Electronic Journal of Statistics
Vol. 8 (2014) 2557–2584
ISSN: 1935-7524
DOI: 10.1214/14-EJS961

A Kolmogorov-Smirnov type test for independence between marks and points of marked point processes*

Tonglin Zhang

Department of Statistics, Purdue University
250 North University Street, West Lafayette, IN 47907-2066
e-mail: tlzhang@purdue.edu

Abstract: Marked point processes are commonly used stochastic models for representing a finite number of natural hazard events located in space and time, because these kinds of data often associate measurements (i.e. marks) with locations (i.e. points) of events. Methods of marked point processes when marks and points are interacting have been proposed, but it is still necessary to know whether the interaction must be considered. This article presents a Kolmogorov-Smirnov type method to test the independence between points and marks of marked point processes. The asymptotic distribution of the test statistic under a few weak regularity conditions is derived. According to the asymptotic result, a specific way to construct the test statistic is recommended as its null distribution can be approximated by the absolute maximum of the two-dimensional standard Brownian pillow. The simulation results and real data analyses demonstrated that the proposed method is powerful in detecting weak dependence between marks and points and performs well with a moderate sample size.

AMS 2000 subject classifications: Primary 62M30; secondary 62G10, 62G20.
Keywords and phrases: Asymptotic distribution, Brownian pillow and Brownian sheet, convergence in distribution, empirical processes, independence between points and marks, Kolmogorov-Smirnov type test, marked point processes.

Received December 2013.

1. Introduction

Marked point processes are important models for a wide variety of scientific disciplines. A typical way to describe a marked point process is to use an artificial order such that data can be represented as

$$\sum_{i=1}^{n} \epsilon_{(S_i, M_i)},$$

where n is the total number of events, S_i are the locations of points, M_i are the associated marks, and ϵ is the Dirac measure. Both spatial and spatiotemporal processes can be represented by the above expression: if S_i represents a spatial

*The author appreciates the suggestive comments from two anonymous referees, Associate Editor, and Editor of the journal, which significantly improved the article.

location, then the marked point process is purely spatial; if S_i represents a spatiotemporal location, then the marked point process is spatiotemporal.

Specific statistical methods include variogram analysis, various kinds of kriging, and geostatistical simulation techniques may be used to model a marked point process [7], but these methods rely on a fundamental assumption that point locations appear independently of marks because the correlation function used in a geostatistical method often ignores the point distributions [14]. However, the independence assumption is often violated in real world data. For instance, the relative positions of trees in a forest have repercussions on their size owing to their competition for light or nutrients [48], indicating that tree sizes and locations of trees may not be independent. Forest wildfire activities exhibit power-law relationships between frequency and burned area [30], indicating that the burned area and the locations of forest wildfires may not be independent. The purpose of this article is to explore a Kolmogorov-Smirnov type method to test the independence between points and marks of marked point processes.

It is especially convenient in modeling, estimation, and prediction in a marked point process if marks and points are independent. Many commonly used Hawkes models, such as the epidemic-type aftershock sequences (ETAS) model [43], may exhibit the independence between marks and points [50]. The **spatstat** [2] and **PtProcess** [18] packages in **R** have been developed for marked point processes. Methods of marked point processes when points and marks are interacted have been proposed [20, 32, 38]. However, it is still important to answer whether it is necessary to account for the interaction between points and marks. Recently, a few methods have been proposed to assess the independence between points and marks, including a test for stationarity and isotropy of a marked point process using variograms [1, 48], a nonparametric kernel-based test to assess the separability of the conditional intensity function [50], and a χ^2-based test to assess the interaction between points and marks for marked point processes with a stationary spatial projection of the marked point process [16]. These methods, however, contain kernel estimation of a nonparametric smoothing function which may result in a loss of power.

In this article, we propose a Kolmogorov-Smirnov type statistic ([53], p. 265) to test the independence between marks and points. Intuitively, the test statistic is derived by computing the absolute maximum difference between the joint empirical process and the product of the marginal empirical processes of marks and points, where the maximum is computed over a collection of subsets of the study area. We show that the choice of the collection of subsets affects the asymptotic distribution of the test statistic. With a careful choice of the collection, the asymptotic null distribution may be distribution-free and approximated by the absolute maximum of the two-dimensional standard Brownian pillow.

The Brownian pillow is an extension of the Brownian bridge from $[0, 1]$ to $[0, 1]^d$ for $d \geq 1$. It is the Brownian bridge if $d = 1$. Many test statistics for multivariate independence converge to the Brownian pillow under the null hypothesis [5, 11, 12, 39, 40, 49]. Many articles have evaluated the probability distribution of the Brownian bridge. However, there are just a few for the Brownian pillow. For example, analytical bounds for the maximum of the Brownian pillow were

derived [9, 15, 27, 29, 52], but there are few known results related to more important features of the Brownian pillow, such as the maximum absolute value [28]. Therefore, a simulation method is natural and convenient to be used.

Marked point processes are often used to model a number of natural hazard events located in space and time. Many successful applications of marked point processes can be found in the literature. These include marked point modeling and prediction of earthquakes, where each earthquake is represented by a magnitude and a space-time coordinate [21, 42, 54]. The three-dimensional space coordinate contains the longitude, latitude and depth of earthquake occurrences. Marked point processes for wildfires have been discussed by [45], where each wildfire is represented by its area burned and space-time coordinate. The two-dimensional space coordinate contains the longitude and latitude of wildfire occurrences. A few statistical methods of marked point processes with the independence assumption may be used [34, 46]. If the independence assumption is violated, one may consider intensity-dependent methods [20, 32, 38]. However, a test for the independence between marks and points is still important in the analysis.

To the best of our knowledge, the present article provides the first Kolmogorov-Smirnov type approach to assess the independence between marks and points of marked point processes. In principle, the Kolmogorov-Smirnov type approach used in this paper is nonparametric. It is easily implemented in all kinds of distributions of marks and points. The proposed method can be applied to both continuous and discrete marks. It is prudent, however, to start with the continuous marks for our asymptotic results because the core derivation does not contain complicated mathematical inferences.

The rest of the article is organized as follows. In Section 2, we briefly review the classical Kolmogorov-Smirnov test for independence. In Section 3, we propose the test statistic and derive its asymptotic distribution. In Section 4, we provide our simulation results, focusing on the marked Poisson process, the marked mixed Poisson process, and the marked Neyman-Scott cluster process. In Section 5, we apply our test statistic to real world data: the *Ambrosia dumosa data* [35] and the *Alberta wildfire data*. Our results showed that the marks and the points were not independent in both data, and a supplementary nonparametric method was proposed and applied to the *Alberta wildfire data* to account for dependence between points and marks. In the end, we provide a discussion.

2. Kolmogorov-Smirnov test for multivariate independence

The Kolmogorov-Smirnov test for multivariate independence has been discussed about fifty years ago (e.g. [5]). It usually considers a p-dimensional multivariate variable $\mathbf{x} = (X_1, \ldots, X_p)$ with the joint CDF $F(x_1, \ldots, x_p)$. Let F_j be the j-th marginal CDF of F. Then, the Kolmogorov-Smirnov test for multivariate independence is to assess the null hypothesis

$$H_0 : F(x_1, \ldots, x_p) = \prod_{j=1}^{p} F_j(x_j). \tag{2.1}$$

Suppose samples distributions of F are observed. Then, the Kolmogorov-Smirnov statistic for multivariate independence is

$$T_n = \sup_{x_1,\ldots,x_p\in\mathbb{R}} \sqrt{n}|\hat{F}(x_1,\ldots,x_p) - \prod_{j=1}^{p} \hat{F}_j(x_j)|, \tag{2.2}$$

where n is the sample size, $\hat{F}$ and $\hat{F}_j$ are the joint sample CDF and marginal sample CDF, respectively. It has been shown that under H_0 the limiting distribution of T_n is a p-dimensional Brownian pillow, which is a Gaussian random field on $[0,1]^p$ with mean 0 and covariance function equal to $\prod_{j=1}^{p}(t_j \wedge t'_j - t_j t'_j)$ for $0 < t_j, t'_j < 1$ with $j = 1,\ldots,p$. To understand the Brownian pillow, we briefly review the case when $p = 2$ below.

A Gaussian random field $W(u,v)$ on $[0,1]^2$ is called the *two-dimensional standard Brownian sheet* if $W[(0,v)] = W[(u,0)] = 0$, $E[W(u,v)] = 0$, and $E[W(u,v)W(u',v')] = (u \wedge u')(v \wedge v')$ for all $(u,v) \in [0,1]^2$, see [56]. Important results on sample path properties of a Brownian sheet were obtained and it has been shown that the sample paths of a Brownian sheet are continuous with probability one and $W(u,v)$ has independent stationary increments on $[0,1]^2$, see [8, 9, 44, 55]. Let

$$\tilde{W}(u,v) = W(u,v) - uW(1,v) - vW(u,1) + uvW(1,1). \tag{2.3}$$

Then, $\tilde{W}(u,v)$ is called the *two-dimensional standard Brownian pillow* (or the *two-dimensional tied-down Brownian sheet*) [12]. In general, the two-dimensional standard Brownian pillow $\tilde{W}(u,v)$ is a Gaussian random field on $[0,1]^2$ with $E[\tilde{W}(u,v)] = 0$, $\tilde{W}(u,0) = \tilde{W}(0,v) = \tilde{W}(u,1) = \tilde{W}(1,v) = 0$, and covariance function given by

$$E[\tilde{W}(u,v)\tilde{W}(u',v')] = (u \wedge u' - uu')(v \wedge v' - vv') \tag{2.4}$$

for $0 \le u,v,u',v' \le 1$. Neither the exact nor the approximate distribution of $\sup_{u,v\in[0,1]} |\tilde{W}(u,v)|$ has not been derived yet [26, 27, 28, 29].

We used a simulation method to evaluate the values of $\tilde{W}_\alpha$, where $\tilde{W}_\alpha$ is the upper α quantile of the distribution of $\sup_{u,v\in[0,1]} |\tilde{W}(u,v)|$. In the method, we independently generated m^2 random variables from $N(0,1/m^2)$. Let them be denoted by X_{ij} for $i,j = 1,\ldots,m$. Let $Y_{kl} = \sum_{i=1}^{k}\sum_{j=1}^{l} X_{ij}$ and $Z_{kl} = Y_{kl} - (l/m)Y_{km} - (k/m)Y_{ml} + (k/m)(l/m)Y_{mm}$. Then, $E(Z_{kl}) = 0$ and $E(Z_{kl}Z_{k'l'}) = (k \wedge k'/m - kk'/m^2)(l \wedge l'/m - ll'/m^2)$. Thus, the distribution of $\sup_{1\le k,l\le m} |Z_{kl}|$ was approximately equal to the distribution of $\sup_{0\le u,v\le 1} |\tilde{W}(u,v)|$ if m is large. In a simulation with 10^5 replications for $m = 1000$, we had $\tilde{W}_{0.1} = 0.7298$, $\tilde{W}_{0.05} = 0.7948$, and $\tilde{W}_{0.01} = 0.9234$. Since we had $P(\sup_{u,v\in[0,1]} |\tilde{W}(u,v)| > 1.5) < 10^{-5}$, we may conclude that the p-value is almost 0 if $T_n > 1.5$.

We attempt to modify the classical test statistic T_n displayed by Equation (2.2) such that the modified test can be used to assess the independence between points and marks of marked point processes. Note that the CDF used in the

definition of T_n cannot be naturally defined in a marked point process. We rephrase the definition of T_n such that it can be easily modified.

Let $\mathcal{R} = \{(-\infty, t], -\infty < t < \infty\}$ and $\mathscr{R}$ be the collection of all Borel sets in $\mathbb{R}$. Then, $\mathscr{R}$ can be generated by $\mathcal{R}$. Let μ be the joint probability measure on $\mathbb{R}^p$ satisfying

$$\mu\{(-\infty, x_1], \ldots, (-\infty, x_p]\} = F(x_1, \ldots, x_p).$$

Then, μ is uniquely determined. Let μ_j be the j-th marginal probability measure of μ. Then, μ_j satisfies

$$\mu_j\{(-\infty, x_j]\} = F_j(x_j).$$

The null hypothesis given by Equation (2.1) can be rephrased as

$$H_0 : \mu\{(-\infty, x_1], \ldots, (-\infty, x_p]\} = \prod_{j=1}^{p} \mu_j\{(-\infty, x_j]\}$$

for all $x_1, \ldots, x_p$, which is also equivalent to

$$H_0 : \mu\{A_1, \ldots, A_p\} = \prod_{j=1}^{p} \mu_j\{A_j\}$$

for all $A_1, \ldots, A_p \in \mathcal{R}$. The test statistic T_n given by Equation (2.2) can be written as

$$T_n = \sup_{A_1, \ldots A_p \in \mathcal{R}} \sqrt{n} |\hat{\mu}(A_1, \ldots, A_p) - \prod_{j=1}^{p} \hat{\mu}_j(A_j)|, \tag{2.5}$$

where $\hat{\mu}$ and $\hat{\mu}_j$ are the sample measures of μ and μ_j, respectively. In the following of this paper, we focus on a modification of T_n given by (2.5) such that it can be used to test the independence between points and marks in a marked point process.

