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False discovery rate control under Archimedean copula

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Abstract: We are concerned with the false discovery rate (FDR) of the linear step-up test φ^{LSU} considered by Benjamini and Hochberg (1995). It is well known that φ^{LSU} controls the FDR at level m_0q/m if the joint distribution of p-values is multivariate totally positive of order 2. In this, m denotes the total number of hypotheses, m_0 the number of true null hypotheses, and q the nominal FDR level. Under the assumption of an Archimedean p-value copula with completely monotone generator, we derive a sharper upper bound for the FDR of φ^{LSU} as well as a non-trivial lower bound. Application of the sharper upper bound to parametric subclasses of Archimedean p-value copulae allows us to increase the power of φ^{LSU} by pre-estimating the copula parameter and adjusting q. Based on the lower bound, a sufficient condition is obtained under which the FDR of φ^{LSU} is exactly equal to m_0q/m, as in the case of stochastically independent p-values. Finally, we deal with high-dimensional multiple test problems with exchangeable test statistics by drawing a connection between infinite sequences of exchangeable p-values and Archimedean copulae with completely monotone generators. Our theoretical results are applied to important copula families, including Clayton copulae and Gumbel-Hougaard copulae.

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Contents

1. Introduction

Control of the false discovery rate (FDR) has become a standard type I error criterion in large-scale multiple hypotheses testing. When the number m of hypotheses to be tested simultaneously is of order $10^3 - 10^6$, as it is prevalent in many modern applications from the life sciences like genetic association analyses, gene expression studies, functional magnetic resonance imaging, or brain-computer interfacing, it is typically infeasible to model or to estimate the full joint distribution of the data. Hence, one is interested in generic procedures that control the FDR under no or only qualitative assumptions regarding this joint distribution. The still by far most popular multiple test for FDR control, the linear step-up test φ^{LSU} (say) considered in the seminal work by Benjamini and Hochberg (1995), operates on marginal p-values $p_1, \ldots, p_m$. As shown by Benjamini and Yekutieli (2001) and Sarkar (2002), φ^{LSU} is generically FDR-controlling over the class of models that lead to positive dependency among the random p-values $P_1, \ldots, P_m$ in the sense of positive regression dependency on subsets (PRDS). All p-value distributions which are multivariate totally positive of order 2 (MTP_2) are PRDS on any subset. Hence, often MTP_2 distributions are considered, because the MTP_2 property is typically more tractable from the mathematical point of view. Under the PRDS assumption, the FDR of φ^{LSU} is upper-bounded by $m_0 q/m$, where m_0 denotes the number of true null hypotheses and q the nominal FDR level.

In this work, we extend these findings by deriving a sharper upper bound for the FDR of φ^{LSU} in the case that the dependency structure among $P_1, \ldots, P_m$ can be expressed by an Archimedean copula. Although copula modeling has become a very important topic in multivariate statistics, the application of copulae in multiple test problems has been discussed only rarely up to now (see, e. g., Sarkar (2008a), Cerqueti, Costantini and Lupi (2012), Dickhaus and Gierl (2013), Stange, Bodnar and Dickhaus (2013)). Our respective contributions are threefold. First, we quantify the magnitude of conservativity (non-exhaustion of the FDR level q) of φ^{LSU} in various copula models as a function of the copula parameter η. This allows for a gain in power in practice by pre-estimating η and adjusting the nominal value of q. Second, we demonstrate by computer simulations that the proposed upper bound leads to a robust procedure in the sense that the variance of this bound over repeated Monte Carlo simulations is much smaller than the corresponding variance of the false discovery proportion (FDP) of φ^{LSU}. This makes the utilization of our upper bound an attractive choice in practice, addressing the issue that the FDP is typically not well concentrated around its mean, the FDR, if p-values are dependent. As a by-product, we directly obtain that the FDR of φ^{LSU} is bounded by $m_0 q/m$ under the assumption of an Archimedean p-value copula, without explicitly relying on the MTP_2 property (which is fulfilled in the class of Archimedean p-value copulae with completely monotone generator functions, cf. Müller and Scarsini (2005)). Third, in an asymptotic setting ($m \to \infty$), we show that the class of Archimedean p-value copulae with completely monotone generators includes certain models with p-values or test statistics, respectively, which are exchangeable under null hypotheses, H_0-exchangeable for short. Such H_0-exchangeable test statistics occur naturally in many multiple test problems, for instance in many-to-one comparisons or if test statistics are given by jointly Studentized means (cf. Finner, Dickhaus and Roters (2007)).

In addition, we also derive and discuss a lower FDR bound for φ^{LSU} in terms of the generator of an Archimedean p-value copula. Application of this lower bound leads to sufficient conditions under which the FDR of φ^{LSU} is exactly equal to $m_0 q/m$, at least asymptotically as m tends to infinity and m_0/m converges to a fixed value. Hence, if the latter conditions are fulfilled, the FDR behaviour of φ^{LSU} is under dependency the same as in the case of jointly stochastically independent p-values.

The paper is organized as follows. In Section 2, we set up the necessary notation, define our class of statistical models for $P_1, \ldots, P_m$, and recall properties and results around the FDR. Our main contributions are presented in Section 3, dealing with FDR control of φ^{LSU} under the assumption of an Archimedean copula. Special parametric copula families are studied in Section 4, where we quantify the realized FDR of φ^{LSU} as a function of η. Section 5 outlines methods for pre-estimation of η. We conclude with a discussion in Section 6. Lengthy proofs are deferred to Section 7.

2. Notation and preliminaries

All multiple test procedures considered in this work depend on the data only via (realized) marginal p-values $p_1, \ldots, p_m$ and their ordered values $p_{(1)} \leq p_{(2)} \leq \cdots \leq p_{(m)}$. Hence, it suffices to model the distribution of the random vector $\mathbf{P} = (P_1, \ldots, P_m)^\top$ of p-values and we consider statistical models of the form $([0,1]^m, \mathcal{B}([0,1]^m), (\mathbb{P}_{\vartheta,\eta} : \vartheta \in \Theta, \eta \in \Xi))$. In this, we assume that ϑ is the (main) parameter of statistical interest and we identify the null hypotheses $H_i : 1 \leq i \leq m$ with non-empty subsets of Θ, with corresponding alternatives $K_i = \Theta \setminus H_i$. The null hypothesis H_i is called true if $\vartheta \in H_i$ and false otherwise. We let $I_0 \equiv I_0(\vartheta) = \{1 \leq i \leq m : \vartheta \in H_i\}$ denote the index set of true hypotheses and $m_0 \equiv m_0(\vartheta) = |I_0|$ the number of true nulls. Without loss of generality, we will assume $I_0(\vartheta) = \{1, \ldots, m_0\}$ throughout the work. Analogously, we define

$I = \{1, \ldots, m\}$, $I_1 \equiv I_1(\vartheta) = I \setminus I_0$ and $m_1 \equiv m_1(\vartheta) = |I_1| = m - m_0$. The intersection hypothesis $H_0 = \bigcap_{i=1}^m H_i$ will be referred to as the global (null) hypothesis.

The parameter η is the copula parameter of the joint distribution of $\mathbf{P}$, thus representing the dependency structure among $P_1, \ldots, P_m$. Throughout the remainder, we will consider different parameter spaces Ξ, depending on the degree of detail of the respective distributional assumption on $\mathbf{P}$. In Section 3, Ξ is taken as an infinite-dimensional functional space, where we consider the class of all Archimedean copulae which are indexed by the generator function ψ. Thus, η is identified with ψ itself in Section 3 and we write $\psi \in \Xi$ instead of $\eta \in \Xi$. However, we sometimes restrict our attention to parametric subclasses of Archimedean copulae, for instance the class of Clayton copulae which can be indexed by a one-dimensional copula parameter $\eta \in \mathbb{R}$ (see Section 4). In such cases, $\Xi \ni \eta$ is of finite dimension and η uniquely determines the generator ψ in the considered subclass. In any case, we will assume that η is a nuisance parameter in the sense that it does not depend on ϑ and that the marginal distribution of each P_i is invariant with respect to η. Therefore, to simplify notation, we will write $\mathbb{P}_\vartheta$ instead of $\mathbb{P}_{\vartheta,\eta}$ if marginal p-value distributions are concerned. Throughout the work, the p-values $P_1, \ldots, P_m$ are assumed to be valid in the sense that

$$\forall 1 \le i \le m : \forall \vartheta \in H_i : \forall t \in [0,1] : \mathbb{P}_\vartheta(P_i \le t) \le t.$$

A (non-randomized) multiple test operating on p-values is a measurable mapping $\varphi = (\varphi_i : 1 \le i \le m) : [0,1]^m \to \{0,1\}^m$ the components of which have the usual interpretation of a statistical test for H_i versus K_i, $1 \le i \le m$. For fixed φ, we let $V_m \equiv V_m(\vartheta) = |\{i \in I_0(\vartheta) : \varphi_i = 1\}|$ denote the (random) number of false rejections (type I errors) of φ and $R_m \equiv R_m(\vartheta) = |\{i \in \{1, \ldots, m\} : \varphi_i = 1\}|$ the total number of rejections. The FDR under (ϑ, η) of φ is then defined by

$$\mathrm{FDR}_{\vartheta,\eta}(\varphi) = \mathbb{E}_{\vartheta,\eta}\left[\frac{V_m}{\max(R_m, 1)}\right],$$

and φ is said to control the FDR at level $q \in (0,1)$ if $\sup_{\vartheta \in \Theta, \eta \in \Xi} \mathrm{FDR}_{\vartheta,\eta}(\varphi) \le q$. The random variable $V_m / \max(R_m, 1)$ is referred to as the false discovery proportion of φ, $\mathrm{FDP}_{\vartheta,\eta}(\varphi)$ for short. Notice that, although the trueness of the null hypotheses is determined by ϑ alone, the FDR depends on ϑ and η, because the dependency structure among the p-values typically influences the distribution of φ when regarded as a statistic with values in $\{0,1\}^m$.

The linear step-up test φ^{LSU}, also referred to as Benjamini-Hochberg test or *the* FDR procedure in the literature, rejects exactly hypotheses $H_{(1)}, \ldots, H_{(k)}$, where the bracketed indices correspond to the order of the p-values and $k = \max\{1 \le i \le m : p_{(i)} \le q_i\}$ for linearly increasing critical values $q_i = iq/m$. If k does not exist, no hypothesis is rejected. The sharpest characterization of FDR control of φ^{LSU} that we are aware of so far is given in the following theorem.

Theorem 2.1 (Finner, Dickhaus and Roters (2009))**.** *Consider the following assumptions.*

(D1) $\forall(\vartheta, \eta) \in \Theta \times \Xi : \forall j \in I : \forall i \in I_0(\vartheta)$*:* $\mathbb{P}_{\vartheta,\eta}(R_m \geq j | P_i \leq t)$ *is non-increasing in* $t \in (0, q_j]$.

(D2) $\forall \vartheta \in \Theta : \forall i \in I_0(\vartheta) : P_i \sim UNI([0, 1])$.

(I1) $\forall(\vartheta, \eta) \in \Theta \times \Xi$*: The p-values* $(P_i : i \in I_0(\vartheta))$ *are independent and identically distributed (iid).*

(I2) $\forall(\vartheta, \eta) \in \Theta \times \Xi$*: The random vectors* $(P_i : i \in I_0(\vartheta))$ *and* $(P_i : i \in I_1(\vartheta))$ *are stochastically independent.*

Then, the following two assertions hold true.

$$\text{Under (D1), } \forall(\vartheta, \eta) \in \Theta \times \Xi : \quad FDR_{\vartheta,\eta}(\varphi^{LSU}) \leq \frac{m_0(\vartheta)}{m} q. \tag{1}$$

$$\text{Under (D2)–(I2), } \forall(\vartheta, \eta) \in \Theta \times \Xi : \quad FDR_{\vartheta,\eta}(\varphi^{LSU}) = \frac{m_0(\vartheta)}{m} q. \tag{2}$$

The crucial assumption (D1) is fulfilled for multivariate distributions of $\mathbf{P}$ which are positively regression dependent on the subset I_0 (PRDS on I_0) in the sense of Benjamini and Yekutieli (2001). In particular, (D1) holds true if the joint distribution of $\mathbf{P}$ is MTP_2. Weaker sufficient conditions for the PRDS on I_0 property have been provided by Benjamini and Yekutieli (2001). To mention also a negative result, Guo and Rao (2008) have shown that there exists a multivariate distribution of $\mathbf{P}$ such that the FDR of φ^{LSU} is equal to $m_0 q/m \sum_{j=1}^{m} j^{-1}$, showing that φ^{LSU} is not generically FDR-controlling over all possible joint distributions of $\mathbf{P}$. The main purpose of the present work (Section 3) is to derive a sharper upper bound on the right-hand side of (1), assuming that Ξ is the space of completely monotone generator functions of Archimedean copulae.

