \geq 0\}|\mathbf{1}_{\{L(0,0)>n\tau\}} \xrightarrow{P} 0$ *as* $n \to \infty$.

PROOF. For part (a), fix $0 < \alpha < 1/2$, $T \geq 0$ and we observe that

$$\begin{aligned}
&\mathbb{P}\big(\sup\big\{\big|\hat{D}_k(0,0) - D_k(0,0)\big| : k \geq 0\big\} \geq n^\alpha, L(0,0) > n\tau\big) \\
&\quad \leq \mathbb{P}\big(\max\big\{\big|\hat{D}_k(0,0) - D_k(0,0)\big| : 0 \leq k \leq n(\tau + T) + 1\big\} \geq n^\alpha, \\
&\qquad L(0,0) > n\tau\big) + \mathbb{P}\big(L(0,0) > n(\tau + T)\big).
\end{aligned}$$

Because of Theorem 1.2, it is enough to show that $\sqrt{n}\mathbb{P}(\max\{|\hat{D}_k(0,0) - D_k(0,0)| : 0 \leq k \leq n(\tau + T) + 1\} \geq n^\alpha, L(0,0) > n\tau) \to 0$ as $n \to \infty$. Here, we present the simple idea behind the proof; the details are available in Roy, Saha and Sarkar [27].

The distance d_k^l between $l_k(0,0)$ and the closest open vertex to the left of $l_k(0,0)$ being n^α or more has a probability $(1-p)^{n^\alpha}$. Thus, the probability that the maximum such difference for $0 \le k \le n(\tau+T)+1$ is bigger that n^α is of the order $n(1-p)^{n^\alpha}$. Similarly, for the distance d_k^r associated with the vertex $r_k(0,0)$. Since $|\hat{D}_k(0,0) - D_k(0,0)| \le d_k^l + d_k^r$, as $n \to \infty$, we have that $\sqrt{n}\mathbb{P}(\max\{|\hat{D}_k(0,0) - D_k(0,0)| : 0 \le k \le n(\tau+T)+1\} \ge n^\alpha, L(0,0) > n\tau)$ converges to 0.

For part (b) of the lemma, we need $D_n^{(0,0)}|\mathbf{1}_{\{L(0,0)>n\tau\}} \Rightarrow \sqrt{2}W^{+,\tau}$ as $n \to \infty$ which follows from part (a) and Lemma 4.5. Hence, $r_k(0,0) - l_k(0,0)$ is of the order $\sqrt{n}$. Also given $l_k(0,0)$ and $r_k(0,0)$, the number of open vertices lying between these vertices has a binomial distribution with parameters $(r_k(0,0) - l_k(0,0) - 1)$ and p. Since these open vertices together with $l_k(0,0)$ and $r_k(0,0)$ constitute $C_k(0,0)$, the proof follows from similar order comparisons as done in (a). □

PROOF OF THEOREM 1.3. We remarked that $W^1|_{[0,1]} = W^{+,1}|_{[0,1]} \overset{d}{=} W^+$. The proof of Theorem 1.3 follows from Lemmas 4.5 and 4.6 and Slutsky's theorem with the choice of $\tau = 1$. □

4.2. *Proof of Theorem* 1.4. For $\lambda > 0$, let $\bar{\lambda} := \lambda^{3/2}(\sqrt{2}\gamma_0 p)^{-1}$. We show that:

LEMMA 4.7. *For* $\tau, \lambda > 0$,

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}\left(L(0,0) > n\tau, \sum_{k=0}^{\infty} \#C_k(0,0) > (\lambda n)^{3/2}\right)$$
$$= \frac{1}{\gamma_0\sqrt{\pi\tau}}\mathbb{P}\left(\sqrt{2}\int_0^\infty W^{+,\tau}(t)\,dt > \bar{\lambda}\right)$$
$$= \frac{1}{2\gamma_0\sqrt{\pi}}\int_\tau^\infty \overline{F}_{I_0^+}(\bar{\lambda}t^{-3/2})t^{-3/2}\,dt.$$

PROOF. For $f \in C[0,\infty)$ let $I(f) := \int_0^\infty H(f)(t)\,dt$. Since $\mathbb{P}(W^\tau \in A) = 1$ where A is defined as in (21), I is almost surely continuous under the measure induced by W^τ on $C[0,\infty)$. The proof follows from Theorem 1.3(ii) and the continuous mapping theorem. □

From the previous lemma, we derive the following.

COROLLARY 4.7.1. *For* $\lambda > 0$, *we have*

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}(\#C(0,0) > (\lambda n)^{3/2}) = \frac{1}{2\gamma_0\sqrt{\pi}}\int_0^\infty \overline{F}_{I_0^+}(\bar{\lambda}t^{-3/2})t^{-3/2}\,dt.$$

PROOF. For any $\tau > 0$, we have $\mathbb{P}(\#C(0,0) > (n\lambda)^{3/2}) \geq \mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{3/2})$, and hence $\liminf_{n\to\infty} \sqrt{n}\mathbb{P}(\#C(0,0) > (n\lambda)^{3/2}) \geq \frac{1}{2\gamma_0\sqrt{\pi}} \int_0^\infty \overline{F}_{I_0^+}(\bar{\lambda}t^{-3/2})t^{-3/2}\,dt$.