3. Method

The definition of marked point processes is well-established and can be found in [10, 25]. In general, a marked point process is a point process defined on the product space of points and marks, but the concept has its own life in applications. To well express our method, we introduce the following notations. Let $\|\cdot\|$ be the L_2–norm and $\|\cdot\|_\gamma$ be the L_γ–norm for $\gamma > 1$ over an Euclidean space. Let ϕ be the empty set, $\bar{C}$ be the complementary set of C. Denote $x \wedge x' = \min(x, x')$ if $x, x' \in \mathbb{R}$, $\mathbf{x} \wedge \mathbf{x}' = (x_1 \wedge x'_1, \ldots, x_k \wedge x'_k)$ if $x, x' \in \mathbb{R}^k$, and write $\mathbf{x} \preceq \mathbf{x}'$ if $x_i \leq x'_i$ for all $i = 1, \ldots, k$, where x_i and x'_i are the i-th component of $\mathbf{x}$ and $\mathbf{x}'$, respectively.

3.1. Statistical formulation

A marked point process $N = (N_s, N_m)$ with points in a complete separable metric space $\mathcal{S}$ and marks in a complete separable metric space $\mathcal{M}$ is a point process on $\mathcal{S} \times \mathcal{M}$ with the additional property that the spatial projection of

marked point process N_s is itself a point process and for any bounded $A \in \mathcal{S}$ there is $N_s(A) = N(A \times \mathcal{M}) < \infty$, where $N(A \times B)$ is number of points in $A \times B$ for Borel sets $A \subseteq \mathcal{S}$ and $B \subseteq \mathcal{M}$. In modeling the occurrence of ecological or geographical events when depth is not involved, we may have $\mathcal{S} = \mathbb{R}^d$ with $d = 2$ if time is not considered or $d = 3$ if time is considered.

The distribution of a marked point process can be mathematically defined using the idea for a purely point process [37]. Let $n = N(\mathcal{S} \times \mathcal{M})$ be the total number of observations and assume n is finite. Then, n is a discrete random variable. Assume $n \geq 1$ and observations are given by an artificial order such that the data can be expressed into $\{(S_i, M_i) : i = 1, \ldots, n\}$. Then, a probability distribution π_n can be defined on the Borel sets of $(\mathcal{S} \times \mathcal{M})^n$ as ([4], p. 232)

$$\pi_n(A_1 \times B_1, \ldots, A_n \times B_n) = P((S_1, M_1) \in A_1 \times B_1, \ldots, (S_n, M_n) \in A_n \times B_n).$$

To be consistent with treating the marked point process as a theory of unordered data, π_n should be permutation invariant as

$$\pi_n(A_1 \times B_1, \ldots, A_n \times B_n) = \pi_n(A_{i_1} \times B_{i_1}, \ldots, A_{i_n} \times B_{i_n}),$$

where $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$. Let

$$\pi_{s,n}(A_1, \ldots, A_n) = \pi_n(A_1 \times \mathcal{M}, \ldots, A_n \times \mathcal{M})$$

and

$$\pi_{m,n}(B_1, \ldots, B_n) = \pi_n(\mathcal{S} \times B_1, \ldots, \mathcal{S} \times B_n).$$

Then, $\pi_{s,n}$ is the marginal distribution of points and $\pi_{m,n}$ is the marginal distribution of marks.

Definition 1. The marked point process N is independent if for all $n \geq 1$ there is

$$\pi_n(A_1 \times B_1, \ldots, A_n \times B_n) = \pi_{s,n}(A_1, \ldots, A_n)\pi_{m,n}(B_1, \ldots, B_n), \tag{3.1}$$

for any Borel sets $A_1, \ldots, A_n \in \mathscr{S}$ and $B_1, \ldots, B_n \in \mathscr{M}$, where $\mathscr{S}$ and $\mathscr{M}$ are the collections of all Borel sets of $\mathcal{S}$ and $\mathcal{M}$, respectively.

In applications, it is often assumed that events appear independently such that data can be modeled by a marked Poisson process. For Poisson point processes with simple ground processes (i.e., with no two points at exactly the same location), the intensity (if it exists) uniquely characterizes all of the finite-dimensional distribution of the process [10, 24]. For a marked Poisson process, one may consider a similar intensity function defined as the expected rate of occurrences at a certain location of points with a certain value of marks [45]. This idea has been used to define a marked Poisson process by [50], which has been summarized in the following.

Definition 2. The marked point process N is called a marked Poisson process if there exists a Borel measure μ on $\mathcal{S} \times \mathcal{M}$ such that for any disjoint $C_1, \ldots, C_k$, $N(C_1), \ldots, N(C_k)$ are independent Poisson distributed with expected values $\mu(C_i)$ for $i = 1, \ldots, k$, respectively.

Proposition 1. *The necessary condition for N to be independent is*

$$\pi(A \times B) = \pi_s(A)\pi_m(B) \tag{3.2}$$

for any $A \in \mathscr{S}$ and $B \in \mathscr{M}$, where $\pi(A \times B) = \pi_1(A \times B)$, $\pi_s(A) = \pi_{s,1}(A)$, and $\pi_m(B) = \pi_{m,1}(B)$. If N is a marked Poisson process, then condition (3.2) is also sufficient.

It can be seen that all marked point processes with intensity-dependent marks violate Equation (3.2). However, marked point processes with intensity-independent marks may also exhibit dependence between points and marks. Therefore, Equation (3.2) is only a special case of Equation (3.1). In applications, it is often assumed that events appear independently so that data can be modeled by a marked Poisson process. For Poisson point processes with simple ground processes, the first-order intensity uniquely characterizes all of the finite-dimensional distribution of the process.

3.2. Derivation of the test statistic

In order to avoid the complexity of Equation (3.1), our interest is to investigate whether Equation (3.2) holds. We consider the null hypothesis

$$H_0 : \varphi(A, B) = 0 \tag{3.3}$$

for any $A \in \mathscr{S}$ and $B \in \mathscr{M}$ against the alternative hypothesis

$$H_1 : \varphi(A, B) \neq 0 \tag{3.4}$$

for some $A \in \mathscr{S}$ and $B \in \mathscr{M}$, where $\varphi(A, B) = \pi(A \times B) - \pi_s(A)\pi_m(B)$.

Lemma 1. *A necessary condition for Equation (3.2) to be held is that $\varphi(A, B) = 0$ for any $A \in \mathcal{A} \subseteq \mathscr{S}$ and $B \in \mathcal{B} \subseteq \mathscr{M}$. If $\mathscr{S}$ and $\mathscr{M}$ can be generated by $\mathcal{A}$ and $\mathcal{B}$, respectively, then the condition is also sufficient.*

Note that $\pi(A \times B)$ is the joint probability distribution for points in A and marks in B, $\pi_s(A)$ is the marginal probability for points in A, and $\pi_m(B)$ is the marginal probability for marks in B. If $n \geq 1$, then they can be estimated by $N(A \times B)/n$, $N_s(A)/n$, and $N_m(B)/n$, respectively. Let

$$\hat{\varphi}_n(A, B) = \hat{\pi}_n(A \times B) - \hat{\pi}_{s,n}(A)\hat{\pi}_{m,n}(B), \tag{3.5}$$

where $\hat{\pi}_n(A \times B) = N(A \times B)/n$, $\hat{\pi}_{s,n}(A) = N_s(A)/n$, and $\hat{\pi}_{m,n}(B) = N_m(B)/n$ if $n > 0$ and $\hat{\pi}_n(A \times B) = \hat{\pi}_{s,n}(A) = \hat{\pi}_{m,n}(B) = 0$ if $n = 0$. Then, $\hat{\varphi}_n(A, B)$ is an estimator of $\varphi(A, B)$. Our test statistic therefore is

$$T_n = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} \sqrt{n} |\hat{\pi}_n(A \times B) - \hat{\pi}_{s,n}(A)\hat{\pi}_{m,n}(B)|, \tag{3.6}$$

where $\mathcal{A} \subseteq \mathscr{S}$ and $\mathcal{B} \subseteq \mathscr{M}$. Then, T_n is a *Kolmogorov-Smirnov* type test statistic with $T_n = 0$ if $n = 0$. It can be seen that the test statistic T_n given by Equation (3.6) is special case of Equation (2.5). It is recommended to reject H_0 if T_n is large.

Even though there are many ways to choose $\mathcal{A}$ and $\mathcal{B}$ in Equation (3.6), it is not necessary to consider all of them. In this article, we only recommend to consider two of them. In the first, we choose $\mathcal{A}$ and $\mathcal{B}$ such that $\mathscr{S}$ and $\mathscr{M}$ can be generated by $\mathcal{A}$ and $\mathcal{B}$, respectively. In the second method, we choose $\mathcal{A}$ and $\mathcal{B}$ such that the p-value can be easily derived using the limiting distribution. Because the second method is more convenient in applications, we focus on the second method in the rest of the paper. The detail derivations of the two methods will be discussed in the next subsection.

3.3. Asymptotics

We derive the asymptotic properties of T_n in this subsection. The asymptotic properties include the asymptotic null distribution, consistency, and asymptotic power functions. These properties are particularly considered in the marked Poisson process as it has independent increments ([25], p. 6), which is an important assumption in our main conclusion for asymptotic power functions given by Theorem 2. If the marked point process is not Poisson, then we can only provide the asymptotic null distribution since the basic theorem that we cite requires N_s be strong stationary.

3.3.1. Asymptotics in marked poisson processes

Let $\varphi_{\mathcal{A},\mathcal{B}}$ be the vector composed by $\varphi(A,B)$ and $\hat{\varphi}_{\mathcal{A},\mathcal{B},n}$ be the vector composed by $\hat{\varphi}_n(A,B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, respectively. Then, $\varphi_{\mathcal{A},\mathcal{B}}$ and $\hat{\varphi}_{\mathcal{A},\mathcal{B},n}$ are finite if both $\mathcal{A}$ and $\mathcal{B}$ are finite; otherwise they are infinite. For example, if $\mathcal{A} = \{A_1, \ldots, A_I\}$ and $\mathcal{B} = \{B_1, \ldots, B_J\}$, then

$$\varphi_{\mathcal{A},\mathcal{B}} = (\varphi(A_1,B_1), \varphi(A_1,B_2), \ldots, \varphi(A_I,B_J)) \tag{3.7}$$

and

$$\hat{\varphi}_{\mathcal{A},\mathcal{B},n} = (\hat{\varphi}_n(A_1,B_1), \hat{\varphi}_n(A_1,B_2), \ldots, \hat{\varphi}_n(A_I,B_J)) \tag{3.8}$$

are both IJ-dimensional vectors.

In order to derive the asymptotic distribution of T_n, we impose the following regularity conditions:

(C1) N is a marked Poisson process.
(C2) The intensity function of N_s has the form of $\lambda_s(\mathbf{s}) = \kappa\lambda_0(\mathbf{s})$, where κ is a finite positive number and $\int_{\mathcal{S}} \lambda_0(\mathbf{s})d\mathbf{s} = 1$.
(C3) Conditioning on points, the marks are continuous random variables with conditional density functions given by $\lambda_1(\mathbf{m}|\mathbf{s}) = \lambda(\mathbf{s},\mathbf{m})/\lambda_s(\mathbf{s})$ for $\mathbf{m} \in \mathcal{M}$ and $\mathbf{s} \in \mathcal{S}$.

Under Conditions (C1)–(C3), we have $N_\kappa(A \times B) \sim Poisson(\kappa\pi(A \times B))$, $N_\kappa(A\times B)|n \sim Bin(n, \pi(A\times B))$ if $n \neq 0$, and $\pi(A\times B) = \int_{A\times B} \lambda(\mathbf{s},\mathbf{m})d\mathbf{s}d\mathbf{m}/\kappa$ if $A \subseteq \mathcal{S}$. We denote $\xrightarrow{P}$ as convergence in probability and $\xrightarrow{D}$ as convergence in distribution as $\kappa \to \infty$.

Lemma 2. *If Conditions (C1)–(C3) hold, then* $\hat{\varphi}_n(A,B) \xrightarrow{P} \varphi(A,B)$ *for any* $A \in \mathscr{S}$ *and* $B \in \mathscr{M}$.