The linear step-up test belongs to the broad class of step-up-down (SUD) multiple tests, introduced by Tamhane, Liu and Dunnett (1998).

Definition 2.1 (Step-up-down test of order λ in terms of p-values, cf. Finner, Gontscharuk and Dickhaus, 2012)**.** Let $p_{(1)} \leq p_{(2)} \leq \cdots \leq p_{(m)}$ denote the ordered p-values for a multiple test problem. For a tuning parameter $\lambda \in \{1, \ldots, m\}$ a step-up-down test $\varphi^{\lambda} = (\varphi_1, \ldots, \varphi_m)$ (say) of order λ based on some critical values $\alpha_{(1)} \leq \cdots \leq \alpha_{(m)}$ is defined as follows: If $p_{(\lambda)} \leq \alpha_{(\lambda)}$, set $k = \max\{j \in \{\lambda, \ldots, m\} : p_{(i)} \leq \alpha_{(i)} \text{ for all } i \in \{\lambda, \ldots, j\}\}$, whereas for $p_{(\lambda)} > \alpha_{(\lambda)}$, put $k = \sup\{j \in \{1, \ldots, \lambda - 1\} : p_{(j)} \leq \alpha_{(j)}\}$ ($\sup \emptyset = -\infty$). Define $\varphi_i = 1$ if $p_i \leq \alpha_{(k)}$ and $\varphi_i = 0$ otherwise ($\alpha_{(-\infty)} = -\infty$).

A step-up-down test of order $\lambda = 1$ or $\lambda = m$, respectively, is called step-down (SD) or step-up (SU) test, respectively. If all critical values are identical, we obtain a single-step test.

In case of φ^{LSU}, $\lambda = m$ and $\alpha_{(i)} = q_i$ for all $1 \leq i \leq m$. In general, the choice of the order λ and of the critical values employed in an SUD test for FDR control depends on model assumptions; cf. Table 5.1 of Dickhaus (2014). Here, we can mention only a few references. Under independence assumptions on $\mathbf{P}$, FDR-controlling SD tests have been derived by Benjamini and Liu (1999) and Gavrilov, Benjamini and Sarkar (2009), and FDR-controlling SUD tests with $1 < \lambda < m$ have been developed by Finner, Dickhaus and Roters (2009) and Finner, Gontscharuk and Dickhaus (2012). Under weak dependence, an adaptive version of φ^{LSU} was considered by Storey, Taylor and Siegmund (2004). FDR-controlling SU tests under general dependence are due to Benjamini and Yekutieli (2001), Blanchard and Roquain (2008, 2009), and Sarkar (2008b). FDR control of single-step tests was considered by Sarkar (2006).

3. FDR control under Archimedean copula

In this section, it is assumed that the joint distribution of $\mathbf{P}$ is given by an Archimedean copula such that

$$F_{\mathbf{P}}(p_1, \ldots, p_m) = \mathbb{P}_{\vartheta,\psi}(P_1 \leq p_1, \ldots, P_m \leq p_m) = \psi\left(\sum_{i=1}^{m} \psi^{-1}\left(F_{P_i}(p_i)\right)\right), \quad (3)$$

where the function $\psi(\cdot)$ is the so-called copula generator and takes the role of η in our general setup. In (3) and throughout the work, F_ξ denotes the cumulative distribution function (cdf) of the variate ξ. The generator ψ fully determines the type of the Archimedean copula; see, e.g. Nelsen (2006). A necessary and sufficient condition under which a function $\psi : \mathbb{R}_+ \to [0, 1]$ with $\psi(0) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$ can be used as a generator for an m-dimensional Archimedean copula is that $\psi(\cdot)$ is an m-altering function, that is, $(-1)^d\psi^{(d)}(\cdot) \geq 0$ for $d \in \{1, 2, \ldots, m\}$, cf. Müller and Scarsini (2005). Throughout the present work, the dimensionality m of the copula coincides with the number m of hypotheses to be tested. Furthermore, for sake of simplicity, we impose a slightly stronger assumption on ψ. Namely, we assume that ψ is completely monotone, i. e. $(-1)^d\psi^{(d)}(\cdot) \geq 0$ for all $d \in \mathbb{N}$.

A very useful property of an Archimedean copula with completely monotone generator ψ is the stochastic representation of $\mathbf{P}$. Namely, there exists a sequence of jointly independent and identically UNI$[0, 1]$-distributed random variables $Y_1, \ldots, Y_m$ such that (cf. Marshall and Olkin (1988), Section 5)

$$\mathbf{P} = (P_i : 1 \leq i \leq m) \stackrel{d}{=} \left(F_{P_i}^{-1}\left(\psi\left(\log\left(Y_i^{-1/Z}\right)\right)\right) : 1 \leq i \leq m\right), \quad (4)$$

where the symbol $\stackrel{d}{=}$ denotes equality in distribution. The random variable Z with Laplace transform $t \mapsto \psi(t) = \mathbb{E}[\exp(-tZ)]$ is independent of $Y_1, \ldots, Y_m$, and its distribution is determined by ψ only. Throughout the remainder, $\mathbb{P}$ and $\mathbb{E}$ refer to the distribution of Z, for ease of presentation. The stochastic representation (4) shows that the type of the Archimedean copula can equivalently be expressed in terms of the random variable Z. Moreover, the p-values $(P_i : 1 \leq i \leq m)$ are conditionally independent given $Z = z$. This second property allows us to establish the following sharper upper bound for the FDR of φ^{LSU}.

Theorem 3.1 (Upper FDR bound)**.** *Let Z be as in (4) and let $\mathbf{P}^{(i)}$ consist of the $(m - 1)$ remaining p-values obtained by dropping P_i from $\mathbf{P}$ so that $P_{(1)}^{(i)} \leq$*

$P^{(i)}_{(2)} \leq \cdots \leq P^{(i)}_{(m-1)}$. *The random set* $D^{(i)}_k$ *is then given by*

$$D^{(i)}_k = \{q_{k+1} \leq P^{(i)}_{(k)}, \ldots, q_m \leq P^{(i)}_{(m-1)}\}.$$

For a given value $Z = z$ *we define the function* $\mathbf{T} : [0,1]^m \to [0,1]^m$ *by* $\mathbf{T}(\mathbf{p}) = (T_1(p_1), \ldots, T_m(p_m))^\top$ *with* $T_j(p_j) = \exp(-z\psi^{-1}(F_{P_j}(p_j)))$ *for* $\mathbf{p} = (p_1, \ldots, p_m)^\top \in [0,1]^m$. *This function transforms, for fixed* $Z = z$, *realizations of* $\mathbf{P}$ *into realizations of* $\mathbf{Y} = (Y_1, \ldots, Y_m)^\top$ *given in* (4). *Let* $D^{(i,z)}_{\mathbf{Y};k}$ *denote the image of the set* $D^{(i)}_k$ *under* $\mathbf{T}$ *for given* $Z = z$ *and let* $G^i_k(z) = \mathbb{P}_{\vartheta,\psi}(D^{(i,z)}_{\mathbf{Y};k})$. *Then it holds*

$$\forall \vartheta \in \Theta : FDR_{\vartheta,\psi}(\varphi^{LSU}) \leq \frac{m_0(\vartheta)}{m} q \Delta(m, \vartheta, \psi),$$

where

$$\Delta(m,\vartheta,\psi) \; = \; 1 - \frac{1}{m_0} \sum_{i=1}^{m_0} \sum_{k=1}^{m-1} \mathbb{E}\left[\left(\frac{\exp\left(-Z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-Z\psi^{-1}(q_k)\right)}{q_k}\right) \times (G^i_k(Z) - G^i_k(z^*_k))\mathbf{1}_{[z^*_k,\infty)}(Z)\right] \qquad (5)$$

with

$$z^*_k = \frac{\log q_{k+1} - \log q_k}{\psi^{-1}(q_k) - \psi^{-1}(q_{k+1})} = \frac{\log\left(1 + 1/k\right)}{\psi^{-1}(kq/m) - \psi^{-1}((k+1)q/m)} \qquad (6)$$

and $\mathbf{1}_A$ *denoting the indicator function of the set* A.

Noticing that $\Delta(m, \vartheta, \psi)$ is always smaller than or equal to one, we obtain the following result as a straightforward corollary of Theorem 3.1.

Corollary 3.1. *Let the copula of* $\mathbf{P} = (P_1, \ldots, P_m)^\top$ *be an Archimedean copula, where* P_i *is continuously distributed on* $[0,1]$ *for* $1 \leq i \leq m$. *Then it holds that*

$$\forall \vartheta \in \Theta : \forall \psi \in \Xi : FDR_{\vartheta,\psi}(\varphi^{LSU}) \leq \frac{m_0(\vartheta)}{m} q,$$

where Ξ *denotes the set of all completely monotone generator functions of Archimedean copulae.*

The result of Corollary 3.1 is in line with the findings obtained by Benjamini and Yekutieli (2001) and Sarkar (2002) that we have recalled in Section 1. Namely, Müller and Scarsini (2005) pointed out that an Archimedean copula possesses the MTP_2 property if the copula generator ψ is completely monotone and, hence, the FDR is controlled by φ^{LSU} in this case.

From the practical point of view, it is problematic that $\Delta(m, \vartheta, \psi)$ depends on the (main) parameter ϑ of statistical interest. In practice, one will therefore often only be able to work with $\sup_{\vartheta \in \Theta}\{m_0(\vartheta) q \Delta(m, \vartheta, \psi)/m\}$. Since $\Delta(m, \vartheta, \psi) \leq 1$ for all $\vartheta \in \Theta$, the latter ϑ-free upper bound will typically still yield an improvement over the "classical" upper bound. The issue of maximization of $\Delta(m, \cdot, \psi)$

over $\vartheta \in \Theta$ is closely related to the challenging task of determining least favorable parameter configurations (LFCs) for the FDR. Under assumptions (I1)–(I2) from Theorem 2.1, so-called Dirac-uniform (DU) configurations (cf., e. g., Blanchard et al. (2014) and references therein) are least favorable (provide upper bounds) for the FDR of φ^{LSU}, see Benjamini and Yekutieli (2001). DU configurations are such that (D2) holds true and the p-values $(P_i : i \in I_1(\vartheta))$ are $\mathbb{P}_\vartheta$-almost surely equal to 0 (Dirac-distributed with point mass 1 in 0). In the case of dependent p-values, general LFC results for the FDR of φ^{LSU} are yet lacking, but it is assumed that Dirac-uniform configurations yield upper FDR bounds for φ^{LSU} also under dependence, at least for large m (cf., e. g., Finner, Dickhaus and Roters (2007), Blanchard et al. (2014)). Troendle (2000) motivated the consideration of Dirac-uniform configurations from the point of view of consistency of marginal tests with respect to the sample size. Throughout the remainder, we write $DU_{m_0,m}$ instead of ϑ if ϑ is a DU configuration with exactly m_0 true null hypotheses.

Based on the aforementioned LFC considerations, Theorem 3.1 suggests to find an adjusted nominal FDR level $q^{\text{adj.}}$ such that

$$\frac{m_0(\vartheta)}{m} q^{\text{adj.}} \Delta(m, DU_{m_0,m}, \psi) = \frac{m_0(\vartheta)}{m} q,$$

where q again stands for the target FDR level. This leads to

$$q^{\text{adj.}} = \frac{q}{\Delta(m, DU_{m_0,m}, \psi)}. \tag{7}$$

Since $m_0 = m_0(\vartheta)$ is an unknown quantity in practice, however, one will typically only be able to use

$$q^{\text{adj.}}_{\min} = \frac{q}{\max\limits_{m_{\text{lower}} \le k \le m} \Delta(m, DU_{k,m}, \psi)}, \tag{8}$$

where m_{lower} denotes a reasonable lower bound for m_0.

Notice that (4) implies exchangeability of the p-values corresponding to the true null hypotheses, provided that (D2) holds true. In particular, all $(P_i : 1 \le i \le m_0)$ are identically uniformly distributed on $[0, 1]$ under (D2). This allows for restricting attention to P_1 in Theorem 3.1 and leads to the expression

$$\begin{aligned} \Delta(m, \vartheta, \psi) \ = \ 1 - \sum_{k=1}^{m-1} \mathbb{E}\Bigg[\Bigg(& \frac{\exp\left(-Z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-Z\psi^{-1}(q_k)\right)}{q_k} \Bigg) \\ & \times (G^1_k(Z) - G^1_k(z^*_k)) \mathbf{1}_{[z^*_k, \infty)}(Z) \Bigg] \end{aligned} \tag{9}$$

for an upper FDR bound of φ^{LSU} under (3) and (D2). Finally, we point out that the expectations in (5) can in general not be calculated analytically. However, they can easily be approximated by means of computer simulations. Namely, the approximation is performed by generating random numbers which behave like independent realizations of Z, which completely specifies the type of the

Archimedean copula, evaluating the functions G_k^i at the generated values and replacing the theoretical expectation of Z by the arithmetic mean of the resulting values of the integrand in (5). Under DU configurations, evaluation of G_k^i can efficiently be performed by means of recursive formulas for the joint cdf of the order statistics of $\mathbf{Y}$. We discuss these points in detail in Section 4.