We observe that

$$\begin{aligned}
&\sqrt{n}\mathbb{P}\big(L(0,0) \leq n\tau, \#C(0,0) > (n\lambda)^{3/2}\big)\\
&\quad \leq \sqrt{n}\mathbb{P}\left(\sum_{k=0}^{\lfloor n\tau\rfloor} \hat{D}_k(0,0) > (n\lambda)^{3/2}\right)\\
&\quad \leq \sqrt{n}\mathbb{E}\left[\sum_{k=0}^{\lfloor n\tau\rfloor} \widehat{D}_k(0,0)\right](n\lambda)^{-3/2}\\
&\quad = \sqrt{n}\big(\lfloor n\tau\rfloor + 1\big)\mathbb{E}\big(\widehat{D}_0(0,0)\big)(n\lambda)^{-3/2},
\end{aligned}$$

where we have used the fact that $\{\hat{D}_k(0,0) = \hat{h}^k(\hat{r}(0,0))(1) - \hat{h}^k(\hat{l}(0,0))(1) : k \geq 0\}$ is a martingale (see Proposition 2.3). From the earlier discussions, it also follows that $\mathbb{E}(\widehat{D}_0(0,0)) \leq 2\mathbb{E}(G) = 2(1-p)p^{-1}$ where G is a geometric random variable. Thus, $\limsup_{n\to\infty} \sqrt{n}\mathbb{P}(L(0,0) \leq n\tau, \#C(0,0) > (n\lambda)^{3/2}) = 0$ as $\tau \to 0$, which completes the proof. $\square$

PROOF OF THEOREM 1.4. We first recall the result Lemma 6.1 of Resnick [25], page 174 which states that for nonnegative Radon measures $\mu, \mu_n, n \geq 1$, on $[0,\infty)^d \setminus \{\mathbf{0}\}$ we have $\mu_n \xrightarrow{v} \mu$ if and only if $\mu_n([0,x_1] \times \cdots \times [0,x_d])^c \to \mu([0,x_1] \times \cdots \times [0,x_d])^c$ for all $x_1,\dots,x_d \geq 0$ with $(x_1,\dots,x_d) \neq \mathbf{0}$. This result implies that Lemma 4.7 together with Corollary 4.7.1 and Theorem 1.2 prove (6).

Fix $\tau > 0$, $\lambda > 0$. For $\alpha < 2/3$, $\delta > 0$ and for all large n, we have $\mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) \leq \mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\delta)^{3/2})$. Fix any $\varepsilon > 0$ and choose $\delta = \delta(\varepsilon) > 0$ so that $\frac{1}{\gamma_0\sqrt{\pi\tau}}\mathbb{P}(\sqrt{2}\int_0^\infty W^{+,\tau}(t)\,dt > \overline{\delta}) < \varepsilon$, where $\overline{\delta} = \delta^{3/2}(\gamma_0 p)^{-1}$. From Lemma 4.7, we have

$$\limsup_{n\to\infty} \sqrt{n}\mathbb{P}\big(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}\big) < \varepsilon.$$

On the other hand, from the properties of W^+ and W^τ, it follows that $\mathbb{P}(\int_0^\infty W^{+,\tau}(t)\,dt > 0) = 1$ for $\tau > 0$. Now for $\alpha > 2/3$ and $\delta > 0$ we have $\mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) \geq \mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\delta)^{3/2})$ for all large n. Again from Lemma 4.7, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \sqrt{n}\mathbb{P}\big(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}\big)\\
&\quad \geq \frac{1}{\gamma_0\sqrt{\pi\tau}}\mathbb{P}\left(\sqrt{2}\int_0^\infty W^{+,\tau}(t)\,dt > \overline{\delta}\right).
\end{aligned}$$

Since

$$\limsup_{n\to\infty} \sqrt{n}\mathbb{P}\big(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}\big)$$
$$\leq \lim_{n\to\infty} \sqrt{n}\mathbb{P}\big(L(0,0) > n\tau\big) = \frac{1}{\gamma_0\sqrt{\pi\tau}},$$

letting $\delta \to 0$, we have $\lim_{n\to\infty} \sqrt{n}\mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) = \frac{1}{\gamma_0\sqrt{\pi\tau}}$ for $\alpha > 2/3$. This completes the proof of (8).

The argument for $(L(0,0), (D_{\max}(0,0))^{1/2})$ being similar is omitted. □

Acknowledgements. Kumarjit Saha is grateful to the Indian Statistical Institute for a fellowship to pursue his Ph.D. The authors also thank the referee for comments which led to a significant improvement of the paper.

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