Lemma 3. *Assume Conditions (C1)–(C3) hold. Let* $\varphi_{\mathcal{A},\mathcal{B}}$ *and* $\hat{\varphi}_{\mathcal{A},\mathcal{B},n}$ *be given by (3.7) and (3.8) for* $\mathcal{A} = \{A_1, \ldots, A_I\} \subseteq \mathscr{S}$ *and* $\mathcal{B} = \{B_1, \ldots, B_J\} \subseteq \mathscr{M}$, *respectively. Let* $\nu_{1i} = \pi(C_i \times D_i)$, $\nu_{2j} = \pi(C_j \times D_j)$, $u_{ij} = \pi[(C_i \cap C_j) \times (D_i \cap D_j)]$ *for* $i,j = 1,2,3,4$, *where* $C_i = A_{i_1}$ *or* $C_j = A_{i_2}$ *if* $i = 1,2$ *or* $j = 1,2$, $C_i = \bar{A}_{i_1}$ *or* $C_j = \bar{A}_{i_2}$ *if* $i = 3,4$ *or* $j = 3,4$, $D_i = B_{j_1}$ *or* $D_j = B_{j_2}$ *if* $i = 1,3$ *or* $j = 1,3$, *and* $D_i = \bar{B}_{j_1}$ *or* $D_j = \bar{B}_{j_2}$ *if* $i = 2,4$ *or* $j = 2,4$ *for* $i_1, i_2 = 1, \ldots, I$ *and* $j_1, j_2 = 1, \ldots, J$. *Then*

$$\sqrt{n}(\hat{\varphi}_{\mathcal{A},\mathcal{B},n} - \varphi_{\mathcal{A},\mathcal{B}}) \xrightarrow{D} \mathcal{N}(0, \Sigma), \tag{3.9}$$

where

$$\Sigma = \begin{pmatrix} \sigma_{A_1,B_1,A_1,B_1} & \sigma_{A_1,B_1,A_1,B_2} & \cdots & \sigma_{A_1,B_1,A_I,B_J} \\ \sigma_{A_1,B_2,A_1,B_1} & \sigma_{A_1,B_2,A_1,B_2} & \cdots & \sigma_{A_1,B_2,A_I,B_J} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{A_I,B_J,A_1,B_1} & \sigma_{A_I,B_J,A_1,B_2} & \cdots & \sigma_{A_I,B_J,A_I,B_J} \end{pmatrix}$$

with

$$\begin{aligned} &\sigma_{A_{i_1},B_{j_1},A_{i_2},B_{j_2}} \\ &= 4\varphi(A_{i_1},B_{j_1})\varphi(A_{i_2},B_{j_2}) + 4\varphi(A_{i_1},B_{j_1})(\nu_{21}\nu_{24} - \nu_{22}\nu_{23}) \\ &\quad + 4\varphi(A_{i_2},B_{j_2})(\nu_{11}\nu_{14} - \nu_{12}\nu_{13}) \\ &\quad + \nu_{14}(u_{11}\nu_{24} - u_{12}\nu_{23} - u_{13}\nu_{22} + u_{14}\nu_{21}) \\ &\quad - \nu_{13}(u_{21}\nu_{24} - u_{22}\nu_{23} - u_{23}\nu_{22} + u_{24}\nu_{21}) \\ &\quad - \nu_{12}(u_{31}\nu_{24} - u_{32}\nu_{23} - u_{33}\nu_{22} + u_{34}\nu_{21}) \\ &\quad + \nu_{11}(u_{41}\nu_{24} - u_{42}\nu_{23} - u_{43}\nu_{22} + u_{44}\nu_{21}). \end{aligned} \tag{3.10}$$

If Equation (3.2) holds, then

$$\begin{aligned} \sigma_{A_{i_1},B_{j_1},A_{i_2},B_{j_2}} &= [\pi_s(A_{i_1} \cap A_{i_2}) - \pi_s(A_{i_1})\pi_s(A_{i_2})] \\ &\quad \times [\pi_m(B_{j_1} \cap B_{j_2}) - \pi_m(B_{j_1})\pi_m(B_{j_2})]. \end{aligned} \tag{3.11}$$

Corollary 1. *If Conditions (C1)–(C3) hold, then for any* $A \in \mathscr{S}$ *and* $B \in \mathscr{M}$ *there is* $\sqrt{n}[\hat{\varphi}_n(A,B) - \varphi(A,B)] \xrightarrow{D} N(0, \sigma^2_{A,B})$, *where*

$$\begin{aligned} \sigma^2_{A,B} &= 4\varphi^2(A,B) + 8\varphi(A,B)[\pi(A \times \bar{B})\pi(\bar{A} \times B) - \pi(A \times B)\pi(\bar{A} \times \bar{B})] \\ &\quad + [\pi^2(A \times B)\pi(\bar{A} \times \bar{B}) + \pi(A \times B)\pi^2(\bar{A} \times \bar{B}) \\ &\quad + \pi^2(A \times \bar{B})\pi(\bar{A} \times B) + \pi(A \times \bar{B})\pi^2(\bar{A} \times B)]. \end{aligned} \tag{3.12}$$

Theorem 1. *If the null hypothesis of Equation (3.3) is violated, then there is* $\lim_{\kappa \to \infty} P(T_n > x) = 1$ *for any* $x > 0$.

Corollary 2 (Consistency)**.** *Suppose Conditions (C1)–(C3) hold. Assume* $\mathscr{S}$ *and* $\mathscr{M}$ *can be generated by* $\mathcal{A}$ *and* $\mathcal{B}$, *respectively. If the null hypothesis given by Equation (3.3) is violated, then* $\lim_{\kappa \to \infty} P(T_n > x) = 1$ *for any* $x > 0$.

Theorem 2 (Asymptotic performance of the power function)**.** *Suppose Conditions (C1)–(C3) hold. Denote* $A_{\mathbf{a}}$ *and* $B_{\mathbf{b}}$ *as sets indexed by* $\mathbf{a} = (a_1, \ldots, a_\alpha) \in \mathbb{R}^\alpha$ *and* $\mathbf{b} = (b_1, \ldots, b_\beta) \in \mathbb{R}^\beta$*, respectively, where* α *and* β *are positive integers. Let* $\mathcal{A} = \{A_{\mathbf{a}} : \mathbf{a} \in \mathbb{R}^\alpha\}$ *and* $\mathcal{B} = \{B_{\mathbf{b}} : \mathbf{b} \in \mathbb{R}^\beta\}$*. Assume*

(A1) $A_{\mathbf{a}'} \subseteq A_{\mathbf{a}}$ *if* $\mathbf{a}' \preceq \mathbf{a}$ *and* $B_{\mathbf{b}'} \subseteq B_{\mathbf{b}}$ *if* $\mathbf{b}' \preceq \mathbf{b}$*;*
(A2) $\lim_{\|\mathbf{a}'-\mathbf{a}\|\to 0} A_{\mathbf{a}'} = A_{\mathbf{a}}$ *and* $\lim_{\|\mathbf{b}'-\mathbf{b}\|\to 0} B_{\mathbf{b}'} = B_{\mathbf{b}}$*;*
(A3) $\lim_{a_i\to-\infty} A_{\mathbf{a}} = \lim_{b_j\to-\infty} B_{\mathbf{b}} = \phi$ *for any* i *and* j*; and*
(A4) $\lim_{a_1\to\infty,\ldots,a_\alpha\to\infty} A_{\mathbf{a}} = \mathcal{S}$ *and* $\lim_{b_1\to\infty,\ldots,b_\beta\to\infty} B_{\mathbf{b}} = \mathcal{M}$*.*

Then,

$$\sup_{A\in\mathcal{A},B\in\mathcal{B}} \sqrt{n}|\hat{\varphi}_{\mathcal{A},\mathcal{B},n} - \varphi_{\mathcal{A},\mathcal{B}}| \xrightarrow{D} \sup_{A\in A,B\in\mathcal{B}} |Z_{A,B}|, \tag{3.13}$$

where $Z_{A,B}$ *is a Gaussian random field with mean* 0 *and covariance function* $\sigma_{A,B,A',B'}$ *for any* $A, A' \in \mathcal{A}$ *and* $B, B' \in \mathcal{B}$*.*

Theorem 2 can be used to derive the asymptotic distribution of T_n under both the null and alternative hypotheses. However, the derivation of T_n under the null hypothesis is more important since it is related to the p-value of the test. If the null hypothesis holds, then the conditional distribution of M does not depend on its location, which implies that $\lambda_1(\mathbf{m}|\mathbf{s})$ is independent of $\mathbf{s}$. Using Condition (C1), an iid sample of the marks is derived.

Theorem 3 (Asymptotic null distribution)**.** *Assume all conditions in Theorem 2 hold. If the null hypothesis given by Equation (3.3) also holds, then*

$$\lim_{\kappa\to\infty} P(T_n \geq x) = P(\sup_{A\in\mathcal{A},B\in\mathcal{B}} |Z_{A,B}| \geq x)$$

for all $x > 0$*, where* $Z_{A,B}$ *is a Gaussian random field with mean* 0 *and covariance function*

$$\tau_{A,B,A'B'} = [\pi_s(A \cap A') - \pi_s(A)\pi_s(A')][\pi_m(B \cap B') - \pi_m(B)\pi_m(B')]$$

for any $A, A' \in \mathcal{A}$ *and* $B, B' \in \mathcal{B}$*.*

It is generally not easy to directly apply Theorem 3 to a real-world application because the asymptotic null distribution of T_n depends on π_s and π_m, which are unknown. If we choose specific $\mathcal{A}$ and $\mathcal{B}$, then the asymptotic null distribution of T_n does not depend on π_s and π_m any more. Then, the p-value of T_n can be easily derived using Theorem 3. The method is given below.

Corollary 3. *Assume all conditions in Theorem 3 hold. If* $\alpha = \beta = 1$ *in Condition (A1) of Theorem 2, then*

$$T_n \xrightarrow{D} \sup_{u,v\in[0,1]} |\tilde{W}(u, v)|,$$

where $\tilde{W}$ *is the standard Brownian pillow on* $[0, 1]^2$*.*

3.3.2. Asymptotic null distribution for stationary marked point processes

Even though we have derived the asymptotic null distribution and asymptotic power functions for marked Poisson process, it is generally hard to derive the asymptotic power functions when the marked point process is not Poisson. Therefore, we decide to only provide the limiting distribution of T_n under the null hypothesis. To derive the limiting distribution of T_n for a marked Point process which is not Poisson, we need to assume that N_s is stationary and satisfies the strong mixing condition.

Let N_s be a spatial point process on $\mathcal{S} = \mathbb{R}^d$. We say N_s is *strong stationary* if the joint distribution of $(N_s(A_1), \ldots, N_s(A_k))$ is equal to the joint distribution of $(N_s(A_1+\mathbf{s}), \ldots, N_s(A_k+\mathbf{s}))$ for any $A_1, \ldots, A_k \in \mathscr{S}$, where $A_i+\mathbf{s} = \{\mathbf{s}'+\mathbf{s} : \mathbf{s}' \in A_i\}$. Assume the kth-order intensity function of N_s exists. Then, the logarithm of the probability generating function of N_s is defined as

$$\psi(\zeta) = \log\left[Ee^{\int_{\mathbb{R}^d} \log \zeta(\mathbf{s}) dN_s(\mathbf{s})}\right],$$

where ζ, $0 \le \zeta \le 1$, is a function with compact support on $\mathbb{R}^d$. The kth-order factorial cumulant measure of N_s is defined as

$$C_{(k)}(A_1 \times \cdots \times A_k) = \lim_{\eta \uparrow 1}\left[\frac{\partial^k}{\partial a_1 \cdots \partial a_k}\psi\left(\eta + \sum_{i=1}^{k} a_i I_{A_i}\right)\right]_{a_1=\cdots=a_k=0}, \quad (3.14)$$

where I_A is the indicator function of $A \in \mathscr{B}(\mathbb{R}^d)$. If $C_{(k)}$ is absolutely continuous, then its kth-order density function, denoted by $Q_k(\mathbf{s}_1, \ldots, \mathbf{s}_k)$ for distinct $\mathbf{s}_1, \ldots, \mathbf{s}_k \in \mathbb{R}^d$, is called the kth-order factorial cumulant density of N_s. Let P be the distribution of N_s. We say N_s satisfies the *strong mixing condition* if for any positive r and d there is

$$\lim_{a \to \infty} \xi(ar, ad) = 0, \quad (3.15)$$

where

$$\xi(r, d) = \sup_{\substack{d(E_1, E_2) \ge r \\ d(E_1) \le d,\, d(E_2) \le d}} \sup_{\substack{U_1 \in \mathscr{F}(E_1) \\ U_2 \in \mathscr{F}(E_2)}} |P(U_1 \cap U_2) - P(U_1)P(U_2)|,$$

$d(E) = \sup_{\mathbf{s},\mathbf{s}' \in E} \rho(\mathbf{s}, \mathbf{s}')$ is the diameter of E, $d(E_1, E_2) = \sup_{\mathbf{s} \in E, \mathbf{s}' \in E'} \rho(\mathbf{s}, \mathbf{s}')$ is the maximum distance between disjoint sets of points and ρ is the Euclidean distance function. The detailed interpretation of those definitions can be found in [23, 37].