Next, we provide a lower bound for the FDR of φ^{LSU} under the assumption of an Archimedean copula.

Theorem 3.2 (Lower FDR bound)**.** *Let the copula of* $\mathbf{P} = (P_1, \ldots, P_m)^T$ *be an Archimedean copula with generator function* ψ*, where* P_i *is continuously distributed on* $[0, 1]$ *for* $i = 1, \ldots, m$*. Then it holds that*

$$\forall \vartheta \in \Theta : FDR_{\vartheta,\psi}(\varphi^{LSU}) \geq \frac{m_0 q}{m} \gamma_{min},$$

where

$$\gamma_{min} \equiv \gamma_{min}(\psi) = \int \min_{k \in \{1,\ldots,m\}} \left\{ \frac{\exp\left(-z\psi^{-1}\left(kq/m\right)\right)}{kq/m} \right\} dF_Z(z). \tag{10}$$

The proof of Theorem 3.2 is given in the appendix. For its application, it is convenient to express γ_{min} from (10) more explicitly.

Lemma 3.1. *The quantity* $\gamma_{min} \equiv \gamma_{min}(\psi)$ *from* (10) *can equivalently be expressed as*

$$\gamma_{min} \quad = \quad 1 - \mathbb{E}\left[\left(g\left(\psi^{-1}(q/m)|Z\right) - g\left(\psi^{-1}(q)|Z\right)\right)\mathbf{1}_{[0,z^*]}(Z)\right], \tag{11}$$

where

$$g(x|z) = \frac{\exp\left(-zx\right)}{\psi(x)} \tag{12}$$

and

$$z^* = \frac{\log m}{\psi^{-1}\left(q/m\right) - \psi^{-1}\left(q\right)}. \tag{13}$$

If the expectation in (11) cannot be calculated analytically, then it can easily be approximated via a Monte Carlo simulation by using the expression on the right-hand side of (11) and replacing the theoretical expectation by its pseudo-sample analogue.

Corollary 3.2. *Under the assumptions of Theorem 3.2, the following two assertions hold true.*

(a) If z^* *from* (13) *does not lie in the support of* F_Z*, i. e., if* $F_Z(z^*) = 0$ *or* $F_Z(z^*) = 1$*, then* $\gamma_{min} = 1$ *and, consequently,* $FDR_{\vartheta,\psi}(\varphi^{LSU}) = m_0 q/m$*.*

(b) Assume that $\pi_0 = \lim_{m\to\infty} m_0/m$ *exists. If* $z^* = z^*(m)$ *is such that* $F_Z(z^*(m)) \to 0$ *or* $F_Z(z^*(m)) \to 1$ *as* $m \to \infty$*, then*

$$\lim_{m\to\infty} FDR_{\vartheta,\psi}(\varphi^{LSU}) = \pi_0 q.$$

Part (b) of Corollary 3.2 motivates a deeper consideration of asymptotic or high-dimensional multiple tests, i. e., the case of $m \to \infty$, under our general setup. This approach has already been discussed widely in previous literature. For instance, it was called "asymptotic multiple test" by Genovese and Wasserman (2002). The case $m \to \infty$ was also considered by Finner and Roters (1998), Storey (2002), Genovese and Wasserman (2004), Finner, Dickhaus and Roters (2007, 2009), Jin and Cai (2007), Sun and Cai (2007), and Cai and Jin (2010), among others. Very interesting connections can be drawn between Archimedean p-value copulae and infinite sequences of H_0-exchangeable p-values defined as follows.

Definition 3.1 (H_0-exchangeability)**.** Let $(P_i)_{i\in\mathbb{N}}$ be an infinite sequence of p-values which are absolutely continuous with marginal cdf F_i of P_i under ϑ. Then we call $(P_i)_{i\in\mathbb{N}}$ an H_0-exchangeable sequence of p-values if $F_1(P_1), \ldots, F_m(P_m), \ldots$ are exchangeable random variables and each P_i is uniformly distributed on $[0, 1]$ under the respective null hypothesis H_i.

Notice that, for an H_0-exchangeable sequence $(P_i)_{i\in\mathbb{N}}$, $F_i = id.$ for all i under the global hypothesis H_0. Consequently, $P_1, \ldots, P_m, \ldots$ themselves are exchangeable under H_0. Sequences of H_0-exchangeable p-values have already been investigated by Finner and Roters (1998) and Finner, Dickhaus and Roters (2007) in special settings. Moreover, the assumption of exchangeability is also pivotal in other areas of statistics, let us mention Bayesian analysis and validity of permutation tests. The problem of exchangeability in population genetics has been discussed by Kingman (1978).

Assuming that $(P_i)_{i\in\mathbb{N}}$ is an H_0-exchangeable sequence, let $\tilde{P}_i = F_i(P_i)$, $i \in \mathbb{N}$, for ease of notation. Because $\tilde{P}_1, \ldots, \tilde{P}_m, \ldots$ is an exchangeable sequence of random variables, it exists a random variable Z with distribution function F_Z such that the joint distribution of $\tilde{P}_1, \ldots, \tilde{P}_m$ is for any fixed $m \in \mathbb{N}$ given by

$$F_{\tilde{P}_1,\ldots,\tilde{P}_m}(p_1, \ldots, p_m) = \int F_{\tilde{P}_1|Z=z}(p_1) \times \cdots \times F_{\tilde{P}_m|Z=z}(p_m) dF_Z(z), \qquad (14)$$

see Olshen (1974) and equation (3.1) of Kingman (1978). Moreover, assuming that $Z \in (0, \infty)$ with probability 1, we obtain for any $i \in \mathbb{N}$ from Marshall and Olkin (1988), p. 834, that

$$p_i = F_{\tilde{P}_i}(p_i) = \int \exp\left(-z\psi^{-1}(p_i)\right) dF_Z(z),$$

where ψ denotes the Laplace transform of Z, i. e., $\psi(t) = \mathbb{E}[\exp(-tZ)]$.

Theorem 3.3 establishes a connection between the finite-dimensional marginal distributions of H_0-exchangeable p-value sequences and Archimedean copulae.

Theorem 3.3. *Assume that the elements in the infinite sequence $(P_i)_{i\in\mathbb{N}}$ are absolutely continuous and H_0-exchangeable. Furthermore, let the following two assumptions be valid.*

(i) The random variable Z from (14) takes values in $(0, \infty)$ with probability 1.
(ii) It holds

$$F_{\tilde{P}_i|Z=z}(p_i) = \exp\left(-z\psi^{-1}(p_i)\right), z \in (0, \infty). \tag{15}$$

Then, for any m,

$$\mathbf{p} = (p_1, \ldots, p_m)^\top \mapsto \psi\left(\sum_{i=1}^{m} \psi^{-1}(p_i)\right)$$

is a copula of $P_1, \ldots, P_m$, where $\psi(t) = \mathbb{E}[\exp(-tZ)]$.

The final result of this section is an immediate consequence of Theorem 3.3 and Corollary 3.1.

Corollary 3.3. *Under the assumptions of Theorem 3.3, it holds:*

a) Any m-dimensional marginal distribution of the sequence $(P_i)_{i \in \mathbb{N}}$ possesses the MTP$_2$ property, $m \geq 2$.
b) The linear step-up test φ^{LSU}, applied to $p_1, \ldots, p_m$, controls the FDR at level q.

4. Examples: Parametric copula families

In this section, we apply the theoretical results of Section 3 to several parametric families of Archimedean copulae. We present computer simulations to validate our findings. To this end, for convenience, we consider the following model for the p-values $(P_i : 1 \leq i \leq m)$.

$$P_i = \begin{cases} U_i & \text{for} \quad i = 1, \ldots, m_0, \\ \Phi\left(\mu_i + \Phi^{-1}(U_i)\right) & \text{for} \quad i = m_0 + 1, \ldots, m, \end{cases} \tag{16}$$

where $\mathbf{U} = (U_1, \ldots, U_m)^\top$ denotes a vector of marginally uniformly distributed random variables following the corresponding Archimedean copula, and Φ denotes the cdf of the univariate standard normal distribution. These p-values correspond to marginal test problems

$$H_i : \mu_i = 0 \qquad \text{versus} \qquad K_i : \mu_i < 0 \quad \text{for} \quad i = 1, \ldots, m,$$

where μ_i denotes the mean of a marginally normally distributed test statistic $T_i \stackrel{d}{=} \mu_i + \Phi^{-1}(U_i)$ with unit variance. It is noted that the p-values from (16) satisfy condition (D2). Moreover, the joint distribution of $\mathbf{P} = (P_1, \ldots, P_m)^\top$ is determined by the same Archimedean copula as for $\mathbf{U}$, because each P_i is an isotonic transformation of U_i. The parameter ϑ is given by $(\mu_1, \ldots, \mu_m)^\top$. Notice that $\mathrm{DU}_{m_0,m}$ is a limiting case of model (16) for $\mu_i = -\infty$ for all $m_0 + 1 \leq i \leq m$.

Below, we apply the results of Section 3 to the independence copula, the Clayton copula, and the Gumbel-Hougaard copula. We assess how strongly the sharper upper bound for the FDR of φ^{LSU} derived in Theorem 3.1 deviates from the traditional upper bound $m_0 q/m$. Furthermore, we investigate the difference between this sharper upper bound and the true value of the FDR of φ^{LSU}. Finally, we compare the empirical power of φ^{LSU} and its improved version where q is replaced by $q^{\text{adj.}}$ from (7).

In case of the Clayton copula and the Gumbel-Hougaard copula, results are obtained by means of Monte Carlo simulations. To this end, pseudo-random vectors which behave like independent relizations of $\mathbf{U}$ have been generated by utilizing the `R` functions `rcopula.clayton` and `rcopula.gumbel`, respectively, from the package `QRM`. The values of the μ_i have been drawn independently from UNI$[-4.5, -2]$ in each Monte Carlo repetition, because φ^{LSU} has non-trivial power in this regime; see, e. g., Section 4 of Dickhaus (2013).

4.1. Independence copula

The generator of the independence copula is given by $\psi(x) = \exp(-x)$. Substituting $\psi^{-1}(x) = -\ln(x)$ in (10), we get

$$\gamma_{min} = \min_{k\in\{1,\dots,m\}} \left\{ \frac{\exp\left(\ln\left(kq/m\right)\right)}{kq/m} \right\} = 1$$

and, hence,

$$\forall\vartheta \in \Theta : FDR_{\vartheta,\psi}(\varphi^{LSU}) = \frac{m_0(\vartheta)}{m} q$$

under the assumption of independence. This result is in line with the previous finding reported in (2).

4.2. Clayton copula

The generator of the Clayton copula is given by

$$\psi(x) = (1 + \eta x)^{-1/\eta}, \quad \eta \in (0, \infty),$$

leading to $\psi^{-1}(x) = (x^{-\eta} - 1)/\eta$ and to the probability density function (pdf)

$$f_Z(z) = \frac{1}{\eta} f_{\Gamma(1/\eta,1)}\left(z/\eta\right) = \frac{1}{\Gamma\left(1/\eta\right)} \eta^{-1/\eta} z^{1/\eta-1} \exp\left(-z/\eta\right)$$

of Z, where Γ denotes Euler's gamma function and $f_{\Gamma(\alpha,\beta)}$ the pdf of the gamma distribution with shape parameter $\alpha \in (0, \infty)$ and scale parameter $\beta \in (0, \infty)$. For the Clayton copula, z^* is given by

$$z^* = \frac{\log m}{\eta^{-1}\left(\left(q/m\right)^{-\eta} - q^{-\eta}\right)} = \frac{\eta \log m}{\left(m/q\right)^{\eta} - \left(1/q\right)^{\eta}}.$$

In Figure 1, we plot $F_Z(z^*)$ as a function of η for $m = 20$ and $q = 0.05$. It is worth mentioning that the Clayton copula converges to the independence copula for $\eta \to 0$. In this case we get $z^* \to 1$ and $f_Z(z^*)$ tends to the Dirac

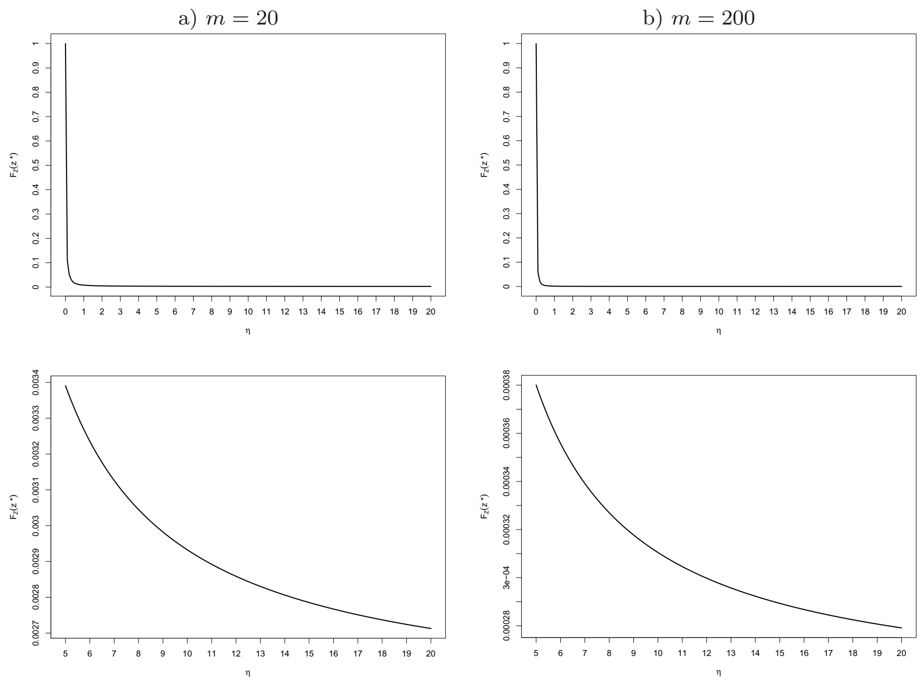

FIG 1. *The value $F_Z(z^*)$ as a function of η for $m \in \{20, 200\}$ and $q = 0.05$ under the assumption of a Clayton copula. The graphs in the lower panel are zoomed.*

delta function concentrated in 1. As a result, we observe that $F_Z(z^*) \to 1$ as $\eta \to 0$ and the FDR of φ^{LSU} approaches $m_0 q/m$. As η increases, $F_Z(z^*)$ steeply decreases and takes values very close to zero for large values of η. Consequently, it is expected that the FDR of φ^{LSU} is close to $m_0 q/m$ for large values of η, too. For η of moderate size, however, the FDR of φ^{LSU} can be much smaller than $m_0 q/m$. This is shown in Figure 2 below and discussed in detail there.

The quantity γ_{min} for the Clayton copula is calculated by

$$\begin{aligned}\gamma_{min} &= 1 - \eta^{-1} \int_0^{z^*} \frac{\exp\left(-z\psi^{-1}(q/m)\right)}{q/m} f_{\Gamma(1/\eta,1)}(z/\eta)\, dz \\ &\quad + \eta^{-1} \int_0^{z^*} \frac{\exp\left(-z\psi^{-1}(q)\right)}{q} f_{\Gamma(1/\eta,1)}(z/\eta)\, dz \\ &= 1 - I_1^C + I_2^C,\end{aligned}$$

where

$$\begin{aligned}I_1^C &= \frac{\eta^{-1/\eta}}{\Gamma(1/\eta)} \frac{m}{q} \int_0^{z^*} z^{1/\eta - 1} \exp\left(-\frac{z}{\eta}\left(\left(\frac{m}{q}\right)^{\eta} - 1\right) - \frac{z}{\eta}\right) dz \\ &= \frac{\eta^{-1/\eta}}{\Gamma(1/\eta)} \frac{m}{q} \int_0^{z^*} z^{1/\eta - 1} \exp\left(-\frac{z}{\eta}\left(\frac{m}{q}\right)^{\eta}\right) dz\end{aligned}$$

a) $m = 20$ b) $m = 200$

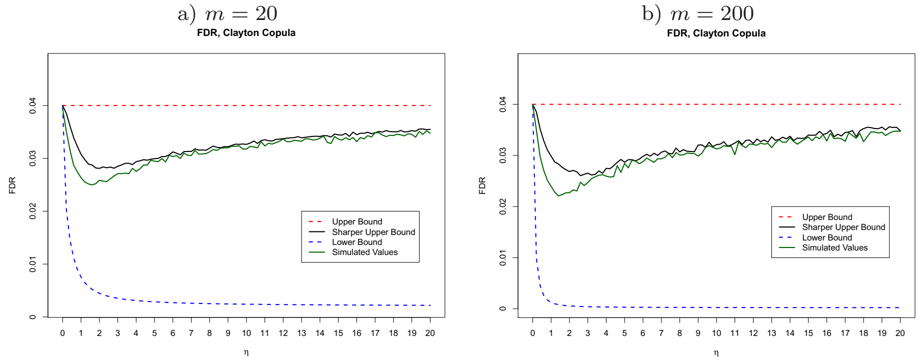

FIG 2. *Lower bound (dashed blue line), upper bound (dashed red line), the sharper upper bound (solid black line), and simulated values of the FDR of φ^{LSU} (solid green line) as functions of η for a Clayton copula. We put $m \in \{20, 200\}$, $q = 0.05$, and $m_0 = 0.8m$. Simulated values are based on 10^5 independent pseudo realizations of Z. The sharper upper bound was calculated under $DU_{m_0,m}$, while simulated data for the solid green line follow the model specified in (16).*

$$
\begin{aligned}
&= F_{\Gamma(1/\eta,\eta^{-1}(m/q)^\eta)}(z^*) = F_{\Gamma(1/\eta,1)}(\eta^{-1}(m/q)^\eta z^*) \\
&= F_{\Gamma(1/\eta,1)}\left(\frac{m^\eta \ln m}{m^\eta - 1}\right)
\end{aligned}
$$

and, similarly,

$$
I_2^C = F_{\Gamma(1/\eta,\eta^{-1}(1/q)^\eta)}(z^*) = F_{\Gamma(1/\eta,1)}(\eta^{-1}(1/q)^\eta z^*) = F_{\Gamma(1/\eta,1)}\left(\frac{\ln m}{m^\eta - 1}\right).
$$

Hence, from Theorem 3.2 we get for all $\vartheta \in \Theta$ that

$$
FDR_{\vartheta,\eta}(\varphi^{LSU}) \geq \frac{m_0 q}{m}\left(1 + F_{\Gamma(1/\eta,1)}\left(\frac{\ln m}{m^\eta - 1}\right) - F_{\Gamma(1/\eta,1)}\left(\frac{m^\eta \ln m}{m^\eta - 1}\right)\right).
$$

Next, we discuss the sharper upper bound for the FDR in the case of Clayton copulae in detail. As outlined in the discussion around Theorem 3.1, we consider DU configurations in connection with maximization of $\Delta(m,\cdot,\eta)$ over $\vartheta \in \Theta$. Due to the Dirac-distribution in 0 of the m_1 p-values corresponding to false nulls, the sharper upper bound for the FDR of φ^{LSU} is then obtained by means of (see (9))

$$
\begin{aligned}
\Delta(m, DU_{m_0,m}, \eta) = 1 - \sum_{k=m_1+1}^{m-1} \mathbb{E}\Bigg[&\left(\frac{\exp\left(-Z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-Z\psi^{-1}(q_k)\right)}{q_k}\right) \\
&\times \left(G_k^1(Z) - G_k^1(z_k^*)\right) \mathbf{1}_{[z_k^*,\infty)}(Z)\Bigg],
\end{aligned}
$$

where z_k^* is given in (6). The random set $D_{\mathbf{Y};k}^{(1,z)}$ the probability of which is given by $G_k^1(z)$ can under $\mathrm{DU}_{m_0,m}$ equivalently be expressed as

$$
D_{\mathbf{Y};k}^{(1,z)} = \left\{\exp\left(-z\psi^{-1}\left(q_{k+1}\right)\right) \leq Y_{(k)}^{(1)}, \ldots, \exp\left(-z\psi^{-1}\left(q_m\right)\right) \leq Y_{(m-1)}^{(1)}\right\}.
$$

This follows from the fact that $Y^{(1)}_{(k)}, \ldots, Y^{(1)}_{(m-1)}$ almost surely correspond to p-values associated with true null hypotheses, i. e.,

$$F_{P^{(1)}_{(k)}}(x) = \cdots = F_{P^{(1)}_{(m-1)}}(x) = x.$$

Moreover, since each of the $Y^{(1)}_{(k)}, \ldots, Y^{(1)}_{(m-1)}$ is obtained by the same isotonic transformation from the corresponding element in the sequence $P^{(1)}_{(k)}, \ldots, P^{(1)}_{(m-1)}$, we get that $Y^{(1)}_{(k)}, \ldots, Y^{(1)}_{(m-1)}$ are the order statistics of independent and identically UNI$[0,1]$-distributed random variables. Hence, the probabilities $G^1_k(z) = \mathbb{P}_{\vartheta,\eta}(D^{(1,z)}_{\mathbf{Y};k})$ for $k \in \{m_1+1, \ldots, m-1\}$ can be calculated recursively, for instance by making use of Bolshev's recursion (see, e. g., Shorack and Wellner (1986), p. 366).

In general, Bolshev's recursion is defined in the following way. Let $0 \le a_1 \le a_2 \le \cdots \le a_n \le 1$ be real constants and let $U_{(1)} \le U_{(2)} \le \cdots \le U_{(n)}$ be the order statistics of independent and identically UNI$[0,1]$-distributed random variables. We let $\bar{P}_n(a_1, \ldots, a_n) = P(a_1 \le U_{(1)}, \ldots, a_n \le U_{(n)})$. Then, the probability $\bar{P}_n(a_1, \ldots, a_n)$ is calculated recursively by

$$\bar{P}_n(a_1, \ldots, a_n) = 1 - \sum_{j=1}^{n} \binom{n}{j} a_j^j \bar{P}_{n-j}(a_{j+1}, \ldots, a_n). \tag{17}$$

Application of (17) with $n = m_0 - 1$ and

$$a_j = \begin{cases} 0 & \text{for} \quad j \in \{1, \ldots, k - m_1 - 1\} \\ \exp\left(-z\psi^{-1}\left(q_{j+m_1+1}\right)\right) & \text{for} \quad j \in \{k - m_1, \ldots, m_0 - 1\} \end{cases}$$

for $k \in \{m_1+1, \ldots, m-1\}$ as well as numerical integration with respect to the distribution of Z over $[z^*_k, \infty]$ lead to a numerical approximation of the sharper upper bound for the FDR of φ^{LSU} under Dirac-uniform configurations.

In Figure 2 we present the lower bound (dashed blue line), the upper bound (dashed red line), the sharper upper bound (solid black line), and the simulated values of the FDR of φ^{LSU} (solid green line) as a function of the parameter $\eta \in [0, 20]$ of a Clayton copula. Larger values for η are not considered, since it is straightforward to show that $\text{FDR}_{\vartheta,\eta}(\varphi^{LSU})$ tends to $m_0 q/m$ when $\eta \to \infty$. In this limiting case the p-values are totally dependent, such that effectively only one single test is performed; cf. the discussion around Figures 2 and 3 in Finner, Dickhaus and Roters (2007). This is also in line with the discussion around our Figure 1. We put $m \in \{20, 200\}$, $q = 0.05$, and $m_0 = 0.8m$. The p-values for deriving the solid green lines have been generated following (16). The simulated values are obtained by using 10^5 independent repetitions. We observe that the FDR of φ^{LSU} starts at $m_0 q/m = 0.04$ for $\eta = 0$ and decreases to a minimum of approximately 0.025 at $\eta \approx 1.6$ for $m = 20$ and to 0.022 at $\eta \approx 1.4$ for $m = 200$. This value is much smaller than the nominal level q, offering some room for improvement of φ^{LSU} for a broad range of values of η. After reaching its minimum, the FDR of φ^{LSU} increases and tends to 0.04 as η increases. This

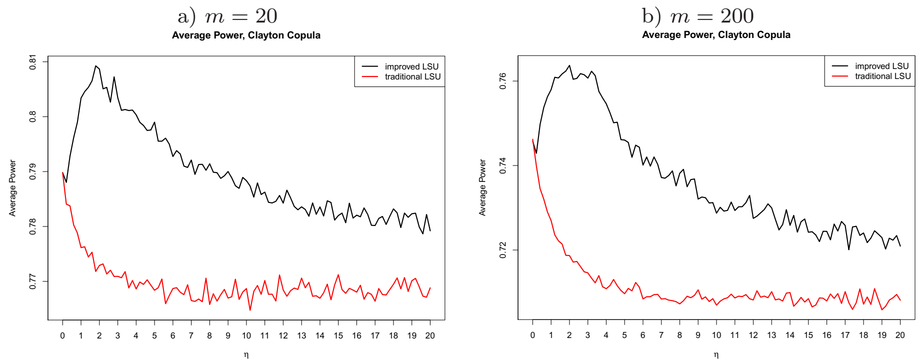

FIG 3. *Average power for φ^{LSU} (solid red line) and the improved φ^{LSU} (solid black line) as functions of η for a Clayton copula under the model specified in (16). We put $m \in \{20, 200\}$, $q = 0.05$, and $m_0 = 0.8m$. Simulated values are based on 10^5 independent pseudo realizations of Z.*

behavior of the FDR of φ^{LSU} is expected from the values of $F_Z(z^*)$, as discussed around Figure 1.