The asymptotic null distribution of T_n is also investigated by assuming $E(n) \to \infty$. To explain the condition $E(n) \to \infty$, we write $\mathcal{S}_\kappa = [-\kappa, \kappa]^d$ and assume points are restricted on $\mathcal{S}_\kappa$. Denote $N_\kappa(A \times B) = N[(A \cap \mathcal{S}_\kappa) \times B]$ and $N_{s,\kappa}(A) = N_\kappa(A \times \mathcal{M})$. The asymptotic properties of T_n is evaluated by considering

$$T_{n_\kappa} = \sup_{A \in \mathcal{A}_\kappa, B \in \mathcal{B}} \sqrt{n_\kappa} |\hat{\varphi}_{n_\kappa}(A, B)|,$$

as $\kappa \to \infty$, where $n_\kappa = N_\kappa(\mathcal{S}_\kappa \times \mathcal{M})$, $\mathcal{A}_\kappa$ is a collection of Borel sets of $\mathcal{S}_\kappa$, $\mathcal{B}$ is a collection of Borel sets of $\mathcal{M}$, $\hat{\varphi}_{n_\kappa}(A, B) = \hat{\pi}_{n_\kappa}(A \times B) - \hat{\pi}_{s,n_\kappa}(A)\hat{\pi}_{m,n_\kappa}(B)$ and $\hat{\pi}_{n_\kappa}$, $\hat{\pi}_{s,n_\kappa}$, $\hat{\pi}_{m,n_\kappa}$ are the restrictions of $\hat{\pi}$, $\hat{\pi}_s$, and $\hat{\pi}_m$ on $\mathcal{S}_\kappa$, respectively. To derive the asymptotic distirbution of T_{n_κ}, we need the following regularity conditions for N_s.

(D1) N_s is strong stationary on $\mathcal{S}$.
(D2) N_s satisfies the strong mixing condition.
(D3) The kth-order factorial cumulant density functions Q_k of N_s satisfies

$$\int_{\mathbf{s}_1,\ldots,\mathbf{s}_k \in \mathcal{S}} |Q_k(\mathbf{s}_1, \ldots, \mathbf{s}_k)| d\mathbf{s}_1 \cdots d\mathbf{s}_k < C_1, k = 2, 3, 4,$$

and

$$\int_{\mathbf{s}_1,\ldots,\mathbf{s}_k \in \mathcal{S}} |Q_k(\mathbf{s}_1, \ldots, \mathbf{s}_k)| d\mathbf{s}_1 \cdots d\mathbf{s}_{k-1} < C_2, k = 2, 3, 4$$

for some constants $C_1, C_2 \in \mathbb{R}$.

Note that Q_k is permutation invariant. The first integral in (D3) is over all of the k-th arguments but the second one is just over $k-1$ of them.

The conditional density of marks in a marked Poisson process given by Condition (C3) can also be used to describe the conditional density of marks in a stationary marked point process. Let the conditional density still be denoted by $\lambda_1(m|\mathbf{s})$. If the null hypothesis holds, then $\lambda_1(m|\mathbf{s})$ does not depend on $\mathbf{s}$, which implies that the conditional distributions of marks given their point locations are identical. If marks are also independent, then they are iid.

Theorem 4. *Assume Conditions (D1)–(D3) hold and the distribution of marks is independent of their point locations. If marks are identically independently distributed, then T_{n_κ} weakly converges to the absolute maximum of a Gaussian random field with mean 0 and covariance function given by Equation (3.11).*

Corollary 4. *Assume all conditions in Theorems 3 and 4 hold. If $\alpha = \beta = 1$ in Condition (A1) of Theorem 2, then*

$$T_n \xrightarrow{D} \sup_{u,v \in [0,1]} |\tilde{W}(u, v)|.$$

To choose $\mathcal{A}$ and $\mathcal{B}$ in Equation (3.6), we need to consider the asymptotic null distribution of T_n given by Theorems 3 and 4 as well as their corollaries. There are two methods recommended. In the first, we choose $\mathcal{A}$ and $\mathcal{B}$ as the minimum collections of Borel sets in $\mathcal{S}$ and $\mathcal{M}$ such that $\mathscr{S}$ and $\mathscr{M}$ can be generated by them, respectively. A significant T_n is enough to conclude H_1 given by Equation (3.4) and an insignificant T_n is enough to conclude H_0 given by Equation (3.3). In the second, we choose $\mathcal{A}$ and $\mathcal{B}$ specifically such that both of them can be generated by a univariate functions from $\mathcal{S}$ to $\mathbb{R}$ or $\mathcal{M}$ to $\mathbb{R}$, respectively. A significant T_n is enough to conclude H_1 given by Equation (3.4) but an insignificant T_n is not enough to conclude H_a given by Equation (3.3).

Because the second method is more convenient than the first method in applications, we focus on the second method in this paper. In the next subsection, we present a few examples about the choice of $\mathcal{S}$ and $\mathcal{M}$ in T_n.

3.4. Examples

We present a few examples to illustrate our ideas. We use α to represent the significance level and N to represent a marked Poisson process (or a stationary marked point process which satisfies the strong mixing condition). In all of these examples, we focus of the choices of $\mathcal{A}$ and $\mathcal{B}$ in T_n.

Example 1. Assume $\mathcal{S} = \mathbb{R}^2$ and $\mathcal{M} = \mathbb{R}$ such that the points are two-dimensional random vectors and the marks are one-dimensional random variables. Let $A_{\mathbf{a}} = A_{a_1,a_2} = (-\infty, a_1] \times (-\infty, a_2]$, $B_b = (-\infty, b]$, $\mathcal{A} = \{A_{\mathbf{a}} : \mathbf{a} \in \mathbb{R}^2\}$, and $\mathcal{B} = \{B_b : b \in \mathbb{R}\}$. If $n = 1$ with the observation being denoted by (S, M), then $F(\mathbf{a}, b) = P(S \in A_{\mathbf{a}}, M \in B_b)$ is the joint CDF of S and M, $F_s(\mathbf{a}) = P(S \in A_{\mathbf{a}})$ is the marginal CDF of S, and $F_m(b) = P(M \in B_b)$ is the marginal CDF of M. If $n \geq 1$, then

$$T_n = \sup_{a\in\mathbb{R}^2, b\in\mathbb{R}} \sqrt{n}\Big|\frac{1}{n}N((-\infty, a_1] \times (-\infty, a_2] \times (-\infty, b]) - \frac{N_s((-\infty, a_1] \times (-\infty, a_2])N_m((-\infty, b])}{n^2}\Big|.$$

If F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^3$, then under H_0 there is

$$\lim_{\kappa\to\infty} P(T_n \geq x) = P(\sup_{\mathbf{u}\in\mathbb{R}^2, 0\leq v\leq 1} |Z(\mathbf{u}, v)| \geq x), \mathbf{u} \in [0, 1]^2, v \in [0, 1],$$

where $Z(\mathbf{u}, v)$ is a mean zero Gaussian random field on $\mathbb{R}^3$ with given by $E[Z(\mathbf{u}, v)Z(\mathbf{u}', v')] = [F_s(\mathbf{u} \wedge \mathbf{u}') - F_s(\mathbf{u})F_s(\mathbf{u}')](v \wedge v' - vv')$, which depends on F_s. If we choose $a_1 = a_2 = a$ such that $A_{\mathbf{a}} = (-\infty, a]^2$ with $\mathbf{a} = (a, a)$, then under H_0 there is

$$\lim_{\kappa\to\infty} P(T_n \geq x) = P(\sup_{0\leq u,v\leq 1} |\tilde{W}(u, v)| \geq x),$$

which does not depend on F_s. Therefore, Equation (3.2) is rejected if $T_n > \tilde{W}_\alpha$.

Example 2. Let $\mathcal{S} = \mathbb{R}^d$ and $\mathcal{M} = \mathbb{R}^k$. Let $A_{\mathbf{a}} = \{\mathbf{x} : \mathbf{x} \preceq \mathbf{a}, \mathbf{a} \in \mathbb{R}^d\}$, $B_{\mathbf{b}} = \{\mathbf{x} : \mathbf{x} \preceq \mathbf{b}, \mathbf{b} \in \mathbb{R}^k\}$, $\mathcal{A} = \{A_{\mathbf{a}} : \mathbf{a} \in \mathbb{R}^d\}$, and $\mathcal{B} = \{B_{\mathbf{b}} : \mathbf{b} \in \mathbb{R}^k\}$. If $n > 0$, then

$$T_n = \sup_{\mathbf{a}\in\mathbb{R}^d, \mathbf{b}\in\mathbb{R}^k} \sqrt{n}\Big|\frac{1}{n}\sum_{i=1}^{n} I_{S_i\preceq\mathbf{a}, M_i\preceq\mathbf{b}} - (\frac{1}{n}\sum_{i=1}^{n} I_{S_i\preceq\mathbf{a}})(\frac{1}{n}\sum_{i=1}^{n} I_{M_i\preceq\mathbf{b}})\Big|.$$

If F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{d+k}$, then under H_0 there is

$$\lim_{\kappa\to\infty} P(T_n \geq x) = P(\sup_{\mathbf{u}\in\mathbb{R}^d, \mathbf{v}\in\mathbb{R}^k} |Z(\mathbf{u}, \mathbf{v})| \geq x),$$

where $Z(\mathbf{u}, \mathbf{v})$ is a mean zero Gaussian random field with the covariance function given by

$$E[Z(\mathbf{u}, \mathbf{v})Z(\mathbf{u}', \mathbf{v}')] = [F_s(\mathbf{u} \wedge \mathbf{u}') - F_s(\mathbf{u})F_s(\mathbf{u}')][F_m(\mathbf{v} \wedge \mathbf{v}') - F_m(\mathbf{v})F_m(\mathbf{v}')],$$

$\mathbf{u} \in \mathbb{R}^d$ and $\mathbf{v} \in \mathbb{R}^k$, which depends on both F_s and F_m. If we choose $A_{\mathbf{a}} = (-\infty, a]^d$, $B_{\mathbf{b}} = (-\infty, b]^k$, $\mathcal{A} = \{A_a : a \in \mathbb{R}\}$, and $\mathcal{B} = \{B_b : b \in \mathbb{R}\}$, then under H_0 there is

$$\lim_{\kappa \to \infty} P(T_n \geq x) = P(\sup_{0 \leq u,v \leq 1} |\tilde{W}(u, v)| \geq x).$$

Therefore, Equation (3.2) is rejected if $T_n > \tilde{W}_\alpha$.

Example 3. In Example 1, let $A_a = \{\mathbf{s} : \|\mathbf{s} - \mathbf{s}_0\| \leq a\}$ and $\mathcal{A} = \{A_a : a \in \mathbb{R}^+\}$ for some $\mathbf{s}_0 = (s_{01}, s_{02}) \in \mathbb{R}^2$. Then under H_0 there is

$$\lim_{\kappa \to \infty} P(T_n \geq x) = P(\sup_{0 \leq u,v \leq 1} |\tilde{W}(u, v)| \geq x).$$

Therefore, we reject Equation (3.2) if $T_n > \tilde{W}_\alpha$.

Example 4. In Example 2, let $A_{a,\gamma} = \{\mathbf{s} : \|\mathbf{s} - \mathbf{s}_0\|_\gamma \leq a\}$, $B_{b,\omega} = \{\mathbf{m} : \|\mathbf{m} - \mathbf{m}_0\|_\omega \leq b\}$, $\mathcal{A}_\gamma = \{A_{a,\gamma} : a \in \mathbb{R}^+\}$, $\mathcal{B}_\omega = \{B_{b,\omega} : b \in \mathbb{R}^+\}$ for some $\mathbf{s}_0 \in \mathbb{R}^d$, $\mathbf{m}_0 \in \mathbb{R}^k$, and $\gamma, \omega \geq 1$, where $\|\cdot\|_\gamma$ is the L_γ-norm. Then under H_0 there is

$$\lim_{\kappa \to \infty} P(T_n \geq x) = P(\sup_{0 \leq u,v \leq 1} |\tilde{W}(u, v)| \geq x).$$

Therefore, we reject Equation (3.2) if $T_n > \tilde{W}_\alpha$.

4. Simulation

We simulated realizations from marked point processes on $[0, 1]^2 \times \mathbb{R}$ for points and marks. We considered marked Poisson processes, marked mixed Poisson processes, and marked Neyman-Scott cluster processes, where only the marked Poisson processes had independent increments. We chose these processes here due to their popularity in modeling ecological data. In order to generate a marked Poisson process specifically, we first generated the purely point process N_s and then generated the associated marks according to a selected conditional distribution function. It is known that N_s in a marked Neyman-Scott cluster process may satisfy the stong mixing condition. However, N_s in a marked mixed Poisson process does not satisfy the strong mixing condition because the dependence between number of points in subregions does not approach to zero if the conditional mean measure is not degenerated. Therefore, we used the marked mixed Poisson process for the robustness of the strong mixing condition in our asymptotics. In our setting, both the Neyman-Scott process and the mixed Poisson process can be thought as special cases of the Cox process (e.g. Chapter 5 in [36]).

We considered both stationary N_s and nonstationary N_s in our simulations. To generate data from a stationary N_s, we used the uniform intensity function

of N_s in $[0,1]^2$ such that the true intensity function of N_s was

$$\lambda_0(u,v) = Y\kappa, \tag{4.1}$$

for $0 \leq u, v \leq 1$, where κ was a constant and Y was a positive random variable. The random variable Y in (4.1) was only used in marked mixed Poisson processes, where we chose $E(Y) = 1$ and $V(Y) = 1/\gamma$, which was determined by the $\Gamma(\gamma,\gamma)$ distribution. For othere cases, we fixed $Y = 1$. Clearly, a marked mixed Poisson process reduced to a marked Poisson process if Y was a constant.