In contrast to the "classical" upper bound, the sharper upper bound reproduces the behavior of the simulated FDR values very well. It provides a good approximation of the true values of the FDR of φ^{LSU} for all considered values of η. In particular, it is much smaller than the "classical" upper bound for moderate values of η. Consequently, application of the sharper upper bound can be used to improve the power of the multiple testing procedure by adjusting the nominal value of q depending on η. This is quantified in Figure 3 where the average power calculated under (16) is plotted for different values of η in case of φ^{LSU} and its improved version employing the adjusted nominal FDR level $q^{\text{adj.}}$ from (7). The average power considered here is defined as the empirical counterpart of

$$\text{power}_{\vartheta,\eta}(\varphi^{LSU}) = \frac{\mathbb{E}_{\vartheta,\eta}[R_m - V_m]}{\max(m_1, 1)}$$

over the Monte Carlo repetitions; cf. Definition 1.4 in Dickhaus (2014). In contrast to the average power of φ^{LSU} which fluctuates slightly around 0.77, the average power of the improved φ^{LSU} increases to 0.81 for $\eta \approx 2$ and then slowly drops as η increases.

It is also remarkable that the difference between the sharper upper bound and the corresponding simulated FDR-values is not large. In contrast, the empirical standard deviations of the sharper upper bound (over repeated simulations) are about five times smaller than the corresponding ones for the simulated values of the FDP of φ^{LSU}, see Figure 4. While these standard deviations are always smaller than 0.03 for the sharper upper bound, they are around 0.15 for almost all of the considered values of η in case of the simulated FDP-values. Finally, we note that the lower bound seems not to be informative in this particular model class. It is close to zero even for moderate values of η.

a) $m = 20$ b) $m = 200$

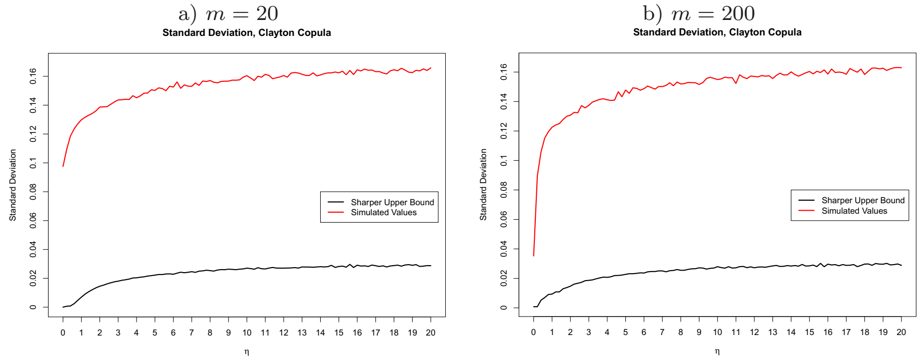

FIG 4. *Empirical standard deviations of the sharper upper bound (solid black line), and of* $FDP_{\vartheta,\eta}(\varphi^{LSU})$ *(solid red line) as functions of the parameter* η *of a Clayton copula. We put* $m \in \{20, 200\}$, $q = 0.05$, *and* $m_0 = 0.8m$. *Simulated values are based on* 10^5 *independent pseudo realizations of* Z.

4.3. Gumbel-Hougaard copula

The generator of the Gumbel-Hougaard copula is given by

$$\psi(x) = \exp\left(-x^{1/\eta}\right), \quad \eta \geq 1, \tag{18}$$

which leads to $\psi^{-1}(x) = (-\ln x)^\eta$ and a stochastic representation

$$Z \stackrel{d}{=} \left(\cos\left(\frac{\pi}{2\eta}\right)\right)^\eta Z_0, \quad \eta > 1, \tag{19}$$

for Z, where the random variable Z_0 has a stable distribution with index of stability $1/\eta$ and unit skewness. The cdf of Z_0 is given by (cf. Chambers, Mallows and Stuck (1976), p. 341)

$$\begin{aligned} F_{Z_0}(z) &= \frac{1}{\pi}\int_0^\pi \exp\left(-z^{-1/(\eta-1)}a(v)\right) dv \quad \text{with} \\ a(v) &= \frac{\sin\left((1-\eta)v/\eta\right)\left(\sin(v/\eta)\right)^{1/(\eta-1)}}{(\sin v)^{\eta/(\eta-1)}}, \quad v \in (0,\pi). \end{aligned}$$

Although (19) in connection with F_{Z_0} characterizes the distribution of Z completely, the integral representation of F_{Z_0} may induce numerical issues with respect to implementation. Somewhat more convenient from this perspective is the following result. Namely, Kanter (1975) obtained a stochastic representation of Z_0, given by

$$Z_0 = (a(U)/W)^{\eta-1}, \tag{20}$$

where U and W are stochastically independent, W is standard exponentially distributed and $U \sim \mathrm{UNI}(0,\pi)$. We used (20) for simulating Z_0 and, consequently, Z.

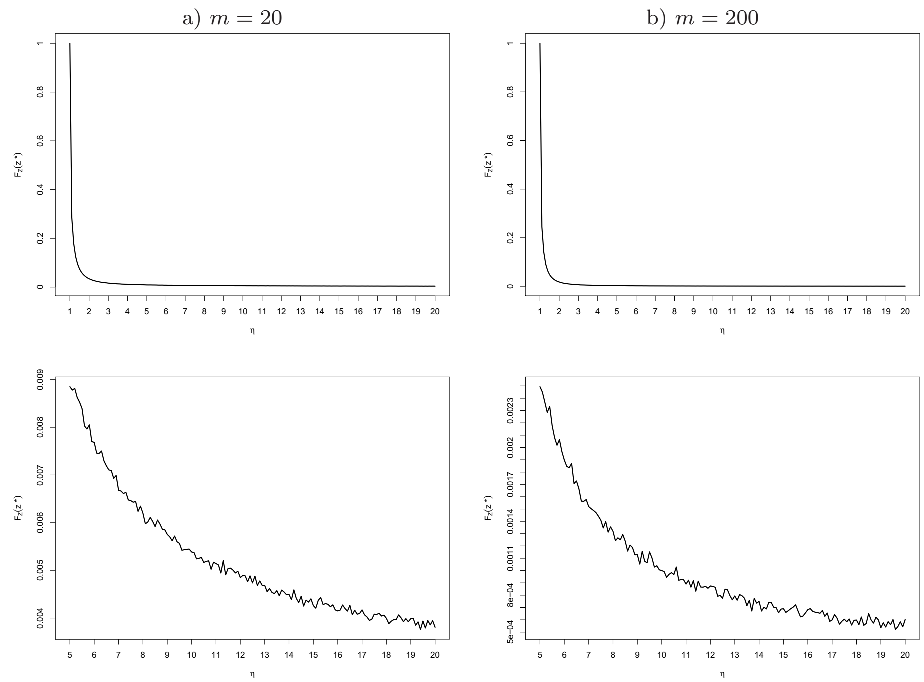

Fig 5. *The value $F_Z(z^*)$ as a function of η for $m \in \{20, 200\}$ and $q = 0.05$ under the assumption of a Gumbel-Hougaard copula. The graph was obtained via simulations by generating 10^6 independent pseudo realizations of Z according to (19) and (20). The graphs in the lower panel are zoomed.*

For the Gumbel-Hougaard copula we get

$$z^* = \frac{\ln m}{\left(-\ln \frac{q}{m}\right)^{\eta} - (-\ln q)^{\eta}} = \frac{\ln m}{\left(\ln \frac{m}{q}\right)^{\eta} - \left(\ln \frac{1}{q}\right)^{\eta}}.$$

In Figure 5, we plot $F_Z(z^*)$ as a function of η for $m = 20$ and $q = 0.05$. A similar behavior as in the case of the Clayton copula is present. If $\eta = 1$ then the Gumbel-Hougaard copula coincides with the independence copula. Hence, $F_Z(z^*) = 1$ and, consequently, the FDR of φ^{LSU} is equal to $m_0 q/m$ in this case. As η increases, $F_Z(z^*)$ decreases and it approaches 0 for larger values of η. Hence, $\mathrm{FDR}_{\vartheta,\eta}(\varphi^{LSU})$ tends to $m_0 q/m$ as η becomes considerably large. For moderate values of η, $\mathrm{FDR}_{\vartheta,\eta}(\varphi^{LSU})$ can again be much smaller than $m_0 q/m$, in analogy to the situation in models with Clayton copulae.

Recall from (11) that

$$\gamma_{min} = 1 - \mathbb{E}\left[g_1(Z)\mathbf{1}_{[0,z^*]}(Z)\right], \tag{21}$$

where $g_1(Z) = g(\psi^{-1}(q/m)|Z) - g(\psi^{-1}(q)|Z)$. For the Gumbel-Hougaard copula, we obtain

a) $m = 20$ b) $m = 200$

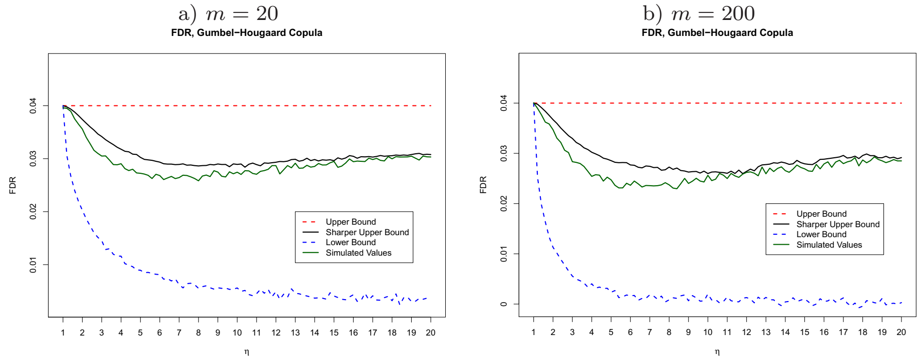

FIG 6. *Lower bound (dashed blue line), upper bound (dashed red line), the sharper upper bound (solid black line), and simulated values of the FDR of φ^{LSU} (solid green line) as functions of the parameter η of a Gumbel-Hougaard copula. We put $m \in \{20, 200\}$, $q = 0.05$, and $m_0 = 0.8m$. Simulated values are based on 10^5 independent pseudo realizations of Z. The sharper upper bound was calculated under $DU_{m_0,m}$, while simulated data for the solid green line follow the model specified in* (16).

$$\begin{aligned} g_1(Z) &= \frac{\exp\left(-Z\psi^{-1}\left(q/m\right)\right)}{q/m} - \frac{\exp\left(-Z\psi^{-1}\left(q\right)\right)}{q} \\ &= \frac{\exp\left(-Z\left(\ln\frac{m}{q}\right)^{\eta}\right)}{q/m} - \frac{\exp\left(-Z\left(\ln\frac{1}{q}\right)^{\eta}\right)}{q}. \end{aligned}$$

The expectation in (21) cannot be calculated analytically. However, it can easily be approximated with Monte Carlo simulations by applying the stochastic representations (19) and (20) for any fixed $\eta > 1$. This leads to a numerical value on the left-hand side of the chain of inequalities

$$\frac{m_0 q}{m}\left(1 - \mathbb{E}\left[g_1(Z)\mathbf{1}_{[0,z^*]}(Z)\right]\right) \leq \mathrm{FDR}_{\vartheta,\eta}(\varphi^{LSU}) \leq \frac{m_0 q}{m}. \tag{22}$$

The sharper upper bound from Theorem 3.1 can be calculated by using Bolshev's recursion similarly to the discussion around (17), but here with ψ as in (18). Figure 6 displays the lower bound (dashed blue line), the upper bound (dashed red line), the sharper upper bound (solid black line), and simulated values of $\mathrm{FDR}_{\vartheta,\eta}(\varphi^{LSU})$ (solid green line) as functions of η. Again, we choose $m \in \{20, 200\}$, $q = 0.05$, and $m_0 = 0.8m$. The p-values are generated according to model (16), as in the case of Clayton copulae. The simulated values were obtained by generating 10^5 independent pseudo realizations of Z.

Similarly to the case of the Clayton copula, the curve of simulated FDR values has a U-shape. It starts at $m_0 q/m = 0.04$ and drops to its minimum of approximately 0.026 for values of η around 8.0 in case of $m = 20$ and to 0.023 at η around 8.4 in case of $m = 200$. For such values of η, the green curve is considerably below the classical upper bound of 0.04. In contrast, the sharper upper bound gives a much tighter approximation of the simulated FDR values

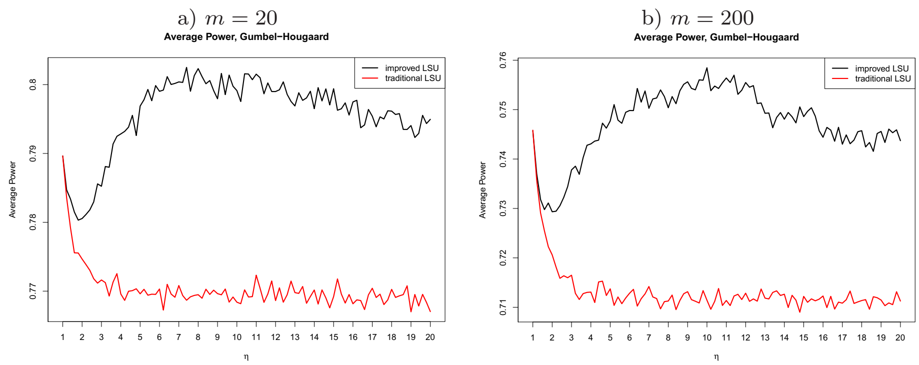

FIG 7. *Average power for φ^{LSU} (solid red line) and the improved φ^{LSU} (solid black line) as functions of η of a Gumbel-Hougaard copula under the model specified in* (16)*. We put $m \in \{20, 200\}$, $q = 0.05$, and $m_0 = 0.8m$. Simulated values are based on 10^5 independent pseudo realizations of Z.*

in such cases and reproduces the U-shape over the entire range of values for the parameter η of the Gumbel-Hougaard copula. As a result, its application can be used to improve power by adjusting the nominal value of q and thereby increasing the probability to detect false null hypotheses. In Figure 7, the average power of φ^{LSU} and its improved version are compared under model (16). As in the case of Clayton copulae, an improvement of about 3% in average power is present for both values of m. Moreover, it is again noted that the empirical standard deviations of the sharper upper bound are much smaller than those of the simulated values of the FDP (see Figure 8). The lower bound from (22) (corresponding to the dashed blue curve in Figure 6) has been obtained by approximating the expectation in (21) via simulations. As in the case of the

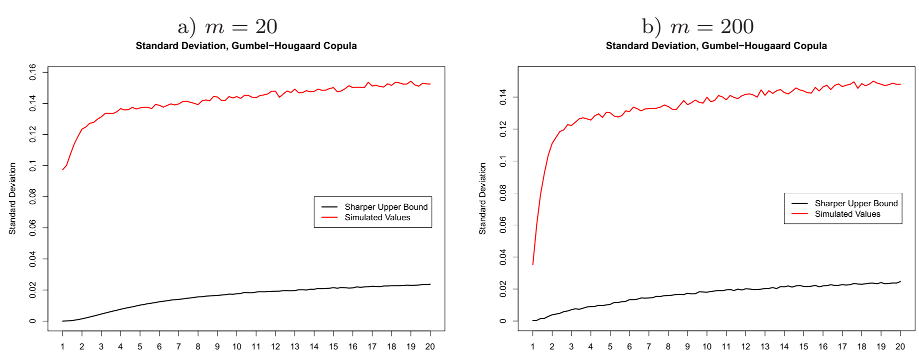

FIG 8. *Empirical standard deviations of the sharper upper bound (solid black line), and of $FDP_{\vartheta,\eta}(\varphi^{LSU})$ (solid red line) as functions of the parameter η of a Gumbel-Hougaard copula. We put $m \in \{20, 200\}$, $q = 0.05$, and $m_0 = 0.8m$. Simulated values are based on 10^5 independent pseudo realizations of Z.*

Clayton copula, the lower bound is not too informative for the model class that we have considered here.

5. Empirical copula calibration

In the previous section we studied the influence of the copula parameter η on the FDR of φ^{LSU} under several parametric families of Archimedean copulae. It turned out that adapting φ^{LSU} to the degree of dependency in the data by adjusting the nominal value of q based on the sharper upper bound from Theorem 3.1 is a promising idea, because the unadjusted procedure may lead to a considerable non-exhaustion of q, cf. Figures 2 and 6. Due to the decision rule of a step-up test, this also entails suboptimal power properties of φ^{LSU} when applied "as is" to models with Archimedean p-value copulae.

In practice, however, often the copula parameter itself is an unknown quantity. Hence, the outlined adaptation of q typically requires some kind of pre-estimation of η before multiple testing is performed. Although this is not in the main focus of the present work, we therefore outline possibilities for estimating η and for quantifying the uncertainty of the estimation in this section.

One class of procedures relies on resampling, namely via the parametric bootstrap or via permutation techniques if $H_1, \ldots, H_m$ correspond to marginal two-sample problems. Pollard and van der Laan (2004) provided an extensive comparison of both approaches and argued that the permutation method reproduces the correct null distribution only under some conditions. However, if these conditions are met, the permutation approach is often superior to bootstrapping (see also Westfall and Young (1993) and Meinshausen, Maathuis and Bühlmann (2011)). Furthermore, it is essential to keep in mind that both bootstrap and permutation-based methods estimate the distribution of the vector $\mathbf{P}$ under the global null hypothesis H_0. Hence, the assumption that η does not depend on ϑ is an essential prerequisite for the applicability of such resampling methods for estimating η. Notice that the latter assumption is an informal description of the "subset pivotality" condition introduced by Westfall and Young (1993). The resampling methods developed by Dudoit and van der Laan (2008) can dispense with subset pivotality in special model classes, but for the particular task of estimating the copula parameter this assumption seems indispensable.

Estimation of η and uncertainty quantification of the estimation based on resampling is generally performed by applying a suitable estimator $\hat{\eta}$ to the re-(pseudo) samples. In the context of Archimedean copulae the two most widely applied estimation procedures are the maximum likelihood method (see, e. g. Joe (2005), Hofert, Mächler and McNeil (2012)) and the method of moments (e. g., the "realized copula" approach by Fengler and Okhrin (2012)).

Hofert, Mächler and McNeil (2012) considered the estimation of the parameter of an Archimedean copula with known margins by the maximum likelihood approach. To this end, they derived analytic expressions for the derivatives of the copula generator for several families of Archimedean copulae, as well as formulas for the corresponding score functions. Using these results and assuming a regular model, an elliptical asymptotic confidence region for the copula parameter η can be obtained by applying general limit theorems for maximum likelihood estimators (see Hofert, Mächler and McNeil (2012) for details and the calculations for different types of Archimedean copulae).

In the context of the method of moments, Kendall's tau is often considered. For a bivariate Archimedean copula with generator ψ of marginally UNI$[0,1]$-distributed variates P_1 and P_2, it is given by

$$\begin{aligned}\tau_{P_1,P_2} &= 4\int_0^1\int_0^1 F_{(P_1,P_2)}(u,v)dF_{(P_1,P_2)}(u,v)-1\\ &= 1-4\int_0^{\psi^{-1}(0)} t[\psi'(t)]^2dt,\end{aligned} \tag{23}$$

cf. McNeil and Nešlehová (2009).

The right-hand side of (23) can analytically be calculated for some families of Archimedean copulae. For instance, for a Clayton copula with parameter η it is given by $\tau(\eta)=\eta/(2+\eta)$, while it is equal to $\tau(\eta)=(\eta-1)/\eta$ for a Gumbel-Hougaard copula with parameter η (see Nelsen (2006), p. 163–164). Based on such moment equations, Fengler and Okhrin (2012) suggested the "realized copula" method for empirical calibration of a one-dimensional parameter η of an m-variate Archimedean copula. The method considers all $m(m-1)/2$ distinct pairs of the m underlying random variables, replaces the population versions of $\tau(\eta)$ by the corresponding sample analogues, and finally aggregates the resulting $m(m-1)/2$ estimates in an appropriate manner. More specifically, consider the functions $g_{ij}(\eta)=\hat{\tau}_{ij}-\tau(\eta)$ for $1\le i<j\le m$ and define $\mathbf{g}(\eta)=(g_{ij}(\eta):1\le i<j\le m)^\top$, where $\hat{\tau}_{ij}$ is the sample estimator of Kendall's tau (see, e. g., Nelsen (2006), Section 5.1.1). The resulting estimator for η is then obtained by

$$\hat{\eta}=\arg\min_{\eta}\left\{\mathbf{g}(\eta)^\top\mathbf{W}\mathbf{g}(\eta)\right\}$$

for an appropriate weight matrix $\mathbf{W}\in\mathbb{R}^{\binom{m}{2}\times\binom{m}{2}}$. An application of the realized copula method to resampled p-values generated by permutations in the context of multiple testing for differential gene expression has been demonstrated by Dickhaus and Gierl (2013). Multivariate extensions of Kendall's tau and central limit theorems for the sample versions have been derived by Genest, Nešlehová and Ben Ghorbal (2011). These results can be used for uncertainty quantification of the moment estimation of η by constructing asymptotic confidence regions. For more details about various estimation procedures for copula parameters and their applications to multiple test problems we defer the reader to Stange, Bodnar and Dickhaus (2013).

6. Discussion

We have derived a sharper upper bound for the FDR of φ^{LSU} in models with Archimedean copulae. This bound can be used to prove that φ^{LSU} controls the FDR for this type of multivariate p-value distributions, a result which is in line with the findings of Benjamini and Yekutieli (2001) and Sarkar (2002). Since certain models with H_0-exchangeable p-values fall into this class at least asymptotically (see Theorem 3.3), our findings complement those of Finner, Dickhaus and Roters (2007) who investigated infinite sequences of H_0-exchangeable p-values in Gaussian models. While our general results in Section 3 qualitatively extend the theory, our results in Section 4 regarding Clayton and Gumbel-Hougaard copulae are quantitatively very much in line with the findings for Gaussian and t-copulae reported by Finner, Dickhaus and Roters (2007). Namely, over a broad class of models with dependent p-values, the FDR of φ^{LSU} as a function of the dependency parameter has a U-shape and becomes smallest for medium strength of dependency among the p-values. This behavior can be exploited by adjusting q in order to adapt to η. We have presented an explicit adaptation scheme based on the upper bound from Theorem 3.1. To the best of our knowledge, this kind of adaptation is novel to FDR theory.

Recall that we have calibrated the improved version of φ^{LSU} by utilizing $q^{\text{adj.}}$ from (7) in Section 4. This choice was motivated by our goal to demonstrate how much gain in power is in principle possible by adjusting φ^{LSU} for the dependency structure among p-values. In practice, however, one has to resort to $q^{\text{adj.}}_{\min}$ from (8). For its calculation, a reasonable lower bound for m_0 is required, because it is easy to show that $\Delta(m, DU_{1,m}, \psi) = 1$. One way to obtain such a lower bound is to consider an additional adaptation to $m_0(\vartheta)$, for instance by pre-estimation as considered, for example, by Schweder and Spjøtvoll (1982) and Storey, Taylor and Siegmund (2004). Future research shall therefore aim at analyzing properties of their (and further) estimators under copula dependency.

It is beyond the scope of the present work to investigate which parametric class of copulae is appropriate for which kind of real-life application. Relatedly, the problem of model misspecification (i. e., quantification of the approximation error if the true model does not belong to the class with Archimedean p-value copulae and is approximated by the (in some suitable norm) closest member of this class) could not be addressed here, but is a challenging topic for future research. One particularly interesting issue in this direction is FDR control for finite sequences of H_0-exchangeable p-values.

Finally, we would like to mention that the empirical variance of the false discovery proportion was large in all our simulations, implying that the random variable $\text{FDP}_{\vartheta,\eta}(\varphi^{LSU})$ was not well concentrated around its expected value $\text{FDR}_{\vartheta,\eta}(\varphi^{LSU})$. This is a known effect for models with dependent p-values (see, e. g., Finner, Dickhaus and Roters (2007), Delattre and Roquain (2011), Blanchard et al. (2014)) and provokes the question if FDR control is a suitable criterion under dependency at all. Maybe more stringent in dependent models is control of the false discovery exceedance rate, meaning to design a multiple test φ ensuring that $\text{FDX}_{\vartheta,\eta}(\varphi) = \mathbb{P}_{\vartheta,\eta}(\text{FDP}_{\vartheta,\eta}(\varphi) > c) \leq \gamma$, for user-defined parameters c and γ. In any case, practitioners should be (made) aware of the fact that controlling the FDR with φ^{LSU} does not necessarily imply that the FDP for their particular experiment is small, at least if dependencies among $P_1, \ldots, P_m$ have to be assumed as it is typically the case in applications. In contrast, the empirical standard deviations of our proposed sharper upper bound are about five times smaller than the empirical standard deviations of the simulated values of the FDP of φ^{LSU}. This provides an additional (robustness) argument for the application of the results presented in Theorem 3.1 in practice.