Let $S_i = (S_{i1}, S_{i2})$, $i = 1,\ldots,n$, be the points that were generated from a simulation. The values of marks, denoted by $M_1,\ldots,M_n$, respectively, were generated independently from the $N(\nu_i, 1)$ distribution, where

$$\nu_i = \nu(S_i) = \eta\sqrt{S_{i1}^2 + S_{i2}^2}. \tag{4.2}$$

To generate data from a nonstationary N_s, we used the intensity function of N_s as

$$\lambda_0(u,v) = Y\kappa\beta(u)\beta(v), u, v \in [0,1], \tag{4.3}$$

where κ is a constant and $\beta(u) = 30u^2(1-u)^2$ is the density of the $Beta(3,3)$-distribution. The values of marks were also generated independently from the $N(\nu_i, 1)$ distribution, where

$$\nu_i = \nu(S_i) = \eta\beta(S_{i1})\beta(S_{i2}), \eta \geq 0. \tag{4.4}$$

It can be seen that ν_i in the stationary N_s did not depend on the intensity of N_s but ν_i in the nonstationary N_s depended on the intensity of N_s.

Before we were able to use Equation (4.2) or Equation (4.4) for marks, we needed to generate N_s according to the spatial Poisson process, the Neyman-Scott cluster process, and the mixed Poisson process, respectively. To generate the marked Neyman-Scott cluster process, we first generated parent points from a Poisson point process with intensity function $\lambda_0(u,v)/k$ and then generated a Poisson number with mean k of offsprings for each parent point, where the position of offspring points relative to its parent point was generated from an independent bivariate normal distribution with standard deviation γ. Therefore, the expected value of the number of parent points was κ/k. In all of our simulation studies, we have $E(n) = \kappa$, $E(M_i|S_i) = \nu_i$ and $V(M_i|S_i) = 1$. If $\eta = 0$, then $\nu_i = 0$ for all i. Overall, in all the cases that we considered, we had that the marked point process was independent if and only if $\eta = 0$.

We had two methods to define T_n: one was given by Theorems 3 and 4 and the other was given by Corollaries 3 and 4. As there were many intensity functions considered in the simulation studies, it was more convenient to focus on the latter one. To define the test statistic T_n recommended by Corollaries 3 and 4, we used $\mathcal{A} = \{A_a : 0 \leq a \leq 1\}$ with $A_a = \{(u,v) : \sqrt{u^2+v^2} \leq a\}$ for points and $\mathcal{B} = \{(-\infty, b] : -\infty < b < \infty\}$ for marks. We tested hypotheses

$$H_0 : \eta = 0 \text{ versus } H_1 : \eta \neq 0$$

TABLE 1
Type I error probabilities and power functions of T_n for stationary N_s at the significance level 0.05, where γ represented the standard deviation for clusters in the Neyman-Scott cluster process or the value of $V^{-1}(Y)$ in the mixed Poisson process

		η							
		$\kappa = 1000$				$\kappa = 5000$			
Process	γ	0.0	0.1	0.2	0.3	0.0	0.1	0.2	0.3
Poisson		0.0490	0.1290	0.3913	0.7248	0.0555	0.4949	0.9831	1.0000
Mixed	1	0.0501	0.1262	0.3389	0.5526	0.0532	0.4235	0.7495	0.8631
	2	0.0473	0.1247	0.3643	0.6133	0.0512	0.4730	0.8594	0.9530
	4	0.0470	0.1325	0.3838	0.6614	0.0517	0.4661	0.9126	0.9857
	8	0.0544	0.1285	0.3859	0.6982	0.0558	0.4781	0.9537	0.9985
	16	0.0493	0.1247	0.3887	0.7072	0.0554	0.4888	0.9689	0.9997
Cluster	0.01	0.0510	0.1377	0.4249	0.7339	0.0575	0.5077	0.9800	1.0000
$k = 10$	0.05	0.0523	0.1381	0.4184	0.7399	0.0565	0.5161	0.9839	1.0000
	0.1	0.0480	0.1395	0.4378	0.7701	0.0544	0.5413	0.9893	1.0000
	0.2	0.0501	0.1582	0.4923	0.8354	0.0526	0.6229	0.9968	1.0000
Cluster	0.01	0.0511	0.2112	0.5752	0.7941	0.0560	0.6593	0.9233	0.9729
$\kappa/k = 10$	0.05	0.0503	0.1967	0.5204	0.7505	0.0525	0.6108	0.9031	0.9630
	0.1	0.0508	0.1877	0.5019	0.7446	0.0558	0.5966	0.9166	0.9758
	0.2	0.0515	0.1804	0.5185	0.7939	0.0539	0.6201	0.9626	0.9943

TABLE 2
Type I error probabilities and power functions of T_n for nonstationary N_s at the significance level 0.05, where γ was the same as that in Table 1

		η							
		$\kappa = 1000$				$\kappa = 5000$			
Process	γ	0.0	0.1	0.2	0.3	0.0	0.1	0.2	0.3
Poisson		0.0514	0.0961	0.3065	0.6295	0.0567	0.3874	0.9709	1.0000
Mixed	1	0.0458	0.1050	0.2860	0.4830	0.0496	0.3589	0.7105	0.8414
	2	0.0498	0.1032	0.2918	0.5536	0.0546	0.3843	0.8063	0.9388
	4	0.0523	0.0996	0.3026	0.5819	0.0513	0.3772	0.8830	0.9805
	8	0.0513	0.1029	0.3066	0.6015	0.0534	0.3886	0.9195	0.9957
	16	0.0491	0.0955	0.2910	0.6184	0.0588	0.3914	0.9451	0.9994
Cluster	0.01	0.0514	0.1127	0.3405	0.6665	0.0529	0.4041	0.9620	0.9999
$\kappa = 10$	0.05	0.0509	0.1194	0.3653	0.6910	0.0584	0.4430	0.9798	1.0000
	0.1	0.0534	0.1310	0.4348	0.7932	0.0539	0.5437	0.9963	1.0000
	0.2	0.0516	0.1723	0.5767	0.9100	0.0577	0.7159	0.9996	1.0000
Cluster	0.01	0.0520	0.1968	0.5475	0.7965	0.0583	0.6367	0.9342	0.9820
$\kappa/k = 10$	0.05	0.0517	0.1896	0.5028	0.7557	0.0550	0.5939	0.9184	0.9739
	0.1	0.0517	0.1850	0.5245	0.7929	0.0555	0.6254	0.9475	0.9889
	0.2	0.0487	0.1928	0.5885	0.8672	0.0558	0.6996	0.9860	0.9988

by T_n. We used 0.05 significance level test. By Corollary 3, we rejected H_0 if $T_n \geq \tilde{W}_{0.05} = 0.7948$.

We simulated 10000 realizations for each selected case in the three types of marked point processes. For each realization, we used T_n to test whether points and marks were independent. We computed the type I error probabilities (when $\eta = 0$) and power functions (when $\eta > 0$) for stationary N_s (Table 1) and nonstationary N_s (Table 2), respectively. The results showed that the type I error probabilities were all close to 0.05. The behaviors for marked Poisson processes and marked point processes were expected because of Corollaries 3 and 4.

To evaluate the power functions, we increased η from 0 to a certain positive value for all the cases that we have considered. We found that the power functions increased as either η or κ increased but they were slightly different within these processes as γ varied. The power functions between the marked Poisson process and the marked mixed Poisson process behaved similarly, but they were slightly different from the power functions of the marked Neyman-Scott cluster processes. This was expected because the dependence between points and marks became stronger as η increased and the expected value of the normalized constant $\sqrt{n}$ in T_n increased as κ increased. However, the value of γ only affected the variability of the normalized constant $\sqrt{n}$ in T_n. The power functions of the marked Neyman-Scott cluster processes behaved differently because the presence of cluster effects might slightly affect the power functions.

5. Applications

We applied the proposed approach to the *Ambrosia Dumosa* data and the *Alberta Forest Wildfire* data. The *Ambrosia Dumosa* data were previously studied by Miriti, Howe, and Wright [35] and Guan and Afshartous [16]. The Alberta Forest Wildfire data consisted of forest wildfire activities in Alberta, Canada, from 1931 to 2012. The test statistic T_n was applied to both data, where the significance level 0.05 was used. For Alberta Forest Wildfire data, we initiated a nonparametric method to assess the dependence between points and marks when the test was significant.

5.1. Ambrosia dumosa data

Ecologists are interested in plant interactions which can cause low productivity in ecosystems. It has been found that both positive and negative plant interactions are common in desert plant communities [6, 22]. To understand these interactions, *Ambrosia dumosa* data were collected within a hectare ($100 \times 100 m^2$) area in the Colorado Desert in 1984 [35]. *Ambrosia dumosa* is a drought deciduous shrub with 20–60cm in height which is abundant on well drained soils below one thousand meter elevation. To formulate hypothesis about plant interaction, we investigated the spatial patterns between plant locations and plant measurements, in which we treated the locations of *Ambrosia dumosa* plants as points and plant measurements as marks.

The *Ambrosia dumosa* data consisted of locations and several important measurements of 4358 *Ambrosia dumosa* (Figure 1(a) and 1(b)), including the height of the plant canopy (M^1), the length of the major axis of the plant canopy (M^2), the length of the minor axis of the plant canopy (M^3), and the volume of the plant canopy (M^4). It is of interest to know whether any of them are independent of their point locations.

It has been pointed out by Guan and Afshartous [16] that the approach of Schlather et. al [48] is not appropriate because the marginal distribution of marks is not normal in this example. The nonparametric approach proposed by

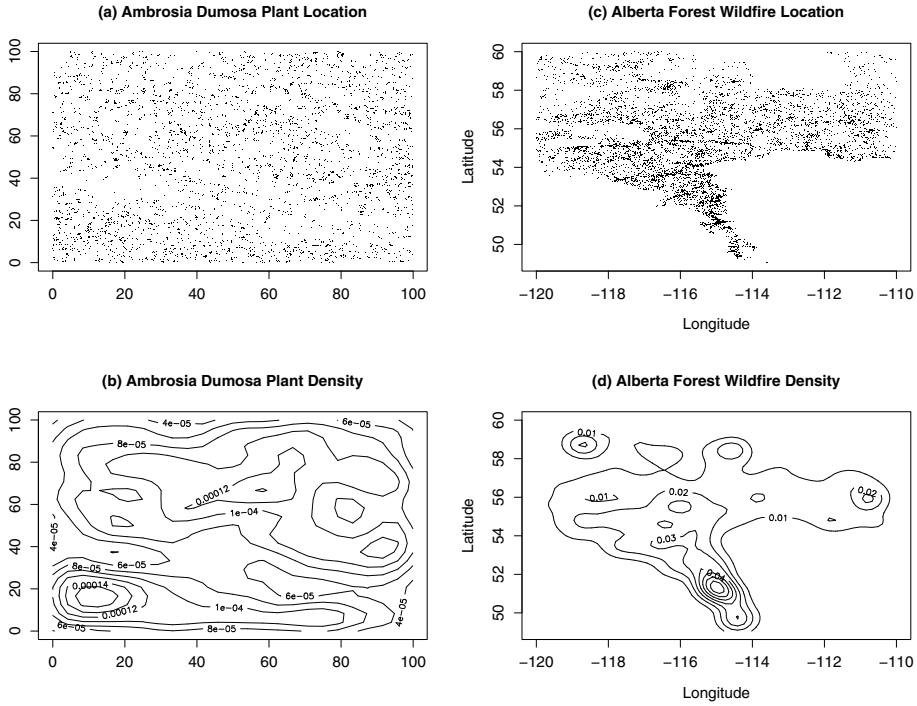

FIG 1. *Tree locations and density in Ambrosia dumosa plant data, and fire locations and density in Alberta Forest Wildfire data.*

[50] requires nonparametric estimates of intensity functions, which relies on the choice of a kernel function and a bandwidth. A kernel-based method proposed by Guan and Afshartous [16] has been used to assess the independence between points and M^4, indicating a significance p-value of 0.0031. Here we used T_n to test the independence between points and every of M^1, M^2, M^3 and M^4, separately.

To define the test statistic T_n, we chose $\mathcal{A} = \{A_a : A_a = \{\|\mathbf{s}\| \leq a\}, a \geq 0\}$ and $\mathcal{B} = \{(-\infty, b] : b \in \mathbb{R}\}$. From Lemma 1, it is enough to conclude that points and marks were not independent if $T_n \geq 0.7948$. The values of T_n for M^1, M^2, M^3, and M^4 were 1.5635, 1.9370, 1.9423, and 2.0861, respectively. All of the p-values were almost equal to 0 (all were $< 10^{-5}$). Therefore, we concluded that the locations of trees were not independent of the tree canopy, the length of the major axis, the length of the minor axis, or the volume of the canopy.