7. Proofs

Proof of Theorem 3.1

Following Benjamini and Yekutieli (2001), an analytic expression for the FDR of φ^{LSU} is given by

$$\mathrm{FDR}_{\vartheta,\psi}(\varphi^{LSU}) = \sum_{i=1}^{m_0}\sum_{k=1}^{m}\frac{1}{k}\mathbb{P}_{\vartheta,\psi}\left\{A_k^{(i)}\right\},$$

where $A_k^{(i)} = \{P_i \le q_k \cap C_k^{(i)}\}$ denotes the event that k hypotheses are rejected one of which is H_i (a true null hypothesis) and $C_k^{(i)}$ is the event that $k-1$ hypotheses additionally to H_i are rejected. It holds that $(C_k^{(i)} : 1 \le k \le m)$ are disjoint and that $\bigcup_{k=1}^m C_k^{(i)} = [0,1]^{m-1}$.

Let $D_k^{(i)} = \bigcup_{j=1}^k C_j^{(i)}$ for $k = 1,\ldots,m$ denote the event that the number of rejected null hypotheses is at most k. In terms of $\mathbf{P}^{(i)}$ introduced in Theorem 3.1, the random set $D_k^{(i)}$ is given by

$$D_k^{(i)} = \{q_{k+1} \le P_{(k)}^{(i)},\ldots,q_m \le P_{(m-1)}^{(i)}\}.$$

Next, we prove that

$$\begin{aligned}\frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_k \cap D_k^{(i)}\right)}{q_k} \le{}& \frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_{k+1} \cap D_k^{(i)}\right)}{q_{k+1}} \\ &- \int_{z_k^*}^{\infty}\left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k}\right) \\ &\times (G_k^i(z) - G_k^i(z_k^*))dF_Z(z).\end{aligned} \tag{24}$$

To this end, we consider the function $\mathbf{T}$ introduced in Theorem 3.1, which transforms a possible realization of the original p-values $\mathbf{P}$ into a realization of $\mathbf{Y}$ for $Z = z$, where $\mathbf{Y} = (Y_1,\ldots,Y_m)^\top$ and Z are as in (4). Because each component of this multivariate transformation is a monotonically increasing function which fully covers the interval $[0,1]$, the resulting transformation bijectively transforms the set $[0,1]^m$ into itself. Let $C_{\mathbf{Y};k}^{(i,z)}$ and $D_{\mathbf{Y};k}^{(i,z)}$ denote the images of the sets $C_k^{(i)}$ and $D_k^{(i)}$ under $\mathbf{T}$ for given $Z = z$. Then

(a) $C_{\mathbf{Y};k}^{(i,z)}$ are disjoint, i. e., $C_{\mathbf{Y};k_1}^{(i,z)} \cap C_{\mathbf{Y};k_2}^{(i,z)} = \emptyset$ for $1 \le k_1 \ne k_2 \le m$,
(b) $D_{\mathbf{Y};k}^{(i,z)} = \bigcup_{j=1}^k C_{\mathbf{Y};j}^{(i,z)}$,
(c) $D_{\mathbf{Y};m}^{(i,z)} = \bigcup_{j=1}^m C_{\mathbf{Y};j}^{(i,z)} = [0,1]^{m-1}$.

Statements (a) - (c) follow directly from the facts that each T_j is a monotonically increasing function and $\mathbf{T}$ is a one-to-one transformation with image equal to $[0,1]^m$. Moreover, we obtain

$$D_{\mathbf{Y};k}^{(i,z)} = \left\{ \forall k \le j \le m-1 : Y_{(j)}^{(i)} \ge \exp\left(-z\psi^{-1}\left(F_{P_{(j)}^{(i)}}(q_{j+1}) \right) \right) \right\},$$

where $\mathbf{Y}^{(i)}$ is the $(m-1)$-dimensional vector obtained from $\mathbf{Y} = (Y_1, \ldots, Y_m)^T$ by deleting Y_i. The last equality shows that $D_{\mathbf{Y};k}^{(i,z_1)} \subseteq D_{\mathbf{Y};k}^{(i,z_2)}$ for $z_1 \le z_2$ and, hence, that G_k^i, given by $G_k^i(z) = \mathbb{P}_{\vartheta,\psi}(D_{\mathbf{Y};k}^{(i,z)})$, is an increasing function in z.

Returning to (24), we obtain

$$\begin{aligned}
&\frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_{k+1} \cap D_k^{(i)}\right)}{q_{k+1}} - \frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_k \cap D_k^{(i)}\right)}{q_k} \\
&= \int \left(\frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_{k+1} \cap D_k^{(i)} | Z = z\right)}{q_{k+1}} \right. \\
&\qquad\qquad \left. - \frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_k \cap D_k^{(i)} | Z = z\right)}{q_k} \right) dF_Z(z) \\
&= \int \left(\frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_{k+1} | Z = z\right) \mathbb{P}_{\vartheta,\psi}\left(D_k^{(i)} | Z = z\right)}{q_{k+1}} \right. \\
&\qquad\qquad \left. - \frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_k | Z = z\right) \mathbb{P}_{\vartheta,\psi}\left(D_k^{(i)} | Z = z\right)}{q_k} \right) dF_Z(z) \\
&= \int \left(\frac{\mathbb{P}_{\vartheta,\psi}\left(Y_i \le \exp\left(-z\psi^{-1}(q_{k+1})\right)\right)}{q_{k+1}} \right. \\
&\qquad\qquad \left. - \frac{\mathbb{P}_{\vartheta,\psi}\left(Y_i \le \exp\left(-z\psi^{-1}(q_k)\right)\right)}{q_k} \right) \mathbb{P}_{\vartheta,\psi}\left(D_{\mathbf{Y};k}^{(i,z)}\right) dF_Z(z) \\
&= \int \left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k} \right) G_k^i(z) dF_Z(z). \qquad (25)
\end{aligned}$$

Next, we analyze the difference under the last integral. It holds that

$$\begin{aligned}
&\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_k} \\
&= \exp\left(-\log q_{k+1} - z\psi^{-1}(q_{k+1})\right) - \exp\left(-\log q_k - z\psi^{-1}(q_k)\right) \\
&= \exp\left(-\log q_k - z\psi^{-1}(q_k)\right) \\
&\quad \times \left(\exp\left(-\log q_{k+1} + \log q_k - z\psi^{-1}(q_{k+1}) + z\psi^{-1}(q_k)\right) - 1\right).
\end{aligned}$$

The last expression is nonnegative if and only if

$$-\log q_{k+1} + \log q_k - z\psi^{-1}(q_{k+1}) + z\psi^{-1}(q_k) \geq 0.$$

Hence, for $z \geq z_k^*$ with z_k^* given in (6), the function under the integral in (25) is positive and for $z \leq z_k^*$ it is negative. Application of this result leads to

$$\begin{aligned}
& \frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \leq q_{k+1} \cap D_k^{(i)}\right)}{q_{k+1}} - \frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \leq q_k \cap D_k^{(i)}\right)}{q_k} \\
= \quad & \int_0^{z_k^*} \left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k} \right) G_k^i(z) dF_Z(z) \\
& + \int_{z_k^*}^{\infty} \left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k} \right) G_k^i(z) dF_Z(z) \\
\geq \quad & G_k^i(z_k^*) \int_0^{z_k^*} \left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k} \right) dF_Z(z) \\
& + \int_{z_k^*}^{\infty} \left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k} \right) G_k^i(z) dF_Z(z).
\end{aligned}$$

Because of

$$\begin{aligned}
& \int \left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k} \right) dF_Z(z) \\
= \quad & \int \frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} dF_Z(z) - \int \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k} dF_Z(z) \\
= \quad & \frac{\psi\left(\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\psi\left(\psi^{-1}(q_k)\right)}{q_k} = \frac{q_{k+1}}{q_{k+1}} - \frac{q_k}{q_k} = 0
\end{aligned}$$

we get

$$\begin{aligned}
& \int_0^{z_k^*} \left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k} \right) dF_Z(z) \\
= \quad & -\int_{z_k^*}^{\infty} \left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k} \right) dF_Z(z)
\end{aligned}$$

and, consequently,

$$\begin{aligned}
& \frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \leq q_{k+1} \cap D_k^{(i)}\right)}{q_{k+1}} - \frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \leq q_k \cap D_k^{(i)}\right)}{q_k} \\
\geq \quad & -G_k^i(z_k^*) \int_{z_k^*}^{\infty} \left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k} \right) dF_Z(z)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{z_k^*}^{\infty}\left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}}-\frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k}\right)G_k^i(z)dF_Z(z)\\
= \quad &\int_{z_k^*}^{\infty}\left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}}-\frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k}\right) \qquad (26)\\
&\qquad\qquad \times (G_k^i(z)-G_k^i(z_k^*))dF_Z(z),
\end{aligned}
$$

which is obviously positive since both the differences under the integral in (26) are positive. This completes the proof of (24).

Using (24), we get for all $1 \le k \le m-1$ that

$$
\begin{aligned}
&\frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_k \cap D_k^{(i)}\right)}{q_k}+\frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_{k+1} \cap C_{k+1}^{(i)}\right)}{q_{k+1}}\\
\le \quad &\frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_{k+1} \cap D_k^{(i)}\right)}{q_{k+1}}+\frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_{k+1} \cap C_{k+1}^{(i)}\right)}{q_{k+1}}\\
&-\int_{z_k^*}^{\infty}\left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}}-\frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k}\right)\\
&\qquad\qquad \times (G_k^i(z)-G_k^i(z_k^*))dF_Z(z)\\
= \quad &\frac{\mathbb{P}_{\vartheta,\psi}\left(P_i \le q_{k+1} \cap D_{k+1}^{(i)}\right)}{q_{k+1}}\\
&-\int_{z_k^*}^{\infty}\left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}}-\frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k}\right)\\
&\qquad\qquad \times (G_k^i(z)-G_k^i(z_k^*))dF_Z(z)
\end{aligned}
$$

and, consequently, starting with $D_1^{(i)} = C_1^{(i)}$ and proceeding step-by-step for all $k \le m-1$, we obtain

$$
\begin{aligned}
&\sum_{k=1}^{m}\frac{\mathbb{P}_{\vartheta,\psi}\left\{P_i \le q_{k+1} \cap C_k^{(i)}\right\}}{q_k} \le \frac{\mathbb{P}_{\vartheta,\psi}\left\{P_i \le q_m \cap D_m^{(i)}\right\}}{q_m}\\
&-\sum_{k=1}^{m-1}\int_{z_k^*}^{\infty}\left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}}-\frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k}\right)\\
&\qquad\qquad \times (G_k^i(z)-G_k^i(z_k^*))dF_Z(z)\\
= \quad & 1-\sum_{k=1}^{m-1}\int_{z_k^*}^{\infty}\left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}}-\frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k}\right)\\
&\qquad\qquad \times (G_k^i(z)-G_k^i(z_k^*))dF_Z(z).
\end{aligned}
$$

Hence,

$$
\mathrm{FDR}_{\vartheta,\psi}(\varphi^{LSU}) = \sum_{i=1}^{m_0}\sum_{k=1}^{m}\frac{1}{k}\mathbb{P}_{\vartheta,\psi}\left\{A_k^{(i)}\right\}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{m_0} \frac{q}{m} \sum_{k=1}^{m} \frac{\mathbb{P}_{\vartheta,\psi}\left\{P_i \le q_{k+1} \cap C_k^{(i)}\right\}}{q_k} \\
&\le \sum_{i=1}^{m_0} \frac{q}{m} - \sum_{i=1}^{m_0} \frac{q}{m} \sum_{k=1}^{m-1} \int_{z_k^*}^{\infty} \left(\frac{\exp\left(-z\psi^{-1}(q_{k+1})\right)}{q_{k+1}} - \frac{\exp\left(-z\psi^{-1}(q_k)\right)}{q_k} \right) \\
&\qquad \times (G_k^i(z) - G_k^i(z_k^*)) dF_Z(z) \\
&= \frac{m_0}{m} q \Delta(m, \vartheta, \psi),
\end{aligned}
$$

where $\Delta(m, \vartheta, \psi)$ is defined in Theorem 3.1. This completes the proof of the theorem. $\square$

Proof of Theorem 3.2

Straightforward calculation yields

$$
\begin{aligned}
\mathrm{FDR}_{\vartheta,\psi}(\varphi^{LSU}) &= \sum_{i=1}^{m_0} \sum_{k=1}^{m} \frac{1}{k} \int \mathbb{P}_{\vartheta,\psi}\left\{A_k^{(i)} | Z = z\right\} dF_Z(z) \\
&= \sum_{i=1}^{m_0} \sum_{k=1}^{m} \frac{1}{k} \int \mathbb{P}_{\vartheta,\psi}\left\{P_i \le q_k | Z = z\right\} \\
&\qquad \times \mathbb{P}_{\vartheta,\psi}\left\{C_k^{(i)} | Z = z\right\} dF_Z(z) \\
&= \sum_{i=1}^{m_0} \frac{q}{m} \sum_{k=1}^{m} \int \frac{\mathbb{P}_{\vartheta,\psi}\left\{P_i \le q_k | Z = z\right\}}{q_k} \\
&\qquad \times \mathbb{P}_{\vartheta,\psi}\left\{C_k^{(i)} | Z = z\right\} dF_Z(z),
\end{aligned}
$$

where the random events $A_k^{(i)}$ and $C_k^{(i)}$ are defined in the proof of Theorem 3.1. Moreover, making use of the notation $C_{\mathbf{Y};k}^{(i,z)}$ introduced in the proof of Theorem 3.1, we can express $\mathrm{FDR}_{\vartheta,\psi}(\varphi^{LSU})$ by

$$
\begin{aligned}
\mathrm{FDR}_{\vartheta,\psi}(\varphi^{LSU}) &= \sum_{i=1}^{m_0} \frac{q}{m} \sum_{k=1}^{m} \int \frac{\mathbb{P}_{\vartheta,\psi}\left\{P_i \le \exp\left(-z\psi^{-1}\left(F_{P_i}\left(\frac{kq}{m}\right)\right)\right)\right\}}{q_k} \\
&\qquad \times \mathbb{P}_{\vartheta,\psi}\left\{C_{\mathbf{Y};k}^{(i,z)}\right\} dF_Z(z) \\
&= \sum_{i=1}^{m_0} \frac{q}{m} \sum_{k=1}^{m} \int \frac{\exp\left(-z\psi^{-1}\left(q_k\right)\right)}{q_k} \mathbb{P}_{\vartheta,\psi}\left\{C_{\mathbf{Y};k}^{(i,z)}\right\} dF_Z(z) \\
&\ge \sum_{i=1}^{m_0} \frac{q}{m} \sum_{k=1}^{m} \int \min_{k \in \{1,\ldots,m\}} \left\{ \frac{\exp\left(-z\psi^{-1}\left(q_k\right)\right)}{q_k} \right\} \\
&\qquad \times \mathbb{P}_{\vartheta,\psi}\left\{C_{\mathbf{Y};k}^{(i,z)}\right\} dF_Z(z),
\end{aligned}
$$

where the latter inequality follows from $Y_\ell \sim \mathrm{UNI}[0,1]$ for all $1 \le \ell \le m$ and the fact that each H_i is a true null hypothesis.

Now, it holds that

$$\begin{aligned}
\mathrm{FDR}_{\vartheta,\psi}(\varphi^{LSU}) &\geq \sum_{i=1}^{m_0} \frac{q}{m} \int \min_{k\in\{1,\ldots,m\}} \left\{ \frac{\exp\left(-z\psi^{-1}\left(\frac{kq}{m}\right)\right)}{kq/m} \right\} \\
&\qquad\qquad \times \sum_{k=1}^{m} \mathbb{P}_{\vartheta,\psi}\left\{C_{\mathbf{Y};k}^{(i,z)}\right\} dF_Z(z) \\
&= \sum_{i=1}^{m_0} \frac{q}{m} \int \min_{k\in\{1,\ldots,m\}} \left\{ \frac{\exp\left(-z\psi^{-1}\left(\frac{kq}{m}\right)\right)}{kq/m} \right\} dF_Z(z) \\
&= \int \min_{k\in\{1,\ldots,m\}} \left\{ \frac{\exp\left(-z\psi^{-1}\left(\frac{kq}{m}\right)\right)}{kq/m} \right\} dF_Z(z) \sum_{i=1}^{m_0} \frac{q}{m} \\
&= \int \min_{k\in\{1,\ldots,m\}} \left\{ \frac{\exp\left(-z\psi^{-1}\left(\frac{kq}{m}\right)\right)}{kq/m} \right\} dF_Z(z) \frac{m_0 q}{m}.
\end{aligned}$$

This completes the proof of the theorem. □

Proof of Lemma 3.1

From the assertion of Theorem 3.2 we conclude that the lower bound for the FDR of φ^{LSU} under the assumption of an Archimedean copula crucially depends on the extreme points of the function $g(\cdot|z)$ given in (12) for $x \in \{\psi^{-1}(q/m), \psi^{-1}(2q/m), \ldots, \psi^{-1}(q)\}$. If for all $z > 0$ the minimum of $g(x|z)$ is always attained for the same index k^* (say), then $\gamma_{min} = 1$ and together with Theorem 3.1 we get $\mathrm{FDR}_{\vartheta,\psi}(\varphi^{LSU}) = m_0(\vartheta)q/m$. This follows directly from the identity

$$\int \frac{\exp\left(-z\psi^{-1}\left(k^*q/m\right)\right)}{k^*q/m} dF_Z(z) = \frac{\psi\left(\psi^{-1}\left(k^*q/m\right)\right)}{k^*q/m} = 1.$$

However, the latter holds true only in some specific cases. To obtain a more explicit constant $\gamma_{min}(\psi)$ in the general case, we notice that, due to the analytic properties of ψ, there exists a point z^* such that $g(\psi^{-1}(q)|z) < g(\psi^{-1}(q/m)|z)$ for $z < z^*$ and $g(\psi^{-1}(q)|z) > g(\psi^{-1}(q/m)|z)$ for $z > z^*$. The point z^* is obtained as the solution of

$$\begin{aligned}
0 &= g\left(\psi^{-1}(q)|z\right) - g\left(\psi^{-1}(q/m)|z\right) \\
&= \frac{\exp\left(-z\psi^{-1}\left(q\right)\right)}{q} - \frac{\exp\left(-z\psi^{-1}\left(q/m\right)\right)}{q/m} \\
&= \frac{1}{q}\left(\exp\left(-z\psi^{-1}\left(q\right)\right) - \exp\left(\log m - z\psi^{-1}\left(\frac{q}{m}\right)\right)\right) \\
&= \frac{\exp\left(-z\psi^{-1}\left(q\right)\right)}{q}\left(1 - \exp\left(\log m + z\left(\psi^{-1}\left(q\right) - \psi^{-1}\left(\frac{q}{m}\right)\right)\right)\right),
\end{aligned}$$

which leads to

$$z^* = \frac{\log m}{\psi^{-1}(q/m) - \psi^{-1}(q)}.$$

Next, we analyze the function $x \mapsto g(x|z)$ for given z. For its derivative with respect to x, it holds that

$$g'(x|z) = -\frac{\exp(-zx)}{(\psi(x))^2}(z\psi(x) + \psi'(x)).$$

Setting this expression to zero, we get that any extreme point of $g(\cdot|z)$ satisfies

$$z\psi(x) + \psi'(x) = 0. \tag{27}$$

Let x_z be a solution of (27). Then, the second derivative of $g(\cdot|z)$ at x_z is given by

$$g''(x_z|z) = -\frac{\exp(-zx_z)}{(\psi(x_z))^2}(z\psi'(x_z) + \psi''(x_z)). \tag{28}$$

Substituting (27) with $x = x_z$ in (28), we obtain

$$\begin{aligned} g''(x_z|z) &= -\frac{\exp(-zx_z)}{(\psi(x_z))^2}\left(-\frac{(\psi'(x_z))^2}{\psi(x_z)} + \psi''(x_z)\right) \\ &= -\frac{\exp(-zx_z)}{(\psi(x_z))^3}\left(\psi(x_z)\psi''(x_z) - (\psi'(x_z))^2\right) \end{aligned}$$

and application of the Cauchy-Schwarz inequality leads to

$$\begin{aligned} \psi(x_z)\psi''(x_z) &= \int \exp(-zx_z)\, dF_Z(z) \int z^2 \exp(-zx_z)\, dF_Z(z) \\ &\geq \left(\int z \exp(-zx_z)\, dF_Z(z)\right)^2 = (\psi'(x_z))^2. \end{aligned}$$

This proves that $g''(x_z|z) \leq 0$ if x_z is an extreme point of $g(x_z|z)$. Thus, any such x_z is a maximum and the minimum in (10) is attained at $\psi^{-1}(q)$ for $z \leq z^*$ as well as at $\psi^{-1}(q/m)$ for $z \geq z^*$. □

Proof of Corollary 3.2

We consider the quantity γ_{min} itself. It holds that $1 \geq \gamma_{min} \geq \underline{\gamma}_{min}$, where

$$\underline{\gamma}_{min} = 1 - \min\left\{\int_0^{z^*} \sup_{z\in[0,z^*]} h(z)dF_Z(z), \int_{z^*}^{\infty} \sup_{z\in[z^*,\infty]} (-h(z))dF_Z(z)\right\}$$

with

$$\begin{aligned} h(z) &= g\left(\psi^{-1}(q/m)|z\right) - g\left(\psi^{-1}(q)|z\right) \\ &= \frac{\exp\left(-z\psi^{-1}\left(\frac{q}{m}\right)\right)}{q/m} - \frac{\exp\left(-z\psi^{-1}(q)\right)}{q}, \end{aligned}$$

because

$$\int_0^{z^*} h(z)dF_Z(z) = -\int_{z^*}^{\infty} h(z)dF_Z(z). \tag{29}$$

However, both of the integrals in (29) can be bounded by different values. To see this, we study the behavior of the function $z \mapsto h(z)$. It holds that

$$\begin{aligned}
h'(z) &= -\psi^{-1}\left(\frac{q}{m}\right)\frac{\exp\left(-z\psi^{-1}\left(\frac{q}{m}\right)\right)}{q/m} + \psi^{-1}(q)\frac{\exp\left(-z\psi^{-1}(q)\right)}{q} \\
&= \psi^{-1}(q)\frac{\exp\left(-z\psi^{-1}(q)\right)}{q} \\
&\quad \times \left(1 - \exp\left(\log m + \log\frac{\psi^{-1}\left(\frac{q}{m}\right)}{\psi^{-1}(q)} + z\left(\psi^{-1}(q) - \psi^{-1}\left(\frac{q}{m}\right)\right)\right)\right).
\end{aligned}$$

Since ψ^{-1} is a non-increasing function, we get that there exists a unique minimum of $h(z)$ at

$$\underline{z}^* = \frac{\log m + \log\psi^{-1}(q/m) - \log\psi^{-1}(q)}{\psi^{-1}(q/m) - \psi^{-1}(q)} \geq z^*.$$

Consequently, we get

$$\begin{aligned}
\int_0^{z^*} \sup_{z\in[0,z^*]} h(z)dF_Z(z) &= \int_0^{z^*} h(0)dF_Z(z) = h(0)F_Z(z^*) \\
&= \frac{m-1}{q}F_Z(z^*), \\
\int_{z^*}^{\infty} \sup_{z\in[z^*,\infty]} (-h(z))dF_Z(z) &= \int_{z^*}^{\infty} h(\underline{z}^*)dF_Z(z) = h(\underline{z}^*)(1-F_Z(z^*)) \\
&= \frac{\exp\left(-\underline{z}^*\psi^{-1}(q)\right)}{q}\left(1 - \frac{\psi^{-1}(q)}{\psi^{-1}(q/m)}\right) \\
&\quad \times (1-F_Z(z^*)).
\end{aligned}$$

$\square$

Proof of Theorem 3.3

We plug (15) into (14) and obtain

$$\begin{aligned}
F_{\tilde{P}_1,\ldots,\tilde{P}_m}(p_1,\ldots,p_m) &= \int \prod_{i=1}^m \exp\left(-z\psi^{-1}(p_i)\right)dF_Z(z) \\
&= \int \exp\left(-z\sum_{i=1}^m \psi^{-1}\left(F_{\tilde{P}_i}(p_i)\right)\right)dF_Z(z) \\
&= \psi\left(\sum_{i=1}^m \psi^{-1}\left(F_{\tilde{P}_i}(p_i)\right)\right),
\end{aligned}$$

since the last integral is the Laplace transform of Z at $\sum_{i=1}^{m} \psi^{-1}(F_{\tilde{P}_i}(p_i))$. Noticing that $P_1, \ldots, P_m$ are obtained by componentwise increasing transformations of $\tilde{P}_1, \ldots, \tilde{P}_m$ we conclude the assertion. $\square$

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