5.2. Alberta wildfire data

The Canadian Alberta Forest Service initiated the modern era of wildfire record keeping in 1931. Over the years, this fire information has been recorded, stored and made available in different formats. Beginning in 1996, paper copies of wild-

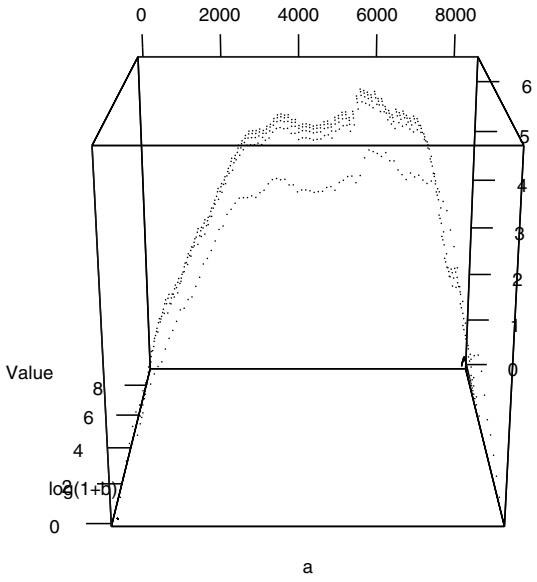

FIG 2. *Values of* $\sqrt{n}\hat{\varphi}_n(A_a, B_b)$ *as functions of* a *and* $\log(1+b)$ *in the Alberta Forest Wildfire data (1996–2010), where the maximum was attained at around* $a = 5438km$ *and* $b = 42(km^2)$.

fire history information were no longer retained. The wildfire history data were entered at the field level on the Fire Information Resource Evaluation System (FIRES), which are available at http://www.srd.alberta.ca/Wildfire.

We collected the historical forest wildfire data from 2006 to 2010 from the website. The dataset contained forest wildfire activities with area burned greater than or equal to 0.01 hectares (Figure 1(c) and 1(d)). The largest wildfire occurred in 2007 and had an area burned of 630km^2. To formulate hypothesis about the interaction between wildfire activities, we investigated the independence between the fire occurrences and area burned, in which we treated the spatial locations of wildfires as points and area burned as marks.

We used T_n to assess the independence for each particular year and all years combined. To define the test statistic T_n, we chose $\mathcal{A} = \{A_a : A_a = \{\rho(\mathbf{s}, \mathbf{s}_0) \leq a\}, a \geq 0\}$ and $\mathcal{B} = \{B_b : B_b = (-\infty, b]\}$, where $\rho(\mathbf{s}, \mathbf{s}_0)$ was the spherical distance between $\mathbf{s}$ and $\mathbf{s}_0$. We chose $\mathbf{s}_0$ at 60^o latitude north and 120^o longitude west because this point was at the upper-left corner of the whole study area. We derived the values $\sqrt{n}\hat{\varphi}_n(A_a, B_b)$ for all possible values of a and b. The combined case is displayed in Figure 2. We computed the values and p-values of T_n and displayed them in Table 3. It showed that the locations and area burned were not independent in all the cases.

To account for the interaction between points and marks, we used a model with intensity-dependent marks. The idea of intensity-dependent marks can be found in [20, 38]. To apply the idea, we needed to derive an estimate of the intensity function of N_s. Therefore, we used a two-step method. In the

TABLE 3
Values T_n for Alberta forest wildfire data, where all of the p-values of T_n were almost 0

Year	Total Area burned (hectares)	Total No. of Fires	T_n
2006	118762	1954	2.4168
2007	103669	1349	3.0343
2008	20787	1712	2.8652
2009	66947	1710	2.6096
2010	83656	1840	3.8439
Combined	393821	8565	6.4260

first step, we used a nonparametric kernel approach to estimate the intensity function $\lambda_0(s)$ of fire occurrences [17, 51]. Let $\mathbf{s}_i$ be the location and m_i be the area burned of the i-th fire given in the dataset. Based on the nonparametric kernel approach, the well-known Berman-Diggle estimator [3, 13] was given by

$$\hat{\lambda}_0(\mathbf{s}) = \frac{1}{\omega^2 C_\omega(\mathbf{s})} \sum_{i=1}^{n} f(\rho(\mathbf{s}, \mathbf{s}_i)/\omega) \tag{5.1}$$

where ρ was the spherical distance function, f was a two-dimensional kernel function, ω was the bandwidth, and $C_\omega(\mathbf{s})$ was the edge correction. We chose f as the bivariate standard normal density function. Based on the L_2-norm cross validation criterion, we had the best ω was about $\hat{\omega} = 77.42$km. In the second step, we used a linear regression method to model area burned. According to the power-law in literature of forest fire research [31], we proposed to consider the following linear regression model

$$\log m_i = \beta_0 + \beta_1 \log \hat{\lambda}_0(\mathbf{s}_i) + N(0, \sigma^2).$$

We had $\hat{\beta}_0 = -8.2845$ with standard error 0.1705, $\hat{\beta}_1 = -1.3825$ with standard error 0.0420, and $\hat{\sigma}^2 = 2.307^2$. As β_1 was significantly different from 0, we concluded there was a strong linear effect between fire occurrence and area burned. Because $\hat{\beta}_1 < 0$, our result implied that large forest fires were dominant if the relative frequency was small, which was consistent with the literature (e.g. [19, 33, 41, 47]).

6. Discussion

This research has proposed a Kolmogorov-Smirnov type method to test the independence between points and marks of marked point processes. The method primarily relies on the test statistic T_n given by Equation (3.6). Two methods for $\mathcal{A}$ and $\mathcal{B}$ are recommended according to Theorems 3 and 4 as well as their corollaries. In the first method, the asymptotic distribution of T_n depends on the unknown intensity functions of the marked point process but in the second method it does not. Therefore, we focus on the second method in this paper. The difference between the two methods is that: it is enough to conclude dependence if the null hypothesis is rejected by either methods but only the first method can accept the null hypothesis if the test is insignificant.

There are several possible extensions of our method. According to the random field model for marked point processes considered by [48], it would be interesting to study the mark-point intensity function and use it to model the local variation of the process. High dimensional nonparametric estimation methods may be used to estimate the mark-point intensity function, which may describe the interaction effect between points and marks. In the future, one could consider partitioning the mark-point intensity function into several important components so that they can represent pure point effects, pure mark effects, and point-mark interaction effects, respectively.

Appendix A: Proofs

Proof of Proposition 1. The necessity is implied by Definition 1. According to the theorem of π-λ system ([4], p. 42), it is enough to consider the proof for disjoint $A_1, \ldots, A_n \in \mathscr{S}$ and $B_1, \ldots, B_n \in \mathscr{M}$. The proof is obvious for a marked Poisson process as we have $\pi_n(A_1 \times B_1, \ldots, A_n \times B_n) = \prod_{i=1}^n \pi(A_i \times B_i)$, $\pi_{s,n}(A_1, \ldots, A_n) = \prod_{i=1}^n \pi_s(A_i)$, and $\pi_{m,n}(B_1, \ldots, B_n) = \prod_{i=1}^n \pi_m(B_i)$ when $n \geq 1$. □

Proof of Lemma 1. It is enough to consider the sufficiency because the necessity can be directly implied by Equation (3.2). If $\mathscr{S}$ and $\mathscr{M}$ can be generated by $\mathcal{A}$ and $\mathcal{B}$, respectively, then for any $\epsilon > 0$, $C \in \mathscr{S}$, and $D \in \mathscr{M}$ there are $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $|\pi_s(A) - \pi_s(C)| \leq \epsilon$, $|\pi_m(B) - \pi_m(D)| \leq \epsilon$, and $|\pi(A \times B) - \pi(C \times D)| \leq \epsilon$ ([4], p. 169). Then,

$$\begin{aligned}\varphi(C, D) &= \pi(C \times D) - \pi_s(C)\pi_m(D) \\ &= [\pi(C \times D) - \pi(A \times B)] + [\pi_s(A) - \pi_s(C)]\pi_m(B) \\ &\quad + \pi_s(C)[\pi_m(B) - \pi_m(D)]\end{aligned}$$

which implies $|\varphi(C, D)| \leq 3\epsilon$. This immediately leads the conclusion by letting $\epsilon \to 0$. □

Proof of Lemma 2. Since $n \sim Poisson(\kappa)$, $E[\hat{\pi}_n(A \times B)|n] = \pi(A \times B)I_{n>0}$ and $V[\hat{\pi}_n(A \times B)|n] = n^{-1}\pi(A \times B)[1 - \pi(A \times B)]I_{n>0}$. Then, $E[\hat{\pi}_n(A \times B)] = E[\pi(A \times B)I_{n>0}] = \pi(A \times B)(1 - e^{-\kappa}) \to \pi(A \times B)$ and

$$\begin{aligned}&V[\hat{\pi}_n(A \times B)] \\ &= V[\pi(A \times B)I_{n>0}] + E\{n^{-1}\pi(A \times B)[1 - \pi(A \times B)]I_{n>0}\} \\ &= \pi^2(A \times B)e^{-\kappa}(1 - e^{-\kappa}) + 2\pi(A \times B)[1 - \pi(A \times B)] \sum_{n=1}^{\infty} \frac{1}{2n} \frac{\kappa^n}{n!} e^{-\kappa} \\ &\leq \pi^2(A \times B)e^{-\kappa}(1 - e^{-\kappa}) + 2\pi(A \times B)[1 - \pi(A \times B)] \sum_{n=1}^{\infty} \frac{\kappa^n}{(n+1)!} e^{-\kappa} \\ &\leq \pi^2(A \times B)e^{-\kappa}(1 - e^{-\kappa}) + 2\pi(A \times B)[1 - \pi(A \times B)]/\kappa \\ &\to 0\end{aligned}$$

as $\kappa \to \infty$. Using the Chebyshev inequality, we have $\hat{\pi}_n(A,B) \xrightarrow{P} \pi(A \times B)$. Similarly, we have $\hat{\pi}_{s,n}(A) \xrightarrow{P} \pi_s(A)$ and $\hat{\pi}_{m,n}(B) \xrightarrow{P} \pi_m(B)$. The final conclusion is drawn from the Continuous Mapping Theorem. □

Proof of Lemma 3. For any $A_{i_1}, A_{i_2} \in \mathcal{A}$ and $B_{j_1}, B_{j_2} \in \mathcal{B}$, let $Y_{11} = N(A_{i_1} \times B_{j_1})/\kappa$, $Y_{12} = N(A_{i_1} \times \bar{B}_{j_1})/\kappa$, $Y_{13} = N(\bar{A}_{i_1} \times B_{j_1})/\kappa$, $Y_{14} = N(\bar{A}_{i_1} \times \bar{B}_{j_1})/\kappa$, $Y_{21} = N(A_{i_2} \times B_{j_2})/\kappa$, $Y_{22} = N(A_{i_2} \times \bar{B}_{j_2})/\kappa$, $Y_{23} = N(\bar{A}_{i_2} \times B_{j_2})/\kappa$, $Y_{24} = N(\bar{A}_{i_2} \times \bar{B}_{j_2})/\kappa$. Let $\mathbf{y}_i = (Y_{i1}, Y_{i2}, Y_{i3}, Y_{i4})^t$, $\nu_i = (\nu_{i1}, \nu_{i2}, \nu_{i3}, \nu_{i4})^t$ for $i = 1, 2$, $\mathbf{y} = (\mathbf{y}_1^t, \mathbf{y}_2^t)^t$, and $\nu = (\nu_1^t, \nu_2^t)^t$. Then,

$$\sqrt{\kappa}(\mathbf{y} - \nu) \xrightarrow{D} N(0, \begin{pmatrix} \mathbf{V}_1 & \mathbf{U} \\ \mathbf{U}^t & \mathbf{V}_2 \end{pmatrix}), \tag{A.1}$$

where $\mathbf{V}_1 = diag(\nu_1)$, $\mathbf{V}_2 = diag(\nu_2)$, and $\mathbf{U} = (u_{ij})_{i,j=1,2,3,4}$. Denote $\mathbf{x} = (\mathbf{x}_1^t, \mathbf{x}_2^t)^t$ with $\mathbf{x}_1 = (x_{11}, x_{12}, x_{13}, x_{14})^t$ and $\mathbf{x}_2 = (x_{21}, x_{22}, x_{23}, x_{24})^t$. Let $H(\mathbf{x}) = (h(\mathbf{x}_1), h(\mathbf{x}_2))^t$, where

$$h(\mathbf{z}) = \frac{z_1}{z_1 + z_2 + z_3 + z_4} - \frac{(z_1 + z_2)(z_1 + z_3)}{(z_1 + z_2 + z_3 + z_4)^2}, \mathbf{z} = (z_1, \ldots, z_8) \in \mathbb{R}^8. \tag{A.2}$$

If $n > 0$, then $H(\mathbf{y}) = (\hat{\varphi}_n(A_{i_1} \times B_{j_1}), \hat{\varphi}_n(A_{i_2} \times B_{j_2}))^t$ and $H(\nu) = (\varphi(A_{i_1} \times B_{j_1}), \varphi(A_{i_2} \times B_{j_2}))^t$. Note that $\lim_{\kappa \to \infty} P(n > 0) = 1$ and $n/\kappa \xrightarrow{P} 1$, we can show the final conclusion using the Δ-Method and Continuous Mapping Theorem. □

Proof of Corollary 1. The conclusion can be directly implied from Lemma 3. □

Proof of Theorem 1. Use the same notations as in Lemma 3. If the null hypothesis of Equation (3.3) is violated, then there are $\tilde{A} \in \mathcal{A}$ and $\tilde{B} \in \mathcal{B}$ such that $\varphi(\tilde{A}, \tilde{B}) = \pi(\tilde{A}, \tilde{B}) - \pi_s(\tilde{A})\pi_m(\tilde{B}) \neq 0$. Without loss of generality, assume $\varphi(\tilde{A}, \tilde{B})$ is positive. Let $Z \sim \mathcal{N}(0, \sigma^2_{\tilde{A},\tilde{B}})$. Then for any $\epsilon > 0$, there are $z_0 > 0$, $n_0 > 0$, and $\kappa_0 > 0$ such that $P(Z > -z_0) \geq 1 - \epsilon/3$, and for any $\kappa > \kappa_0$ there are $P(n \leq n_0) \leq \epsilon/3$ and

$$|P\{\sqrt{n}[\hat{\varphi}_n(\tilde{A}, \tilde{B}) - \varphi(\tilde{A}, \tilde{B})] > -z_0\} - P(Z > -z_0)| \leq \epsilon/3.$$

Then, we have

$$P\{\sqrt{n}[\hat{\varphi}_n(\tilde{A}, \tilde{B}) - \varphi(\tilde{A}, \tilde{B})] > -z_0\} \geq P(Z > -z_0) - \epsilon/3 \geq 1 - 2\epsilon/3.$$

Note that the way to choose n_0 depends on κ_0 and n_0 can be arbitrary large if κ_0 becomes large. Therefore, we can increase κ_0 such that $\sqrt{n_0}\varphi(\tilde{A}, \tilde{B}) - z_0 > x$. Then

$$\begin{aligned} P\{T_n > x\} &= P\{\max_{A \in \mathcal{A}, B \in \mathcal{B}} \sqrt{n}|\hat{\varphi}_n(A, B)| \geq x\} \\ &\geq P\{\sqrt{n}\hat{\varphi}_n(\tilde{A}, \tilde{B}) > x\} \\ &\geq P\{\sqrt{n}[\hat{\varphi}_n(\tilde{A}, \tilde{B}) - \varphi(\tilde{A}, \tilde{B})] > -z_0, n > n_0\} \end{aligned}$$

$$
\begin{aligned}
&+ P\{\sqrt{n}[\hat{\varphi}_n(\tilde{A},\tilde{B}) - \varphi(\tilde{A},\tilde{B})] > -z_0, n \le n_0\} - P(n \le n_0)\\
&\ge P\{\sqrt{n}[\hat{\varphi}_n(\tilde{A},\tilde{B}) - \varphi(\tilde{A},\tilde{B})] > -z_0\} - \epsilon/3\\
&\ge 1 - \epsilon.
\end{aligned}
$$

Therefore, $\lim_{\kappa\to\infty} P(T_n > x) = 1$ for all $x > 0$. □

Proof of Corollary 2. The conclusion can be directly drawn using Lemma 1 and Theorem 1. □

Proof of Theorem 2. We use the empirical process approach to show the conclusion of the theorem. Let (S, M) be a single observation in the data. Then, N can be understood as being independently obtained from the distribution of (S, M) with total n observations, where $n \sim Poisson(\kappa)$. Let $F(x) = \pi(A_{\mathbf{a}} \times B_{\mathbf{b}}) = P[(S, M) \in A_{\mathbf{a}} \times B_{\mathbf{b}}]$ for $\mathbf{x} = (\mathbf{a}, \mathbf{b})$ with $\mathbf{a} \in \mathbb{R}^\alpha$ and $\mathbf{b} \in \mathbb{R}^\beta$. Then, $F(\mathbf{x})$ is continuous in $\mathbf{a}$ and $\mathbf{b}$ because its density is $\lambda_1(m|\mathbf{s})\lambda_0(\mathbf{s})$ according to Conditions (C2) and (C3) as well as Assumptions (A1)–(A3). Let $I_{\mathbf{a}} = \{\mathbf{s} : \mathbf{s} \in A_{\mathbf{a}}\}$ be the indicator function for the point in $A_{\mathbf{a}}$ and $I_{\mathbf{b}} = \{\mathbf{m} : \mathbf{m} \in B_{\mathbf{b}}\}$ be the indicator function for the mark in $B_{\mathbf{b}}$. Then, $\hat{F}(\mathbf{x}) = \hat{\pi}_n(A_{\mathbf{a}} \times B_{\mathbf{b}})$ is the empirical distribution of $F(x)$. Let $\mathcal{G}_1 = \{I_{\mathbf{a}} I_{\mathbf{b}} : \mathbf{a} \in \mathbb{R}^\alpha, \mathbf{b} \in \mathbb{R}^\beta\}$, $\mathcal{G}_2 = \{I_{\mathbf{a}}(1 - I_{\mathbf{b}}) : \mathbf{a} \in \mathbb{R}^\alpha, \mathbf{b} \in \mathbb{R}^\beta\}$, $\mathcal{G}_3 = \{(1 - I_{\mathbf{a}})I_{\mathbf{b}} : \mathbf{a} \in \mathbb{R}^\alpha, \mathbf{b} \in \mathbb{R}^\beta\}$, and $\mathcal{G}_4 = \{(1 - I_{\mathbf{a}})(1 - I_{\mathbf{b}}) : \mathbf{a} \in \mathbb{R}^\alpha, \mathbf{b} \in \mathbb{R}^\beta\}$. To apply the empirical process approach, it is enough to show $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$ are F-Donsker, where the definition can be found on page 270 in [53]. In the following, we only display the proof for $\mathcal{G}_1$ because the proofs for $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$ are similar.

Let F_i be the i-th marginal CDF of F. For any $\epsilon \in (0, 1)$, we can find $J + 2$ points denoted by $x_{0i0}, x_{0i1}, \ldots, x_{0iJ}, x_{0i(J+1)}$, $i = 1, \ldots, \alpha+\beta$, with $x_{0i0} = -\infty$ and $x_{0i(J+1)} = \infty$ such that

$$
\frac{\epsilon^2}{\epsilon + \alpha + \beta} \le F_i(x_{0i(j+1)}) - F_i(x_{0ij}) \le \frac{\epsilon^2}{\alpha + \beta},\ j = 0, \ldots, J,
$$

with $(\alpha + \beta)/\epsilon^2 \le J \le (\alpha + \beta)/\epsilon^2 + 1$. Let

$$
\mathcal{X}_\epsilon = \{\mathbf{x} = (x_1, \ldots, x_{\alpha+\beta}) : x_i = x_{0ij} \text{ for some } j = 0, 1, \ldots, J + 1\}.
$$

Then $\#\mathcal{X}_\epsilon = (J + 2)^{\alpha+\beta} \le (\alpha + \beta + 3)^{\alpha+\beta}/\epsilon^{2(\alpha+\beta)}$. For any $g_y \in \mathcal{G}_1$, we can find $\mathbf{x}', \mathbf{x}'' \in \mathcal{X}_\epsilon$ such that $\mathbf{x}' \preceq \mathbf{y} \preceq \mathbf{x}''$ but there is no $\mathbf{x}^* \in \mathcal{X}_\epsilon$ satisfying $x_i' < x_i^* < x_i''$ for some $i = 1, \ldots, \alpha + \beta$, where x_i', x_i^*, and x_i'' are the i-th component of $\mathbf{x}'$, $\mathbf{x}^*$, and $\mathbf{x}''$, respectively. Then, $g_{\mathbf{x}'} \le g_{\mathbf{y}} \le g_{\mathbf{x}''}$ and

$$
\|g_{\mathbf{x}'} - g_{\mathbf{x}''}\|^2 = \int_{\mathbb{R}^{\alpha+\beta}} |g_{\mathbf{x}''}(\mathbf{x}) - g_{\mathbf{x}'}(\mathbf{x})|^2 F(d\mathbf{x}) \le \sum_{i=1}^{\alpha+\beta} [F_i(x_i'') - F_i(x_i')] \le \epsilon^2.
$$

Because

$$
\int_0^1 \sqrt{\log(\#\mathcal{X}_\epsilon)}d\epsilon \le \int_0^1 \sqrt{(\alpha + \beta)\log(\alpha + \beta + 3) - 2(\alpha + \beta)\log\epsilon}d\epsilon < \infty,
$$

$\mathcal{G}_1$ is F-Donsker. Similarly, we can show $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$ are also F-Donsker. The conclusion is drawn using the method in proofs of Corollary 1 and Lemma 3

with the functional Δ-method ([53], p. 291) because the gradient of h defined in (A.2) is uniformly bounded. □

Proof of Theorem 3. The conclusion can be directly implied by Lemma 3 and Theorem 2. □

Proof of Corollary 3. Let $u(a) = \pi_s(A_a)$ and $v(b) = \pi_m(A_b)$ for $-\infty < a, b < \infty$. Then, $\lim_{a\to-\infty} u(a) = \lim_{b\to-\infty} v(b) = 0$ and $\lim_{a\to\infty} u(a) = \lim_{b\to\infty} v(b) = 1$. Denote $u = u(a)$, $u' = u(a')$, $v = v(b)$, and $v' = v(b')$. Then, $\pi_s(A_a \cap A_{a'}) - \pi_s(A_a)\pi_s(A_{a'}) = u \wedge u' - uu'$ and $\pi_m(B_b \cap B_{b'}) - \pi_m(B_a)\pi_s(B_{a'}) = v \wedge v' - vv'$. Thus, $\sigma_{A_a,B_b,A_{a'},B_{b'}} = (u \wedge u' - uu')(v \wedge v' - vv')$, which is the covariance function of the two-dimensional standard Brownian pillow given by (2.4). □

Proof of Theorem 4. Let $B_1, \ldots, B_J$ be a partition of $\mathcal{M}$. Conditioning on $N_{s,\kappa}(A)$ for a given Borel $A \in \mathcal{S}_\kappa$, $N_\kappa(A \times B_1), \ldots, N_\kappa(A \times B_J)$ are multinomially distributed with the total count equal to $N_{s,\kappa}(A)$ and the proportion vector equal to $(\pi_m(B_1), \ldots, \pi_m(B_J))^t$. For any measurable $A, A' \in \mathcal{S}_1 = [-1, 1]^d$, let $\mathbf{p}_{B,B'} = (\pi_m(B \cap B'), \pi_m(B \cap \bar{B}'), \pi_m(\bar{B} \cap B'), \pi_m(\bar{B} \cap \bar{B}'))^t$ and

$$\tilde{\mathbf{y}}_{\kappa,A,A',B,B'} = (N_\kappa[(A_\kappa \cap A'_\kappa) \times (B \cap B')], N_\kappa[(A_\kappa \cap A'_\kappa) \times (B \cap \bar{B}')], \\ N_\kappa[(A_\kappa \cap A'_\kappa) \times (\bar{B} \cap B')], N_\kappa[(A_\kappa \cap A'_\kappa) \times (\bar{B} \cap \bar{B}')])^t,$$

where $A_\kappa = \kappa A$ and $A'_\kappa = \kappa A'$. Then,

$$\sqrt{N_{s,\kappa}(A_\kappa \cap A'_\kappa)}[\frac{\tilde{\mathbf{y}}_{\kappa,A,A'B,B'}}{N_{s,\kappa}(A_\kappa \cap A'_\kappa)} - \mathbf{p}_{B,B'}] \xrightarrow{D} N[0, diag(\mathbf{p}_{B,B'}) - \mathbf{p}_{B,B'}\mathbf{p}^t_{B,B'}],$$

as $\kappa \to \infty$, which is equivalent to

$$\frac{n_\kappa}{\sqrt{N_{s,\kappa}(A_\kappa \cap A'_\kappa)}}[\frac{\tilde{\mathbf{y}}_{\kappa,A,A',B,B'}}{n_\kappa} - \mathbf{p}_{B,B'}\frac{N_{s,\kappa}(A_\kappa \cap A'_\kappa)}{n_\kappa}] \\ \xrightarrow{D} N[0, diag(\mathbf{p}_{B,B'}) - \mathbf{p}_{B,B'}\mathbf{p}^t_{B,B'}].$$

Note that $N_{s,\kappa}(A_\kappa \cap A'_\kappa)/n_\kappa \xrightarrow{P} \pi_s(A \cap A')$. We have

$$\sqrt{n_\kappa}(\frac{\tilde{\mathbf{y}}_{\kappa,A,A',B,B'}}{n_\kappa} - \tilde{\mu}_{A,A'B,B'}) \xrightarrow{D} N(0, \tilde{\Sigma}_{A,A'B,B'}),$$

where $\tilde{\mu}_{A,A',B,B'} = \mathbf{p}_{B,B'}\pi_s(A \cap A')$ and $\tilde{\Sigma}_{A,A',B,B'} = \pi_s(A \cap A')[diag(\mathbf{p}_{B,B'}) - \mathbf{p}_{B,B'}\mathbf{p}^t_{B,B'}]$. Let

$$\mathbf{y}_{\kappa,A,A',B,B'} = (\tilde{\mathbf{y}}_{\kappa,A,A',B,B'}, \tilde{\mathbf{y}}_{\kappa,A,\bar{A}',B,B'}, \tilde{\mathbf{y}}_{\kappa,\bar{A},A',B,B'}, \tilde{\mathbf{y}}_{\kappa,\bar{A},\bar{A}',B,B'})^t,$$

$$\mu_{A,A',B,B'} = (\tilde{\mu}_{A,A',B,B'}, \tilde{\mu}_{A,\bar{A}',B,B'}, \tilde{\mu}_{\bar{A},A',B,B'}, \tilde{\mu}_{\bar{A},\bar{A}',B,B'})^t$$

and

$$\Sigma_{A,A'B,B'} = diag(\tilde{\Sigma}_{A,A',B,B'}, \tilde{\Sigma}_{A,\bar{A}',B,B'}, \tilde{\Sigma}_{\bar{A},A',B,B'}, \tilde{\Sigma}_{\bar{A},\bar{A}',B,B'}).$$

From Theorem 4.2 in [23], we have

$$\sqrt{n_\kappa}(\mathbf{y}_{\kappa,A,A',B,B'} - \mu_{A,A',B,B'}) \xrightarrow{D} N(0, \Sigma_{A,A',B,B'}).$$

Let $H(z_1,\dots,z_{16}) = (h_1(z_1,\dots,z_{16}), h_2(z_1,\dots,z_{16}))^t$ with $h_1(z_1,\dots,z_{16}) = (z_1+z_2+z_5+z_6) - (z_1+z_2+z_5+z_6+z_9+z_{10}+z_{13}+z_{14})(z_1+z_2+z_3+z_4+z_5+z_6+z_7+z_8)$, $h_2(z_1,\dots,z_{16}) = (z_1+z_3+z_5+z_7) - (z_1+z_3+z_5+z_7+z_9+z_{11}+z_{13}+z_{15})(z_1+z_2+z_3+z_4+z_9+z_{10}+z_{11}+z_{12})$. Then, $H(\mathbf{y}_{\kappa,A,A',B,B'}) = (\hat{\varphi}(A,B), \hat{\varphi}(A',B'))^t$, $H(\mu_{AA'BB'}) = (0,0)^t$. Using the Δ-theorem, we have

$$\sqrt{n}\begin{pmatrix} \hat{\varphi}(A,B) \\ \hat{\varphi}(A',B') \end{pmatrix} \xrightarrow{D} N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{A,B,A,B} & \sigma_{A,B,A',B'} \\ \sigma_{A,B,A',B'} & \sigma_{A',B',A',B'} \end{pmatrix}\right],$$

where $\sigma_{A,B,A',B'}$ is given in Equation (3.11). We finally have the conclusion of the theorem using Corollary 7.2 in [23]. □

Proof of Corollary 4. The conclusion can be proven using exactly the same way displayed in proof of Corollary 3. □

References

[1] Assuncao, R. and Maia, A. (2007). A note on testing separability in spatial-temporal marked point processes. *Biometrics*, **63**, 290–294. MR2345599

[2] Baddeley, A. and Turner, R. (2005). **spatstat**: An R package for analyzing spatial point patterns. *Journal of Statistical Software*, **12**, 1–42.

[3] Berman, M. and Diggle, P.J. (1989). Estimating weighted integrals of the second-order intensity of a spatial point process. *Journal of Royal Statistical Society Series B*, **51**, 81–92. MR0984995

[4] Billingsley, P. (1995). *Probability and Measure*, Wiley, New York. MR1324786

[5] Blum, J.R., Kiefer, J., and Rosenblatt, M. (1961). Distribution free tests of independence based on the sample distribution function. *Annals of Mathematical Statistics*, **32**, 485–496. MR0125690

[6] Casper, B.B. (1996). Demographic consequences of drought in the herbaceous perennial *Cryptantha flava*: Effects of density, associations with shrubs, and plant size. *Oecologia*, **106**, 144–152.

[7] Cressie, N. (1993). *Statistics for Spatial Data*, Wiley, New York. MR1239641

[8] Czörgö, M. and Révécs, P. (1981). On the nondifferentiability of the Weiner sheet. In: *Contributions to Probability*, Academic Press, New York–London, 143–150.

[9] Dalang, R.C. (2003). Level sets, bubbles and excursions of a Brownian sheet. *Lecture Notes in Mathematics*, **1802/2003**, 167–208. MR1975520

[10] Daley, D.J. and Vere-Jones, D. (2003). *An Introduction to Theory of Point Processes: Volume I: Elementary Theory and Methods, Second Edition*, Springer, New York. MR1950431

[11] Deheuvels, P. (1981). An asymptotic decomposition for multivariate distribution-free tests of independence. *Journal of Multivariate Analysis*, **11**, 102–113. MR0612295

[12] Deheuvels, P., Peccati, G., and Yor, M. (2006). On quadratic functionals of the Brownian sheet and related processes. *Stochastic Processes and Their Applications*, **116**, 493–538. MR2199561

[13] Diggle, P.J. (1985). A kernel method for smoothing point process data. *Applied Statistics*, **34**, 138–147.

[14] Diggle, P.J., Ribeiro, J.P.J., and Christensen, O. (2003). An introduction to model-based geostatistics. In *Spatial Statistics and Computational Methods* (ed. J. Moller), pp. 43–86. Springer, New York. MR2001385

[15] Florit, C. and Nualart, D. (1995). A local criterion for smoothness of densities and application to the supremum of the Brownian sheet. *Statistics and Probability Letters*, **22**, 25–31. MR1327725

[16] Guan, Y. and Afshartous, D.R. (2007). Test for independence between marks and points of marked point processes: A subsampling approach. *Environmental Ecological Statistics*, **14**, 101–111. MR2370957

[17] Hall, P., Watson, G.S., and Cabrera, J. (1987). Kernel density estimation with spherical data. *Biometrika*, **74**, 751–762. MR0919843

[18] Harte, D. (2010). **PtProcess**: An R Package for modelling marked point processes indexed by time. *Journal of Statistical Software*, **35**, 1–32.

[19] He, Y., Zhuang, Q., and Chen, M. (2011). Modeling the carbon dynamics of Alaskan boreal forest ecosystems with a process-based biogeochemistry model at a species level from 1922 to 2099. In review at Global Change Biology.

[20] Ho, L.P. and Stoyan, D. (2008). Modeling marked point patterns by intensity-marked Cox processes. *Statistics and Probability Letters*, **78**, 2831–2842. MR2441462

[21] Holden, L., Sannan, S. and Bungum, H. (2003). A stochastic marked point process model for earthquakes. *Natural Hazards and Earth System Sciences*, **3**, 95–101.

[22] Holzapfel, C. and Mahall, B.E. (1999). Bidirectional facilitation and interference between shrubs and annuals in the Majave Desert. *Ecology*, **80**, 1747–1761.

[23] Ivanoff, G. (1982). Central limit theorems for point processes. *Stochastic Processes and Their Applications*, **12**, 171–186. MR0651902

[24] Jacod, P.J. (1975). Multivariate point processes: Predictable projection, Radon-Nikodym Derivatives, Representation of Martingales. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, **31**, 235–253. MR0380978

[25] Karr, A.F. (1991). *Point Processes and Their Statistical Inference, 2nd Edition*, Marcel Dekker, Inc. New York. MR1113698

[26] Keifer, J. (1958). On the derivations of the empirical distribution function of vector chance variables. *Transactions of the American Mathematical Society*, **87**, 173–186. MR0099075

[27] Koning, A.J. and Protasov, V. (2003). Tail behavior of Gaussian processes with applications to the Brownian pillow. *Journal of Multivariate Analysis*, **87**, 370–397. MR2016944

[28] Kruglova, N.V. (2008). Distribution of the maximum of the Chentsov random field. *Theory of Stochastic Processes*, **14**, 76–81. MR2479708

[29] Kruglova, N.V. (2010). Asymptotic behavior of the distribution of the maximum of a Chentsov field on polygonal lines. *Theory of Probability and Mathematical Statistics*, **81**, 101–115. MR2667313

[30] Malamud, B.D., Millington, J.D.A., and Perry, G.L.W. (2005). Characterizing wildfire regimes in the United States. *Proceedings of the National Academy of Sciences of the United States of America*, **102**, 4694–4699.

[31] Malamud, B.D. and Turcotte, D.L. (2006). An inverse cascade explanation for the power-law frequency-area statistics of earthquakes, landslides and wildfires. *Geological Society*, **261**, 1–9.

[32] Malinowski, A., Schlather, M., and Zhang, Z. (2012). Intrinsically weighted means of marked point processes. arXiv:1201.1335.

[33] McCarthy, M.A. and Gill, A.M. (1997). Fire modeling and biodiversity. In 'Frontiers in Ecology: Building the Links', *Proceedings of the 1997 Conference of the Ecological Society of Australia*, 1–3 October 1997, Albury, Australia (Eds. N. Klomp, I. Lunt), pp. 79–88. Elsevier, Oxford, UK.

[34] McElroy, T. and Politis, D.N. (2007). Stable marked point processes. *Annals of Statistics*, **35**, 393–419. MR2332280

[35] Miriti, M.N., Howe, H.F., and Wright, S.J. (1998). Spatial patterns of mortality in a Colorado desert plant community. *Plant Ecology*, **136**, 41–51.

[36] Møller, J. and Waageoetersen, R.P. (2004). *Statistical Inference and Simulation for Spatial Point Processes*, Chapman & Hall/CRC, London. MR2004226

[37] Moyal, J.E. (1962). The general theory of Stochastic population processes. *Acta Mathematics*, **108**, 1–31. MR0148107

[38] Myllymäki, M. and Penttinen, A. (2009). Conditionally heteroscedastic intensity-dependent marking of log Gaussian Cox processes. *Statistica Neerlandica*, **63**, 450–473. MR2598981

[39] Nazarov, A.I. and Nikitin, Y.Y. (2000). Some extremal problems for Gaussian and empirical random fields. In *Proceedings of the Saint-Petersburg Mathematical Society* (ed. N.N. Uraltseva), **8**, pp. 214–230 (in Russian). English translation in *Translations of the American Mathematical Society*, **2**, 205, 189–202. MR1868035

[40] Nikitin, Y. (1995). *Asymptotic Efficiency of Nonparametric Tests*. Cambridge University Press. MR1335235

[41] Niklasson, M. and Granstrom, A. (2000). Numbers and sizes of fires: Long-term spatially explicit fire history in a Swedish boreal landscape. *Ecology*, **81**, 1484–1499.

[42] Ogata, Y. (1988). Statistical models for earthquake occurrences and residual analysis for point processes. *Journal of the American Statistical Association*, **83**, 9–27.

[43] Ogata, Y. (1998). Space-time point process models for earthquake occurrences. *Annals of the Institution Statistical Mathematics*, **50**, 379–402.

[44] Orey, S. and Pruitt, W.E. (1973). Sample functions of the N-parameter Weiner process. *Annals of Probability*, **1**, 138–163. MR0346925

[45] Peng, R.D., Schoenberg, F.P., and Woods, J.A. (2005). A space-time conditional intensity model for evaluating a wildfire hazard index. *Journal of the American Statistical Association*, **100**, 26–35. MR2166067

[46] Poliltis, D.N. and Sherman, M. (2001). Moment estimation for statistics from marked point processes. *Journal of Royal Statistical Society B*, **63**, 261–275. MR1841414

[47] Ricotta, C., Avena, G., and Marchetti, M. (1999). The flaming sand pile: Self organized criticality and wildfires. *Ecological Modelling*, **119**, 73–77.

[48] Schlather, M., Ribeiro, P.J. and Diggle, P.J. (2004). Detecting dependence between marks and locations of marked point processes. *Journal of the Royal Statistical Society B*, **66**, 79–93. MR2035760

[49] Schmid, F. and Schmidt, R. (2007). Multivariate rank statistics for independence and their asymptotic efficiency. *Mathematical Methods of Statistics*, **12**, 197–217.

[50] Schoenberg, F.P. (2004). Testing separability in spatial-temporal marked point processes. *Biometrics*, **60**, 471–481. MR2066282

[51] Stone, C.J. (1984). An asymptotically optimal window selection rule for kernel density estimates. *Annals of Statistics*, **12**, 1285–1297. MR0760688

[52] Toth, D. (2004). Adding interior points to an existing Brownian sheet lattice. *Statistics and Probability Letters*, **66**, 221–227. MR2044907

[53] van der Vaart, A.W. (1998). *Asymptotic Statistics*, Cambridge University Press, Cambridge, UK. MR1652247

[54] Vere-Jones, D. (1995). Statistical methods for the description and display of earthquake catalogs. In *Statistics in Environmental and Earth Sciences* (eds. A. Walden, P. Guttorp), pp. 220–236. Edward Arnold, London.

[55] Walsh, J.B. (1960). Propagation of singularities in the Brownian sheet. *Annals of Probability*, **10**, 279–288. MR0647504

[56] Yeh, J. (1960). Wiener measure in a space of functions of two variables. *Transactions of the American Mathematical Society*, **95**, 433–450. MR0125433